BLM2502 Theory of Computation

Spring 2016

BLM2502 Theory of Computation

- **Course Outline**
- Week — Content
- 1 — Introduction to Course
- 2 — Computability Theory, Complexity Theory, Automata Theory, Set Theory, Relations, Proofs, Pigeonhole Principle
- 3 — Regular Expressions
- 4 — Finite Automata
- 5 — Deterministic and Nondeterministic Finite Automata
- 6 — Epsilon Transition, Equivalence of Automata
- 7 — Pumping Theorem
- 8 — April 10 - 14 week is the first midterm week
- 9 — Context Free Grammars
- 10 — Parse Tree, Ambiguity,
- **11 — Pumping Theorem**
- 12 — Turing Machines, Recognition and Computation, Church-Turing Hypothesis
- 13 — Turing Machines, Recognition and Computation, Church-Turing Hypothesis
- 14 — May 22 – 27 week is the second midterm week
- 15 — Review
- 16 — Final Exam date will be announced

The Pumping Lemma for CFL's

Pumping Lemma

- Recall the pumping lemma for regular languages.
- It told us that if there was a string long enough to cause a cycle in the DFA for the language, then we could "pump" the cycle and discover an infinite sequence of strings that had to be in the language.
- For CFL's the situation is a little more complicated.
- We can always find two pieces of any sufficiently long string to "pump" in tandem.
  - That is: if we repeat each of the two pieces the same number of times, we get another string in the language.

Statement of the CFL Pumping Lemma

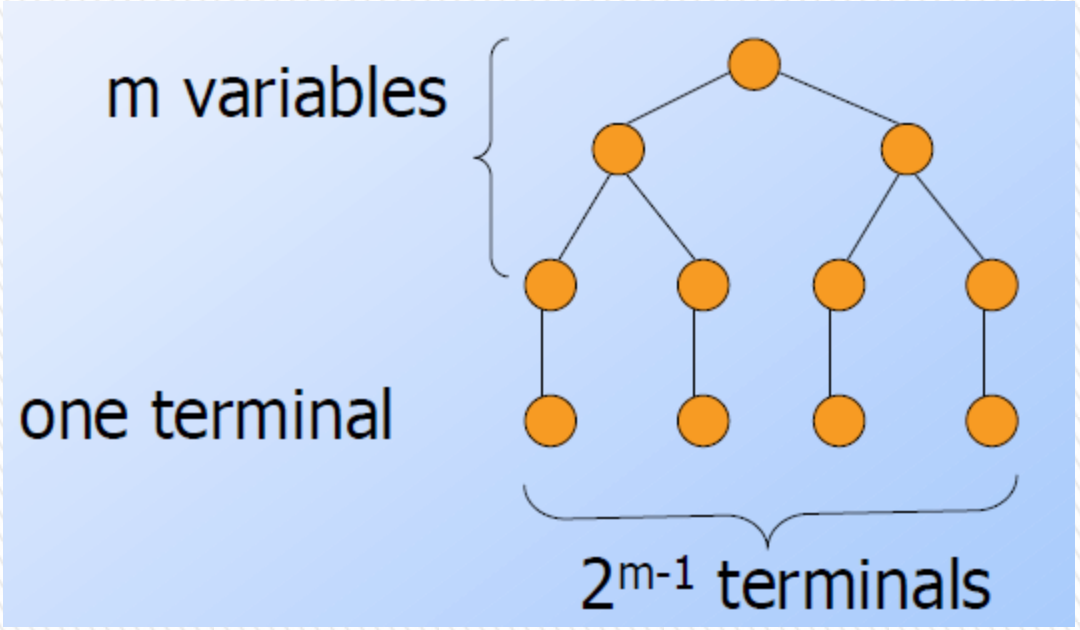
- For every context-free language L There is an integer n, such that For every string z in L of length > n There exists z = uvwxy such that:
  1. $|vwx| < n$.
  2. $|vx| > 0$.
  3. For all $i > 0$, uv^iwx^iy is in L.

Proof of the Pumping Lemma

- Start with a CNF grammar for L - {ε}.
- Let the grammar have m variables.
  - Pick $n = 2^m$.
  - Let $|z| > n$.
- We claim ("Lemma 1") that a parse tree with yield z must have a path of length m+2 or more.

Proof of Lemma 1

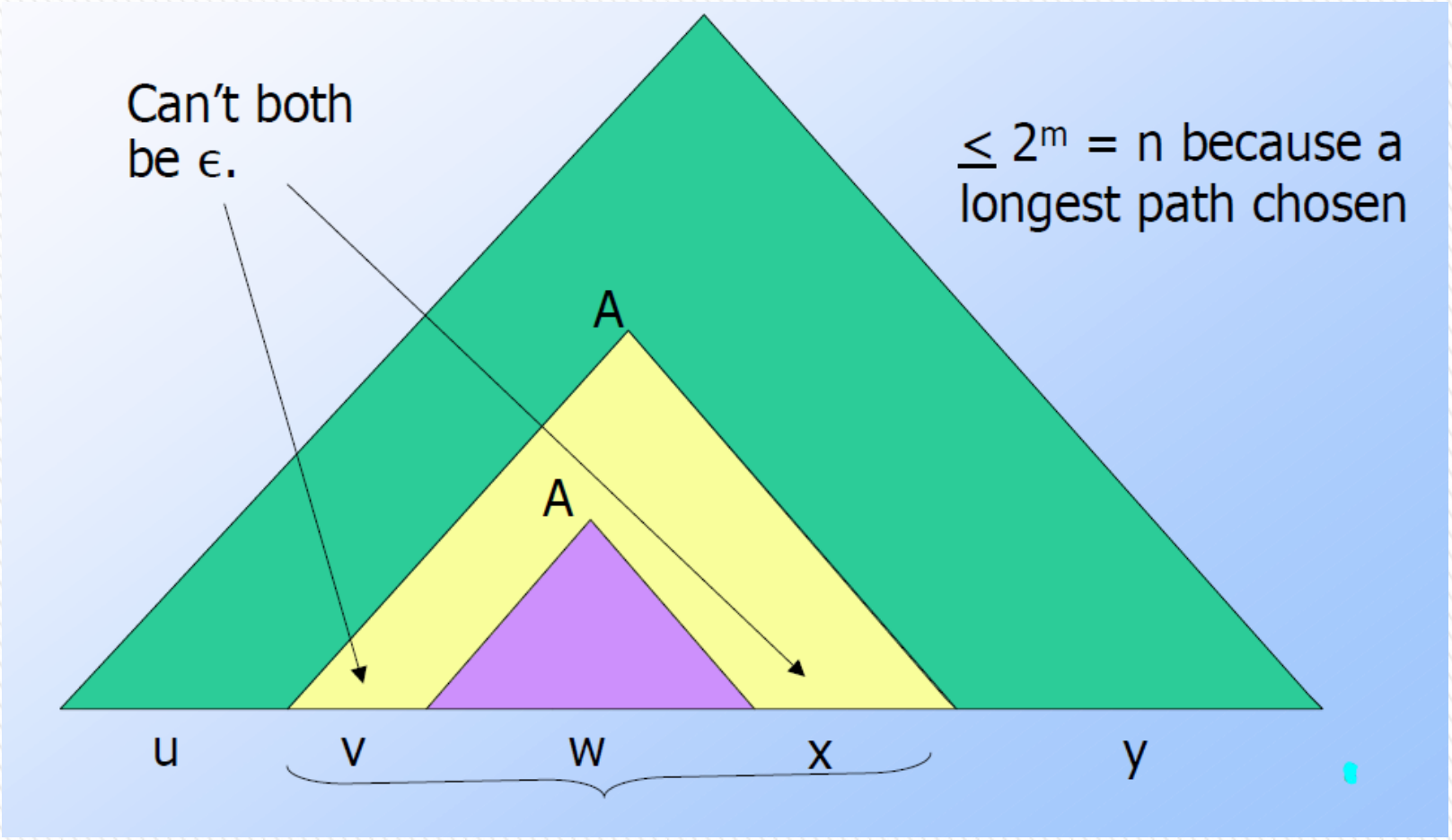
» If all paths in the parse tree of a CNF grammar are of length < m+1, then the longest yield has length 2^{m-1}, as in figure:



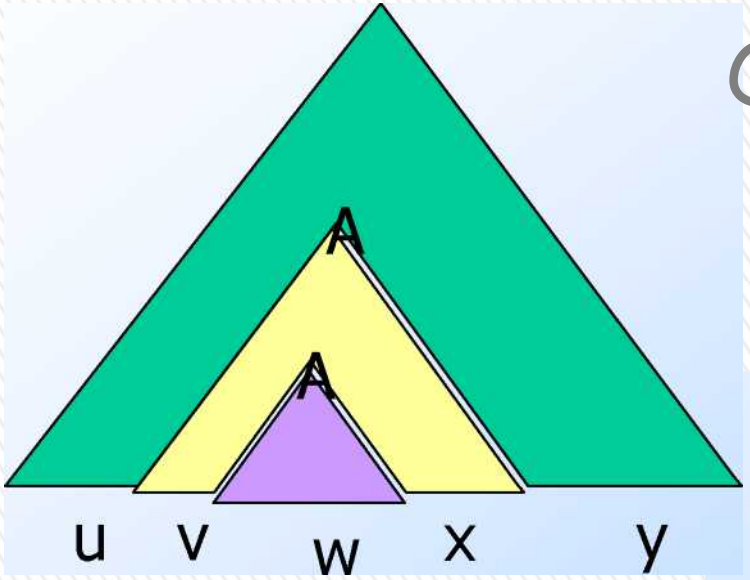
Proof of the Pumping Lemma

- Now we know that the parse tree for z has a path with at least m+1 variables.
- Consider some longest path.
- There are only m different variables, so among the lowest m+1 we can find two nodes with the same label, say A.
- The parse tree thus looks like:

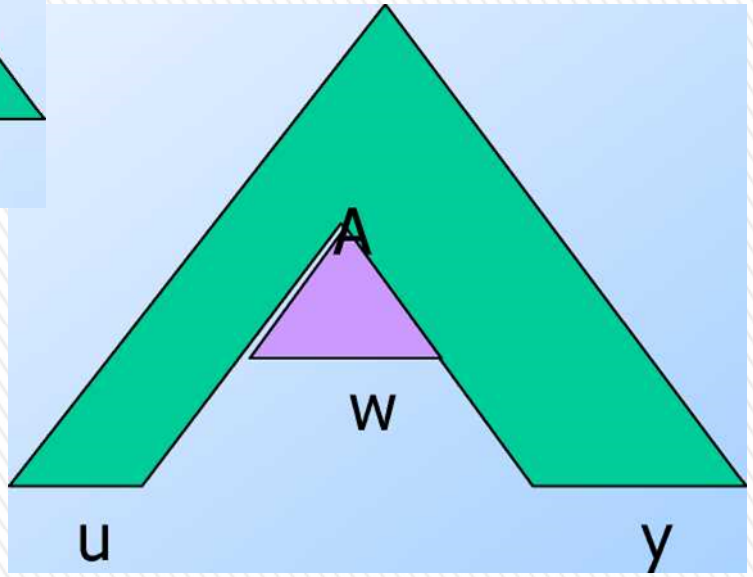
Parse Tree in the Pumping-Lemma



Pumping



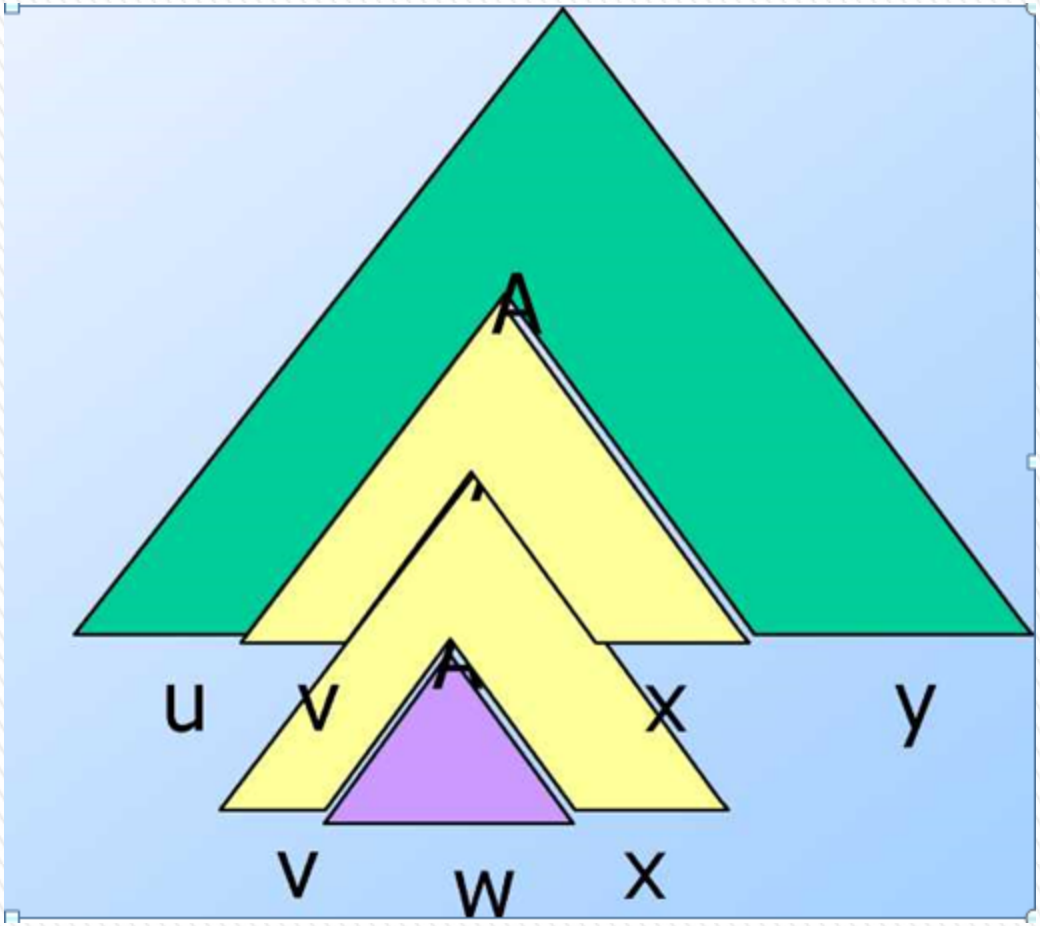
Once



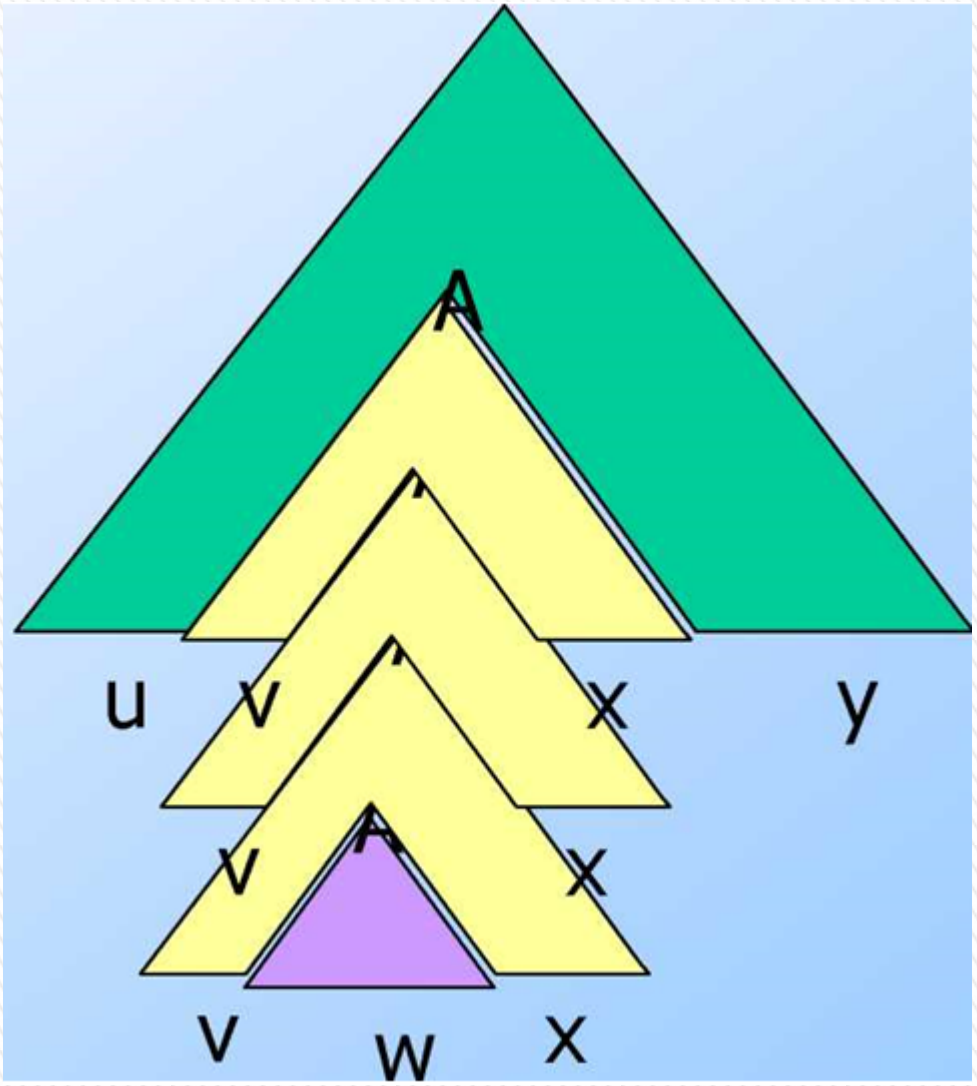
Zero times

Pumping

Twice



Pumping



Thrice, …

Using the Pumping Lemma

- Non-CFL's typically involve trying to match two pairs of counts or match two strings.
- Example: pumping lemma can be used to show that L = {ww | w in (0+1)*} is not a CFL.
- {0'10' | i >1} is a CFL.
  - We can match one pair of counts.

Using the Pumping Lemma

- L = $\{0^i10^i10^i \mid i > 1\}$ is not a CFL
  - We can't match two pairs, or three counts as a group.
  - Proof using the pumping lemma.
  - Suppose L were a CFL.
  - Let n be L's pumping-lemma constant.
  - Consider z = $0^n10^n10^n$.
  - We can write z = uvwxy, where $|vwx| < n$, and $|vx| > 1$.
  - Case 1: vx has no 0's.
  - Then at least one of them is a 1, and uwy has at most one 1, which no string in L does.

Using the Pumping Lemma

- Still considering $z = 0^n10^n10^n$.
- Case 2: vx has at least one 0.
- vwx is too short (length < n) to extend to all three blocks of 0's in $0^n10^n10^n$.
- Thus, uwy has at least one block of n 0's, and at least one block with fewer than n 0's.
- Thus, uwy is not in L.

Simplifications of Context-Free Grammars

A Substitution Rule

$S \to aB$

$A \to aaA$

$A \to abBc$

$B \to aA$

$B \to b$

Substitute

$B \to b$

Equivalent grammar

$S \to aB \mid ab$

$A \to aaA$

$A \to abBc \mid abbc$

$B \to aA$

$$S \rightarrow aB \mid ab$$

$$A \rightarrow aaA$$

$$A \rightarrow abBc \mid abbc$$

$$B \rightarrow aA$$

Substitute

$$B \rightarrow aA$$

$$S \rightarrow \cancel{aB} \mid ab \mid aaA$$

$$A \rightarrow aaA$$

$$A \rightarrow \cancel{abBc} \mid abbc \mid abaAc$$

Equivalent grammar

In general: $A \to xBz$

$B \to y_1$

Substitute

$B \to y_1$

$A \to xBz \mid xy_1z$

equivalent grammar

Nullable Variables

ε – production : $X \to \varepsilon$

Nullable Variable: $Y \Rightarrow \ldots \Rightarrow \varepsilon$

Example: $S \to aMb$

$M \to aMb$

$M \to \varepsilon$

Nullable variable

ε – production

Removing ε – productions

$S \rightarrow aMb$

$M \rightarrow aMb$

~~$M \rightarrow \varepsilon$~~

Substitute

$M \rightarrow \varepsilon$

$S \rightarrow aMb \mid ab$

$M \rightarrow aMb \mid ab$

After we remove all the ε - productions all the nullable variables disappear (except for the start variable)

Unit-Productions

Unit Production: $X \rightarrow Y$

(a single variable in both sides)

Example:

$S \rightarrow aA$

$A \rightarrow a$

$A \rightarrow B$

$B \rightarrow A$

Unit Productions

$B \rightarrow bb$

Removal of unit productions:

$S \rightarrow aA$

$A \rightarrow a$

~~$A \rightarrow B$~~

$B \rightarrow A$

$B \rightarrow bb$

Substitute

$A \rightarrow B$

$S \rightarrow aA \mid aB$

$A \rightarrow a$

$B \rightarrow A \mid B$

$B \rightarrow bb$

Unit productions of form $X \to X$ can be removed immediately

$S \to aA \mid aB$

$A \to a$

$B \to A \mid B$

$B \to bb$

Remove

$B \to B$

$S \to aA \mid aB$

$A \to a$

$B \to A$

$B \to bb$

$S \rightarrow aA \mid aB$

$A \rightarrow a$

~~$B \rightarrow A$~~

$B \rightarrow bb$

Substitute

$B \rightarrow A$

$S \rightarrow aA \mid aB \mid aA$

$A \rightarrow a$

$B \rightarrow bb$

Remove repeated productions

$$S \rightarrow aA \mid aB \mid aA$$

$$A \rightarrow a$$

$$B \rightarrow bb$$

Final grammar

$$S \rightarrow aA \mid aB$$

$$A \rightarrow a$$

$$B \rightarrow bb$$

Useless Productions

$$S \rightarrow aSb$$

$$S \rightarrow \lambda$$

$$S \rightarrow A$$

$$A \rightarrow aA$$ Useless Production

Some derivations never terminate…

$$S \Rightarrow A \Rightarrow aA \Rightarrow aaA \Rightarrow \ldots \Rightarrow aa\ldots aA \Rightarrow \ldots$$

Another grammar:

$$S \rightarrow A$$

$$A \rightarrow aA$$

$$A \rightarrow \lambda$$

$$B \rightarrow bA$$ Useless Production

Not reachable from S

In general:

If there is a derivation

$$S \Rightarrow \ldots \Rightarrow xAy \Rightarrow \ldots \Rightarrow w \in L(G)$$

consists of terminals

Then variable A is useful

Otherwise, variable A is useless

A production $A \to x$ is useless
if any of its variables is useless

$S \to aSb$

$S \to \varepsilon$

Variables		Productions
	$S \to A$	useless
useless	$A \to aA$	useless
useless	$B \to C$	useless
useless	$C \to D$	useless

Removing Useless Variables and Productions

Example Grammar:

$$S \rightarrow aS \mid A \mid C$$

$$A \rightarrow a$$

$$B \rightarrow aa$$

$$C \rightarrow aCb$$

First: find all variables that can produce strings with only terminals or ε (possible useful variables)

$S \rightarrow aS \mid A \mid C$

$A \rightarrow a$

$B \rightarrow aa$

$C \rightarrow aCb$

Round 1: $\{A, B\}$

(the right hand side of production that has only terminals)

Round 2: $\{A, B, S\}$

(the right hand side of a production has terminals and variables of previous round)

This process can be generalized

Then, remove productions that use variables other than $\{A, B, S\}$

$S \rightarrow aS \mid A \mid C$

$A \rightarrow a$

$B \rightarrow aa$

$C \rightarrow aCb$

$S \rightarrow aS \mid A$

$A \rightarrow a$

$B \rightarrow aa$

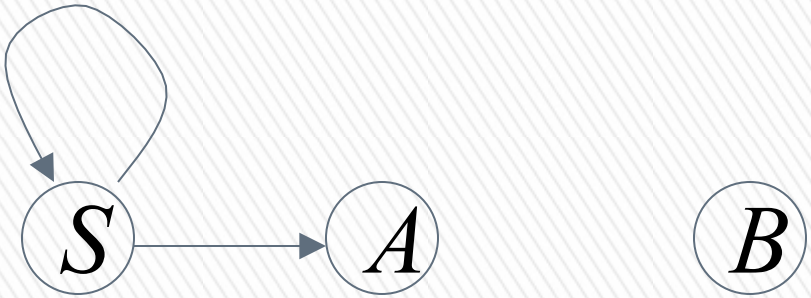
Second: Find all variables reachable from S

Use a Dependency Graph where nodes are variables

$$S \rightarrow aS \mid A$$

$$A \rightarrow a$$

$$B \rightarrow aa$$



unreachable

Keep only the variables reachable from S

$S \rightarrow aS \mid A$

$A \rightarrow a$

~~$B \rightarrow aa$~~

Final Grammar

$S \rightarrow aS \mid A$

$A \rightarrow a$

Contains only useful variables

Removing All

- **Step 1:** Remove Nullable Variables
- **Step 2:** Remove Unit-Productions
- **Step 3:** Remove Useless Variables

This sequence guarantees that unwanted variables and productions are removed

Normal Forms for Context-free Grammars

Chomsky Normal Form

Each productions has form:

$A \rightarrow BC$ or $A \rightarrow a$

Examples:

$$S \to AS$$
$$S \to a$$
$$A \to SA$$
$$A \to b$$

Chomsky Normal Form

$$S \to AS$$
$$S \to AAS$$
$$A \to SA$$
$$A \to aa$$

Not Chomsky Normal Form

Conversion to Chomsky Normal Form

» Example:

$$S \rightarrow ABa$$

$$A \rightarrow aab$$

$$B \rightarrow Ac$$

Not in Chomsky Normal Form

We will convert it to Chomsky Normal Form

Introduce new variables for the terminals:

$$T_a, T_b, T_c$$

$$S \rightarrow ABa$$

$$A \rightarrow aab$$

$$B \rightarrow Ac$$

$$S \rightarrow ABT_a$$

$$A \rightarrow T_a T_a T_b$$

$$B \rightarrow AT_c$$

$$T_a \rightarrow a$$

$$T_b \rightarrow b$$

$$T_c \rightarrow c$$

Introduce new intermediate variable V_1 to break first production:

$$S \rightarrow ABT_a$$

$$A \rightarrow T_a T_a T_b$$

$$B \rightarrow AT_c$$

$$T_a \rightarrow a$$

$$T_b \rightarrow b$$

$$T_c \rightarrow c$$

$$S \rightarrow AV_1$$

$$V_1 \rightarrow BT_a$$

$$A \rightarrow T_a T_a T_b$$

$$B \rightarrow AT_c$$

$$T_a \rightarrow a$$

$$T_b \rightarrow b$$

$$T_c \rightarrow c$$

Introduce intermediate variable: V_2

$$S \rightarrow AV_1$$
$$V_1 \rightarrow BT_a$$
$$A \rightarrow T_aT_aT_b$$
$$B \rightarrow AT_c$$
$$T_a \rightarrow a$$
$$T_b \rightarrow b$$
$$T_c \rightarrow c$$

$$S \rightarrow AV_1$$
$$V_1 \rightarrow BT_a$$
$$A \rightarrow T_aV_2$$
$$V_2 \rightarrow T_aT_b$$
$$B \rightarrow AT_c$$
$$T_a \rightarrow a$$
$$T_b \rightarrow b$$
$$T_c \rightarrow c$$

›

Final grammar in Chomsky Normal Form:

Initial grammar

$$S \rightarrow ABa$$

$$A \rightarrow aab$$

$$B \rightarrow Ac$$

$$S \rightarrow AV_1$$

$$V_1 \rightarrow BT_a$$

$$A \rightarrow T_aV_2$$

$$V_2 \rightarrow T_aT_b$$

$$B \rightarrow AT_c$$

$$T_a \rightarrow a$$

$$T_b \rightarrow b$$

$$T_c \rightarrow c$$

›

In general:

From any context-free grammar
(which doesn't produce ε)
not in Chomsky Normal Form

we can obtain:

an equivalent grammar
in Chomsky Normal Form

The Procedure

First remove:

Nullable variables

Unit productions

(Useless variables optional)

Then, for every symbol a :

New variable: T_a

Add production $T_a \rightarrow a$

In productions with length at least 2 replace a with T_a

Productions of form $A \rightarrow a$ do not need to change!

Replace any production $A \to C_1 C_2 \cdots C_n$

with

$$A \to C_1 V_1$$
$$V_1 \to C_2 V_2$$
$$\cdots$$
$$V_{n-2} \to C_{n-1} C_n$$

New intermediate variables: $V_1, V_2, \ldots, V_{n-2}$

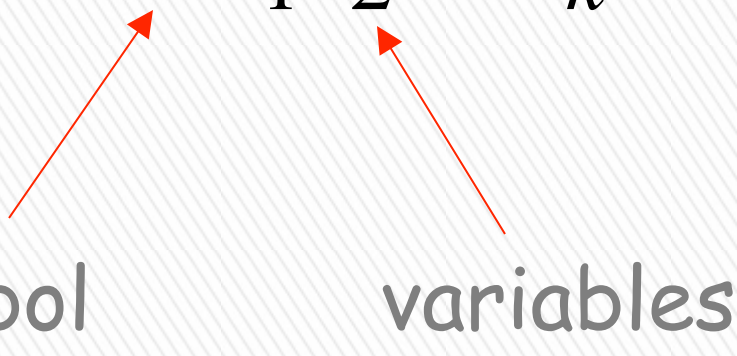
Observations

- Chomsky normal forms are good for parsing and proving theorems
- It is easy to find the Chomsky normal form for any context-free grammar

Greinbach Normal Form

All productions have form:

$$A \rightarrow a V_1 V_2 \cdots V_k \qquad k \geq 0$$



Examples:

$$S \rightarrow cAB$$
$$A \rightarrow aA \mid bB \mid b$$
$$B \rightarrow b$$

Greinbach Normal Form

$$S \rightarrow abSb$$
$$S \rightarrow aa$$

Not Greinbach Normal Form

Conversion to Greinbach Normal Form:

$$S \rightarrow abSb$$

$$S \rightarrow aa$$

$$S \rightarrow aT_bST_b$$

$$S \rightarrow aT_a$$

$$T_a \rightarrow a$$

$$T_b \rightarrow b$$

Greinbach Normal Form

Observations

- Greinbach normal forms are very good for parsing strings (better than Chomsky Normal Forms)
- However, it is difficult to find the Greinbach normal of a grammar

BLM2502 Theory of Computation