Multivariate distributions

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Bivariate distributions

Bivariate distributions (Review from STAT2011/2911)

- Consider two random variables X, Y defined on a sample space with probability measure P. Its joint distribution is defined as

$$F_{X,Y}(x,y) = P(X \le x,\ Y \le y),\ (x,y) \in \mathbb{R}^2.$$

- If both X and Y are discrete, then they have a joint probability mass function

$$p_{X,Y}(x,y) = P(X = x, Y = y).$$

- If both X and Y are continuous, then they have a joint density function $f_{X,Y}(x,y$.

Marginal and conditional distributions

	(X, Y) discrete	(X, Y) continuous
Joint dist	$p_{X,Y}(x,y)$	$f_{X,Y}(x,y)$
Marginal dist	$p_X(x) = \sum_y p_{X,Y}(x,y)$	$f_X(x) = \int_y f_{X,Y}(x,y)dy$
Conditional dist	$p_{Y\|X}(y\|x) = \dfrac{p_{X,Y}(x,y)}{p_X(x)}$	$f_{Y\|X}(y\|x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)}$
Independence	$p_{X,Y}(x,y) = p_X(x)p_Y(y)$	$f_{X,Y}(x,y) = f_X(x)f_Y(y)$

Checking independence

Two continuous random variables X and Y are independent if and only if there exist functions $g(x)$ and $h(y)$ such that

$$f_{X,Y}(x,y) = g(x)h(y).$$

Replacing $f_{X,Y}(x,y)$ by $p_{X,Y}(x,y)$ gives the similar results for two discrete random variables.

Example

Expectation and conditional expectation

	(X, Y) discrete	(X, Y) continuous
$E\{g(X, Y)\}$	$\sum_x \sum_y g(x, y) p_{X,Y}(x, y)$	$\int_x \int_y g(x, y) f_{X,Y}(x, y) dx dy$
$E\{g(Y) \mid X = x\}$	$\sum_y g(y) p_{Y\mid X}(y \mid x)$	$\int g(y) f_{Y\mid X}(y \mid x) dy$

Conditional expectation as a random variable

- $h(x) = E(Y|X = x)$ is always a function of x (scalar). Nevertheless, since X is a random variable. $h(X) = E(Y|X)$ is another random variable.

Example:

Property of conditional expectation

$$E\{E(Y|X)\} = E(Y).$$

Proof:

Conditional variances

- Similarly, the conditional variance $\text{Var}(Y|X)$ is also another random variable depending on X.

Property of conditional variance:

$$\text{Var}\{X\} = E\{\text{Var}(X|Y)\} + \text{Var}\{E(X|Y)\}$$

Proof:

A hierarchical model

Instead of specifying a joint distribution of X and Y, we can specify a marginal distribution and a conditional distribution.

Example:

Examples

Bivariate normal distribution

Joint density

Marginal and conditional distribution

Correlatedness and independence

Multivariate distribution

Random vector

- $\mathbf{X} = (X_1, \ldots, X_n)$ is a n-dimensional random vector if each component $X_1, \ldots, X_n$ is a random variable. Its realized values is denoted as $\mathbf{x} \in \mathbb{R}^n$
- We can extend the idea of joint distributions, marginal distributions, conditional distributions in the bivariate case to the multivariate case.

Multivariate distribution

Example

Multinomial as a generalization of binomial distribution

- Consider m independent trials, each trial having n possible (distinct) outcomes with probabilities $p_1, \ldots, p_n$ such that $\sum_{i=1}^{n} p_i = 1$.
- Let X_i be the number of trials with the ith outcome, then the random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is said to follow the multinomial distribution with m trials and cell probabilities $(p_1, \ldots, p_n)$, with the joint pdf given by

Example: Unbalanced die

Multinomial theorem

Let m and n be positive integers. Let $\mathcal{A}$ be a set of vectors $\mathbf{x} = (x_1, \ldots, x_n)$ such that each x_i is a nonnegative integer and $\sum_{i=1}^{n} x_i = m$. Then for any real numbers $p_1, \ldots, p_n$, we have

$$(p_1 + p_2 + \ldots + p_n)^m = \sum_{\mathbf{x} \in \mathcal{A}} \frac{m!}{x_1! \ldots x_n!} p_1^{x_1} \ldots p_n^{x_n}$$

Marginal and conditional distribution

Let $\mathbf{X} = (X_1, \ldots, X_n)$ follow the multinomial distribution with m trials and cell probabilities $\mathbf{p} = (p_1, \ldots, p_n)$. Then

- $X_i \sim \mathrm{Bin}(m, 1 - p_i)$.
- Given $X_k = x_k$, $\mathbf{X}_{-k} = (X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_n)$ follow the multinomial distribution with $m - x_k$ trials and cell probabilities

$$\mathbf{p}_{-k} = \left(\frac{p_1}{1 - p_k}, \ldots, \frac{p_{k-1}}{1 - p_k}, \frac{p_{k+1}}{1 - p_k} \ldots, \frac{p_n}{1 - p_k} \right)$$

.

Proof: