THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solution to Tutorial Week 4

STAT3023: Statistical Inference — Semester 2, 2023

1. The transformation is one-to-one, i.e we can solve for X and Y uniquely from W and Z, i.e $X = W$ and $Y = Z - W$. The Jacobian of the transformation is

$$\begin{bmatrix} \frac{\partial X}{\partial W} & \frac{\partial X}{\partial Z} \\ \frac{\partial Y}{\partial W} & \frac{\partial Y}{\partial Z} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix},$$

with the determinant $\det(J) = 1$. Hence, the joint density of W and Z is given by

$$f_{W,Z}(w,z) = 24w(z-w)|\det(J)| = 24w(z-w).$$

We need to specify the range of z and w. Because $0 < x, y < 1$ and $x + y < 1$, this above density is only defined in the region with $0 < w < 1$, $0 < z < 1$ and $w < z$.

2. Since X and Y are independent, the joint density of X and Y is given by

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) = \left(\frac{1}{\beta^\alpha\Gamma(\alpha)}\right)^2 e^{-(x+y)/\beta}(xy)^{\alpha-1},\ x, y > 0.$$

(a) The transformation between (X, Y) and (U, V) is one-to-one, i.e we can solve $X = UV$ and $Y = V - UV$. The Jacobian of the transformation is

$$\begin{bmatrix} \frac{\partial X}{\partial U} & \frac{\partial X}{\partial V} \\ \frac{\partial Y}{\partial U} & \frac{\partial Y}{\partial V} \end{bmatrix} = \begin{bmatrix} V & U \\ -V & 1-U \end{bmatrix},$$

with the determinant $\det(J) = V$. Hence, the joint density of U and V is given by

$$\begin{aligned} f_{U,V}(u,v) &= \left(\frac{1}{\beta^\alpha\Gamma(\alpha)}\right)^2 e^{-v/\beta}\left\{uv(v-uv)\right\}^{\alpha-1} v \\ &= \left(\frac{1}{\beta^\alpha\Gamma(\alpha)}\right)^2 e^{-v/\beta}u^{\alpha-1}(1-u)^{\alpha-1}v^{2\alpha-1},\ v > 0,\ 0 < u < 1. \end{aligned}$$

(b) The marginal density of U is obtained by integrating v out of the joint density

$$f_U(u) = \int_0^\infty f_{U,V}(u,v)dv = \left(\frac{1}{\beta^\alpha\Gamma(\alpha)}\right)^2 u^{\alpha-1}(1-u)^{\alpha-1}\int_0^\infty e^{-v/\beta}v^{2\alpha-1}dv$$

The last integral is the unnormalized density of the Gamma$(2\alpha, \beta)$, so it equals

$$\int_0^\infty e^{-v/\beta}v^{2\alpha-1}dv = \beta^{2\alpha}\Gamma(2\alpha),$$

hence the marginal density of U is given by

$$f_U(u) = \left(\frac{1}{\beta^\alpha \Gamma(\alpha)}\right)^2 u^{\alpha-1}(1-u)^{\alpha-1}\beta^{2\alpha}\Gamma(2\alpha) = \frac{\Gamma(2\alpha)}{\Gamma(\alpha)\Gamma(\alpha)}u^{\alpha-1}(1-u)^{\alpha-1},\ 0 < u < 1.$$

Comparing it with the general form of the beta pdf, we can see that $U \sim$ beta(α, α).

3. (a) The expectation of T is given by

$$E(T) = \int_{-\infty}^{\infty} t f_T(t)dt = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{d\pi}\Gamma\left(\frac{d}{2}\right)} \int_{-\infty}^{\infty} t\left(1+\frac{t^2}{d}\right)^{-(d+1)/2} dt.$$

For the last integral, we write

$$\int_{-\infty}^{\infty} t\left(1+\frac{t^2}{d}\right)^{-(d+1)/2} dt = \int_{-\infty}^{0} t\left(1+\frac{t^2}{d}\right)^{-(d+1)/2} dt + \int_{0}^{\infty} t\left(1+\frac{t^2}{d}\right)^{-(d+1)/2} dt.$$

Let $u = 1 + t^2/d$, so $du = (2/d)tdt$. Then, we have

$$\begin{aligned}\int_{0}^{\infty} t\left(1+\frac{t^2}{d}\right)^{-(d+1)/2} dt &= \frac{d}{2}\int_{1}^{\infty} u^{-(d+1)/2}du \\ &= \frac{d}{1-d}u^{-(d-1)/2}\Big|_{u=1}^{u=\infty} = \frac{d}{1-d}(-1 + \lim_{u\to\infty} u^{-(d-1)/2});\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{0} t\left(1+\frac{t^2}{d}\right)^{-(d+1)/2} dt &= \frac{d}{2}\int_{\infty}^{1} u^{-(d+1)/2}du \\ &= \frac{d}{1-d}u^{-(d-1)/2}\Big|_{u=\infty}^{u=1} = \frac{d}{1-d}(1 - \lim_{u\to\infty} u^{-(d-1)/2}).\end{aligned}$$

Hence, the expectation only exists when $\lim_{u\to\infty} u^{-(d-1)/2}$ is finite, which occurs if and only if $-(d-1)/2 < 0$, or $d > 1$. In this case, the limit is 0, so

$$\int_{-\infty}^{\infty} t\left(1+\frac{t^2}{d}\right)^{-(d+1)/2} dt = \frac{d}{d-1} + \frac{d}{1-d} = 0,$$

i.e., $E(T) = 0$ when $d > 1$.

(b) When $d > 2 > 1$, by part (a), we have $E(T) = 0$, hence $\mathrm{Var}(T) = E(T^2)$. We have

$$E(T^2) = \int_{-\infty}^{\infty} t^2 f_T(t)dt = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{d\pi}\Gamma\left(\frac{d}{2}\right)} \int_{-\infty}^{\infty} t^2\left(1+\frac{t^2}{d}\right)^{-(d+1)/2} dt. \quad (1)$$

Note that the function in the integrand (i.e., the function $t^2(1+t^2/d)^{-(d+1)/2}$) is an even function, so we can write

$$\int_{-\infty}^{\infty} t^2\left(1+\frac{t^2}{d}\right)^{-(d+1)/2} dt = 2\int_{0}^{\infty} t^2\left(1+\frac{t^2}{d}\right)^{-(d+1)/2} dt.$$

Now let $u = t^2/d$. Since the integral is taken from $(0, \infty)$, we have $t = \sqrt{ud}$ and $dt = (1/2)\sqrt{d/u}du$. Therefore,

$$
\begin{aligned}
2\int_0^\infty t^2\left(1+\frac{t^2}{d}\right)^{-(d+1)/2} dt &= \int_0^\infty (ud)\,(1+u)^{-(d+1)/2}\sqrt{\frac{d}{u}}du \\
&= d^{3/2}\int_0^\infty u^{1/2}(1+u)^{-(d+1)/2}du \\
&= d^{3/2}\int_0^\infty \left(\frac{u}{1+u}\right)^{1/2}\left(\frac{1}{1+u}\right)^{d/2} du.
\end{aligned}
$$

Now let $v = u/(1+u)$. Then we have $u = \dfrac{v}{1-v}$ and $du = (1-v)^{-2}dv$. Hence, we have

$$
\int_0^\infty \left(\frac{u}{1+u}\right)^{1/2}\left(\frac{1}{1+u}\right)^{d/2} du = d^{3/2}\int_0^1 v^{1/2}(1-v)^{d/2-2}dv.
$$

We can see the last integral is the unnormalized density of a beta distribution with parameters $\alpha = 3/2$ and $\beta = d/2 - 1$. Hence, we have

$$
\int_0^\infty \left(\frac{u}{1+u}\right)^{1/2}\left(\frac{1}{1+u}\right)^{d/2} du = \int_0^1 v^{1/2}(1-v)^{d/2-2}dv = \frac{\Gamma(3/2)\Gamma(d/2-1)}{\Gamma\left(\dfrac{d+1}{2}\right)}.
$$

Substituting it into (1), we then have

$$
\begin{aligned}
E(T^2) &= \frac{\Gamma\left(\dfrac{d+1}{2}\right)}{\sqrt{d\pi}\Gamma\left(\dfrac{d}{2}\right)} \times d^{3/2}\frac{\Gamma(3/2)\Gamma(d/2-1)}{\Gamma\left(\dfrac{d+1}{2}\right)} = d\frac{\Gamma(3/2)\Gamma(d/2-1)}{\Gamma(1/2)\Gamma(d/2)} \\
&= d\frac{(1/2)\Gamma(1/2)\Gamma(d/2-1)}{\Gamma(1/2)\Gamma(d/2-1)(d/2-1)} = \frac{d}{d-2},
\end{aligned}
$$

where the second-to-last step uses the property of the gamma function that $\Gamma(x+1) = x\Gamma(x)$.

(c) As $d \to \infty$, we have

$$
\lim_{d\to\infty} f_T(t) = \lim_{d\to\infty}\frac{\Gamma\left(\dfrac{d+1}{2}\right)}{\sqrt{d\pi}\Gamma\left(\dfrac{d}{2}\right)}\lim_{d\to\infty}\left(1+\frac{t^2}{d}\right)^{-(d+1)/2}.
$$

For the second limit, we have

$$
\begin{aligned}
\lim_{d\to\infty}\left(1+\frac{t^2}{d}\right)^{-(d+1)/2} &= \lim_{d\to\infty}\left(1+\frac{t^2}{d}\right)^{-1/2}\lim_{d\to\infty}\left(1+\frac{t^2}{d}\right)^{-d/2} \\
&= \frac{1}{\lim_{d\to\infty}\left(1+\dfrac{t^2/2}{d/2}\right)^{d/2}} = e^{-t^2/2}.
\end{aligned}
$$

For the first limit, using the Stirling approximation, we have

$$\Gamma\left(\frac{d+1}{2}\right) \approx \sqrt{\frac{2\pi}{\frac{d+1}{2}}}\left(\frac{(d+1)/2}{e}\right)^{(d+1)/2} = \sqrt{2\pi}\left(\frac{d+1}{2}\right)^{d/2} e^{-(d+1)/2}$$

and

$$\Gamma\left(\frac{d}{2}\right) \approx \sqrt{\frac{2\pi}{\frac{d}{2}}}\left(\frac{d/2}{e}\right)^{d/2} = \sqrt{2\pi}\left(\frac{d}{2}\right)^{-1/2}\left(\frac{d}{2}\right)^{d/2} e^{-d/2}.$$

Hence,

$$\begin{aligned}
\lim_{d\to\infty} \frac{\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{d\pi}\Gamma\left(\frac{d}{2}\right)} &= \lim_{d\to\infty} \frac{1}{\sqrt{2\pi}}\left(1+\frac{1}{d}\right)^{d/2} e^{-1/2} = \frac{1}{\sqrt{2\pi}} \lim_{d\to\infty}\left(1+\frac{1/2}{d/2}\right)^{d/2} e^{-1/2} \\
&= \frac{1}{\sqrt{2\pi}} e^{1/2} e^{-1/2} = \frac{1}{\sqrt{2\pi}}.
\end{aligned}$$

Hence, together

$$\lim_{d\to\infty} f_T(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}.$$

4. The general form of an exponential family is

$$f(x|\theta) = h(x)\exp\left(\sum_{i=1}^{k} w_i(\theta)t_i(x) - A(\theta)\right),$$

and in the natural parameter form, it is written in the form

$$f(x|\eta) = h(x)\exp\left(\sum_{i=1}^{k} \eta_i t_i(x) - A^*(\eta)\right).$$

(a)

$$\begin{aligned}
f(x|\beta) &= \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{\frac{-x}{\beta}} I_{(0,\infty)}(x) \\
&= \frac{1}{\Gamma(\alpha)} x^{\alpha-1} I_{(0,\infty)}(x) \exp\left(\frac{-x}{\beta} - \alpha\log(\beta)\right)
\end{aligned}$$

so it is a full exponential family with $k = d = 1$, $h(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} I_{(0,\infty)}(x)$, $w_1(\beta) = \frac{1}{\beta}$, $t_1(x) = -x$, and $A(\beta) = \alpha\log(\beta)$. The natural parameter is $\eta = 1/\beta$. Since $\beta > 0$, the natural parameter space is $\{\eta : \eta > 0\}$.

(b)

$$\begin{aligned}
f(x|\alpha) &= \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{\frac{-x}{\beta}} I_{(0,\infty)}(x) \\
&= e^{-x/\beta} I_{(0,\infty)}(x) \exp\left\{(\alpha-1)\log x - \alpha\log(\beta) - \log\Gamma(\alpha)\right\},
\end{aligned}$$

so it is a full exponential family with $k = d = 1$, $h(x) = e^{-x/\beta} I_{(0,\infty)}(x)$, $w_1(\alpha) = \alpha - 1$, $t_1(x) = \log(x)$, and $A(\alpha) = \alpha \log(\beta) + \log \Gamma(\alpha)$. The natural parameter is $\eta = \alpha - 1$. Since $\alpha > 0$, the natural parameter space is $\{\eta : \eta > -1\}$.

(c)

$$\begin{aligned} f(x|\alpha, \beta) &= \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{\frac{-x}{\beta}} I_{(0,\infty)}(x) \\ &= I_{(0,\infty)}(x) \frac{1}{\Gamma(\alpha)\beta^\alpha} \exp\left\{(\alpha - 1)\log x - \frac{x}{\beta} - \alpha \log(\beta) - \log \Gamma(\alpha)\right\}, \end{aligned}$$

so it is a full exponential family with $k = d = 2$, $h(x) = I_{(0,\infty)}(x)$, $w_1(\alpha, \beta) = \alpha - 1$, $t_1(x) = \log(x)$, $w_2(\alpha, \beta) = \frac{1}{\beta}$, $t_2(x) = -x$, and $A(\alpha, \beta) = \alpha \log(\beta) + \log \Gamma(\alpha)$. The natural parameters are $\eta_1 = \alpha - 1$, $\eta_2 = \frac{1}{\beta}$. Since $\alpha, \beta > 0$, the natural parameter space is $\{(\eta_1, \eta_2) : \eta_1 > -1, \eta_2 > 0\}$.

(d)

$$\begin{aligned} f(x|\alpha, \beta) &= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1 - x)^{\beta-1} I_{(0,1)}(x) \\ &= I_{(0,1)}(x) \exp\{(\alpha - 1)\log(x) + (\beta - 1)\log(1 - x) + \\ &\quad \log \Gamma(\alpha + \beta) - \log \Gamma(\alpha) - \log \Gamma(\beta)\} \end{aligned}$$

so it is a full exponential family with $k = d = 2$, $h(x) = I_{(0,1}(x)$, $w_1(\alpha, \beta) = \alpha - 1$, $t_1(x) = \log(x)$, $w_2(\alpha, \beta) = \beta - 1$, $t_2(x) = \log(1 - x)$, and $A(\alpha, \beta) = -\log \Gamma(\alpha + \beta) + \log \Gamma(\alpha) + \log \Gamma(\beta)$. The natural parameters are $\eta_1 = \alpha - 1$, $\eta_2 = \beta - 1$. Since $\alpha, \beta > 0$, the natural parameter space is $\{(\eta_1, \eta_2) : \eta_1 > -1, \eta_2 > -1\}$.

(e)

$$\begin{aligned} f(x|\theta) &= \frac{1}{\sqrt{2\pi}\theta^{1/2}} \exp\left(-\frac{1}{2\theta}(x - \theta)^2\right) \\ &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\theta} + 2x - \frac{\theta}{2} - \frac{1}{2}\log\theta\right) \\ &= \frac{1}{\sqrt{2\pi}} e^{2x} I_{(-\infty,\infty)}(x) \exp\left(-\frac{x^2}{2\theta} - \frac{\theta}{2} - \frac{1}{2}\log\theta\right) \end{aligned}$$

so it is a full exponential family with $k = d = 1$, $h(x) = \frac{1}{\sqrt{2\pi}} e^{2x} I_{(-\infty,\infty)}(x)$, $w_1(\theta) = -1/(2\theta)$, $t_1(x) = x^2$, and $A(\theta) = \theta/2 + (1/2)\log\theta$. The natural parameter is $\eta = -1/(2\theta)$. Since $\theta > 0$, the natural parameter space is $\{\eta : \eta < 0\}$.

(f)

$$\begin{aligned} f(x|\alpha) &= \frac{1}{\Gamma(\alpha)\alpha^\alpha} x^{\alpha-1} e^{\frac{-x}{\alpha}} I_{(0,\infty)}(x) \\ &= I_{(0,\infty)}(x) \exp\left\{(\alpha - 1)\log x - \frac{x}{\alpha} - \alpha \log(\alpha) - \log \Gamma(\alpha)\right\}, \end{aligned}$$

so it is a full exponential family with $k = 2$, $d = 1$, $h(x) = I_{(0,\infty)}(x)$, $w_1(\alpha) = (\alpha - 1)$, $t_1(x) = \log(x)$, $w_2(\alpha) = 1/\alpha$, $t_2(x) = -x$ and $A(\alpha) = \alpha \log(\alpha) + \log \Gamma(\alpha)$. The natural parameter is $\eta_1 = \alpha - 1$, $\eta_2 = 1/\alpha$. Since $\alpha > 0$, the natural parameter space is $\{(\eta_1, \eta_2) : \eta_1 = 1/(\eta_2 + 1), \eta_2 > 0\}$.

Finally, for the distribution in part (e), we see that X^2 is the sufficient statistic for the natural parameter $\eta = -1/(2\theta)$. The corresponding $A^*(\eta)$ function in the natural parameter form is

$$A^*(\eta) = \frac{-1}{4\eta} + \frac{1}{2} \log\left(\frac{1}{-2\eta}\right) = \frac{-1}{4\eta} - \frac{1}{2}\log(-2\eta).$$

Hence,

$$E(X^2) = \frac{dA^*(\eta)}{d\eta} = \frac{1}{4\eta^2} - \frac{1}{2\eta} = \theta^2 + \theta.$$