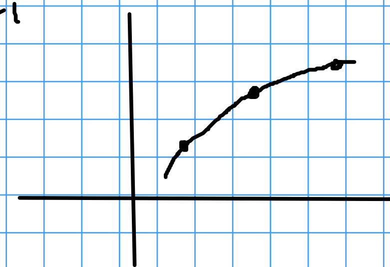
Worksheet 5 – 1D Classification

Interpolation: $f(x_i) = \sum_{j=1}^{N} a_j \phi_j(x_i) = f_i \quad , i = 1, \dots, N$

$$= \sum_{j=1}^{N} G_{ij} a_j = f_i$$

$$\Leftrightarrow \quad G\vec{a} = \vec{f}$$



Approximation:

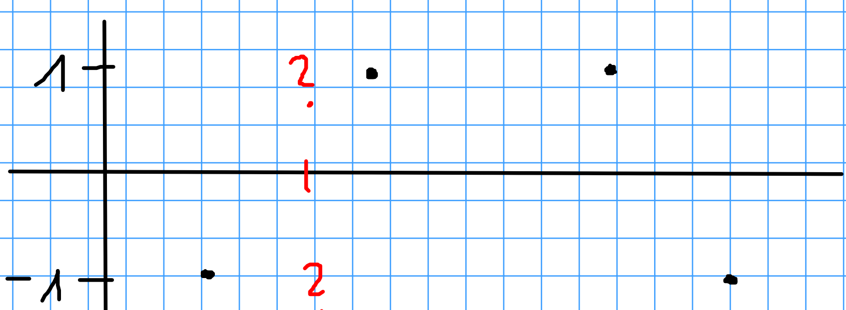
$$f(x) \approx f_N(x) = \sum_{j=1}^{N} v_j \phi_j(x)$$

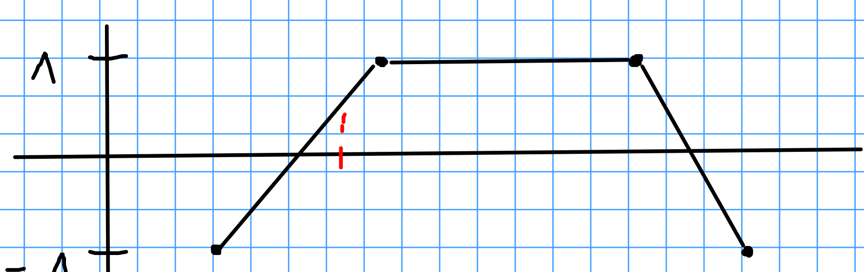
↑ Ansatz (piecewise liner functions)

$$\Leftrightarrow \quad G^T G v = G^T y$$

Exercise 1: Interpolation-like classification

Motivation:

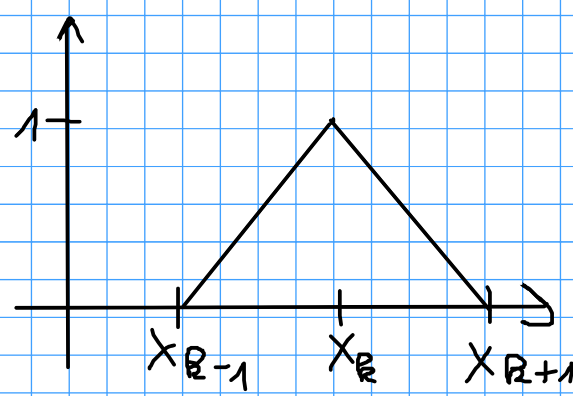




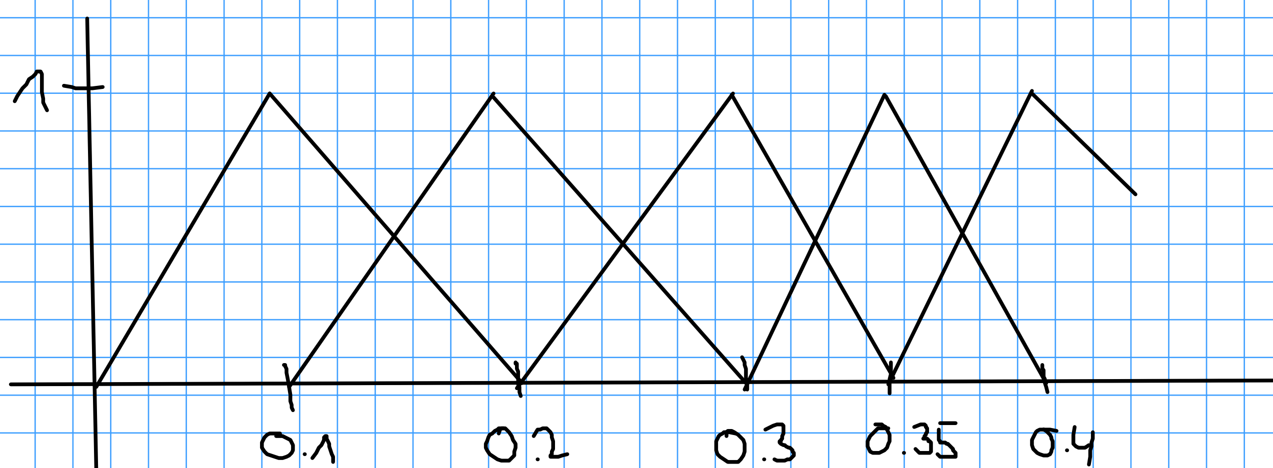
Example naive linear interpolation

Nodal basis

$$\phi_k = \begin{cases} \frac{1}{h_{k-1}} (x - x_{k-1}) & x_{k-1} < x < x_k \\ \frac{1}{h_{k-1}} (x_{k+1} - x) & x_k < x < x_{k+1} \\ 0 & \end{cases}$$

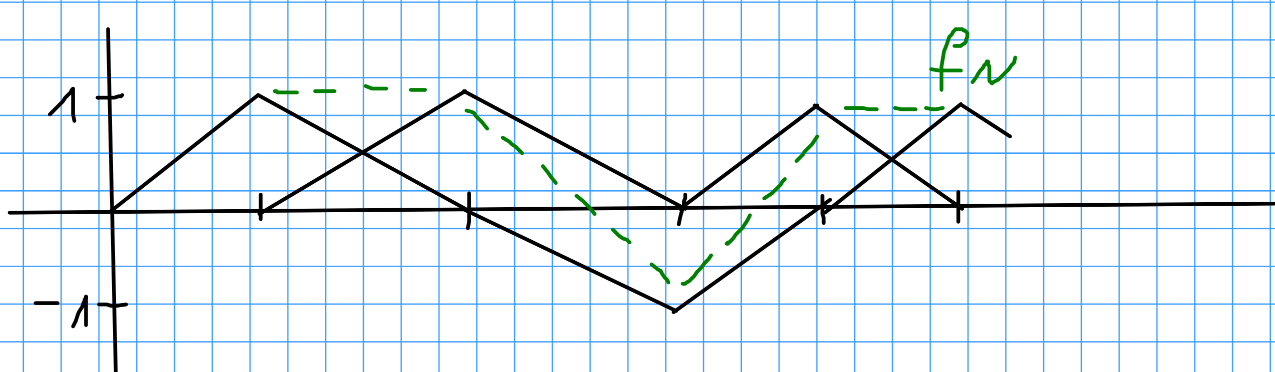


a)



b) $v_j = f(x_j) \Rightarrow v_j = y_i$

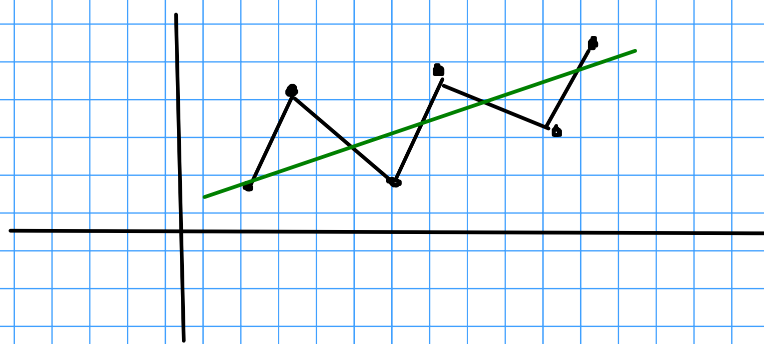
because $\phi_j(x_j) = 1$



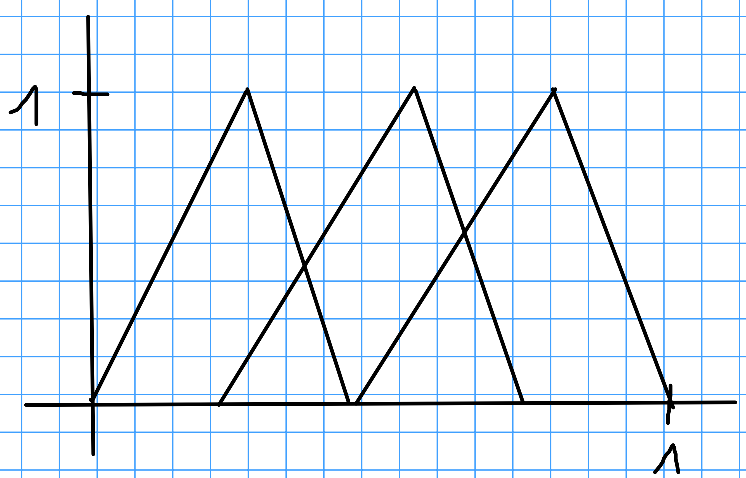
c) Evaluate $f_N(0.5)$

$f_N(0.5) = -0.33\ldots < 0 \Rightarrow$ classify as -1

d) Problems : - Curse of dimensionality
$\Rightarrow$ We need N^d basis functions
- Noise / outlier



Exercise 2 : Equidistant nodal basis



$\Omega = [0, 1]$

$f(x) \approx f_N(x) = \sum_{j=1}^{N} v_j \phi_j(x)$

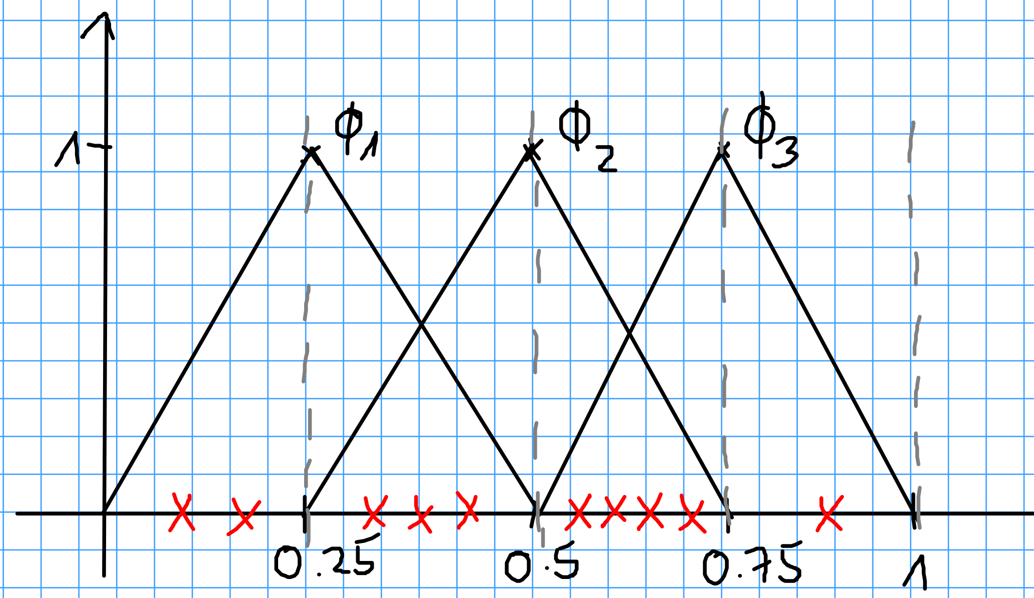
Need to solve a least-squares problem

$$\underset{v}{\operatorname{argmin}} \left(\| Gv - y \|_2^2 \right) \quad \text{with} \quad G_{\underset{r}{i}\underset{c}{j}} = \phi_{\underset{c}{j}}(x_{\underset{r}{i}})$$

Equivalent to solving the normal eq. for v

$$G^T G v = G^T y$$

a) Construct G



10 rows
3 columns

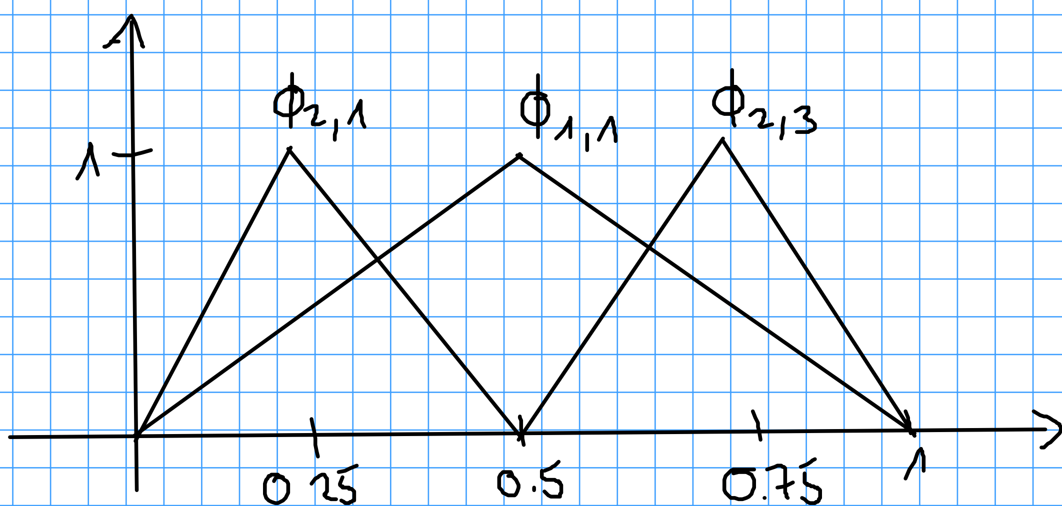
$$
G = \begin{array}{c}
\begin{array}{ccc} \phi_1 & \phi_2 & \phi_3 \end{array} \\
\begin{pmatrix}
x & 0 & 0 \\
x & 0 & 0 \\
x & x & 0 \\
x & x & 0 \\
x & x & 0 \\
0 & x & x \\
0 & x & x \\
0 & x & x \\
0 & x & x \\
0 & 0 & x
\end{pmatrix}
\end{array}
$$

b) Solve $G^T G v = G^T y$

$\uparrow$ to solve for

c) $f_N(0.5) < 0 \Rightarrow$ classify as -1

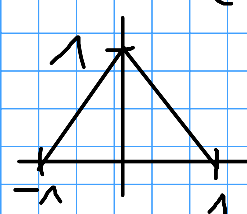
Exercise 3: Hierarchical classification



Mother of all hat functions: $\phi(x) = \max\{1 - |x|, 0\}$

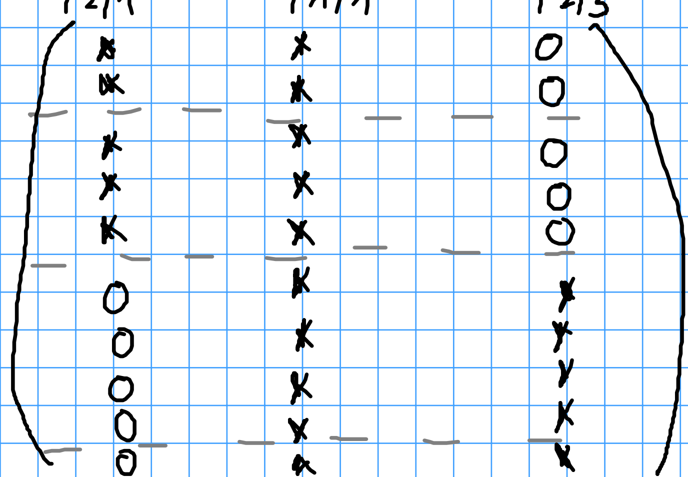
$$\phi_{n,i} = \phi\left(\frac{x - \overset{\text{shift}}{x_{n,i}}}{\underset{\text{scaling}}{h_n}}\right), \quad h_n = 2^{-n}$$

$$x_{n,i} = i \cdot h_n$$

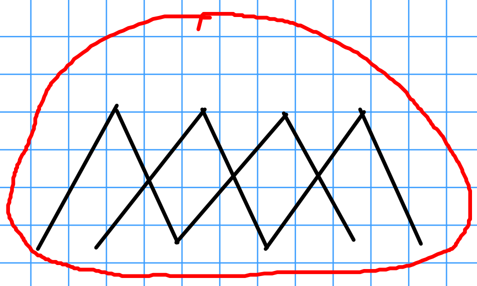


a) $\phi_{2,1}$ $\phi_{1,1}$ $\phi_{2,3}$

$G =$



Nodal basis



Hierarchical basis