Lattice Theory

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1 Ordered Sets

An ordering is a binary relation on a set of objects.

Definition 1.1. *Let $\boldsymbol{P}$ be a set. An **order** (or **partial order**) on $\boldsymbol{P}$ is a binary relation $\leq$ on $\boldsymbol{P}$ such that for all $x, y, z \in \boldsymbol{P}$.*

- *(reflexivity)* $x \leq x$
- *(antisymmetry)* $x \leq y \wedge y \leq x \implies x = y$
- *(transitivity)* $x \leq y \wedge y \leq z \implies x \leq z$

$$
\begin{aligned}
\langle \boldsymbol{P}; \leq \rangle &\coloneqq \text{"ordered set" or "partially ordered set", poset.} \\
&\coloneqq \text{A set } \boldsymbol{P} \text{ equipped with an order relation } \leq .
\end{aligned}
$$

discrete order $\coloneqq$ "=" is an order on any set.
guasi-order or pre-order $\coloneqq$
induced order $\coloneqq x, y \in \boldsymbol{Q}, x \leq y \iff x \leq y \in \boldsymbol{P},$ where $\boldsymbol{Q} \subseteq \boldsymbol{P}$.

Definition 1.2. *Let $\boldsymbol{P}$ be an ordered set. Then $\boldsymbol{P}$ is a **chain** if for all $, y \in \boldsymbol{P}$, either $x \leq y$ or $y \leq x$.*

Chain can be called, *linearly ordered set*, or *totally ordered set*.
Antichain $\coloneqq$ if $x \leq y$ in $\boldsymbol{P}$ only if $x = y$. Hence, any set $\boldsymbol{S}$ may be converted into an antichain $\bar{\boldsymbol{S}}$ by giving $\boldsymbol{S}$ the discrete order.

Definition 1.3. *Order-isomorphisms $\boldsymbol{P} \cong \boldsymbol{Q}$, if there exists a map φ from $\boldsymbol{P}$ onto $\boldsymbol{Q}$ such that $x \leq y$ in $\boldsymbol{P} \iff \varphi(x) \leq \varphi(y)$ in $\boldsymbol{Q}$. Then φ is called an "order-isomorphism".*

Note: order-isomorphism $\Rightarrow$ bijection (one-to-one and onto). Hence,

$$\varphi \colon \boldsymbol{P} \to \boldsymbol{Q}, \quad \varphi^{-1} \colon \boldsymbol{Q} \to \boldsymbol{P}$$

.

Definition 1.4. *Powerset $\wp(\boldsymbol{X})$, consisting of all subsets of x, is ordered by set inclusion: for $A, B \in \wp(\boldsymbol{X})$, we define: $A \leq B \iff A \subseteq B$.*

Note: any subset of $\wp(\boldsymbol{X})$ inherits the inclusion order.

$$
\begin{aligned}
\mathit{Predicate} &\coloneqq \text{ A statement taking value } T(\text{true}) \text{ or value } F(\text{false}). \\
&\coloneqq x \to \{T, F\}.
\end{aligned}
$$

Example 1.5.

$$p\colon \mathbb{R} \to \{T, F\}$$

$$p(x) = \begin{cases} T, & \textit{if } x \geq 0 \\ F, & \textit{if } x < 0 \end{cases}$$

$\mathbb{P}(\boldsymbol{X}) \coloneqq$ The set of predicates on $\boldsymbol{X}$.
Let p and q are predicate. Then we denote by

$$p \Rrightarrow q \iff \{x \in \boldsymbol{X} \mid p(x) = T\} \subseteq \{x \in \boldsymbol{X} \mid g(x) = T\}.$$

Definition 1.6. *The convering relation Let $\boldsymbol{P}$ be an ordered set, $x, y \in \boldsymbol{P}$*

$$\left\{ \begin{array}{l} x \textit{ is covered by } y \\ y \textit{ covers } x \end{array} \right\}.$$

$$x \prec y \textit{ or } y \succ x$$

Note: $x < y \wedge x \leq z < y \to x = z$.
Note: if $\boldsymbol{P}$ is finite,

$$x < y \iff \text{there exists a finite sequence of covering relations}$$
$$x = x_0 \prec x_1 \prec x_2 \cdots \prec x_n = y$$

Example 1.7. *In the chain $\mathbb{N}$, we have $m \prec n \iff n = m + 1$.*

Example 1.8. *In $\mathbb{R}$, there are no pairs x, y such that $x \prec y$.*

Example 1.9. *In $\wp(\boldsymbol{X})$, we have*

$$\boldsymbol{A} \prec \boldsymbol{B} \iff \boldsymbol{B} = \boldsymbol{A} \cup \{b\},$$

for same $b \in \boldsymbol{X} \setminus \boldsymbol{A}$.

Diagrams Rules: (Hasse diagram)

1. To each point $x \in \boldsymbol{P}$, associate a point $p(x)$ of the Euclidean plane $\mathbb{R}^2$, depicted by a small circle with centre at $p(x)$.
2. For each covering pair $x \prec y$ in $\boldsymbol{P}$, take a line segment $\ell(x, y)$ joining the circle at $p(x)$ to the circle at $p(y)$.
3. Carry out 1. and 2. in such a way that:
   (a) if $x \prec y$, then $p(x)$ is "lower" than $p(y)$.
   (b) the circle at $p(z)$ does not intersect the line segment $\ell(x, y)$ if $z \neq x \wedge z \neq y$.

By diagram,

$$x < y \iff \text{there is a sequence of connected line segments moving upwards from } x \text{ to } y.$$

Lemma 1.10. *Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be finite ordered sets and let $\varphi \colon \boldsymbol{P} \to \boldsymbol{Q}$ be a bijective map. Then the following are equivalent:*

1. *φ is an order-isomorphism.*
2. *$x < y$ in $\boldsymbol{P} \iff \varphi(x) < \varphi(y)$ in $\boldsymbol{Q}$.*
3. *$x \prec y$ in $\boldsymbol{P} \iff \varphi(x) \prec \varphi(y)$ in $\boldsymbol{Q}$.*

Proposition 1. *Two finite ordered sets $\boldsymbol{P}$ and $\boldsymbol{Q}$ are order-isomorphic iff they can be drawn with identical diagrams.*

Definition 1.11. *The dual of an ordered set Given any ordered set $\boldsymbol{P}$, we can form a new ordered set $\boldsymbol{P}^\partial$ (the dual of $\boldsymbol{P}$) by defining $x \leq y$ to hold in $\boldsymbol{P}^\partial \iff y \leq x$ holds in $\boldsymbol{P}$.*

The Duality Principle:
Given a statement Φ about ordered sets which is true in all ordered sets, the dual statement Φ^∂ is also true in all ordered set.

Note:

$$\begin{aligned} \perp \in \boldsymbol{P}&: \text{ bottom, if } \perp \leq x, \forall x \in \boldsymbol{P}. \\ \top \in \boldsymbol{P}&: \text{ top, if } \top \geq x, \forall x \in \boldsymbol{P}. \end{aligned}$$

PS $\perp$ and $\top$ are unique, by duality principle and antisymmetry.

Definition 1.12. *Maximal (MaxQ) Let $\boldsymbol{P}$ be an ordered set and let $\boldsymbol{Q} \subseteq \boldsymbol{P}$. Then $a \in \boldsymbol{Q}$ is a maximal element of $\boldsymbol{Q}$ if $a \leq x \wedge x \in \boldsymbol{Q} \implies a = x$.*

Definition 1.13. *Minimal (MinQ) Let $\boldsymbol{P}$ be an ordered set and let $\boldsymbol{Q} \subseteq \boldsymbol{P}$. Then $a \in \boldsymbol{Q}$ is a minimal element of $\boldsymbol{Q}$ if $a \geq x \wedge x \in \boldsymbol{Q} \implies a = x$.*

Definition 1.14. *Maximum ($\top_Q = maxQ$) If $\boldsymbol{Q}$ (with the order inherited from $\boldsymbol{P}$) has a top element, $\top_Q$, then $MaxQ = \{\top_Q\}$.*

Definition 1.15. *Minimum ($\bot_Q = minQ$) If $\boldsymbol{Q}$ (with the order inherited from $\boldsymbol{P}$) has a bottom element, $\bot_Q$, then $MinQ = \{\bot_Q\}$.*