

**Tutorial Week 5**

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STAT3023: Statistical Inference

Semester 2, 2022

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1. Suppose  $X_1, \dots, X_n$  is a random sample.

- (a) If  $X_i$  are iid Bernoulli( $\theta$ ) with  $\theta \in (0, 1)$ , show that the likelihood function can be written in the canonical form of the exponential family

$$L(\theta) = h(x) \exp \{ \eta T(\mathbf{x}) - A^*(\eta) \}$$

for  $\mathbf{X} = (X_1, \dots, X_n)$ ,  $x_i \in \{0, 1\}$ . Identify a sufficient statistic  $T$  for  $\theta$ , and find  $E(T)$  and  $\text{Var}(T)$  using  $A^*$ .

- (b) Using the same argument, show that if  $X_i$  are iid  $N(\theta, 1)$ , then  $T = \sum_{i=1}^n X_i$  is a sufficient statistic for  $\theta$ . Identify  $E(T)$  and  $\text{Var}(T)$ .
- (c) Using the same argument, if  $X_i$  are iid  $N(0, \theta)$ , find a sufficient statistic for  $\theta$ .

2. This question reviews the method of moment and maximum likelihood estimates from STAT2011/2911.

Suppose  $X_1, \dots, X_n$  is a random sample from the Gamma distribution  $\text{Gamma}(\alpha, \beta)$ , i.e the density of each  $X_i$  is

$$f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \quad x > 0$$

- (a) Assuming  $\alpha$  is known, find a method of moment and maximum likelihood estimates for  $\beta$ .
- (b) If both  $\alpha$  and  $\beta$  are unknown, write out an equation that is used to solve for the maximum likelihood of  $\alpha$ .

3. Suppose  $X_1, \dots, X_n$  is a random sample from Poisson( $\lambda$ ) distribution. Note that

$$\theta = P(X_i = 0) = e^{-\lambda}, \text{ for } i = 1, \dots, n$$

- (a) Find the maximum likelihood estimator for  $\lambda$ , then the corresponding maximum likelihood estimator for  $\theta$ . Denote this estimator to be  $\hat{\theta}_1$ .
- (b) Find the bias and variance of  $\hat{\theta}_1$ .
- (c) Now, let  $Y$  be the number of zeros among  $X_1, \dots, X_n$ . What is the distribution of  $Y$ ?
- (d) Based on part (c), identify an unbiased estimator for  $\theta$ . Denote this estimator to be  $\hat{\theta}_2$ . What is the variance of this estimator?
- (e) (For advanced students STAT3923/4023 only) Using the Delta method, compare the asymptotic relative efficiency of  $\hat{\theta}_1$  and  $\hat{\theta}_2$ .