

**Solution to Tutorial Week 2**

STAT3023: Statistical Inference

Semester 2, 2022

1. First we have  $|X_i| = |1 - (1/2)^i| < 1$  for all  $i = 1, \dots, n$ , so  $X_i$  is uniformly bounded.

Next, we have

$$E(X_i) = \frac{1}{2} \left(1 - \frac{1}{2^i}\right) + \frac{1}{2} \left(\frac{1}{2^i} - 1\right) = 0,$$

so its variance equals

$$\begin{aligned} \text{Var}(X_i) &= E(X_i^2) - \{E(X_i)\}^2 \\ &= \frac{1}{2} \left(1 - \frac{1}{2^i}\right)^2 + \frac{1}{2} \left(\frac{1}{2^i} - 1\right)^2 - 0 \\ &= 1 - 2 \left(\frac{1}{2}\right)^i + \left(\frac{1}{4}\right)^i \end{aligned}$$

Hence, the sum  $Y_n = \sum_{i=1}^n X_i$  has the variance

$$\begin{aligned} \text{Var}(Y_n) &= \text{Var}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \text{Var}(X_i) = \sum_{i=1}^n \left\{1 - 2 \left(\frac{1}{2}\right)^i + \left(\frac{1}{4}\right)^i\right\} \\ &= n - 2 \sum_{i=1}^n \left(\frac{1}{2}\right)^i + \sum_{i=1}^n \left(\frac{1}{4}\right)^i \end{aligned}$$

As  $n \rightarrow \infty$ , the sum  $\sum_{i=1}^n a^i \rightarrow \sum_{i=1}^{\infty} a^i = \frac{1}{1-a}$  for any  $a < 1$  (geometric series).

Hence the last two terms on the right-hand side converge to constants. However,  $\lim_{n \rightarrow \infty} n = \infty$ , so  $\lim_{n \rightarrow \infty} \text{Var}(Y_n) = \infty$  as required.

2. We have

$$\begin{aligned} P(X = x | S = s) &= \frac{P(X = x, S = s)}{P(S = s)} \\ &= \frac{P(X = x, Y = s - x)}{P(S = s)} \\ &\stackrel{(i)}{=} \frac{P(X = x)P(Y = s - x)}{P(S = s)} \\ &= \frac{\frac{e^{-\lambda_1} \lambda_1^x}{x!} \frac{e^{-\lambda_2} \lambda_2^{s-x}}{(s-x)!}}{\frac{e^{-(\lambda_1 + \lambda_2)} (\lambda_1 + \lambda_2)^s}{s!}} \\ &= \frac{s!}{x!(s-x)!} \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^x \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^{s-x}, \end{aligned}$$

where step (i) follows from the independence of  $X$  and  $Y$ . This is the pmf of a binomial distribution with  $s$  trials and success probability  $\lambda_1/(\lambda_1 + \lambda_2)$ . Hence,

$$X|S \sim \text{Bin}\left(S, \frac{\lambda_1}{\lambda_1 + \lambda_2}\right).$$

By a similar argument, we have

$$Y|S \sim \text{Bin}\left(S, \frac{\lambda_2}{\lambda_1 + \lambda_2}\right).$$

3. Since  $X|Y \sim N(0, 2Y)$ , then we have  $E(e^{tX}|Y) = e^{(1/2)t^2(2Y)} = e^{t^2Y}$ . Hence, by the property of the conditional expectation, we can compute the moment generating function of  $X$  to be

$$\begin{aligned} E(e^{tX}) &= E_Y[E(e^{tX}|Y)] \\ &= E_Y[e^{t^2Y}] \\ &= \int_0^\infty e^{-y} e^{t^2y} dy \\ &= \int_0^\infty e^{(t^2-1)y} dy \\ &= \frac{1}{t^2-1} e^{(t^2-1)y} \Big|_{y=0}^{y=\infty} \\ &= \frac{1}{1-t^2}, \text{ for } |t| < 1, \end{aligned}$$

which is the MGF of the standard Laplace distribution. Hence the marginal distribution of  $X$  is standard Laplace.

Comment: Due to this hierarchical structure, then the Laplace distribution belongs to the class of *scale mixture of normal distributions*. This class of distributions includes many other distributions (such as the  $t$  distribution) and can be used to model different behaviours of variances.

4. (a) Note that  $Y$  is discrete, so we can find the probability mass function of  $Y$  to be

$$\begin{aligned} P(Y = y) &= P(Y = y, 0 \leq \Lambda < \infty) \\ &= \int_0^\infty P(Y = y|\Lambda = \lambda) f_\Lambda(\lambda) d\lambda \\ &= \int_0^\infty e^{-\lambda} \frac{\lambda^y}{y!} \frac{1}{\beta} e^{-\lambda/\beta} d\lambda \\ &= \frac{1}{\beta y!} \int_0^\infty e^{-\lambda(1+1/\beta)} \lambda^y d\lambda \end{aligned} \tag{1}$$

The integral above is the unnormalized density of a Gamma $\left(y+1, \frac{1}{1+1/\beta}\right)$  (see tutorial week 1 for this density). In other words, we have

$$\frac{1}{\Gamma(y+1) \left(\frac{1}{1+1/\beta}\right)^{y+1}} \int_0^\infty e^{-\lambda(1+1/\beta)} \lambda^y d\lambda = 1,$$

so

$$\int_0^\infty e^{-\lambda(1+1/\beta)} \lambda^y d\lambda \frac{\Gamma(y+1)}{(1+1/\beta)^{y+1}} \stackrel{(i)}{=} \frac{y!}{(1+1/\beta)^{y+1}}$$

where step (i) follows from  $\Gamma(y+1) = y!$  for  $y \in \mathbb{N}$ . Substituting it into (1), we have

$$P(Y = y) = \frac{1}{\beta} \frac{1}{(1+1/\beta)^{y+1}} = \frac{1}{\beta} \left( \frac{\beta}{\beta+1} \right)^{y+1}.$$

Comparing it with the general form of a negative binomial distribution, then we can see  $Y \sim \text{NB} \left( r = 1, p = \frac{\beta}{1+\beta} \right)$ .

(b) From the marginal distribution of  $Y$ , we can get

$$E(Y) = \frac{pr}{1-p} = \frac{\beta/(1+\beta)}{1/(1+\beta)} = \beta.$$

$$\text{Var}(Y) = \frac{pr}{(1-p)^2} = \beta(1+\beta).$$

We can verify it by using the properties of conditional expectations and variances as

$$\begin{aligned} E(Y) &= E[E(Y|\Lambda)] = E(\Lambda) = \beta, \\ \text{Var}(Y) &= E[\text{Var}(Y|\Lambda)] + \text{Var}[E(Y|\Lambda)] \\ &= E(\Lambda) + \text{Var}(\Lambda) \\ &= \beta + \beta^2 = \beta(1+\beta). \end{aligned}$$

5. (a) The exponent in the bivariate normal density has the form

$$\begin{aligned} & -\frac{1}{2(1-\rho^2)} \left[ \left( \frac{x-\mu_X}{\sigma_X} \right)^2 + \left( \frac{y-\mu_Y}{\sigma_Y} \right)^2 - 2\rho \frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} \right] \\ &= -\frac{1}{2(1-\rho^2)\sigma_X^2} (x-\mu_X)^2 - \frac{1}{2(1-\rho^2)\sigma_Y^2} (y-\mu_Y)^2 + \frac{2\rho}{(1-\rho^2)\sigma_X\sigma_Y} (x-\mu_X)(y-\mu_Y) \end{aligned}$$

The given exponent is

$$\frac{-1}{102} [(x+2)^2 - 2.8(x+2)(y-1) + 4(y-1)^2],$$

so by matching components, we have

$$\mu_X = -2, \mu_Y = 1, \frac{1}{2(1-\rho^2)\sigma_Y^2} = \frac{4}{102}, \frac{1}{2(1-\rho^2)\sigma_X^2} = \frac{1}{102}, \frac{2\rho}{2(1-\rho^2)\sigma_X\sigma_Y} = \frac{2.8}{102}.$$

From the third and fourth equations, we have

$$\sigma_Y = \sqrt{\frac{102}{8(1-\rho^2)}}, \quad \sigma_X = \sqrt{\frac{102}{2(1-\rho^2)}}.$$

Substituting into the last equation gives

$$\frac{2\rho}{\frac{102}{4}} = \frac{2.8}{102}, \quad 8\rho = 2.8, \quad \rho = 0.35.$$

Hence  $\sigma_X \approx 7.62$  and  $\sigma_Y \approx 3.82$ .

(b)

$$\mu_{Y|x} = \mu_Y + \rho \frac{\sigma_Y}{\sigma_X} (x - \mu_X) = 1 + (0.35) \times \frac{1}{2} (x + 2) = 1.35 + \frac{1}{2} x$$

$$\sigma_{Y|x}^2 = \sigma_Y^2 (1 - \rho^2) = 12.75.$$