

# Mathematical Toolkit Assignment 1

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1. (a)

$$\dim(A) = \text{rank}(A) + \text{null}(A)$$

$$n = m + \text{null}(A)$$

$$\text{null}(A) = n - m$$

(b)

$$\text{null}(A) = \dim(\ker(A))$$

Then,  $\ker(A)$  can have a basis  $B$ .

$$\text{i.e. } \forall \mathbf{v} \in \ker(A), \exists a_1, \dots, a_{n-m} \in \mathbb{F}_2, \mathbf{v} = a_1 \mathbf{b}_1 + \dots + a_{n-m} \mathbf{b}_{n-m} \ (\mathbf{b}_i \in B)$$

$\therefore$  The answer is  $2^{n-m}$ .

(c)

$$\forall \mathbf{x} \text{ s.t. } A\mathbf{x} = \mathbf{b}, A(\mathbf{x} - \mathbf{x}_0) = 0 \because A\mathbf{x}_0 = \mathbf{b}$$

$$\therefore \mathbf{x} - \mathbf{x}_0 \in \ker(A)$$

Then, choosing each element of  $\mathbf{x}$  carefully (1 or 0),  $\mathbf{x} - \mathbf{x}_0$  can be any element of  $\mathbb{F}_2^n$ .

$$\therefore \{\mathbf{x} - \mathbf{x}_0 | A\mathbf{x} = \mathbf{b}\} = \ker(A)$$

$\therefore \mathbf{x} - \mathbf{x}_0$  has  $2^{n-m}$  solutions.

$\therefore \mathbf{x}$  has  $2^{n-m}$  solutions.

2. (a)

$$\forall \mathbf{v}, f(c\mathbf{v} + (-c)\mathbf{v}) \geq \min\{f(\mathbf{v}), f(\mathbf{v})\}$$

$$f(\mathbf{0}_V) \geq f(\mathbf{v})$$

(b) Because every element  $\mathbf{v}_t \in V_t$  is in  $V$  by definition.

$$V_t \subseteq V$$

3.

$$\begin{aligned}
 p(x) &= x^2 + bx + c \\
 &= (x - r_1)(x - r_2) \\
 &= x^2 - (r_1 + r_2)x + r_1 r_2 \\
 \therefore b &= -r_1 - r_2, c = r_1 r_2
 \end{aligned}$$

I'm lost.

4.

$$\begin{aligned}
 \mu(P, Q) &= \text{degree}(PQ) \\
 &= \text{degree}(QP) \\
 &= \mu(Q, P) \\
 \mu(0, 0) &= \text{degree}(0) \\
 &= 0 \\
 \forall P \neq 0, \mu(P, P) &= \text{degree}(P^2) \\
 &= 2\text{degree}(P) \\
 &> 0 \\
 \mu(P + Q, R) &= \text{degree}((P + Q)R) \\
 &= \max\{\text{degree}(P), \text{degree}(Q)\} + \text{degree}(R) \\
 \mu(P, R) + \mu(Q, R) &= \max\{\text{degree}(P) + \text{degree}(R), \text{degree}(Q) + \text{degree}(R)\} \\
 &= \max\{\text{degree}(P), \text{degree}(Q)\} + \text{degree}(R) \\
 \therefore \mu(P + Q, R) &= \mu(P, R) + \mu(Q, R) \\
 c \in \mathbb{R}, \mu(cP, R) &= \text{degree}(cPR) \\
 &= \text{degree}(PR) \\
 &\neq c \cdot \text{degree}(P, R)
 \end{aligned}$$

$\therefore \mu(\cdot, R)$  is not a LT.

$\therefore \mu$  is not a IP.

5.

$$\begin{aligned}
 \alpha\beta\mathbf{x} &= \lambda\mathbf{x} \\
 \beta\alpha\beta\mathbf{x} &= \beta(\lambda\mathbf{x}) \\
 \beta\alpha(\beta\mathbf{x}) &= \lambda(\beta\mathbf{x})
 \end{aligned}$$

$\therefore \lambda$  is an eigenvalue of  $\beta\alpha$ .

6. (a)

$$\begin{aligned}
\varphi(\mathbf{v}) &= \lambda \mathbf{v} \\
\varphi(\mathbf{v}) &= \lambda \varphi(\mathbf{v}) \\
(\lambda - 1)\varphi(\mathbf{v}) &= 0 \\
\lambda = 1_{\mathbb{F}} \vee \varphi(\mathbf{v}) &= 0 \\
\lambda = 1_{\mathbb{F}} \vee \varphi(\mathbf{v}) &= 0_{\mathbb{F}} \mathbf{v} \\
\therefore \lambda &\in \{0_{\mathbb{F}}, 1_{\mathbb{F}}\}
\end{aligned}$$

(b) Let  $\varphi$  be a projection s.t.  $\forall \mathbf{v}, \varphi(\mathbf{v}) = 1$  and assume  $\varphi = \varphi^*$ .

$$\begin{aligned}
\forall \mathbf{v}, \mathbf{w} \in \mathbb{R} \text{ or } \mathbb{C} \text{ s.t. } \mathbf{v} &\neq \mathbf{w}, \\
\langle \varphi(\mathbf{v}), \mathbf{w} \rangle &= \langle \mathbf{v}, \varphi^*(\mathbf{w}) \rangle \\
\langle 1, \mathbf{w} \rangle &= \langle \mathbf{v}, 1 \rangle \\
\mathbf{v} &= \mathbf{w} \text{ } \nexists
\end{aligned}$$

$\therefore$  Not always  $\varphi = \varphi^*$

7. (a)

$$\begin{aligned}
\langle \varphi(\mathbf{v}), \mathbf{w} \rangle &= \langle \mathbf{v}, \varphi^*(\mathbf{w}) \rangle \\
\langle \varphi^*(\mathbf{w}), \mathbf{v} \rangle &= \langle \mathbf{w}, \varphi(\mathbf{v}) \rangle \\
\therefore (\varphi^*)^* &= \varphi
\end{aligned}$$

(b)

$$\begin{aligned}
\forall \mathbf{v} \in \ker(\varphi), \varphi(\mathbf{v}) &= 0 \\
\langle \varphi(\mathbf{v}), \mathbf{w} \rangle &= 0 \\
\langle \mathbf{v}, \varphi^*(\mathbf{w}) \rangle &= 0 \\
\mathbf{v} &\in (\text{im}(\varphi^*))^\perp \\
\forall \mathbf{v} \in (\text{im}(\varphi^*))^\perp, \forall \mathbf{w} \in W, \langle \mathbf{v}, \varphi^*(\mathbf{w}) \rangle &= 0 \\
\langle \varphi(\mathbf{v}), \mathbf{w} \rangle &= 0 \\
\varphi(\mathbf{v}) &= 0 \\
\mathbf{v} &\in \ker(\varphi) \\
\therefore \ker(\varphi) &= (\text{im}(\varphi^*))^\perp
\end{aligned}$$

(c)

$$\begin{aligned}
\forall \mathbf{w} \in \text{im}(\varphi), \exists \mathbf{v} \in V \text{ s.t. } \mathbf{w} = \varphi(\mathbf{v}), \forall \mathbf{w}' \in \ker(\varphi^*), \langle \mathbf{w}, \mathbf{w}' \rangle &= \langle \varphi(\mathbf{v}), \mathbf{w}' \rangle \\
&= \langle \mathbf{v}, \varphi^*(\mathbf{w}') \rangle \\
&= 0 \\
\mathbf{w} &\in (\ker(\varphi^*))^\perp
\end{aligned}$$

$$\begin{aligned}
\forall \mathbf{v} \in V, \forall \mathbf{w} \in \ker(\varphi^*), \varphi^*(\mathbf{w}) &= 0 \\
\langle \mathbf{v}, \varphi^*(\mathbf{w}) \rangle &= 0 \\
\therefore \text{im}(\varphi) &= (\ker(\varphi^*))^\perp
\end{aligned}$$

(d)

$$\begin{aligned}
\text{rank}(\varphi) &= \dim(\text{im}(\varphi)) \\
&= \dim((\ker(\varphi^*))^\perp) \\
&= \dim(W) - \dim(\ker(\varphi^*)) \\
&= \dim(\text{im}(\varphi^*)) \\
&= \text{rank}(\varphi^*)
\end{aligned}$$

(e) In discussion with Tomoki Tsujimura,

$$\begin{aligned}
\text{rank}(A) &= \text{rank}(A^*) \because \text{(d)} \\
\text{rank}_{\text{row}}(A) &= \text{rank}_{\text{row}}(A^*) \\
\text{rank}_{\text{row}}(A) &= \text{rank}_{\text{column}}(\overline{A}) \\
\forall n \in \mathbb{N}, \forall \mathbf{v}, \mathbf{w} \in \mathbb{C}^n \text{ s.t. } \langle \mathbf{v}, \mathbf{w} \rangle &= 0, \\
\overline{\langle \mathbf{v}, \mathbf{w} \rangle} &= 0 \\
\overline{\mathbf{v}^T \mathbf{w}} &= 0 \\
\overline{\mathbf{v}^T} \overline{\mathbf{w}} &= 0 \\
\therefore \text{rank}_{\text{row}}(A) &= \text{rank}_{\text{column}}(A)
\end{aligned}$$