Exponential family

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Exponential family

A pdf or pmf is said to belong to an exponential family if it can be written in the form

$$f(x|\theta) = h(x) \exp\left(\sum_{i=1}^{k} w_i(\theta) t_i(x) - A(\theta)\right), \; x \in \mathbb{R},$$

with $h(\cdot), w_i(\cdot), t_i(\cdot), A(\cdot)$ being real-valued functions.

- $h(x)$ often contains information about the support (or the range of X) through the use of indicator function.
  - An indicator function is of the form $I_{\mathcal{A}}(x) = 1$ if $x \in \mathcal{A}$ and $I_{\mathcal{A}}(x) = 0$ if $x \notin \mathcal{A}$.
- $A(\theta)$ is the normalizing constant to make f a valid pdf or pmf.
- Let d be the number of elements in θ. If $d = k$, the pdf or pmf belongs to a full exponential family; if $d < k$, it belongs to the curved exponential family.

Examples

Let X be a Poisson random variable with pmf $p_X(x) = e^{-\lambda}\frac{\lambda^x}{x!}$ for $x = 0, 1, 2, \ldots$ Show X belongs to the exponential family.

$$f(x|\theta) = h(x)\exp\left(\sum_{i=1}^{k} w_i(\theta)t_i(x) - A(\theta)\right)$$

Examples

Let X be a binomial random variable with pmf $p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$ for $x = 0, 1, \ldots, n$. Show X belongs to the exponential family for known n and unknown p.

Examples

Let X be a normal random variable $N(\mu, \sigma^2)$ with pdf $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$. Assuming σ is known, show X belongs to the exponential family.

Examples

Let X be a normal random variable $N(\theta, \theta^2)$ with pdf $f_X(x) = \frac{1}{\sqrt{2\pi}\theta} \exp\left(-\frac{1}{2\theta^2}(x-\theta)^2\right)$ for some $\theta > 0$. Show X belongs to the exponential family.

Examples

Again, let X be a normal random variable $N(\mu, \sigma^2)$ with pdf $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$. Assuming both μ and σ are unknown, show X belongs to the exponential family.

Natural parameters

The exponential family can be parameterized into the **canonical form**

$$f(x|\theta) = h(x) \exp\left(\sum_{i=1}^{k} \eta_i t_i(x) - A^*(\boldsymbol{\eta})\right)$$
$$= h(x) \exp\left(\boldsymbol{\eta}^\top T(x) - A^*(\boldsymbol{\eta})\right),\ x \in \mathbb{R},$$

where $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_k)$ is the set of natural parameters, and $T(x) = (t_1(x), \ldots, t_k(x))^\top$ is the set of sufficient statistics (more on it later).

Compare the canonical form with

$$f(x|\theta) = h(x) \exp\left(\sum_{i=1}^{k} w_i(\theta) t_i(x) - A(\theta)\right),\ x \in \mathbb{R}.$$

Examples

Write down the canonical form for $X \sim \text{Bin}(n, p)$ for n known and p unknown.

Examples

Write down the canonical form for $X \sim \text{Poisson}(\lambda)$.

Examples

Sometimes, we need to pay attention to the natural parameter space for the exponential family in the canonical form.

Write down the canonical form for $X \sim N(\theta, \theta^2)$ for $\theta > 0$.

Properties

- $E\left\{T(X)\right\} = \dfrac{\partial A^*}{\partial \boldsymbol{\eta}^\top}$
- $\mathrm{Var}\left\{T(X)\right\} = \dfrac{\partial^2 A^*}{\partial \boldsymbol{\eta} \partial \boldsymbol{\eta}^\top}$
- $\log M_{T(X)}(\boldsymbol{s}) = A^*(\boldsymbol{s} + \boldsymbol{\eta}) - A^*(\boldsymbol{\eta})$

Proofs

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Examples

Verify the above three properties with $\text{Bin}(n, p)$.

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Verify the above three properties with $\mathrm{Bin}(n, p)$.

Examples

Verify the above three properties with Poisson(λ).