

# Mathematical Toolkit Assignment 1

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1. (a)

$$\dim(A) = \text{rank}(A) + \text{null}(A)$$

$$n = m + \text{null}(A)$$

$$\text{null}(A) = n - m$$

(b)

$$\text{null}(A) = \dim(\ker(A))$$

Then,  $\ker(A)$  can have a basis  $B$ .

i.e.  $\forall \mathbf{v} \in \ker(A), \exists a_1, \dots, a_{n-m} \in \mathbb{F}_2, \mathbf{v} = a_1 \mathbf{b}_1 + \dots + a_{n-m} \mathbf{b}_{n-m} \ (\mathbf{b}_i \in B)$

$\therefore$  The answer is  $2^{n-m}$ .

(c)

$$\forall \mathbf{x} \text{ s.t. } A\mathbf{x} = \mathbf{b},$$

$$\therefore A(\mathbf{x} - \mathbf{x}_0) = 0 \because A\mathbf{x}_0 = \mathbf{b}$$

$$\mathbf{x} - \mathbf{x}_0 \in \ker(A)$$

Then, choosing each element of  $\mathbf{x}$  carefully (1 or 0),  $\mathbf{x} - \mathbf{x}_0$  can be any element of  $\mathbb{F}_2^n$ .

$$\therefore \{\mathbf{x} - \mathbf{x}_0 | A\mathbf{x} = \mathbf{b}\} = \ker(A)$$

$\therefore \mathbf{x} - \mathbf{x}_0$  has  $2^{n-m}$  solutions.

$\therefore \mathbf{x}$  has  $2^{n-m}$  solutions.

2. (a)

$$f(c\mathbf{v} + (-c)\mathbf{v}) \geq \min\{f(\mathbf{v}), f(\mathbf{v})\}$$

$$\therefore f(\mathbf{0}_V) \geq f(\mathbf{v})$$

(b) Because every element  $\mathbf{v}_t \in V_t$  is in  $V$  by definition.

$$V_t \subseteq V$$

3.

$$\begin{aligned} p(x) &= x^2 + bx + c \\ &= (x - r_1)(x - r_2) \\ &= x^2 - (r_1 + r_2)x + r_1r_2 \end{aligned}$$

$$\therefore b = -r_1 - r_2, c = r_1r_2$$

4.

$$\begin{aligned} \mu(P, Q) &= \text{degree}(PQ) \\ &= \text{degree}(QP) \\ &= \mu(Q, P) \end{aligned}$$

$$\begin{aligned} \mu(0, 0) &= \text{degree}(0) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \forall P \neq 0, \\ \mu(P, P) &= \text{degree}(P^2) \\ &= 2\text{degree}(P) \\ &> 0 \end{aligned}$$

$$\begin{aligned} \mu(P + Q, R) &= \text{degree}((P + Q)R) \\ &= \max \{ \text{degree}(P), \text{degree}(Q) \} + \text{degree}(R) \end{aligned}$$

$$\begin{aligned} \mu(P, R) + \mu(Q, R) &= \max \{ \text{degree}(P) + \text{degree}(R), \text{degree}(Q) + \text{degree}(R) \} \\ &= \max \{ \text{degree}(P), \text{degree}(Q) \} + \text{degree}(R) \end{aligned}$$

$$\therefore \mu(P + Q, R) = \mu(P, R) + \mu(Q, R)$$

$$\begin{aligned} c &\in \mathbb{R}, \\ \mu(cP, R) &= \text{degree}(cPR) \\ &= \text{degree}(PR) \\ &\neq c \cdot \text{degree}(P, R) \end{aligned}$$

$\therefore \mu(\cdot, R)$  is not a LT.  
 $\therefore \mu$  is not a IP.

5.

$$\begin{aligned}\alpha\beta\mathbf{x} &= \lambda\mathbf{x} \\ \beta\alpha\beta\mathbf{x} &= \beta(\lambda\mathbf{x}) \\ \beta\alpha(\beta\mathbf{x}) &= \lambda(\beta\mathbf{x})\end{aligned}$$

$\therefore \lambda$  is an eigenvalue of  $\beta\alpha$ .

6. (a)

$$\begin{aligned}\varphi(\mathbf{v}) &= \lambda\mathbf{v} \\ \varphi(\mathbf{v}) &= \lambda\varphi(\mathbf{v}) \\ (\lambda - 1)\varphi(\mathbf{v}) &= 0 \\ \lambda = 1_{\mathbb{F}} \vee \varphi(\mathbf{v}) &= 0 \\ \lambda = 1_{\mathbb{F}} \vee \varphi(\mathbf{v}) &= 0_{\mathbb{F}}\mathbf{v} \\ \therefore \lambda &\in \{0_{\mathbb{F}}, 1_{\mathbb{F}}\}\end{aligned}$$

(b) Let  $\forall \mathbf{v}, \varphi(\mathbf{v}) = \mathbf{v}_0$ . ( $\mathbf{v}_0$  is fixed.)  
Then assume  $\varphi = \varphi^*$ .  
If  $\mathbf{v} \neq \mathbf{w} \in V$ ,

$$\begin{aligned}\langle \mathbf{v}_0, \mathbf{w} \rangle &= \langle \mathbf{v}, \varphi^*(\mathbf{w}) \rangle \\ \langle \mathbf{v}_0, \mathbf{w} \rangle &= \langle \mathbf{v}, \mathbf{v}_0 \rangle \\ \langle \mathbf{w}, \mathbf{v}_0 \rangle &= \langle \mathbf{v}, \mathbf{v}_0 \rangle \\ \mathbf{v} &= \mathbf{w}\end{aligned}$$

This is contradiction.  
 $\therefore$  not always  $\varphi = \varphi^*$ .

7. (a)

$$\begin{aligned}\langle \varphi(\mathbf{v}), \mathbf{w} \rangle &= \langle \mathbf{v}, \varphi^*(\mathbf{w}) \rangle \\ \langle \varphi^*(\mathbf{w}), \mathbf{v} \rangle &= \langle \mathbf{w}, \varphi(\mathbf{v}) \rangle \\ \therefore (\varphi^*)^* &= \varphi\end{aligned}$$

(b)

$$\begin{aligned}\forall \mathbf{v} \in \ker(\varphi), \varphi(\mathbf{v}) &= 0 \\ \forall \mathbf{v}' \in (\text{im}(\varphi^*))^\perp, \forall \mathbf{w} \in W, \\ \langle \mathbf{v}, \varphi^*(\mathbf{w}) \rangle &= 0 \\ \langle \varphi(\mathbf{v}), \mathbf{w} \rangle &= 0 \\ \therefore \text{ if } \mathbf{v} \in \ker(\varphi), \\ \varphi(\mathbf{v}) &= 0 \therefore \forall \mathbf{w}, \langle \varphi(\mathbf{v}), \mathbf{w} \rangle = 0 \\ \langle \mathbf{v}, \varphi^*(\mathbf{w}) \rangle &= 0 \\ \mathbf{v} &\in (\text{im}(\varphi^*))^\perp\end{aligned}$$

(c)

$$\begin{aligned}\forall \mathbf{v} \in V \\ \text{If } \mathbf{w} \in \text{im}(\varphi), w &= \varphi(\mathbf{v}) \\ \langle \varphi(\mathbf{v}), \mathbf{w}' \rangle &= \langle \mathbf{v}, \varphi(\mathbf{w}') \rangle = 0 \\ \therefore \mathbf{w} &\in (\ker(\varphi^*))^\perp \\ \text{If } \mathbf{w} \in (\ker(\varphi^*))^\perp, \\ \forall \mathbf{w}' \in W \text{ s.t. } \varphi^*(\mathbf{w}') &= 0_V, \langle \mathbf{w}, \mathbf{w}' \rangle = 0 \\ \forall \mathbf{v} \in V, \langle \mathbf{v}, \varphi^*(\mathbf{w}') \rangle &= 0 \\ \langle \varphi(\mathbf{v}), \mathbf{w}' \rangle &= 0 \therefore \mathbf{w} \in \text{im}(\varphi) \\ \varphi(\mathbf{v}) &= \mathbf{w}\end{aligned}$$

(d)

$$\begin{aligned}\text{rank}(\varphi) &= \dim(\text{im}(\varphi)) \\ &= \dim((\ker(\varphi^*))^\perp) \\ &= \dim(W) - \dim(\ker(\varphi^*)) \\ &= \dim(\text{im}(\varphi^*)) \\ &= \text{rank}(\varphi^*)\end{aligned}$$

(e)

$$\begin{aligned}\text{Let } A &= BC, B \in \mathbb{C}^{m \times r}, C \in \mathbb{C}^{r \times n} \\ \text{then } A_{i,:} &= \sum_{j=1}^r B_{i,j} C_{j,:} \\ A_{:,i} &= \sum_{j=1}^r C_{j,i} B_{:,j} \\ \therefore \begin{cases} \text{rank}_{\text{row}}(A) \leq \text{rank}_{\text{row}}(C) \leq r \\ \text{rank}_{\text{column}}(A) \leq \text{rank}_{\text{column}}(B) \leq r \end{cases}\end{aligned}$$

Choose a minimal  $r$ .

Then rows of  $C$  form a minimal spanning set of rows of  $A$ .

And, columns of  $C$  form a minimal spanning set of columns of  $A$ .

$\therefore r$  is the rank of both row and column spaces of  $A$ .

$$\therefore \text{rank}_{\text{row}}(A) = \text{rank}_{\text{column}}(A)$$