# Hypothesis Testing

## STAT3023

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# Multivariate case

Suppose we have a family of distributions indexed by more than 1 parameter. GLRT provides a useful general method of testing $H_0 : \theta \in \Theta_0$ vs. $H_1 : \theta \in \Theta \backslash \Theta_0$:

log of likelihood ratio:

$$\ell(\hat{\underset{\sim}{\theta}} ; \underset{\sim}{x}) - \ell(\hat{\underset{\sim}{\theta}}_0 ; \underset{\sim}{x})$$

$\hat{\underset{\sim}{\theta}}$ = "unrestricted" MLE

$$\hat{\underset{\sim}{\theta}}_0 = \underset{\underset{\sim}{\theta} \in \Theta_0}{\arg\max}\ \ell(\underset{\sim}{\theta} ; \underset{\sim}{x})$$

Many commonly used statistical tests are equivalent to GLRT.

# Multivariate case

Example: 1-way ANOVA F-test

$X_{ij} \sim N(\mu_i, \sigma^2)$, $i = 1, \cdots, g$ groups

For each $i$, $j = 1, \cdots, n_i$

All $X_{ij}$ indep.

$N = \sum_{i=1}^{g} n_i$ total sample size.

$H_0: \mu_1 = \mu_2 \cdots = \mu_g$

$H_1: \mu_i$'s not all equal.

$\underset{\sim}{\mu} = (\mu_1, \cdots, \mu_g)$

log likelihood is

$$\ell(\underset{\sim}{\mu}, \sigma^2; \underset{\sim}{x}) = \log\left(\prod_{i=1}^{g} \prod_{j=1}^{n_i} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x_{ij} - \mu_i)^2}\right)$$

$\vdots$

$$= -N\log\sqrt{2\pi} - N\log\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{g}\sum_{j=1}^{n_i}(X_{ij}-\mu_i)^2$$

score functions:

$$\frac{\partial l}{\partial \mu_i} = \frac{1}{\sigma^2}\sum_{j=1}^{n_i}(X_{ij}-\mu_i) = 0$$

$$\hat{\mu}_i = \bar{X}_{i\cdot} = \frac{1}{n_i}\sum_{j=1}^{n_i} X_{ij}$$

group $i$ average.

$$\frac{\partial l}{\partial \sigma} = -\frac{N}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{g}\sum_{j=1}^{n_i}(X_{ij}-\mu_i)^2$$

$$= \frac{N}{\sigma^3}\left\{\frac{1}{N}\sum_{i=1}^{g}\sum_{j=1}^{n_i}(X_{ij}-\mu_i)^2 - \sigma^2\right\} = 0$$

$$\hat{\sigma}^2 = \frac{1}{N}\sum_{i=1}^{g}\sum_{j=1}^{n_i}(X_{ij}-\mu_i)^2$$

$$= \frac{1}{N}\sum_{i=1}^{g}\sum_{j=1}^{n_i}(X_{ij}-\bar{X}_{i\cdot})^2$$

$$\frac{\text{residual sum of squares}}{N}$$

Under $H_0$, the data consists of a large sample of size $N$ from $N(\mu_0, \sigma^2)$ for some $\mu_0 = \mu_1 = \cdots = \mu_g$

$$\hat{\mu}_0 = \frac{1}{N}\sum_{i=1}^{g}\sum_{j=1}^{n_i} X_{ij} = \bar{X}_{\cdot\cdot}$$

$$\hat{\sigma}_0 = \frac{1}{N}\sum_{i=1}^{g}\sum_{j=1}^{n_i}(X_{ij}-\bar{X}_{\cdot\cdot})^2 \qquad \frac{\text{total sum of squares}}{N}$$

$$\ell(\hat{\mu}, \hat{\sigma}^2; \underset{\sim}{X}) - \ell(\hat{\mu}_0, \hat{\sigma}_0^2; \underset{\sim}{X})$$

$$= -N \log \hat{\sigma} - \underbrace{\frac{1}{2\hat{\sigma}^2} \sum_{i=1}^{g} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_{i\cdot})^2}_{\frac{N}{2}}$$

$$+ N \log \hat{\sigma}_0 + \underbrace{\frac{1}{2\hat{\sigma}_0^2} \sum_{i=1}^{g} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_{\cdot\cdot})^2}_{\frac{N}{2}}$$

$$= N \cdot \log \frac{\hat{\sigma}_0}{\hat{\sigma}} = \frac{N}{2} \log \frac{\hat{\sigma}_0^2}{\hat{\sigma}^2}$$

increasing function of $\frac{\hat{\sigma}_0^2}{\hat{\sigma}^2}$

Recall the ANOVA decomposition.

$$\sum_i \sum_j (X_{ij} - \bar{X}_{\cdot\cdot})^2 \qquad \text{total s.s.}$$

$$= \sum_i \sum_j (X_{ij} - \bar{X}_{i\cdot} + \bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot})^2$$

$$= \sum_i \sum_j (X_{ij} - \bar{X}_{i\cdot})^2 + \sum_i \sum_j (\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot})^2$$

$$+ 2 \sum_i \underbrace{\sum_j (X_{ij} - \bar{X}_{i\cdot})}_{=0} (\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot})$$

$$= \text{residual s.s.} + \text{treatment s.s.}$$

$$\frac{\hat{\sigma}_0^2}{\hat{\sigma}^2} = \frac{\text{Total s.s.}/N}{\text{residual s.s.}/N} = \frac{\text{residual s.s.} + \text{treatment s.s.}}{\text{residual s.s.}}$$

$$= 1 + \boxed{\frac{\text{treatment s.s.}}{\text{residual s.s.}}} \rightarrow \frac{g-1}{N-g} \cdot F$$

Equivalent to the ANOVA F-test.

# Multivariate case

Example: one-sided $t$-test. $X_1, \ldots, X_n$ iid $N(\mu, \sigma^2)$,
$\Theta = \{(\mu, \sigma^2) : \mu \geq 0, \sigma^2 > 0\}$
Consider testing $H_0 : \mu = 0$ vs. $H_1 : \mu > 0$.

$$l(\mu, \sigma; \underset{\sim}{x}) = \log\left\{\prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x_i-\mu)^2}\right\}$$

$$= -n\log(\sqrt{2\pi}) - n\log\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

Under the full model,

$$\max_{\substack{\mu\geq 0\\ \sigma^2>0}} l(\mu, \sigma; \underset{\sim}{x}) = \max_{\mu\geq 0}\left(\max_{\sigma^2>0} l(\mu, \sigma; \underset{\sim}{x})\right)$$

$$\frac{\partial}{\partial\sigma} l(\mu, \sigma; \underset{\sim}{x}) = -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$= \frac{n}{\sigma^3}\left\{\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2 - \sigma^2\right\} = 0$$

$$\hat{\sigma}^2(\mu) = \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2$$

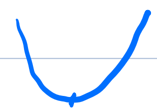
$$\left[\text{Under } H_0:\ \hat{\sigma}^2(0) = \frac{1}{n}\sum_{i=1}^{n}X_i^2 = \hat{\sigma}_0^2\ \right] \leftarrow$$

$$\ell(\mu, \hat{\sigma}^2(\mu); \underset{\sim}{x}) = -n\log(\hat{\sigma}(\mu)) - \frac{n}{2}\log(2\pi) - \underbrace{\frac{1}{2\hat{\sigma}^2(\mu)}\sum_{i=1}^{n}(X_i-\mu)^2}_{\frac{n}{2}}$$

$$= -\frac{n}{2}\log\left(\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2\right) - \frac{n}{2}(\log 2\pi + 1)$$

maximising $\ell$ is equivalent to minimising $\sum_{i=1}^{n}(X_i-\mu)^2$ over $\mu \geq 0$.

$$\sum_{i=1}^{n}(X_i-\mu)^2 = \sum_{i=1}^{n}X_i^2 - 2\mu\sum_{i=1}^{n}X_i + n\cdot\mu^2$$

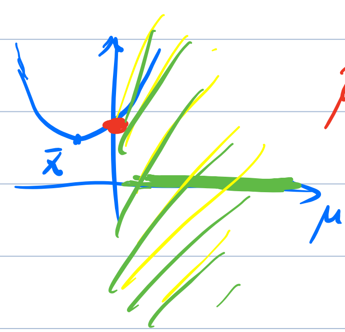


$\hat{\mu} = \bar{X}$.

$$\hat{\mu} = \frac{\cancel{2}\sum_{i=1}^{n}X_i}{\cancel{2}n} = \bar{X}$$

Case 1 $\quad \bar{X} \geq 0, \quad \hat{\mu} = \bar{X}$

Case 2 $\quad \bar{X} < 0$




$\hat{\mu} = 0$

$$\hat{\sigma}^2 = \hat{\sigma}^2(0) = \frac{1}{n}\sum_{i=1}^{n}X_i^2$$

$$L_n = \ell(\hat{\mu}, \hat{\sigma}; \underset{\sim}{X}) - \ell(0, \hat{\sigma}_0; \underset{\sim}{X})$$

$$= -n\log\hat{\sigma} - \underbrace{\frac{1}{2\hat{\sigma}^2}\sum_{i=1}^{n}(X_i-\hat{\mu})^2}_{\frac{n}{2}}$$

$$+ n\log\hat{\sigma}_0 + \underbrace{\frac{1}{2\hat{\sigma}_0^2}\sum_{i=1}^{n}X_i^2}_{\frac{n}{2}}$$

$$= n\log\frac{\hat{\sigma}_0}{\hat{\sigma}} = \frac{n}{2}\log\frac{\hat{\sigma}_0^2}{\hat{\sigma}^2}$$

reject for large values of $\frac{\hat{\sigma}_0^2}{\hat{\sigma}^2}$

$$\frac{\hat{\sigma}_0^2}{\hat{\sigma}^2} = \begin{cases} \dfrac{\sum_{i=1}^{n} X_i^2}{\sum_{i=1}^{n}(X_i-\bar{X})^2} & \leftarrow \quad \bar{X} \geq 0 \\ 1 & \quad \bar{X} < 0 \end{cases}$$

Under $H_0$, $\bar{X} \sim N(0, \frac{\sigma^2}{n})$

$$P(L_n = 0) = 0.5 \quad (P_0(\bar{X} < 0) = 0.5)$$

For any $\alpha < 0.5$

reject for $\frac{\sum_{i=1}^{n} X_i^2}{\sum_{i=1}^{n}(X_i-\bar{X})^2} \geq c$ for some $c > 1$

check

$$\frac{\sum_{i=1}^{n} X_i^2}{\sum_{i=1}^{n}(X_i-\bar{X})^2} = \frac{\sum_{i=1}^{n}(X_i-\bar{X})^2 + n\bar{X}^2}{\sum_{i=1}^{n}(X_i-\bar{X})^2}$$

$$= 1 + \frac{n\bar{X}^2}{\sum_{i=1}^{n}(X_i-\bar{X})^2} = 1 + \frac{1}{n-1}\cdot T^2$$

$$T = \frac{\sqrt{n} \cdot \overline{X}}{\sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2}} \qquad t\text{-statistic.}$$

reject for large values of $T^2$.

but only when $\overline{X} > 0$, i.e. $T > 0$

For $\alpha < 0.5$, 1-sided t-test.