YASSER TAIMA
hw#1 Math 270C
Sp'11 yasser@ucla.edu

1) Attempt to show $\underline{r}_{k+1} = \underline{r}_k - \underline{A}\, q_{k+1} \hat{\lambda}_{k+1} \quad \forall\, k \geq 0$

with $\underline{r}_0 = \underline{b} - \underline{A}\underbrace{\underline{x}_0}_{0} = \underline{b}$ so $\underline{r}_1 = \underline{b} - \underline{A}\underline{x}_1 = \underline{r}_0 - \underline{A} q_1 \hat{\lambda}_1$

$$k \geq 1 \Rightarrow \underline{r}_{k+1} := \underline{b} - \underline{A}\underline{x}_{k+1} = \underline{b} - \underline{A}\underline{Q}^{k+1}\hat{\lambda}_{k+1} = \underline{b} - \underbrace{\underline{Q}^{k+2}\tilde{\underline{H}}^{k+1}}_{\in \mathbb{R}^{n\times(k+1)}}\hat{\lambda}_{k+1} = \underline{b} - \Big[\underbrace{\underline{Q}^{k+1}}_{\in\mathbb{R}^{n\times(k+1)}} \;\vdots\; \underbrace{q_{k+2}}_{\in\mathbb{R}^{n\times 1}}\Big]\underbrace{\begin{bmatrix} \underbrace{\tilde{\underline{H}}^k}_{\in\mathbb{R}^{(k+1)\times k}} & \underbrace{\underline{h}_{k+1}}_{\in\mathbb{R}^{(k+1)\times 1}} \\ \underbrace{\underline{0}^T}_{\in\mathbb{R}^{1\times k}} & \underbrace{h_{k+2,k+1}}_{\in\mathbb{R}} \end{bmatrix}}_{\mathbb{R}^{(k+2)\times(k+1)}}\begin{bmatrix} \underbrace{\hat{\lambda}_k}_{\in\mathbb{R}^{k\times 1}} \\ \underbrace{\lambda^{k+1}_{k+1}}_{\in\mathbb{R}} \end{bmatrix}$$

$$= \underline{b} - \underbrace{\Big[\underbrace{\underline{Q}^{k+1}\tilde{\underline{H}}^k}_{\in\mathbb{R}^{n\times k}} \;\vdots\; \underbrace{\underline{Q}^{k+1}\underline{h}_{k+1} + h_{k+2,k+1}\, q_{k+2}}_{\in\mathbb{R}^{n\times 1}}\Big]}_{\in\mathbb{R}^{n\times(k+1)}}\begin{bmatrix} \hat{\lambda}_k \in\mathbb{R}^{k\times 1} \\ \lambda^{k+1}_{k+1} \in\mathbb{R} \end{bmatrix}$$

$$= \underline{b} - \underline{Q}^{k+1}\tilde{\underline{H}}^k\hat{\lambda}_k - \underbrace{\left(\underline{Q}^{k+1}\underline{h}_{k+1} + h_{k+2,k+1}\, q_{k+2}\right)}_{\text{last column of } \underline{Q}^{k+2}\tilde{\underline{H}}^{k+1} = \underline{A}\underline{Q}^{k+1}}\lambda^{k+1}_{k+1}, \quad \lambda^{k+1}_{k+1} = \lambda^{j}_{k+1} = \cdots = \lambda^{1}_{k+1} \quad \forall\, 1\leq j\leq k+1$$

so $\lambda_{k+1} := \lambda^{k+1}_{k+1}$

$$= \underline{b} - \underbrace{\underline{Q}^{k+1}\tilde{\underline{H}}^k}_{\underline{A}\underline{Q}^k}\hat{\lambda}_k - \underline{A} q_{k+1}\lambda_{k+1}$$

$$= \underline{b} - \underline{A}\underbrace{(\underline{Q}^k\hat{\lambda}_k)}_{\underline{x}_k} - \underline{A} q_{k+1}\lambda_{k+1} = \underbrace{(\underline{b} - \underline{A}\underline{x}_k)}_{\underline{r}_k} - \underline{A} q_{k+1}\lambda_{k+1}$$

so $\underline{r}_{k+1} = \underline{r}_k - \underline{A} q_{k+1}\lambda_{k+1} \quad \forall\, k \geq 0$

i.e. can get residual at current step k from the previous residual r_{k-1} & $\underline{A}$ × current search direction q_k × current λ^k_k, all w/out assembling $\underline{x}_k$ explicitly.

We note that for GMRES $\hat{\lambda}_k = \begin{pmatrix} \hat{\lambda}_{k-1} \\ \lambda_k \end{pmatrix}$ and $\tilde{\underline{H}}^{k+1} = \left[\begin{array}{c|c} \tilde{\underline{H}}^k & \underline{h}_{k+1} \\ \hline \underline{0}^T & h_{k+2,k+1} \end{array}\right]$

where $\underline{h}_{k+1} = \begin{bmatrix} h_{1,k+1} \\ h_{2,k+1} \\ \vdots \\ h_{k+1,k+1} \end{bmatrix} \quad \forall\, k \geq 1$

3)

2|

$$\underline{L}^k = \begin{pmatrix} 1 & & & 0 \\ \mu_1 & 1 & & \\ & \mu_2 & 1 & \\ 0 & & \ddots & \\ & & \mu_{k-1} & 1 \end{pmatrix} \quad \text{so} \quad \underline{L}^{k^T} = \begin{pmatrix} 1 & \mu_1 & & 0 \\ & 1 & \mu_2 & \\ & & \ddots & \\ 0 & & & \mu_{k-1} \\ & & & 1 \end{pmatrix}$$

$$\underline{L}^{k^T} \underline{\lambda}_k = \underline{f}_k \qquad \underline{f}_k = \begin{pmatrix} p_1^k \\ p_2^k \\ \vdots \\ p_{k-1}^k \\ p_k^k \end{pmatrix} \qquad p_j^k = (-1)^{j-1} \left(\prod_{i=1}^{j-1} \mu_i\right) \frac{\|b\|}{d_j} \quad 1 \le j \le k \quad \forall k \ge 1$$

μ_j, d_j obtained from α_j, β_j

$\alpha_j = q_j^T A q_j \quad \beta_j = q_j^T A q_{j+1}$

so unchanged at each iteration

$$\underline{L}^{(k+1)T} \underline{\lambda}_{k+1} = \underline{f}_{k+1} = \begin{pmatrix} p_1^{k+1} \\ p_2^{k+1} \\ \vdots \\ p_k^{k+1} \\ p_{k+1}^{k+1} \end{pmatrix} \qquad p_j^{k+1} = (-1)^{j-1} \left(\prod_{i=1}^{j-1} \mu_i\right) \frac{\|b\|}{d_j} = p_j^k \quad \forall 1 \le j \le k$$

only $p_{k+1}^{k+1} = (-1)^k \left(\prod_{i=1}^{k} \mu_i\right) \frac{\|b\|}{d_k}$ is new

so $\underline{L}^{(k+1)T} \lambda_{k+1} = \begin{pmatrix} \underline{f}_k \\ p_{k+1}^{k+1} \end{pmatrix}$.

$$\underline{L}^{(k+1)T} \underline{\lambda}_{k+1} = \left(\begin{array}{c|c} \underline{L}^{k^T} & \begin{matrix} 0 \\ \vdots \\ 0 \\ \mu_k \end{matrix} \\ \hline \underset{\in \mathbb{R}^{k \times 1}}{\underline{0}^T} & 1 \end{array}\right) \begin{pmatrix} \lambda_1^{k+1} \\ \lambda_2^{k+1} \\ \vdots \\ \lambda_k^{k+1} \\ \lambda_{k+1}^{k+1} \end{pmatrix} = \left[\begin{array}{c} \underline{L}^{k^T} \begin{pmatrix} \lambda_1^{k+1} \\ \lambda_2^{k+1} \\ \vdots \\ \lambda_k^{k+1} \end{pmatrix} \\ \hline \underset{\in \mathbb{R}^{k \times 1}}{\underline{0}^T} \end{array}\right] + \lambda_{k+1}^{k+1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mu_k \\ 1 \end{bmatrix}$$

<u>Suppose</u> that $\forall k \ge 1 \quad \underline{\lambda}_{k+1} = \begin{pmatrix} \underline{\lambda}_k \\ \lambda_{k+1}^{k+1} \end{pmatrix}$ (#)

i.e. $\lambda_1^{k+1} = \lambda_1^k, \lambda_2^{k+1} = \lambda_2^k, \ldots, \lambda_k^{k+1} = \lambda_k^k$.

then $\underline{L}^{(k+1)T} \underline{\lambda}_{k+1} = \begin{bmatrix} \underline{f}_k \; \underline{L}^{k^T} \underline{\lambda}_k \\ \hline \underline{0}^T \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mu_k \lambda_{k+1}^{k+1} \\ \lambda_{k+1}^{k+1} \end{bmatrix} = \begin{pmatrix} \underline{f}_k \\ \cdots \\ p_{k+1}^{k+1} \end{pmatrix}$

$\Rightarrow \lambda_{k+1}^{k+1} = p_{k+1}^{k+1}$ and $\mu_k \lambda_{k+1}^{k+1} = 0$. $\mu_k = \frac{\beta_k}{d_k} = 0 \Rightarrow V_k = 0 \Rightarrow q_k = 0$

i.e. Krylov space ran out: $K^l = K^{k-1} \; \forall l > k$: contradicting having $\underline{T}^k$ at k^{th} step,

or $0 = \lambda_{k+1}^{k+1} = p_{k+1}^{k+1} = (-1)^k \left(\underbrace{\prod_{i=1}^{k} \mu_i}_{\neq 0 \text{ up to } k}\right) \frac{\|b\|}{d_k} \Rightarrow \|b\| = 0 \Rightarrow b = 0$ but we chose

A invertible then $b \neq 0$ (otherwise trivial system $x = 0$) $\rightarrow\leftarrow$ (#)

hence $\exists k \ge 1$, $\underline{\lambda}_k \neq \begin{pmatrix} \underline{\lambda}_k \\ \lambda_k^k \end{pmatrix}$.

3) A spd $\overset{M^{G^{-1}}N_G v}{=} -(D+L)^{-1}L^T v = \lambda v$ for v eigenvector of $M^{G-1}N^G$

λ its eigenvalue . $v \neq 0$

$\Rightarrow -v^T L^T v = \lambda v^T(D+L)v$ $\quad D>0$ as A spd

$\Rightarrow -v^T L v = \lambda(v^T D v + v^T L v)$ $\quad v^T(L^T v) = (L^T v)^T v = v^T L v$

$0 < v^T A v = v^T(D+L+L^T)v = \underset{>0}{v^T D v} + 2v^T L v$

So $v^T L v$ real. Also if $v^T D v + v^T L v = 0$ then $v^T L v > 0$

$\ast$ So $|\lambda|^2 = \dfrac{(v^T L v)^2}{(v^T D v)^2 + 2(v^T D v)(v^T L v) + (v^T L v)^2}$

$= \dfrac{(v^T L v)^2}{\underset{>0}{(v^T D v)}\underbrace{(v^T D v + 2v^T L v)}_{=v^T A v > 0} + (v^T L v)^2}$

$\Rightarrow |\lambda|^2 = \dfrac{(v^T L v)^2}{\alpha + (v^T L v)^2} = \dfrac{1}{\frac{\alpha}{(v^T L v)^2} + 1}$ where $\alpha > 0$

$\Rightarrow |\lambda|^2 < 1 \Rightarrow |\lambda| < 1 \quad \forall \lambda$ eigenvalue of $M^{G-1}N^G$.

YASSER TAIMA
hw #2 MATH 270C
Sp'11 yasser@ucla.edu

1) $\mathrm{span}\{f_1, f_2, \dots, f_k\} \stackrel{?}{=} \mathrm{span}\{f_1, \underline{A}^T f_1, (\underline{A}^T)^2 f_1, \dots, (\underline{A}^T)^{k-1} f_1\}$ if no catastrophic failure

WTS $\langle\{f_1, f_2, \dots, f_k\}\rangle = \langle\{f_1, \underline{A}^T f_1, \dots (\underline{A}^T)^{k-1} f_1\}\rangle \quad \forall k \geq 1$

$k=1$ $\langle\{f_1\}\rangle = \langle\{f_1\}\rangle$ trivially.

$k \to k+1$:
suppose $K'^k := \langle\{f_1, (\underline{A}^T) f_1, (\underline{A}^T)^2 f_1, \dots, (\underline{A}^T)^{k-1} f_1\}\rangle = \langle\{f_1, f_2, \dots, f_k\}\rangle$

WTS ① $K'^{k+1} \subseteq \langle\{f_1, f_2, \dots, f_{k+1}\}\rangle$ and ② $\langle\{f_1, \dots, f_{k+1}\}\rangle \subseteq K'^{k+1}$ given $K'^{k+1} \neq K'^k$ (i.e. not stopped at k)

① $(\underline{A}^T)^j f_1 \in K'^k = \langle\{f_1, \dots, f_k\}\rangle \subset \langle\{f_1, f_2, \dots, f_{k+1}\}\rangle$ by the induction hypothesis, $\forall\ 1 \leq j \leq k-1$.

$$(\underline{A}^T)^k f_1 = \underline{A}^T \underbrace{(\underline{A}^T)^{k-1} f_1}_{\in K'^k} = \underline{A}^T \Big(\sum_{i=1}^{k} \gamma_i' f_i\Big) = \underline{A}^T \gamma_k' f_k + \underbrace{\sum_{i=1}^{k-1} \gamma_i' \underline{A}^T f_i}_{=\sum_{i=1}^{k} \beta_i' f_i}$$

by induction hypothesis

as $f_i \in K'^{k-1}$
$\Rightarrow f_i = \sum_{j=0}^{k-2} \alpha_j' (\underline{A}^T)^j f_1$
$\Rightarrow \underline{A}^T f_i = \sum_{j=0}^{k-2} \alpha_j' (\underline{A}^T)^{j+1} f_1 = \sum_{j=1}^{k-1} \alpha_j' (\underline{A}^T)^j f_1 \in K'^k$

$$\Rightarrow (\underline{A}^T)^k f_1 = \gamma_k' \underline{A}^T f_k + \sum_{i=1}^{k} \beta_i' f_i \quad (\dagger)$$

By def. of f_{k+1} $\quad \underline{w}_{k+1} = \gamma_k f_{k+1} = \underline{A}^T f_k - \beta_{k-1} f_{k-1} - \alpha_k f_k$

so $\underline{A}^T f_k = \beta_{k-1} f_{k-1} + \alpha_k f_k + \gamma_k f_{k+1}$

then $(\underline{A}^T)^k f_1 = \gamma_k' \beta_{k-1} f_{k-1} + \alpha_k f_k + \sum_{i=1}^{k} \beta_i' f_i + \gamma_k f_{k+1} \in \langle\{f_1, f_2, \dots, f_{k+1}\}\rangle$

② from $(\dagger)$, if $\gamma_k' = 0$ then $(\underline{A}^T)^k f_1 \in K'^k \Rightarrow K'^{k+1} \subset K'^k \subset K'^{k+1} \Rightarrow$
$K'^{k+1} = K'^k$ ✗ as we're not stopped at k, by hypothesis.

So can assume $\gamma_k' \neq 0$. Then

$$\underline{A}^T f_k = \frac{1}{\gamma_k'} \Big((\underline{A}^T)^k f_1 - \sum_{i=1}^{k} \beta_i' f_i\Big)$$

$$\text{so } \gamma_k f_{k+1} = \frac{1}{\gamma_k'} (\underline{A}^T)^k f_1 - \sum_{i=1}^{k} \frac{\beta_i'}{\gamma_k'} f_i - \beta_{k-1} f_{k-1} - \alpha_k f_k$$

Assumed no catastrophic failure then $\gamma_k \neq 0$

so $f_{k+1} \in \langle\{\underbrace{(\underline{A}^T)^k f_1}_{K'^{k+1}}, \underbrace{f_1, f_2, \dots, f_k}_{\in K'^{k+1}}\}\rangle = \langle K'^{k+1} \rangle = K'^{k+1}$ as $\langle\{f_1, f_2, \dots, f_k\}\rangle = K'^k \subset K'^{k+1}$

then $\langle\{f_1, f_2, \dots, f_{k+1}\}\rangle \subset K'^{k+1}$

so by ①, ② $\quad K'^{k+1} = \langle\{f_1, f_2, \dots, f_{k+1}\}\rangle$.

2) Bidirectional or two-sided G-Schmidt is, given

$\underline{F} = [\underline{f}_1, \underline{f}_2, \dots, \underline{f}_n] \in \mathbb{R}^{n\times n}$ $\quad \underline{G} = [\underline{g}_1, \underline{g}_2, \dots, \underline{g}_n] \in \mathbb{R}^{n\times n}$

w/ $\det \underline{F} \neq 0$, $\det \underline{G} \neq 0$,

construct two new matrices $\underline{V} = [\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n]$ and $\underline{W} = [\underline{w}_1, \underline{w}_2, \dots, \underline{w}_n]$

with $\underline{W}^T\underline{V} = \underline{V}^T\underline{W} = \operatorname{diag}(\psi_1, \dots, \psi_n)$ where $\psi_j = \underline{w}_j^T\underline{v}_j \quad 1 \leq j \leq n$

and $\langle\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_j\}\rangle = \langle\{\underline{f}_1, \underline{f}_2, \dots, \underline{f}_j\}\rangle$

and $\langle\{\underline{w}_1, \underline{w}_2, \dots, \underline{w}_j\}\rangle = \langle\{\underline{g}_1, \underline{g}_2, \dots, \underline{g}_n\}\rangle$.

It's done by setting $\underline{v}_1 = \underline{f}_1$, $\underline{w}_1 = \underline{g}_1$,

and inductively $\underline{v}_j := \underline{f}_j - \sum_{i=1}^{j-1}\left(\frac{\underline{w}_i^T\underline{f}_j}{\psi_i}\right)\underline{v}_i$, $\underline{w}_j = \underline{g}_j - \sum_{i=1}^{j-1}\left(\frac{\underline{v}_i^T\underline{g}_j}{\psi_i}\right)\underline{w}_i$

for $1 \leq j \leq n$.

Choose $\underline{F} = [\underline{q}_1, \underline{A}\underline{q}_1, \underline{A}^2\underline{q}_1, \dots, \underline{A}^{n-1}\underline{q}_1]$, $\underline{G} = [\underline{p}_1, \underline{A}^T\underline{p}_1, (\underline{A}^T)^2\underline{p}_1, \dots, (\underline{A}^T)^{n-1}\underline{p}_1]$, $\underline{p}_1 = \underline{q}_1 = \underline{b}/\|\underline{b}\|$.

By question ① $K^k = \langle\{\underline{q}_1, \underline{q}_2, \dots, \underline{q}_k\}\rangle$ and $K'^k = \langle\{\underline{p}_1, \underline{p}_2, \dots, \underline{p}_k\}\rangle$

where $K^k = \langle\{\underline{q}_1, \underline{A}\underline{q}_1, \dots, \underline{A}^{k-1}\underline{q}_1\}\rangle$, $K'^k = \langle\{\underline{p}_1, \underline{A}\underline{p}_1, \dots, \underline{A}^{k-1}\underline{p}_1\}\rangle$, $\forall 1 \leq k \leq n$

Then starting from $k=2$ and going up to n, we see that

each $\underline{v}_{k} \in \langle\{\underline{w}_1, \underline{w}_2, \dots, \underline{w}_{k-1}\}\rangle^\perp = \langle\{\underline{p}_1, \underline{A}^T\underline{p}_1, \dots, (\underline{A}^T)^{k-2}\underline{p}_1\}\rangle^\perp = \langle\{\underline{p}_1, \underline{p}_2, \dots, \underline{p}_{k-1}\}\rangle^\perp$

and $\underline{w}_k \in \langle\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_{k-1}\}\rangle^\perp = \langle\{\underline{q}_1, \underline{A}\underline{q}_1, \dots, \underline{A}^{k-2}\underline{q}_1\}\rangle^\perp = \langle\{\underline{q}_1, \underline{q}_2, \dots, \underline{q}_{k-1}\}\rangle^\perp$.

Going down from n, $\underline{w}_n \in \langle\{\underline{q}_1, \underline{q}_2, \dots, \underline{q}_{n-1}\}\rangle^\perp = \langle\{\underline{q}_1, \underline{A}\underline{q}_1, \underline{A}^2\underline{q}_1, \dots, \underline{A}^{n-2}\underline{q}_1\}\rangle^\perp = K^{n-1\perp}$

which has dimension 1 as $\dim(K^{n-1}) = n-1$, because $\mathbb{R}^n = K^{n-1} \oplus K^{n-1\perp}$.

But $\underline{p}_n \in \langle\{\underline{q}_1, \underline{q}_2, \dots, \underline{q}_{n-1}\}\rangle^\perp = K^{n-1\perp}$ also so $\underline{w}_n$ & $\underline{p}_n$ are colinear,

and similarly $\underline{v}_n$ and $\underline{q}_n$ are colinear, i.e. equivalent, respectively.

Now $\underline{w}_{n-1} \in K^{n-2\perp}$ and $\underline{p}_{n-1} \in K^{n-2\perp} \supset K^{n-1\perp}$ as $K^{n-2} \subset K^{n-1}$

so $\underline{w}_{n-1}$ and $\underline{p}_{n-1}$ are in the same plane $K^{n-2\perp} = K^{n-1\perp} \oplus E_1$

but $\underline{w}_{n-1}$ and $\underline{p}_{n-1}$ are orthogonal to $\underline{p}_n$ (colinear w/ $\underline{w}_n$) so $\underline{w}_{n-1}$ and $\underline{p}_{n-1}$

both are in E_1, of dimension 1, hence also are colinear, and

similarly for $\underline{v}_{n-1}$ and $\underline{q}_{n-1}$.

Given that $\mathbb{R}^n = \bigoplus_{i=1}^n \langle\{\underline{p}_i\}\rangle = \bigoplus_{i=1}^n \langle\{\underline{q}_i\}\rangle$ as $\{\underline{p}_i\}_{i=1}^n$ & $\{\underline{q}_i\}_{i=1}^n$ are

orthonormal basis of $\mathbb{R}^n = \bigoplus_{i=1}^n \langle\{\underline{w}_i\}\rangle = \bigoplus_{i=1}^n \langle\{\underline{v}_i\}\rangle$, we can keep applying

this argument to conclude that $\underline{q}_i$ & $\underline{v}_i$, and $\underline{p}_i$ and $\underline{w}_i$, are colinear, respectively.

Apply $\underline{x}_k$: $1 \le j,k \le n-2$

3)

$$\underline{A}\,\underline{x}_k = \frac{1}{\Delta x^2}\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & 0 & \\ & -1 & 2 & -1 & & \\ 0 & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2\end{pmatrix}\begin{pmatrix} x_1^k \\ x_2^k \\ \vdots \\ x_{n-3}^k \\ x_{n-2}^k\end{pmatrix}$$

$$= \begin{pmatrix} \frac{2x_1^k - x_2^k}{\Delta x^2} \\ \frac{-x_1^k + 2x_2^k - x_3^k}{\Delta x^2} \\ \vdots \\ \frac{-x_{n-4}^k + 2x_{n-3}^k - x_{n-2}^k}{\Delta x^2} \\ \frac{-x_{n-3} + 2x_{n-2}^k}{\Delta x^2} \end{pmatrix} = -\begin{pmatrix} \frac{\frac{x_2^k - x_1^k}{\Delta x} - \frac{x_1^k - x_0^k}{\Delta x}}{\Delta x} \\ \frac{\frac{x_3^k - x_2^k}{\Delta x} - \frac{x_2^k - x_1^k}{2}}{\Delta x} \\ \vdots \\ \frac{\frac{x_{n-2}^k - x_{n-3}^k}{\Delta x} - \frac{x_{n-3}^k - x_{n-2}^k}{\Delta x}}{\Delta x} \\ \frac{\frac{x_{n-1}^k - x_{n-2}^k}{\Delta x} - \frac{x_{n-2}^k - x_{n-3}^k}{\Delta x}}{\Delta x} \end{pmatrix} = -\left(\frac{\frac{x_{j+1}^k - x_j^k}{\Delta x} - \frac{x_j^k - x_{j-1}^k}{\Delta x}}{\Delta x}\right)_{j=1}^{n-2}$$

where $x_0^k = x_{n-1}^k = 0$

$$= -\frac{1}{2i}\left[\left(\frac{\frac{e^{ik\pi(j+1)\Delta x}-e^{ik\pi j\Delta x}}{\Delta x} - \frac{\left(e^{ik\pi j\Delta x}-e^{ik\pi(j-1)\Delta x}\right)}{\Delta x}}{\Delta x}\right) - \left(\frac{\frac{e^{-ik\pi(j+1)\Delta x}-e^{-ik\pi j\Delta x}}{\Delta x} - \frac{\left(e^{-ik\pi j\Delta x}-e^{-ik\pi(j-1)\Delta x}\right)}{\Delta x}}{\Delta x}\right)\right]_{j=1}^{n-2}$$

$$= -\frac{1}{2i}e^{ik\pi j\Delta x}\left(\frac{\frac{e^{ik\pi\Delta x}-1}{\Delta x} - \frac{(1-e^{-ik\pi\Delta x})}{\Delta x}}{\Delta x}\right) + \frac{1}{2i}e^{-ik\pi j\Delta x}\left(\frac{\frac{e^{-ik\pi\Delta x}-1}{\Delta x}-\frac{(1-e^{ik\pi\Delta x})}{\Delta x}}{\Delta x}\right)_{j=1}^{n-2}$$

$$= \left(-\frac{1}{2i}e^{ik\pi j\Delta x}\left(\frac{\frac{e^{ik\pi\Delta x}-1}{\Delta x}-\frac{(1-e^{-ik\pi\Delta x})}{\Delta x}}{\Delta x}\right)+\frac{1}{2i}e^{-ik\pi j\Delta x}\left[\frac{\frac{(e^{ik\pi\Delta x}-1)}{\Delta x}-\frac{(1-e^{-ik\pi\Delta x})}{\Delta x}}{\Delta x}\right]\right)_{j=1}^{n-2}$$

$$= \left(-\frac{(e^{ik\pi j\Delta x}-e^{-ik\pi j\Delta x})}{2i}\left(\frac{\frac{e^{ik\pi\Delta x}-1}{\Delta x}-\frac{(1-e^{-ik\pi\Delta x})}{\Delta x}}{\Delta x}\right)\right)_{j=1}^{n-2}$$

$$= -\left(\frac{e^{ik\pi\Delta x}+e^{-ik\pi\Delta x}-2}{\Delta x^2}\right)\left(\sin(2\pi kj\Delta x)\right)_{j=1}^{n-2} = \frac{-2(\cos k\pi\Delta x - 1)}{\Delta x^2}\,\underline{x}_k = \frac{2(1-\cos k\pi\Delta x)}{\Delta x^2}\,\underline{x}_k$$

so $\underline{A}\,\underline{x}_k = \frac{2(1-\cos k\pi\Delta x)}{\Delta x^2}\,\underline{x}_k$ so $\underline{x}_k$ $1 \le k \le n-2$ eigenvectors

with $\boxed{\lambda_k = 2\left(\frac{1-\cos k\pi\Delta x}{\Delta x^2}\right)}$. Since $\{\underline{x}_k\}_{k=1}^{n-2}$ are orthogonal and there are $n-2 = \dim(A)$ of them, these are the only eigenvectors thus also the only eigenvalues.

Programming comment on problem 3:

Maxiterations set to 2 = $n-1$ where $n=3$ since A is 3×3.

GMRES(1) reached a better estimate than GMRES(2). This may be because the residual which was used in the second outer iteration is a better guess than $x_0 = 0$.