The University of Sydney
School of Mathematics and Statistics

Tutorial Week 7

STAT3023: Statistical Inference — Semester 2, 2023

1. Suppose $\mathbf{X} = (X_1, \ldots, X_n)$ is a random sample from the Gamma distribution with shape α and rate θ, i.e.,

$$f_X(x) = \frac{\theta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-x\theta}, \quad x > 0,\ \alpha > 0,\ \theta > 0.$$

Each X_i has expectation $E(X) = \alpha/\theta$, variance $\mathrm{Var}(X) = \alpha/\theta^2$, and moment generating function $M_X(t) = (1 - t/\theta)^{-\alpha}$.

(a) Assuming both α and θ to be unknown, write down the log likelihood function $\ell(\theta, \alpha; \mathbf{X})$ and the corresponding score functions $\dfrac{\partial \ell}{\partial \theta}$ and $\dfrac{\partial \ell}{\partial \alpha}$.

(b) Verify that each score function has zero expectation.

(c) Assuming α to be known:

(i) Calculate the Cramer Rao Lower Bound (CRLB) for the variance of an unbiased estimator of θ.

(ii) Is there any unbiased estimator of θ whose variance attain the CRLB?

(iii) Show that

$$S = \frac{\alpha - 1}{n} \sum_{i=1}^{n} \left(\frac{1}{X_i} \right)$$

is an unbiased estimator for θ. What is the MVU estimator for θ?

(iv) Identify a change of parameter $\eta = \eta(\theta)$ for which there exists an unbiased estimator with variance attained the CRLB.

(v) For the parameter in the previous part, identify the MVU estimator whose variance attains the CRLB. Compute this variance.

2. Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$ distribution.

(a) Show that $(\bar{X}, S^2)$ is a sufficient statistic for (μ, σ^2), where $\bar{X} = n^{-1} \sum_{i=1}^n X_i$ and $S^2 = (n-1)^{-1} \sum_{i=1}^n (X_i - \bar{X})^2$.

(b) Find the best unbiased estimator for σ^p, where $p > 0$ and it is not necessarily an integer.

3. Let $X_1, \ldots, X_n$ be a random sample from a distribution having pdf $f(x; \theta) = \theta x^{\theta - 1}$ for $0 < x < 1$. Using the likelihood ratio statistic, show the critical region for testing $H_0 : \theta = 1$ against $H_1 : \theta = 2$ takes the form $C = \{(x_1, \ldots, x_n) : c \leq \prod_{i=1}^n x_i\}$ for some constant c.

4. Let $X_1, \ldots, X_n$ be a random sample from $N(\theta, 100)$. Show that the likelihood ratio statistic leads to the critical region $C = \{(x_1, \ldots, x_n) : c \leq \bar{x} = \sum_{i=1}^n x_i/n\}$, where c is some constant, for testing $H_0 : \theta = 75$ against $H_1 : \theta = 78$. Using this critical region, find n and c such that the type I error is 0.05 and the power is 0.9 approximately.