



BLM2502

Theory of

Computation

Spring 2015

BLM2502 Theory of Computation

» Course Outline

- | » Week | Content |
|-------------|---|
| » 1 | Introduction to Course |
| » 2 | Computability Theory, Complexity Theory, Automata Theory, Set Theory, Relations, Proofs, Pigeonhole Principle |
| » 3 | Regular Expressions |
| » 4 | Finite Automata |
| » 5 | Deterministic and Nondeterministic Finite Automata |
| » 6 | Epsilon Transition, Equivalence of Automata |
| » 7 | Pumping Theorem |
| » 8 | April 10 - 14 week is the first midterm week |
| » 9 | Context Free Grammars, Parse Tree, Ambiguity |
| » 10 | Pumping Theorem, Normal Forms |
| » 11 | Pushdown Automata |
| » 12 | Turing Machines, Recognition and Computation, Church-Turing Hypothesis |
| » 13 | Turing Machines, Recognition and Computation, Church-Turing Hypothesis |
| » 14 | May 22 – 27 week is the second midterm week |
| » 15 | Review |
| » 16 | Final Exam date will be announced |



2



Pushdown Automata

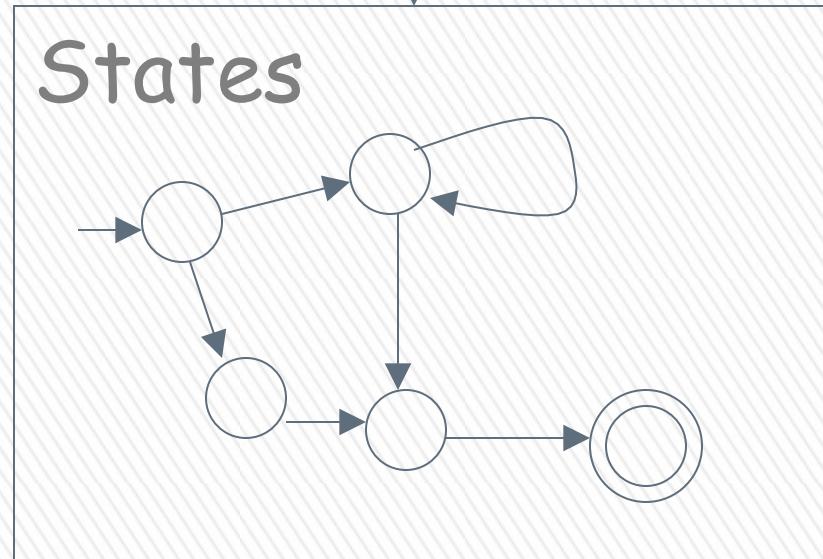
PDA

Pushdown Automaton -- PDA

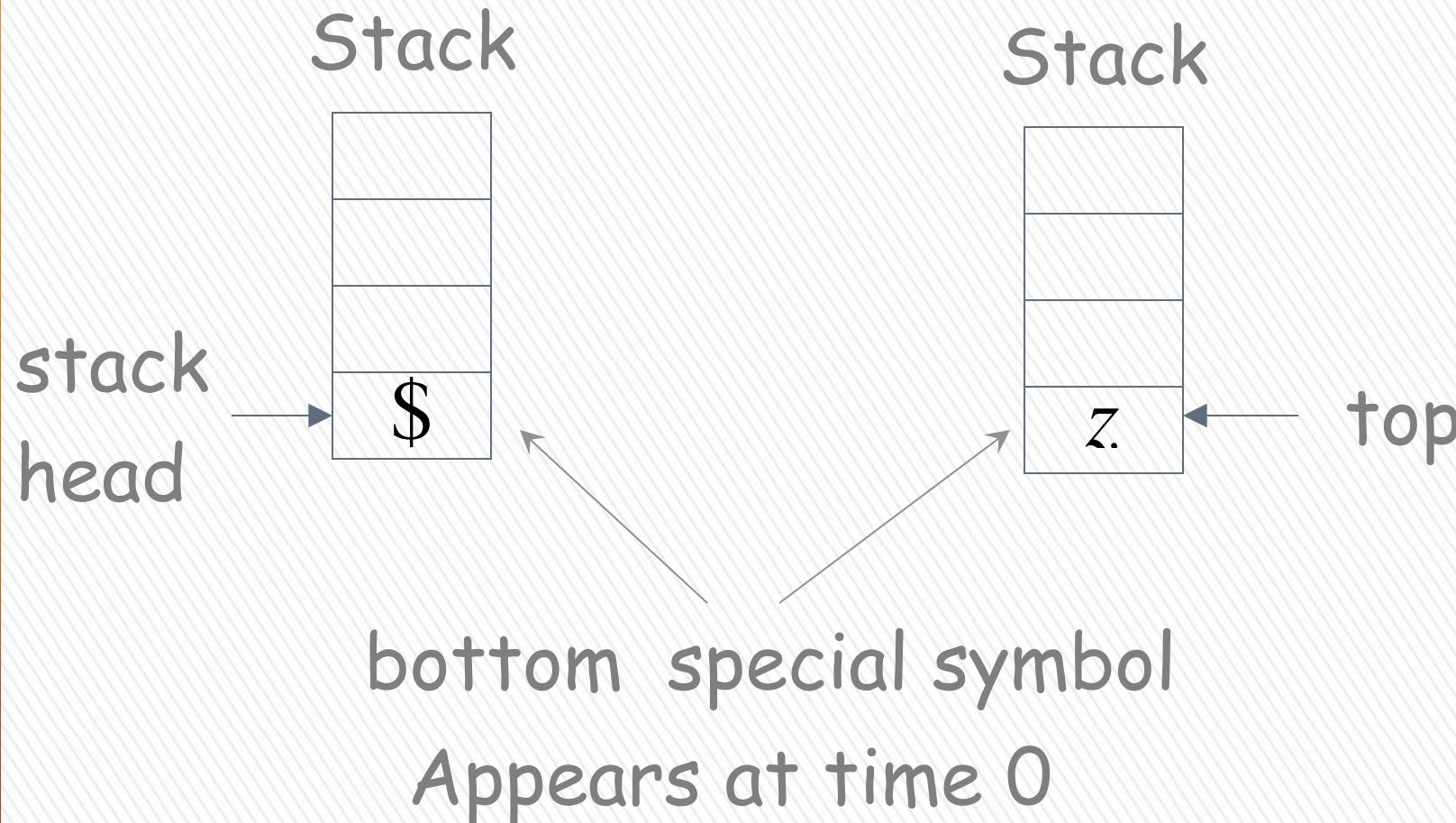
Input String



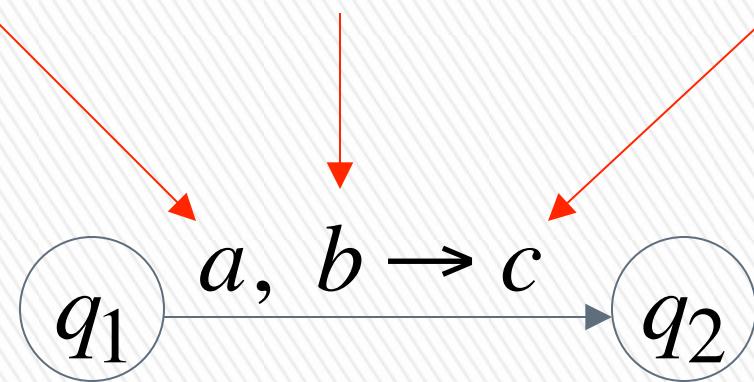
Stack



Initial Stack Symbol

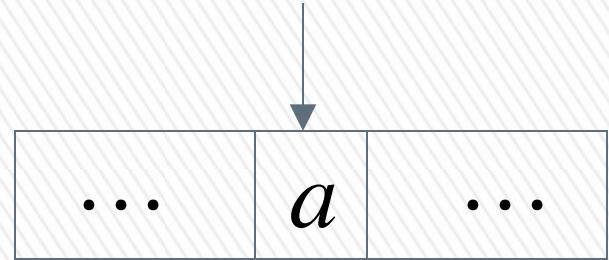
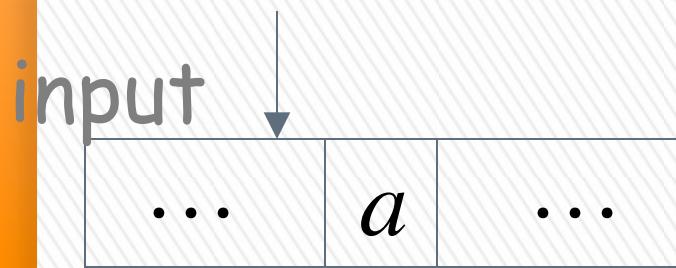
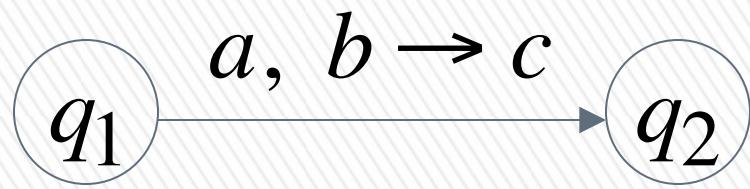


Input symbol Pop symbol Push symbol



The States

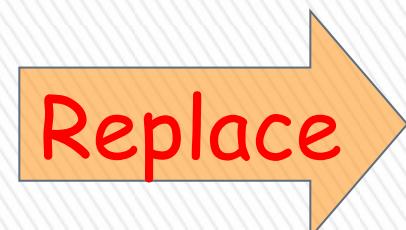




stack

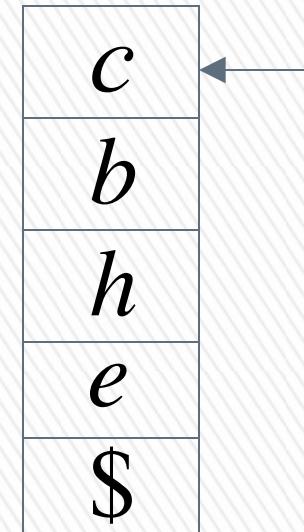
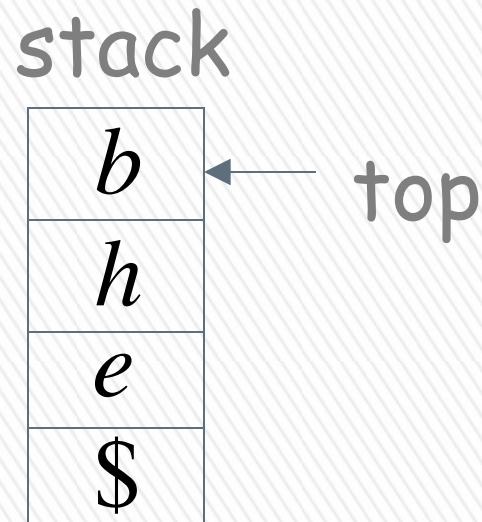
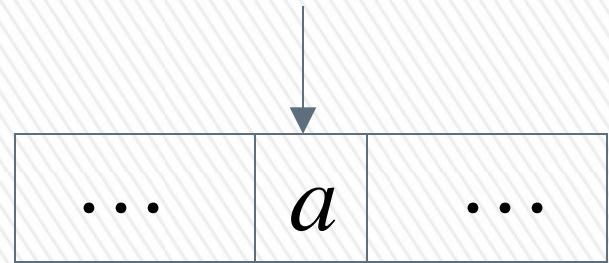
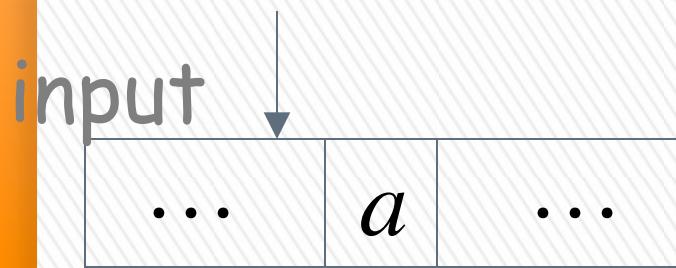
b
h
e
$\$$

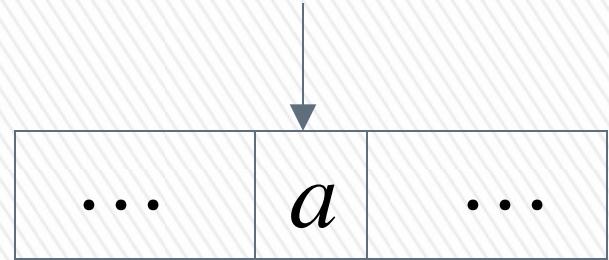
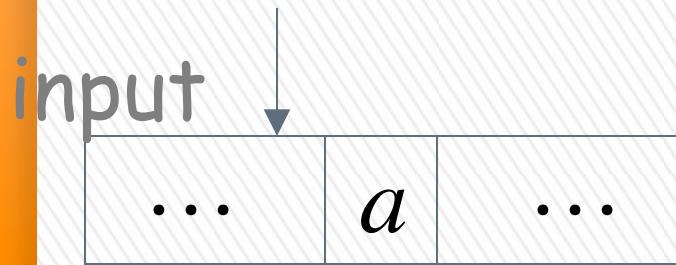
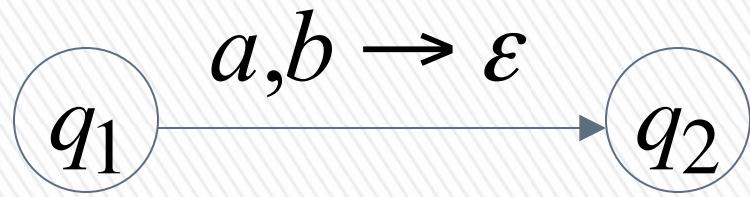
top



c
h
e
$\$$







stack

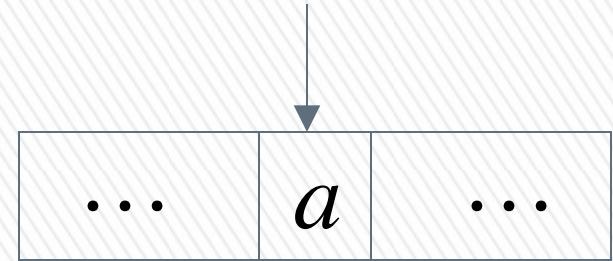
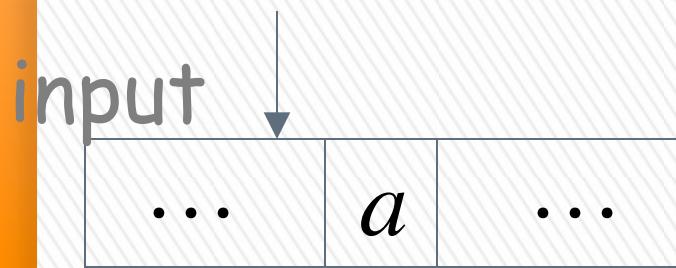
b
h
e
$\$$

top



h
e
$\$$





stack

b
h
e
$\$$

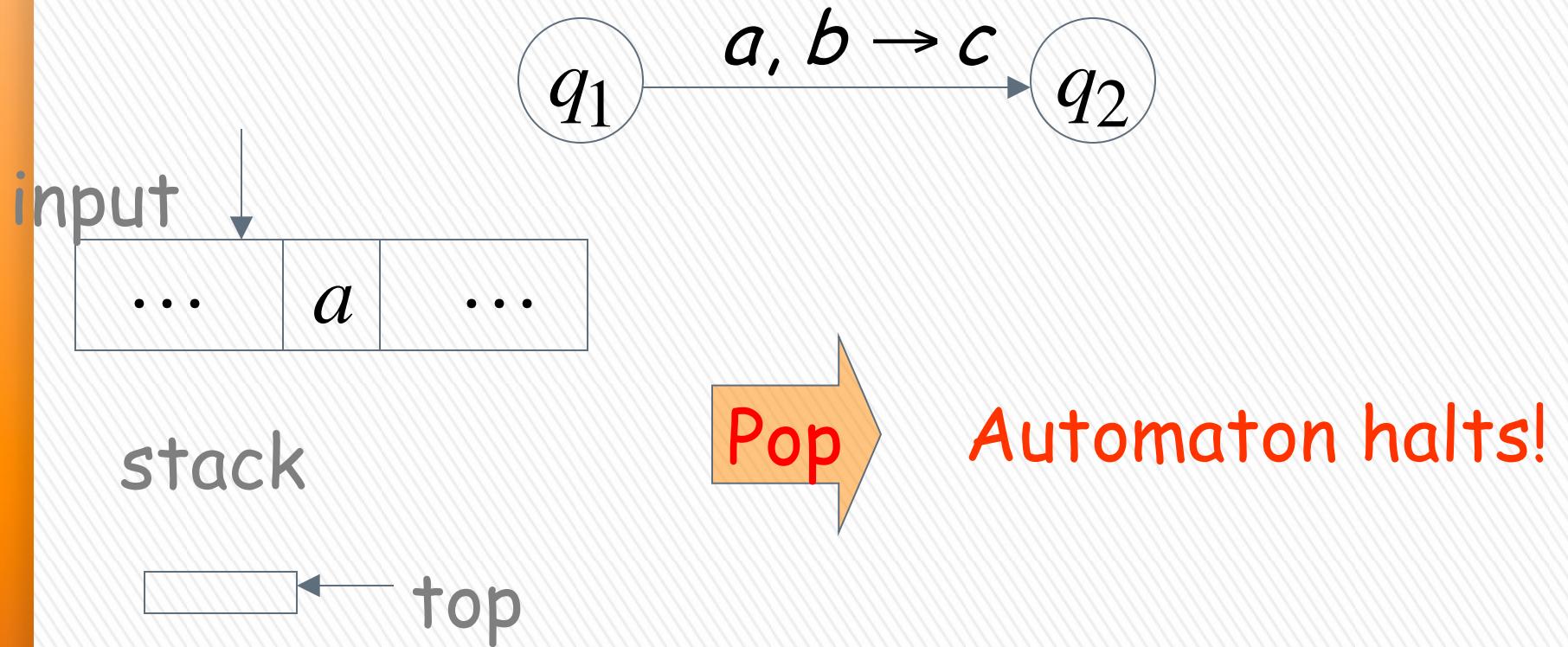
top

No Change

b
h
e
$\$$



Pop from Empty Stack

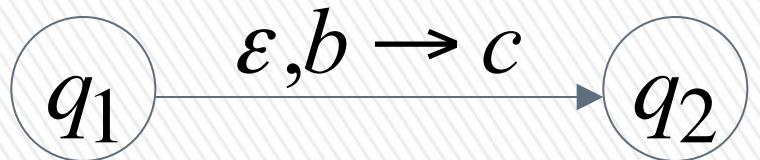
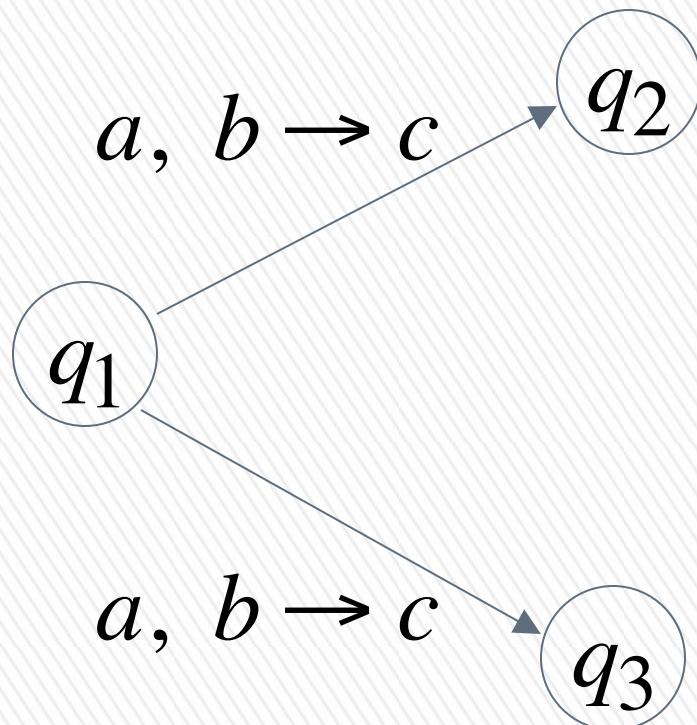


If the automaton attempts to pop from empty stack then it halts and rejects input

Non-Determinism

PDAs are non-deterministic

Allowed non-deterministic transitions



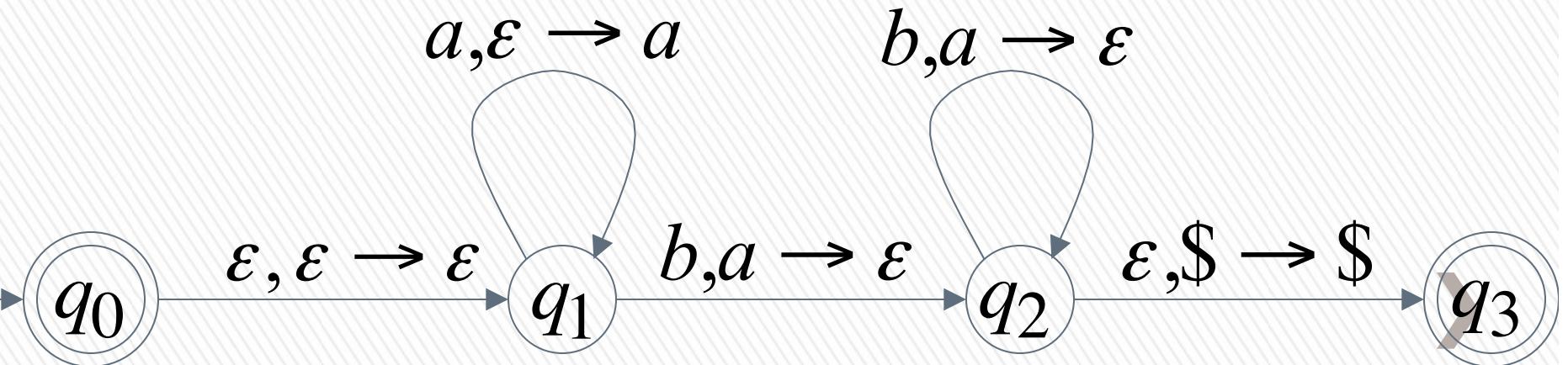
ϵ – transition



Example PDA

PDA M :

$$L(M) = \{a^n b^n : n \geq 0\}$$



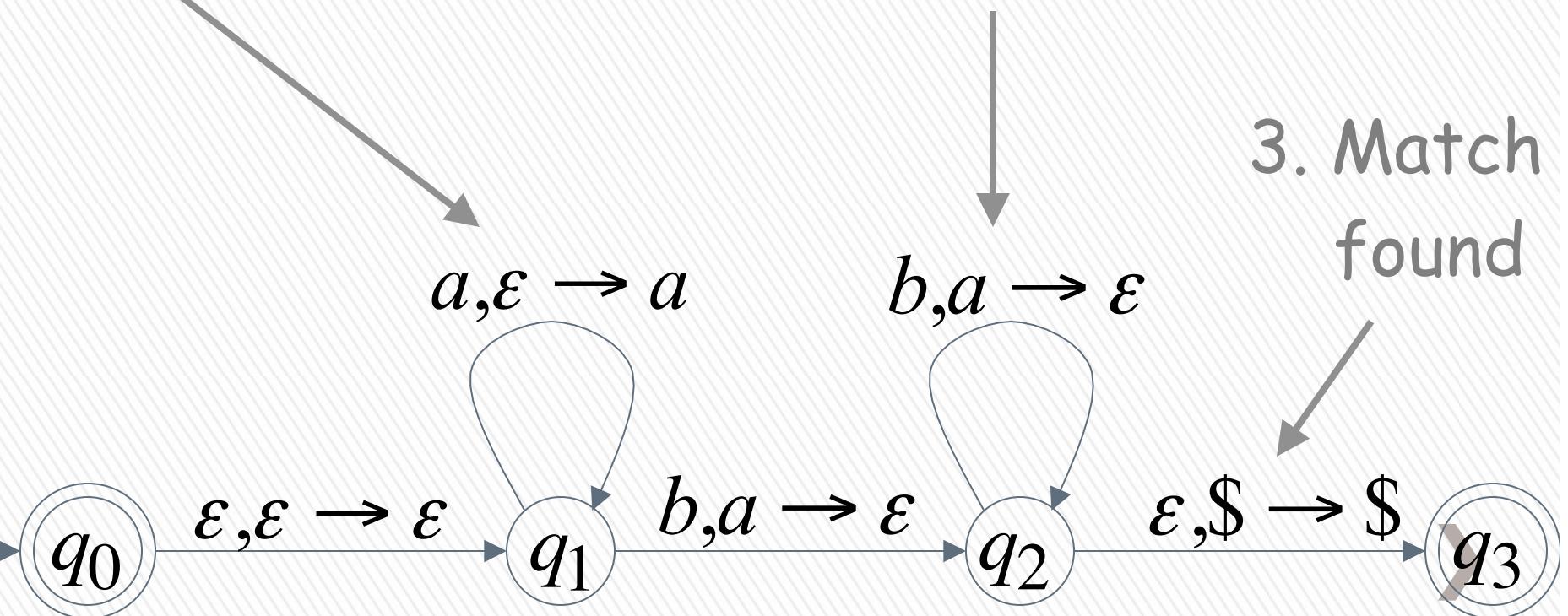
$$L(M) = \{a^n b^n : n \geq 0\}$$

Basic Idea:

1. Push the a's
on the stack

2. Match the b's on input
with a's on stack

3. Match
found



Execution Example: Time 0

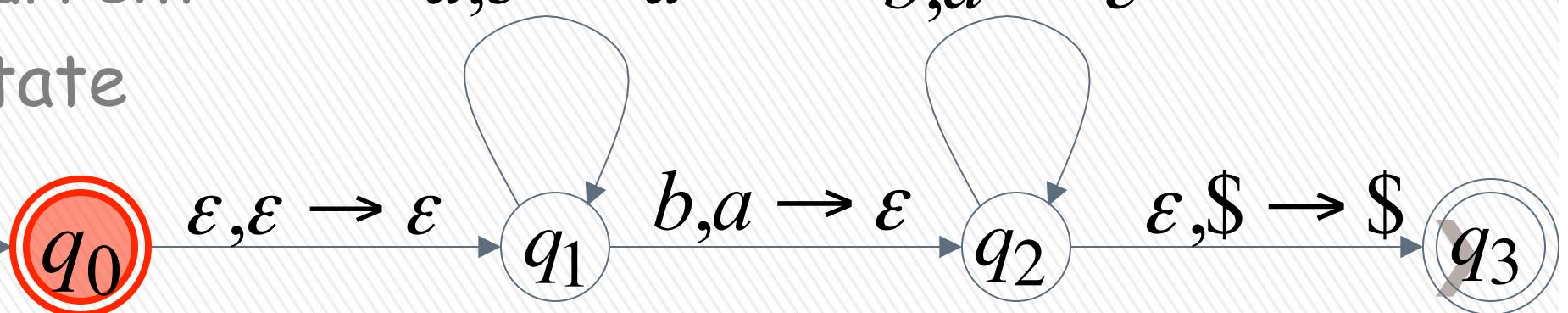
Input

a	a	a	b	b	b
-----	-----	-----	-----	-----	-----



Stack

current
state



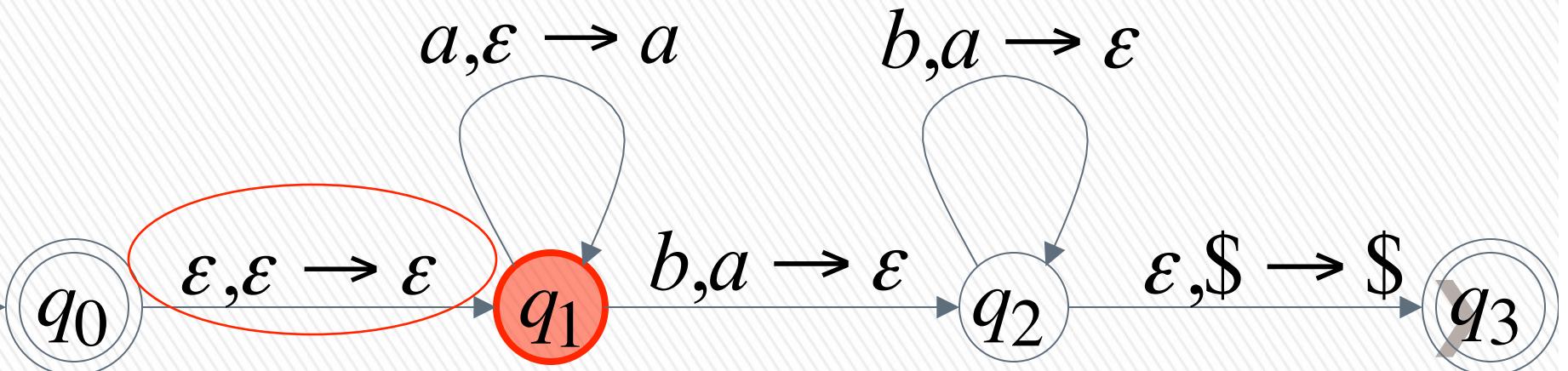
Input

a	a	a	b	b	b
-----	-----	-----	-----	-----	-----

Time 1

\$

Stack



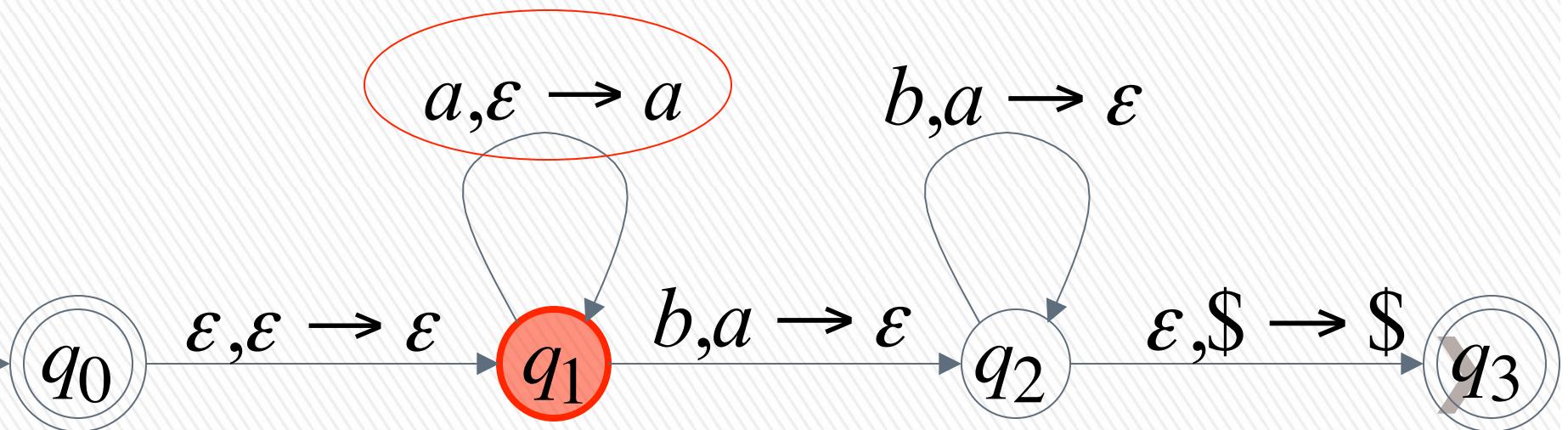
Input

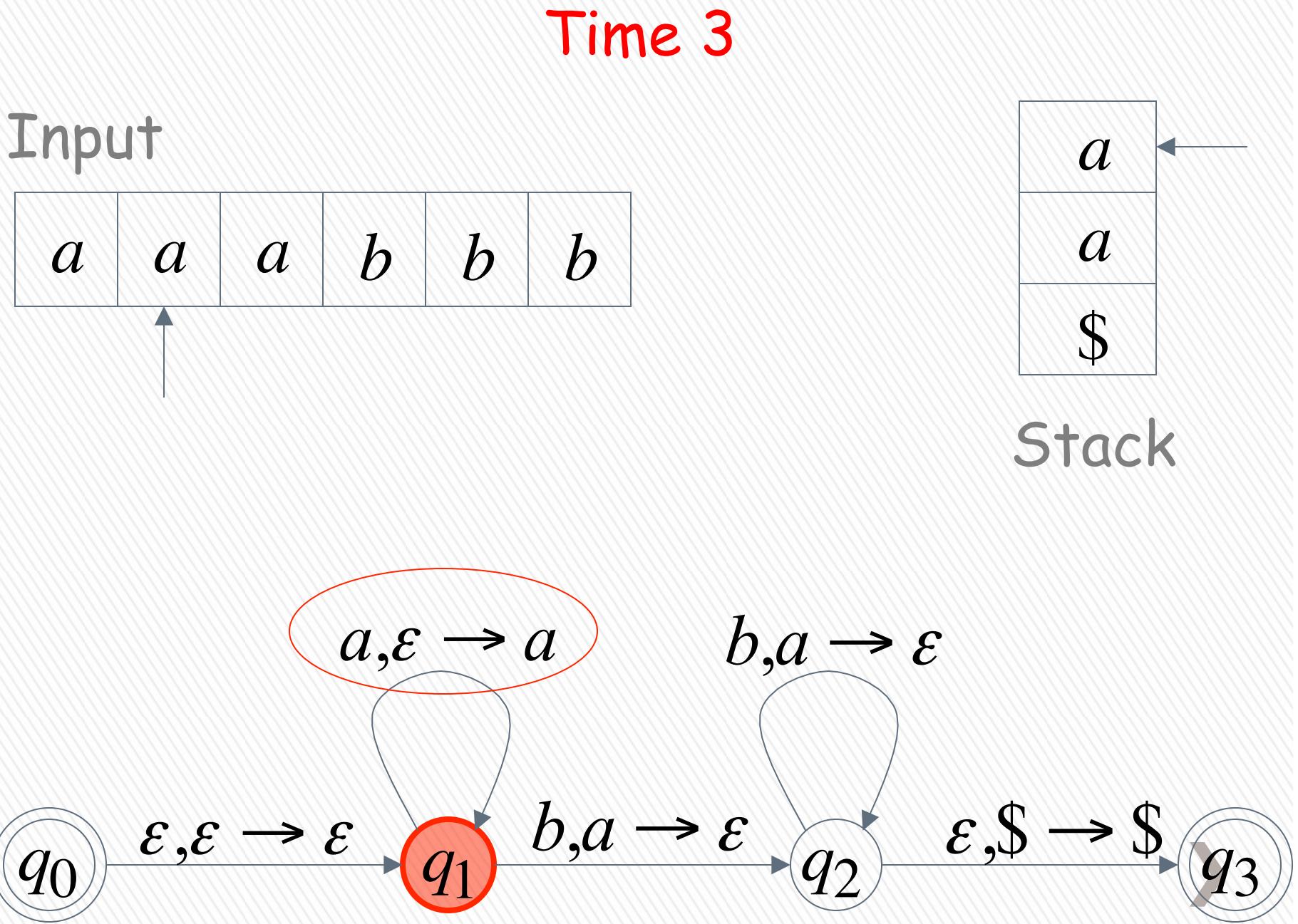
a	a	a	b	b	b
-----	-----	-----	-----	-----	-----

Time 2

a
\$

Stack

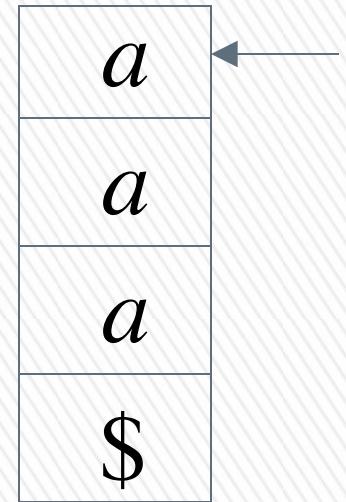




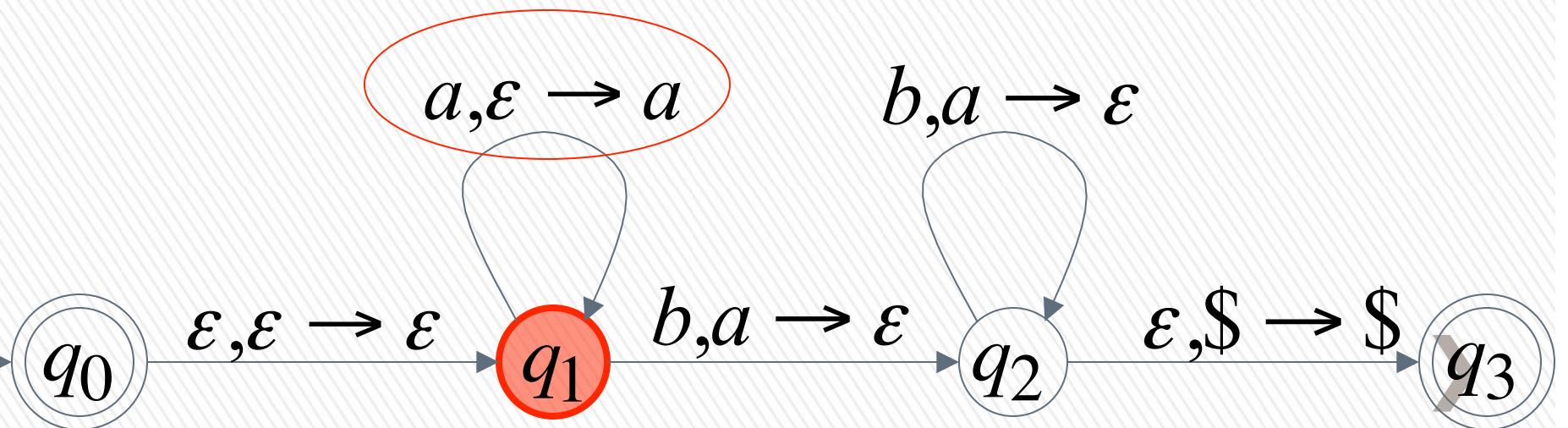
Input

a	a	a	b	b	b
---	---	---	---	---	---

Time 4



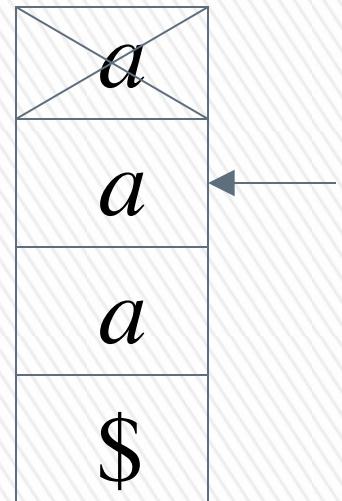
Stack



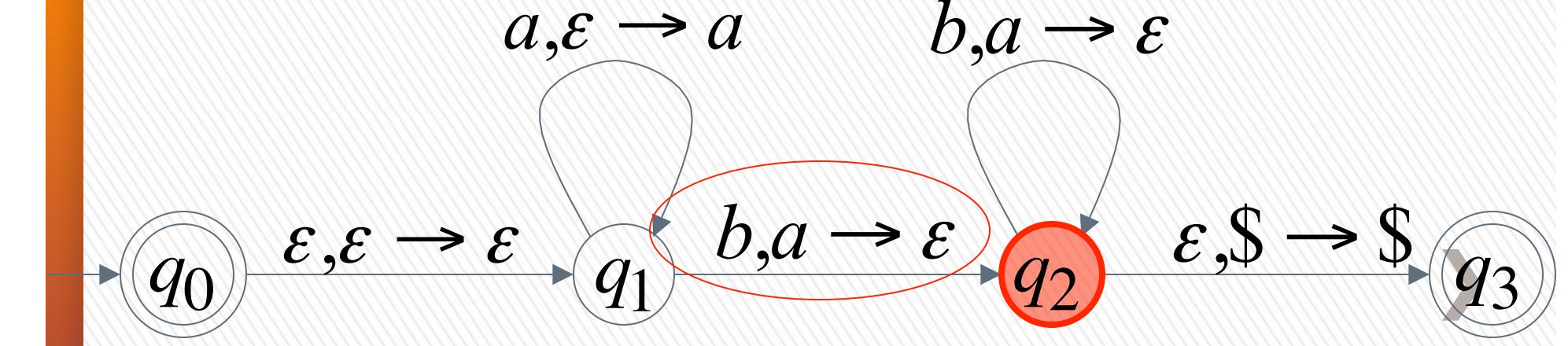
Input

a	a	a	b	b	b
---	---	---	---	---	---

Time 5



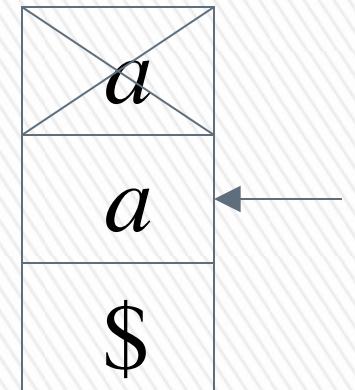
Stack



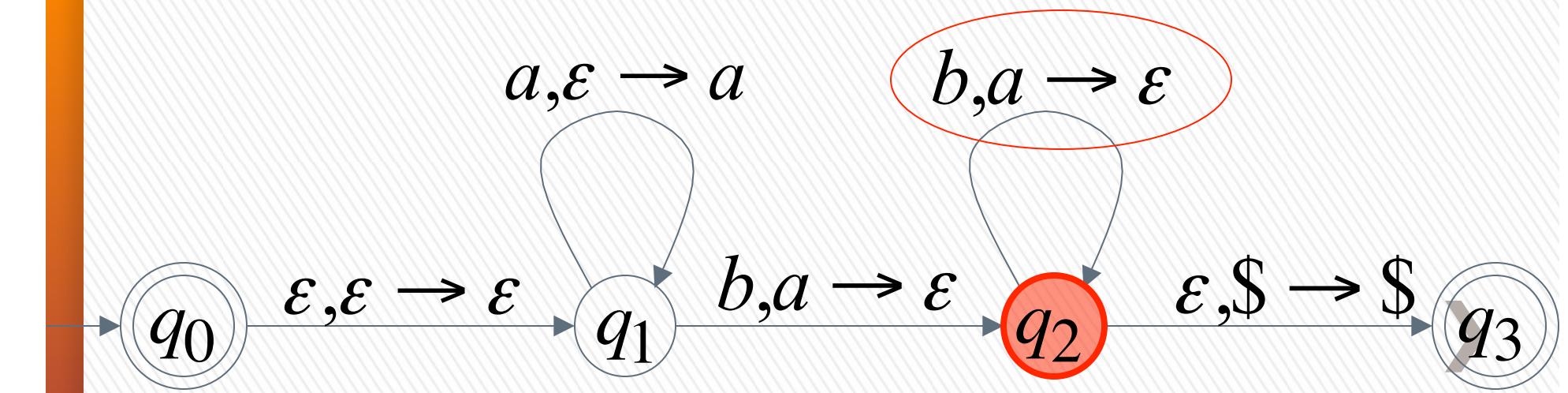
Input

a	a	a	b	b	b
-----	-----	-----	-----	-----	-----

Time 6



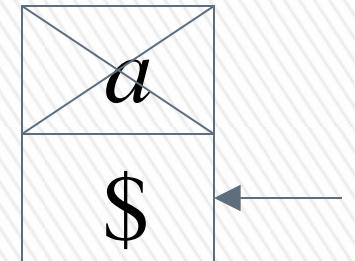
Stack



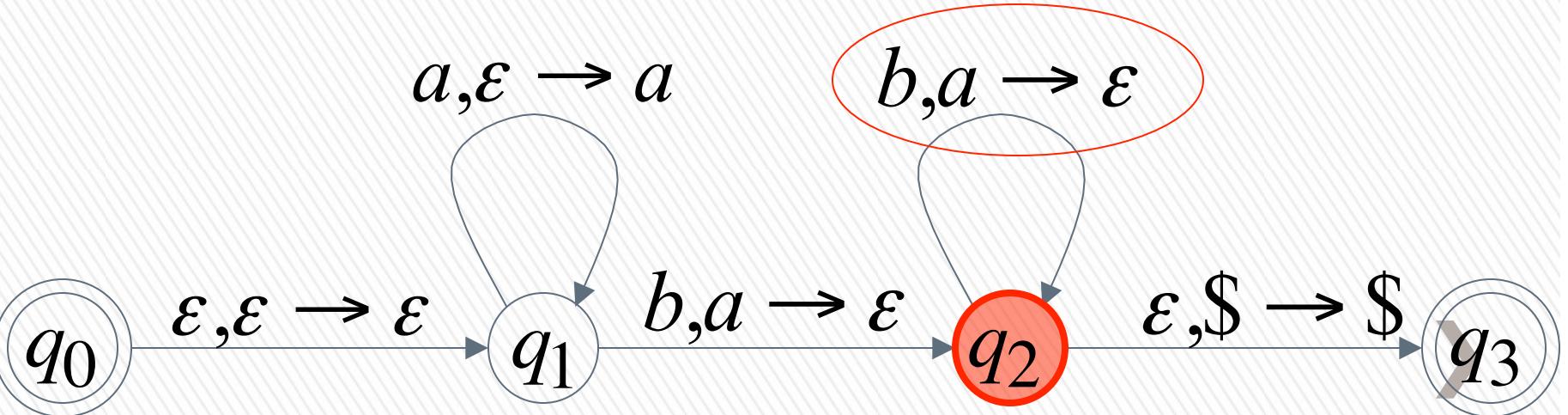
Time 7

Input

a	a	a	b	b	b
-----	-----	-----	-----	-----	-----



Stack



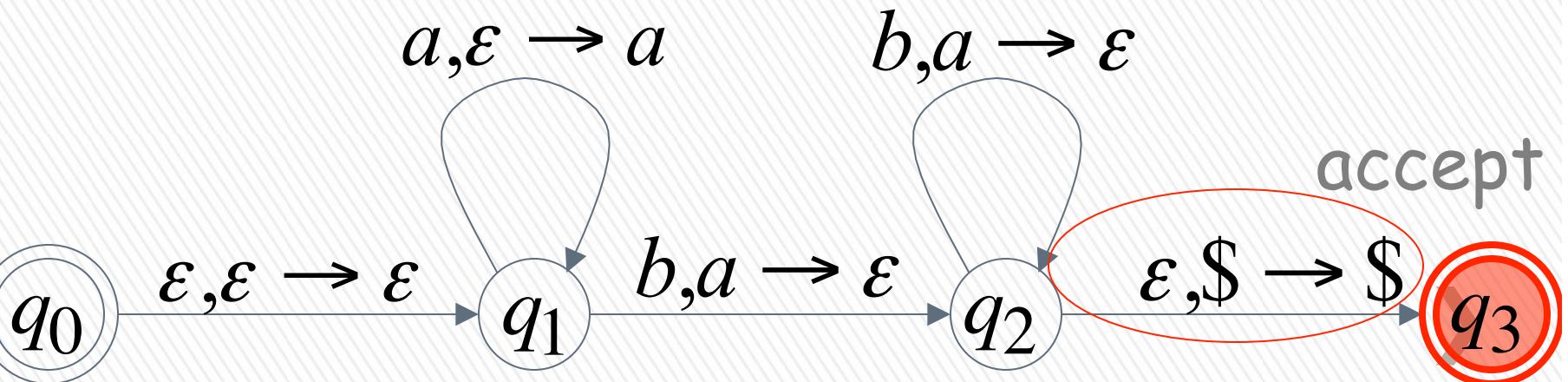
Time 8

Input

a	a	a	b	b	b
-----	-----	-----	-----	-----	-----



Stack



A string is accepted if there is
a computation such that:

All the input is consumed
AND

The last state is an accepting state

we do not care about the stack contents
at the end of the accepting computation ➤

Rejection Example: Time 0

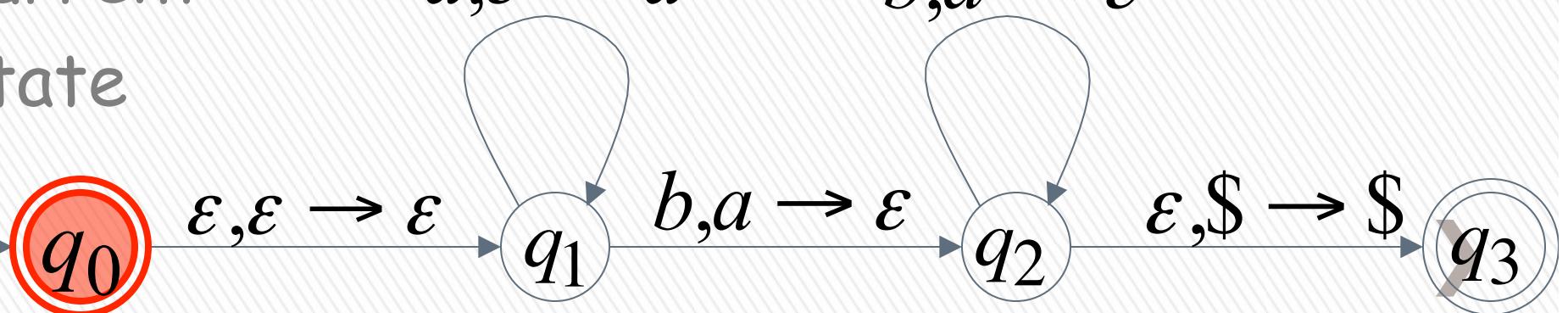
Input

a	a	b
-----	-----	-----



Stack

current
state



Rejection Example: Time 1

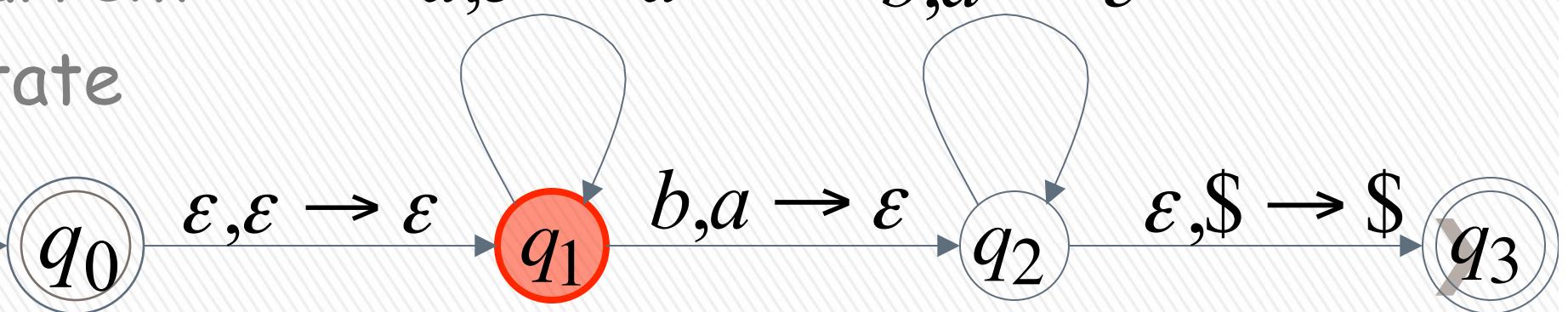
Input

a	a	b
-----	-----	-----



Stack

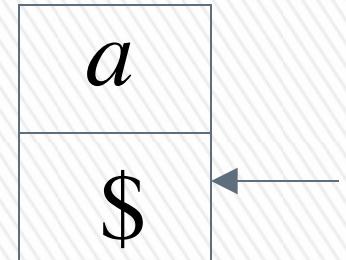
current
state



Rejection Example: Time 2

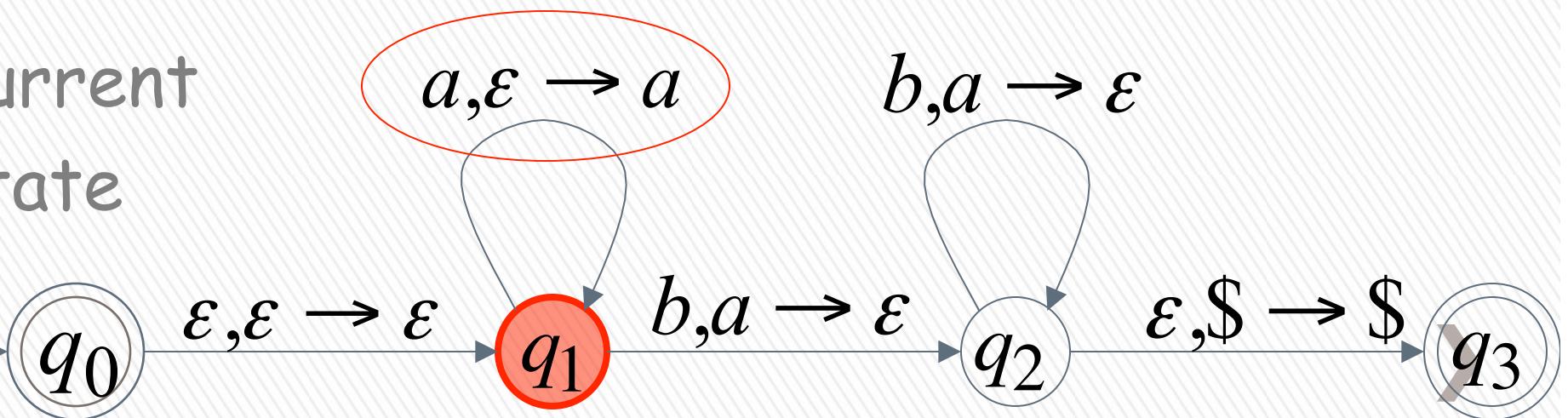
Input

a	a	b
-----	-----	-----



Stack

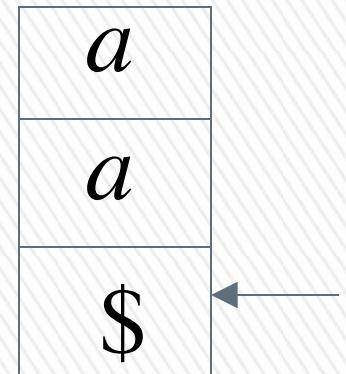
current
state



Rejection Example: Time 3

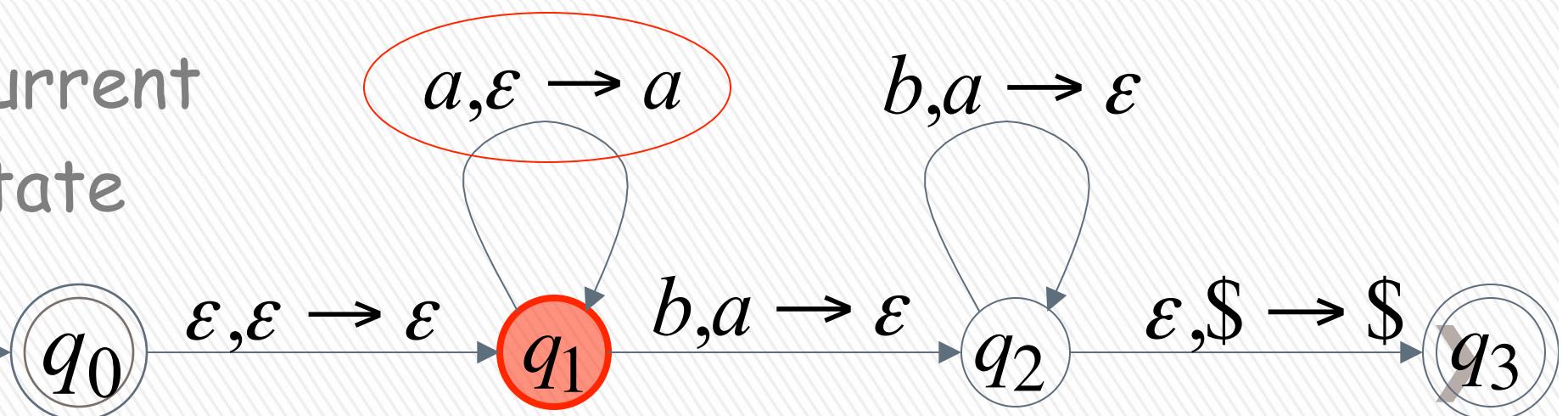
Input

a	a	b
-----	-----	-----



Stack

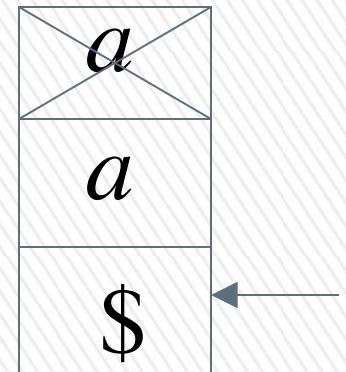
current
state



Rejection Example: Time 4

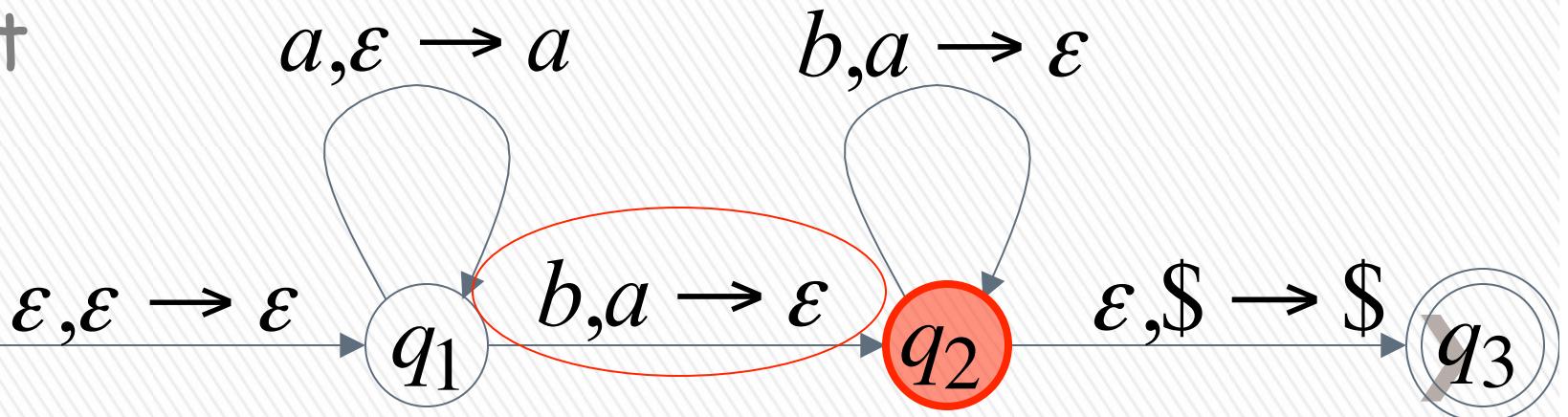
Input

a	a	b
-----	-----	-----



Stack

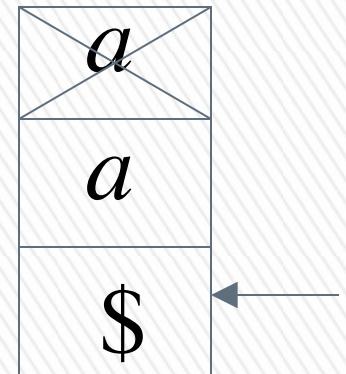
current
state



Rejection Example: Time 4

Input

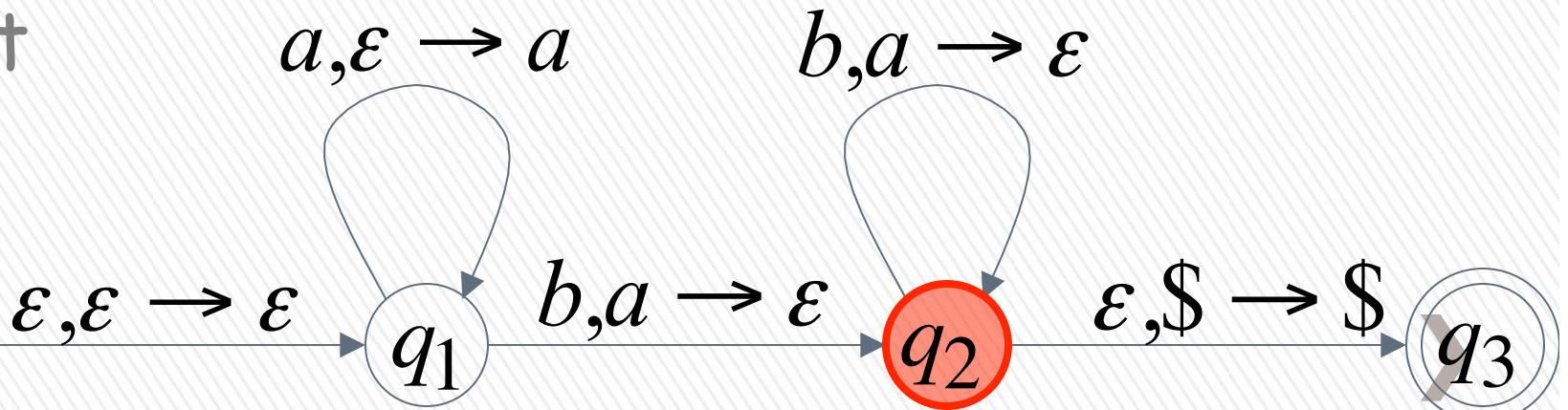
a	a	b
-----	-----	-----



Stack

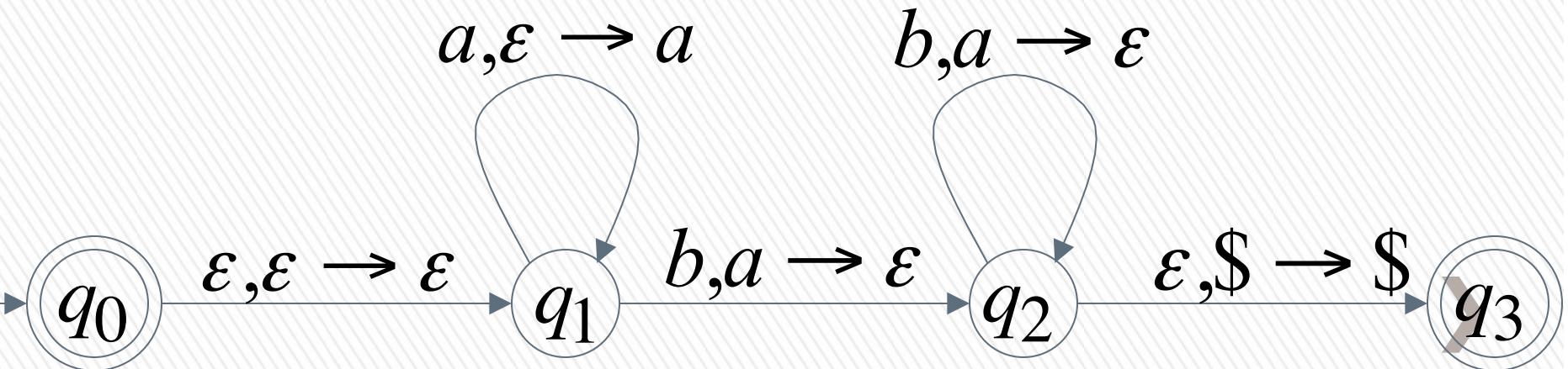
reject

current
state



There is no accepting computation for aab

The string aab is rejected by the PDA



Another PDA example

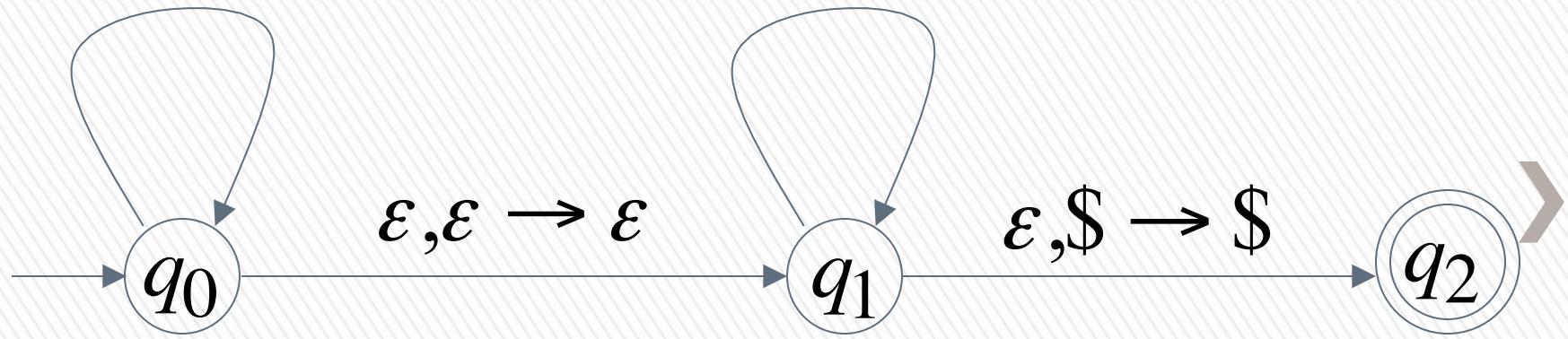
PDA M : $L(M) = \{vv^R : v \in \{a,b\}^*\}$

$a, \epsilon \rightarrow a$

$b, \epsilon \rightarrow b$

$a, a \rightarrow \epsilon$

$b, b \rightarrow \epsilon$



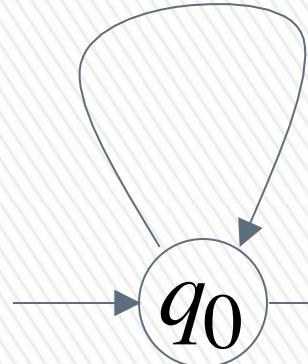
Basic Idea:

$$L(M) = \{vv^R : v \in \{a,b\}^*\}$$

1. Push v
on stack



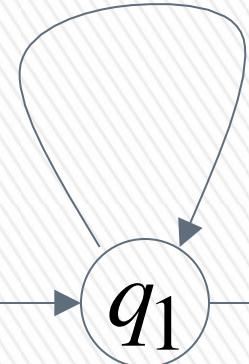
$$\begin{array}{l} a, \varepsilon \rightarrow a \\ b, \varepsilon \rightarrow b \end{array}$$



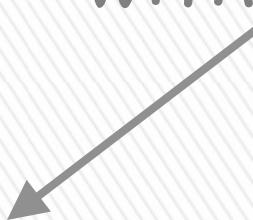
2. Guess
middle
of input



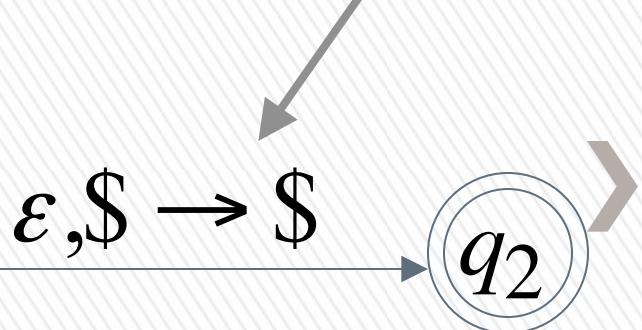
$$\begin{array}{l} a, a \rightarrow \varepsilon \\ b, b \rightarrow \varepsilon \end{array}$$



3. Match v^R on input
with v on stack



4. Match
found



Execution Example: Time 0

Input

a	b	b	a
-----	-----	-----	-----



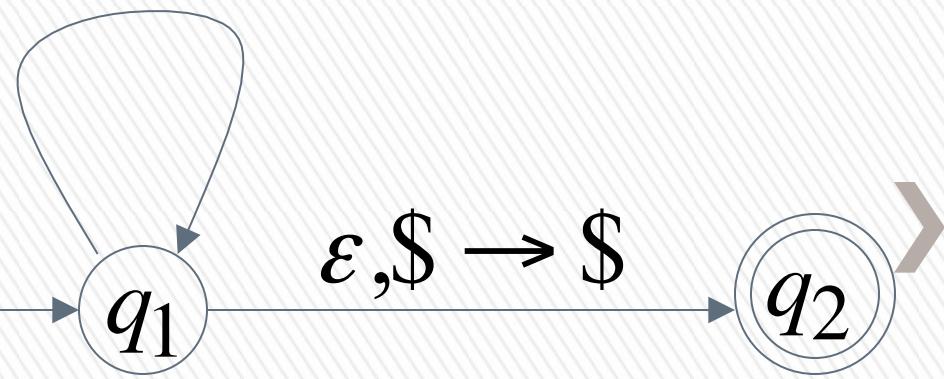
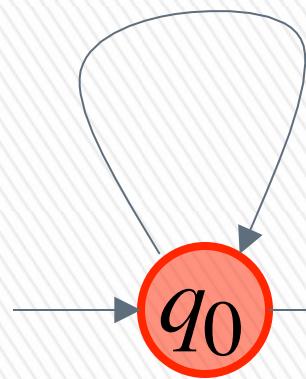
Stack

$$a, \varepsilon \rightarrow a$$

$$a, a \rightarrow \varepsilon$$

$$b, \varepsilon \rightarrow b$$

$$b, b \rightarrow \varepsilon$$



$$\varepsilon, \varepsilon \rightarrow \varepsilon$$

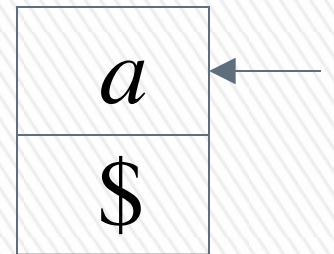
$$\varepsilon, \$ \rightarrow \$$$



Time 1

Input

a	b	b	a
-----	-----	-----	-----



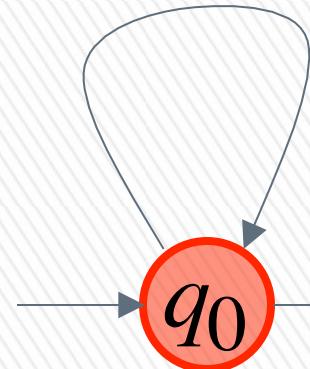
Stack

$$a, \varepsilon \rightarrow a$$

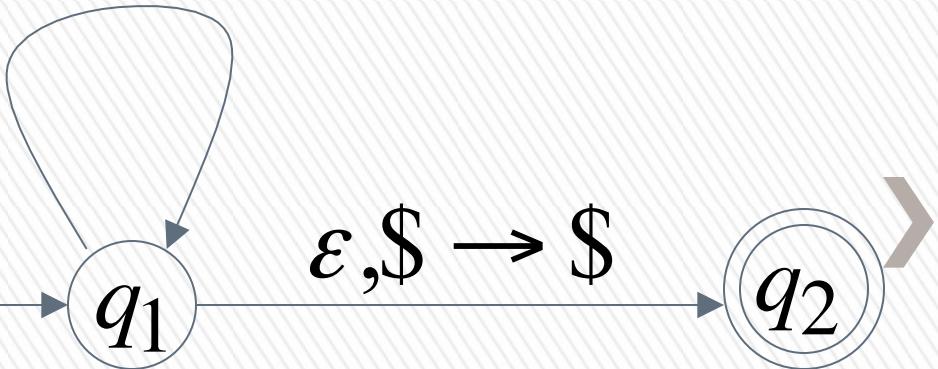
$$b, \varepsilon \rightarrow b$$

$$a, a \rightarrow \varepsilon$$

$$b, b \rightarrow \varepsilon$$



$$\varepsilon, \varepsilon \rightarrow \varepsilon$$



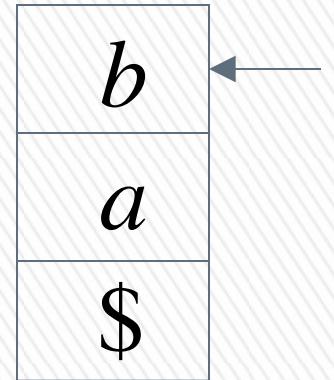
$$\varepsilon, \$ \rightarrow \$$$



Time 2

Input

a	b	b	a
-----	-----	-----	-----



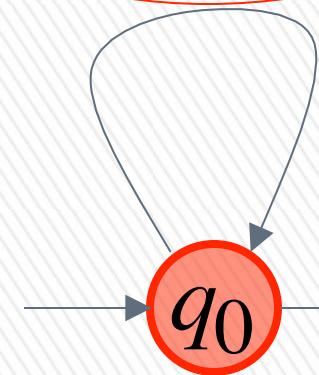
Stack

$$a, \varepsilon \rightarrow a$$

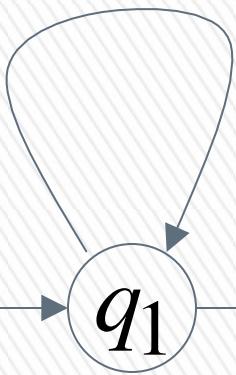
$$b, \varepsilon \rightarrow b$$

$$a, a \rightarrow \varepsilon$$

$$b, b \rightarrow \varepsilon$$



$$\varepsilon, \varepsilon \rightarrow \varepsilon$$



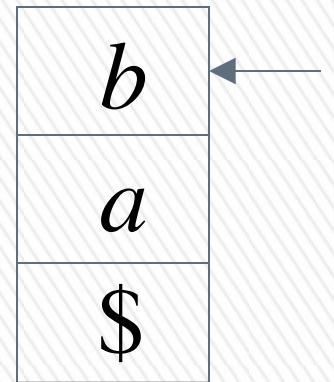
$$\varepsilon, \$ \rightarrow \$$$



Time 3

Input

a	b	b	a
-----	-----	-----	-----



Guess the middle
of string

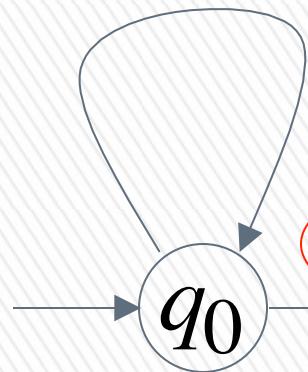
Stack

$$a, \epsilon \rightarrow a$$

$$b, \epsilon \rightarrow b$$

$$a, a \rightarrow \epsilon$$

$$b, b \rightarrow \epsilon$$



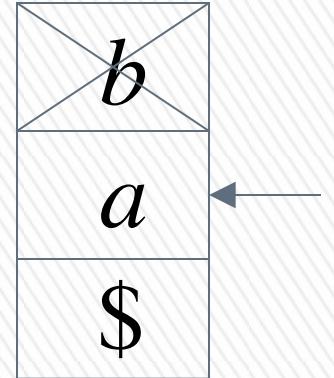
q_1

q_2

Time 4

Input

a	b	b	a
-----	-----	-----	-----



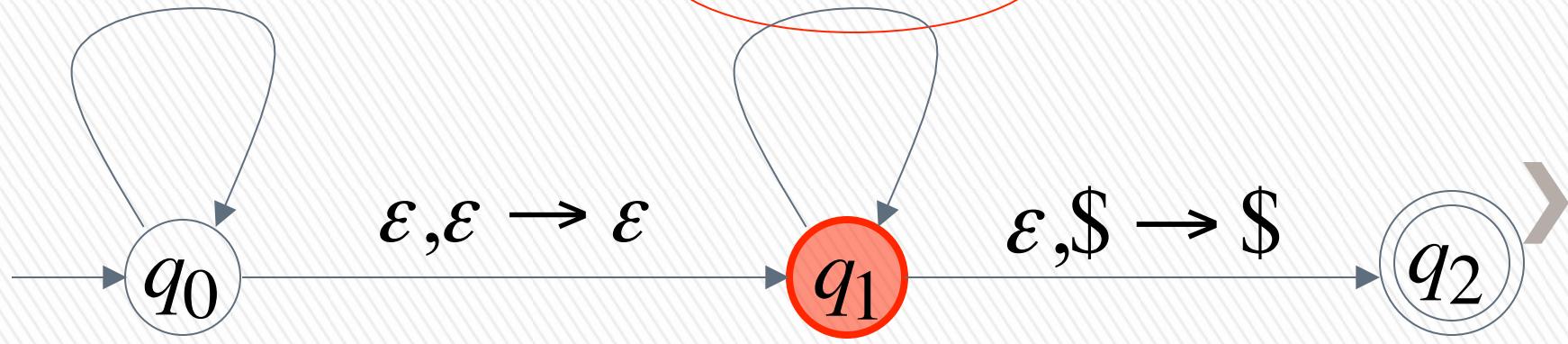
Stack

$$a, \varepsilon \rightarrow a$$

$$b, \varepsilon \rightarrow b$$

$$a, a \rightarrow \varepsilon$$

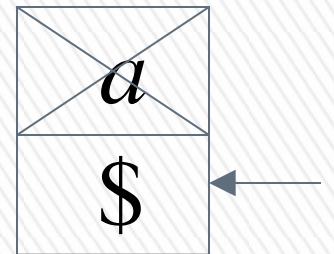
$$b, b \rightarrow \varepsilon$$



Time 5

Input

a	b	b	a
-----	-----	-----	-----



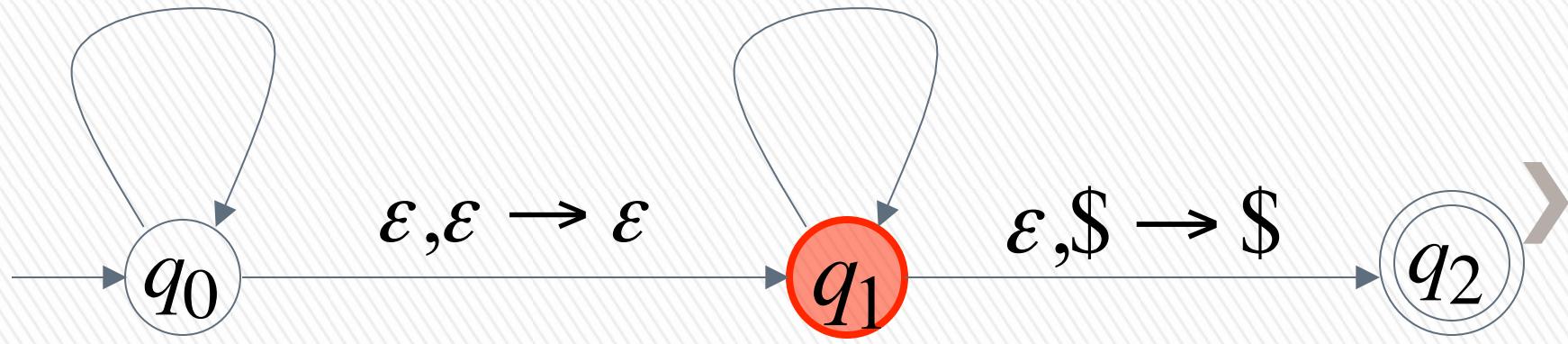
Stack

$$a, \varepsilon \rightarrow a$$

$$b, \varepsilon \rightarrow b$$

$$a, a \rightarrow \varepsilon$$

$$b, b \rightarrow \varepsilon$$



Time 6

Input

a	b	b	a
-----	-----	-----	-----



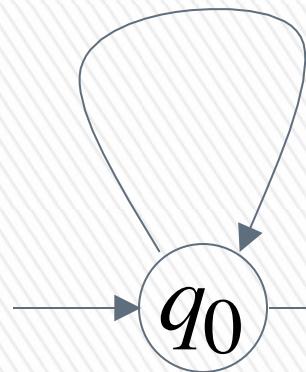
Stack

$$a, \varepsilon \rightarrow a$$

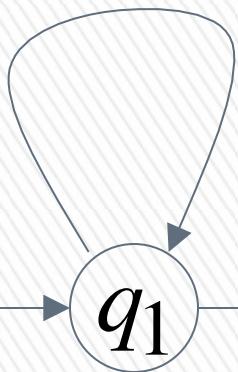
$$a, a \rightarrow \varepsilon$$

$$b, \varepsilon \rightarrow b$$

$$b, b \rightarrow \varepsilon$$



$$\varepsilon, \varepsilon \rightarrow \varepsilon$$



$$\varepsilon, \$ \rightarrow \$$$



accept

Rejection Example: Time 0

Input

a	b	b	b
-----	-----	-----	-----



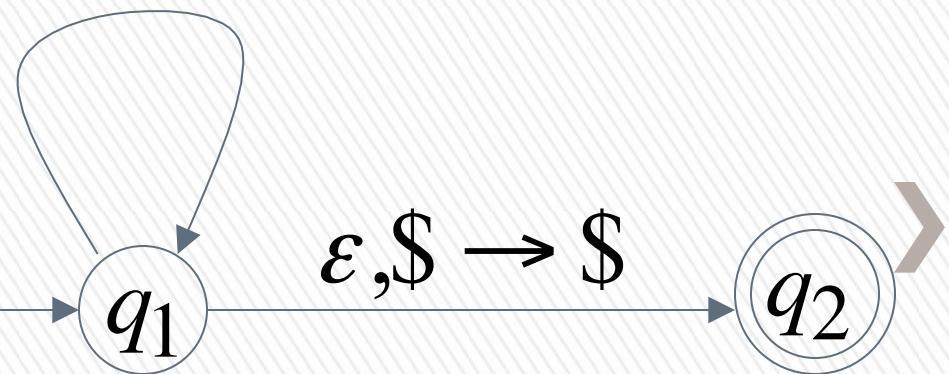
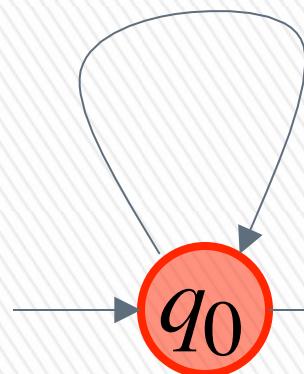
Stack

$$a, \varepsilon \rightarrow a$$

$$a, a \rightarrow \varepsilon$$

$$b, \varepsilon \rightarrow b$$

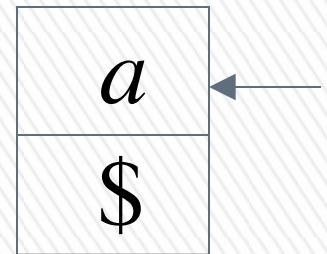
$$b, b \rightarrow \varepsilon$$



Time 1

Input

a	b	b	b
-----	-----	-----	-----



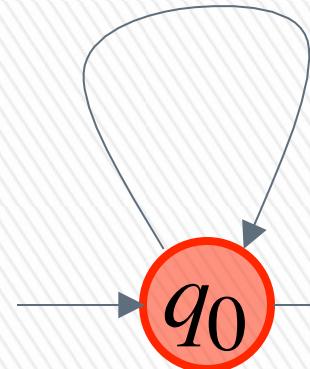
Stack

$$a, \varepsilon \rightarrow a$$

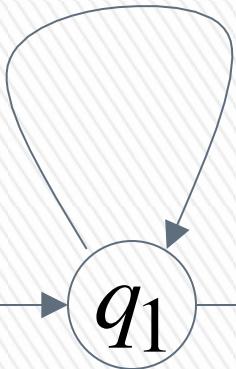
$$b, \varepsilon \rightarrow b$$

$$a, a \rightarrow \varepsilon$$

$$b, b \rightarrow \varepsilon$$



$$\varepsilon, \varepsilon \rightarrow \varepsilon$$



$$\varepsilon, \$ \rightarrow \$$$



q_1

q_2

Time 2

Input

a	b	b	b
-----	-----	-----	-----

$$a, \varepsilon \rightarrow a$$

$$b, \varepsilon \rightarrow b$$



q_0

$$a, a \rightarrow \varepsilon$$

$$b, b \rightarrow \varepsilon$$



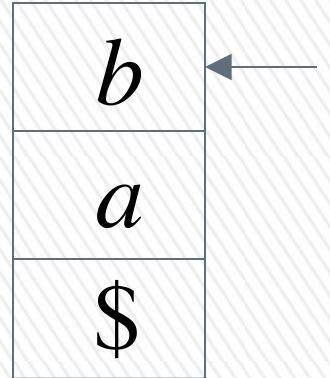
q_1

$$\varepsilon, \varepsilon \rightarrow \varepsilon$$

$$\varepsilon, \$ \rightarrow \$$$

q_2

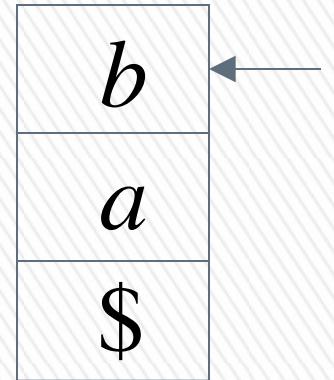
Stack



Time 3

Input

a	b	b	b
---	---	---	---



Guess the middle
of string

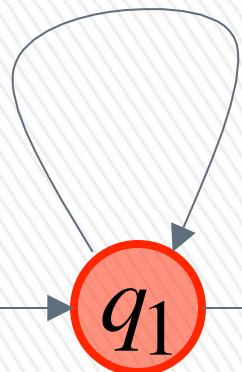
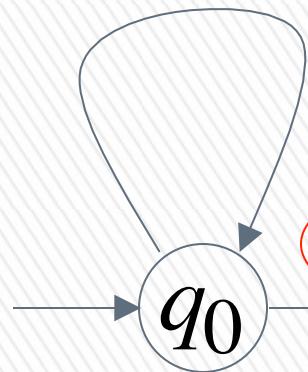
Stack

$$a, \epsilon \rightarrow a$$

$$b, \epsilon \rightarrow b$$

$$a, a \rightarrow \epsilon$$

$$b, b \rightarrow \epsilon$$



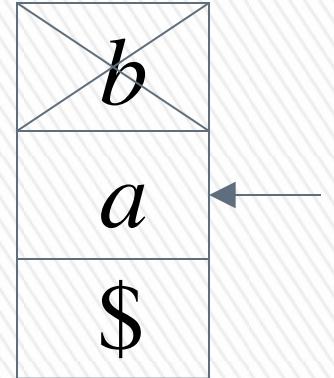
$$\epsilon, \$ \rightarrow \$$$



Time 4

Input

a	b	b	b
-----	-----	-----	-----



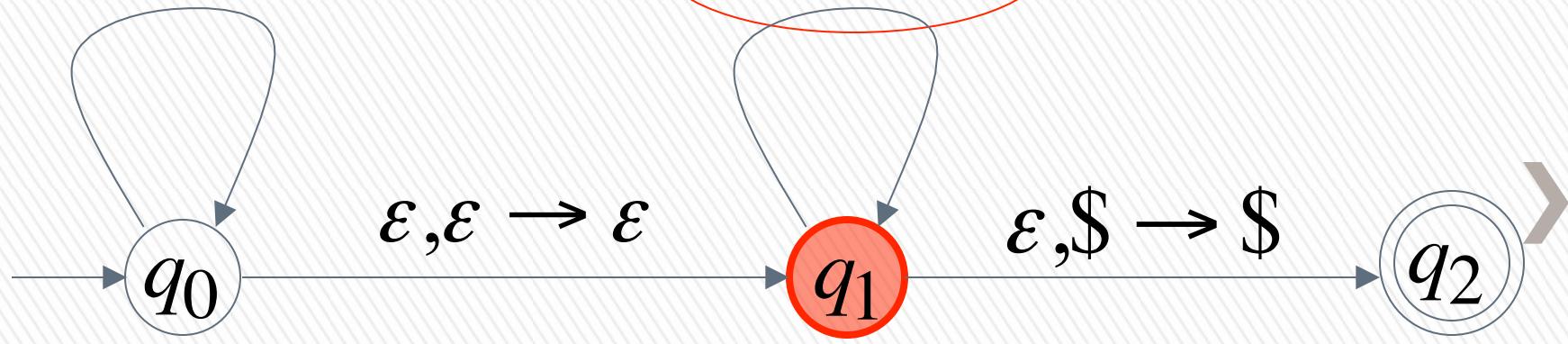
Stack

$$a, \varepsilon \rightarrow a$$

$$b, \varepsilon \rightarrow b$$

$$a, a \rightarrow \varepsilon$$

$$b, b \rightarrow \varepsilon$$



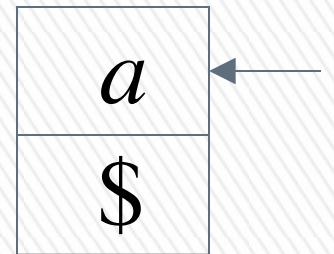
Time 5

Input

a	b	b	b
-----	-----	-----	-----

There is no possible transition.

Input is not consumed



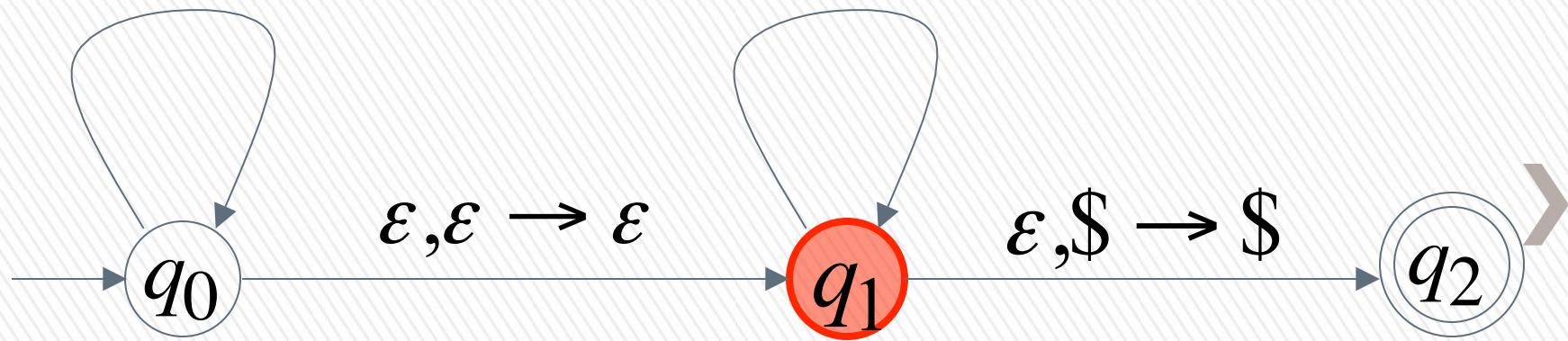
Stack

$$a, \varepsilon \rightarrow a$$

$$a, a \rightarrow \varepsilon$$

$$b, \varepsilon \rightarrow b$$

$$b, b \rightarrow \varepsilon$$



Another computation on same string:

Input

a	b	b	b
-----	-----	-----	-----

Time 0

\$

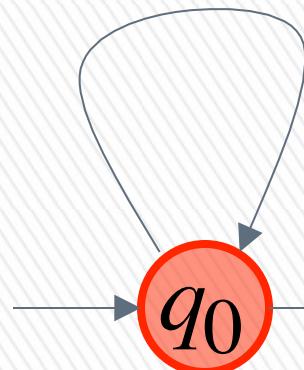
Stack

$$a, \epsilon \rightarrow a$$

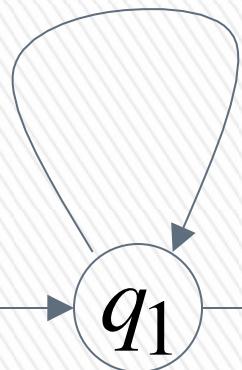
$$b, \epsilon \rightarrow b$$

$$a, a \rightarrow \epsilon$$

$$b, b \rightarrow \epsilon$$



$$\epsilon, \epsilon \rightarrow \epsilon$$



$$\epsilon, \$ \rightarrow \$$$



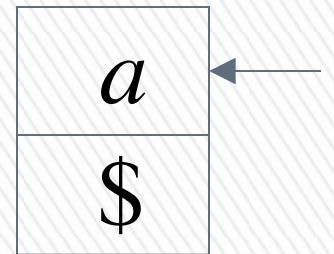
q_1

q_2

Time 1

Input

a	b	b	b
-----	-----	-----	-----



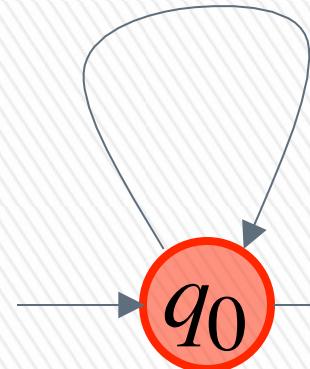
Stack

$$a, \varepsilon \rightarrow a$$

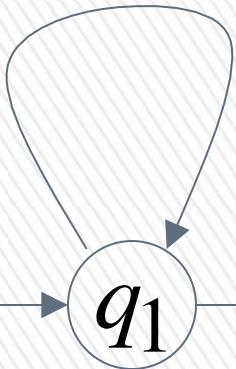
$$b, \varepsilon \rightarrow b$$

$$a, a \rightarrow \varepsilon$$

$$b, b \rightarrow \varepsilon$$



$$\varepsilon, \varepsilon \rightarrow \varepsilon$$



$$\varepsilon, \$ \rightarrow \$$$



Time 2

Input

a	b	b	b
-----	-----	-----	-----

$a, \epsilon \rightarrow a$

$b, \epsilon \rightarrow b$

q_0

$a, a \rightarrow \epsilon$

$b, b \rightarrow \epsilon$

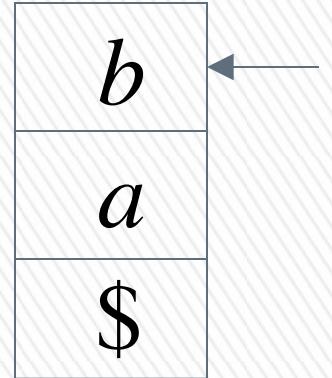
q_1

$\epsilon, \epsilon \rightarrow \epsilon$

$\epsilon, \$ \rightarrow \$$

q_2

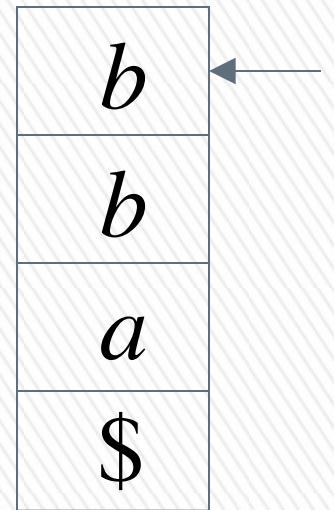
Stack



Time 3

Input

a	b	b	b
-----	-----	-----	-----



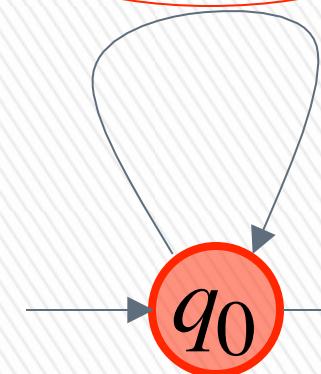
Stack

$$a, \varepsilon \rightarrow a$$

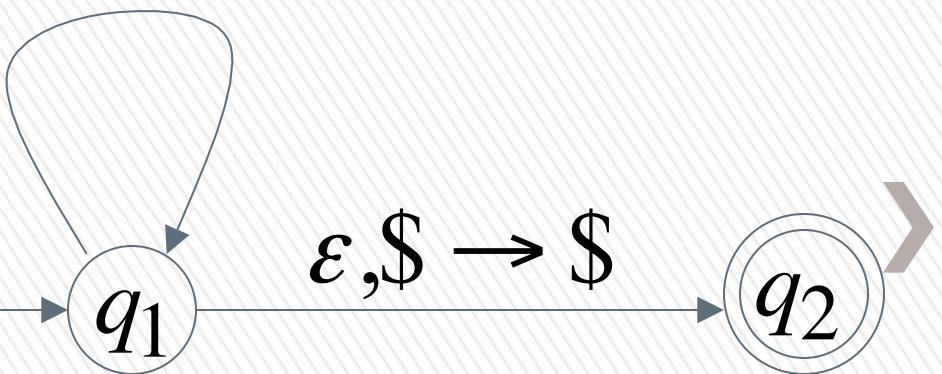
$$b, \varepsilon \rightarrow b$$

$$a, a \rightarrow \varepsilon$$

$$b, b \rightarrow \varepsilon$$



$$\varepsilon, \varepsilon \rightarrow \varepsilon$$



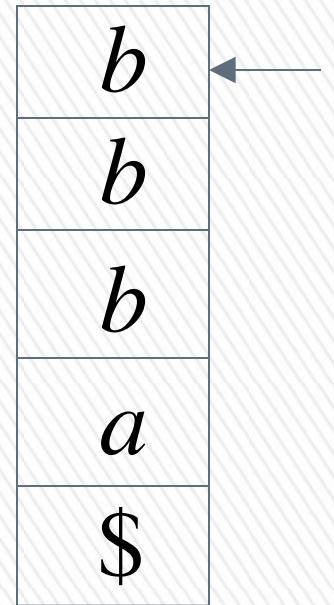
$$\varepsilon, \$ \rightarrow \$$$



Input

a	b	b	b
-----	-----	-----	-----

Time 4



Stack

$$a, \varepsilon \rightarrow a$$

$$b, \varepsilon \rightarrow b$$

$$a, a \rightarrow \varepsilon$$

$$b, b \rightarrow \varepsilon$$

q_0

q_1

q_2

$$\varepsilon, \varepsilon \rightarrow \varepsilon$$

$$\varepsilon, \$ \rightarrow \$$$

q_0

q_1

q_2

q_0

q_1

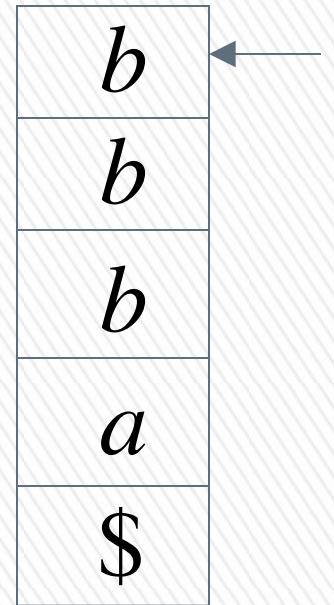
q_2

Input

a	b	b	b
-----	-----	-----	-----

Time 5

No accept state
is reached



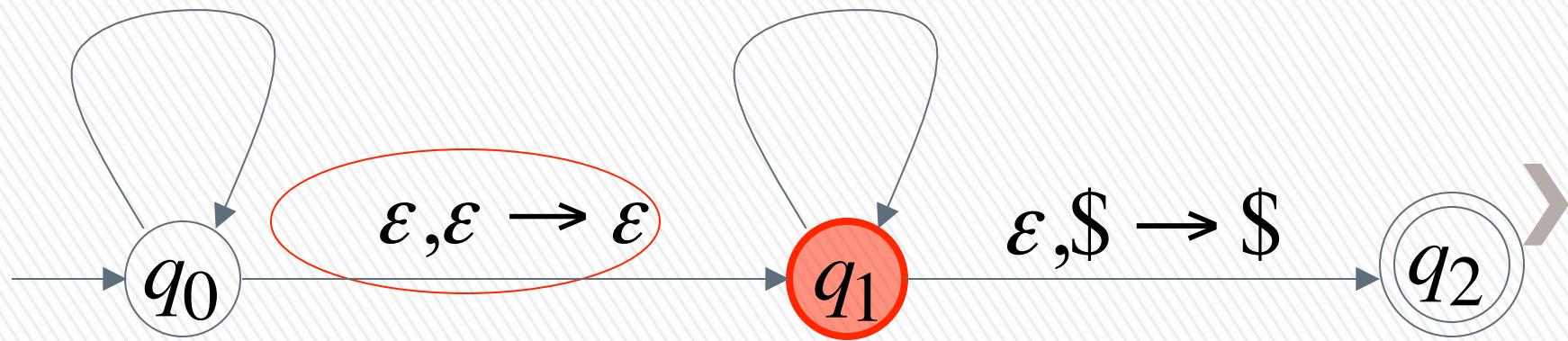
Stack

$$a, \epsilon \rightarrow a$$

$$a, a \rightarrow \epsilon$$

$$b, \epsilon \rightarrow b$$

$$b, b \rightarrow \epsilon$$



There is no computation
that accepts string $abbb$

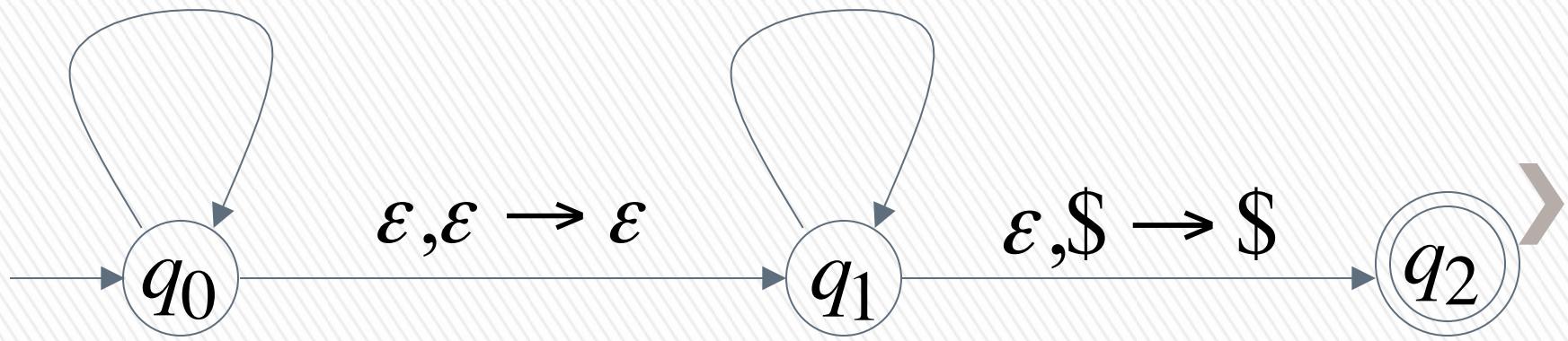
$$abbb \notin L(M)$$

$$a, \varepsilon \rightarrow a$$

$$b, \varepsilon \rightarrow b$$

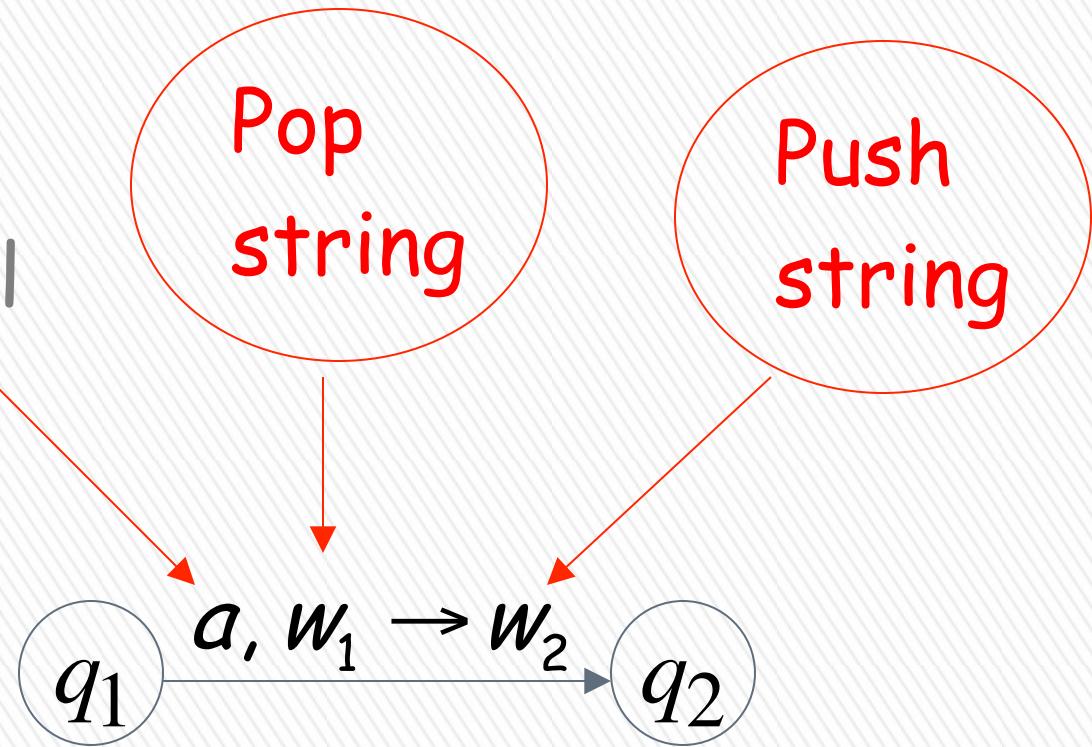
$$a, a \rightarrow \varepsilon$$

$$b, b \rightarrow \varepsilon$$

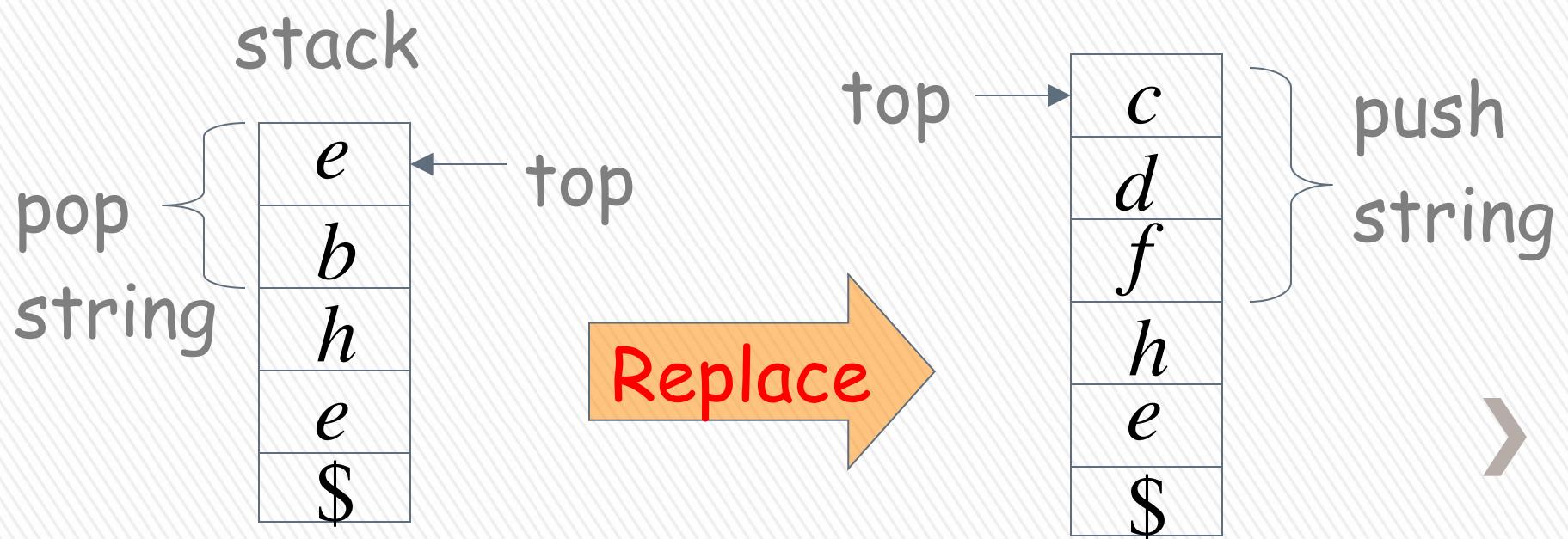
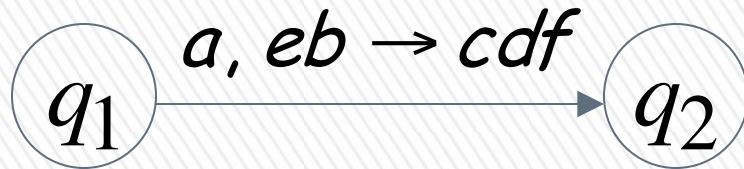


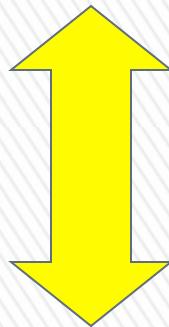
Pushing & Popping Strings

Input
symbol



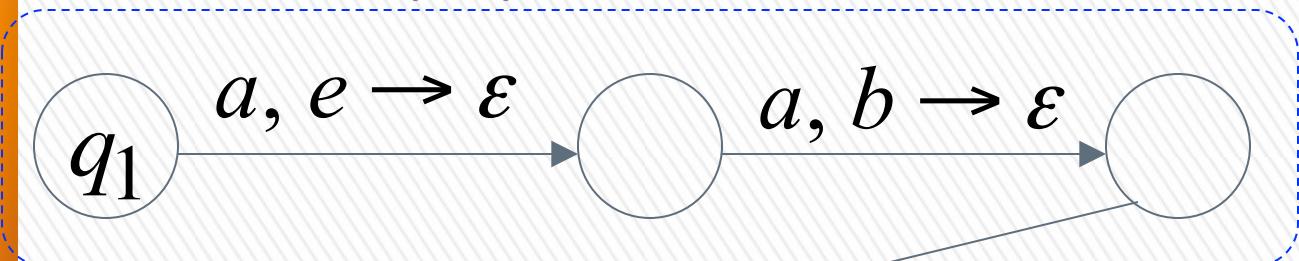
Example:





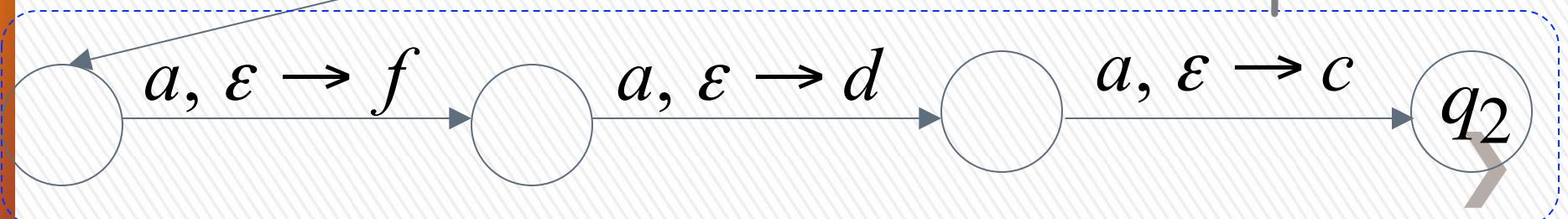
Equivalent
transitions

pop



$\varepsilon, \varepsilon \rightarrow \varepsilon$

push



Another PDA example

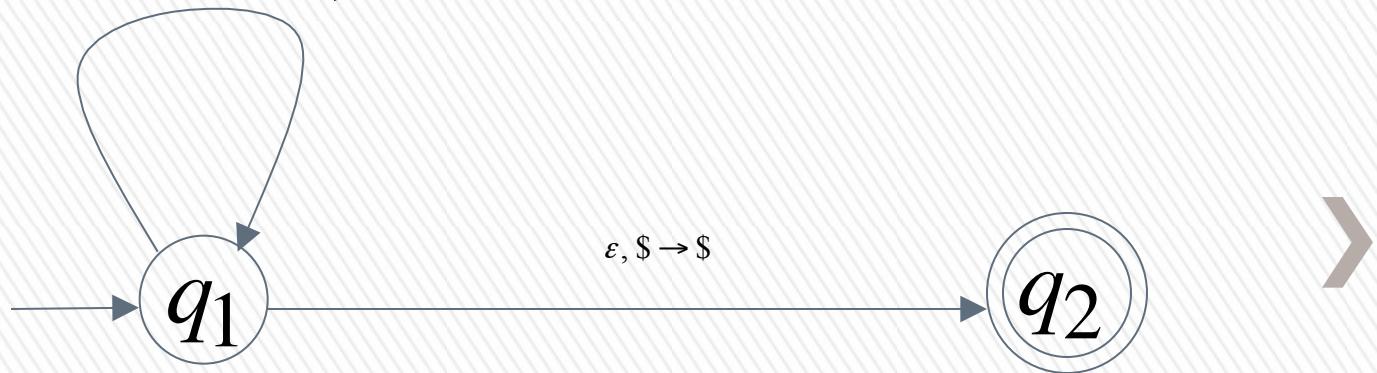
$$L(M) = \{w \in \{a,b\}^*: n_a(w) = n_b(w)\}$$

PDA M

$$a, \$ \rightarrow 0\$ \quad b, \$ \rightarrow 1\$$$

$$a, 0 \rightarrow 00 \quad b, 1 \rightarrow 11$$

$$a, 1 \rightarrow \epsilon \quad b, 0 \rightarrow \epsilon$$



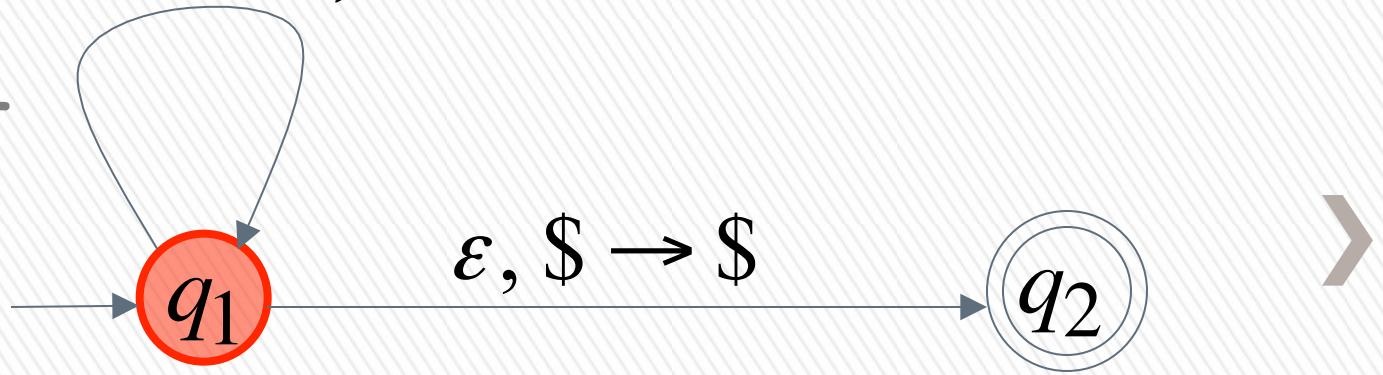
Execution Example: Time 0

Input

a	b	b	b	a	a
-----	-----	-----	-----	-----	-----

$$\begin{array}{ll} a, \$ \rightarrow 0\$ & b, \$ \rightarrow 1\$ \\ a, 0 \rightarrow 00 & b, 1 \rightarrow 11 \\ a, 1 \rightarrow \epsilon & b, 0 \rightarrow \epsilon \end{array}$$

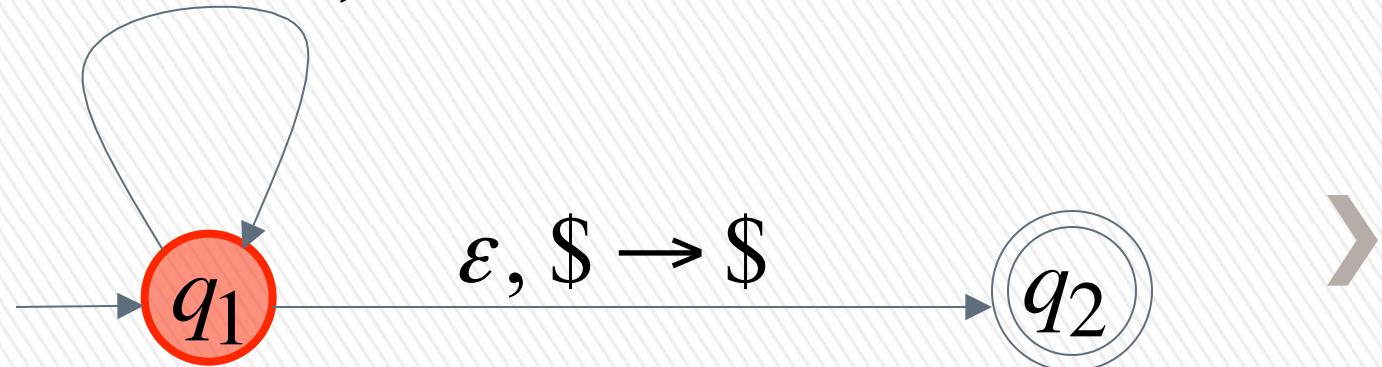
current
state



Time 1
Input

a	b	b	b	a	a
-----	-----	-----	-----	-----	-----

$$\begin{array}{ll} a, \$ \rightarrow 0\$ & b, \$ \rightarrow 1\$ \\ a, 0 \rightarrow 00 & b, 1 \rightarrow 11 \\ a, 1 \rightarrow \epsilon & b, 0 \rightarrow \epsilon \end{array}$$

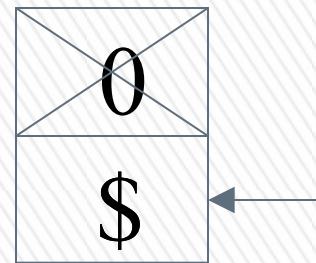


Time 3

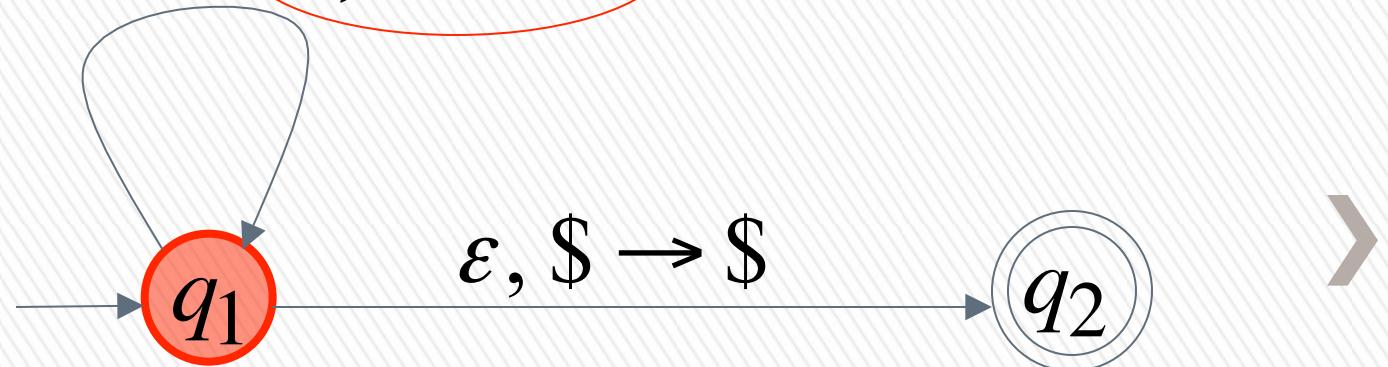
Input

a	b	b	b	a	a
-----	-----	-----	-----	-----	-----

$$\begin{array}{ll} a, \$ \rightarrow 0\$ & b, \$ \rightarrow 1\$ \\ a, 0 \rightarrow 00 & b, 1 \rightarrow 11 \\ a, 1 \rightarrow \epsilon & b, 0 \rightarrow \epsilon \end{array}$$



Stack



Time 4

Input

a	b	b	b	a	a
-----	-----	-----	-----	-----	-----



$a, \$ \rightarrow 0\$$

$a, 0 \rightarrow 00$

$a, 1 \rightarrow \epsilon$

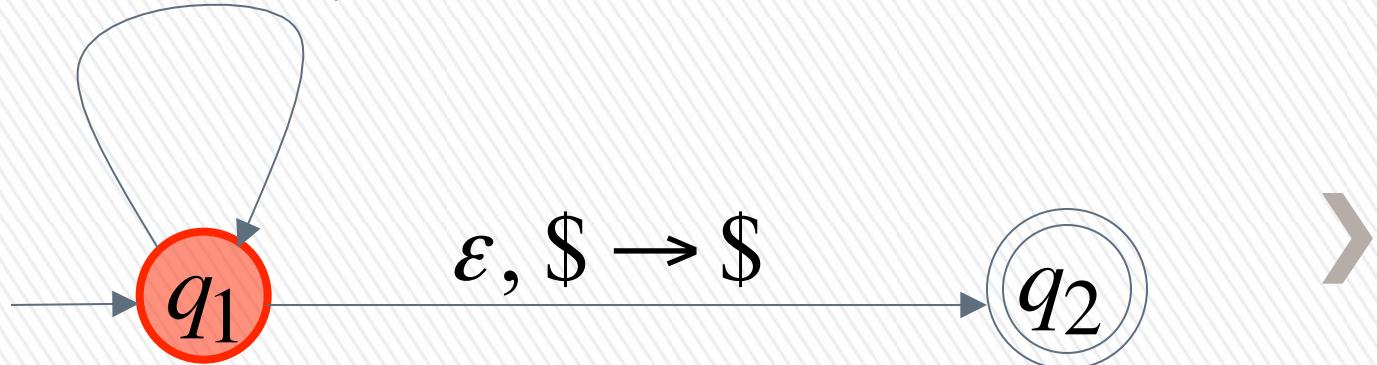
$b, \$ \rightarrow 1\$$

$b, 1 \rightarrow 11$

$b, 0 \rightarrow \epsilon$

1
\$

Stack



Time 5

Input

a	b	b	b	a	a
-----	-----	-----	-----	-----	-----



$a, \$ \rightarrow 0\$$

$a, 0 \rightarrow 00$

$a, 1 \rightarrow \epsilon$

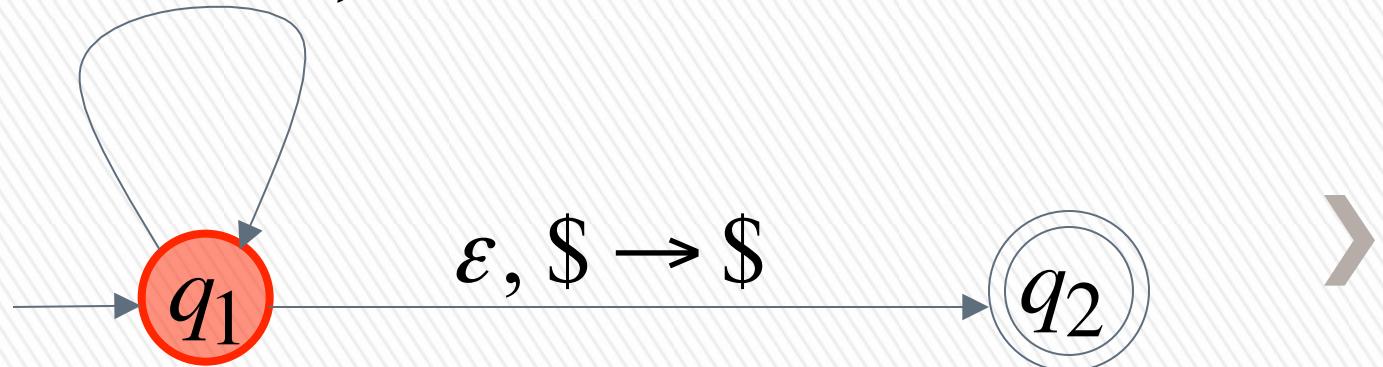
$b, \$ \rightarrow 1\$$

$b, 1 \rightarrow 11$

$b, 0 \rightarrow \epsilon$

1
1
\$

Stack



Time 6

Input

a	b	b	b	a	a
-----	-----	-----	-----	-----	-----



$a, \$ \rightarrow 0\$$

$a, 0 \rightarrow 00$

$a, 1 \rightarrow \epsilon$

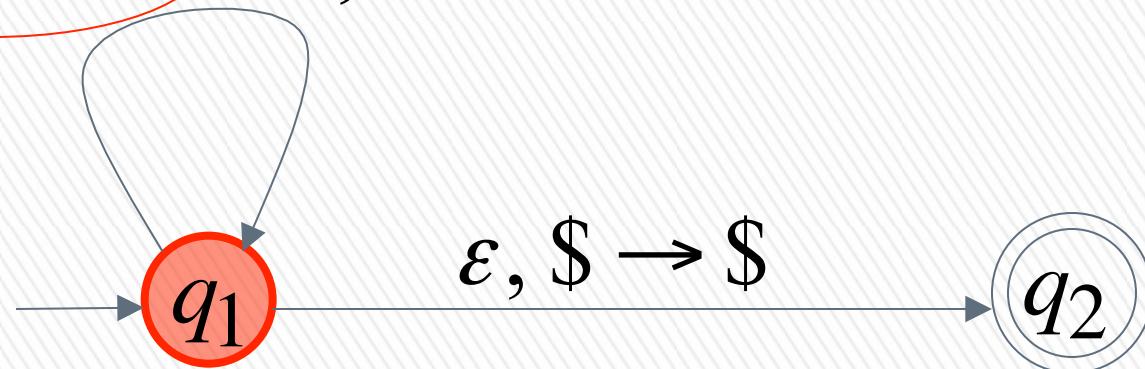
$b, \$ \rightarrow 1\$$

$b, 1 \rightarrow 11$

$b, 0 \rightarrow \epsilon$

1
1
\$

Stack



Time 7

Input

a	b	b	b	a	a
-----	-----	-----	-----	-----	-----



$a, \$ \rightarrow 0\$$

$a, 0 \rightarrow 00$

$a, 1 \rightarrow \epsilon$

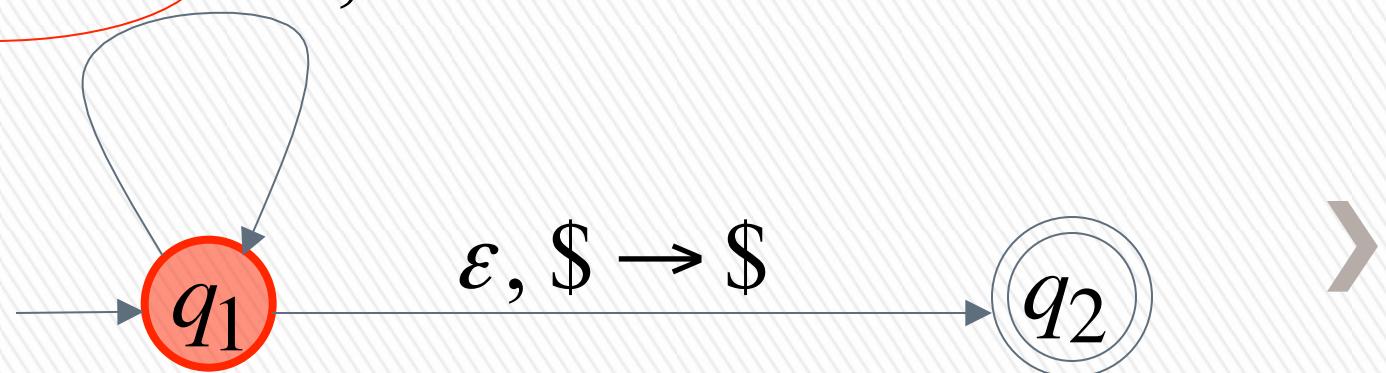
$b, \$ \rightarrow 1\$$

$b, 1 \rightarrow 11$

$b, 0 \rightarrow \epsilon$

1
$\$$

Stack



Time 8

Input

a	b	b	b	a	a
-----	-----	-----	-----	-----	-----



$a, \$ \rightarrow 0\$$

$a, 0 \rightarrow 00$

$a, 1 \rightarrow \epsilon$

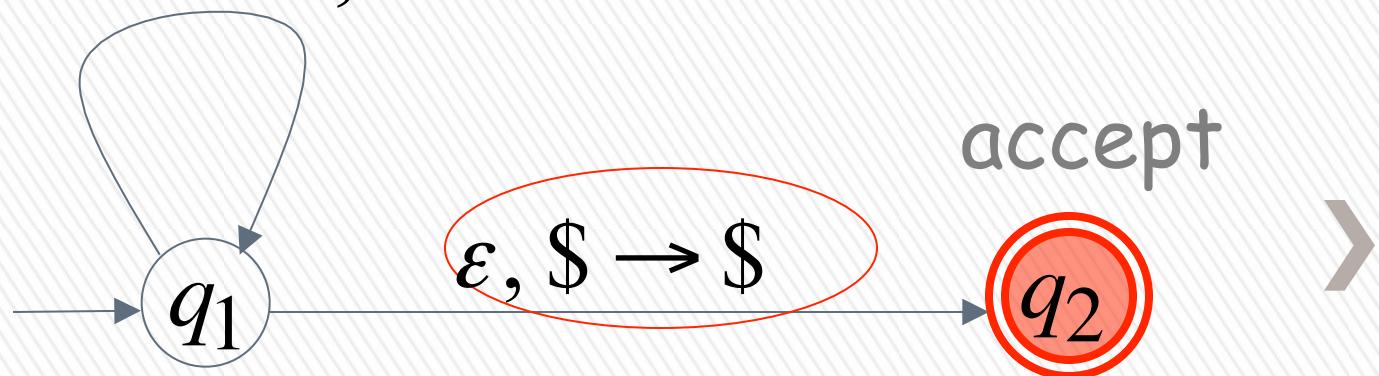
$b, \$ \rightarrow 1\$$

$b, 1 \rightarrow 11$

$b, 0 \rightarrow \epsilon$

\$

Stack



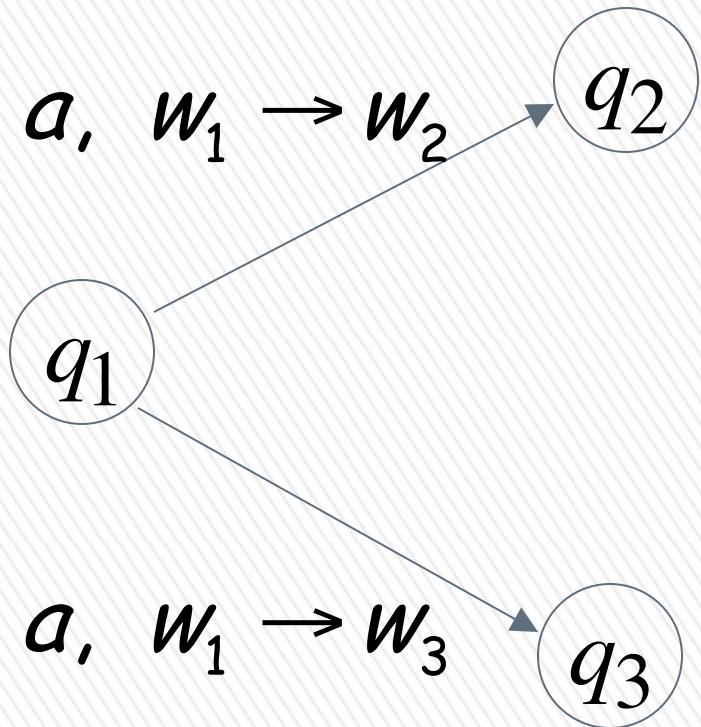
Formalities for PDAs



Transition function:

$$\delta(q_1, a, w_1) = \{(q_2, w_2)\}$$





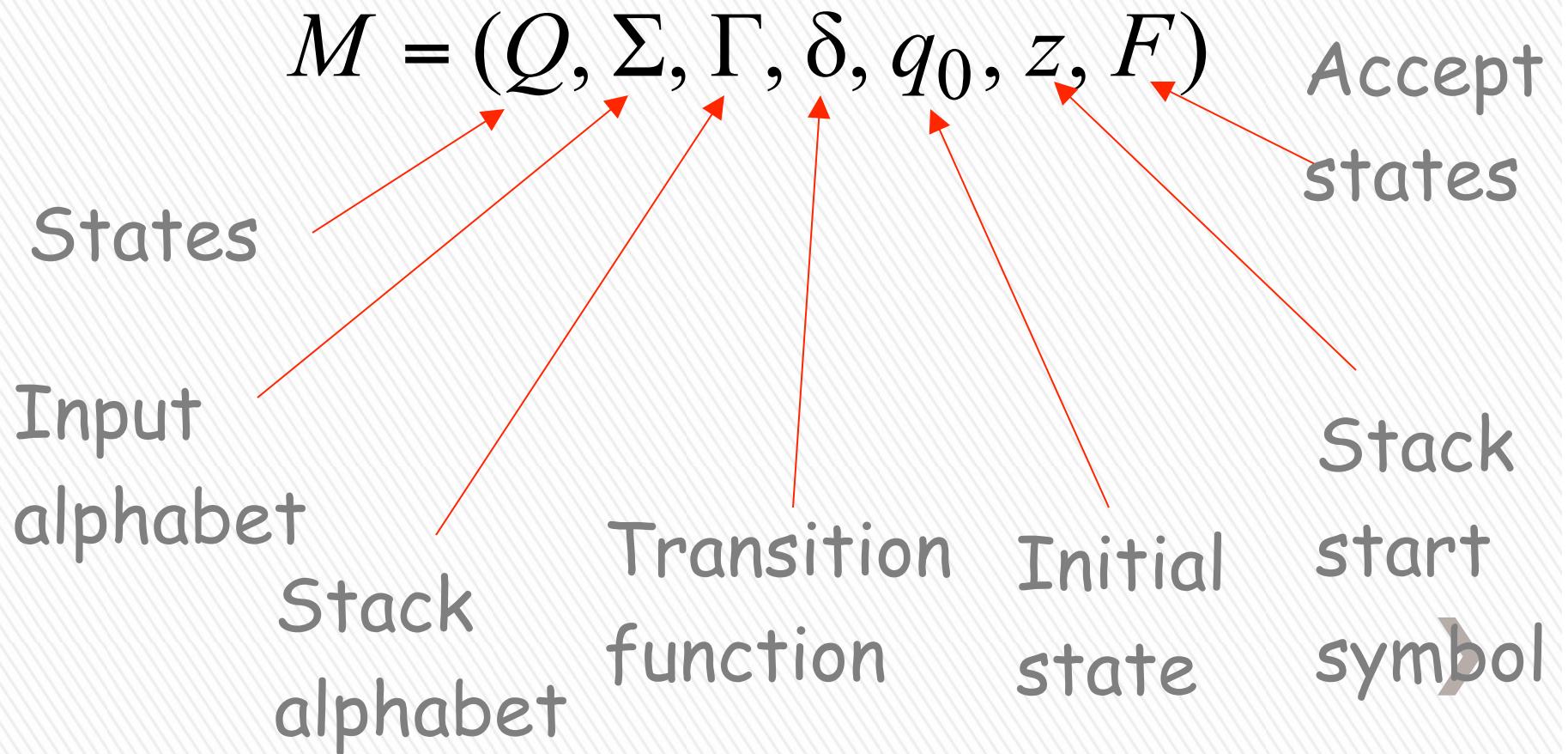
Transition function:

$$\delta(q_1, a, w_1) = \{(q_2, w_2), (q_3, w_3)\}$$

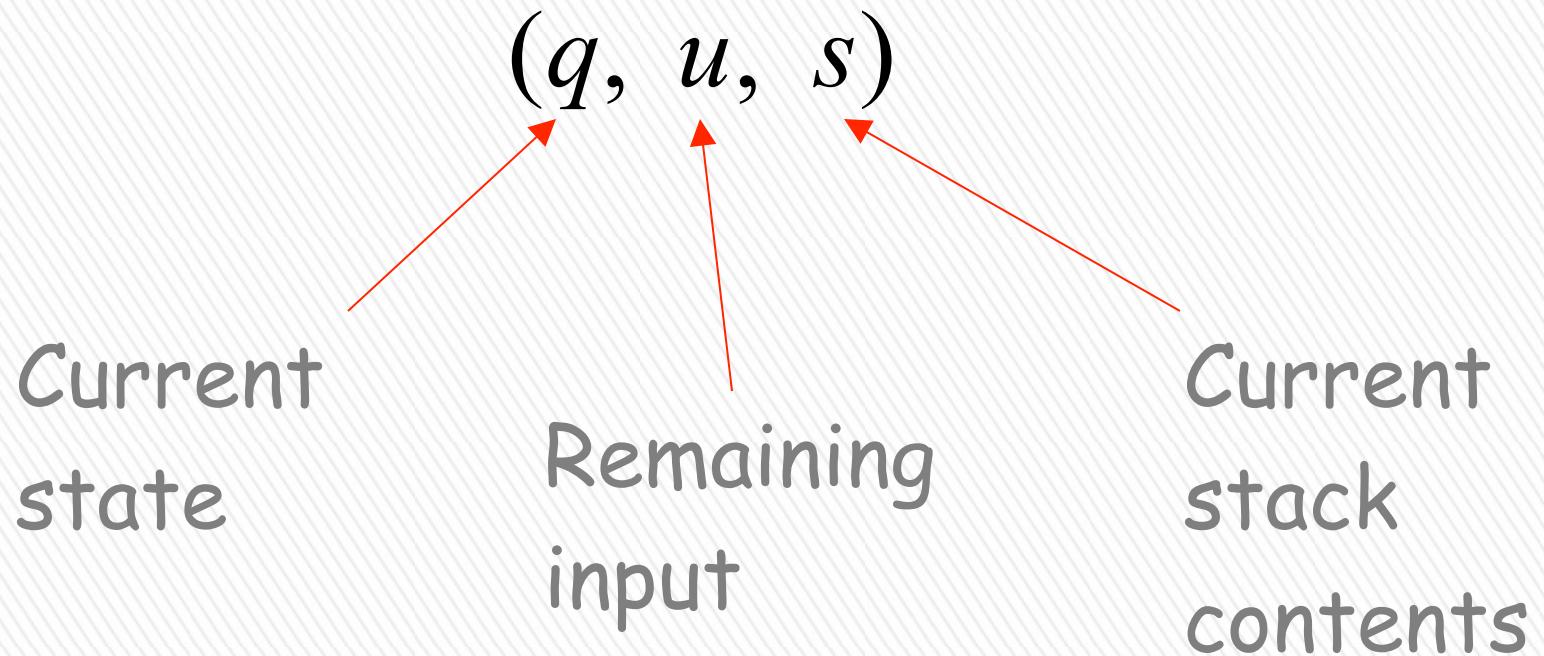


Formal Definition

Pushdown Automaton (PDA)



Instantaneous Description



Example:

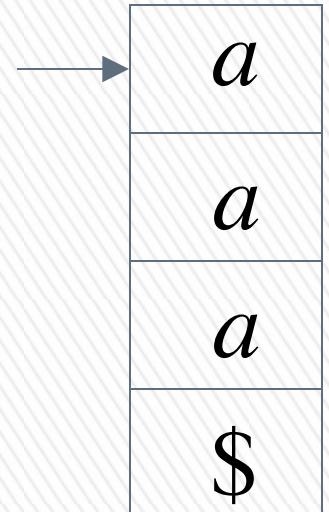
Instantaneous Description

$$(q_1, bbb, aaa\$)$$

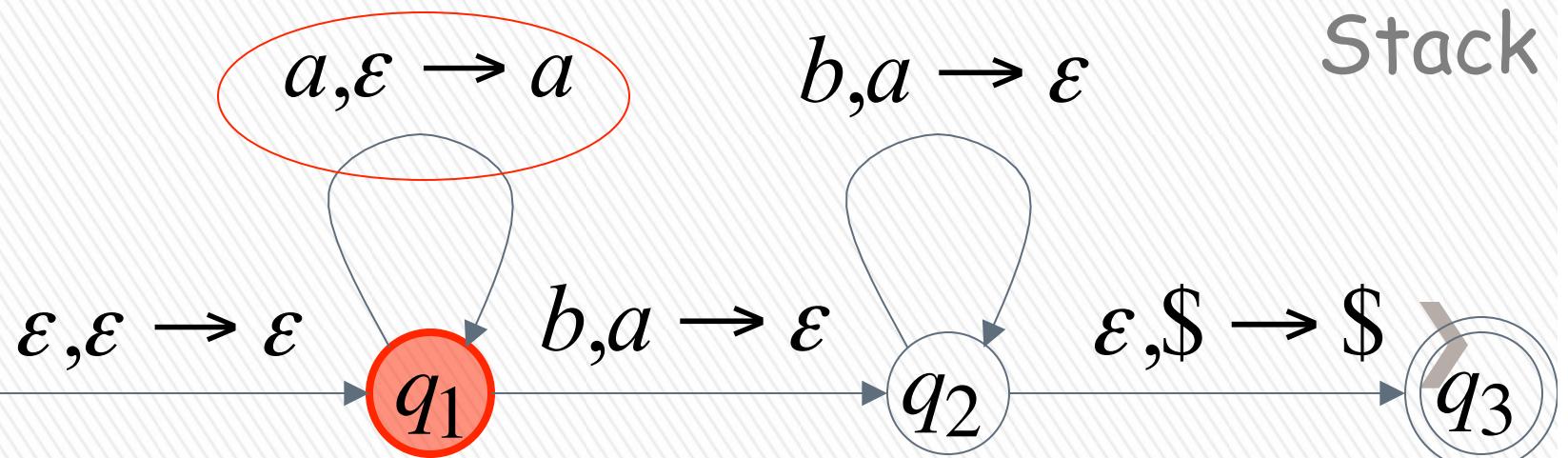
Time 4:

Input

a	a	a	b	b	b
---	---	---	---	---	---



Stack



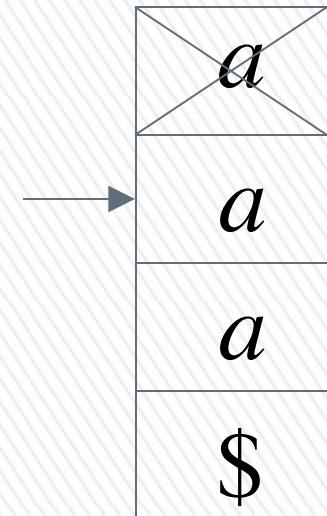
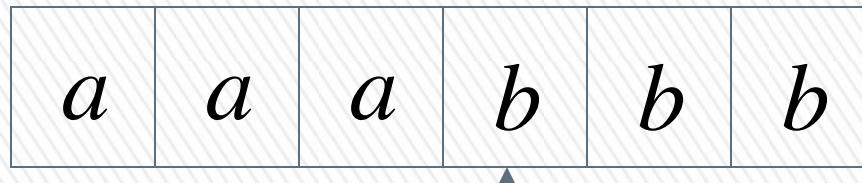
Example:

Instantaneous Description

($q_2, bb, aa\$$)

Time 5:

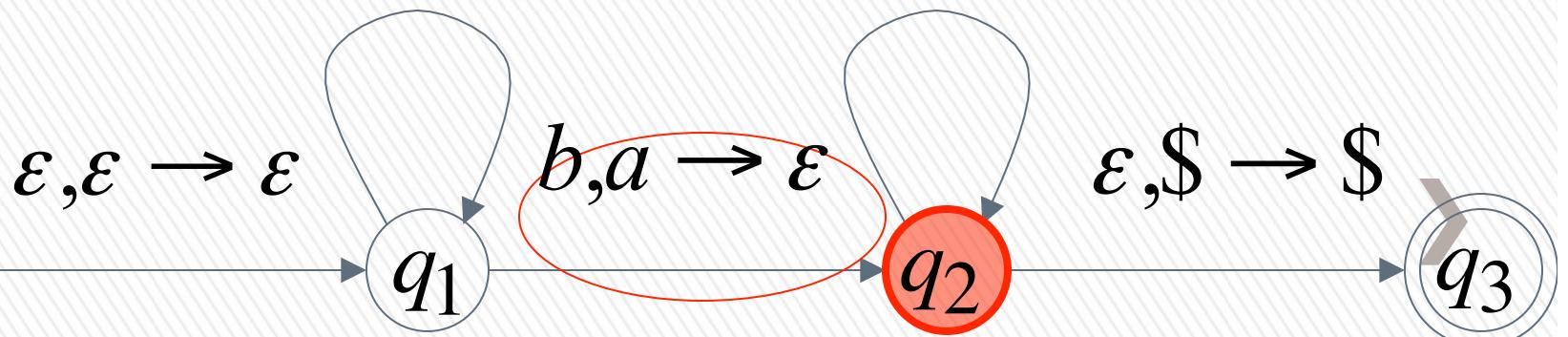
Input



$a, \mathcal{E} \rightarrow a$

$b,a \rightarrow \varepsilon$

Stack



We write:

$$(q_1, bbb, aaa\$) \succ (q_2, bb, aa\$)$$

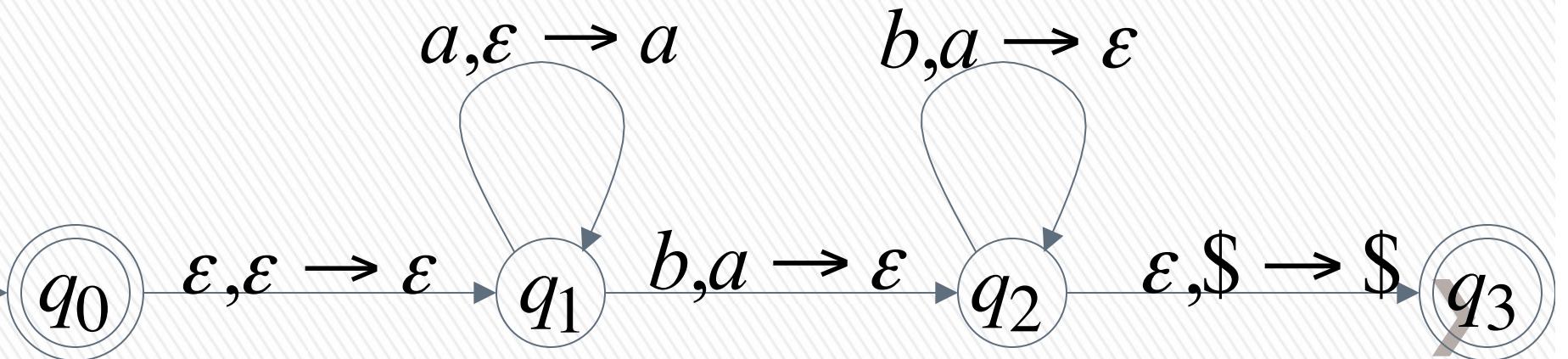
Time 4

Time 5



A computation:

$(q_0, aaabbb, \$) \succ (q_1, aaabbb, \$) \succ$
 $(q_1, aabbb, a\$) \succ (q_1, abbb, aa\$) \succ (q_1, bbb, aaa\$) \succ$
 $(q_2, bb, aa\$) \succ (q_2, b, a\$) \succ (q_2, \epsilon, \$) \succ (q_3, \epsilon, \$)$



$(q_0, aaabbb, \$) \succ (q_1, aaabbb, \$) \succ$
 $(q_1, aabbb, a\$) \succ (q_1, abbb, aa\$) \succ (q_1, bbb, aaa\$) \succ$
 $(q_2, bb, aa\$) \succ (q_2, b, a\$) \succ (q_2, \varepsilon, \$) \succ (q_3, \varepsilon, \$)$

For convenience we write:

$$(q_0, aaabbb, \$) \stackrel{*}{\succ} (q_3, \varepsilon, \$)$$


Language of PDA

Language $L(M)$ accepted by PDA M :

$$L(M) = \{w : (q_0, w, z) \xrightarrow{*} (q_f, \epsilon, s)\}$$

The equation $L(M) = \{w : (q_0, w, z) \xrightarrow{*} (q_f, \epsilon, s)\}$ is displayed. Two blue arrows point from the text "Initial state" to the initial state q_0 and from the text "Accept state" to the final state q_f .



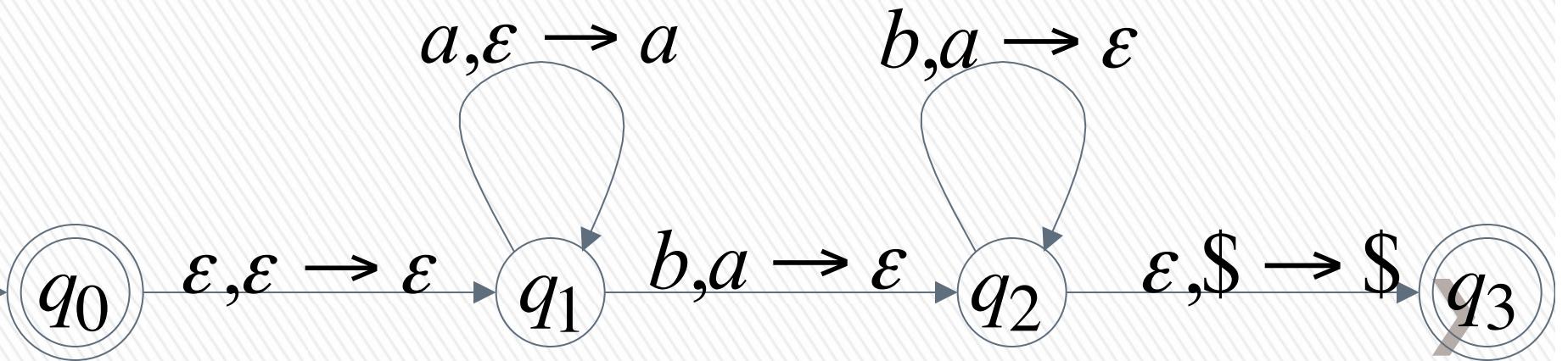
Example:

$$(q_0, aaabbb, \$) \xsucc{*} (q_3, \epsilon, \$)$$



$$aaabbb \in L(M)$$

PDA M :

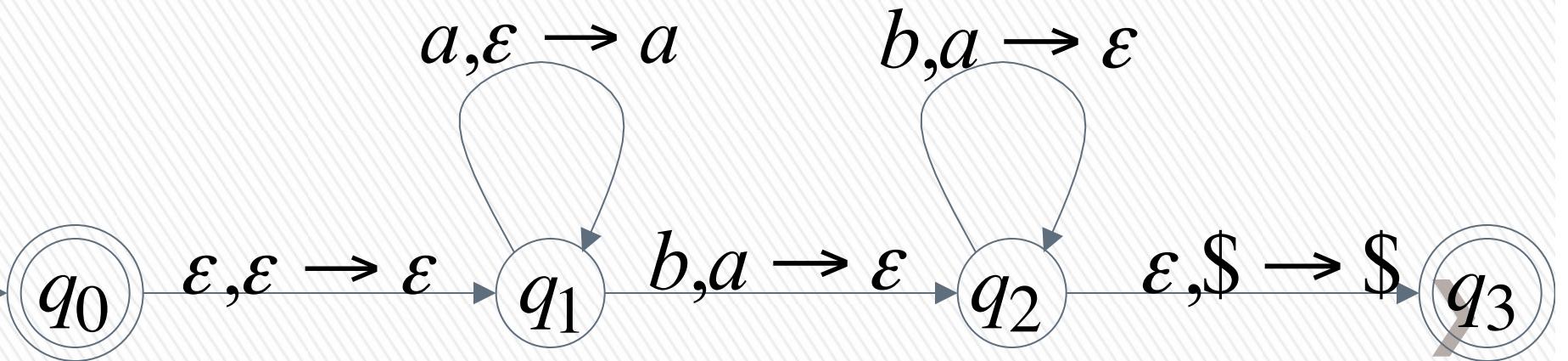


$$(q_0, a^n b^n, \$) \xrightarrow{*} (q_3, \epsilon, \$)$$



$$a^n b^n \in L(M)$$

PDA M :



Therefore:

$$L(M) = \{a^n b^n : n \geq 0\}$$

PDA M :

