The University of Sydney
School of Mathematics and Statistics

Tutorial Week 2

STAT3023: Statistical Inference — Semester 2, 2022

1. (Problem 8.7 in Freund's) The following is a sufficient condition for the central limit theorem: If the random variables $X_1, \ldots, X_n$ are independent and uniformly bounded (i.e $P(|X_i| < M) = 1$ for some constants M) and if the variance of the sum

$$Y_n = X_1 + X_2 + \ldots + X_n$$

is infinite when $n \to \infty$, then the standardized mean of X_i approaches the standard normal distribution.

Show that this sufficient condition holds for a sequence of independent random variables X_i with the respective distribution

$$p_{X_i}(x_i) = \begin{cases} 1/2, & \text{for } x_i = 1 - (1/2)^i \\ 1/2, & \text{for } x_i = (1/2)^i - 1. \end{cases}$$

2. Let $X_1 \sim \text{Poisson}(\lambda_1)$ and $X_2 \sim \text{Poisson}(\lambda_2)$ be independent random variables. Using the moment generating function, we have shown in the lecture that $S = X + Y \sim \text{Poisson}(\lambda_1 + \lambda_2)$. Derive the conditional distributions $X_1|S$ and $X_2|S$.

3. A random variable X is said to have a standard Laplace (or double exponential) a distribution if its joint pdf is given by

$$f_X(x) = \frac{1}{2}e^{-|x|}, \; x \in \mathbb{R}.$$

In Tutorial 1, we have shown that the MGF of X is given by $M_X(t) = \dfrac{1}{1-t^2}$ for $|t| < 1$. This distribution can be obtained from the following hierarchical model. Let $Y \sim \text{Exp}(1)$ be the standard exponential distribution, and $X|Y \sim N(0, 2Y)$ be the standard normal distribution with mean 0 and variance $2Y$. Prove that the marginal distribution of X is standard Laplace.

4. Verify the example given in the lecture. Let $Y|\Lambda \sim \text{Poisson}(\Lambda)$, and $\Lambda \sim \text{Exp}(\beta)$, where β is the scale of the exponential distribution.

(a) Prove that the marginal distribution of Y is a negative binomial distribution.

(b) Calculate the marginal mean and variance of Y from (1) this marginal distribution, (2) properties of conditional expectation and conditional variances.

5. (Problem 6.45 in Freund's) If the exponent part of the bivariate normal density function is

$$\frac{-1}{102}\left[(x+2)^2 - 2.8(x+2)(y-1) + 4(y-1)^2\right],$$

find

(a) $\mu_X, \mu_Y, \sigma_X, \sigma_Y$ and ρ.

(b) $\mu_{Y|x}$ and $\sigma^2_{Y|x}$.

6. (Move to week 3) Let $\mathbf{X} = (X_1, \ldots, X_n)$ follow a multinomial distribution with m trials and cell probabilities $\mathbf{p} = (p_1, \ldots, p_n)$ such that $\sum_{i=1}^n p_i = 1$.

(a) Show that $\mathrm{Cov}(X_i, X_j) = -mp_ip_j$.

(b) Derive the marginal distribution of (X_1, X_2).

(c) Derive the conditional distribution $(X_3, \ldots X_n | X_1 = x_1, X_2 = x_2)$.

Some useful formulas

Exponential distribution A continuous random variable X is said to follow an exponential distribution with scale parameters $\beta > 0$ if its density is given by

$$f_X(x) = \frac{1}{\beta} e^{-x/\beta}, \ x > 0.$$

The corresponding mean and variance of X are $E(X) = \beta$ and $\mathrm{Var}(X) = \beta^2$.

Negative binomial distribution A discrete random variable X is said to follow a negative binomial distribution with parameters (r, p) if its probability mass function is given by

$$p_X(x) = \binom{x + r - 1}{x} (1-p)^r p^k.$$

The mean and variance of X are $E(X) = (pr)/(1-p)$ and $\mathrm{Var}(X) = pr/(1-p)^2$.