

1. (a)

$$\dim(A) = \text{rank}(A) + \text{null}(A) \quad (1)$$

$$n = m + \text{null}(A) \quad (2)$$

$$\text{null}(A) = n - m \quad (3)$$

(b)

$$\text{null}(A) = \dim(\ker(A)) \quad (4)$$

Then, $\ker(A)$ can have a basis B s.t. $\text{Span}(B) = \ker(A)$. i.e.

$$\forall \mathbf{v} \in \ker(A), \exists a_1, \dots, a_{n-m} \in \mathbb{F}_2, \quad (5)$$

$$\mathbf{v} = a_1 \mathbf{b}_1 + \dots + a_{n-m} \mathbf{b}_{n-m} (b_i \in B) \quad (6)$$

\therefore The answer is 2^{n-m} .

(c)

$$\forall \mathbf{x} \text{ s.t. } \begin{cases} A\mathbf{x} = \mathbf{b} \\ A\mathbf{x}_0 = \mathbf{b} \end{cases} \quad (7)$$

$$\therefore A(\mathbf{x} - \mathbf{x}_0) = 0 \quad (8)$$

$$\mathbf{x} - \mathbf{x}_0 \in \ker(A) \quad (9)$$

Then, choosing each element of \mathbf{x} carefully (1 or 0), $\mathbf{x} - \mathbf{x}_0$ can be any element of \mathbb{F}_2^n .

$$\therefore \mathbf{x} - \mathbf{x}_0 | A\mathbf{x} = \mathbf{b} = \ker(A) \quad (10)$$

$\therefore \mathbf{x} - \mathbf{x}_0$ has 2^{n-m} solutions.

$\therefore \mathbf{x}$ has 2^{n-m} solutions.

2.

$$f(c\mathbf{v} + (-c)\mathbf{v}) \leq \min f(\mathbf{v}), f(\mathbf{v}) \quad (11)$$

$$\therefore f(\mathbf{0}_V) \leq f(\mathbf{v}) \quad (12)$$