Chair of Scientific Computing in Computer Science
Department of Informatics
Technical University of Munich

Eexam
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- During the attendance check a sticker containing a unique code will be put on this exam.
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# Algorithms for Scientific Computing

| | | | |
|---|---|---|---|
| **Exam:** | IN2001 / Tech Test | **Date:** | Monday 19th July, 2021 |
| **Examiner:** | Michael Bader | **Time:** | 08:00 – 18:00 |

## Working instructions

- This exam consists of **10 pages** with a total of **4 problems**.
  Please make sure now that you received a complete copy of the exam.
- The total amount of achievable credits in this exam is 44 credits.
- Detaching pages from the exam is prohibited.
- Allowed resources:
  - one **non-programmable pocket calculator**
  - one **analog dictionary** English $\leftrightarrow$ native language
- Subproblems marked by * can be solved without results of previous subproblems.
- **Answers are only accepted if the solution approach is documented.** Give a reason for each answer unless explicitly stated otherwise in the respective subproblem.
- Do not write with red or green colors nor use pencils.
- Physically turn off all electronic devices, put them into your bag and close the bag.

# Problem 1 Quarter-Wave Discrete Sine Transform (11 credits)

Given is a real-valued input data-set $f_0, ..., f_{2N-1} \in \mathbb{R}$ (i.e., length $2N$) with the symmetry conditions $f_{2N-n-1} = -f_n$. Complete the tasks below.

0 1 2 3 4 5

a) Show that the corresponding Quarter-Wave Fourier coefficients

$$F_k = \frac{1}{2N} \sum_{n=0}^{2N-1} f_n \omega_{2N}^{-k\left(n+\frac{1}{2}\right)} \tag{1.1}$$

have only imaginary values and can be written as

$$F_k = -\frac{i}{N} \sum_{n=0}^{N-1} f_n \sin\left(\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right) \tag{1.2}$$

The proof is done in the following steps:

① Isolate the symmetry condition

② Insert the symmetry condition

③ Assemble terms to a sum over $f_n$

④ Make terms "imaginary"

$$\begin{aligned}
F_k &= \frac{1}{2N} \sum_{n=0}^{2N-1} f_n \omega_{2N}^{-k\left(n+\frac{1}{2}\right)} \\
&\overset{①}{=} \frac{1}{2N} \left( \sum_{n=0}^{N-1} f_n \omega_{2N}^{-k\left(n+\frac{1}{2}\right)} + \sum_{n=N}^{2N-1} f_n \omega_{2N}^{-k\left(n+\frac{1}{2}\right)} \right) \\
&\overset{②}{=} \frac{1}{2N} \left( \sum_{n=0}^{N-1} f_n \omega_{2N}^{-k\left(n+\frac{1}{2}\right)} - \sum_{n=N}^{2N-1} f_{2N-n-1} \omega_{2N}^{-k\left(n+\frac{1}{2}\right)} \right) \\
&= \frac{1}{2N} \left( \sum_{n=0}^{N-1} f_n \omega_{2N}^{-k\left(n+\frac{1}{2}\right)} - \sum_{n=0}^{N-1} f_n \omega_{2N}^{-k\left(2N-n-1+\frac{1}{2}\right)} \right) \\
&= \frac{1}{2N} \left( \sum_{n=0}^{N-1} f_n \omega_{2N}^{-k\left(n+\frac{1}{2}\right)} - \sum_{n=0}^{N-1} f_n \omega_{2N}^{k\left(n+\frac{1}{2}\right)} \right) \\
&\overset{③}{=} \frac{1}{2N} \sum_{n=0}^{N-1} f_n \underbrace{\left( \omega_{2N}^{-k\left(n+\frac{1}{2}\right)} - \omega_{2N}^{k\left(n+\frac{1}{2}\right)} \right)}_{=\omega_{2N}^{-k\left(n+\frac{1}{2}\right)} - \left(\omega_{2N}^{k\left(n+\frac{1}{2}\right)}\right)^* = -2i\mathrm{Im}\{\omega_{2N}^{-k\left(n+\frac{1}{2}\right)}\}} \\
&\overset{④}{=} \frac{-2i}{2N} \sum_{n=0}^{N-1} f_n \underbrace{\mathrm{Im}\{e^{i2\pi k\left(n+\frac{1}{2}\right)/2N}\}}_{=\mathrm{Im}\left\{\cos\left(\frac{\pi k\left(n+\frac{1}{2}\right)}{N}\right) + i\sin\left(\frac{\pi k\left(n+\frac{1}{2}\right)}{N}\right)\right\}} \\
&= -\frac{i}{N} \sum_{n=0}^{N-1} f_n \sin\left(\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right) \in \mathbb{C} \qquad \text{q.e.d.}
\end{aligned}$$

b)
Show that the coefficients $F_k$ of the QW-DST again justify a symmetry condition!

0 1 2 3

Since there are only sine terms, we assume (and get) an odd symmetry:

$$
\begin{aligned}
F_{2N-k} &= -\frac{i}{N}\sum_{n=0}^{N-1} f_n \sin\left(\frac{\pi}{N}(2N-k)\left(n+\frac{1}{2}\right)\right) \\
&= -\frac{i}{N}\sum_{n=0}^{N-1} f_n \sin\left(2\pi - \frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right) \\
&= \frac{i}{N}\sum_{n=0}^{N-1} f_n \sin\left(\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right) \\
&= -F_k
\end{aligned}
$$

Since all $F_k = -F_{2N-k}$, we need the $F_k$ only for $k = 0, \dots, N$ for a Sine Transform.

c) Let `real-FFT(g,N)` be a procedure that computes the Fourier coefficients $G_k$ efficiently from a real data-set `g` that consists of $2N$ values $g_n$.
Describe a **procedure** `QW-DST(g,N)` that uses the given procedure `real-FFT` to compute the coefficients $F_k$ for $k = 0, \dots, N-1$ from equation (1.2) for the (non-symmetrical) real data $f_0, \dots, f_{N-1}$, stored in the parameter field `g`.
Note: `real-FFT(g,N)` **does not** compute a QW-RDFT!

0 1 2 3

QW-DST algorithm reduced to real FFT:

1. For $n = 0, \dots, N-1$ set

$$
\begin{aligned}
g_n &= f_n, \\
g_{2N-n-1} &= -f_n.
\end{aligned}
$$

2. Use $2N$ `real-FFT` to compute $G_k$ from $g_n$ (for $k = 0, \dots, N$).

3. Calculate the QW-DST coefficients $F_k = G_k \omega_{2N}^{-\frac{k}{2}}$

## Problem 2 Wavelets Approximation (12 credits)

The cubic Battle-Lemarié wavelet is given by the coefficients $\{c_k\}$:

$$c_0 = \frac{1}{8} \qquad c_1 = \frac{1}{2} \qquad c_2 = \frac{3}{4} \qquad c_3 = \frac{1}{2} \qquad c_4 = \frac{1}{8}. \tag{2.1}$$

Starting with the mother hat function

$$\phi_0(t) = \max\{1 - |t|, 0\}, \tag{2.2}$$

we can compute an approximation of the scaling and wavelet functions by iterating over their dilation equations. Ignoring the scaling factor such as $\frac{1}{\sqrt{2}}$, the dilation equation for the scaling (father) function is given by

$$\phi_{n+1}(t) = \sum_{k=0}^{4} c_k \cdot \phi_n(2t - k), \tag{2.3}$$

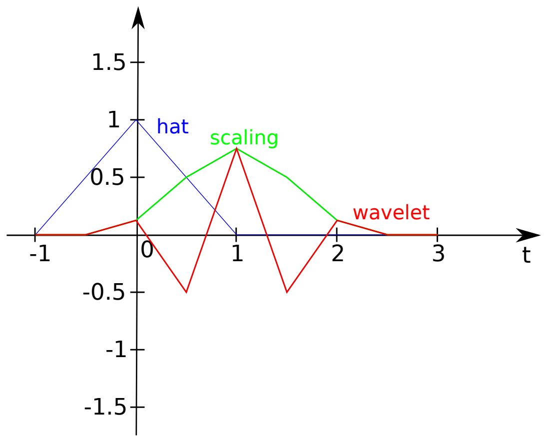
and the dilation equation for the wavelet (mother) function is given by

$$\psi_{n+1}(t) = \sum_{k=0}^{4} (-1)^k c_{K-k} \cdot \phi_n(2t - k), \quad \text{where } K = 4. \tag{2.4}$$

0 1 2 3 4 5 6 7 8

a) In the given figure below, sketch the first approximation of the scaling function $\phi_1(t)$ and the wavelet function $\psi_1(t)$ on the interval $[-1, 3]$.

**Hint 1**: Use the given helper table on the right to compute the values of $\phi_1(t)$ and $\psi_1(t)$ at given points to help for your sketching.
**Hint 2**: Backup figure and helper table is provided in case you need to correct your solutions.




| | $\phi_1(t)$ | $\psi_1(t)$ |
|---|---|---|
| $t = -1.0$ | | |
| $t = -0.5$ | | |
| $t = 0.0$ | | |
| $t = 0.5$ | | |
| $t = 1.0$ | | |
| $t = 1.5$ | | |
| $t = 2.0$ | | |
| $t = 2.5$ | | |
| $t = 3.0$ | | |

b) By observation, what conclusion can you draw for the integral of the obtained wavelet, i.e., $\int_{-\infty}^{+\infty} \psi_1(t)$? Explain, what this implies for wavelets in general.

0 1 2 3 4

Observation:

- Area above x-axis equals area below x-axis,
  i.e., $\rightarrow \int_{-\infty}^{+\infty} \psi_1(t) = 0$

What it implies:

- The mean is zero $\rightarrow$ The wavelet basis functions will represent fluctuations to the average of the coarser level.
- (admissibility condition is met)

## Problem 3 Sparse Grids (10 credits)

0 1 2 3 4 5 6

a) Name **two** different data structures for sparse grids. For each of them, discuss the following points:

(i) Hierarchization/dehierarchization (Hint: consider data access and traversal complexity)

(ii) Spatial adaptivity

(iii) Memory consumption

Discuss any two of these data structures:

**Arrays**

(i) Access by index $O(1)$, need mapping between hierarchical index $(\vec{l}, \vec{i})$ and “flat“ (1D) index $j$. Hierarchical neighbors can be deduced from the current node’s hierarchical index $(\vec{l}, \vec{i})$.

(ii) Cannot add or delete element $\rightarrow$ Does not support spatial adaptivity. However, dimensional adaptivity can be accommodated.

(iii) Store only data, i.e., $v_j$, at each grid point. No need to store $(\vec{l}, \vec{i})$ if correct mapping is provided. Cache efficient!

**Trees**

(i) One hierarchical neighbour is the parent node, the other one (ancestor) can be found from the root node. The neighbours can be passed as additional parameters in the algorithm.

(ii) Spatial Adaptivity is done straightforward by adding the nodes corresponding to the adaptive points.

(iii) On each node: $2d$ pointers to children, $d$ optional pointers for parent (ancestors can be trace back through parent pointer) $\rightarrow$ relatively high memory consumption.

**Hash tables**

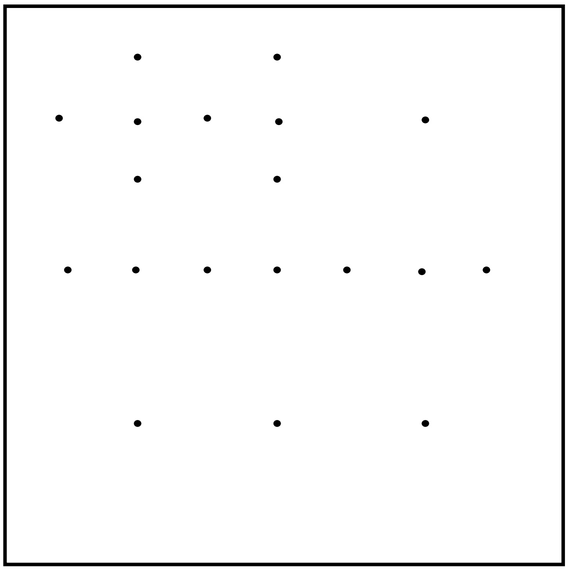
(i) Access by Key—hierarchical index $(\vec{l}, \vec{i})$ *rightarrow* complexity $O(1)$, computation of indices (hashing).

(ii) Spatial Adaptivity poses no problem, additional points can be inserted independently of existing points in the data structure.

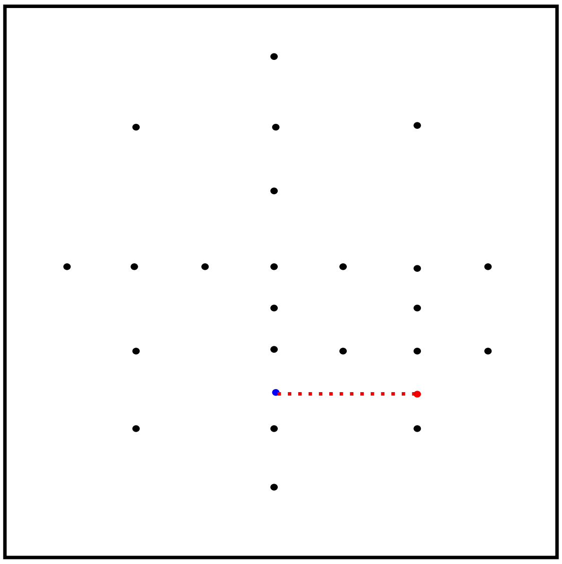
(iii) Memory consumption is low, because for each node, we only need to store a Key $(\vec{l}, \vec{i})$ and a data $v_j$. Cache inefficient due to hashing.

b) Discuss whether the following two grids are valid sparse grids. Explain your reasons. You can annotate the grids directly.

0
1
2
3
4



Valid: adaptive refinement, no missing parent



Invalid: point in red missing a parent (blue)

## Problem 4 Space-filling Curves - Peano-Meander Curve (11 credits)

Figure 4.1 depicts the first two iterations of a Meander type Peano curve.

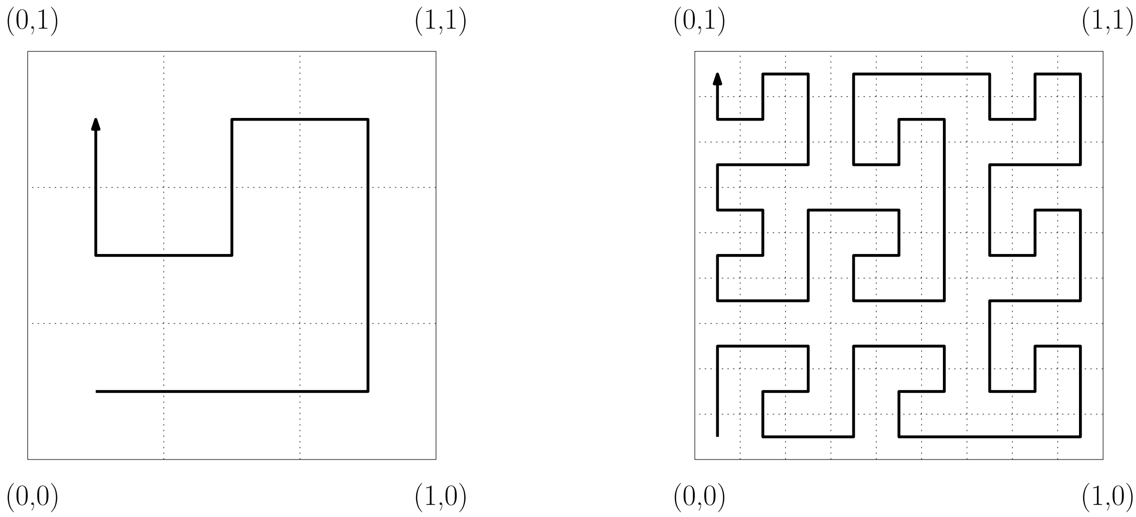
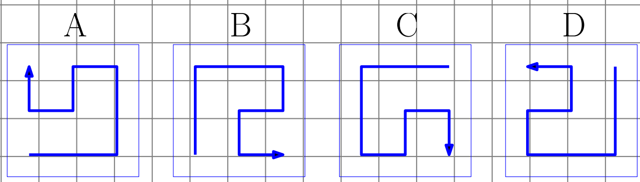



Figure 4.1: First two iterations of a Peano-Meander curve.

a) Find a grammar which constructs the Peano-Meander curve from figure 4.1.

Terminals: $\uparrow, \downarrow, \leftarrow, \rightarrow$
Nonterminals: $A, B, C, D$




Startsymbol: $A$
Productions:

- $A \longrightarrow B \rightarrow B \rightarrow A \uparrow A \uparrow A \leftarrow C \downarrow D \leftarrow D \uparrow A$
- $B \longrightarrow A \uparrow A \uparrow B \rightarrow B \rightarrow B \downarrow D \leftarrow C \downarrow C \rightarrow B$
- $C \longrightarrow D \leftarrow D \leftarrow C \downarrow C \downarrow C \rightarrow A \uparrow B \rightarrow B \downarrow C$
- $D \longrightarrow C \downarrow C \downarrow D \leftarrow D \leftarrow D \uparrow B \rightarrow A \uparrow A \leftarrow D$

b) Given is a parametrization $q(t)$ of the Peano-Meander curve

$$q(0_9.n_1 n_2 n_3 n_4 \ldots) = Q_{n_1} \circ Q_{n_2} \circ Q_{n_3} \circ Q_{n_4} \circ \ldots \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{4.1}$$

where $t = 0_9.n_1 n_2 n_3 n_4 \ldots$ is the representation of $t$ in a base nine system.

1) Determine the operators $Q_1$, $Q_4$ and $Q_6$.

$$Q_1 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix}$$

$$Q_4 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$Q_6 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$$

2) Compute the coordinates of $q(\frac{2}{3})$ and $q(\frac{1}{2})$.

$\frac{2}{3} = 0_9.6$

$q\left(\frac{2}{3}\right) = Q_6\binom{0}{0} = \begin{pmatrix}\frac{2}{3}\\ \frac{2}{3}\end{pmatrix}$

$\frac{1}{2} = 0_9.44444\ldots$

$q\left(\frac{1}{2}\right) = Q_4 \circ Q_4 \circ \ldots \binom{0}{0} = \lim_{n\to\infty} Q_4^n \binom{0}{0}$

Write $Q_4$ in the following form:

$$Q_4 = \underbrace{\begin{pmatrix}\frac{1}{3} & 0\\ 0 & \frac{1}{3}\end{pmatrix}}_{=:A}\underbrace{\begin{pmatrix}x\\ y\end{pmatrix}}_{=:v} + \underbrace{\begin{pmatrix}\frac{2}{3}\\ \frac{2}{3}\end{pmatrix}}_{=:b} = Av + b.$$

From this, we get
$Q_4^n v = A^n v + A^{n-1}b + \ldots + Ab + b$
and
$(I - A)\left(A^{n-1}b + \ldots + Ab + b\right) = b - A^n b = (I - A^n)\, b$
Now we can compute

$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} =: \lim_{n\to\infty} Q_4^n \binom{0}{0} = \lim_{n\to\infty}\left(\underbrace{A^n v}_{\to 0}(I-A)^{-1}(I - \underbrace{A^n}_{\to 0})b\right) = (I-A)^{-1}\, b$

$\Leftrightarrow (I - A)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = b \Leftrightarrow \begin{pmatrix}\frac{2}{3} & 0\\ 0 & \frac{2}{3}\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\frac{2}{3}\\ \frac{2}{3}\end{pmatrix}$

Solution: $\alpha = 1$ , $\beta = 1$.

**Additional space for solutions–clearly mark the (sub)problem your answers are related to and strike out invalid solutions.**