The University of Sydney
School of Mathematics and Statistics

Tutorial Week 1

STAT3023: Statistical Inference | Semester 2, 2022

1. Exercises from the textbook (Freund's, 8th ed.)
4.36, 4.37, 4.40

2. (Based on 6.12) Find the moment generating function of $X \sim \text{Gamma}(\alpha, \beta)$. Recall the density function is

$$f_X(x) = \begin{cases} \frac{e^{-x/\beta} x^{\alpha-1}}{\beta^\alpha \Gamma(\alpha)}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

3. (Based on 7.43) If n independent random variables have the same gamma distribution with the parameters α and β, find the moment generating function of their sum and identify its distribution.

4. A continuous (positive) random variable X is said to have a standard log normal distribution if $Y = \log(X) \sim N(0, 1)$.

(a) Using the fact that $M_Y(t) = \exp\left(\dfrac{t^2}{2}\right)$, derive $E(X^r)$ for any $r = 1, 2, \ldots$

(b) It can be proved that (the proof will be covered in Week 3), the pdf of X is given by

$$f_X(x) = \frac{1}{x\sqrt{2\pi}} \exp\left(-\frac{\{\log(x)\}^2}{2}\right), \quad x > 0.$$

Prove that the moment generating function of X does not exist (in the way it is defined in the lecture). Specifically, prove that for any positive t, the expectation $E\{\exp(tX)\}$ is not finite.

5. Let $X_i,\ i = 1, 2 \ldots, n$ be a sequence of independent Rademacher random variables, i.e $P(X_i = 1) = P(X_i = -1) = 0.5$. Let $S = \sum_{i=1}^n X_i$ for $i = 1, 2, \ldots, n$. Using the Chernoff bound, prove that for any $x \in (-1, 1)$, we have

$$P(S \geq nx) \leq \exp\{-nF(x)\},$$

where

$$F(x) = \frac{1}{2}(1 - x)\log(1 - x) + \frac{1}{2}(1 + x)\log(1 + x).$$

$$P(X \geqq a) \leqq \frac{\mu}{a}$$

This inequality is called **Markov's inequality**, and we have given it here mainly because it leads to a relatively simple alternative proof of Chebyshev's theorem.

30. Use the inequality of Exercise 29 to prove Chebyshev's theorem. [*Hint*: Substitute $(X-\mu)^2$ for X.]

31. What is the smallest value of k in Chebyshev's theorem for which the probability that a random variable will take on a value between $\mu - k\sigma$ and $\mu + k\sigma$ is

(a) at least 0.95;

(b) at least 0.99?

32. If we let $k\sigma = c$ in Chebyshev's theorem, what does this theorem assert about the probability that a random variable will take on a value between $\mu - c$ and $\mu + c$?

33. Find the moment-generating function of the continuous random variable X whose probability density is given by

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

and use it to find μ_1', μ_2', and σ^2.

34. Find the moment-generating function of the discrete random variable X that has the probability distribution

$$f(x) = 2\left(\frac{1}{3}\right)^x \quad \text{for } x = 1, 2, 3, \ldots$$

and use it to determine the values of μ_1' and μ_2'.

35. If we let $R_X(t) = \ln M_X(t)$, show that $R_X'(0) = \mu$ and $R_X''(0) = \sigma^2$. Also, use these results to find the mean and the variance of a random variable X having the moment-generating function

$$M_X(t) = e^{4(e^t - 1)}$$

36. Explain why there can be no random variable for which $M_X(t) = \frac{t}{1-t}$.

37. Show that if a random variable has the probability density

$$f(x) = \frac{1}{2}\, e^{-|x|} \quad \text{for } -\infty < x < \infty$$

its moment-generating function is given by

$$M_X(t) = \frac{1}{1-t^2}$$

38. With reference to Exercise 37, find the variance of the random variable by

(a) expanding the moment-generating function as an infinite series and reading off the necessary coefficients;

(b) using Theorem 9.

39. Prove the three parts of Theorem 10.

40. Given the moment-generating function $M_X(t) = e^{3t+8t^2}$, find the moment-generating function of the random variable $Z = \frac{1}{4}(X-3)$, and use it to determine the mean and the variance of Z.

6 Product Moments

To continue the discussion of Section 3, let us now present the **product moments** of two random variables.

DEFINITION 7. PRODUCT MOMENTS ABOUT THE ORIGIN. *The* **r*th* and s*th* product moment about the origin** *of the random variables* X *and* Y*, denoted by* $\mu'_{r,s}$*, is the expected value of* X^rY^s*; symbolically,*

$$\mu'_{r,s} = E(X^rY^s) = \sum_x \sum_y x^r y^s \cdot f(x,y)$$

for $r = 0, 1, 2, \ldots$ *and* $s = 0, 1, 2, \ldots$ *when* X *and* Y *are discrete, and*

$$\mu'_{r,s} = E(X^rY^s) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^r y^s \cdot f(x,y)\,dx\,dy$$

when X *and* Y *are continuous.*