Tutorial Week 4

STAT3023: Statistical Inference | Semester 2, 2022

1. (Problem 7.37 from Freund's) Let X and Y be two continuous random variables with the joint density

$$f_{X,Y}(x,y) = 24xy,\ 0 < x, y < 1,\ x + y < 1.$$

Find the joint density of $Z = X + Y$ and $W = X$.

2. (Problem 7.38 from Freund's) Let X and Y be two independent random variables, each having Gamma(α, β) distribution.

(a) Find the joint density of $U = \dfrac{X}{X+Y}$ and $V = X + Y$.

(b) Find the marginal density of U.

3. The density of a random variable T that follows a t_d distribution is given by

$$f_T(t) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{d\pi}\,\Gamma\left(\frac{d}{2}\right)}\left(1 + \frac{t^2}{d}\right)^{-\frac{d+1}{2}}.$$

(a) Show that the expectation of T is only well-defined when $d > 1$. In this case, $E(T) = 0$.

(b) Show that if $d > 2$, $\mathrm{Var}(T) = \dfrac{d}{d-2}$.

(c) Using Stirling's approximation that for any $k \to \infty$,

$$\Gamma(k) \approx \sqrt{\frac{2\pi}{k}}\left(\frac{k}{e}\right)^k,$$

show that if $d \to \infty$, then

$$f_T(t) \to \frac{1}{\sqrt{2\pi}} e^{-t^2/2},$$

that is, if $d \to \infty$, the t distribution converges to the standard normal distribution.

4. For each of the following distributions, verify it belongs to an exponential family, indicate whether it belongs to a full or a curved one, identify its natural parameter and its natural parameter space.

(a) The gamma distribution with shape parameter α known, but the scale parameter β unknown.

(b) The gamma distribution with shape parameter α unknown, but the scale parameter β known.

(c) The gamma distribution with both the shape α and the scale parameters β unknown.

(d) The beta distribution (pdf is given below) with both parameters α and β unknown.

(e) The normal distribution $N(\theta, \theta)$ for $\theta > 0$, i.e mean and variance are the same.

(f) The gamma distribution with the shape parameter α and the scale parameter $\beta = \alpha$, α unknown.

For the distribution in part(e), from the exponential family form, verify that $E(X^2) = \theta^2 + \theta$.

Some useful formulas

The beta distribution A random variable X is said to follow a binomial distribution with parameters α and β if its density is given by

$$f_X(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 \leq x \leq 1.$$

The gamma distribution A random variable X is said to follow a gamma distribution with shape parameter α and scale parameter β if its the density function is

$$f_X(x) = \frac{e^{-x/\beta} x^{\alpha-1}}{\beta^\alpha \Gamma(\alpha)}, \quad x > 0$$

Limit definition of e^a

$$\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = e^a.$$

Properties of Gamma function

$$\Gamma(x+1) = x\Gamma(x), \quad \Gamma(1/2) = \sqrt{\pi}, \quad \Gamma(n+1) = n!$$

for any $x > 0$, $n \in \mathbb{N}$.