# Greedy Algorithms

Meng-Tsung Tsai

02/05/2018

# A common technique

Here is a common technique to design greedy algorithms.

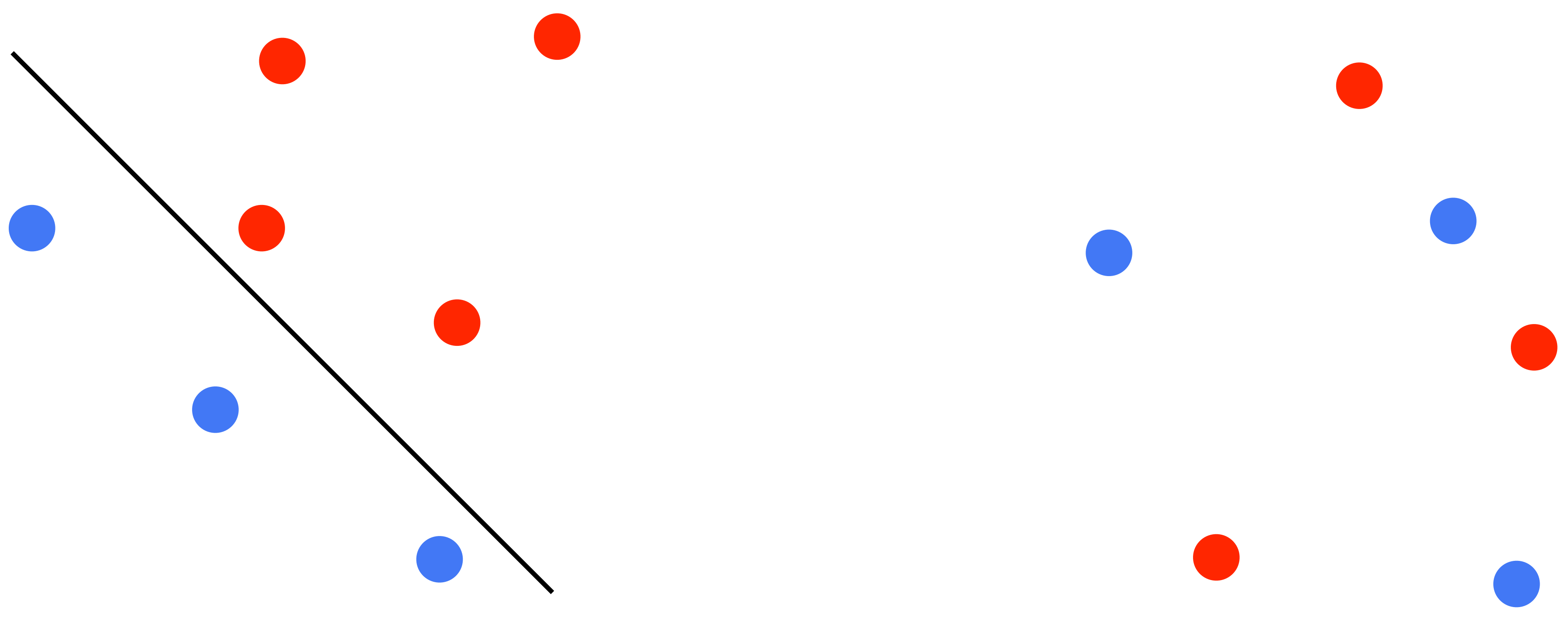
(1) Imagine what the optimum solution is.
(2) Give an initial guess.
(3) If the guess $\neq$ the optimum solution, find a way to **morph** the guess to the solution.

**G. W. Busch**

**A. Schwarzenegger**

# Seperate colored points by a line

Given n points on a 2D plane. The color of each point is blue or red. Decide whether there exists a line that seperates the points into two halves so that each half contains points of same color.



Yes

No

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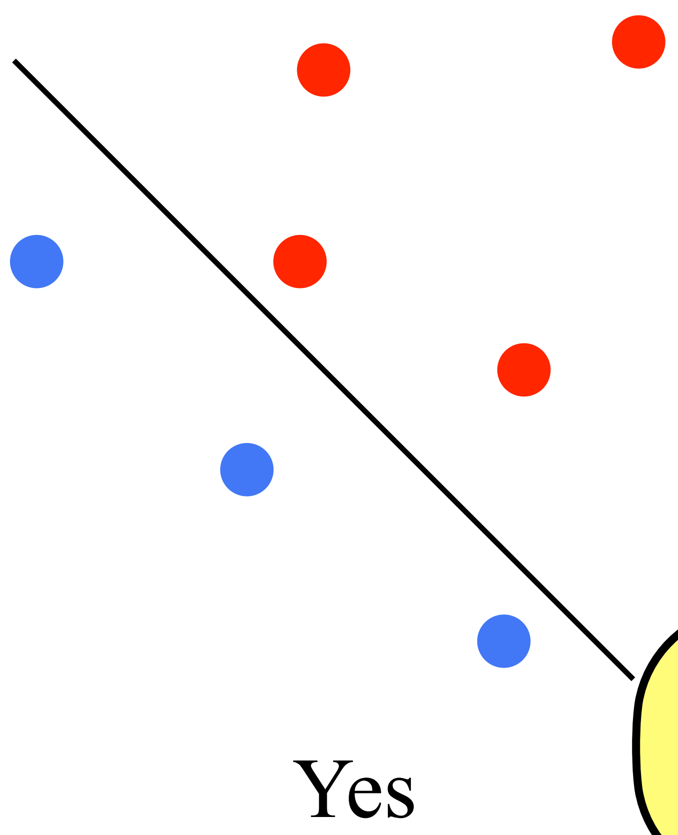
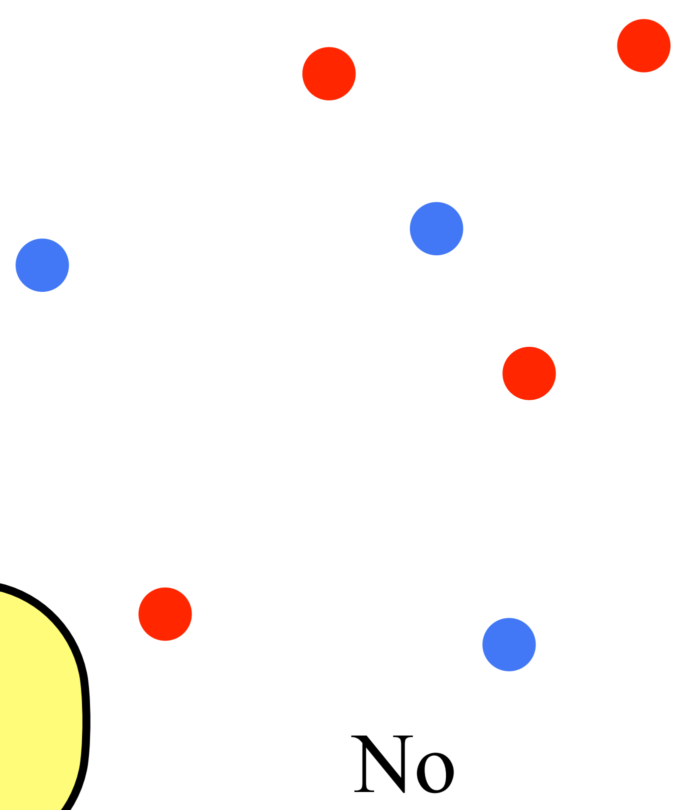
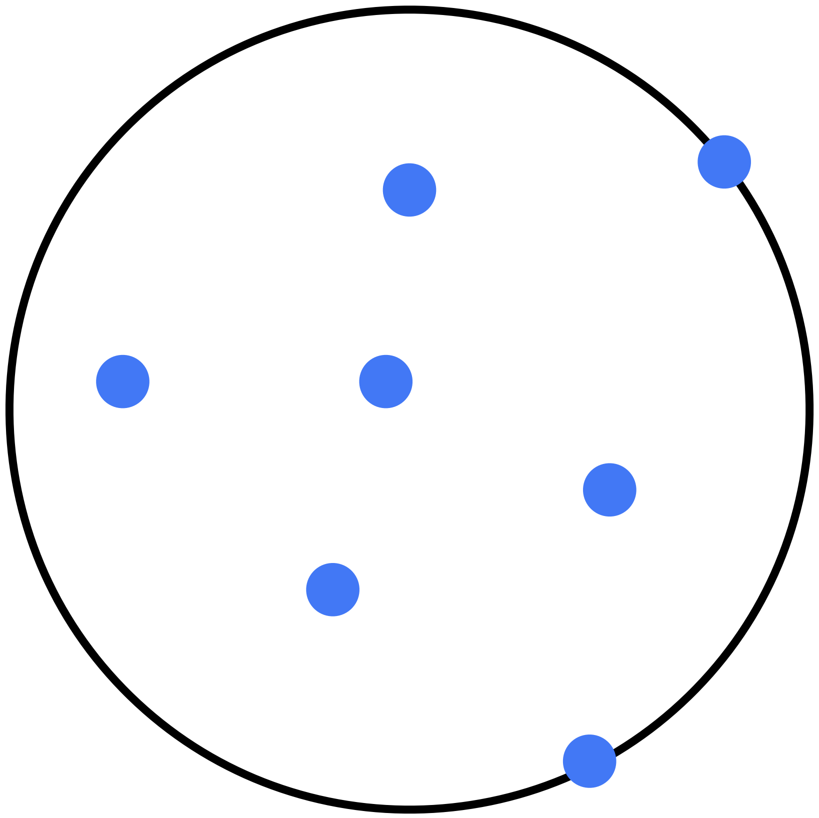



O($n^3$) time.
Optimal?



# The minimum enclosing ball problem

Given n points on a 2D plane, find the smallest circle so that each of the n point is either on the boundary of the circle or in its interior.



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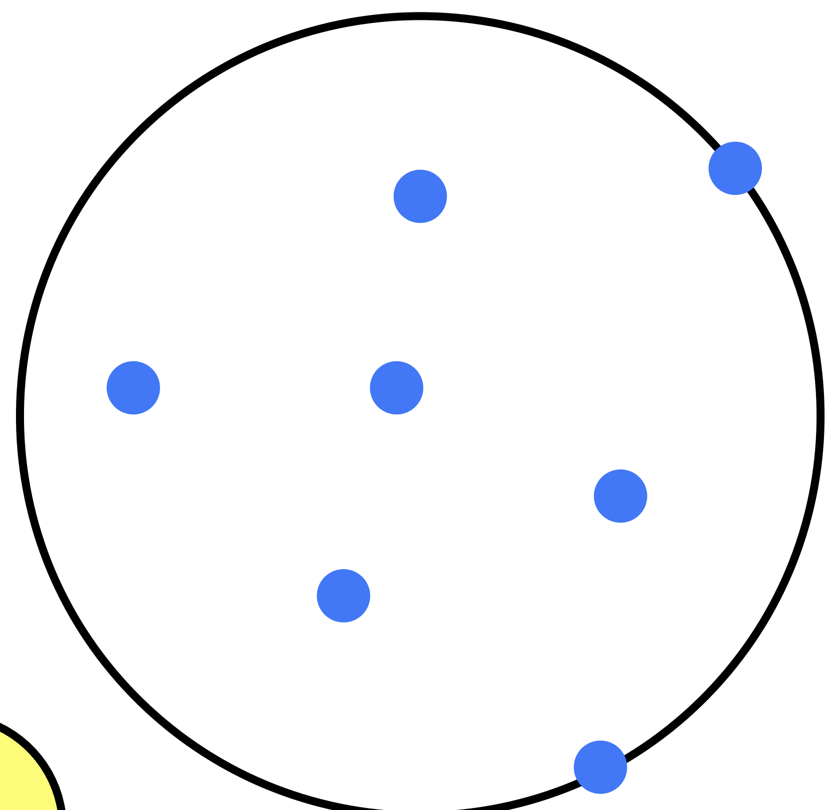
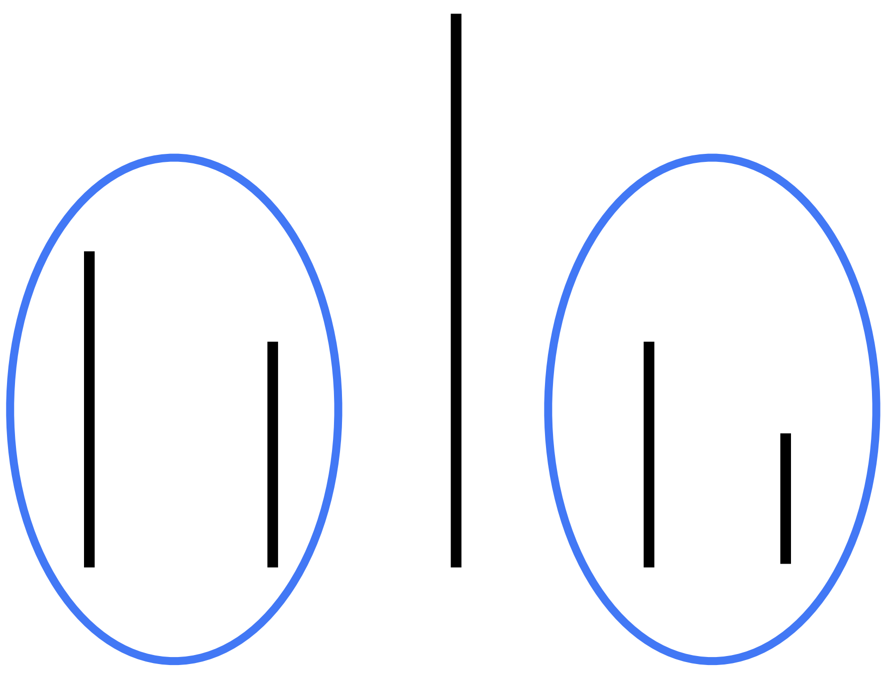


An enclosing ball

$O(n^4)$ time.
Optimal?

# Pair Chopsticks

Given n chopsticks that have length $\ell_1, \ell_2, \ldots, \ell_n \geq 0$. If two chopsticks have length difference smaller than d, then we can pair the two chopsticks. Maximize the number of paired chopsticks, noting that each chopstick can join at most 1 pair.



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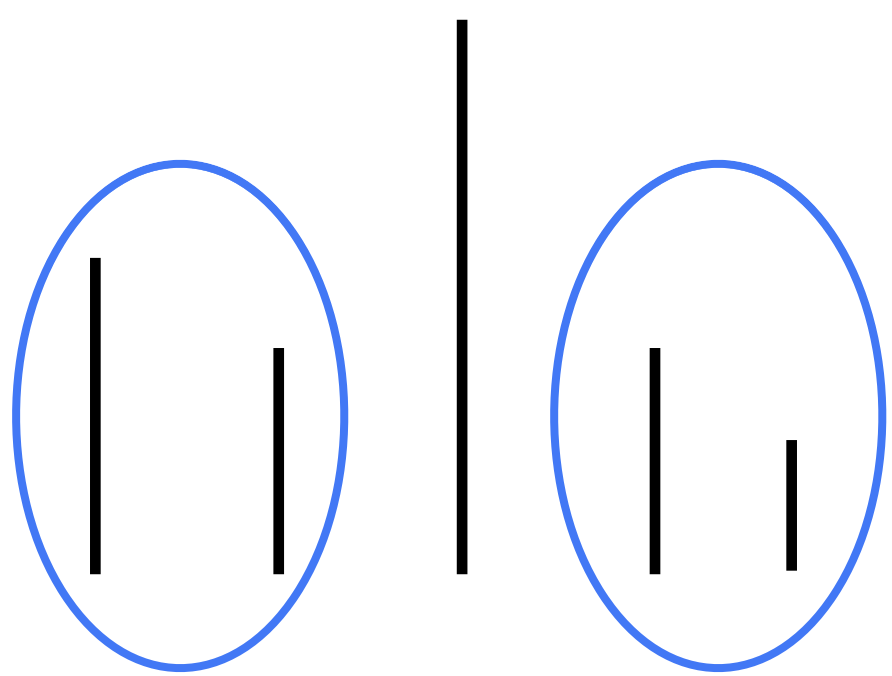


O(n log n) time.

# Another common technique

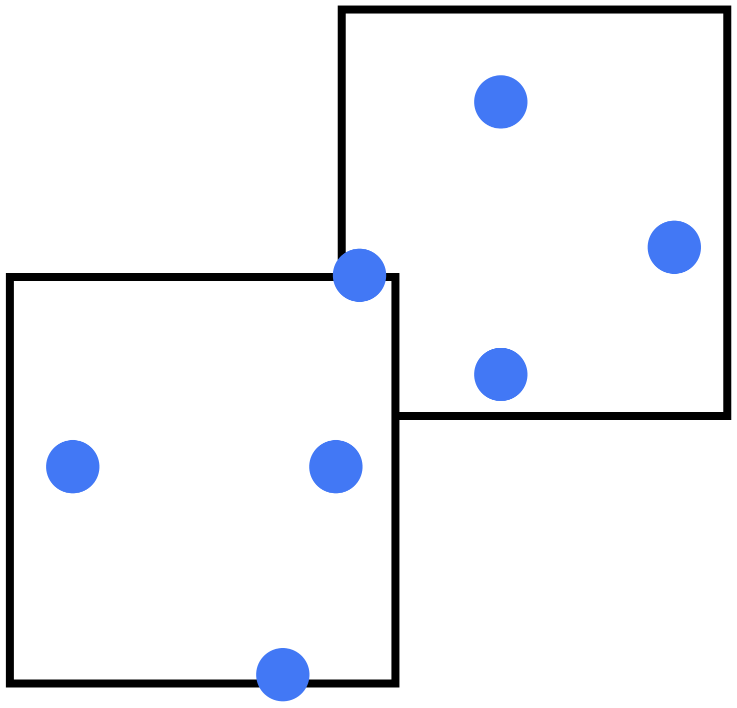
Here is another common technique to design greedy algorithms.

(1) Let some parameter be a fixed value and see whether you can solve the problem.

(2) Observe the relationship (e.g. monotonicity) between the solutions for two different fixed values.

# Cover points by two sqaures of length L

Given n points on a 2D plane, cover all the points by two squares of length L so that L is minimized. These two squares may overlap.



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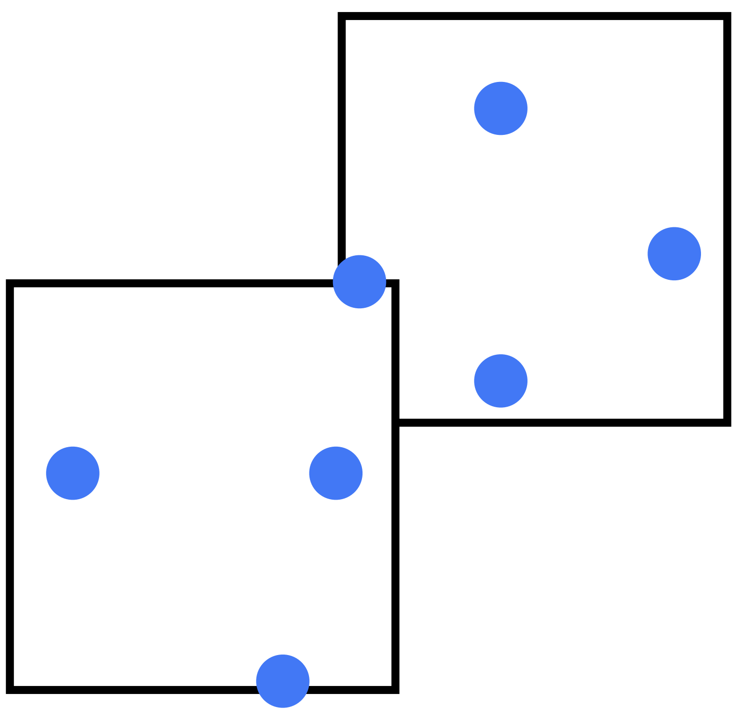
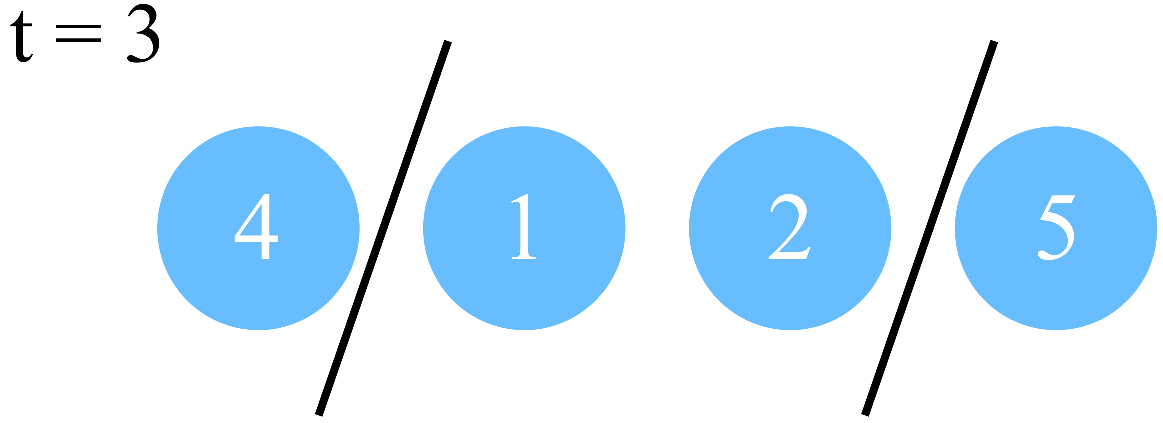


$O(n^2 \log n)$ time.
What if L must be an integer?

# Partition a sequence

Given a sequence of n positive integers $a_1, a_2, \ldots, a_n$. Define weight of a subsequence to be the sum of elements in it. Partition these n integers into t consecutive subsequences so that the maximum weight of the t subsequences is minimized.



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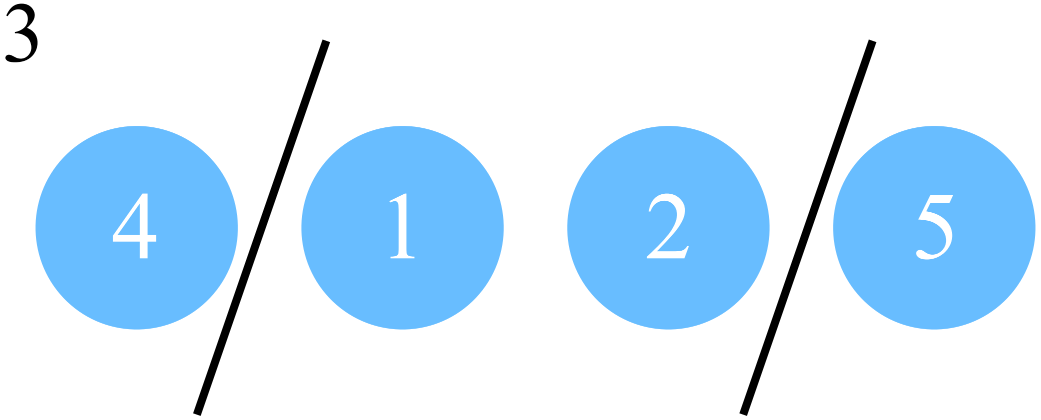
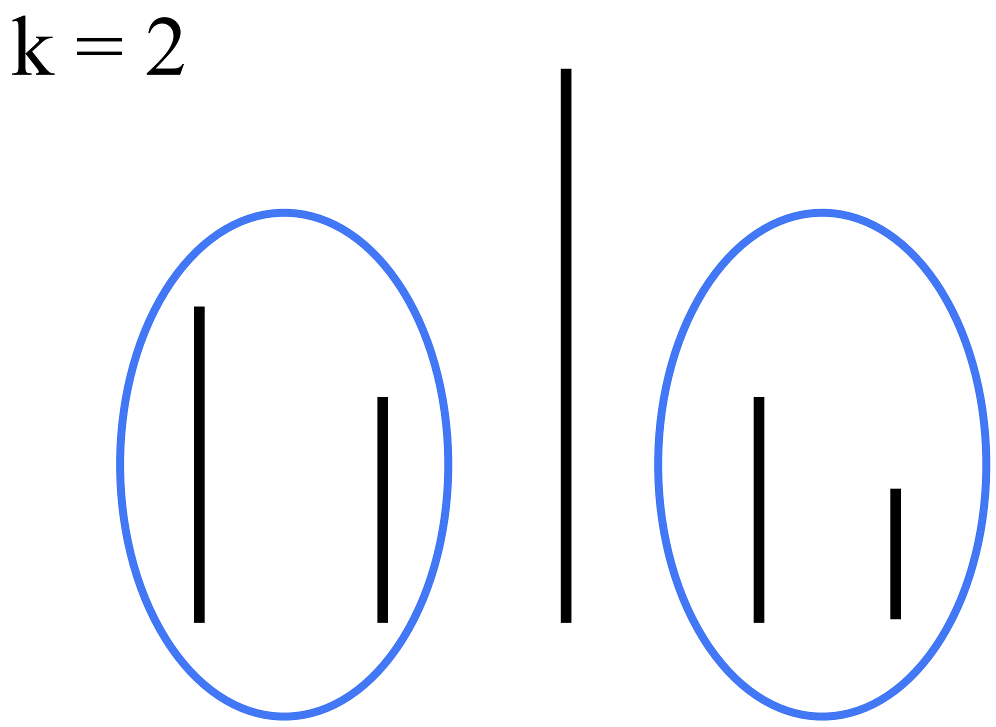
$t = 3$




$O(n \log(\max_i a_i))$ time.

# Pair Chopsticks (A variation)

Given n chopsticks that have **integral** length $\ell_1, \ell_2, \ldots, \ell_n \geq 0$. Find k pairs of chopsticks from the n given ones so that the maximum length difference in a pair is minimized.



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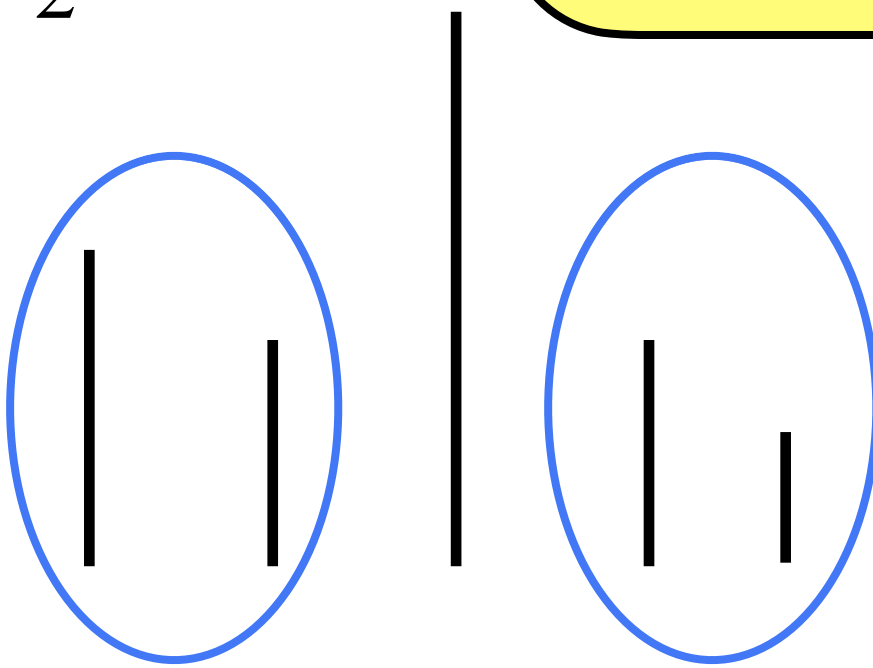
$O(n\,(\log n + \log(\max_i \ell_i)))$ time.

k = 2



# Bridge

n people would like to cross a bridge. A group of at most two people may cross the bridge at any time with a flashlight. There is only one flashlight. People $p_i$ needs $t_i$ seconds to cross the bridge. If $p_i$ and $p_j$ cross the bridge together, then they needs $\max(p_i, p_j)$ seconds. Find an arrangment so that it takes the least time for all to reach the other side of the bridge.

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O(n log n) time.

# Matroid

# What is a matriod?

It is a structure $M = (S, F)$, where:

(1) $S$ is a set of $n$ elements
(2) $F$ is a non-empty collection of subsets of $S$

<u>Example</u>.

Given $S = \{1, 2, .., n\}$

F could be {
$S_1 = \{2\}$,
$S_2 = \{1, n\}$,
$S_3 = \varnothing$,
…}

# What is a matriod?

It is a structure $M = (S, F)$, where:

(1) S is a set of n elements
(2) F is a non-empty collection of subsets of S

(3) F is **hereditary**. That is, if $A \subset B$ and $B \in F$, then $A \in F$.

If $F = \{$
$S_1 = \{1, n\}, \ldots\}$

Then, F contains also
$\varnothing$, $\{1\}$ **and** $\{n\}$.

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(4) $M$ satisfies the **exchange property**. That is, for any $A, B \in F$, if $|A| < |B|$, then exists $x \in B \setminus A$ so that $A \cup \{x\} \in F$.

If $F = \{$
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Then, $F$ contains also $\{1, 2\}$ **or** $\{2, n\}$.

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Such a structure $M$ is called a **matroid**.
We say any subset in $F$ is an **independent** subset.

# The Weighted Matriod Problem

# A non-negative weight function

A function w that assigns a **non-negative** weight w(x) to each element x in S.

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We say M = (S, F) is a **weighted matroid** if its S is associated with a weight function $w: S \rightarrow R_+ \cup \{0\}$.

# The weighted matroid problem

Input: a matroid $M = (S, F)$ and a weight function $w: S \rightarrow R_+ \cup \{0\}$.

Output: an independent subset $R$ in $F$ so that $w(R)$ is maximized.

# The weighted matroid problem

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Many problems can be encoded as a weighted matroid problem. We can solve all such problems in a unified way.

# The greedy algorithm that solves the weighted matroid problem optimally

```
Greedy(M = (S, F), w){
    Sort the elements in S into the order of non-increasing
    weight w. Let the sorted order be s1, s2, …, sn.

    A = ∅; // let the answer be an empty set initially
    for(i=1; i≤n; ++i){
        if(A ∪ {si} in F){
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That is it. Now we have a unified solution for any weighted matroid problem.

# Applications

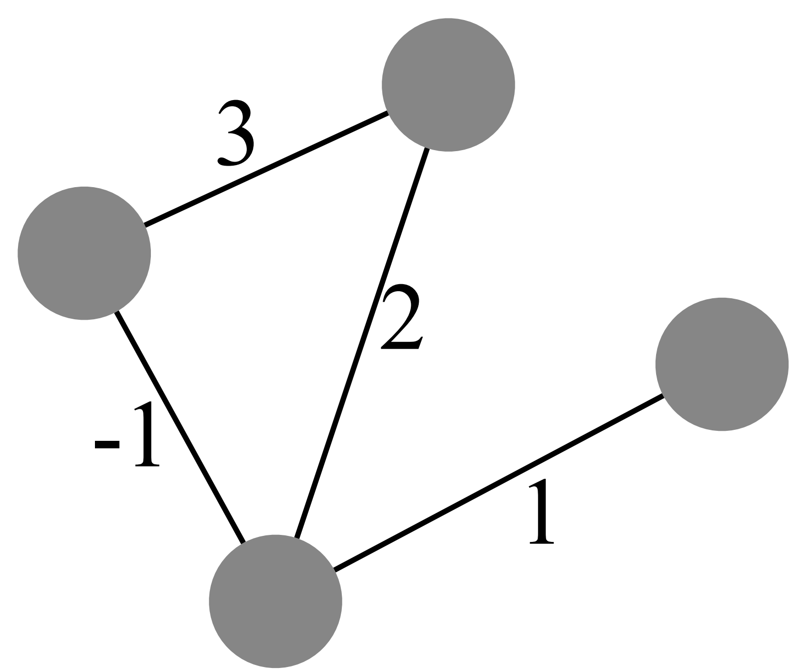
# The Minimum Spanning Tree Problem

Input: a connected undirected graph $G = (V, E)$ and a weight function $w: E \rightarrow R$.

Ouput: a spanning tree $T$ of $G$, i.e. a tree containing all nodes, so that $w(T) = \sum_{e \in T} w(e)$ is minimized.

<u>Example</u>.

graph G



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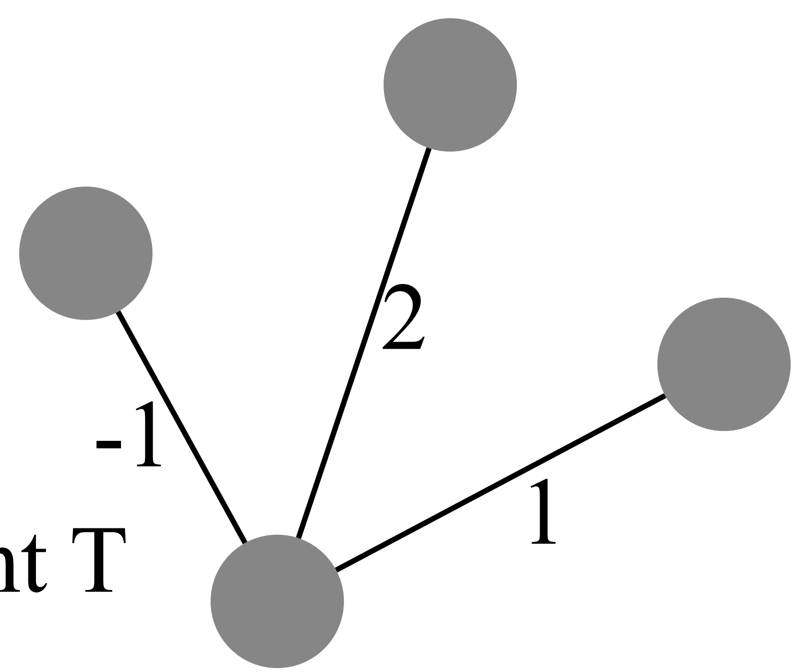


an optimum-weight T

# MST is a weighted matroid problem (1/3)

Plan: encoding the MST as a weighted matriod problem.
Thus, we can solve it by the unified solution of WMP.

We define a matriod as follows.

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Given the edge set B of a spanning forest of G (i.e. a subgraph containing no cycles), then of course any subset A of B is a spanning forest.

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Proof. Let A, B be the edge set of two spanning forests and $|A| < |B|$. Thus, # connected components in A is more than # connected components in B. Then, B must contain a tree T whose nodes are in two different trees in A. Since T is connected, then it must contain an edge $\{u, v\}$ that connects two different components in A. In other words $A \cup \{u, v\}$ is a spanning forest, i.e. in F.

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In MST, we want to find a spanning tree of the **minimum weight**. In a weighted matroid problem, it returns an independent subset of the **maximum weight**. Since they are mismatched, we need a fix here.

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Solution. We can let $w_0$ be the $\max_{e \in E} w(e)$, and define a new weight function $z(e) = w_0 - w(e)$. In this way, the weighted matriod problem with input $M_G = (E, F)$ and $z$ is equivalent to the MST problem.

# A task scheduling problem

Input: a set S of unit-time tasks $\{s_1, s_2, \ldots, s_n\}$. Each task $s_i$ has a deadline $d_i$ and a non-negative penalty $w_i$.

Output: a permutation of S so that the first task begins at time 0 and finishes at time 1, the second task begins at time 1 and finishes at time 2, etc. and the total **penalty** incurred by the tasks not finished by their dealines is **minimized**.

Example.

| $s_i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $d_i$ | 4 | 2 | 4 | 3 | 1 | 4 | 6 |
| $w_i$ | 70 | 60 | 50 | 40 | 30 | 20 | 10 |

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penalty = 50

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# TSP is a weighted matroid problem (1/3)

Plan: encoding the TSP as a weight matriod problem. Thus, we can solve it by the unified solution of WMP.

We define a matriod as follows.

Let $M_S = (S, F)$.

(1) S is the set of all tasks.
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If you can solve a set of tasks all by their deadlines, then of course you can solve any subset of the tasks all by their deadlines.

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Proof. Yes, see the proof on Page 445 in I2A.

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Since $M_S$ is a matriod, solving the corresponding weighted matriod problem $M_S = (S, F)$ and $w(s_i) = d_i$ gives a subset of the n tasks, all of which can be solved by their deadlines and the total penalty is maximized.

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In other words, the penalty incurred by those tasks not finished by their deadlines is minimized, as required.

# The Correctness of the Greedy Algorithm for the Weighted Matroid Problem

# Recall the greedy algorithm

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Greedy(M = (S, F), w){
    Sort the elements in S into the order of non-increasing
    weight w. Let the sorted order be s1, s2, …, sn.

    A = ∅; // let the answer be an empty set initially
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The returned subset $A = \{a_1, a_2, \ldots, a_x\}$ must be independent. Let $r(a_1) < r(a_2) < \ldots < r(a_x)$, where $r(e)$ denotes the rank of $e$ in the sorted $S$.

# Proof (1/3)

Claim 1. A is maximal. In other words, A is not a subset of any other independent subset in F.

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Proof. Suppose that $A \subset B$ for some $B \in F$ and $|B| > |A|$. Let z be some element in $B \setminus A$ so that $A \cup \{z\}$ in F. Such z exists due to the exchange property. By the hereditary property, $\{a_1, a_2, \ldots, a_t\} \cup \{z\}$ in F for any $t \in [1, x]$.

If $r(a_x) < r(z)$, then Greedy should have added some e into A so that $r(a_x) < r(e) \le r(z)$. $\rightarrow\leftarrow$

Otherwise $r(a_{t-1}) < r(z) < r(a_t)$ for some $t \in [1, x]$, then Greedy should have added some e into A so that $r(a_{t-1}) < r(e) \le r(z)$.

$\rightarrow\leftarrow$

# Proof (2/3)

Claim 2. Every maximal subset in F has the same size.

Proof. Let A and B be two maximal subsets in F and $|A| < |B|$. By the exchange property, there exists some z in $B \setminus A$ so that $A \cup \{z\}$ in F, contradicting the maximality of A.

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Claim 3. Some maximal subset in F is a max-weight subset in F.

Proof. Let A be a max-weight subset but A is a proper subset of some B in F. Then, B also has the max-weight because the weight function w is non-negative. Such B is a witness.

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Proof. Let A be a max-weight subset but A is a proper subset of some B in F. Then, B also has the max-weight because the weight function w is non-negative. Such B is a witness.

Both A and a max-weight subset are maximal.

# Proof (3/3)

Let the max-weight subset be $B = \{b_1, b_2, \ldots, b_x\}$ where $r(b_1) < r(b_2) < \ldots < r(b_x)$. Since $A \neq B$, there exists some $t \in [1, x]$ so that $r(a_1) = r(b_1)$, $r(a_2) = r(b_2)$, …, $r(a_{t-1}) = r(b_{t-1})$, $r(a_t) < r(b_t)$. If there are multiple max-weight susbets, we pick B as the one whose t is the largest.

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Theorem. A = B, and therefore A is a max-weight subset.

# Proof (3/3)

Let the max-weight subset be $B = \{b_1, b_2, \ldots, b_x\}$ where $r(b_1) < r(b_2) < \ldots < r(b_x)$. Since $A \neq B$, there exists some $t \in [1, x]$ so that $r(a_1) = r(b_1)$, $r(a_2) = r(b_2)$, …, $r(a_{t-1}) = r(b_{t-1})$, $r(a_t) < r(b_t)$. If there are multiple max-weight susbets, we pick B as the one whose t is the largest.

Theorem. A = B, and therefore A is a max-weight subset.

Proof. Let $A' = \{a_1, a_2, \ldots, a_t\}$. By exchange property, we could iteratively add some element from $B \setminus A'$ to $A'$ until $A'$ has the same size as B. Finally, $A'$ becomes $B \cup \{a_t\} \setminus \{z\}$ for some z in $\{b_t, b_{t+1}, \ldots, b_x\}$. Since $r(a_t) < r(b_t) \leq r(z)$, $w(a_t) \geq w(z)$ and A' is another max-weight subset, violating the condition that B is the one whose t is the largest.

# Submodularity

# Submodular functions

Let S be a set of n elements. Let f be a function that maps any subset of S to a real number.

We say such f is **submodular** if

$$f(X \cup \{e\}) - f(X) \geq f(Y \cup \{e\}) - f(Y)$$

for any $X \subseteq Y \subseteq S$.

# Monotone functions

Let S be a set of n elements. Let f be a function that maps any subset of S to a real number.

We say such f is **monotone** if

$$f(X) \leq f(Y)$$

for any $X \subseteq Y \subseteq S$.

# Submodular Maximization

Given a set S, a submodular function f: $2^S \rightarrow R$, and an integer k, find a subset X of S so that:

(1) $|X| \le k$
(2) f(X) is maximum among all subsets of S that has cardinality $\le$ k.

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Thm. If S is submodular, monotone, and non-negative, then there exists a greedy algorithm that can find a (1-1/e)-approximation.

# The greedy algorithm

```
Greedy(S, f, k){

    A = ∅; // let the answer be an empty set initially
    for(i=1; i≤k; ++i){
        for(j=1; j≤|S|; ++j){
            if(f(A ∪ {s_j}) ≥ f(A ∪ {s_k}) for all k's){
                A = A ∪ {s_j};
            }
        }
    }
    return A;
}
```

# The maximum coverage problem

Given n points. Place k circles, each of which has radius d and is centered at some given point, so that the number of covered points is maximized.