Final Project of SEEM5360

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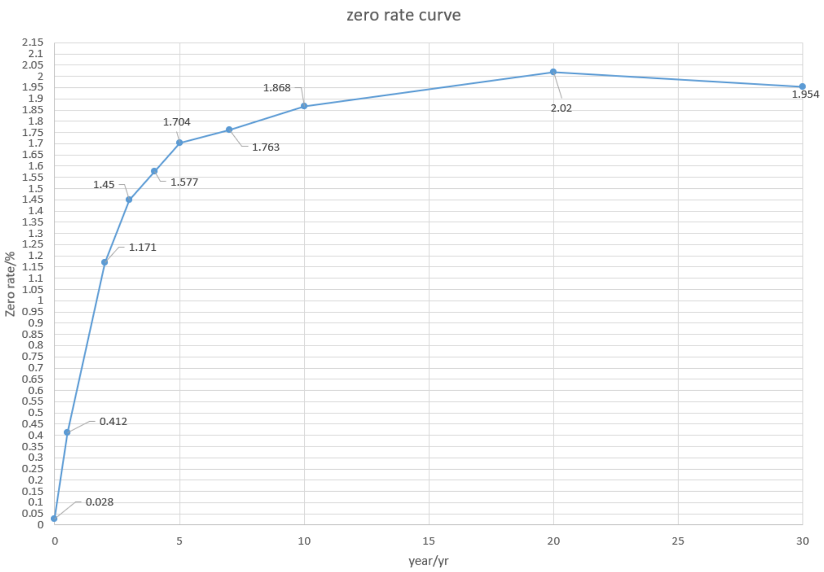
1. *Overview of the project*

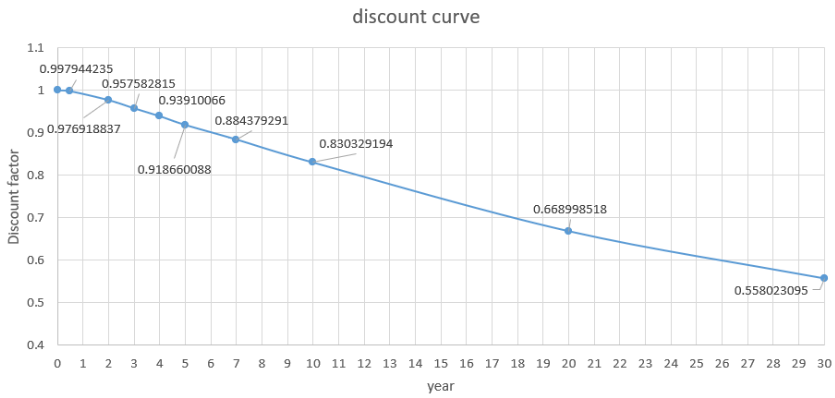
The whole project contains the formulation of zero rate curve and discount factor, the construction of calibrated lattice tree, pricing the EU/AM type coupon bond option, the impact of parameters, obtaining results via Monte Carlo and other self-thinking.

First is the zero rate and discount curve, we will implement the linear extrapolation in part one and try the cubic spline extrapolation or other extrapolation and analysis the change it makes for pricing in part three; part three would also have another thinking about the reason of so many -ve parts of part 2 interest tree, the method to obtain the -ve parts as small as possible in our interest tree (change the sigma in model), and how this effect the pricing procedure; and also the analysis about how we can decrease the variance in Monte Carlo simulation.

2. *Do the linear extrapolation*

After doing the ***linear extrapolation***, we obtain the corresponding zero rate and discount curve as (from Excel):





3. *Construct the calibrated lattice interest rate tree for part two*

Next, let us firstly assume some parameters in part two and part three and then construct the calibrated lattice tree for the Hull-White Model.

We set the question situation as pricing a $T_0 = 1\,yr$ maturity EU/AM call option on the post-coupon price of a $T_N = 4yr$ coupon paying bond (face value 100 dollars) with the payoff of the form, where $K = 99.2$

$$V_{T_0} = \max\left(B^{c}_{\cdot,T_0} - K, 0\right).$$

For part two (a), we will assume $(1)\,\Delta t = \Delta T = 1yr$, now the EU and AM type option is the same because the maturity of option is the same as the time step of lattice tree and the coupon payment period; For part two (b), we will assume $(2)\,\Delta t = 0.5yr, \Delta T = 1yr$, where the EU and AM may not be the same because the time step of lattice tree is smaller than the maturity of option and the coupon payment period, so AM option may be exercised before the 1yr maturity (under the below coupon rate assumption (3)).

For part two (a), we will assume the $(3)\,c_1 = c_2 = c_3 = 2\%$, and thus the $\overline{r}$ and $\underline{r}$ can be any arbitrarily non-negative numbers, thus the lattice tree for part two (a) has $t_n = t_4$ and we do not need to know each interest rate on every t_4 nodes; for part two (b), we will change $(4)\,c_1,\ c_2,\ c_3$ as three different non-negative numbers and thus we would also set the $\overline{r} = 10\%$ and $\underline{r} = 3\%$ (under the above time step assumption (1)) as:

$$c(T_j) = \begin{cases} c_1 = 5\%, & if\ f_{\Delta T}(T_j, T_j) \geq \bar{r} \\ c_2 = 3\%, & if\ \underline{r} \leq f_{\Delta T}(T_j, T_j) \leq \bar{r} \\ c_3 = 1\%, & if\ f_{\Delta T}(T_j, T_j) \leq \bar{r} \end{cases}$$

and, we also need to check the tree up to $t_n = t_5$ in order to get each coupon rate for all t_4 nodes. And the $f_{\Delta T}(T_j, T_j) = 1/\Delta T * (\exp(r(t)dt) - 1)$ (under assumption (1)) and under assumption (2), $f_{\Delta T}(T_j, T_j) = 1/\Delta T * (\exp(r(t)dt + \mathbb{E}(r(t+dt))dt) - 1)$, which is our own assumption for calculating the forward rate.

For part two (b), we can also change other parameters such like $K = 98\ or\ 99.85$ (under above time step assumption (1) and coupon rate assumption (3)) and $T_0 = 1, T_N = 5$ (under the same assumption of (1), (3), and (a)'s $K = 99.2$).

Now we construct the calibrated lattice tree for part two (a) first, with assumption (1) and (3), the most important is obtaining the calibrated $\theta\,(t)$, since the $k, \delta r, P(*), \sigma, \kappa$ are known or can be calculated via current known numbers in these formulas (in Lecture Note 05) [1]:

$$J = \left[\frac{1}{2\kappa\Delta t}\right] + 1, \quad k(j) = \begin{cases} -J+1, & \text{if } j = -J, \\ j, & \text{if } |j| < J, \\ J-1, & \text{if } j = J. \end{cases}, \quad \delta r = \sigma\sqrt{3\Delta t}.$$

$$p^{(k+1)}_{j,n} = \frac{1}{2}\left(\frac{\sigma^2\Delta t + \eta^2_{j,n}}{(\delta r)^2} + \frac{\eta_{j,n}}{\delta r}\right),$$

$$p^{(k-1)}_{j,n} = \frac{1}{2}\left(\frac{\sigma^2\Delta t + \eta^2_{j,n}}{(\delta r)^2} - \frac{\eta_{j,n}}{\delta r}\right),$$

$$p^{(k)}_{j,n} = 1 - \frac{\sigma^2\Delta t + \eta^2_{j,n}}{(\delta r)^2},$$

where

$$\begin{aligned} \eta_{j,n} &= \kappa\left(r_{0,n} - r_{j,n}\right)\Delta t - \left(k(j) - j\right)\delta r \\ &= \kappa\left(r_{0,n} - \left(r_{0,n} + j\delta r\right)\right)\Delta t - \left(k(j) - j\right)\delta r \\ &= \left(\left(1 - \kappa\Delta t\right)j - k(j)\right)\delta r. \end{aligned}$$

By using the part one discount curve, we obtain the corresponding calibrated $\theta(t), t = 0, 1, 2, 3$ for part two (a) as, via the algorithm in "Answer to Question 2" part in P41 of Lecture Note 05[2]

$$\{\theta(t), t = 0, 1, 2, 3\} = \{0.155557, 0.142642, 0.150107, 0.177970\}$$

Then we have the final interest tree with each probability for part 2 (a) as:

The interest tree follows up → middle → down in each row in this matrix figure 1; the same order

	1	2	3	4	5	6	7	8	9
1	2.800000000000000e-04	0	0	0	0	0	0	0	0
2	0.077265061350481	0.023571486315846	-0.030122088718789	0	0	0	0	0	0
3	0.128985799902704	0.075292224868068	0.021598649833433	-0.032094925201202	-0.085788500235837	0	0	0	0
4	0.183768742874441	0.130075167839805	0.076381592805170	0.022688017770535	-0.031005557264100	-0.084699132298735	-0.138392707333371	0	0
5	0.241615946708898	0.187922371674263	0.134228796639628	0.080535221604992	0.026841646570357	-0.026851928464278	-0.080545503498913	-0.134239078533548	-0.187932653568184

Figure 1 the interest rate tree for part 2 (a)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	0.166666...	0.666666...	0.166666...	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0.102916...	0.644166...	0.252916...	0.166666...	0.666666...	0.166666...	0.252916...	0.644166...	0.102916...	0	0	0	0	0	0	0	0	0	0	0	0
3	0.061666...	0.576666...	0.361666...	0.102916...	0.644166...	0.252916...	0.166666...	0.666666...	0.166666...	0.252916...	0.644166...	0.102916...	0.361666...	0.576666...	0.061666...	0	0	0	0	0	0
4	0.042916...	0.464166...	0.492916...	0.061666...	0.576666...	0.361666...	0.102916...	0.644166...	0.252916...	0.166666...	0.666666...	0.166666...	0.252916...	0.644166...	0.102916...	0.361666...	0.576666...	0.061666...	0.492916...	0.464166...	0.042916...

Figure 2 the probability from each node for part 2 (a)

is in every three probabilities in each row in figure 2.

The tree of part 2 (a) is under such condition: Discount curve in part 1/ (1)/ (3)/ Strick price K=99.2. And then we can obtain the EU&AM option price (they are the same in part 2 (a) under such condition and the reason is stated in assumption (1)) via the same procedure in class activity on 21 Feb, 2022.

4. Obtain each case results from different interest rate lattice trees

Part 2 (a) (EU&AM option price, same) result is 2.56 dollars, it comes from

	1	2	3	4	5	6	7	8	9
1	2.555762270276741	0	0	0	0	0	0	0	0
2	86.771779634115430	99.376829310639850	1.138315506608865e+02	0	0	0	0	0	0
3	81.795826946413240	90.251538631036210	99.585491201953600	1.098891848309941e+02	1.212636781894975e+02	0	0	0	0
4	84.877078150007260	89.559001744181360	94.499185978559650	99.711876826391660	1.052121060862607e+02	1.110157347291721e+02	1.171394986367194e+02	0	0
5	102	102	102	102	102	102	102	102	102

Figure 3 the value of each node under assumption (1)+(3)

Then we move to part 2 (b), here we change several different parameters to analysis the influence they cause in pricing procedure. We will change to assumption ①(1)+(4), ②(2)+(3), ③(1)+(3)+different K, and ④(1)+(3)+different T_0, T_N and ⑤(2)+(4) to analysis the change.

Part 2 (b) under above ① (interest rate tree and probability matrix are the same as part 2 (a)) (EU&AM option price, same) result is 1.96 dollars, it comes from

	1	2	3	4	5	6	7	8	9
1	1.955056251237486	0	0	0	0	0	0	0	0
2	88.113286734001630	97.831960682037770	1.109336224617992e+02	0	0	0	0	0	0
3	85.151713840664870	91.326161126120580	98.445414197491490	1.078845740748875e+02	1.190066069730351e+02	0	0	0	0
4	87.373462801478070	91.557982077229600	95.147709803902710	99.060165213147290	1.041806148501209e+02	1.099273451730038e+02	1.159910721794967e+02	0	0
5	105	105	105	103	101	101	101	101	101

Figure 4 the value of each node under assumption (1)+(4)

Part 2 (b) under above ② (interest rate tree and probability matrix are different from part 2 (a)) (EU and AM option price, maybe different) results are 3.67 dollars (for EU option) and $max\,(3.67, 3.73) = 3.73$ dollars (for AM option), they are from

$$\{\theta(t), t = 0, 1, 2, 3, 4, 5, 6\}$$
$$= \{0.33255, 0.14491, 0.15236, 0.28243, 0.29453, 0.29625, 0.30958\}$$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	3.667989...	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0.281300...	2.766988...	10.661764152509384	0	0	0	0	0	0	0	0	0	0	0	0
3	83.55632...	91.68048...	1.006026846641974e+02	1.104016994830190e+02	1.211641157448663e+02	0	0	0	0	0	0	0	0	0	0
4	79.54551...	86.14272...	93.296425856139320	1.010538117026404e+02	1.094660937809864e+02	1.185888368231521e+02	1.284823323588495e+02	0	0	0	0	0	0	0	0
5	76.11309...	81.39773...	87.051066402884430	93.098890821948600	99.568809185010280	1.064903609682756e+02	1.138951568610988e+02	1.218170...	1.302921...	0	0	0	0	0	0
6	77.53806...	81.67794...	86.041004499734480	90.639324845358000	95.485634625453510	1.005933536306088e+02	1.059766289427552e+02	1.116503...	1.176303...	1.239330...	1.305759...	0	0	0	0
7	80.06873...	83.04883...	86.139854644669010	89.345916227255160	92.671304244352380	96.120460152149040	99.697990743072000	1.034086...	1.072574...	1.112495...	1.153901...	1.196848...	1.241394...	0	0
8	88.26975...	89.96143...	91.685536621368370	93.442678803543570	95.233496402389240	97.058634803178920	98.918751759916650	1.008145...	1.027466...	1.047157...	1.067226...	1.087679...	1.108524...	1.129769...	1.151421...
9	102	102	102	102	102	102	102	102	102	102	102	102	102	102	102

Figure 5 the value of each node under assumption (2)+(3) (EU option)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	2.800000000000000e-04	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0.063167322327860	0.025200231314720	-0.012766859698419	0	0	0	0	0	0	0	0	0	0	0	0
3	0.087041679508673	0.049074588495534	0.011107497482395	-0.026859593530744	-0.064826684543884	0	0	0	0	0	0	0	0	0	0
4	0.125549571470192	0.087582480457053	0.049615389443913	0.011648298430774	-0.026318792582365	-0.064285883595504	-0.102252974608644	0	0	0	0	0	0	0	0
5	0.173255725650666	0.135288634637526	0.097321543624387	0.059354452611248	0.021387361598109	-0.016579729415031	-0.054546820428170	-0.09251...	-0.13048...	0	0	0	0	0	0
6	0.212114556016641	0.174147465003502	0.136180373990362	0.098213282977223	0.060246191964084	0.022279100950945	-0.015687990062195	-0.05365...	-0.09162...	-0.12958...	-0.16755...	0	0	0	0
7	0.250196362401228	0.212229271388089	0.174262180374949	0.136295089361810	0.098327998348671	0.060360907335532	0.022393816322392	-0.01557...	-0.05354...	-0.09150...	-0.12947...	-0.16744...	-0.20540...	0	0
8	0.289150524865391	0.251183433852252	0.213216342839112	0.175249251825973	0.137282160812834	0.099315069799694	0.061347978786555	0.023380...	-0.01458...	-0.05255...	-0.09052...	-0.12848...	-0.16645...	-0.20442...	-0.24238...

Figure 6 the interest rate tree for part 2 (b) under assumption (2)+(3)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	0.166...	0.666...	0.166...	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0.131...	0.661...	0.206...	0.166...	0.666...	0.166...	0.206...	0.661...	0.131...	0	0	0	0	0	0	0	0
3	0.102...	0.644...	0.252...	0.131...	0.661...	0.206...	0.166...	0.666...	0.166...	0.206...	0.661...	0.131...	0.252...	0.644...	0.102...	0	0
4	0.079...	0.616...	0.304...	0.102...	0.644...	0.252...	0.131...	0.661...	0.206...	0.166...	0.666...	0.166...	0.206...	0.661...	0.131...	0.252...	0.644...
5	0.061...	0.576...	0.361...	0.079...	0.616...	0.304...	0.102...	0.644...	0.252...	0.131...	0.661...	0.206...	0.166...	0.666...	0.166...	0.206...	0.661...
6	0.049...	0.526...	0.424...	0.061...	0.576...	0.361...	0.079...	0.616...	0.304...	0.102...	0.644...	0.252...	0.131...	0.661...	0.206...	0.166...	0.666...
7	0.042...	0.464...	0.492...	0.049...	0.526...	0.424...	0.061...	0.576...	0.361...	0.079...	0.616...	0.304...	0.102...	0.644...	0.252...	0.131...	0.661...
8	0.516...	0.441...	0.041...	0.042...	0.464...	0.492...	0.049...	0.526...	0.424...	0.061...	0.576...	0.361...	0.079...	0.616...	0.304...	0.102...	0.644...

	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0.102...	0.304...	0.616...	0.079...	0	0	0	0	0	0	0	0	0	0	0	0
5	0.131...	0.252...	0.644...	0.102...	0.304...	0.616...	0.079...	0.361...	0.576...	0.061...	0	0	0	0	0	0
6	0.166...	0.206...	0.661...	0.131...	0.252...	0.644...	0.102...	0.304...	0.616...	0.079...	0.361...	0.576...	0.061...	0.424...	0.526...	0.049...
7	0.206...	0.166...	0.666...	0.166...	0.206...	0.661...	0.131...	0.252...	0.644...	0.102...	0.304...	0.616...	0.079...	0.361...	0.576...	0.061...
8	0.252...	0.131...	0.661...	0.206...	0.166...	0.666...	0.166...	0.206...	0.661...	0.131...	0.252...	0.644...	0.102...	0.304...	0.616...	0.079...

Figure 7 the probability matrix for part 2 (b) under assumption (2)+(3) (8*45 matrix)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	3.725981...	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	91.51835...	1.014622...	1.125098...	0	0	0	0	0	0	0	0	0	0	0	0
3	83.55632...	91.68048...	1.006026...	1.104016...	1.211641...	0	0	0	0	0	0	0	0	0	0
4	79.54551...	86.14272...	93.29642...	1.010538...	1.094660...	1.185888...	1.284823...	0	0	0	0	0	0	0	0
5	76.11309...	81.39773...	87.05106...	93.09889...	99.56880...	1.064903...	1.138951...	1.218170...	1.302921...	0	0	0	0	0	0
6	77.53806...	81.67794...	86.04100...	90.63932...	95.48563...	1.005933...	1.059766...	1.116503...	1.176303...	1.239330...	1.305759...	0	0	0	0
7	80.06873...	83.04883...	86.13985...	89.34591...	92.67130...	96.12046...	99.69799...	1.034086...	1.072574...	1.112495...	1.153901...	1.196848...	1.241394...	0	0
8	88.26975...	89.96143...	91.68553...	93.44267...	95.23349...	97.05863...	98.91875...	1.008145...	1.027466...	1.047157...	1.067226...	1.087679...	1.108524...	1.129769...	1.151421...
9	102	102	102	102	102	102	102	102	102	102	102	102	102	102	102

Figure 8 the value of each node under assumption (2)+(3) (AM option early exercise case)

Part 2 (b) under above ③ (interest rate tree and probability matrix are the same as part 2 (a)) (EU&AM option price, same) results are 3.56 dollars (for $K = 98$) and 2.33 dollars (for $K = 99.85$), they are from

	1	2	3	4	5	6	7	8	9
1	2.329606062454519	0	0	0	0	0	0	0	0
2	86.771779634115430	99.376829310639850	1.138315506608865e+02	0	0	0	0	0	0
3	81.795826946413240	90.251538631036210	99.585491201953600	1.098891848309941e+02	1.212636781894975e+02	0	0	0	0
4	84.877078150007260	89.559001744181360	94.499185978559650	99.711876826391660	1.052121060862607e+02	1.110157347291721e+02	1.171394986367194e+02	0	0
5	102	102	102	102	102	102	102	102	102

Figure 9 the value of each node under assumption (1)+(3)+K = 98

	1	2	3	4	5	6	7	8	9
1	3.555482309473085	0	0	0	0	0	0	0	0
2	86.771779634115430	99.376829310639850	1.138315506608865e+02	0	0	0	0	0	0
3	81.795826946413240	90.251538631036210	99.585491201953600	1.098891848309941e+02	1.212636781894975e+02	0	0	0	0
4	84.877078150007260	89.559001744181360	94.499185978559650	99.711876826391660	1.052121060862607e+02	1.110157347291721e+02	1.171394986367194e+02	0	0
5	102	102	102	102	102	102	102	102	102

Figure 10 the value of each node under assumption (1)+(3)+K = 99.85

Part 2 (b) under above ④ (interest rate tree is the same as part 2 (a) and probability matrix is different from part 2 (a)) (EU&AM option price, same) result is 2.97 dollars, it comes from

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	0.1666...	0.6666...	0.1666...	0	0	0	0	0	0	0	0	0	0	0
2	0.1029...	0.6441...	0.2529...	0.1666...	0.6666...	0.1666...	0.2529...	0.6441...	0.1029...	0	0	0	0	0
3	0.0616...	0.5766...	0.3616...	0.1029...	0.6441...	0.2529...	0.1666...	0.6666...	0.1666...	0.2529...	0.6441...	0.1029...	0.3616...	0.5766...
4	0.0429...	0.4641...	0.4929...	0.0616...	0.5766...	0.3616...	0.1029...	0.6441...	0.2529...	0.1666...	0.6666...	0.1666...	0.2529...	0.6441...
5	0.4466...	0.5066...	0.0466...	0.0429...	0.4641...	0.4929...	0.0616...	0.5766...	0.3616...	0.1029...	0.6441...	0.2529...	0.1666...	0.6666...

	15	16	17	18	19	20	21	22	23	24	25	26	27
1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0.0616...	0	0	0	0	0	0	0	0	0	0	0	0
4	0.1029...	0.3616...	0.5766...	0.0616...	0.4929...	0.4641...	0.0429...	0	0	0	0	0	0
5	0.1666...	0.2529...	0.6441...	0.1029...	0.3616...	0.5766...	0.0616...	0.4929...	0.4641...	0.0429...	0.0466...	0.5066...	0.4466...

Figure 11 the probability matrix for part 2 (b) under assumption (1)+(3)+different T0/TN

	1	2	3	4	5	6	7	8	9
1	2.967147376099423	0	0	0	0	0	0	0	0
2	83.822201447758030	99.016158978005180	1.170078697621266e+02	0	0	0	0	0	0
3	75.546894828501200	86.505369903974970	99.070551737367440	1.134794742726778e+02	1.300043457238121e+02	0	0	0	0
4	73.754731561470350	81.375372863037100	89.787268808082250	99.072835223159100	1.093230950221301e+02	1.206385797017255e+02	1.331303256046501e+02	0	0
5	80.106489364191130	84.525261437586660	89.187778390973760	94.107485490488640	99.298569659616320	1.047760003899183e+02	1.105555729084466e+02	1.166539537263240e+02	1.230887286998417e+02
6	102	102	102	102	102	102	102	102	102

Figure 12 the value of each node under assumption (1)+(3)+different T0/TN

We will illustrate these four parameters changed cases with the Monte Carlo simulation results in one list followed below.

Part 2 (b) under above ⑤ (interest rate tree and probability matrix are different from part 2 case) (EU and AM option price, maybe different) results are 3.71 dollars (for EU option) and $max\ (3.71, 3.43) =$ 3.71 dollars (for AM option), they are from

$$\{\theta\ (t), t = 0,1,2,3,4,5,6,7,8\}$$

$$= \{0.33255, 0.14491, 0.15236, 0.28243, 0.29453, 0.29625, 0.30958, 0.35053, 0.36829\}$$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	2.800000...	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0.063167...	0.025200...	-0.01276...	0	0	0	0	0	0	0	0	0	0	0	0
3	0.087041...	0.049074...	0.011107...	-0.02685...	-0.06482...	0	0	0	0	0	0	0	0	0	0
4	0.125549...	0.087582...	0.049615...	0.011648...	-0.02631...	-0.06428...	-0.10225...	0	0	0	0	0	0	0	0
5	0.173255...	0.135288...	0.097321...	0.059354...	0.021387...	-0.01657...	-0.05454...	-0.09251...	-0.13048...	0	0	0	0	0	0
6	0.212114...	0.174147...	0.136180...	0.098213...	0.060246...	0.022279...	-0.01568...	-0.05365...	-0.09162...	-0.12958...	-0.16755...	0	0	0	0
7	0.250196...	0.212229...	0.174262...	0.136295...	0.098327...	0.060360...	0.022393...	-0.01557...	-0.05354...	-0.09150...	-0.12947...	-0.16744...	-0.20540...	0	0
8	0.289150...	0.251183...	0.213216...	0.175249...	0.137282...	0.099315...	0.061347...	0.023380...	-0.01458...	-0.05255...	-0.09052...	-0.12848...	-0.16645...	-0.20442...	-0.24238...
9	0.330176...	0.292209...	0.254242...	0.216275...	0.178308...	0.140341...	0.102374...	0.064406...	0.026439...	-0.01152...	-0.04949...	-0.08746...	-0.12542...	-0.16339...	-0.20136...
10	0.331013...	0.293046...	0.255079...	0.217111...	0.179144...	0.141177...	0.103210...	0.065243...	0.027276...	-0.01069...	-0.04865...	-0.08662...	-0.12459...	-0.16255...	-0.20052...

Figure 13 the interest rate tree for part 2 (b) under assumption (2)+(4)

	1	2	3	4	5	6	7	8	9	10	11
1	0.166666...	0.666666...	0.166666...	0	0	0	0	0	0	0	0
2	0.131979...	0.661041...	0.206979...	0.166666...	0.666666...	0.166666...	0.206979...	0.661041...	0.131979...	0	0
3	0.102916...	0.644166...	0.252916...	0.131979...	0.661041...	0.206979...	0.166666...	0.666666...	0.166666...	0.206979...	0.661041...
4	0.079479...	0.616041...	0.304479...	0.102916...	0.644166...	0.252916...	0.131979...	0.661041...	0.206979...	0.166666...	0.666666...
5	0.061666...	0.576666...	0.361666...	0.079479...	0.616041...	0.304479...	0.102916...	0.644166...	0.252916...	0.131979...	0.661041...
6	0.049479...	0.526041...	0.424479...	0.061666...	0.576666...	0.361666...	0.079479...	0.616041...	0.304479...	0.102916...	0.644166...
7	0.042916...	0.464166...	0.492916...	0.049479...	0.526041...	0.424479...	0.061666...	0.576666...	0.361666...	0.079479...	0.616041...
8	0.516979...	0.441041...	0.041979...	0.042916...	0.464166...	0.492916...	0.049479...	0.526041...	0.424479...	0.061666...	0.576666...
9	0.175846...	0.666341...	0.157812...	0.158270...	0.666374...	0.175355...	0.159649...	0.666463...	0.173886...	0.161036...	0.666536...

	12	13	14	15	16	17	18	19	20	21	2
1	0	0	0	0	0	0	0	0	0	0	
2	0	0	0	0	0	0	0	0	0	0	
3	0.131979...	0.252916...	0.644166...	0.102916...	0	0	0	0	0	0	
4	0.166666...	0.206979...	0.661041...	0.131979...	0.252916...	0.644166...	0.102916...	0.304479...	0.616041...	0.079479...	
5	0.206979...	0.166666...	0.666666...	0.166666...	0.206979...	0.661041...	0.131979...	0.252916...	0.644166...	0.102916...	0.304
6	0.252916...	0.131979...	0.661041...	0.206979...	0.166666...	0.666666...	0.166666...	0.206979...	0.661041...	0.131979...	0.252
7	0.304479...	0.102916...	0.644166...	0.252916...	0.131979...	0.661041...	0.206979...	0.166666...	0.666666...	0.166666...	0.206
8	0.361666...	0.079479...	0.616041...	0.304479...	0.102916...	0.644166...	0.252916...	0.131979...	0.661041...	0.206979...	0.166
9	0.172426...	0.162431...	0.666593...	0.170974...	0.163835...	0.666634...	0.169530...	0.165246...	0.666658...	0.168094...	0.166

Figure 14 the probability matrix for part 2 (b) under assumption (2)+(4) (9*45 matrix)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	3.706462...	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0.391437...	2.979983...	9.930521...	0	0	0	0	0	0	0	0	0	0	0	0
3	87.88777...	94.29058...	1.011518...	1.094991...	1.193182...	0	0	0	0	0	0	0	0	0	0
4	86.08360...	90.95301...	96.27216...	1.014064...	1.083931...	1.166620...	1.258154...	0	0	0	0	0	0	0	0
5	80.82906...	86.16073...	91.19451...	95.71048...	1.004926...	1.063117...	1.128944...	1.201066...	1.280832...	0	0	0	0	0	0
6	82.46381...	86.77614...	91.29170...	95.42288...	98.36512...	1.013444...	1.056952...	1.105397...	1.158117...	1.217358...	1.282297...	0	0	0	0
7	82.42369...	85.49145...	88.67337...	91.97373...	95.30026...	98.35922...	1.011022...	1.041100...	1.070038...	1.103243...	1.142588...	1.185114...	1.229224...	0	0
8	90.86592...	92.60736...	94.38217...	96.19099...	98.03448...	99.91330...	1.014266...	1.021323...	1.034880...	1.042084...	1.056763...	1.077015...	1.097656...	1.118693...	1.140132...
9	105	105	105	105	105	105	105	103	103	101	101	101	101	101	101

Figure 15 the value of each node under assumption (2)+(4) (EU case)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	3.425912...	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	94.42195...	1.014465...	1.107721...	0	0	0	0	0	0	0	0	0	0	0	0
3	87.88777...	94.29058...	1.011518...	1.094991...	1.193182...	0	0	0	0	0	0	0	0	0	0
4	86.08360...	90.95301...	96.27216...	1.014064...	1.083931...	1.166620...	1.258154...	0	0	0	0	0	0	0	0
5	80.82906...	86.16073...	91.19451...	95.71048...	1.004926...	1.063117...	1.128944...	1.201066...	1.280832...	0	0	0	0	0	0
6	82.46381...	86.77614...	91.29170...	95.42288...	98.36512...	1.013444...	1.056952...	1.105397...	1.158117...	1.217358...	1.282297...	0	0	0	0
7	82.42369...	85.49145...	88.67337...	91.97373...	95.30026...	98.35922...	1.011022...	1.041100...	1.070038...	1.103243...	1.142588...	1.185114...	1.229224...	0	0
8	90.86592...	92.60736...	94.38217...	96.19099...	98.03448...	99.91330...	1.014266...	1.021323...	1.034880...	1.042084...	1.056763...	1.077015...	1.097656...	1.118693...	1.140132...
9	105	105	105	105	105	105	105	103	103	101	101	101	101	101	101

Figure 16 the value of each node under assumption (2)+(4) (AM option early exercise case)

5. Obtain each case results from Monte Carlo simulation

Now, we can follow the Monte Carlo method to obtain the estimated payoff and calculate the corresponding EU option price with part 2 assumption (1)/(2) + (3) situation. We will implement the Monte Carlo simulation method in P54, Lecture Note 04 Ⅰ [3], this is only for $(c_1 = c_2 = c_3)$ case, but for $(c_1 \neq c_2 \neq c_3)$ case, MC is to obtain each path for interest rate, via above $\theta\ (t)$, as:

$$\Delta r_t = \kappa\left(\theta_t - r_t\right)\Delta t + \sigma \Delta W_t.$$

Monte-Carlo simulation algorithm

1. Construct the discount curve, P_0^T, $\forall T$, with the U.S. Treasury data at $t=0$.
2. Simulate a number of Brownian paths and calculate the discounted prices according to the scheme

$$\frac{P_{t+\Delta t}^{T_i}}{B_{t+\Delta t}} = \frac{P_t^{T_i}}{B_t}\exp\left(-\frac{1}{2}\left\|\Sigma(t,T_i)\right\|^2 \Delta t + \Sigma^{\mathrm{T}}(t,T_i)\Delta\tilde{\mathbf{W}}_t\right) \qquad (4.65)$$

for $i=0,1,\ldots,n$, until $t+\Delta t$ equals T_0.

3. Average the payoffs:

$$\left(\sum_{i=1}^{n}\Delta T c \cdot \frac{P_{T_0}^{T_i}}{B_{T_0}} + \frac{P_{T_0}^{T_n}}{B_{T_0}} - \frac{K}{B_{T_0}}\right)^{+}. \qquad (4.66)$$

For ($c_1 = c_2 = c_3$), we set the initial $t = 0$ and $\Delta_t = 1/360$ to obtain the Brownian paths (we set 3*1000 paths) and the discount prices. Then corresponding EU option price of the Monte Carlo simulation with part 2 (1)/(2) + (3) situation is $3.62 \sim 3.71$ dollars (fluctuate with calculation of round and round). In part three we will do the Monte Carlo simulation again after changing the sigma value in model from 0.013 to 0.0001 (reason details in part 3), then we will figure out that the Monte Carlo method with smaller sigma model will be much more stable during calculation of round and round and will be much closer to the result calculating from the corresponding interest rate tree. And for ($c_1 \neq c_2 \neq c_3$), we experiment the algorithm on case ⑤ in part 2 (b) for 10000 paths interest rate tree, $\Delta_t = 1/2$, the result is $1.90 \sim 1.98$ dollars.

Compare each result in the part 2 (a), (b) and (c), and we also contain the result we will obtain in part three, we can easily figure out the difference when the parameters change:

Case	Coupon Bond Option Details			Monte Carlo Simulation	Interest Rate Tree		
	ΔT	Maturity	Tenor	Hull-White Model $\sigma = 0.031 (c_1 =/\neq c_2 =/\neq c_3)$	Δt	K	Result ($c_1 =/\neq c_2 =/\neq c_3$)
1.	1yr	1yr	1yr-4yr	3.62~3.71(EU)	1yr	99.2	2.56(EU&AM)/1.96(EU&AM)
2.	1yr	1yr	1yr-4yr	3.62~3.71(EU)/1.90~1.98(EU)	0.5yr	99.2	3.67(EU)&3.73(AM)/3.71(EU)&3.71(AM)
3.	1yr	1yr	1yr-5yr	4.11~ 4.32(EU)	1yr	99.2	2.97(EU&AM)
4.	1yr	1yr	1yr-4yr	4.86~4.88(EU)	1yr	98	3.56(EU&AM)
5.	1yr	1yr	1yr-4yr	3.03~ 3.06(EU)	1yr	99.85	2.33(EU&AM)
	ΔT	Maturity	Tenor	Hull-White Model $\sigma = 0.0001$	Δt	K	Result ($c_1 = c_2 = c_3$)
6.	1yr	1yr	1yr-4yr	1.7065~1.7082(EU)	0.5yr	99.2	1.7071(EU)2.45(AM)
7.	1yr	1yr	1yr-4yr	1.7065~1.7082(EU)	1yr	99.2	0.48(EU&AM)

6. Discussion and comments

From the chart, we say the most important feature when changing the parameter is when decreasing the time step of interest rate tree, i.e., from case 1 to case 2 and case 7 to case 6, because the price of EU option is much closer to the value we calculate from Monte Carlo simulation (which we believe is the more "correct" answer), and we see also in case 1 and 2, under the assumption (2) ($c_1 \neq c_2 \neq c_3$), the decrease of the lattice tree time step significantly make the result become larger and closer to the simulated answer; When the tenor extends, like case 1 to case 3, we see naturally the price of the option is larger due to calculate more times coupon and we also see that the value we obtain from the interest rate tree in case 3 is not as close to the value obtained from Monte Carlo as case 1, I think it may because of the interest rate tree contains more -ve value and larger value in case 3; For case 2 we can see that when the time step of the lattice tree is smaller than the coupon payment period, the AM option price would be different from the EU option due to we can early exercise, but in this case, they are the same, in case 6, they are different, I think it's due to the gap of interest rate between each time step of lattice tree is much larger than case 6, thus case 2 would always be better to exercise at the maturity; For case 1 to case 4 and 5, we see the effect of parameter K, when K becomes smaller (larger), the price naturally becomes larger (smaller), and an interesting appearance is the distance of the value from interest rate tree to the Monte Carlo simulation, when K becomes larger, the distance is larger, we think it may because of the special of call option and also because of the interest rate tree in part 2 contains so many -ve values. The analysis of -ve in tree and each value after changing the model parameters will be through the whole part 3. For the Monte Carlo of different coupon rate in case 2 ($c_1 =/\neq c_2 =/\neq c_3$), we see the simulation of interest rate via known $\theta(t)$ is not so accurate as the original simulation in future discount factor, I think it is due to the interest rate Monte Carlo shows the simulation of interest rate is so sensitive with the time step period and for $\Delta_t = 1/2$, there are huge different in the first 2 period for 10000 cases and thus the accurate level of this simulation is so bad.

7. Further own additional items

Finally is the part 3 of this project and here we will firstly try the cubic spline extrapolation to obtain the discount factor curve and implement this curve in some cases in part 2, and then secondly, we will illustrate why the -ve part of tree is so large in part 2, change the sigma of the model to obtain the interest rate tree contains -ve part as small as possible and find its impact to the pricing procedure, and eventually we discuss the method to decrease the variance in Monte Carlo simulation.

After implement the same idea in HW1 of this course, we obtain the discount factor curve of the cubic spline extrapolation method and use these numbers in part 2

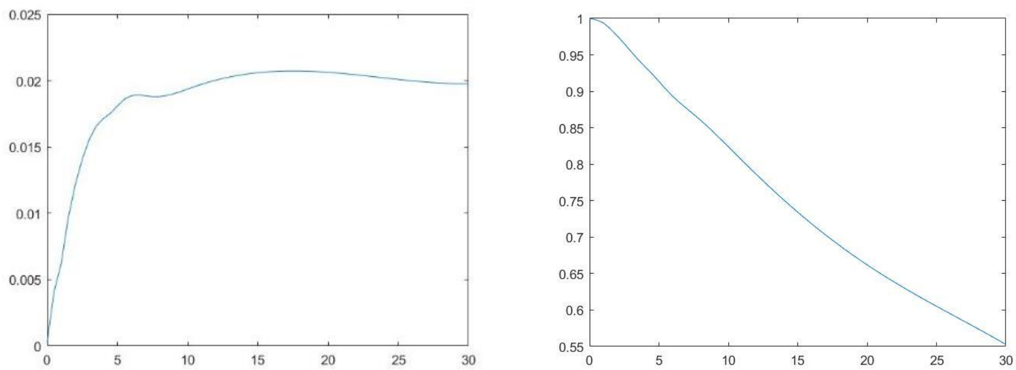


Figure 13 the spot rate curve and discount curve for cubic spline method in part 3

Case	Coupon Bond Option Details			Monte Carlo Simulation	Interest Rate Tree		
	ΔT	Maturity	Tenor	Hull-White Model $\sigma = 0.031$	Δt	K	Result ($c_1 = c_2 = c_3$)
1.	1yr	1yr	1yr-4yr	3.12~3.15	1yr	99.2	2.33(EU&AM)
2.	1yr	1yr	1yr-4yr	3.12~3.15	0.5yr	99.2	2.76(EU)&2.76(AM)

We can easily figure out that when the discount factor curve changes from linear extrapolation to cubic spline extrapolation, the result obtained from interest rate tree is much closer with the Monte Carlo simulation when the interest rate tree time step equals to the coupon payment period, but the effect of taking the smaller time step of interest rate tree doesn't have as much influence as the part 2 on making the price closer with the Monte Carlo simulation. The reason we think is when taking the cubic spline extrapolation, the discount factors are all larger than that for liner extrapolation, so when the interest rate tree time step goes smaller, the effect of it will be declined due to a larger discount factor value (the level of the price of each node going larger becomes smaller).

Then, according to figure 1 and figure 6, we see there are so many -ve values in the interest rate tree, after reviewing the algorithm, we think this is due to a significant large sigma in the Hull-White model (3.1%) we use in part 2, and the initial of interest rate tree is also super small (0.028%), thus when taking a time step in interest rate tree, the largest and smallest of the lattice tree would be such far away from the initial which is close to zero, that would cause so many -ve values in the lattice tree. Hence, we decide to make the sigma of the model smaller, from 3.1% to 0.01%, thus we will obtain a new interest rate tree as (time-step of lattice tree is 0.5yr):

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	2.800000...	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0.025202...	0.025080...	0.024957...	0	0	0	0	0	0	0	0	0	0	0	0
3	0.010907...	0.010784...	0.010662...	0.010539...	0.010417...	0	0	0	0	0	0	0	0	0	0
4	0.011086...	0.010964...	0.010841...	0.010719...	0.010597...	0.010474...	0.010352...	0	0	0	0	0	0	0	0
5	0.020344...	0.020222...	0.020099...	0.019977...	0.019854...	0.019732...	0.019609...	0.019487...	0.019364...	0	0	0	0	0	0
6	0.020666...	0.020543...	0.020421...	0.020298...	0.020176...	0.020053...	0.019931...	0.019808...	0.019686...	0.019563...	0.019441...	0	0	0	0
7	0.020147...	0.020025...	0.019902...	0.019780...	0.019658...	0.019535...	0.019413...	0.019290...	0.019168...	0.019045...	0.018923...	0.018800...	0.018678...	0	0
8	0.020460...	0.020338...	0.020215...	0.020093...	0.019970...	0.019848...	0.019725...	0.019603...	0.019480...	0.019358...	0.019235...	0.019113...	0.018990...	0.018868...	0.018746...

Figure 14 the interest rate tree after changing the sigma of the model

We can see clearly there is no -ve value inside the interest rate tree figure 14 so now let us check some cases' value and compare with the corresponding Monte Carlo. Surprisingly it works so well for pricing option under interest rate tree, we can see for time step 0.5yr, the Monte Carlo simulation result does not fluctuate more than 0.001 with calculation of round and round, and the value of EU option fits so well for the Monte Carlo simulation, i.e., from lattice tree, it is 1.7071, from Monte Carlo, a large amount of simulations are around 1.7070 to 1.7073, only few of it goes out of this range, but for $\sigma = 0.031$, the lattice tree result (3.67) is just a few cases appeared in Monte Carlo, many of them are up to around 3.70. I think it is due to the significant impact of all the nodes of interest rate tree are +ve, which makes the calculation from lattice tree more accurate.

In the last part, we will discuss the method to decrease the variance in Monte Carlo simulation, because we need the Monte Carlo result become closer and closer to the "correct" answer for option pricing, the only thing we can do is to decrease the volatility of the calculation as much as possible.

All below materials are from the Chapter Four of the "*Monte Carlo Methods in Financial Engineering*", P. Glasserman [4].

Control Variates method is what we always implement to reduce the variance of the Monte Carlo, and the key idea is we set $\{Y_1, Y_2, Y_3, \ldots, Y_n\}$ be output of the simulation, i.e., the Y_i can be the discounted payoff of a derivative security on the $i-th$ simulated path, i.i.d., then we always set the average of Y_i as $\bar{Y} = (Y_1 + Y_2 + Y_3 + \cdots + Y_n)/n$ as the final simulated result, which is unbiased and converges with probability 1. But now on each replication we consider another output X_i along with the Y_i, they are i.i.d. and the expectation of X is known yet, then for any fixed b, we can calculate

$$Y_i(b) = Y_i - b(X_i - \mathbb{E}(X))$$

then compute the sample mean as

$$\bar{Y}(b) = \bar{Y} - b(\bar{X} - \mathbb{E}(X))$$

where here is a control variate estimator, and $\bar{X} - \mathbb{E}(X)$ is called as a control in estimating $\bar{Y}$, we can also confirm that the $\bar{Y}(b)$ is unbiased and consistent, with the variance of each component as

$$Var\big(Y_i(b)\big) = \sigma_Y^2 - 2b\sigma_X\sigma_Y\rho_{XY} + b^2\sigma_X^2$$

where $\sigma_X^2 = Var(X)$ and $\sigma_Y^2 = Var(Y)$. Solve the equation for finding the minimizer $b^* = \frac{\sigma_Y\rho_{XY}}{\sigma_X} = \frac{Cov(X,Y)}{Var(X)}$ and substitute it back we see

$$\frac{Var\left(\bar{Y} - b^*\big(\bar{X} - \mathbb{E}(X)\big)\right)}{Var(\bar{Y})} = 1 - \rho_{XY}^2$$

which could naturally decrease the variance of obtaining the sample mean after the Monte Carlo simulation, next is an example 4.1.1 from textbook [4] to vividly explain how this method works.

For derivative pricing simulations, suppose the underlying asset is $S(t)$, and we are working in the risk-neutral measure and the interest rate is a constant r, we select $X_i = S_i(T)$ where i stands for the $i-th$ sample in Monte Carlo simulation, then $\mathbb{E}(e^{-rT}X_i) = S_i(0) = S(0)$, and we also set $Y_i = e^{-rT}(S_i(T) - K)^+$ as the derivative pricing formula for each path of S we simulate, now we can form the control variate estimator as

$$\frac{1}{n}\sum_{i=1}^{n}(Y_i - b\big(S_i(T) - e^{rT}S(0)\big))$$

the effectiveness of the control variate depends on the strike price. If $S \sim GBM(r, \sigma^2)$, the textbook gives a numerical example,

K	40	45	50	55	60	65	70
$\hat{\rho}$	0.995	0.968	0.895	0.768	0.604	0.433	0.286
$\hat{\rho}^2$	0.99	0.94	0.80	0.59	0.36	0.19	0.08

Table 4.1. Estimated correlation $\hat{\rho}$ between $S(T)$ and $(S(T) - K)^+$ for various values of K, with $S(0) = 50$, $\sigma = 30\%$, $r = 5\%$, and $T = 0.25$. The third row measures the fraction of variance in the call option payoff eliminated by using the underlying asset as a control variate.

We can see the control variate significantly decrease the variance of call option payoff estimating, thus it will achieve the aim of reducing the variance of Monte Carlo simulation.

Reference

[1] $J, k, \delta r, P(*)$ *calculation method*, Lecture Note 05, p38-p39.
[2] *The algorithm in "Answer to Question 2"*, Lecture Note 05, p41-p43.
[3] *The Monte Carlo simulation method for coupon bond option*, Lecture Note 04 Ⅰ, p54.
[4] P. Glasserman. *Monte Carlo Methods in Financial Engineering*. Springer 2004.