THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solution to Tutorial Week 2

STAT3023: Statistical Inference — Semester 2, 2023

1. First we have $|X_i| = |1-(1/2)^i| < 1$ for all $i = 1, \ldots, n$, so X_i is uniformly bounded. Next, we have

$$E(X_i) = \frac{1}{2}\left(1 - \frac{1}{2^i}\right) + \frac{1}{2}\left(\frac{1}{2^i} - 1\right) = 0,$$

so its variance equals

$$\begin{aligned}\mathrm{Var}(X_i) &= E\left(X_i^2\right) - \{E(X_i)\}^2 \\ &= \frac{1}{2}\left(1 - \frac{1}{2^i}\right)^2 + \frac{1}{2}\left(\frac{1}{2^i} - 1\right)^2 - 0 \\ &= 1 - 2\left(\frac{1}{2}\right)^i + \left(\frac{1}{4}\right)^i\end{aligned}$$

Hence, the sum $Y_n = \sum_{i=1}^n$ has the variance

$$\begin{aligned}\mathrm{Var}(Y_n) = \mathrm{Var}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \mathrm{Var}(X_i) &= \sum_{i=1}^n \left\{1 - 2\left(\frac{1}{2}\right)^i + \left(\frac{1}{4}\right)^i\right\} \\ &= n - 2\sum_{i=1}^n \left(\frac{1}{2}\right)^i + \sum_{i=1}^n \left(\frac{1}{4}\right)^i\end{aligned}$$

As $n \to \infty$, the sum $\sum_{i=1}^n a^i \to \sum_{i=1}^\infty a^i = \frac{1}{1-a}$ for any $a < 1$ (geometric series). Hence the last two terms on the right-hand side converge to constants. However, $\lim_{n\to\infty} n = \infty$, so $\lim_{n\to\infty} \mathrm{Var}(Y_n) = \infty$ as required.

2. We have

$$\begin{aligned}P(X = x|S = s) &= \frac{P(X = x, S = s)}{P(S = s)} \\ &= \frac{P(X = x, Y = s - x)}{P(S = s)} \\ &\overset{(i)}{=} \frac{P(X = x)P(Y = s - x)}{P(S = s)} \\ &= \frac{\dfrac{e^{-\lambda_1}\lambda_1^x}{x!}\,\dfrac{e^{-\lambda_2}\lambda_2^{s-x}}{(s-x)!}}{\dfrac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^s}{s!}} \\ &= \frac{s!}{x!(s-x)!}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^x \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{s-x},\end{aligned}$$

where step (i) follows from the independence of X and Y. This is the pmf of a binomial distribution with s trials and success probability $\lambda_1/(\lambda_1+\lambda_2)$. Hence,

$$X|S \sim \text{Bin}\left(S, \frac{\lambda_1}{\lambda_1+\lambda_2}\right).$$

By a similar argument, we have

$$Y|S \sim \text{Bin}\left(S, \frac{\lambda_2}{\lambda_1+\lambda_2}\right).$$

3. Since $X|Y \sim N(0, 2Y)$, then we have $E(e^{tX}|Y) = e^{(1/2)t^2(2Y)} = e^{t^2 Y}$. Hence, by the property of the conditional expectation, we can compute the moment generating function of X to be

$$\begin{aligned}
E\left(e^{tX}\right) &= E_Y\left[E(e^{tX}|Y)\right] \\
&= E_Y\left[e^{t^2 Y}\right] \\
&= \int_0^\infty e^{-y} e^{t^2 y} dy \\
&= \int_0^\infty e^{(t^2-1)y} dy \\
&= \frac{1}{t^2-1} e^{(t^2-1)y}\Big|_{y=0}^{y=\infty} \\
&= \frac{1}{1-t^2}, \text{ for } |t| < 1,
\end{aligned}$$

which is the MGF of the standard Laplace distribution. Hence the marginal distribution of X is standard Laplace.

Comment: Due to this hierarchical structure, then the Laplace distribution belongs to the class of *scale mixture of normal distributions*. This class of distributions includes many other distributions (such as the t distribution) and can be used to model different behaviours of variances.

4. (a) Note that Y is discrete, so we can find the probability mass function of Y to be

$$\begin{aligned}
P(Y = y) &= P(Y = y, 0 \le \Lambda < \infty) \\
&= \int_0^\infty P(Y = y|\Lambda = \lambda) f_\Lambda(\lambda) d\lambda \\
&= \int_0^\infty e^{-\lambda} \frac{\lambda^y}{y!} \frac{1}{\beta} e^{-\lambda/\beta} d\lambda \\
&= \frac{1}{\beta y!} \int_0^\infty e^{-\lambda(1+1/\beta)} \lambda^y d\lambda
\end{aligned} \tag{1}$$

The integral above is the unnormalized density of a Gamma$\left(y+1, \dfrac{1}{1+1/\beta}\right)$ (see tutorial week 1 for this density). In other words, we have

$$\frac{1}{\Gamma(y+1)\left(\dfrac{1}{1+1/\beta}\right)^{y+1}} \int_0^\infty e^{-\lambda(1+1/\beta)} \lambda^y d\lambda = 1,$$

so

$$\int_0^\infty e^{-\lambda(1+1/\beta)}\lambda^y d\lambda = \frac{\Gamma(y+1)}{(1+1/\beta)^{y+1}} \overset{(i)}{=} \frac{y!}{(1+1/\beta)^{y+1}}$$

where step (i) follows from $\Gamma(y+1) = y!$ for $y \in \mathbb{N}$. Substituting it into (1), we have

$$P(Y=y) = \frac{1}{\beta}\frac{1}{(1+1/\beta)^{y+1}} = \frac{1}{\beta}\left(\frac{\beta}{\beta+1}\right)^{y+1}.$$

Comparing it with the general form of a negative binomial distribution, then we can see $Y \sim \text{NB}\left(r=1, p=\dfrac{\beta}{1+\beta}\right)$.

(b) From the marginal distribution of Y, we can get

$$E(Y) = \frac{pr}{1-p} = \frac{\beta/(1+\beta)}{1/(1+\beta)} = \beta,$$

$$\text{Var}(Y) = \frac{pr}{(1-p)^2} = \beta(1+\beta).$$

We can verify it by using the properties of conditional expectations and variances as

$$\begin{aligned} E(Y) &= E\left[E(Y|\Lambda)\right] = E(\Lambda) = \beta, \\ \text{Var}(Y) &= E\left[\text{Var}(Y|\Lambda)\right] + \text{Var}\left[E(Y|\Lambda)\right] \\ &= E\left(\Lambda\right) + \text{Var}\left(\Lambda\right) \\ &= \beta + \beta^2 = \beta(1+\beta). \end{aligned}$$

5. (a) The exponent in the bivariate normal density has the form

$$\begin{aligned} &-\frac{1}{2\left(1-\rho^2\right)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\frac{\left(x-\mu_X\right)\left(y-\mu_Y\right)}{\sigma_X\sigma_Y}\right] \\ =&-\frac{1}{2\left(1-\rho^2\right)\sigma_X^2}\left(x-\mu_X\right)^2 - \frac{1}{2\left(1-\rho^2\right)\sigma_Y^2}\left(y-\mu_Y\right)^2 \\ &+\frac{\rho}{(1-\rho^2)\sigma_X\sigma_Y}\left(x-\mu_X\right)\left(y-\mu_Y\right) \end{aligned}$$

The given exponent is

$$\frac{-1}{102}\left[(x+2)^2 - 2.8(x+2)(y-1) + 4(y-1)^2\right],$$

so by matching components, we have

$$\mu_X = -2,\ \mu_Y = 1,\ \frac{1}{2(1-\rho^2)\sigma_Y^2} = \frac{4}{102},\ \frac{1}{2(1-\rho^2)\sigma_X^2} = \frac{1}{102},\ \frac{\rho}{(1-\rho^2)\sigma_X\sigma_Y} = \frac{2.8}{102}.$$

From the third and fourth equations, we have

$$\sigma_Y = \sqrt{\frac{102}{8(1-\rho^2)}},\ \sigma_X = \sqrt{\frac{102}{2(1-\rho^2)}}.$$

Substituting into the last equation gives

$$\frac{\rho}{\frac{102}{4}} = \frac{2.8}{102}, \; 4\rho = 2.8, \quad \rho = 0.70.$$

Hence $\sigma_X = 10$ and $\sigma_Y = 5$.

(b)

$$\mu_{Y|x} = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X) = 1 + 0.70 \times \frac{1}{2}(x+2) = 1.70 + 0.35x;$$
$$\sigma^2_{Y|x} = \sigma^2_Y(1-\rho^2) = 12.75.$$