The University of Sydney
School of Mathematics and Statistics

Tutorial Week 5

STAT3023: Statistical Inference | Semester 2, 2023

1. Suppose $X_1, \ldots, X_n$ is a random sample.

(a) If X_i are iid Bernoulli(θ) with $\theta \in (0,1)$, show that the likelihood function can be written in the canonical form of the exponential family

$$L(\theta) = h(x) \exp\{\eta T(\mathbf{x}) - A^*(\eta)\}$$

for $\mathbf{X} = (X_1, \ldots, X_n)$, $x_i \in \{0,1\}$. Identify a sufficient statistic T for θ, and find $E(T)$ and $\mathrm{Var}(T)$ using A^*.

(b) Using the same argument, show that if X_i are iid $N(\theta, 1)$, then $T = \sum_{i=1}^n X_i$ is a sufficient statistic for θ. Identify $E(T)$ and $\mathrm{Var}(T)$.

(c) Using the same argument, if X_i are iid $N(0, \theta)$, find a sufficient statistic for θ.

2. This question reviews the method of moment and maximum likelihood estimates from STAT2011/2911.

Suppose $X_1, \ldots, X_n$ is a random sample from the Gamma distribution Gamma(α, β), i.e., the density of each X_i is

$$f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \quad x > 0.$$

(a) Assuming α is known, find a method of moment and maximum likelihood estimates for β.

(b) If both α and β are unknown, write out an equation that is used to solve for the maximum likelihood of α.

3. Suppose $X_1, \ldots, X_n$ is a random sample from Poisson(λ) distribution. Note that

$$\theta = P(X_i = 0) = e^{-\lambda}, \text{ for } i = 1, \ldots, n$$

(a) Find the maximum likelihood estimator for λ, then the corresponding maximum likelihood estimator for θ. Denote this estimator to be $\hat{\theta}_1$.

(b) Find the bias and variance of $\hat{\theta}_1$.

(c) Now, let Y be the number of zeros among $X_1, \ldots, X_n$. What is the distribution of Y?

(d) Based on part (c), identify an unbiased estimator for θ. Denote this estimator to be $\hat{\theta}_2$. What is the variance of this estimator?

(e) (For advanced students STAT3923/4023 only) Using the Delta method, compare the asymptotic relative efficiency of $\hat{\theta}_1$ and $\hat{\theta}_2$.