THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Tutorial Week 7 Solution

STAT3023: Statistical Inference — Semester 2, 2023

1. (a) The likelihood function of (α, θ) is

$$L(\alpha, \theta; \mathbf{X}) = \prod_{i=1}^{n} f_{X_i}(X_i) = \prod_{i=1}^{n} \frac{\theta^\alpha}{\Gamma(\alpha)}(X_i)^{\alpha-1}e^{-X_i\theta}$$
$$= \theta^{n\alpha}\left\{\Gamma(\alpha)\right\}^{-n}\left(\prod_{i=1}^{n} X_i\right)^{\alpha-1} e^{-\sum_{i=1}^{n} X_i\theta}.$$

The corresponding log likelihood is

$$\begin{aligned}\ell(\alpha, \theta; \mathbf{X}) &= \log L(\beta) \\ &= n\alpha\log(\theta) - n\log\Gamma(\alpha) + (\alpha-1)\log\left(\prod_{i=1}^{n} X_i\right) - \theta\sum_{i=1}^{n} X_i \\ &= n\alpha\log(\theta) - n\log\Gamma(\alpha) + (\alpha-1)\sum_{i=1}^{n}\log(X_i) - \theta\sum_{i=1}^{n} X_i.\end{aligned}$$

Hence, the score functions are given by

$$\frac{\partial\ell}{\partial\alpha} = n\log\theta - n\psi(\alpha) + \sum_{i=1}^{n}\log(X_i),\ \ \psi(\alpha) = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)},$$
$$\frac{\partial\ell}{\partial\theta} = \frac{n\alpha}{\theta} - \sum_{i=1}^{n} X_i.$$

(b) Note that $E(X_i) = \alpha/\theta$, so

$$E\left(\frac{\partial\ell}{\partial\theta}\right) = \frac{n\alpha}{\theta} - \sum_{i=1}^{n} E(X_i) = \frac{n\alpha}{\theta} - n\frac{\alpha}{\theta} = 0.$$

Now to compute the expectation for the score function $\partial\ell/\partial\alpha$, we will need to compute the expectation $E\left\{\log(X)\right\}$, where $X \sim f_X(x)$. Note that $f_X(x)$ belongs to the class of exponential family, i.e., we can write

$$f_X(x) = \frac{1}{x}\exp\left\{\alpha\log(x) - \theta x + \alpha\log(\theta) - \log\Gamma(\alpha)\right\},$$

so $\eta = (\alpha, \theta)$ is the natural parameter and $T(x) = (\log(x), -x)$ is a sufficient statistic. The function $A^*(\alpha, \theta) = -\alpha\log(\theta) + \log\Gamma(\alpha)$. By the properties of exponential family, we then have

$$E\left\{\log(X)\right\} = \frac{\partial A^*(\alpha, \theta)}{\partial\alpha} = -\log\theta + \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} = -\log\theta + \psi(\alpha).$$

Therefore, we have

$$
\begin{aligned}
E\left\{\frac{\partial \ell}{\partial \alpha}\right\} &= n\log\theta - n\psi(\alpha) + \sum_{i=1}^{n} E\left\{\log(X_i)\right\} \\
&= n\log\theta - n\psi(\alpha) - \{n\log\theta - n\psi(\alpha)\} = 0.
\end{aligned}
$$

(c) If α is known, then the score function is only $\dfrac{\partial \ell}{\partial \theta} = n\alpha/\theta - \sum_{i=1}^{n} X_i$.

(i) If $\hat{\theta}$ is unbiased for θ, then $E(\hat{\theta}) = \theta$ and $(d/d\theta)E(\hat{\theta}) = 1$. Hence, the CRLB for the variance of an unbiased estimator of θ is

$$
\frac{1}{\mathrm{Var}_\theta\left(\dfrac{\partial \ell}{\partial \theta}\right)} = \frac{1}{\mathrm{Var}_\theta\left(\sum_{i=1}^{n} X_i\right)} = \frac{1}{n\alpha/\theta^2} = \frac{\theta^2}{n\alpha}.
$$

(ii) No. If there was an unbiased estimator $W(\mathbf{X})$ for θ that achieves the CRLB, then the score function has to be written in the form $C_\theta(W(\mathbf{X})-\theta)$ for some constant C_θ. Here we couldn't write the score function in that form.

(iii) First, we have

$$
\begin{aligned}
E\left(\frac{1}{X}\right) &= \frac{\theta^\alpha}{\Gamma(\alpha)}\int_0^\infty \frac{1}{x} x^{\alpha-1} e^{-x\theta} dx = \frac{\theta^\alpha}{\Gamma(\alpha)}\int_0^\infty x^{(\alpha-1)-1} e^{-x\theta} dx \\
&= \frac{\theta^\alpha}{\Gamma(\alpha)}\frac{\Gamma(\alpha-1)}{\theta^{\alpha-1}}\int_0^\infty \frac{\theta^{\alpha-1}}{\Gamma(\alpha-1)} x^{(\alpha-1)-1} e^{-x\theta} dx \overset{(i)}{=} \frac{\theta}{\alpha-1},
\end{aligned}
$$

where step (i) follows from the fact that the last integral is the integral of the pdf of a gamma distribution with shape $\alpha-1$ and rate θ, and that $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$. Therefore,

$$
E(S) = \frac{\alpha-1}{n}\sum_{i=1}^{n} E\left(\frac{1}{X_i}\right) = \frac{\alpha-1}{n}\frac{n\theta}{\alpha-1} = \theta,
$$

so S is an unbiased estimator for θ.
A sufficient statistic for θ is $T = \sum_{i=1}^{n} X_i$, and since this distribution is the full exponential family, then any function of the sufficient statistic would be the best unbiased estimator for its expectation. Hence, the best unbiased estimator for θ is

$$
\hat{\theta} = E(S \mid T) = \frac{\alpha-1}{n} E\left(\sum_{i=1}^{n}(1/X_i) \mid \sum_{i=1}^{n} X_i\right).
$$

(iv) From the form of the score function $\dfrac{\partial \ell}{\partial \theta} = n\alpha/\theta - \sum_{i=1}^{n} X_i = -n\left(\bar{X} - \alpha/\theta\right)$, with $\bar{X} = n^{-1}\sum_{i=1}^{n} X_i$, we can make $\eta = \eta(\theta) = \alpha/\theta$. The score function with respect to η is given by

$$
\frac{\partial \ell}{\partial \eta} = \frac{\partial \ell}{\partial \theta}\frac{\partial \theta}{\partial \eta} = -n(\bar{X} - \eta)\left(-\frac{\alpha}{\eta^2}\right) = \frac{n\alpha}{\eta^2}(\bar{X} - \eta)
$$

which is in the special form $C_\eta(\bar{X} - \eta)$.

(v) The CRLB for estimating η is

$$\frac{\left(\frac{\partial}{\partial\theta}\eta\right)^2}{\mathrm{Var}_\theta\left(\frac{\partial\ell}{\partial\theta}\right)} = \frac{(-\alpha/\theta^2)^2}{n\alpha/\theta^2} = \frac{\alpha}{n\theta^2} = \mathrm{Var}_\theta(\bar{X}),$$

so $\bar{X}$ is MVU estimator for η.

2. (a) The likelihood of (μ, σ^2) based on $\mathbf{X} = (X_1, \ldots, X_n)$ is given by

$$\begin{aligned} L(\mu, \sigma^2; \mathbf{X}) &= \prod_{i=1}^n f_{X_i}(X_i) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(X_i - \mu)^2\right\} \\ &= (2\pi)^{-n/2}(\sigma^2)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (X_i - \mu)^2\right\}. \end{aligned}$$

Now we are using the same algebra manipulation as in the proof of the independence between the sample mean and the sample variance (Lecture week 3). We have

$$\begin{aligned} \sum_{i=1}^n (X_i - \mu)^2 &= \sum_{i=1}^n (X_i - \bar{X} + \bar{X} - \mu)^2 \\ &= \sum_{i=1}^n (X_i - \bar{X})^2 + n(\bar{X} - \mu)^2 - 2(\bar{X} - \mu)\underbrace{\sum_{i=1}^n (X_i - \bar{X})}_{=0} \\ &= (n-1)S^2 + n(\bar{X} - \mu)^2. \end{aligned}$$

Hence, the likelihood is

$$\begin{aligned} L(\mu, \sigma^2; \mathbf{X}) &= (2\pi)^{-n/2}(\sigma^2)^{-n/2} \exp\left\{-\frac{(n-1)S^2 + n(\bar{X} - \mu)^2}{2\sigma^2}\right\} \\ &= g(\bar{X}, S^2; \mu, \sigma^2)h(\mathbf{X}) \end{aligned}$$

with $g(\bar{X}, S^2; \mu, \sigma^2) = (\sigma^2)^{-n/2} \exp\left\{-\frac{(n-1)S^2 + n(\bar{X} - \mu)^2}{2\sigma^2}\right\}$ and $h(\mathbf{X}) = (2\pi)^{-n/2}$. Hence, by the factorization theorem, $(\bar{X}, S^2)$ is a sufficient statistic for (μ, σ^2).

(b) The result from part (a), along with the fact that this normal distribution is a full exponential family, suggests the best unbiased estimator for σ^p should be obtained from the sufficient statistic $(\bar{X}, S^2)$. In other words, we only need to find a function of this sufficient statistic that is unbiased for σ^p.

To do it, recall that $T = (n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$. Hence,

$$
\begin{aligned}
E\left(T^{p/2}\right) &= \int_0^\infty t^{p/2} f_T(t)dt = \underbrace{\frac{1}{2^{(n-1)/2}\Gamma\left(\dfrac{n-1}{2}\right)}}_{=K} \int_0^\infty t^{p/2} t^{(n-1)/2-1} e^{-t/2} dt \\
&= K \int_0^\infty t^{(n-1+p)/2-1} e^{-t/2} dt \\
&= K \times 2^{(n-1+p)/2} \Gamma\left(\frac{n-1+p}{2}\right) \\
&\quad \times \underbrace{\int_0^\infty \frac{1}{2^{(n-1+p)/2}\Gamma\left(\dfrac{n-1+p}{2}\right)} t^{(n-1+p)/2-1} e^{-t/2} dt}_{\text{pdf of } \chi^2_{n-1+p}} \\
&= 2^{p/2} \frac{\Gamma\left(\dfrac{n-1+p}{2}\right)}{\Gamma\left(\dfrac{n-1}{2}\right)} := C_{p,n}.
\end{aligned}
$$

Furthermore,

$$
T^{p/2} = \frac{\{(n-1)S^2\}^p}{\sigma^p},
$$

so we have

$$
E\left(\frac{\{(n-1)S^2\}^p}{\sigma^p}\right) = C_{p,n}.
$$

Therefore,

$$
E\left(\frac{\{(n-1)S^2\}^p}{C_{p,n}}\right) = \sigma^p,
$$

and hence $V = \dfrac{\{(n-1)S^2\}^p}{C_{p,n}}$ is an unbiased estimator for σ^p. Note V is also a function of the sufficient statistic S^2, so it is the MVU estimator for σ^p.

3. $\mathbf{X} = (X_1, \ldots, X_n)$, the likelihood ratio is given by

$$
\begin{aligned}
\frac{f_1(\mathbf{X})}{f_0(\mathbf{X})} &= \frac{\theta_1^n \prod_{i=1}^n X_i^{\theta_1 - 1}}{\theta_0^n \prod_{i=1}^n X_i^{\theta_0 - 1}} = \left(\frac{\theta_1}{\theta_0}\right)^n \left(\prod_{i=1}^n X_i\right)^{\theta_1 - \theta_0} \\
&= 2^n \prod_{i=1}^n X_i,
\end{aligned}
$$

where $\theta_1 = 2$, $\theta_0 = 1$. The critical region at level α is given by

$$
\begin{aligned}
C &= \left\{ \mathbf{x} = (x_1, \ldots, x_n) : \frac{f_1(\mathbf{x})}{f_0(\mathbf{x})} \geq y_\alpha \right\} \\
&= \left\{ \mathbf{x} = (x_1, \ldots, x_n) : \prod_{i=1}^n x_i \geq \underbrace{2^{-n} y_\alpha}_{c} \right\}.
\end{aligned}
$$

4. $\mathbf{X} = (X_1, \ldots, X_n)$, the likelihood ratio is given by

$$\begin{aligned}
\frac{f_1(\mathbf{X})}{f_0(\mathbf{X})} &= \frac{\prod_{i=1}^n \exp\left(-\frac{(X_i-\theta_1)^2}{2\sigma^2}\right)}{\prod_{i=1}^n \exp\left(-\frac{(X_i-\theta_0)^2}{2\sigma^2}\right)} \\
&= \exp\left(-\frac{1}{2\sigma^2}\left(\sum_{i=1}^n (X_i-\theta_1)^2 - \sum_{i=1}^n (X_i-\theta_0)^2\right)\right) \\
&= \exp\left(\frac{(\theta_1-\theta_0)}{\sigma^2}\sum_{i=1}^n X_i\right)\exp\left(-\frac{n(\theta_1^2-\theta_0^2)}{2\sigma^2}\right),
\end{aligned}$$

where $\theta_0 = 75$, $\theta_1 = 78$, $\sigma^2 = 100$. The critical region at level α is given by

$$\begin{aligned}
C &= \left\{(x_1, \ldots, x_n) : \frac{f_1(\mathbf{x})}{f_0(\mathbf{x})} \geq y_\alpha\right\} \\
&= \left\{(x_1, \ldots, x_n) : \sum_{i=1}^n x_i \geq \frac{\sigma^2}{\theta_1-\theta_0}\left(\log y_\alpha + \frac{n(\theta_1^2-\theta_0^2)}{2\sigma^2}\right)\right\} \\
&= \left\{(x_1, \ldots, x_n) : \bar{x} \geq \underbrace{\frac{\sigma^2}{n(\theta_1-\theta_0)}\left(\log y_\alpha + \frac{n(\theta_1^2-\theta_0^2)}{2\sigma^2}\right)}_{c}\right\}.
\end{aligned}$$

Under H_0, $\bar{X} \sim N(75, 100/n)$; under H_1, $\bar{X} \sim N(78, 100/n)$. It follows then

$$\begin{aligned}
P_{f_0}(\bar{X} \geq c) &= P\left(Z \geq \frac{c-75}{\sqrt{100/n}}\right) = 0.05, \quad Z \sim N(0,1); \\
P_{f_1}(\bar{X} \geq c) &= P\left(Z \geq \frac{c-78}{\sqrt{100/n}}\right) = 0.9,
\end{aligned}$$

from which we get

$$\frac{c-75}{\sqrt{100/n}} = q_0, \quad \frac{c-78}{\sqrt{100/n}} = q_1$$

where $q_0 = \texttt{qnorm}(0.95) \approx 1.645$, $q_1 = \texttt{qnorm}(0.1) \approx -1.282$. Solving the above two equations for n and c yields $n \approx 95$, $c \approx 76.69$.