Functions of random variables

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Transformation of a random variable

Distribution of functions of a random variable

Let X be a random variable with cdf $F_X(x) = P(X \leq x)$, then for any function g, we have $Y = g(X)$ is also a random variable. The distribution of Y can be determined from

$$F_Y(y) = P(Y \leq y) = P\left(g(X) \leq y\right) = P(X \in \mathcal{A})$$

where $\mathcal{A} = \{x : g(x) \leq y\}$.

Example

Distribution of functions of a random variable

If g is **monotone increasing**, the calculation can be simplified. In this case, the function g has also a monotone increasing inverse function g^{-1}. Then we have

If g is **monotone decreasing**, then we have

Density of $g(X)$

Example

Bivariate transformation

Joint distribution

Let X and Y be two random variables with known joint distribution. Consider $U = g_1(X,Y)$ and $V = g_2(X,Y)$ for some functions g_1 and g_2. Using the cdf argument, we can also find the joint distribution of U and V.

Example:

Jacobian technique

Consider a case when both X and Y are continuous with joint pdf $f_{X,Y}(x,y)$, and there is a one-to-one transformation between (X,Y) and (U,V). In this case, we can write $X = h_1(U,V)$ and $Y = h_2(U,V)$, and the joint density of U and V is given by

Examples

Examples

Extension to multivariate distributions

Sample from normal distributions

Sample mean and variances

Let $X_1, \ldots, X_n$ denote iid samples from $N(\mu, \sigma^2)$ distribution. Let $\bar{X}_n = n^{-1} \sum_{i=1}^n X_i$ and $s_n^2 = \sum_{i=1}^n (X_i - \bar{X})^2 / (n-1)$.

Then:

Proofs

The t-distribution

Let $Z \sim N(0, 1)$ and $V \sim \chi^2_v$ independent of Z. Then the random variable

$$T = \frac{Z}{\sqrt{V/v}} \sim t_v,$$

the t distribution with v degrees of freedom.

Question: Derive the density of T?

The t-distribution

Let $X_1, \ldots, X_n$ denote iid samples from $N(\mu, \sigma^2)$ distribution, with $\bar{X}_n$ and s_n^2 the sample mean and sample variance. Then

$$T = \frac{\bar{X}_n - \mu}{s_n / \sqrt{n}} \sim t_{n-1}$$