THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solution to Tutorial Week 1

STAT3023: Statistical Inference — Semester 2, 2023

1. (4.36) If $M_X(t)$ is a mgf of a random variable, then $M_X(0) = 1$. Clearly, the given function has $M_X(0) = 0$, so it can't be a mgf of a random variable.

(4.37) By definition, we have

$$\begin{aligned} M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx &= \frac{1}{2}\int_{-\infty}^{\infty} e^{tx}e^{-|x|}dx \\ &= \frac{1}{2}\int_{-\infty}^{0} e^{tx}e^{x}dx + \frac{1}{2}\int_{0}^{\infty} e^{tx}e^{-x}dx \\ &= \frac{1}{2}\int_{-\infty}^{0} e^{(t+1)x}dx + \frac{1}{2}\int_{0}^{\infty} e^{(t-1)x}dx \\ &= \frac{1}{2(t+1)}e^{(t+1)x}\Big|_{x=-\infty}^{x=0} + \frac{1}{2(t-1)}e^{(t-1)x}\Big|_{x=0}^{x=\infty} \\ &= \frac{1}{2(t+1)} - \frac{1}{2(t-1)} = \frac{1}{1-t^2},\ |t| < 1. \end{aligned}$$

(4.40) $Z = \frac{1}{4}(X-3) = \frac{1}{4}X - \frac{3}{4}$, so

$$\begin{aligned} M_Z(t) &= \exp\left(-\frac{3t}{4}\right) M_X\left(\frac{t}{4}\right) \\ &= \exp\left(-\frac{3t}{4}\right) \exp\left(\frac{3t}{4} + 8\left(\frac{t}{4}\right)^2\right) = \exp\left(\frac{t^2}{2}\right), \end{aligned}$$

which is the mgf of the standard normal random variable $N(0,1)$. Hence, $Z \sim N(0,1)$, so $E(Z) = 0$ and $\mathrm{var}(Z) = 1$.

2. By definition, we have

$$\begin{aligned} M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx &= \int_{0}^{\infty} e^{tx}\frac{e^{-x/\beta}x^{\alpha-1}}{\beta^\alpha\Gamma(\alpha)}dx \\ &= \frac{1}{\beta^\alpha\Gamma(\alpha)}\int_0^\infty e^{-x(1/\beta-t)}x^{\alpha-1}dx \end{aligned}$$

Let $u = (1/\beta - t)^{-1}$, then if $t < 1/\beta$, the integral $\int_0^\infty e^{-x(1/\beta-t)}x^{\alpha-1}dx = \int_0^\infty e^{-x/u}x^{\alpha-1}dx$ is the unnormalized density of a Gamma(α, u) distribution, so we have

$$\int_0^\infty e^{-x/u}x^{\alpha-1}dx = u^\alpha\Gamma(\alpha).$$

Hence, the mgf of X is equal to

$$M_X(t) = \left(\frac{u}{\beta}\right)^\alpha = \left\{\beta\left(\frac{1}{\beta} - t\right)\right\}^{-\alpha} = (1 - t\beta)^{-\alpha},\ t < 1/\beta.$$

3. Let $S = \sum_{i=1}^n X_i$, where $X_i \sim \text{Gamma}(\alpha, \beta)$ distribution. By the previous problem, we have $M_{X_i}(t) = (1 - t\beta)^{-\alpha}$ for $i = 1, \ldots, n$. Because X_i's are mutually independent, we have

$$M_S(t) = \prod_{i=1}^n M_{X_i}(t) = (1 - t\beta)^{-\alpha n},$$

which is the mgf of the Gamma$(\alpha n, \beta)$ distribution. Hence $S \sim \text{Gamma}(\alpha n, \beta)$.

4. (a) We have $Y = \log(X)$, so $X = e^Y$. As a result, we can write

$$E(X^r) = E(e^{Yr}) = M_Y(r) = e^{r^2/2}.$$

(b) For any $t > 0$, we have

$$\begin{aligned} E(e^{tX}) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{1}{x} \exp\left(tx - \frac{\{\log(x)\}^2}{2}\right) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp\left(tx - \log(x) - \frac{\{\log(x)\}^2}{2}\right) dx. \end{aligned}$$

This integral is infinite if we can prove

$$\lim_{x\to\infty} tx - \log(x) - \frac{\{\log(x)\}^2}{2} = \infty.$$

To prove it, we can consider the limit

$$\lim_{x\to\infty} \frac{tx - \log(x) - \frac{\{\log(x)\}^2}{2}}{tx}.$$

We have

$$\begin{aligned} \lim_{x\to\infty} \frac{tx - \log(x) - \frac{\{\log(x)\}^2}{2}}{tx} &= 1 - \frac{1}{t}\lim_{x\to\infty}\frac{\log(x)}{x} - \frac{1}{2t}\lim_{x\to\infty}\frac{\{\log(x)\}^2}{x} \\ &\overset{(i)}{=} 1 - \frac{1}{t}\lim_{x\to\infty}\frac{1}{x} - \frac{1}{t}\lim_{x\to\infty}\frac{\log(x)}{x} \\ &\overset{(ii)}{=} 1 - \frac{1}{t}\lim_{x\to\infty}\frac{1}{x} = 1, \end{aligned}$$

where steps (i) and (ii) follow from the L'Hopital rule. Therefore, we obtain

$$\lim_{x\to\infty} tx - \log(x) - \frac{\{\log(x)\}^2}{2} = \lim_{x\to\infty}(tx) = \infty, \text{ for any } t > 0.$$

5. Each X_i has MGF $M_{X_i}(t) = \frac{1}{2}(e^t + e^{-t})$, so the sum $S = \sum_{i=1}^n X_i$ has MGF

$$M_S(t) = \prod_{i=1}^n M_{X_i}(t) = \frac{1}{2^n}\left(e^t + e^{-t}\right)^n.$$

Using the argument as derived in Chernoff bound, for any $t > 0$ we have

$$P(S \geq nx) = P\left\{\exp(St) \geq \exp(ntx)\right\} \leq \exp(-ntx) M_S(t).$$

Since this above inequality holds for any $t > 0$, then the lowest upper bound can be obtained by finding t to minimize the right-hand side

$$\exp(-ntx) M_S(t) = \frac{1}{2^n} \exp(-ntx) \left(e^t + e^{-t}\right)^n. \tag{1}$$

Equivalently, we can take the log and find t^* to minimize

$$g(t) = -ntx + n \log(e^t + e^{-t}).$$

The derivative of $g(t)$ is equal to

$$g'(t) = -nx + \frac{n(e^t - e^{-t})}{e^t + e^{-t}}.$$

Setting $g'(t) = 0$, we solve the equation

$$e^t - e^{-t} = x(e^t + e^{-t}), \text{ i.e. } e^t(1-x) = e^{-t}(1+x).$$

Multiplying both sides by e^t, the above equation is equivalent to

$$e^{2t} = \frac{1+x}{1-x}, \quad t^* = \frac{1}{2} \log \frac{1+x}{1-x}.$$

Now, we only need to substitute this t^* to (1). At $t = t^*$, we have $e^{t^*} + e^{-t^*} = \sqrt{\frac{1+x}{1-x}} + \sqrt{\frac{1-x}{1+x}} = \frac{2}{\sqrt{(1+x)(1-x)}}$, hence,

$$\begin{aligned}
P(S \geq nx) &\leq \frac{1}{2^n} \exp(-nt^*x) \left(e^{t^*} + e^{-t^*}\right)^n \\
&= \frac{1}{2^n} \exp\left(-\frac{nx}{2} \log \frac{1+x}{1-x}\right) 2^n (1+x)^{-n/2} (1-x)^{-n/2} \\
&= \exp\left(-\frac{n}{2}(1+x)\log(1+x) - \frac{n}{2}(1-x)\log(1-x)\right) \\
&= \exp\left\{-nF(x)\right\},
\end{aligned}$$

as claimed.