



Lattice Theory for Parallel Programming

Solutions for exercises

1 Theoretical Exercises

Exercise 1

Show that if (X, \leq) is a poset, then (X, \leq^∂) is also a poset.

Solution 1

Exercise 2

Show that $(\mathbb{P}(X), \Rightarrow)$ is a poset.

Solution 2

Exercise 3

If (X, \leq_X) and (Y, \leq_Y) are chains, then their linear sum $(X \oplus Y, \leq)$ is also a chain.

Solution 3

Exercise 4

The pointwise order of two posets is an order.

Solution 4

Exercise 5

If (X, \leq_X) and (Y, \leq_Y) are chains, then the lexicographic order on product is also a chain. However, this is not necessarily true for the pointwise order on the product.

Solution 5**Exercise 6**

Let (X, \leq) and (Y, \leq) be posets, and let $f: X \rightarrow Y$ be a function. The following conditions are equivalent.

1. f is an order isomorphism.
2. f is a monotone bijective map and the map f^{-1} is monotone.

Solution 6**Exercise 7**

Prove that the following statements are true.

1. The function $f: (\mathbb{N}, \leq) \rightarrow (\mathbb{N}, \leq)$ defined by $f(n) = 2n$ is order-preserving.
2. The function $g: (\mathbb{R}, \leq) \rightarrow (\mathbb{R}, \leq)$ defined by $g(x) = x^2$ is not order-preserving. For instance, $-2 < -1$, but $g(-2) = 4 > 1 = g(-1)$.
3. The inclusion map $i: (\mathbb{N}, \leq) \rightarrow (\mathbb{Z}, \leq)$ defined by $i(n) = n$ is an order-embedding.
4. The function $f: (\mathbb{N}, \leq) \rightarrow (\mathbb{N}, \leq)$ defined by $f(n) = 2n$ is an order embedding.
5. The function $f: (\mathbb{R}, \leq) \rightarrow (\mathbb{R}, \leq)$ defined by $f(x) = \lfloor x \rfloor$ is order-preserving but is not an order embedding.
6. The function $f: (\mathbb{N}, \leq) \rightarrow (\mathbb{N}^*, \leq)$ defined by $f(n) = n + 1$ is an order isomorphism.
7. The function $h: (\mathbb{N} \times \mathbb{N}, \leq) \rightarrow (\mathbb{N}, \leq)$, where $\mathbb{N} \times \mathbb{N}$ is equipped with the pointwise order, defined by $f(x, y) = x + y$ is order-preserving but not an embedding.
8. If (X_1, \leq) and (X_2, \leq) are two posets that the projections maps $\pi_1, \pi_2: X_1 \times X_2 \rightarrow X_i$ defined as $\pi_i(x_1, x_2) = x_i$ is order-preserving if $X_1 \times X_2$ is equipped with the pointwise order.

Solution 7**Exercise 8**

Let $X = \{a, b, c, d\}$. Prove that the powerset poset $(2^X, \subset)$ is isomorphic to the predicate poset $(\mathbb{P}, \Rightarrow)$. Then, prove that the previous statement holds for any set X (even an infinite one).

Solution 8**Exercise 9**

Prove that (\mathbb{Z}, \leq) is isomorphic to $(\mathbb{Z}, \leq^\partial)$. Is (\mathbb{N}, \leq) isomorphic to $(\mathbb{N}, \leq^\partial)$?

Solution 9

Exercise 10

Characterize the $n \in \mathbb{N}$ whose divisor poset is isomorphic to $(2^{\{0,1\}}, \subset)$.

Solution 10**Exercise 11**

Prove that (\mathbb{Z}, \leq) is isomorphic to $(\mathbb{N}, \leq^\partial) \oplus (\mathbb{N}, \leq)$.

Solution 11**Exercise 12**

Let C be the set of subset X of \mathbb{N} such that $\mathbb{N} \setminus X$ is finite. Show that C is a filter in $(2^{\mathbb{N}}, \subset)$.

Solution 12**Exercise 13**

Let (X, \leq) be a poset and $Q \subset X$. We set

1. Show that $\downarrow Q$ is an down-set that contains Q . Deduce that $\uparrow Q$ is an up-set that contains Q .
2. Let $(Up(X), \supseteq)$ be the set of up-sets of (X, \leq) ordered by reverse inclusion. Show that the map $f: X \rightarrow Up(X)$ defined as $f(x) = \uparrow x$ is an order-embedding.

Solution 13**Exercise 14**

Prove that if (X, \leq) has a top element, then it is unique. Similarly, if (X, \leq) has a bottom element, then it is unique.

Solution 14**Exercise 15**

Let (X, \leq) be a poset and S be a subset of X . Prove that if S has a least upper bound then it is unique. Deduce that if S has a greatest lower bound, then it is unique.

Solution 15

Exercise 16

Prove that in any lattice (L, \leq) , for all $x, y, z \in L$:

1. $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$ (commutativity)
2. $(x \vee y) \vee z = x \vee (y \vee z)$ and $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ (associativity)
3. $x \vee x = x$ and $x \wedge x = x$ (idempotence)
4. $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$ (absorption)

Solution 16**Exercise 17**

Prove that if $(L, \vee, \wedge, 0, 1)$ is an algebraic bounded lattice, then in the corresponding order-theoretic lattice (L, \leq) , the element 0 is the bottom element and 1 is the top element.

Solution 17**Exercise 18**

Consider the following statements about lattice operations and constructions:

1. The disjoint union of lattices is a lattice.
2. The linear sum of lattices is a lattice.
3. The lexicographic order on the product of lattices might not be a lattice.
4. The pointwise order on the product of lattices is always a lattice, with join and meet operations computed pointwise.

For each statement, provide a proof or a counterexample to justify why the statement is true or false.

Solution 18**Exercise 19**

If (L, \wedge, \vee) is a lattice and S is a sublattice of L , then (S, \vee, \wedge) is a lattice.

Solution 19**Exercise 20**

Let $(L, \vee, \wedge, 0, 1)$ be a bounded lattice and S be a sublattice of L .

1. Prove that if S is a 0-sublattice, then 0 is the bottom element of S .
2. Prove that if S is a 1-sublattice, then 1 is the top element of S .
3. Give an example of a sublattice that has a bottom element different from 0 and a top element different from 1.

Solution 20**Exercise 21**

Let $f: L_1 \rightarrow L_2$ be a lattice isomorphism. Prove that f^{-1} is a lattice isomorphism.

Solution 21**Exercise 22**

Prove that if $f: L_1 \rightarrow L_2$ is a bounded lattice isomorphism, then $f^{-1}: L_2 \rightarrow L_1$ is also a bounded lattice isomorphism.

Solution 22**Exercise 23**

Give an example of bounded lattices L_1 and L_2 and lattice homomorphism $f: L_1 \rightarrow L_2$ which is not a bounded lattice homomorphism.

Solution 23**Exercise 24**

Give a precise inductive definition of the interpretation of a (bound) lattice term on a (bounded) lattice.

Solution 24**Exercise 25**

Any lattice satisfies the equation $(x \vee y) \vee z = x \vee (y \vee z)$. If $X = \{a, b\}$ then the lattice $(2^X, \cup, \cap)$ satisfies the equation $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ while the lattice N_5 depicted Fig. 2 doesn't satisfy it.

Solution 25**Exercise 26**

Let \mathcal{K} be a class of (bounded) lattices and $t = s$ be a class of (bounded) lattice equations. Show that if every element of \mathcal{K} satisfies $t = s$, then every element of $\mathbb{S}(\mathcal{K}), \mathbb{P}(\mathcal{K})$ and $\mathbb{H}(\mathcal{K})$ satisfies $t = s$.

Solution 26**Exercise 27**

Give an example of a non-distributive lattice whose every element has a complement, but there is at least one element which has two complements.

Solution 27**Exercise 28**

Show that the class of complemented bounded distributive lattices is not equational.

Solution 28**Exercise 29**

Prove that in a Boolean algebra, De Morgan's laws hold:

$$(x \vee y)' = x' \wedge y' \text{ and } (x \wedge y)' = x' \vee y'$$

Solution 29**Exercise 30**

Prove that in a Boolean algebra B , the complementation operation is a bounded lattice isomorphism between the underlying bounded lattice of B and its order dual. (*Hint:* Prove that (i) B^∂ equipped with the complementation operation from B is a Boolean algebra, (ii) the complementation operation $\iota: B \rightarrow B^\partial$ is a monotone map which is its own inverse.)

Solution 30**Exercise 31**

Let B be the set of subsets X of \mathbb{N} that satisfy X is finite or $\mathbb{N} \setminus X$ is finite. Show that B is a Boolean algebra.

Solution 31**Exercise 32**

We adopt the notation of the previous paragraphs.

1. Show that $\downarrow f$ is an ideal in $X \multimap Y$.
2. Show that $\downarrow f$ is a complete bounded lattice in its own right, with \emptyset and f as bottom and top element, respectively.

Solution 32

Exercise 33

Let $c: 2^{\mathbb{R}^\infty} \rightarrow 2^{\mathbb{R}^\infty}$ be defined as $c(X) = (\bigvee X]$.

1. Show that c is a closure operator.
2. Describe the set K of closed elements.
3. Describe the complete lattice operation on K .
4. Describe $c(x)$ in terms of closed elements for every $x \in 2^{\mathbb{R}^\infty}$.

Solution 33**Exercise 34**

Let $(L, \wedge, \vee, \perp, \top)$ be a bounded lattice, and X be an infinite set of variables. For any $n \geq 0$, let $T_n(X)$ the set of n -ary bounded lattice terms over X and set $T(X) := \cup_{n \geq 0} T_n(X)$. Define $c: 2^L \rightarrow 2^L$ by

$$c(A) = \{t^L(a_1, \dots, a_n) \mid n \geq 0, t \in T_n(X) \text{ and } a_1, \dots, a_n \in A\}.$$

1. Show that c is a closure operator.
2. Describe the set K of closed elements.
3. Describe the complete lattice operations on K .
4. Describe $c(A)$ in terms of closed elements for every $A \in 2^L$.

Solution 34**Exercise 35**

Show that the constructions of Table 1 define are mutually inverse.

Solution 35**Exercise 36**

Let (X, \leq) be a poset. Define $U := \{\uparrow x \mid x \in X\}$ and $D := \{\downarrow x \mid x \in X\}$, as well as the maps $\alpha: D \rightarrow U$ and $\gamma: U \rightarrow D$ defined as $\gamma(\uparrow x) := \downarrow x$ and $\alpha(\downarrow x) := \uparrow x$. Prove that

$$(L, \leq) \xrightleftharpoons[\alpha]{\gamma} (M, \sqsubseteq)$$

Solution 36**Exercise 37**

Let (C, \leq) be a complete totally ordered set and \rightarrow be the binary operation defined on C by $a \rightarrow b := \bigvee\{x \in C \mid a \wedge x \leq b\}$. Prove that for any $a \in C$, we have

Solution 37**Exercise 38**

(From Galois Connection to Closure Operator). Assume that

Solution 38**Exercise 39**

Identify the Galois connections among the Galois connections we have introduced so far.

Solution 39