Moments and moment generating function

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Review of random variables (STAT2011/2911)

Consider a sample space Ω with a probability measure P. Let X be a random variable defined on this sample space.

- Any random variable X has a cumulative distribution function (cdf), $F_X(x) = P(X \leq x)$.
- Discrete random variables:
- Continuous random variables:

Moments

For any random variable X and a function $g : \mathbb{R} \to \mathbb{R}$, define the **expectation** of $g(X)$ to be

Examples:

- rth moment:
- rth central moment:

Moment generating functions

- Moment generating function: encoding the sequence of moments $\{E(X^r)\}$, $r = 1, 2 \ldots, \infty$ into the coefficients of a power series.

- Choose $g(x) = \exp(tx)$, then the moment generating function (mgf) of a random variable X is defined to be

$$M_X(t) = E\{g(X)\} = E\{\exp(tX)\},$$

provided this expectation exists for t in some open interval containing zero.

Getting moments from mgf

Examples

Examples

Uniqueness of mgf

- If the moment generating functions exists, and $M_X(t) = M_Y(t)$, then X and Y have the same distributions.
- Nevertheless, if two random variables have all the same moments, $E(X^r) = E(Y^r)$ for all $r = 1, 2, \ldots$ then X and Y do not necessarily have the same distributions.

Example:

Properties of mgf

Let X be a RV with mgf $M_X(t)$. Then the random variable $Z = aX + b$ has mgf

Properties of mgf

Recall for two **independent** random variables X and Y, we have

$$E\{g(X)h(Y)\} = E\{g(X)\}\,E\{h(Y)\}$$

for any two functions g and h. Let $M_X(t)$ and $M_Y(t)$ be mgfs of X and Y respectively, then the mgf of $Z = X + Y$ is given by

Sum of independent random variables

More generally, if $X_1, \ldots, X_n$ be mutually independent random variables with mgfs $M_{X_i}(t)$ for $i = 1, \ldots, n$, then the mgf of $Z = \sum_{i=1}^{n} X_i$ is given by

Examples

Probability bounds

Markov's inequality: For any non-negative random variable X and any $a > 0$, we have

$$P(X \geq a) \leq \frac{E(X)}{a}.$$

Proof:

Probability bounds

Chebyshev's inequality: For any random variable X and any $a > 0$, we have

$$P\left(|X - E(X)| \geq a\right) \leq \frac{\mathsf{Var}(X)}{a^2}$$

Proof:

Probability bounds

Chernoff's bounds:

Examples

Convergence of mgfs implies convergences of cdfs

Suppose $X_1, X_2, \ldots,$ is a sequence of random variables, each with mgf $M_{X_n}(t)$. Furthermore, suppose that

$$\lim_{n\to\infty} M_{X_n}(t) = M_X(t)$$

for all t in an open interval containing zero, and $M_X(t)$ is the mgf of a random variable X. Then for any x such that $F_X(x)$ is continuous, we have

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x).$$

We also say that the sequence $X_1, X_2, \ldots X_n$ converges to X in distribution.

Application: Poisson approximation to binomial distribution

Application: Central limit theorem

Convergence in probability

Weak law of large numbers