Decision Theory: Part 3

Dr. Qiuzhuang Sun

STAT3023

Review: Asymptotically minimax procedures

We have learned that $d(\cdot)$ is asymptotically minimax if it minimises

$$\lim_{n\to\infty} \max_{\theta\in[a,b]} R_n(\theta|d).$$

An asymptotic minimax procedure can be found with 2 steps.

(i) Determine a lower bound on

$$\lim_{n\to\infty} \max_{a\leq\theta\leq b} R_n(\theta|d)$$

for any procedure $d(\cdot)$.

(ii) Show that a given procedure attains the lower bound.

Review: Asymptotic minimax lower bound

We introduce the following theorem last week about a lower bound to the limiting **maximum** (rescaled) risk for any estimator.

Theorem: Suppose that for sequence $\{L_n(\cdot|\theta)\}$ of loss functions and any $\theta_0 < \theta_1$, the corresponding sequence of Bayes procedures $\{\tilde{d}_n(\cdot)\}$ based on the Unif(θ_0, θ_1) prior $w(\theta) = (\theta_1 - \theta_0)^{-1} 1\{\theta_0 < \theta < \theta_1\}$ is such that for each $\theta_0 < \theta < \theta_1$,

$$\lim_{n\to\infty} E_\theta[L_n(\tilde{d}_n(\mathbf{X})|\theta)] = S(\theta),$$

where $S(\cdot)$ is a continuous function. Then, for any other sequence of procedures $\{d_n(\cdot)\}$ and any $a < b$,

$$\lim_{n\to\infty} \max_{a\le\theta\le b} E_\theta[L_n(d_n(\mathbf{X})|\theta)] \ge \max_{a\le\theta\le b} S(\theta).$$

Asymptotic minimax lower bound

We illustrated the use of the Asymptotic Minimax Lower Bound Theorem using an interval estimation of a Poisson distribution last week. Here is another example.

Example (**Interval estimation of a normal mean**): Suppose $\mathbf{X} = (X_1, \ldots, X_n)$ consists of iid $N(\theta, 1)$ random variables for some unknown $\theta \in \Theta = \mathbb{R}$. Consider the decision space $\mathcal{D} = \mathbb{R}$ and the loss function:

$$L(d|\theta) = L_n(d|\theta) = 1\left\{|d - \theta| > \frac{c}{\sqrt{n}}\right\} = 1\{\sqrt{n}|d - \theta| > c\},$$

for some positive constant c. Find asymptotically minimax procedures over $[a, b]$.

Asymptotic minimax lower bound

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Asymptotic minimax lower bound

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Asymptotic minimax lower bound

We have shown $\bar{X}$ is asymptotically minimax in the last example. Indeed, there are other estimators that are also asymptotically minimax.

Consider the Bayes procedure using a conjugate $N(\mu_0, \sigma_0^2)$ prior:

$$w(\theta) = \frac{1}{\sigma_0\sqrt{2\pi}} e^{-\frac{1}{\sigma^2}(\theta-\mu_0)^2}.$$

We know the posterior is also normal, namely

$$N\left(\frac{1}{1+n\sigma_0^2}\mu_0 + \frac{n\sigma_0^2}{1+n\sigma_0^2}\bar{X}, \frac{\sigma_0^2}{1+n\sigma_0^2}\right).$$

Use this prior to find another asymptotically minimax procedure.

Asymptotic minimax lower bound

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Asymptotic minimax lower bound

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Asymptotic minimax lower bound

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Asymptotic minimax lower bound

(Example continued)

Asymptotically minimax procedures

Example (**Interval estimation of a Uniform scale parameter**): Suppose $\mathbf{X} = (X_1, \ldots, X_n)$ consists of iid $U(0, \theta)$ random variables for some unknown $\theta > 0$. Consider 0-1 loss $L(d|\theta) = 1\{|d - \theta| > \frac{c}{n}\}$ for some $c > 0$. Find asymptotically minimax procedures over $[a, b]$.

Asymptotic minimax lower bound

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