UNIVERSITÉ DU LUXEMBOURG

Lattice Theory for Parallel Programming
Solutions for exercises

1 Theoretical Exercises

Exercise 1
Show that if $(X, \leq)$ is a poset, then $(X, \leq^{\partial})$ is also a poset. **Solution 1**

Let $x, y, z \in X$

1. $x \leq^{\partial} x \Leftrightarrow x \leq x$
2. $x \leq^{\partial} y \wedge y \leq^{\partial} x \Leftrightarrow y \leq x \wedge x \leq y \implies x = y$
3. $x \leq^{\partial} y \wedge y \leq^{\partial} z \Leftrightarrow y \leq x \wedge z \leq y \implies z \leq x \Leftrightarrow x \leq^{\partial} z$

Therefore, if $(X, \leq)$ is a poset, then $(X, \leq^{\partial})$ is also a poset.

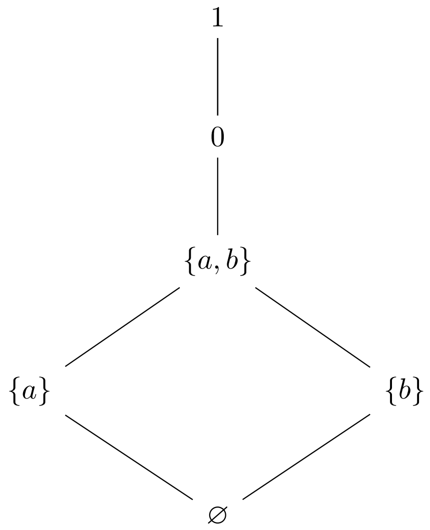
Exercise 2
Show that $(\mathbb{P}(X), \Rightarrow)$ is a poset. **Solution 2**

Let $P, Q, R \in \mathbb{P}(X)$

1. $P \Rightarrow P \Leftrightarrow \{x \in X \mid P(x)\} \subseteq \{x \in X \mid P(x)\}$
2. $P \Rightarrow Q \wedge Q \Rightarrow P \Leftrightarrow \{x \in X \mid P(x)\} \subseteq \{x \in X \mid Q(x)\} \wedge \{x \in X \mid Q(x)\} \subseteq \{x \in X \mid P(x)\} \Leftrightarrow \{x \in X \mid P(x)\} = \{x \in X \mid Q(x)\} \Leftrightarrow P = Q$
3. $P \Leftarrow Q \wedge Q \Leftarrow R \Leftrightarrow \{x \in X \mid P(x)\} \subseteq \{x \in X \mid Q(x)\} \wedge \{x \in X \mid Q(x)\} \subseteq \{x \in X \mid R(x)\} \Leftarrow \{x \in X \mid P(x)\} \subseteq \{x \in X \mid R(x)\} \Leftrightarrow P \Leftarrow Q$

Exercise 3
If $(X, \leq_X)$ and $(Y, \leq_Y)$ are chains, then their linear sum $(X \oplus Y, \leq)$ is also a chain.

Solution 3



Exercise 4

If $(X, \leq_X)$ and $(Y, \leq_Y)$ are chains, then their linear sum $(X \oplus Y, \leq)$ is also a chain.

Solution 4

Since $(X, \leq_X)$ and $(Y, \leq_Y)$ are chains, it means that all elements in each of which are comparable.
Also, by linear sum, all elements of X have to be placed below all elements of Y, while preserving the original orders within X and Y. Therefore, $\forall x \in X, \forall y \in Y, x \leq y$.
So, $(X \oplus Y, \leq)$ is a chain.

Exercise 5

The pointwise order of two posets is an order.

Solution 5

Let $(X, \leq_X)$ and $(Y, \leq_Y)$ be two posets. Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$.

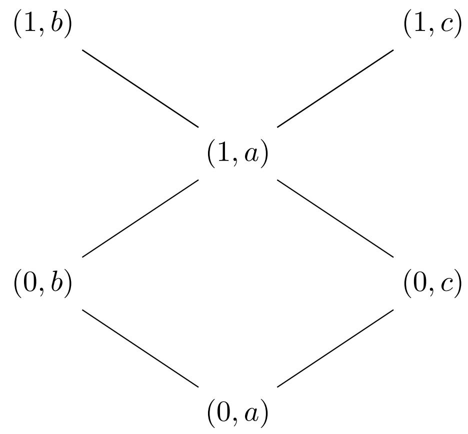
1. $(x_1, y_1) \leq (x_1, y_1) \Leftrightarrow x_1 \leq_X x_1 \wedge y_1 \leq_Y y_1$
2. $(x_1, y_1) \leq (x_2, y_2) \wedge (x_2, y_2) \leq (x_1, y_1) \Leftrightarrow (x_1 \leq_X x_2 \wedge y_1 \leq_Y y_2) \wedge (x_2 \leq_X x_1 \wedge y_2 \leq_Y y_1) \Leftrightarrow x_1 = x_2 \wedge y_1 = y_2 \Leftrightarrow (x_1, y_1) = (x_2, y_2)$
3. $(x_1, y_1) \leq (x_2, y_2) \wedge (x_2, y_2) \wedge (x_3, y_3) \Leftrightarrow (x_1 \leq_X x_2 \wedge y_1 \leq_Y y_2) \wedge (x_2 \leq_X x_3 \wedge y_2 \leq_Y y_3) \implies x_1 \leq_X x_3 \wedge y_1 \leq_Y y_3 \Leftrightarrow (x_1, y_1) \leq (x_3, y_3)$

Exercise 6

What is the Hase diagram of $B \times A$ with lexicographic order?

Solution 6

$B \times A = \{(0, a), (0, b), (0, c), (1, a), (1, b), (1, c)\}$



Exercise 7
If $(X, \leq_X)$ and $(Y, \leq_Y)$ are chains, then the lexicographic order on product is also a chain. However, this is not necessarily ture for the pointwise order on the product.

Solution 7
By definition of lexicographic order. If $(X, \leq_X)$ and $(Y, \leq_Y)$ are chains, then all elements in their product are comparable. Because it only depends on either $\leq_X$ or $\leq_Y$. However, in pointwise order, the elements in the product of $(X, \leq_X)$ and $(Y, \leq_Y)$ might be incomparable.
For example, $X = \{1, 2\}, Y = \{a, b, c\}$, their order is defined as natural and alphabetic.
Hence $X \times Y = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$, where $(1, b)$ and $(2, a)$ are incomparable.
Since $(1, b) \not\leq (2, a), 1 \leq_X 2$, but $b \not\leq a$, $(X \times Y, \leq)$ is not a chain.

Exercise 8
Let $(X, \leq)$ and $(Y, \leq)$ be posets, and let $f \colon X \to Y$ be a function. The following conditions are equivalent.

1. f is an order isomorphism.
2. f is a monotone bijective map and the map f^{-1} is monotone.

Solution 8

- Assume (i) is correct. So f is an onto order embedding. It means that:

$$x_1 \leq x_2 \Leftrightarrow f(x_1) \leq f(x_2)$$

  which is a monotone bijective map. $f^{-1} \colon Y \to X$, by (i), f is an onto function. Thus, every element in Y can be mapped by at least one element in X with $f(x)$. Also, f is one-to-one map, it means that 1 element in X will map to exactly one element in Y. By considering 2 properties, $\forall x_1, x_2 \in X, f^{-1}(f(x_1))$ and $f^{-1}(f(x_2))$ will map back to x_1 and x_2, respectively.

- Assume (ii) is correct. Because f is a monotone bijective map, it is order-embedding. Also, when f^{-1} is monotone, it implies that every element in Y can find the corresponding element in X meaning f^{-1} is an onto. Therefore, by combing order-embedding and onto, we can conclude that f is order-isomorphism.

Exercise 9
Prove that the following statements are true.

1. The function $f\colon (\mathbb{N}, \leq) \to (\mathbb{N}, \leq)$ defined by $f(n) = 2n$ is order-preserving.
2. The function $g\colon (\mathbb{R}, \leq) \to (\mathbb{R}, \leq)$ defined by $g(x) = x^2$ is not order-preserving. For instance, $-2 < -1$, but $g(-2) = 4 > 1 = g(-1)$.
3. The inclusion map $i\colon (\mathbb{N}, \leq) \to (\mathbb{Z}, \leq)$ defined by $i(n) = n$ is an order-embedding.
4. The function $f\colon (\mathbb{N}, \leq) \to (\mathbb{N}, \leq)$ defined by $f(n) = 2n$ is an order embedding.
5. The function $f\colon (\mathbb{R}, \leq) \to (\mathbb{R}, \leq)$ defined by $f(x) = \lfloor x \rfloor$ is order-preserving but is not an order embedding.
6. The function $f\colon (\mathbb{N}, \leq) \to (\mathbb{N}^*, \leq)$ defined by $f(n) = n + 1$ is an order isomorphism.
7. The function $h\colon (\mathbb{N} \times \mathbb{N}, \leq) \to (\mathbb{N}, \leq)$, where $\mathbb{N} \times \mathbb{N}$ is equipped with the pointwise order, defined by $f(x, y) = x + y$ is order-preserving but not an embedding.
8. If $(X_1, \leq)$ and $(X_2, \leq)$ are two posets that the projections maps $\pi_1, \pi_2\colon X_1 \times X_2 \to X_i$ defined as $\pi_i(x_1, x_2) = x_i$ is order-preserving if $X_1 \times X_2$ is equipped with the pointwise order.

Solution 9
(1)-(4) and (6) are trivial.

1. Let $f(x) = 1, f(y) = 1$, then: $f(x) \leq f(y)$, but (x, y) can be $(1.5, 1.3)$ which means $x \nleq y$. So, f is not order-embedding.
2. f is a monotone function. By pointwise-order, when $(x_1, y_1) \dot{\leq} (x_2, y_2) \Leftrightarrow x_1 \leq x_2 \wedge y_1 \leq y_2$. So, $x_1 + y_1 \leq x_2 + y_2 \Leftrightarrow f((x_1, y_1)) \leq f((x_2, y_2))$. f is not order-embedding function. If $f((x_1, y_1)) \leq f((x_2, y_2)) \equiv 3 \leq 3$ (x_1, y_1) can be $(2, 1)$ and (x_2, y_2) can be $(1, 2)$. In such a case, $(x_1, y_1) \nleq (x_2, y_2)$, so it is not order-embedding.
3. $\forall x_{11}, x_{12} \in X_1, \forall x_{21}, x_{22} \in X_2$, $(x_{11}, x_{12}) \leq (x_{12}, x_{22}) \Leftrightarrow x_{11} \leq x_{12} \wedge x_{21} \leq x_{22}$. So, by π_i, it will reserve one of dimensions, it means that the order relation will reserve the corresponding one.
   Therefore, $\begin{cases} (x_{11}, x_{12}) \leq (x_{12}, x_{22}) \Rightarrow \pi_1((x_{11}, x_{12})) \leq \pi_1((x_{12}, x_{22})) & \Leftrightarrow x_{11} \leq x_{12} \\ (x_{11}, x_{12}) \leq (x_{12}, x_{22}) \Rightarrow \pi_2((x_{11}, x_{12})) \leq \pi_2((x_{12}, x_{22})) & \Leftrightarrow x_{21} \leq x_{22} \end{cases}$.
   For lexicographic order, π_1 is order-preserving $\Leftrightarrow x_{21} \leq x_{22}$. Yet, π_2 is not.

Exercise 10
Let $X = \{a, b, c, d\}$. Prove that the powerset poset $(2^X, \subseteq)$ is isomorphic to the predicate post $(\mathbb{P}, \Rightarrow)$. Then, prove that the previous statement hods for any set X (even an infinite one).

Solution 10
Define $\phi\colon 2^X \to \mathbb{P}(X)$. Let $g \in 2^X, \phi(y) = \{x \in X \mid x \in y\}$.
By this definition, ϕ is order-isomorphism, bijective. Therefore, $(2^X, \subseteq) = (\mathbb{P}(X), \Rightarrow)$

Exercise 11
Prove that $(\mathbb{Z}, \leq)$ is isomorphic to $(\mathbb{Z}, \leq^\partial)$. Is $(\mathbb{N}, \leq)$ isomorphic to $(\mathbb{N}, \leq^\partial)$?

Solution 11
We define $f\colon (\mathbb{Z}, \leq) \to (\mathbb{Z}, \leq^\partial)$, $f(x) \triangleq -x$, such that $x_1 \leq x_2 \Leftrightarrow f(x_1) \leq^\partial f(x_2)$ Therefore, $(\mathbb{Z}, \leq) = (\mathbb{Z}, \leq^\partial)$.
Because, in $(\mathbb{N}, \leq)$, there is a bottom element, but top element. While, in $(\mathbb{N}, \leq^\partial)$, there is no bottom element but top element. So, we cannot find an order-isomorphism between them, they're not isomorphic.

Exercise 12
Characterize the $n \in \mathbb{N}$ whose divisor poset is isomorphic to $(2^{\{0,1\}}, \subset)$.

Solution 12
Prime number

Exercise 13
Prove that $(\mathbb{Z}, \leq)$ is isomorphic to $(\mathbb{N}, \leq^{\partial}) \oplus (\mathbb{N}, \leq)$.

Solution 13
Define $f\colon \mathbb{Z} \to \mathbb{N}$

$$f \triangleq \begin{cases} x-1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ |x|, & \text{if } x < 0 \end{cases}$$

Exercise 14
Let C be the set of subset X of $\mathbb{N}$ such that $\mathbb{N} \setminus X$ is finite. Show that C is a filter in $(2^{\mathbb{N}}, \subseteq)$.

Solution 14
Let $X \in C, Y \supseteq X$ such that $\mathbb{N} \setminus X$ is finite (by given condition). Then $\mathbb{N} \setminus X \supseteq \mathbb{N} \setminus Y$, therefore, $\mathbb{N} \setminus Y$ is also finite. It means that $Y \in C$. Hence, C is an up-set (upward-closed). In addition, for any $X, Y \in C$, there is $Z \in C$ such that $Z \subseteq X \wedge Z \subseteq Y$. This Z would be $\{\infty\}$. Assume that $X \wedge Y = Z, X \supseteq Z \wedge Y \supseteq Z$.$\mathbb{N} \setminus X$ is finite , $\mathbb{N} \setminus Y$ is finite , $\mathbb{N} \setminus Z = \mathbb{N} \setminus (X \cap Y) = \mathbb{N} \setminus X \cup \mathbb{N} \setminus Y$. Therefore, $C = \{X \in 2^{\mathbb{N}} \mid \{\infty\} \subseteq X\}$.

Exercise 15
Let $(X, \leq)$ be a poset and $Q \subseteq X$. We set

1. Show that $\downarrow Q$ is an down-set that contains Q. Deduce that $\uparrow Q$ is an up-et that contains Q.
2. Let $(Up(X), \supseteq)$ be the set of up-sets of $(X, \leq)$ ordered by reverse inclusion. Show that the map $f\colon X \to Up(X)$ defined as $f(x) = \uparrow x$ is an order-embedding.

Solution 15
$\downarrow Q = \{x \in X \mid \exists q \in Q x \leq q\}$, by definition. Down-set means that if $y \in Y \wedge x \leq y$, then $x \in Y$, where $Y \subseteq X$. Since $Q \subseteq X \wedge Q \neq \varnothing$, meaning that $(Q, \leq)$ is a poset as well. By definition of $\downarrow Q$, since $(X, \leq)\&(Q, \leq)$ are posets, $\forall x \in X \exists q \in Q, x \leq q, x$ can be q (reflexive) such that $\downarrow Q$ contains Q. Moreover, $\downarrow Q$ contains every element that is lower than at least one element in Q. It means that if we have an element $x \in X$ and an element $y \in \downarrow Q$ and their relation is $x \leq y$. Then it implies x must in $\downarrow Q$, by the definition of $\downarrow Q$. Therefore, $\downarrow Q$ is a down-set. Dually, $\uparrow Q$ is an up-set that contains Q.

- Show that the map $f\colon X \to UP(X)$ defined as $f(x) = \uparrow x$ is an order-embedding.
- Order embedding: $x \leq y \Leftrightarrow f(x) \supseteq f(y)$.
- $\uparrow x = \{y \in X \mid y \geq x\}$.

$(\Rightarrow)$
Since $x \leq y, |f(x)| \geq |f(y)|$. Also, by the definition of $\uparrow x$. $\uparrow x$ constains every element that above x (including y, and the elements above y). But x is below y, it means that $\uparrow y$ does not contain x. Still $\uparrow y$ contains every element that y. Hence $\uparrow y$ is a subset of $\uparrow x$ such that $\uparrow x \supseteq \uparrow y \Leftrightarrow f(x) \subseteq f(y)$.
$(\Leftarrow)$
When $f(x) \supseteq f(y) \Leftrightarrow \uparrow x \supset \uparrow y$ is true. We can find the minimal elements x, y in $f(x), f(y)$, respectively. Both elements are in X such that $x \leq y$. Therefore, the map f is an order-embedding, it is an injective function.

Exercise 16

Prove that if $(X, \leq)$ has a top element, then it is unique. Similarly, if $(X, \leq)$ has a bottom element, then it is unique.

Solution 16

Suppose that we have 2 top elements, x, y, in $(X, \leq)$. By the definition of top element, $x\&y$ are the upper bounds of X. By the definition of upper bound, we know that $x \leq \wedge y \leq x$. By antisymmetric, we can therefore conclude $x = y$. Therefore, if $(X, \leq)$ has a top element, it is unique. Dually, if $(X, \leq)$ has a bottom element, it is also unique.

Exercise 17

Prove that in any lattice $(L, \leq)$, for all $x, y, z \in L$:

1. $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$ (commutativity)
2. $(x \vee y) \vee z = x \vee (y \vee z)$ and $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ (associativity)
3. $x \vee x = x$ and $x \wedge x = x$ (idempotence)
4. $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$ (absorption)

Solution 17

1. $x \vee y = \{x, y\}^u = y \vee x$
2. $(x \vee y) \vee x = \{x, y\}^u \vee z = \{\{x, y\}^u, z\}^u = \{x, y, z\}^u, x \vee (y \vee z) = x \vee \{y, z\}^u = \{x, \{y, z\}^u\}^u = \{x, y, z\}^u$
3. $x \vee x = \{x, x\}^u$
4. $x \wedge y \leq x \Leftrightarrow x \vee (x \wedge y) \leq x \vee x \Leftrightarrow x \vee (x \wedge y) \leq x$. By join definition, $x \leq x \vee (x \wedge y)$. Therefore, $x \vee (x \wedge y) = x$.

Exercise 18

Prove that if $(L, \vee, \wedge, 0, 1)$ is an algebraic bounded lattice, then in the corresponding order-theoretic lattice $(L, \leq)$, the element 0 is the bottom element and 1 is the top element.

Solution 18

By the definition of bounded lattice algebra $\mathcal{L} = (L, \vee, \wedge, 0, 1)$.

1. $\forall x \in L, x \vee 0 = x,$ and $x \wedge 1 = x$.
2. $\forall x \in L, x \wedge 0 = 0,$ and $x \vee 1 = 1$.

By connecting lemma, we know that when $x \vee 0 = x, x \wedge 0 = 0$, it implies that $0 \leq x$, since every element $x \in L$ is greater than 0. In other words, 0 is a lower bound for all $x \in L$. Therefore, 0 is the bottom element in L. Similarly, when $x \vee 1 = 1, x \wedge 1 = x$, it means that $x \leq 1$, since every $x \in L$ is lower than 1. It is saying that 1 is an upper bound for all $x \in L$. Therefore, 1 is the top element in L.

Exercise 19

Consider the following statements about lattice operations and constructions:

1. The disjoint union of lattices is a lattice.
2. The linear sum of lattices is a lattice.
3. The lexicographic order on the product of lattices might not be a lattice.
4. The pointwise order on the product of lattices is always a lattice, with join and meet operations computed pointwise.

For each statement, provide a proof or a counterexample to justify why the statement is true or false.

Solution 19

1. false, when $x \in P, y \in Q, x\&y$ are incomparable. Also, meaning $x\&y$ do not have the least ub and the greatest lb.
2. true, by definition of linear sum, every element can always find $lub\&glb$.
3. true, in lexicographic order, the second dimension might not comparable such that it might not be a lattice. So $lub\&glb$ are defined by pointwise join/meet. For example, $\mathbb{M}_3 \times \mathbb{Z}, lub\&glb$ do not exist!
4. true.

Exercise 20

If $(L, \wedge, \vee)$ is a lattice and S is a sublattice of L, then $(S, \vee, \wedge)$ is a lattice.

Solution 20

When S is a sublattice of L, by definition, $\forall x, y \in S, x \vee y \in S, x \wedge y \in S$. It implies that every pair of element in S has the least upper bound and the greatest lower bound which are defined in S. Therefore, S itself is a lattice.

Exercise 21

Let $(L, \vee, \wedge, 0, 1)$ be a bounded lattice and S be a sublattice of L.

1. Prove that if S is a 0-sublattice, then 0 is the bottom element of S.
2. Prove that if S is a 1-sublattice, then 1 is the top element of S.
3. Give an example of a sublattice that has a bottom element different from 0 and a top element different from 1.

Solution 21

(1) & (2) are similar to exercise 63.
(3)
Let L is a bounded chain, $\{0, 1, 2, 3\}$, where top is 3 and bottom is 0. There exists a sublattice $S = \{1, 2\}$. 1 is the bottom of S, but not $0-$element in L, 2 is the top of S, but not $1-$element in L.

Exercise 22

Let $f\colon L_1 \to L_2$ be a lattice isomorphism. Prove that f^{-1} is a lattice isomorphism.

Solution 22

$f^{-1}\colon L_2 \to L_1$. Let $x_1, y_1 \in L_1, x_2, y_2 \in L_2$. We know $f(x_1 \vee y_1) = f(x_1) \vee f(y_1), f(x_1 \wedge y_1) = f(x_1) \wedge f(y_1)$. Also, f is bijective and monotone. We have to show that

1. f^{-1} is lattice homomorphism
2. f^{-1} is bijective

[join]
$f(x_1) \vee f(y_1) = f(x_1 \vee y_1) \Leftrightarrow f^{-1}(f(x_1)) \vee f^{-1}(f(y_1)) = f^{-1}(f(x_1 \vee y_1)) \Leftrightarrow f^{-1}(f(x_1)) \vee f^{-1}(f(y_1)) = f^{-1}(f(x_1) \vee f(y_1))$.
[meet]
$f(x_1) \wedge f(y_1) = f(x_1 \wedge y_1) \Leftrightarrow f^{-1}(f(x_1)) \wedge f^{-1}(f(y_1)) = f^{-1}(f(x_1 \wedge y_1)) \Leftrightarrow f^{-1}(f(x_1)) \wedge f^{-1}(f(y_1)) = f^{-1}(f(x_1) \wedge f(y_1))$.
Since f is bijective, f^{-1} will be bijective as well.

Exercise 23
Prove that if $f\colon L_1 \to L_2$ is a bounded lattice isomorphism, then $f^{-1}\colon L_2 \to L_1$ is also a bounded lattice isomorphism.

Solution 23
Given that $f\colon L_1 \to L_2$ is a bounded lattice isomorphism. So, we know f is bounded lattice isomorphism and bijective. Hence, let $x_1, y_1 \in L_1, x_2, y_2 \in L_2$.
[join]
$f(x_1 \vee y_1) = f(x_1) \vee f(y_1) \Leftrightarrow f^{-1}(f(x_1 \vee y_1)) = f^{-1}(f(x_1)) \vee f^{-1}(f(y_1)) \Leftrightarrow f^{-1}(f(x_1) \vee f(y_1)) = f^{-1}(f(x_1)) \vee f^{-1}(f(y_1)) \Leftrightarrow f^{-1}(x_2 \vee y_2) = f^{-1}(x_2) \vee f^{-1}(y_2)$
[meet]
$f(x_1 \wedge y_1) = f(x_1) \wedge f(y_1) \Leftrightarrow f^{-1}(f(x_1 \wedge y_1)) = f^{-1}(f(x_1)) \wedge f^{-1}(f(y_1)) \Leftrightarrow f^{-1}(f(x_1) \wedge f(y_1)) = f^{-1}(f(x_1)) \wedge f^{-1}(f(y_1)) \Leftrightarrow f^{-1}(x_2 \wedge y_2) = f^{-1}(x_2) \wedge f^{-1}(y_2)$
Since f is bijective, f^{-1} is bijective as well.

Exercise 24
Give an example of bounded lattices L_1 and L_2 and lattice homomorphism $f\colon L_1 \to L_2$ which is not a bounded lattice homomorphism.

Solution 24
Let $L_1 = \{\bot, a, b, \top\}$, where $a||b$. Let $L_2 = \{0, 1, 2, 3\}$, where its order is defined as natural. f is defined as $f(\bot) = f(a) = 1, f(b) = f(\top) = 2$. In this case, two tops and bottoms from L_1 and L_2 are not mapped.

Exercise 25
Give a precise inductive definition of the interpretation of a (bounde) lattice term on a (bounded) lattice.

Solution 25
It is very similar with the operations used in L and the elements shown in L.

1. $\forall x \in X, t^L(x) = x$
2. $t^L(0) = 0, t^L(1) = 1$
3. $t(x, y) = x \vee y, t^L(x, y) = x \vee y.t(x, y) = x \wedge y, t^L(x, y) = x \wedge y$

Exercise 26
Any lattice satisfies the equation $(x \vee y) \vee z = x \vee (y \vee z)$. If $X = \{a, b\}$ then the lattice $(2^X, \cup, \cap)$ satisfies the equation $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ while the lattice N_5 depicted Fig. 2 doesn't satisfy it.

Solution 26
$a \vee (b \wedge c) = a \neq (a \vee b) \wedge (a \vee c) = c$

Exercise 27
Let $\mathcal{K}$ be a class of (bounded) lattices and $t = s$ be a class of (bounded) lattice equations. Show that if every element of $\mathcal{K}$ satisfies $t = s$, then every element of $\mathbb{S}(\mathcal{K}), \mathbb{P}(\mathcal{K})$ and $\mathbb{H}(\mathcal{K})$ satisfies $t = s$.

Solution 27
$\forall L \in K \colon t^L(\overrightarrow{a} = s^L(\overrightarrow{a}))$, where $\overrightarrow{a} \in L$.

- sublattices: Let $subL$ as the set of some sublattices of L. $\forall SL \in subL \colon t^{SL}(\overrightarrow{b}) = t^L(\overrightarrow{b} = s^L(\overrightarrow{b}) = s^{SL}(\overrightarrow{b}))$. Therefore, $SL \in L$, meaning that to interpret $t^{SL}(\overrightarrow{b})$, we can use $t^L(\overrightarrow{b})$. Similar for s.
- Homomorphism image: Let $L_1, L_2 \in K, f \colon L_1 \to L_2$ is surjective homomorphism. $f(t^{L_1}(\overrightarrow{a})) = t^{L_2}(f(a_1), \ldots, f(a_n))$. $t^{L_2}(f(\overrightarrow{a})) = f(s^{L_1}(\overrightarrow{a})) = s^{L_2}(\overrightarrow{a})$.
- Products: Let $(L_i)_{i \in I}$ be a family of lattices. $\pi_{i \in I} L_i$ is the set of all tuples $(a_i)_{i \in I}$ with $a_i \in L_i$, equipped with pointwise operators. $t^L(\overrightarrow{a}) = (t^{L_i}(\overrightarrow{a}_i))_i = (s^{L_i}(\overrightarrow{a}_i)_i) = s^L(\overrightarrow{a})$

Exercise 28
Give an example of a non-distributive lattice whose every element has a complement, but there is a at least one element which has tow complements.

Solution 28
$\mathbb{M}_3$ or $\mathbb{N}_5$

Exercise 29
Show that the class of complemented bounded distributive lattices is not equational.

Solution 29
This is a complemental bounded distributive lattices. Yet, its sublattices are not, e.g., $\{\perp, a, \top\}$. In general, any chain in this lattice is not satisified. Therefore, a does not have a complement.

Exercise 30
Prove that in a Boolean algebra, De Morgan's laws hold:

$$(x \vee y)' = x' \wedge y' \text{ and } (x \wedge y)' = x' \vee y'$$

Solution 30
$(x \vee y) \vee (x' \wedge y') = (x \vee y \vee x') \wedge (x \vee y \vee y') = (1 \vee y) \wedge (x \vee 1) = 1 \wedge 1 = 1$
$(x \wedge y) \wedge (x' \vee y') = (x \wedge y \wedge x') \vee (x \wedge y \wedge y') = (0 \wedge y) \vee (x \wedge 0) = 0 \vee 0 = 0$

Exercise 31
Prove that in a Boolean algebra B, the complementation operation is a bounded lattice isomorphism between the underlying bounded lattice of B and its order dual. (*Hint:* Prove that (i) B^∂ equipped with the complementation operation from B is a Boolean algebra, (ii) the complementation operation $' \colon B \to B^\partial$ is a monotone map which is its own inverse.)

Solution 31

Exercise 32
Let B be the set of subsets X of $\mathbb{N}$ that satisfy B is finite or $\mathbb{N} \setminus B$ is finite. Show that B is a Boolean algebra.

Solution 32

Exercise 33
We adopt the notation of the previous paragraphs.

1. Show that $\downarrow f$ is an ideal in $X \multimap Y$.
2. Show that $\downarrow f$ is a complete bounded lattice in its own right, with $\varnothing$ and f as bottom and tope element, respectively.

Solution 33

Exercise 34
Let $c\colon 2^{\mathbb{R}^\infty} \to 2^{\mathbb{R}^\infty}$ be defined as $c(X) = (\bigvee X]$.

1. Show that c is a closure operator.
2. Describe the set K of closed elements.
3. Describe the complete lattice operation on K.
4. Describe $c(x)$ in terms of closed elements for every $x \in 2^{\mathbb{R}^\infty}$.

Solution 34

Exercise 35
Let $(L, \wedge, \vee, \perp, \top)$ be a bounded lattice, and X be an infinite set of variables. For any $n \geq 0$, let $T_n(X)$ the set of $n-$ary bounded lattice terms over X and set $T(X) \coloneqq \cup_{n\geq 0} T_n(X)$. Define $c\colon 2^L \to 2^L$ by

$$c(A) = \{t^L(a_1, ..., a_n) \mid n \geq 0, t \in T_n(X) \text{ and } a_1, ..., a_n \in A\}.$$

1. Show that c is a closure operator.
2. Describe the set K of closed elements.
3. Describe the complete lattice operations on K.
4. Describe $c(A)$ in terms of closed elements for every $A \in 2^L$.

Solution 35

Exercise 36
Show that the constructions of Table 1 define are mutually inverse.

Solution 36

Exercise 37
Let $(X, \leq)$ be a poset. Define $U := \{\uparrow x \mid x \in X\}$ and $D := \{\downarrow x \mid x \in X\}$, as well as the maps $\alpha\colon D \to U$ and $\gamma\colon U \to D$ defined as $\gamma(\uparrow x) :=\downarrow x$ and $\alpha(\downarrow x) :=\uparrow x$. Prove that

$$(L, \leq) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} (M, \sqsubseteq)$$

Solution 37

Exercise 38
Let $(C, \leq)$ be a complete totally ordered set and $\to$ be the binary operation defined on C by $a \to b := \bigvee\{x \in C \mid a \wedge x \leq b\}$. Prove that for any $a \in C$, we have

Solution 38

Exercise 39
(From Galois Connection to Closure Operator). Assume that

Solution 39

Exercise 40
Identify the Galois connections among the Galois connections we have introduced so far.

Solution 40