THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Tutorial Week 8

STAT3023: Statistical Inference — Semester 2, 2023

1. Suppose X has a gamma distribution with known shape $\gamma_0 > 0$ and unknown scale parameter θ (see the computer exercise). Then X has PDF

$$f(x; \gamma_0, \theta) = \frac{x^{\gamma_0 - 1} e^{-x/\theta}}{\theta^{\gamma_0} \Gamma(\gamma_0)}$$

for $x > 0$, and 0 otherwise. Since this is an exponential family with sufficient statistic X (and X has a continuous distribution, so no randomisation required), the UMPU test of $H_0\colon \theta = \theta_0$ against $H_1\colon \theta \neq \theta_0$ is of the form

$$\delta(X) = \begin{cases} 1 & \text{for } X < c \text{ or } X > d \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

where the constants c and d ($c < d$) are chosen so that both the following equalities hold:

$$E_{\theta_0}[\delta(X)] = \alpha\,; \tag{2}$$

$$E_{\theta_0}[X\delta(X)] = \alpha E_{\theta_0}(X)\,. \tag{3}$$

(a) Write down a formula for $E_\theta(X)$.

(b) Show that the conditions (2) and (3) above imply

$$\int_c^d f(x; \gamma_0, \theta_0)\, dx = \int_c^d f(x; \gamma_0 + 1, \theta_0)\, dx\,. \tag{4}$$

(c) By integrating the right-hand side of (4) by parts, show that c and d in (1) satisfy

$$c^{\gamma_0} e^{-c/\theta_0} = d^{\gamma_0} e^{-d/\theta_0}\,. \tag{5}$$

(d) Explain why the UMPU test of $H_0\colon \theta = \theta_0$ against $H_1\colon \theta \neq \theta_0$ rejects for large values of

$$\frac{X}{\mu_0} - \log\left\{\frac{X}{\mu_0}\right\}$$

where $\mu_0 = \gamma_0\theta_0 = E_{\theta_0}(X)$, the expected value of X under H_0.

2. Suppose $X_1, \ldots, X_n$ (for $n \geq 2$) are iid $N(\mu, \sigma^2)$ and we are interesting in testing

$$H_0\colon \sigma^2 = \sigma_0^2 \text{ against } H_1\colon \sigma^2 \neq \sigma_0^2 \tag{6}$$

(a) The statistic $Y = \frac{1}{2}\sum_{i=1}^n (X_i - \bar{X})^2 \sim \frac{\sigma^2}{2}\chi^2_{n-1}$ which is the same as a gamma random variable with (known) shape $\frac{n-1}{2}$ and (unknown) scale parameter σ^2. It turns out that the UMPU test of (6) is the same as the test from question 1 applied to the statistic Y (which makes sense since Y is sufficient for the scale parameter σ^2). Show that this test rejects for large values of

$$\frac{S^2}{\sigma_0^2} - \log\left(\frac{S^2}{\sigma_0^2}\right) \tag{7}$$

where $S^2 = \frac{2Y}{n-1} = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$ is the sample variance.

(b) Is the GLRT of the hypotheses (6) above equivalent to the test rejecting for large values of (7)?

3. Suppose $X_1, \ldots, X_5$ are iid Poisson(θ) random variables. Determine the UMP test of $H_0\colon \theta = 1$ against $H_1\colon \theta < 1$ at level 0.05. **Hint:** you will need to calculate a few Poisson probabilities!

4. Suppose that $X_1, \ldots, X_n$ are iid $U(0, \theta)$, with common density

$$f_\theta(x) = \frac{1\{0 \le x \le \theta\}}{\theta}.$$

(a) Show that the sample maximum $X_{(n)}$ is a sufficient statistic for θ.

(b) Derive the maximum likelihood estimator $\hat{\theta}_{\mathrm{ML}}$ and show that it is biased.

(c) For what value of the multiplier c_n is the estimator $c_n\hat{\theta}_{\mathrm{ML}}$

(i) unbiased;

(ii) have the smallest possible MSE, i.e. for which $E_\theta\left[\left(c_n\hat{\theta}_{ML} - \theta\right)^2\right]$ is minimised?

(d) Determine the form of the GLRT for testing $H_0\colon \theta = \theta_0$ against $H_1\colon \theta \neq \theta_0$ at level α.