Lattice Theory for Parallel Programming
Solutions for exercises

1 Theoretical Exercises

Exercise 1
Show that if $(X, \leq)$ is a poset, then $(X, \leq^{\partial})$ is also a poset.

Exercise 2
Show that $(\mathbb{P}(X), \Rightarrow)$ is a poset.

Exercise 3
If $(X, \leq_X)$ and $(Y, \leq_Y)$ are chains, then their linear sum $(X \bigoplus Y, \leq)$ is also a chain.

Exercise 4
The pointwise order of two posets is an order.

Exercise 5
If $(X, \leq_X)$ and $(Y, \leq_Y)$ are chains, then the lexicographic order on product is also a chain. However, this is not necessarily ture for the pointwise order on the product.

Exercise 6
Let $(X, \leq)$ and $(Y, \leq)$ be posets, and let $f\colon X \to Y$ be a function. The following conditions are equivalent.

1. f is an order isomorphism.
2. f is a monotone bijective map and the map f^{-1} is monotone.

Exercise 7
Prove that the following statements are true.

1. The function $f\colon (\mathbb{N}, \leq) \to (\mathbb{N}, \leq)$ defined by $f(n) = 2n$ is order-preserving.
2. The function $g\colon (\mathbb{R}, \leq) \to (\mathbb{R}, \leq)$ defined by $g(x) = x^2$ is not order-preserving. For instance, $-2 < -1$, but $g(-2) = 4 > 1 = g(-1)$.
3. The inclusion map $i\colon (\mathbb{N}, \leq) \to (\mathbb{Z}, \leq)$ defined by $i(n) = n$ is an order-embedding.

4. The function $f\colon (\mathbb{N}, \leq) \to (\mathbb{N}, \leq)$ defined by $f(n) = 2n$ is an order embedding.
5. The function $f\colon (\mathbb{R}, \leq) \to (\mathbb{R}, \leq)$ defined by $f(x) = \lfloor x \rfloor$ is order-preserving but is not an order embedding.
6. The function $f\colon (\mathbb{N}, \leq) \to (\mathbb{N}^*, \leq)$ defined by $f(n) = n + 1$ is an order isomorphism.
7. The function $h\colon (\mathbb{N} \times \mathbb{N}, \leq) \to (\mathbb{N}, \leq)$, where $\mathbb{N} \times \mathbb{N}$ is equipped with the pointwise order, defined by $f(x, y) = x + y$ is order-preserving but not an embedding.
8. If $(X_1, \leq)$ and $(X_2, \leq)$ are two posets that the projections maps $\pi_1, \pi_2\colon X_1 \times X_2 \to X_i$ defined as $\pi_i(x_1, x_2) = x_i$ is order-preserving if $X_1 \times X_2$ is equipped with the pointwise order.

Exercise 8
Let $X = \{a, b, c, d\}$. Prove that the powerset poset $(2^X, \subset)$ is isomorphic to the predicate post $(\mathbb{P}, \Rightarrow)$. Then, prove that the previous statement hods for any set X (even an infinite one).

Exercise 9
Prove that $(\mathbb{Z}, \leq)$ is isomorphic to $(\mathbb{Z}, \leq^\partial)$. Is $(\mathbb{N}, \leq)$ isomorphic to $(\mathbb{N}, \leq^\partial)$?

Exercise 10
Characterize the $n \in \mathbb{N}$ whose divisor poset is isomorphic to $(2^{\{0,1\}}, \subset)$.

Exercise 11
Prove that $(\mathbb{Z}, \leq)$ is isomorphic to $(\mathbb{N}, \leq^\partial) \bigoplus (\mathbb{N}, \leq)$.

Exercise 12
Let C be the set of subset X of $\mathbb{N}$ such that $\mathbb{N} \setminus X$ is finite. Show that C is a filter in $(2^{\mathbb{N}}, \subset)$.

Exercise 13
Let $(X, \leq)$ be a poset and $Q \subset X$. We set

1. Show that $\downarrow Q$ is an down-set that contains Q. Deduce that $\uparrow Q$ is an up-et that contains Q.
2. Let $(Up(X), \supseteq)$ be the set of up-sets of $(X, \leq)$ ordered by reverse inclusion. Show that the map $f\colon X \to Up(X)$ defined as $f(x) = \uparrow x$ is an order-embedding.

Exercise 14
Prove that if $(X, \leq)$ has a top element, then it is unique. Similarly, if $(X, \leq)$ has a bottom element, then it is unique.

Exercise 15
Let $(X, \leq)$ be a poset and S be a subset of X. Prove that if S has a least upper bound then it is unique. Deduce that if S has a greatest lower bound, then it is unique.

Exercise 16
Prove that in any lattice $(L, \leq)$, for all $x, y, z \in L$:

1. $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$ (commutativity)
2. $(x \vee y) \vee z = x \vee (y \vee z)$ and $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ (associativity)
3. $x \vee x = x$ and $x \wedge x = x$ (idempotence)
4. $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$ (absorption)

Exercise 17
Prove that if $(L, \vee, \wedge, 0, 1)$ is an algebraic bounded lattice, then in the corresponding order-theoretic lattice $(L, \leq)$, the element 0 is the bottom element and 1 is the top element.

Exercise 18
Consider the following statements about lattice operations and constructions:

1. The disjoint union of lattices is a lattice.
2. The linear sum of lattices is a lattice.
3. The lexicographic order on the product of lattices might not be a lattice.
4. The pointwise order on the product of lattices is always a lattice, with join and meet operations computed pointwise.

For each statement, provide a proof or a counterexample to justify why the statement is true or false.

Exercise 19
If $(L, \wedge, \vee)$ is a lattice and S is a sublattice of L, then $(S, \vee, \wedge)$ is a lattice.

Exercise 20
Let $(L, \vee, \wedge, 0, 1)$ be a bounded lattice and S be a sublattice of L.

1. Prove that if S is a 0-sublattice, then 0 is the bottom element of S.
2. Prove that if S is a 1-sublattice, then 1 is the top element of S.
3. Give an example of a sublattice that has a bottom element different from 0 and a top element different from 1.

Exercise 21
Let $f\colon L_1 \to L_2$ be a lattice isomorphism. Prove that f^{-1} is a lattice isomorphism.

Exercise 22
Prove that if $f\colon L_1 \to L_2$ is a bounded lattice isomorphism, then $f^{-1}\colon L_2 \to L_1$ is also a bounded lattice isomorphism.

Exercise 23
Give an example of bounded lattices L_1 and L_2 and lattice homomorphism $f\colon L_1 \to L_2$ which is not a bounded lattice homomorphism.

Exercise 24
Give a precise inductive definition of the interpretation of a (bounde) lattice term on a (bounded) lattice.

Exercise 25
Any lattice satisfies the equation $(x \vee y) \vee z = x \vee (y \vee z)$. If $X = \{a, b\}$ then the lattice $(2^X, \cup, \cap)$ satisfies the equation $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ while the lattice N_5 depicted Fig. 2 doesn't satisfy it.

Exercise 26
Let $\mathcal{K}$ be a class of (bounded) lattices and $t = s$ be a class of (bounded) lattice equations. Show that if every element of $\mathcal{K}$ satisfies $t = s$, then every element of $\mathbb{S}(\mathcal{K}), \mathbb{P}(\mathcal{K})$ and $\mathbb{H}(\mathcal{K})$ satisfies $t = s$.

Exercise 27
Give an example of a non-distributive lattice whose every element has a complement, but there is a at least one element which has tow complements.

Exercise 28
Show that the class of complemented bounded distributive lattices is not equational.

Exercise 29
Prove that in a Boolean algebra, De Morgan's laws hold:

$$(x \vee y)' = x' \wedge y' \text{ and } (x \wedge y)' = x' \vee y'$$

Exercise 30
Prove that in a Boolean algebra B, the complementation operation is a bounded lattice isomorphism between the underlying bounded lattice of B and its order dual. (*Hint:* Prove that (i) B^{∂} equipped with the complementation operation from B is a Boolean algebra, (ii) the complementation operation $\prime \colon B \to B^{\partial}$ is a monotone map which is its own inverse.)

Exercise 31
Let B be the set of subsets X of $\mathbb{N}$ that satisfy B is finite or $\mathbb{N} \setminus B$ is finite. Show that B is a Boolean algebra.

Exercise 32
We adopt the notation of the previous paragraphs.

1. Show that $\downarrow f$ is an ideal in $X \mathrel{-\!\!\ominus\!\!\rightarrow} Y$.
2. Show that $\downarrow f$ is a complete bounded lattice in its own right, with $\varnothing$ and f as bottom and tope element, respectively.

Exercise 33
Let $c \colon 2^{\mathbb{R}^{\infty}} \to 2^{\mathbb{R}^{\infty}}$ be defined as $c(X) = (\bigvee X]$.

1. Show that c is a closure operator.
2. Describe the set K of closed elements.
3. Describe the complete lattice operation on K.
4. Describe $c(x)$ in terms of closed elements for every $x \in 2^{\mathbb{R}^{\infty}}$.

Exercise 34
Let $(L, \wedge, \vee, \bot, \top)$ be a bounded lattice, and X be an infinite set of variables. For any $n \geq 0$, let $T_n(X)$ the set of $n-$ary bounded lattice terms over X and set $T(X) := \cup_{n \geq 0} T_n(X)$. Define $c \colon 2^L \to 2^L$ by

$$c(A) = \{t^L(a_1, ..., a_n) \mid n \geq 0, t \in T_n(X) \text{ and } a_1, ..., a_n \in A\}.$$

1. Show that c is a closure operator.
2. Describe the set K of closed elements.
3. Describe the complete lattice operations on K.
4. Describe $c(A)$ in terms of closed elements for every $A \in 2^L$.

Exercise 35
Show that the constructions of Table 1 define are mutually inverse.

Exercise 36
Let $(X, \leq)$ be a poset. Define $U \coloneqq \{\uparrow x \mid x \in X\}$ and $D \coloneqq \{\downarrow x \mid x \in X\}$, as well as the maps $\alpha \colon D \to U$ and $\gamma \colon U \to D$ defined as $\gamma(\uparrow x) \coloneqq \downarrow x$ and $\alpha(\downarrow x) \coloneqq \uparrow x$. Prove that

Exercise 37
Let $(C, \leq)$ be a complete totally ordered set and $\to$ be the binary operation defined on C by $a \to b \coloneqq \bigvee\{x \in C \mid a \wedge x \leq b\}$. Prove that for any $a \in C$, we have

Exercise 38
(From Galois Connection to Closure Operator). Assume that

Exercise 39
Identify the Galois connections among the Galois connections we have introduced so far.