THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Tutorial Week 12

STAT3023: Statistical Inference — Semester 2, 2023

1. Suppose $X \sim B(n, \theta)$ and that $\tilde{d}(X)$ is the Bayes procedure based on a $U[\theta_0, \theta_1]$ prior under squared-error loss. Suppose also that for all $\theta_0 < \theta < \theta_1$,

$$\lim_{n\to\infty} nE_\theta\left\{\left[\tilde{d}(X) - \theta\right]^2\right\} \to \theta(1-\theta)\,.$$

Use the Asymptotic Minimax Lower Bound Theorem to show that the maximum likelihood estimator of θ is asymptotically minimax (over any interval $[a, b]$ for $0 < a < b < 1$).

2. Suppose $\mathbf{X} = (X_1, \ldots, X_n)$ consists of iid random variables with a gamma distribution with known shape α_0 but unknown *scale* parameter $\theta = \Theta = (0, \infty)$. Consider the decision problem where the decision space is $\mathcal{D} = \Theta$ and loss is $L(d|\theta) = (d - \theta)^2$. Write $T = \sum_{i=1}^n X_i$.

(a) Define the family of estimators $\{d_{k\ell}(\cdot)\colon k, \ell \in \mathbb{R}\}$ according to

$$d_{k\ell}(\mathbf{X}) = \frac{T + k}{n\alpha_0 + \ell}\,.$$

Determine the risk

$$R(\theta|d_{k\ell}) = E_\theta\left\{\left[d_{k\ell}(\mathbf{X}) - \theta\right]^2\right\}\,.$$

(b) Determine $d_{\text{flat}}(\mathbf{X})$, the Bayes procedure using the "flat prior" $w(\theta) \equiv 1$.

(c) Show that for any $k, \ell \in \mathbb{R}$, $d_{k\ell}(\mathbf{X})$ is asymptotically minimax. You may assume that for any $0 \leq \theta_0 < \theta_1 < \infty$, the Bayes procedure $\widetilde{d}(\mathbf{X})$ based on the $U[\theta_0, \theta_1]$ prior has the same limiting (rescaled) risk as $d_{\text{flat}}(\mathbf{X})$: for all $\theta_0 < \theta < \theta_1$,

$$\lim_{n\to\infty} nR(\theta|\widetilde{d}) = \lim_{n\to\infty} nR(\theta|d_{\text{flat}})\,. \tag{1}$$

(d) Show that

(i) the maximum likelihood estimator;

(ii) $d_{\text{flat}}(\mathbf{X})$;

(iii) any Bayes procedure based on an Inverse Gamma (conjugate) prior

are all asymptotically minimax.

3. The beta function is given by

$$\text{beta}(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

(where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$ is the gamma function, satisfying $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, for all $\alpha > 0$), and is the normalising constant in the beta(α, β) density:

$$f_X(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\text{beta}(\alpha, \beta)} \text{ for } 0 < x < 1.$$

Suppose X has the density $f_X(\cdot)$ above, and then define $Y = 1/X$.

(a) For $\alpha > 1$, determine $E(Y)$.

(b) Determine the density of Y.

4. If Y has a geometric distribution with

$$P(Y = y) = (1-p)^{y-1}p \text{ for } y = 1, 2, \cdots$$

then $E(Y) = 1/p$ and $\mathrm{Var}(Y) = (1-p)/p^2$. Suppose $\mathbf{X} = (X_1, \ldots, X_n)$ consists of iid geometric random variables with unknown *mean* $\theta \in \Theta = (1, \infty)$. Consider the decision problem with decision space $\mathcal{D} = \Theta$ and loss $L(d|\theta) = (d - \theta)^2$. Assume that $n \geq 3$.

(a) Determine $E_\theta(T)$ and $\mathrm{Var}_\theta(T)$ where $T = \sum_{i=1}^n X_i$ as functions of θ.

(b) Define the family of estimators $\{d_{k\ell}(\cdot)\colon k, \ell \in \mathbb{R}\}$ according to

$$d_{k\ell}(\mathbf{X}) = \frac{T+k}{n+\ell}\,.$$

Determine the risk

$$R(\theta|d_{k\ell}) = E_\theta\left\{[d_{k\ell}(\mathbf{X}) - \theta]^2\right\}\,.$$

(c) Write down the probability mass function of X_1 as a function of θ.

(d) Write out the likelihood.

(e) Determine the Bayes procedure $d_{\text{flat}}(\mathbf{X})$ using a flat prior $w(\theta) \equiv 1$ (Question 3 may prove useful here).

(f) Show that

(i) the maximum likelihood estimator;

(ii) $d_{\text{flat}}(\mathbf{X})$;

(iii) any Bayes procedure based on a (conjugate) prior of the form

$$w(\theta) = \frac{1}{\mathrm{beta}(\alpha_0, \beta_0)} \frac{(\theta-1)^{\beta_0 - 1}}{\theta^{\alpha_0 + \beta_0}}\,, \quad \text{for } \theta > 1 \tag{2}$$

are all asymptotically minimax. You may assume that for any $1 < \theta_0 < \theta_1 < \infty$, the Bayes procedure $\widetilde{d}(\mathbf{X})$ based on the $U[\theta_0, \theta_1]$ prior has the same limiting (rescaled) risk as $d_{\text{flat}}(\mathbf{X})$: for all $\theta_0 < \theta < \theta_1$,

$$\lim_{n\to\infty} nR(\theta|\widetilde{d}) = \lim_{n\to\infty} nR(\theta|d_{\text{flat}})\,.$$

Hint: determine the forms of all the estimators first.