

Solution to Tutorial Week 1

STAT3023: Statistical Inference

Semester 2, 2022

1. (4.36) If $M_X(t)$ is a mgf of a random variable, then $M_X(0) = 1$. Clearly, the given function has $M_X(0) = 0$, so it can't be a mgf of a random variable.

(4.37) By definition, we have

$$\begin{aligned} M_X(t) &= \int_{-\infty}^{\infty} e^{tx} f(x) dx = \frac{1}{2} \int_{-\infty}^{\infty} e^{tx} e^{-|x|} dx \\ &= \frac{1}{2} \int_{-\infty}^0 e^{tx} e^x dx + \frac{1}{2} \int_0^{\infty} e^{tx} e^{-x} dx \\ &= \frac{1}{2} \int_{-\infty}^0 e^{(t+1)x} dx + \frac{1}{2} \int_0^{\infty} e^{(t-1)x} dx \\ &= \frac{1}{2(t+1)} e^{(t+1)x} \Big|_{x=-\infty}^{x=0} + \frac{1}{2(t-1)} e^{(t-1)x} \Big|_{x=0}^{x=\infty} \\ &= \frac{1}{2(t+1)} - \frac{1}{2(t-1)} = \frac{1}{1-t^2}, \quad |t| < 1. \end{aligned}$$

(4.40) $Z = \frac{1}{4}(X - 3) = \frac{1}{4}X - \frac{3}{4}$, so

$$\begin{aligned} M_Z(t) &= \exp\left(-\frac{3t}{4}\right) M_X\left(\frac{t}{4}\right) \\ &= \exp\left(-\frac{3t}{4}\right) \exp\left(\frac{3t}{4} + 8\left(\frac{t}{4}\right)^2\right) = \exp\left(\frac{t^2}{2}\right), \end{aligned}$$

which is the mgf of the standard normal random variable $N(0, 1)$. Hence, $Z \sim N(0, 1)$, so $E(Z) = 0$ and $\text{var}(Z) = 1$.

2. By definition, we have

$$\begin{aligned} M_X(t) &= \int_{-\infty}^{\infty} e^{tx} f(x) dx = \int_0^{\infty} e^{tx} \frac{e^{-x/\beta} x^{\alpha-1}}{\beta^{\alpha} \Gamma(\alpha)} dx \\ &= \frac{1}{\beta^{\alpha} \Gamma(\alpha)} \int_0^{\infty} e^{-x(1/\beta-t)} x^{\alpha-1} dx \end{aligned}$$

Let $u = (1/\beta - t)^{-1}$, then if $t < 1/\beta$, the integral $\int_0^{\infty} e^{-x(1/\beta-t)} x^{\alpha-1} dx = \int_0^{\infty} e^{-x/u} x^{\alpha-1} dx$ is the unnormalized density of a Gamma(u, α) distribution, so we have

$$\int_0^{\infty} e^{-x/u} x^{\alpha-1} dx = u^{\alpha} \Gamma(\alpha).$$

Hence, the mgf of X is equal to

$$M_X(t) = \left(\frac{u}{\beta}\right)^{\alpha} = \left\{\beta \left(\frac{1}{\beta} - t\right)\right\}^{-\alpha} = (1 - t\beta)^{-\alpha}, \quad t < 1/\beta.$$

3. Let $S = \sum_{i=1}^n X_i$, where $X_i \sim \text{Gamma}(\alpha, \beta)$ distribution. By the previous problem, we have $M_{X_i}(t) = (1 - t\beta)^{-\alpha}$ for $i = 1, \dots, n$. Because X_i 's are mutually independent, we have

$$M_S(t) = \prod_{i=1}^n M_{X_i}(t) = (1 - t\beta)^{-\alpha n},$$

which is the mgf of the $\text{Gamma}(\alpha n, \beta)$ distribution. Hence $S \sim \text{Gamma}(\alpha n, \beta)$.

4. (a) We have $Y = \log(X)$, so $X = e^Y$. As a result, we can write

$$E(X^r) = E(e^{Yr}) = M_Y(r) = e^{r^2/2}.$$

- (b) For any $t > 0$, we have

$$\begin{aligned} E(e^{tX}) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{1}{x} \exp\left(tx - \frac{\{\log(x)\}^2}{2}\right) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp\left(tx - \log(x) - \frac{\{\log(x)\}^2}{2}\right) dx. \end{aligned}$$

This integral is infinite if we can prove

$$\lim_{x \rightarrow \infty} tx - \log(x) - \frac{\{\log(x)\}^2}{2} = \infty.$$

To prove it, we can consider the limit

$$\lim_{x \rightarrow \infty} \frac{tx - \log(x) - \frac{\{\log(x)\}^2}{2}}{tx}$$

and by the l'Hopital rule, we can take the derivative of both numerator and denominator with respect to x to obtain

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{tx - \log(x) - \frac{\{\log(x)\}^2}{2}}{tx} &= \lim_{x \rightarrow \infty} \frac{t - \frac{1}{x} - \frac{\log(x)}{x}}{t} \\ &= 1 - \frac{1}{t} \lim_{x \rightarrow \infty} \frac{\log(x)}{x} = 1 - \frac{1}{t} \lim_{x \rightarrow \infty} \frac{1}{x} = 1. \end{aligned}$$

Therefore, we have

$$\lim_{x \rightarrow \infty} \frac{tx - \log(x) - \frac{\{\log(x)\}^2}{2}}{tx} = \lim_{x \rightarrow \infty} (tx) = \infty, \text{ for any } t > 0.$$

5. Each X_i has MGF $M_{X_i}(t) = \frac{1}{2}(e^t + e^{-t})$, so the sum $S = \sum_{i=1}^n X_i$ has MGF

$$M_S(t) = \prod_{i=1}^n M_{X_i}(t) = \frac{1}{2^n} (e^t + e^{-t})^n.$$

Using the argument as derived in Chernoff bound, for any $t > 0$ we have

$$P(S \geq nx) = P\{\exp(St) \geq \exp(ntx)\} \leq \exp(-ntx)M_S(t).$$

Since this above inequality holds for any $t > 0$, then the lowest upper bound can be obtained by finding t to minimize the right-hand side

$$\exp(-ntx)M_S(t) = \frac{1}{2^n} \exp(-ntx) (e^t + e^{-t})^n. \quad (1)$$

Equivalently, we can take the log and find t^* to minimize

$$g(t) = -ntx + n \log(e^t + e^{-t}).$$

The derivative of $g(t)$ is equal to

$$g'(t) = -nx + \frac{n(e^t - e^{-t})}{e^t + e^{-t}}.$$

Setting $g'(t) = 0$, we solve the equation

$$e^t - e^{-t} = x(e^t + e^{-t}), \text{ i.e } e^t(1 - x) = e^{-t}(1 + x).$$

Multiplying both sides by e^t , the above equation is equivalent to

$$e^{2t} = \frac{1+x}{1-x}, \quad t^* = \frac{1}{2} \log \frac{1+x}{1-x}.$$

Now, we only need to substitute this t^* to (1). At $t = t^*$, we have $e^{t^*} + e^{-t^*} = \sqrt{\frac{1+x}{1-x}} + \sqrt{\frac{1-x}{1+x}} = \frac{2}{\sqrt{(1+x)(1-x)}}$, hence,

$$\begin{aligned} P(S \geq nx) &\leq \frac{1}{2^n} \exp(-nt^*x) (e^{t^*} + e^{-t^*})^n \\ &= \frac{1}{2^n} \exp\left(-\frac{nx}{2} \log \frac{1+x}{1-x}\right) 2^n (1+x)^{-n/2} (1-x)^{-n/2} \\ &= \exp\left(-\frac{n}{2}(1+x) \log(1+x) - \frac{n}{2}(1-x) \log(1-x)\right) \\ &= \exp\{-nF(x)\}, \end{aligned}$$

as claimed.