Multivariate distributions

Dr. Qiuzhuang Sun

STAT3023

Bivariate distributions

Bivariate distributions (Review from STAT2011/2911)

- Consider two random variables X, Y defined on a sample space with probability measure P. Its joint distribution is defined as

$$F_{X,Y}(x,y) = P(X \leq x,\ Y \leq y),\ (x,y) \in \mathbb{R}^2.$$

- If both X and Y are discrete, then they have a joint probability mass function

$$p_{X,Y}(x,y) = P(X = x, Y = y).$$

- If both X and Y are continuous, then they have a joint density function $f_{X,Y}(x,y)$.

Marginal and conditional distributions

	(X, Y) discrete	(X, Y) continuous
Joint dist	$p_{X,Y}(x,y)$	$f_{X,Y}(x,y)$
Marginal dist	$p_X(x) = \sum_y p_{X,Y}(x,y)$	$f_X(x) = \int_y f_{X,Y}(x,y)dy$
Conditional dist	$p_{Y\|X}(y\|x) = \dfrac{p_{X,Y}(x,y)}{p_X(x)}$	$f_{Y\|X}(y\|x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)}$
Independence	$p_{X,Y}(x,y) = p_X(x)p_Y(y)$	$f_{X,Y}(x,y) = f_X(x)f_Y(y)$

Checking independence

Two continuous random variables X and Y are independent if and only if there exist functions $g(x)$ and $h(y)$ such that

$$f_{X,Y}(x,y) = g(x)h(y).$$

Replacing $f_{X,Y}(x,y)$ by $p_{X,Y}(x,y)$ gives the similar results for two discrete random variables.

Proof:

Proof and Example

(Proof continued)

Assume X and Y follow a joint density function $f_{X,Y}(x,y) = \frac{1}{384}x^2y^4e^{-(y+\frac{x}{2})}$. Show X and Y are independent.

Expectation and conditional expectation

	(X, Y) discrete	(X, Y) continuous
$E\{g(X, Y)\}$	$\sum_x \sum_y g(x, y) p_{X,Y}(x, y)$	$\int_x \int_y g(x, y) f_{X,Y}(x, y) dx dy$
$E\{g(Y) \| X = x\}$	$\sum_y g(y) p_{Y\|X}(y\|x)$	$\int g(y) f_{Y\|X}(y\|x) dy$

Conditional expectation as a random variable

- $h(x) = E(Y|X = x)$ is always a function of x (scalar). However, since X is a random variable, $h(X) = E(Y|X)$ is another random variable.

Example: Let X and Y be discrete random variables with PMFs in the table. Find $E(Y|X = 1)$, $E(Y|X = 2)$, and $E(Y|X)$.

Property of conditional expectation

Law of iterated expectation: $E\{E(Y|X)\} = E(Y)$.

Proof:

Conditional variances

- Similarly, the conditional variance $\text{Var}(Y|X)$ is also another random variable depending on X.

Property of conditional variance:

$$\text{Var}\{X\} = E\{\text{Var}(X|Y)\} + \text{Var}\{E(X|Y)\}$$

Proof:

A hierarchical model

Instead of specifying a joint distribution of X and Y, we can specify a marginal distribution and a conditional distribution.

Example: Consider a hierarchical model: $(X|N) \sim \text{Binomial}(N, p)$ and $N \sim \text{Poisson}(\lambda)$. What is the marginal distribution of X?

Examples

Examples

Example: Consider a hierarchical model: $(Y|\Omega) \sim \text{Poisson}(\Omega)$ and $\Omega \sim \text{Exp}(\beta)$. What is the marginal distribution of Y?

Bivariate normal distribution

Joint density

Let X and Y be continuous random variables with joint density function

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 \right.\right.$$
$$\left.\left. -2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)\right]\right], \quad -\infty < x < \infty, \ -\infty < y < \infty.$$

We have

1. $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$;
2. $(Y|X) \sim N(\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(X-\mu_X), \sigma_Y^2(1-\rho^2))$;
3. $\text{Cor}(X,Y) = \rho$ $(-1 < \rho < 1)$.

Marginal and conditional distribution

Marginal and conditional distribution

Correlatedness and independence

Correlatedness and independence

Correlatedness and independence

X and Y are independent $\Rightarrow \text{Cov}(X, Y) = \text{Cor}(X, Y) = 0$.

If $\text{Cor}(X, Y) = 0$, are X and Y independent?

If X and Y are bivariate normal with $\rho = \text{Cor}(X, Y) = 0$, are X and Y independent?

Multivariate distribution

Random vector

- $\mathbf{X} = (X_1, \ldots, X_n)$ is a n-dimensional random vector if each component $X_1, \ldots, X_n$ is a random variable. Its realized value is denoted as $\mathbf{x} \in \mathbb{R}^n$.
- We can extend the idea of joint distributions, marginal distributions, conditional distributions in the bivariate case to the multivariate case.
  - Joint CDF: $F_{\mathbf{X}}(x_1, \ldots, x_n) = P(X_1 \leq x_1, \ldots, X_n \leq x_n)$
  - Joint PMF for discrete random vectors: $P_{\mathbf{X}}(x_1, \ldots, x_n) = P(X_1 = x_1, \ldots, X_n = x_n)$
  - Joint PDF for continuous random vectors: $f_{\mathbf{X}}(x_1, \ldots, x_n)$

$$f_{\mathbf{X}}(x_1, \ldots, x_n) \geq 0;$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{\mathbf{X}}(x_1, \ldots, x_n) dx_1 \cdots dx_n = 1.$$

Multivariate distribution

- Marginal distribution:

$$f_{(X_1,\dots,X_k)}(x_1,\dots,x_k) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{\mathbf{X}}(x_1,\dots,x_n)dx_{k+1}\cdots dx_n$$

- Conditional distribution:

$$f_{X_{k+1},\dots,X_n|X_1,\dots,X_k}(x_{k+1},\dots,x_n \mid X_1 = x_1,\dots,X_k = x_k) = \frac{f_{\mathbf{X}}(x_1,\dots,x_n)}{f_{(X_1,\dots,X_k)}(x_1,\dots,x_k)}$$

Example

Consider a random vector $\mathbf{X} = (X_1, X_2, X_3, X_4)$ with joint PDF

$$f_{\mathbf{X}}(x_1, x_2, x_3, x_4) = \frac{3}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2), \quad x_1, x_2, x_3, x_4 \in (0,1).$$

Compute $f_{(X_1,X_4)}(x_1, x_4)$, $E[X_1 X_4]$, and $f_{X_2,X_3|X_1,X_4}(x_2, x_3 | X_1 = x_1, X_4 = x_4)$.

Multinomial as a generalization of binomial distribution

- Consider m independent trials, each trial having n possible (distinct) outcomes with probabilities $p_1, \ldots, p_n$ such that $\sum_{i=1}^n p_i = 1$.
- Let X_i be the number of trials with the ith outcome, then the random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is said to follow the multinomial distribution with m trials and cell probabilities $(p_1, \ldots, p_n)$, with the joint pmf given by

$$P(X_1 = x_1, \ldots, X_n = x_n) = \frac{m!}{x_1! x_2! \cdots x_n!} p_1^{x_1} p_2^{x_2} \cdots p_n^{x_n},$$

for $x_1 + x_2 + \cdots + x_n = m$.

Example: Unbalanced die

Consider a die such that $P(i\text{th side come up in a toss}) = \frac{i}{21} = p_i$. Toss the die $m = 10$ times and let X_i be the number of tosses showing the ith side. What is the probability $P(X_1 = 0, X_2 = 0, X_3 = 1, X_4 = 2, X_5 = 3, X_6 = 4)$?

Multinomial theorem

Let m and n be positive integers. Let $\mathcal{A}$ be a set of vectors $\mathbf{x} = (x_1, \ldots, x_n)$ such that each x_i is a nonnegative integer and $\sum_{i=1}^{n} x_i = m$. Then for any real numbers $p_1, \ldots, p_n$, we have

$$(p_1 + p_2 + \cdots + p_n)^m = \sum_{\mathbf{x} \in \mathcal{A}} \frac{m!}{x_1! \cdots x_n!} p_1^{x_1} \cdots p_n^{x_n}.$$

Compare it with the binomial theorem:

$$(p_1 + p_2)^m = \sum_{x_1=0}^{m} \binom{m}{x_1} p_1^{x_1} p_2^{m-x_1}.$$

We can use the multinomial theorem to show the PMFs of multinomial distributions sum to 1.

Marginal and conditional distribution

Let $\mathbf{X} = (X_1, \ldots, X_n)$ follow the multinomial distribution with m trials and cell probabilities $\mathbf{p} = (p_1, \ldots, p_n)$. Then

- $X_i \sim \mathrm{Bin}(m, p_i)$.
- Given $X_k = x_k$, $\mathbf{X}_{-k} = (X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_n)$ follow the multinomial distribution with $m - x_k$ trials and cell probabilities

$$\mathbf{p}_{-k} = \left(\frac{p_1}{1-p_k}, \ldots, \frac{p_{k-1}}{1-p_k}, \frac{p_{k+1}}{1-p_k} \ldots, \frac{p_n}{1-p_k} \right).$$

Proof:

Marginal and conditional distribution

(Proof continued)

Conditional distribution example

Consider the unbalanced die before such that $P(i\text{th side come up in a toss}) = \frac{i}{21} = p_i$. Out of 10 tosses, we observe the 6th side in 3 tosses ($X_6 = 3$). What is the joint distribution of $(X_1, \ldots, X_5)$?