

# Mathematical Toolkit Assignment 1

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1. (a)

$$\dim(A) = \text{rank}(A) + \text{null}(A)$$

$$n = m + \text{null}(A)$$

$$\text{null}(A) = n - m$$

(b)

$$\text{null}(A) = \dim(\ker(A))$$

Then,  $\ker(A)$  can have a basis  $B$ .

i.e.  $\forall \mathbf{v} \in \ker(A), \exists a_1, \dots, a_{n-m} \in \mathbb{F}_2, \mathbf{v} = a_1 \mathbf{b}_1 + \dots + a_{n-m} \mathbf{b}_{n-m} \ (\mathbf{b}_i \in B)$

$\therefore$  The answer is  $2^{n-m}$ .

(c)

$$\forall \mathbf{x} \text{ s.t. } A\mathbf{x} = \mathbf{b}, A(\mathbf{x} - \mathbf{x}_0) = 0 \because A\mathbf{x}_0 = \mathbf{b}$$

$$\therefore \mathbf{x} - \mathbf{x}_0 \in \ker(A)$$

Then, choosing each element of  $\mathbf{x}$  carefully (1 or 0),  $\mathbf{x} - \mathbf{x}_0$  can be any element of  $\mathbb{F}_2^n$ .

$$\therefore \{\mathbf{x} - \mathbf{x}_0 | A\mathbf{x} = \mathbf{b}\} = \ker(A)$$

$\therefore \mathbf{x} - \mathbf{x}_0$  has  $2^{n-m}$  solutions.

$\therefore \mathbf{x}$  has  $2^{n-m}$  solutions.

2. (a)

$$\forall \mathbf{v}, f(c\mathbf{v} + (-c)\mathbf{v}) \geq \min\{f(\mathbf{v}), f(\mathbf{v})\}$$

$$f(\mathbf{0}) \geq f(\mathbf{v})$$

(b)

$$\begin{aligned}
V_t &\subseteq V \text{ by definition} \\
\mathbf{0} &\in V_t \because \text{(a)} \\
\forall t \in [0, f(\mathbf{0})], \forall \mathbf{v}, \mathbf{w} \in V_t, \\
f(\mathbf{v} + \mathbf{w}) &\geq \min \{f(\mathbf{v}), f(\mathbf{w})\} \\
&\geq t \\
\therefore \mathbf{v} + \mathbf{w} &\in V_t \\
\text{and } \forall c \in \mathbb{F}, f(c\mathbf{v}) &\geq \min \{f(\mathbf{v}), f(\mathbf{0})\} \\
&\geq f(\mathbf{v}) \\
&\geq t \\
\therefore c \cdot \mathbf{v} &\in V_t \\
f(-\mathbf{v}) &\geq t \text{ as above} \\
\therefore -\mathbf{v} &\in V_t
\end{aligned}$$

$\therefore V_t$  is a subspace of  $V$ .

3.

$$\begin{aligned}
p(x) &= x^2 + bx + c \\
&= (x - r_1)(x - r_2) \\
&= x^2 - (r_1 + r_2)x + r_1r_2 \\
\therefore b &= -r_1 - r_2, c = r_1r_2 \\
\therefore \varphi_1(x_1, x_2, x_3) &= x_1 - (r_1 + r_2)x_2 + r_1r_2x_3 \\
\forall f \in \mathbb{R}^{\mathbb{N}} \text{ of an eigenvector of } \varphi_{left} &\text{ with an eigenvalue } \lambda, \\
(\varphi_2(f))(n) &= \varphi_1(f(n), f(n+1), f(n+2)) \\
&= f(n) - (r_1 + r_2)f(n+1) + r_1r_2f(n+2) \\
&= f(n) - (r_1 + r_2)(\varphi_{left}(f))(n) + r_1r_2(\varphi_{left}^2(f))(n) \\
&= f(n) - (r_1 + r_2)(\lambda f)(n) + r_1r_2(\lambda^2 f)(n) \\
\varphi_2(f) &= f - (r_1 + r_2)\lambda f + r_1r_2\lambda^2 f \\
&= (1 - \lambda r_1 - \lambda r_2 + \lambda^2 r_1 r_2)f \\
&= (1 - \lambda r_1)(1 - \lambda r_2)f
\end{aligned}$$

Let  $f_1, f_2 \in \mathbb{R}^{\mathbb{N}}$  be 2 eigenvectors of  $\varphi_{left}$  with eigenvalues  $r_1, r_2$  s.t.  $\forall i \in \{1, 2\}, f_i(n) = r_i^n$ .

$$\begin{aligned}
\varphi_2(f_1) &= \varphi_2(f_2) = 0 \\
\{f_1, f_2\} &\subseteq \ker(\varphi_2) \\
\dim(\ker(\varphi_2)) &\geq 2
\end{aligned}$$

4.

$$\begin{aligned}
\mu(P, Q) &= \text{degree}(PQ) \\
&= \text{degree}(QP) \\
&= \mu(Q, P) \\
\mu(0, 0) &= \text{degree}(0) \\
&= 0 \\
\forall P \neq 0, \mu(P, P) &= \text{degree}(P^2) \\
&= 2\text{degree}(P) \\
&> 0 \\
\mu(P + Q, R) &= \text{degree}((P + Q)R) \\
&= \max \{ \text{degree}(P), \text{degree}(Q) \} + \text{degree}(R) \\
\mu(P, R) + \mu(Q, R) &= \max \{ \text{degree}(P) + \text{degree}(R), \text{degree}(Q) + \text{degree}(R) \} \\
&= \max \{ \text{degree}(P), \text{degree}(Q) \} + \text{degree}(R) \\
\therefore \mu(P + Q, R) &= \mu(P, R) + \mu(Q, R) \\
c \in \mathbb{R}, \mu(cP, R) &= \text{degree}(cPR) \\
&= \text{degree}(PR) \\
&\neq c \cdot \text{degree}(P, R)
\end{aligned}$$

$\therefore \mu(\cdot, R)$  is not a LT.  
 $\therefore \mu$  is not a IP.

5.

$$\begin{aligned}
\alpha\beta\mathbf{x} &= \lambda\mathbf{x} \\
\beta\alpha\beta\mathbf{x} &= \beta(\lambda\mathbf{x}) \\
\beta\alpha(\beta\mathbf{x}) &= \lambda(\beta\mathbf{x})
\end{aligned}$$

$\therefore \lambda$  is an eigenvalue of  $\beta\alpha$ .

6. (a)

$$\begin{aligned}
\varphi(\mathbf{v}) &= \lambda\mathbf{v} \\
\varphi(\mathbf{v}) &= \lambda\varphi(\mathbf{v}) \\
(\lambda - 1)\varphi(\mathbf{v}) &= 0 \\
\lambda = 1_{\mathbb{F}} \vee \varphi(\mathbf{v}) &= 0 \\
\lambda = 1_{\mathbb{F}} \vee \varphi(\mathbf{v}) &= 0_{\mathbb{F}}\mathbf{v} \\
\therefore \lambda &\in \{0_{\mathbb{F}}, 1_{\mathbb{F}}\}
\end{aligned}$$

- (b) Let  $\varphi$  is a projection on  $\mathbb{F}^n = \mathbb{R}^n$  or  $\mathbb{C}^n$  over  $\mathbb{F}$ .  
 $\varphi$  is p.s.d.  $\therefore$  (a)

$$\begin{aligned}\exists \alpha : \mathbb{F}^n &\rightarrow \mathbb{F}^n \text{ s.t. } \varphi = \alpha^* \alpha \\ \forall \mathbf{v}, \mathbf{w} &\in \mathbb{F}^n, \\ \langle \varphi \mathbf{v}, \mathbf{w} \rangle &= \langle \alpha^* \alpha \mathbf{v}, \mathbf{w} \rangle \\ &= \langle \mathbf{v}, \alpha^* \alpha \mathbf{w} \rangle \\ &= \langle \mathbf{v}, \varphi^* \mathbf{w} \rangle \\ \therefore \varphi &= \varphi^*\end{aligned}$$

7. (a)

$$\begin{aligned}\langle \varphi(\mathbf{v}), \mathbf{w} \rangle &= \langle \mathbf{v}, \varphi^*(\mathbf{w}) \rangle \\ \langle \varphi^*(\mathbf{w}), \mathbf{v} \rangle &= \langle \mathbf{w}, \varphi(\mathbf{v}) \rangle \\ \therefore (\varphi^*)^* &= \varphi\end{aligned}$$

(b)

$$\begin{aligned}\forall \mathbf{v} \in \ker(\varphi), \varphi(\mathbf{v}) &= 0 \\ \langle \varphi(\mathbf{v}), \mathbf{w} \rangle &= 0 \\ \langle \mathbf{v}, \varphi^*(\mathbf{w}) \rangle &= 0 \\ \mathbf{v} &\in (\text{im}(\varphi^*))^\perp \\ \forall \mathbf{v} \in (\text{im}(\varphi^*))^\perp, \forall \mathbf{w} \in W, \langle \mathbf{v}, \varphi^*(\mathbf{w}) \rangle &= 0 \\ \langle \varphi(\mathbf{v}), \mathbf{w} \rangle &= 0 \\ \varphi(\mathbf{v}) &= 0 \\ \mathbf{v} &\in \ker(\varphi) \\ \therefore \ker(\varphi) &= (\text{im}(\varphi^*))^\perp\end{aligned}$$

(c)

$$\begin{aligned}\forall \mathbf{w} \in \text{im}(\varphi), \exists \mathbf{v} \in V \text{ s.t. } \mathbf{w} &= \varphi(\mathbf{v}), \forall \mathbf{w}' \in \ker(\varphi^*), \langle \mathbf{w}, \mathbf{w}' \rangle = \langle \varphi(\mathbf{v}), \mathbf{w}' \rangle \\ &= \langle \mathbf{v}, \varphi^*(\mathbf{w}') \rangle \\ &= 0 \\ \mathbf{w} &\in (\ker(\varphi^*))^\perp \\ \forall \mathbf{v} \in V, \forall \mathbf{w} \in \ker(\varphi^*), \varphi^*(\mathbf{w}) &= 0 \\ \langle \mathbf{v}, \varphi^*(\mathbf{w}) \rangle &= 0 \\ \therefore \text{im}(\varphi) &= (\ker(\varphi^*))^\perp\end{aligned}$$

(d)

$$\begin{aligned}\text{rank}(\varphi) &= \dim(\text{im}(\varphi)) \\ &= \dim((\ker(\varphi^*))^\perp) \\ &= \dim(W) - \dim(\ker(\varphi^*)) \\ &= \dim(\text{im}(\varphi^*)) \\ &= \text{rank}(\varphi^*)\end{aligned}$$

(e)

$$\begin{aligned}\text{rank}(A) &= \text{rank}(A^*) \because \text{(d)} \\ \text{rank}_{\text{row}}(A) &= \text{rank}_{\text{row}}(A^*) \\ \text{rank}_{\text{row}}(A) &= \text{rank}_{\text{column}}(\overline{A}) \\ \forall n \in \mathbb{N}, \forall \mathbf{v}, \mathbf{w} \in \mathbb{C}^n \text{ s.t. } \langle \mathbf{v}, \mathbf{w} \rangle &= 0, \\ \overline{\langle \mathbf{v}, \mathbf{w} \rangle} &= 0 \\ \overline{\mathbf{v}^T \mathbf{w}} &= 0 \\ \overline{\mathbf{v}}^T \overline{\mathbf{w}} &= 0 \\ \langle \overline{\mathbf{v}}, \overline{\mathbf{w}} \rangle &= 0 \\ \therefore \text{rank}_{\text{row}}(A) &= \text{rank}_{\text{column}}(A)\end{aligned}$$

## References

- [1] Madhur Tulsiani, Mathematical Toolkit - Autumn 2016,  
<http://ttic.uchicago.edu/~madhurt/courses/toolkit2016/index.html>
- [2] Eugenia Cheng, How to write proofs: a quick guide,  
<http://cheng.staff.shef.ac.uk/proofguide/proofguide.pdf>
- [3] Discussion with Tomoki Tsujimura