The University of Sydney
School of Mathematics and Statistics

Tutorial Week 10

STAT3023: Statistical Inference — Semester 2, 2023

1. If Y has a gamma distribution with shape α and **rate**† λ its PDF is

$$f_Y(y) = \frac{y^{\alpha-1} e^{-\lambda y} \lambda^{\alpha}}{\Gamma(\alpha)},$$

for $y > 0$.

(a) Determine a formula for $E\left\{Y^{-k}\right\}$ which is valid for all (positive) integers k such that $0 < k < \alpha$.

(b) The random variable $X = Y^{-1}$ is said to have an *inverse Gamma* distribution. Use the previous part to determine the mean and variance of X (for $\alpha > 2$).

(c) Use the CDF method to derive the PDF of X.

2. Suppose now $X_1, \ldots, X_n$ are iid exponential random variables with **mean** θ, so the common PDF is, for $x, \theta > 0$, given by

$$f_\theta(x) = \frac{1}{\theta} e^{-x/\theta}.$$

(a) Determine the maximum likelihood estimator $\hat{\theta}_{\mathrm{ML}}(\mathbf{X})$.

(b) Determine the Bayes estimator $\hat{\theta}_{\mathrm{flat}}(\mathbf{X})$ under squared-error loss using the weight function $w(\theta) \equiv 1$ (the "flat prior").

(c) Determine the Bayes estimator $\hat{\theta}_{\mathrm{conj}}(\mathbf{X})$ under squared-error loss using the conjugate prior

$$w(\theta) = \frac{\lambda_0^{\alpha_0} e^{-\lambda_0/\theta}}{\theta^{\alpha_0+1}\Gamma(\alpha_0)},$$

for $\theta > 0$.

(d) Determine the risk $R(\theta|d)$ of the estimator

$$d(\mathbf{X}) = \frac{\ell + \sum_{i=1}^n X_i}{n+k}$$

under squared-error loss and hence also determine the limiting (rescaled) risk $\lim_{n\to\infty} nR(\theta|d)$.

(e) Determine the risk $R(\theta|d)$ and limiting (rescaled) risk $\lim_{n\to\infty} nR(\theta|d)$ where d is replaced by each of the 3 estimators in the questions (a)–(c) above.

3. Suppose $X_1, \ldots, X_n$ are iid $U[0,\theta]$ and it is desired to estimate θ using squared-error loss.

(a) Write down the CDF $F_\theta(x) = P_\theta\left\{X_1 \le x\right\}$.

(b) For any n iid random variables $Y_1, \ldots, Y_n$ the CDF of the maximum $Y_{(n)} = \max_{i=1,\ldots,n} Y_i$ is given by

$$P(Y_{(n)} \le y) = P(Y_1 \le y, \ldots, Y_n \le y) = P(Y_1 \le y) \cdots P(Y_n \le y) \quad \text{(by independence)}.$$

Use this to derive the CDF of $X_{(n)}$ above (the $U[0,\theta]$ sample maximum).

(c) Derive the PDF of $X_{(n)}$ and hence a formula for $E\left\{X_{(n)}^k\right\}$, for $k = 1, 2, \ldots$

†Note that the gamma *rate* parameter is the reciprocal of the gamma *scale* parameter.

(d) Determine the bias, variance and thus mean-squared error (risk) of the maximum likelihood estimator $X_{(n)}$.

(e) Determine the limiting (rescaled) risk $\lim_{n\to\infty} n^2 E_\theta \left\{ \left[X_{(n)} - \theta\right]^2 \right\}$.

(f) Defining the unbiased estimator $\hat{\theta}_{\text{unb}}(\boldsymbol{X}) = \left(\frac{n+1}{n}\right) X_{(n)}$, determine

$$\lim_{n\to\infty} n^2 E_\theta \left\{ \left[\hat{\theta}_{\text{unb}}(\boldsymbol{X}) - \theta\right]^2 \right\} .$$