THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Tutorial Week 6

STAT3023: Statistical Inference — Semester 2, 2023

1. Let $X_1, \ldots, X_n$ be a random sample from a Poisson(λ) distribution. Let $T = \sum_{i=1}^n X_i$. Show that T is a sufficient statistic for λ by:

(a) Showing the conditional distribution of $X_1, \ldots, X_n$ given $T = t$ is independent of λ.

(b) Using the Neyman factorization theorem.

2. (Problem 10.43 & 10.44 of Freund's) Let $X_1 \sim \text{Bin}(n_1, \theta)$ and $X_2 \sim \text{Bin}(n_2, \theta)$. Show that $T_1 = \dfrac{X_1 + X_2}{n_1 + n_2}$ is a sufficient statistic for θ. Is $T_2 = \dfrac{X_1 + 2X_2}{n_1 + n_2}$ sufficient for θ?

3. (Problem 10.41 of Freund's) To show that an estimator can be consistent without being unbiased nor asymptotically unbiased, consider the following estimation procedure. Let $X_1, \ldots, X_n$ be a random sample from a population with mean μ and finite variance σ^2. Consider the following estimation procedure: First, we randomly draw one slip of n papers numbered from 1 to n. If we get $2, 3, \ldots, n$, we use $\bar{X} = n^{-1} \sum_{i=1}^n$ as our estimate for μ; otherwise, we use n^2 as the estimate for μ. Show that this estimation procedure is

(a) consistent.

(b) neither unbiased nor asymptotically unbiased.

4. (a) The exponential distribution with mean $\theta > 0$ has pdf

$$f_\theta(x) = \frac{1}{\theta} e^{-x/\theta}, \ x > 0.$$

This distribution has mean θ and variance θ^2. Show that the sample mean $\bar{X} = n^{-1} \sum_{i=1}^n X_i$ is a minimum-variance unbiased estimator of θ. Verify that $\bar{X}$ achieves the CRLB for estimating θ.

(b) The Pareto distribution with *mean* $\theta > 1$ has pdf

$$f_\theta(x) = \left(\frac{\theta}{\theta - 1} \right) x^{-\left(\frac{2\theta - 1}{\theta - 1}\right)}, \ x > 1.$$

Suppose $X_1, \ldots, X_n$ are iid with common pdf $f_\theta(\cdot)$.

(i) The sample mean $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$ is an unbiased estimator of θ. Derive a formula for its variance as a function of θ.

(ii) Write down the log-likelihood $\ell_\theta(\theta; \mathbf{X})$ based on $\mathbf{X} = (X_1, \ldots, X_n)$.

(iii) Write down the score function $\frac{\partial}{\partial \theta} \ell(\theta; \mathbf{X})$. Can it be written in the form

$$C_\theta \left[\hat{\theta}(\mathbf{X}) - \theta \right] ?$$

(iv) Show that $\log(X_1)$ has an exponential distribution with a rate depending on θ. Hence determine the CRLB for estimating θ

(v) Identify a change of parameters from θ to $\eta = \eta(\theta)$ such that a MVU estimator $\hat{\eta}(\mathbf{X})$ exists for η. Determine the form of this $\hat{\eta}(\mathbf{X})$.

5. Consider a random sample $X_1, \ldots, X_n \sim N(\theta, 1)$. Let $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$.

(a) Show that $T_n = \bar{X}^2 - 1/n$ is the MVU estimator for θ^2.

(b) Determine the CRLB for estimating θ^2 and show that this lower bound is not achieved by T_n. (hint: for a normal distribution $X \sim N(\mu, \sigma^2)$, then $E(X^4) = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$).