Hypothesis Testing: Part 2

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Simple vs. composite: UMP tests

Two-sided tests

Now consider testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$. It can be shown (proof omitted) that if the family has monotone likelihood ratio then the power functions of the UMP 1-sided tests are strictly monotone.

Each of the 1-sided UMP test works well for their stated 1-sided H_1. However, for the 2-sided H_1, there exists θ such that the tests are biased in the sense that

$$E_\theta[\delta_0(\mathbf{X})] < E_{\theta_0}[\delta_0(\mathbf{X})].$$

Simple vs. composite: UMPU tests

An unbiased test has power function no small than α for all θ under H_1.

For 1-parameter exponential families with PDF $f_\theta(x) = h(x)e^{\theta T(x) - A(\theta)}$, a UMP-Unbiased (UMPU) test exists.

Theorem: For a 1-parameter exponential family with sufficient statistic $T(\mathbf{X})$, a UMPU test at level α for testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$ exists and is given by:

$$\delta(\mathbf{X}) = \begin{cases} 1, & T(\mathbf{X}) > c_2 \text{ or } T(\mathbf{X}) < c_1 \\ \gamma_i, & T(\mathbf{X}) = c_i,\ i = 1, 2 \\ 0, & c_1 < T(\mathbf{X}) < c_2 \end{cases}$$

where c_i, γ_i, $i = 1, 2$, are chosen such that $E_{\theta_0}[\delta(\mathbf{X})] = \alpha$ and $E_{\theta_0}[T(\mathbf{X})\delta(\mathbf{X})] = E_{\theta_0}[T(\mathbf{X})]E_{\theta_0}[\delta(\mathbf{X})] = \alpha E_{\theta_0}[T(\mathbf{X})]$.

Simple vs. composite: UMPU tests

Example: Suppose we can observe a sample $X \sim \exp(\theta)$, with PDF $f_\theta(x) = \frac{1}{\theta}e^{-x/\theta}$, $x > 0$. Determine the UMPU test for testing $H_0 : \theta = 1$ vs. $H_1 : \theta \neq 1$.

Simple vs. composite: UMPU tests

Example: Suppose $X \sim \exp(\theta)$, $f_\theta(x) = \frac{1}{\theta} e^{-x/\theta}$, $x > 0$. Determine the UMPU test for testing $H_0 : \theta = 1$ vs. $H_1 : \theta \neq 1$.

Simple vs. Composite

- We have mainly presented methods in this case applicable to 1-parameter exponential families.
- The general properties of MLE also lead to generally applicable testing methods.
- Suppose we have $\mathbf{X} = (X_1, \ldots, X_n)$ iid with common $f_\theta(\cdot)$ for the parametric family $\{f_\theta(\cdot) : \theta \in \Theta\}$, $\Theta \in \mathbb{R}$.
- Recall the most power test for $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$ is given by the Neyman-Pearson likelihood ratio (NPLR) test statistic

$$\frac{\prod_{i=1}^n f_{\theta_1}(X_i)}{\prod_{i=1}^n f_{\theta_0}(X_i)},$$

  which requires knowing both θ_0 and θ_1.
- If we are testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \in \Theta \backslash \{\theta_0\}$, we could try to first "estimate" a θ_1 value and plug it into the NPLR statistic.

Simple vs. Composite: GLRT

Definition: The Generalised likelihood ratio test (GLRT) for testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \in \Theta \backslash \{\theta_0\}$ uses the statistic:

$$\frac{\prod_{i=1}^{n} f_{\hat{\theta}}(X_i)}{\prod_{i=1}^{n} f_{\theta_0}(X_i)},$$

where $\hat{\theta} = \arg\max_{\theta \in \Theta} \prod_{i=1}^{n} f_{\theta}(X_i)$.

Simple vs. Composite: GLRT

The limiting distribution of the log-GLRT statistic under H_0 is $\frac{1}{2}\chi_1^2$ (under some regularity conditions).

Sketch proof:

Simple vs. Composite: GLRT

(Proof continued)

Simple vs. Composite: GLRT

(Proof continued)

Composite vs. Composite

Suppose $\mathbf{X} \sim f_\theta(\mathbf{x})$ for a 1-parameter family $\{f_\theta(\cdot) : \theta \in \Theta\}$, $\Theta \in \mathbb{R}$. We are testing $H_0 : \theta \in \Theta_0$ vs. $H_1 : \theta \in \Theta \backslash \Theta_0$, $\Theta_0 \subseteq \Theta$.

For certain composite H_0, optimal tests exist.

Proposition: If the family has monotone likelihood ratio in a statistic $T(\mathbf{X})$, then the UMP test of $H_0 : \theta \leq \theta_0$ vs. $H_1 : \theta > \theta_0$ is of the same form as for $H_0 : \theta = \theta_0$ vs. $H_1 : \theta > \theta_0$:

$$\delta(\mathbf{X}) = \begin{cases} 1, & T(\mathbf{X}) > c \\ \gamma, & T(\mathbf{X}) = c \\ 0, & T(\mathbf{X}) < c, \end{cases}$$

where c, γ are chosen such that $E_{\theta_0}[\delta(\mathbf{X})] = \alpha$.

Composite vs. Composite

Proposition: For a 1-parameter exponential family:

$$f_\theta(x) = h(x)\exp(w(\theta)T(x) - A(\theta))$$

where $w(\theta)$ is strictly increasing in θ. We are testing $H_0 : \theta \leq \theta_1$ or $\theta \geq \theta_2$ $(\theta_1 < \theta_2)$ vs. $H_1 : \theta_1 < \theta < \theta_2$. The UMPU test exists and is of the form:

$$\delta(\mathbf{X}) = \begin{cases} 1, & c_1 < T(\mathbf{X}) < c_2 \\ \gamma_i, & T(\mathbf{X}) = \gamma_i,\ i = 1,2 \\ 0, & T(\mathbf{X}) < c_1 \text{ or } T(\mathbf{X}) > c_2, \end{cases}$$

where c_i, γ_i, $i = 1, 2$, are selected such that $E_{\theta_1}[\delta(\mathbf{X})] = E_{\theta_2}[\delta(\mathbf{X})] = \alpha$. (Such tests are of interest when trying to show a new drug is "effectively equivalent" to some standard.)

Multivariate case

Suppose we have a family of distributions indexed by more than 1 parameter. The GLRT provides a useful general method of testing $H_0 : \theta \in \Theta_0$ vs. $H_1 : \theta \in \Theta \backslash \Theta_0$. Compute the test statistic:

$$\log\left(\frac{L(\hat{\theta}; \mathbf{X})}{L(\hat{\theta}_0; \mathbf{X})}\right)$$

where

$$\hat{\theta} = \arg\max_{\theta \in \Theta} L(\theta; \mathbf{X})$$

is the "unrestricted" MLE, and

$$\hat{\theta}_0 = \arg\max_{\theta \in \Theta_0} L(\theta; \mathbf{X}).$$

Multivariate case

Many commonly used statistical tests are equivalent to the GLRT.

Example: **1-way ANOVA F-test.** Consider $X_{ij} \sim N(\mu_i, \sigma^2)$, for $i = 1, \ldots, g$ being indexes for groups and $j = 1, \ldots, n_i$, and all X_{ij} are independent. The total sample size is denoted by $N = \sum_{i=1}^{g} n_i$. Consider testing $H_0 : \mu_1 = \mu_2 = \cdots = \mu_g$ vs. $H_1 : \mu_i$'s are not all equal.

Multivariate case

(Example continued)

Multivariate case

(Example continued)

Multivariate case

(Example continued)

Multivariate case

Example: **one-sided t-test.** Let $X_1, \ldots, X_n$ be iid $N(\mu, \sigma^2)$, and $\Theta = \{(\mu, \sigma^2) : \mu \geq 0,\ \sigma^2 > 0\}$. Consider testing $H_0 : \mu = 0$ vs. $H_1 : \mu > 0$.

Multivariate case

(Example continued)

Multivariate case

(Example continued)

Multivariate case

(Example continued)

Multivariate case

(Example continued)

Multivariate case

(Example continued)