The University of Sydney
School of Mathematics and Statistics

Tutorial Week 9

STAT3023: Statistical Inference — Semester 2, 2023

1. For the simple prediction problem where Y has a strictly increasing, continuous CDF $F(\cdot)$ and $\mu = E(Y)$ exists and is finite and the decision space is $\mathcal{D} = \mathbb{R}$, determine the decision d that minimises the risk

$$R(d) = E\left[L(d|Y)\right]$$

for the asymmetric piecewise-linear loss function given by

$$L(d|y) = \begin{cases} p(y-d) & \text{for } d < y, \\ (1-p)(d-y) & \text{for } d > y \end{cases}$$

and some $0 < p < 1$ (**hint**: we have already seen the case $p = 0.5$).

2. Determine the optimal decision $d \in \mathcal{D} = \mathbb{R}$ for the simple prediction problem where Y has a continuous distribution on $(0, \infty)$ with density $f(\cdot)$ satisfying

- $f(x) = 0$ for $x \leq 0$;
- $f(x) > 0$ and decreasing in x for $x > 0$

and the loss function $L(d|y)$ is given by

$$L(d|y) = \begin{cases} 0 & \text{if } |d-y| \leq C \\ 1 & \text{if } |d-y| > C, \end{cases}$$

for some known $0 < C < \infty$.

3. Suppose $Z \sim N(0,1)$.

(a) Show that for any constant c,

$$E\left\{|c+Z|\right\} = c\left[1 - 2\Phi(-c)\right] + \frac{2e^{-\frac{1}{2}c^2}}{\sqrt{2\pi}}\,.$$

where $\Phi(\cdot)$ is the cdf of $N(0,1)$.

(b) Suppose $c_n \to 0$ as $n \to \infty$. Determine $\lim_{n\to\infty} E\left\{|c_n + Z|\right\}$.

4. Suppose $\boldsymbol{X} = (X_1, \ldots, X_n)$ consists of iid $N(\theta, 1)$ random variables and that it is desired to determine Bayes procedures using the weight function/prior is given by $w(\theta) \equiv 1$ (the "flat prior"). Show that the resultant posterior density is the normal density with mean $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$ and variance $\frac{1}{n}$.

5. Suppose $\boldsymbol{X} = (X_1, \ldots, X_n)$ consists of iid $N(\theta, 1)$ random variables and that it is desired to determine Bayes procedures using the weight function/prior $w(\cdot)$ given by the $N(\mu_0, \sigma_0^2)$ density, that is

$$w(\theta) = \frac{1}{\sigma_0\sqrt{2\pi}} e^{-\frac{1}{2\sigma_0^2}(\theta - \mu_0)^2}\,.$$

Show that the resultant posterior density is the normal density with mean

$$\left(\frac{1}{1+n\sigma_0^2}\right)\mu_0 + \left(\frac{n\sigma_0^2}{1+n\sigma_0^2}\right)\bar{X}$$

and variance

$$\frac{\sigma_0^2}{1+n\sigma_0^2}\,.$$

6. Suppose $\mathbf{X} = (X_1, \ldots, X_n)$ consists of iid $N(\theta, 1)$ random varibles. We are interested in finding Bayes decisions/procedures for various loss functions using each of the two weight functions/priors used in questions 4 and 5 above: the "flat prior" and the "normal prior" respectively.

(a) When the loss function is $L(d|\theta) = (d-\theta)^2$, the Bayes procedure in each case is the posterior mean. Determine for both decisions $d(\cdot)$,

  (i) the risk $R(\theta|d) = E_\theta\left[L\left(d(\mathbf{X})|\theta\right)\right]$;

  (ii) the *limiting* risk $\lim_{n\to\infty} nE_\theta\left[L\left(d(\mathbf{X})|\theta\right)\right]$.

(b) When the loss function is $L(d|\theta) = |d-\theta|$, the Bayes procedure in each case is the posterior median. Determine for both decisions $d(\cdot)$.

  (i) the risk $R(\theta|d) = E_\theta\left[L\left(d(\mathbf{X})|\theta\right)\right]$;

  (ii) the *limiting* risk $\lim_{n\to\infty} \sqrt{n}E_\theta\left[L\left(d(\mathbf{X})|\theta\right)\right]$.

Hint: in each case write the risk in the form $k_n E_\theta\left\{\left|c_n + \sqrt{n}(\bar{X}-\theta)\right|\right\}$ for sequences $\{k_n\}$ and $\{c_n\}$ and use question 3 above.

(c) When the loss function is $L(d|\theta) = 1\left\{|d-\theta| > C/\sqrt{n}\right\}$ the Bayes procedure in each case is the level set of the posterior density of width $\frac{2C}{\sqrt{n}}$. Because the posterior density is symmetric about the posterior mean/median (and unimodal) in each case, this is simply of the form

$$\text{posterior mean } \pm \frac{C}{\sqrt{n}}.$$

Determine for both decisions $d(\cdot)$

  (i) the risk $R(\theta|d) = E_\theta\left[L\left(d(\mathbf{X})|\theta\right)\right]$;

  (ii) the *limiting* risk $\lim_{n\to\infty} E_\theta\left[L\left(d(\mathbf{X})|\theta\right)\right]$.