Mathematical Toolkit Assignment 1

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1. (a)

$$\begin{aligned} \dim(A) &= \operatorname{rank}(A) + \operatorname{null}(A) \\ n &= m + \operatorname{null}(A) \\ \operatorname{null}(A) &= n - m \end{aligned}$$

(b)

$$\operatorname{null}(A) = \dim(\ker(A))$$

Then, $\ker(A)$ can have a basis B.

i.e. $\forall \mathbf{v} \in \ker(A), \exists a_1, ..., a_{n-m} \in \mathbb{F}_2, \mathbf{v} = a_1\mathbf{b}_1 + \cdots + a_{n-m}\mathbf{b}_{n-m}$ ($\mathbf{b}_i \in B$)

$\therefore$ The answer is 2^{n-m}.

(c)

$$\begin{aligned} \forall \mathbf{x} \text{ s.t. } A\mathbf{x} = \mathbf{b}, A(\mathbf{x} - \mathbf{x}_0) = 0 \because A\mathbf{x}_0 = \mathbf{b} \\ \therefore\ \mathbf{x} - \mathbf{x}_0 \in \ker(A) \end{aligned}$$

Then, choosing each element of $\mathbf{x}$ carefully (1 or 0), $\mathbf{x} - \mathbf{x}_0$ can be any element of $\mathbb{F}_2^n$.

$$\therefore\ \{\mathbf{x} - \mathbf{x}_0 | A\mathbf{x} = \mathbf{b}\} = \ker(A)$$

$\therefore$ $\mathbf{x} - \mathbf{x}_0$ has 2^{n-m} solutions.
$\therefore$ $\mathbf{x}$ has 2^{n-m} solutions.

2. (a)

$$\begin{aligned} \forall \mathbf{v}, f(c\mathbf{v} + (-c)\mathbf{v}) &\geq \min\{f(\mathbf{v}), f(\mathbf{v})\} \\ f(\mathbf{0}) &\geq f(\mathbf{v}) \end{aligned}$$

(b)

$$\begin{aligned}
V_t &\subseteq V \text{ by definition} \\
\mathbf{0} &\in V_t \because \text{ (a)} \\
\forall t \in [0, f(\mathbf{0})], &\forall \mathbf{v}, \mathbf{w} \in V_t, \\
f(\mathbf{v}+\mathbf{w}) &\geq \min\{f(\mathbf{v}), f(\mathbf{w})\} \\
&\geq t \\
\therefore\ \mathbf{v}+\mathbf{w} &\in V_t \\
\text{and } \forall c \in \mathbb{F}, f(c\mathbf{v}) &\geq \min\{f(\mathbf{v}), f(\mathbf{0})\} \\
&\geq f(\mathbf{v}) \\
&\geq t \\
\therefore\ c \cdot \mathbf{v} &\in V_t \\
f(-\mathbf{v}) &\geq t \text{ as above} \\
\therefore\ -\mathbf{v} &\in V_t
\end{aligned}$$

$\therefore V_t$ is a subspace of V.

3.

$$\begin{aligned}
p(x) &= x^2 + bx + c \\
&= (x - r_1)(x - r_2) \\
&= x^2 - (r_1 + r_2)x + r_1 r_2 \\
\therefore\ b &= -r_1 - r_2, c = r_1 r_2 \\
\therefore\ \varphi_1(x_1, x_2, x_3) &= x_1 - (r_1 + r_2)x_2 + r_1 r_2 x_3 \\
\forall f \in \mathbb{R}^{\mathbb{N}} \text{ of an eigenvector of } &\varphi_{left} \text{ with an eigenvalue } \lambda\ , \\
(\varphi_2(f))(n) &= \varphi_1(f(n), f(n+1), f(n+2)) \\
&= f(n) - (r_1 + r_2)f(n+1) + r_1 r_2 f(n+2) \\
&= f(n) - (r_1 + r_2)(\varphi_{left}(f))(n) + r_1 r_2 (\varphi_{left}^2(f))(n) \\
&= f(n) - (r_1 + r_2)(\lambda f)(n) + r_1 r_2 (\lambda^2 f)(n) \\
\varphi_2(f) &= f - (r_1 + r_2)\lambda f + r_1 r_2 \lambda^2 f \\
&= (1 - \lambda r_1 - \lambda r_2 + \lambda^2 r_1 r_2) f \\
&= (1 - \lambda r_1)(1 - \lambda r_2) f
\end{aligned}$$

Let $f_1, f_2 \in \mathbb{R}^{\mathbb{N}}$ be 2 eigenvectors of φ_{left} with eignevalues r_1, r_2 s.t. $\forall i \in \{1, 2\}, f_i(n) = r_i^n$.

$$\begin{gathered}
\varphi_2(f_1) = \varphi_2(f_2) = 0 \\
\{f_1, f_2\} \subseteq \ker(\varphi_2) \\
\dim(\ker(\varphi_2)) \geq 2
\end{gathered}$$

4.

$$\begin{aligned}
\mu(P, Q) &= \text{degree}(PQ) \\
&= \text{degree}(QP) \\
&= \mu(Q, P) \\
\mu(0, 0) &= \text{degree}(0) \\
&= 0 \\
\forall P \neq 0, \mu(P, P) &= \text{degree}(P^2) \\
&= 2\text{degree}(P) \\
&> 0 \\
\mu(P + Q, R) &= \text{degree}((P + Q)R) \\
&= \max\{\text{degree}(P), \text{degree}(Q)\} + \text{degree}(R) \\
\mu(P, R) + \mu(Q, R) &= \max\{\text{degree}(P) + \text{degree}(R), \text{degree}(Q) + \text{degree}(R)\} \\
&= \max\{\text{degree}(P), \text{degree}(Q)\} + \text{degree}(R) \\
\therefore\ \mu(P + Q, R) &= \mu(P, R) + \mu(P, Q) \\
c \in \mathbb{R}, \mu(cP, R) &= \text{degree}(cPR) \\
&= \text{degree}(PR) \\
&\neq c \cdot \text{degree}(P, R)
\end{aligned}$$

$\therefore \mu(\cdot, R)$ is not a LT.
$\therefore \mu$ is not a IP.

5.

$$\begin{aligned}
\alpha\beta\mathbf{x} &= \lambda\mathbf{x} \\
\beta\alpha\beta\mathbf{x} &= \beta(\lambda\mathbf{x}) \\
\beta\alpha(\beta\mathbf{x}) &= \lambda(\beta\mathbf{x})
\end{aligned}$$

$\therefore \lambda$ is an eigenvalue of $\beta\alpha$.

6. (a)

$$\begin{aligned}
\varphi(\mathbf{v}) &= \lambda\mathbf{v} \\
\varphi(\mathbf{v}) &= \lambda\varphi(\mathbf{v}) \\
(\lambda - 1)\varphi(\mathbf{v}) &= 0 \\
\lambda = 1_{\mathbb{F}} &\vee \varphi(\mathbf{v}) = 0 \\
\lambda = 1_{\mathbb{F}} &\vee \varphi(\mathbf{v}) = 0_{\mathbb{F}}\mathbf{v} \\
\therefore\ \lambda &\in \{0_{\mathbb{F}}, 1_{\mathbb{F}}\}
\end{aligned}$$

(b) Let φ is a projection on V over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.
φ is p.s.d. $\because$ (a)
$\exists \alpha : V \to V$ s.t. $\varphi = \alpha^* \alpha$

$$\begin{aligned} \forall \mathbf{v}, \mathbf{w} \in V, \langle \varphi(\mathbf{v}), \mathbf{w} \rangle &= \langle \alpha^*(\alpha(\mathbf{v})), \mathbf{w} \rangle \\ &= \langle \mathbf{v}, \alpha^*(\alpha(\mathbf{w})) \rangle \\ &= \langle \mathbf{v}, \varphi^*(\mathbf{w}) \rangle \\ \therefore \ \varphi &= \varphi^* \end{aligned}$$

7. (a)

$$\begin{aligned} \langle \varphi(\mathbf{v}), \mathbf{w} \rangle &= \langle \mathbf{v}, \varphi^*(\mathbf{w}) \rangle \\ \langle \varphi^*(\mathbf{w}), \mathbf{v} \rangle &= \langle \mathbf{w}, \varphi(\mathbf{v}) \rangle \\ \therefore \ (\varphi^*)^* &= \varphi \end{aligned}$$

(b)

$$\begin{aligned} \forall \mathbf{v} \in \ker(\varphi), \varphi(\mathbf{v}) &= 0 \\ \langle \varphi(\mathbf{v}), \mathbf{w} \rangle &= 0 \\ \langle \mathbf{v}, \varphi^*(\mathbf{w}) \rangle &= 0 \\ \mathbf{v} &\in (\operatorname{im}(\varphi^*))^\perp \\ \forall \mathbf{v} \in (\operatorname{im}(\varphi^*))^\perp, \forall \mathbf{w} \in W, \langle \mathbf{v}, \varphi^*(\mathbf{w}) \rangle &= 0 \\ \langle \varphi(\mathbf{v}), \mathbf{w} \rangle &= 0 \\ \varphi(\mathbf{v}) &= 0 \\ \mathbf{v} &\in \ker(\varphi) \\ \therefore \ \ker(\varphi) &= (\operatorname{im}(\varphi^*))^\perp \end{aligned}$$

(c)

$$\begin{aligned} \forall \mathbf{w} \in \operatorname{im}(\varphi), \exists \mathbf{v} \in V \text{ s.t. } \mathbf{w} = \varphi(\mathbf{v}), \forall \mathbf{w}' \in \ker(\varphi^*), \langle \mathbf{w}, \mathbf{w}' \rangle &= \langle \varphi(\mathbf{v}), \mathbf{w}' \rangle \\ &= \langle \mathbf{v}, \varphi^*(\mathbf{w}') \rangle \\ &= 0 \\ \mathbf{w} &\in (\ker(\varphi^*))^\perp \\ \forall \mathbf{v} \in V, \forall \mathbf{w} \in \ker(\varphi^*), \varphi^*(\mathbf{w}) &= 0 \\ \langle \mathbf{v}, \varphi^*(\mathbf{w}) \rangle &= 0 \\ \therefore \ \operatorname{im}(\varphi) &= (\ker(\varphi^*))^\perp \end{aligned}$$

(d)

$$\begin{aligned}
\operatorname{rank}(\varphi) &= \dim(\operatorname{im}(\varphi)) \\
&= \dim((\ker(\varphi^*))^{\perp}) \\
&= \dim(W) - \dim(\ker(\varphi^*)) \\
&= \dim(im(\varphi^*)) \\
&= \operatorname{rank}(\varphi^*)
\end{aligned}$$

(e)

$$\begin{aligned}
\operatorname{rank}(A) = \operatorname{rank}(A^*) &\because \text{ (d)} \\
\operatorname{rank}_{\text{row}}(A) &= \operatorname{rank}_{\text{row}}(A^*) \\
\operatorname{rank}_{\text{row}}(A) &= \operatorname{rank}_{\text{column}}(\overline{A}) \\
\forall n \in \mathbb{N}, \forall \mathbf{v}, \mathbf{w} \in \mathbb{C}^n \text{ s.t. } \langle \mathbf{v}, \mathbf{w} \rangle &= 0, \\
\overline{\langle \mathbf{v}, \mathbf{w} \rangle} &= 0 \\
\overline{\mathbf{v}^T \overline{\mathbf{w}}} &= 0 \\
\overline{\mathbf{v}}^T \overline{\overline{\mathbf{w}}} &= 0 \\
\langle \overline{\mathbf{v}}, \overline{\mathbf{w}} \rangle &= 0 \\
\therefore \operatorname{rank}_{\text{row}}(A) &= \operatorname{rank}_{\text{column}}(A)
\end{aligned}$$

References

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[4] Discussion with Tomoki Tsujimura