# Moments and moment generating function

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#### Review of random variables (STAT2011/2911)

of function of so

Consider a sample space  $\Omega$  with a probability measure P. Let X be a random variable defined on this sample space.

- Any random variable X has a cumulative distribution function  $(cdf)(F_X(x)) = P(X \le x). \times random \ variable$   $+ random \ Variable$
- Discrete random variables:

#### Moments

For any random variable X and a function  $g: \mathbb{R} \to \mathbb{R}$ , define the **expectation** of g(X) to be

Examples:

examples:

• rth moment: 
$$q(x) = x^{r}$$
,  $y_{r} = E(x^{r})$ ;  $y_{r} = E(x^{r})$ ;  $y_{r} = E(x^{r})$ 

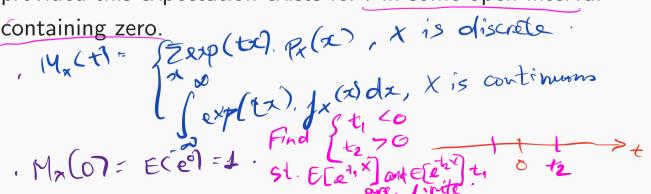
• rth central moment:  $g(x) = (x - \mu)^{-1}$ ;  $\mu r = E[(x - \mu)]^{-1}$  r = 2;  $E[(x - \mu)^{-2}] = Var(x)$   $= E(x^{2}) - \mu^{2}$ 

### Moment generating functions

- Moment generating function: encoding the sequence of moments  $\{E(X^r)\}$ ,  $r=1,2\ldots,\infty$  into the coefficients of a power series. Generating function  $\{E(X^r)\}$ , then the moment generating function
- (mgf) of a random variable X is defined to be

$$M_X(t) = E\{g(X)\} = E\{\exp(tX)\}, \text{ furtion } g t.$$

provided this expectation exists for t in some open interval containing zero.

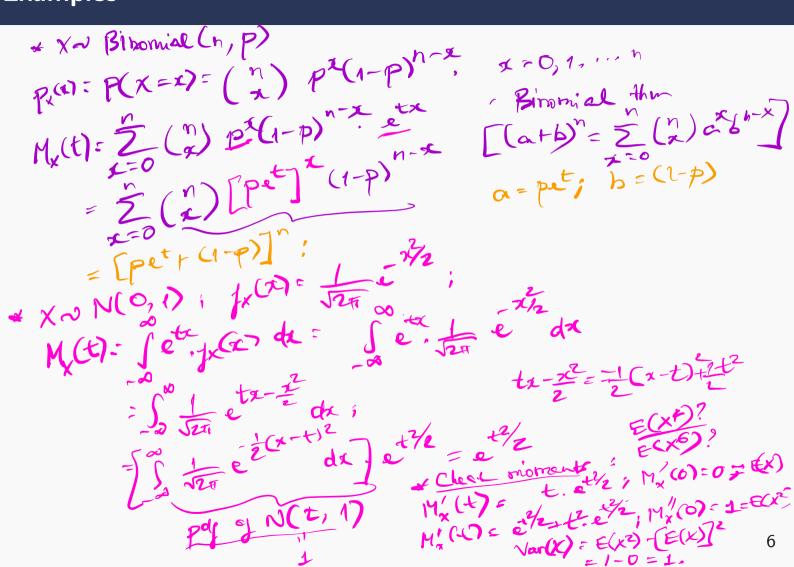


### Getting moments from mgf

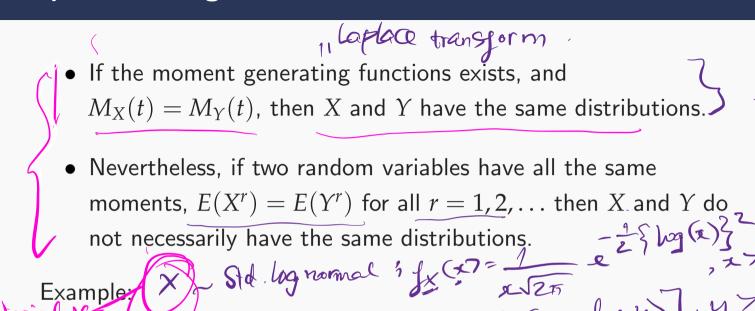
Calculus: 
$$a = \frac{2}{2}$$
 of  $a = t \times 1$ 
 $M_{x}(t) = E[e^{tx}]$   $E[x]$   $E[x]$ 

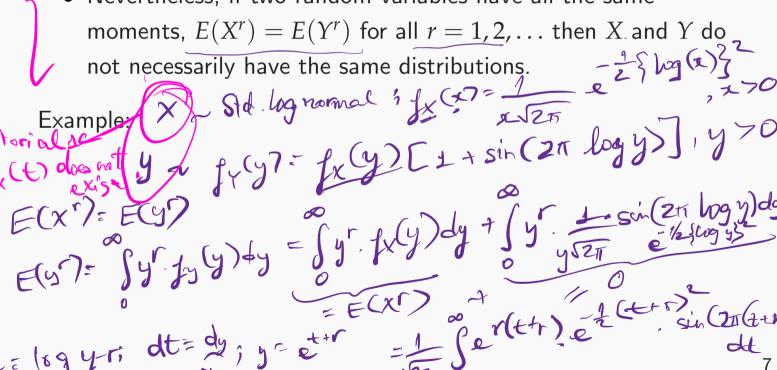
### **Examples**

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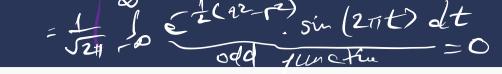


# Uniqueness of mgf





### Properties of mgf



Let X be a RV with  $mgf M_X(t)$ . Then the random variable

$$Z = aX + b \text{ has mgf}$$

$$M_Z(x) = E[e^{t}] = E[e^{t}] + e^{t}X$$

$$= e^{t}b E[e^{t}X] - e^{t}b M_X(e^{t})$$

$$= e^{t}b E[e^{t}X] - e^{t}b M_X(e^{t}X]$$

#### Properties of mgf

Recall for two independent random variables X and Y, we have

$$E\left\{g(X)h(Y)\right\} = E\left\{g(X)\right\}E\left\{h(Y)\right\}$$

for any two functions g and h. Let  $M_X(t)$  and  $M_Y(t)$  be mgfs of X and Y respectively, then the mgf of Z = X + Y is given by

$$|M_{z}(t)| = E[e^{zt}] = E[e^{(x+y)t}]$$

$$= E(e^{x+t}, e^{y+t})$$

$$= E(e^{x+t}, e^{x+t})$$

$$= E(e^{x+t}$$

### Sum of independent random variables

More generally, if  $X_1, \ldots, X_n$  be mutually independent random variables with mgfs  $M_{X_i}(t)$  for  $i=1,\ldots,n$ , then the mgf of  $Z = \sum_{i=1}^{n} X_i$  is given by Eg:  $\chi_1$ :  $\chi_1$ :  $\chi_1$ :  $\chi_2$ :  $\chi_3$ :  $\chi_4$ :  $\chi_5$  $M_{z}(t) = \prod_{i=1}^{n} M_{x_{i}}(t) =$   $P(x_{i}=2) = e^{x_{i}} x_{i}/x_{i}; x=0,1,...$ Mx(4)= 2 = \( \) \

#### **Examples**

$$M_{Z}(t) = \prod_{i=1}^{n} e^{\chi_{i}} (e^{t} - 1) = \underbrace{e^{\sum_{i=1}^{n} \chi_{i}}}_{\text{MGF}} (e^{t} - 1)$$

$$\Rightarrow Z \sim \text{Poisson}(\sum_{i=1}^{n} \chi_{i})$$

$$\star \chi_{i} \sim N(\mu_{i}, \sigma_{i}^{2}); \quad \chi_{i} \sim \chi_{n} : \text{independent}$$

$$Z = \underbrace{\sum_{i=1}^{n} \chi_{i}^{n}}_{\text{MGF}} ?$$

$$M_{Z}(t) = \underbrace{\prod_{i=1}^{n} M_{X_{i}}(t)}_{\text{MGF}} = \underbrace{\underbrace{\sum_{i=1}^{n} \chi_{i}^{n}}_{\text{MGF}} (e^{t} - 1)}_{\text{MGF}} (e^{t} - 1)$$

$$\Rightarrow Z \sim N(\underbrace{\sum_{i=1}^{n} M_{X_{i}}(t)}_{\text{MGF}} (e^{t} - 1)$$

### **Probability bounds**

Markov's inequality: For any non-negative random variable X and any a > 0, we have

Proof: 
$$X$$
: Continuous,  $\int_{X} (x) dx + \int_{X} (x) dx + \int_{X} (x) dx = \int_{X} (x) dx + \int_{X} (x) dx + \int_{X} (x) dx = \int_{X} (x) dx + \int_{X} (x) dx = \int_{X} (x) dx + \int_{X} (x) dx = \int_{X} (x)$ 

#### **Probability bounds**

Chebyshev's inequality: For any random variable X and any a>0, we have

Proof: 
$$E(X) = \mu$$
;

$$|X - E(X)| = |X - \mu| \neq 0$$

$$|X - \mu| \neq \alpha = P(|X - \mu|^2 \neq \alpha^2)$$

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$$|X - \mu|^2 \Rightarrow \alpha^2$$

$$|X - \mu| \neq \alpha^2$$

$$|X$$

#### **Probability bounds**

## **Examples**

X-Bin (n, Plip(x7, xn) yor PCXC1.  $P(x = xn) \in E(x) = \frac{np}{xn} = \frac{p}{x} = \frac{2}{3}$   $P = \frac{1}{2} : x = \frac{3}{4}$  Chil = 0\* Marbori P(x7 an) = P(x-np 7 an -np) \* Chebyshov: 5 P( )x-np) 7, an - np) =  $\frac{p(1-p)}{(\alpha n-np)^2} = \frac{p(1-p)}{n(\alpha-p)^2} = \frac{p(1-p)}{n(\alpha-p)^2}$ P= /21 x= /qp => P(x>xn) = 4/n. & Chernoff: P(X // dn) & min étan [ Epeta (1-P)] Find to minimize h(t) { etacpet + (1-P)} > t minimize a(t) 9(14) = 0 74. h(x) = (1-2)

#### Convergence of mgfs implies convergences of cdfs

Suppose  $X_1, X_2, \ldots$ , is a sequence of random variables, each with mgf  $M_{X_n}(t)$ . Furthermore, suppose that

$$\lim_{n\to\infty} M_{X_n}(t) = M_X(t)$$

for all t in an open interval containing zero, and  $M_X(t)$  is the mgf of a random variable X. Then for any x such that  $F_X(x)$  is continuous, we have

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x).$$

We also say that the sequence  $X_1, X_2, ... X_n$  converges to X in distribution.

### Application: Poisson approximation to binomial distribution

# **Application: Central limit theorem**

# **Convergence in probability**

# Weak law of large numbers