The University of Sydney
School of Mathematics and Statistics

Tutorial Week 13

STAT3023: Statistical Inference — Semester 2, 2023

Review exercises based on the geometric distribution

1. Recall that if a discrete random variable X has probability mass function (PMF) in the exponential family form

$$P_\theta(X = x) = e^{\theta t(x) - K(\theta) - M(x)} \tag{1}$$

then $E_\theta\left[t(X)\right] = K'(\theta)$ and $Var_\theta\left[t(X)\right] = K''(\theta)$. We call the parameter θ the "natural" or "canonical" parameter of the exponential family.

Suppose X has a geometric(p) distribution so that $P(X = x) = (1-p)^{x-1}p$ for $x = 1, 2, \ldots$ By writing the PMF of X in exponential family form (1), deduce $E(X)$ and $\mathrm{Var}(X)$ as functions of p.

2. Suppose $X_1, \ldots, X_n$ are iid geometric with $P(X_1 = x) = (1-p)^{x-1}p$ for $x = 1, 2, \ldots$, but it is desired to estimate the *natural/canonical* parameter θ rather than p.

(a) Write down the likelihood $f_\theta(\mathbf{X})$ in terms of the natural parameter θ and hence obtain the score function $\ell'(\theta; \mathbf{X}) = \frac{\partial}{\partial\theta} \log f_\theta(\mathbf{X})$.

(b) Determine the Cramér-Rao lower bound to the variance of an unbiased estimator of θ.

(c) Derive the maximum-likelihood estimator $\hat{\theta}_{\mathrm{ML}}$ of θ.

3. Suppose $X_1, \ldots, X_{10}$ are iid geometric with $P(X = x) = (1-p)^{x-1}p$. Derive the UMP test at level 0.05 for testing $H_0\colon p = 0.5$ against the alternative $H_1\colon p > 0.5$. You may use the `R` output below and the fact that $T = \sum_{i=1}^n X_i$ has a *negative binomial* distribution and the CDF of $T - n$; specifically

$$P(T - n \le x)$$

is given by the `R` function `pnbinom(x, n, p)`.

```
> x = 0:20
> cbind(x, pnbinom(x, 10, .5))
       x
 [1,]  0 0.0009765625
 [2,]  1 0.0058593750
 [3,]  2 0.0192871094
 [4,]  3 0.0461425781
 [5,]  4 0.0897827148
 [6,]  5 0.1508789063
 [7,]  6 0.2272491455
 [8,]  7 0.3145294189
 [9,]  8 0.4072647095
[10,]  9 0.5000000000
[11,] 10 0.5880985260
[12,] 11 0.6681880951
[13,] 12 0.7382664680
[14,] 13 0.7975635529
[15,] 14 0.8462718725
[16,] 15 0.8852385283
[17,] 16 0.9156812280
[18,] 17 0.9389609396
[19,] 18 0.9564207233
[20,] 19 0.9692858271
[21,] 20 0.9786130274
```

4. Suppose $X_1, \ldots, X_n$ are iid geometric with $P(X_1 = x) = (1-p)^{x-1}p$. Derive the Bayes estimator of p under squared-error loss using the $U[0,1]$ prior.