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Worksheet 2
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Algorithms for Scientific Computing

Exercise 1

In accordance with standard FFT-splitting, we form the sum formula for the c_k in:

$$
\begin{aligned}
c_k &= \frac{1}{12}\sum_{l=0}^{11} X_l e^{-i2\pi kl/12} \\
&= \frac{1}{12}\sum_{l=0}^{5}\left(X_{2l}e^{-i2\pi k(2l)/12} + X_{2l+1}e^{-i2\pi k(2l+1)/12}\right) \\
&= \frac{1}{12}\left(\sum_{l=0}^{5} X_{2l}e^{-i2\pi kl/6} + e^{-i2\pi k/12}\sum_{l=0}^{5} X_{2l+1}e^{-i2\pi kl/6}\right)
\end{aligned}
$$

We use this to compute the c_k for $k = 0, \ldots, 5$. To compute c_6 we reformulate as following:

$$
\begin{aligned}
c_6 = c_{0+6} &= \frac{1}{12}\left(\sum_{l=0}^{5} X_{2l}e^{-i2\pi(0+6)l/6} + e^{-i2\pi(0+6)/12}\sum_{l=0}^{5} X_{2l+1}e^{-i2\pi(0+6)l/6}\right) \\
&= \frac{1}{12}\left(\sum_{l=0}^{5} X_{2l}e^{-i2\pi 0l/6} + e^{-i2\pi 0/12}\cdot e^{-i2\pi 6/12}\sum_{l=0}^{5} X_{2l+1}e^{-i2\pi 0l/6}\right) \\
&= \frac{1}{12}\left(\sum_{l=0}^{5} X_{2l}e^{-i2\pi 0l/6} - \underbrace{e^{-i2\pi 0/12}}_{=1}\sum_{l=0}^{5} X_{2l+1}e^{-i2\pi 0l/6}\right).
\end{aligned}
$$

This is now the butterfly scheme for c_0 and c_6!

Now we define the coefficients for the required length 6 DFTs

$$
\begin{aligned}
\widetilde{c}_k &:= \sum_{l=0}^{5} X_{2l}e^{-i2\pi kl/6} \\
\widehat{c}_k &:= \sum_{l=0}^{5} X_{2l+1}e^{-i2\pi kl/6},
\end{aligned}
$$

for each $k = 0, \ldots, 5$, the coefficients c_k are calculated:

$$\begin{aligned} c_k &= \frac{1}{12}\left(\widetilde{c}_k + e^{-i2\pi k/12} \cdot \widehat{c}_k\right) \quad \text{for} \quad k = 0, \ldots, 5 \\ c_6 &= \frac{1}{12}\left(\widetilde{c}_0 - \widehat{c}_0\right). \end{aligned}$$

We calculate the Fourier transform of the 12 real data, dividing them into 2 real Fourier transforms of length 6.

Calculation of the length 6-DFTs using length 3 DFTs

In exactly the same was, the coefficients $\widetilde{c}_k$ and $\widehat{c}_k$ are calculated according to the FFT-Butterfly scheme:

$$\begin{aligned} \widetilde{c}_k &= \sum_{l=0}^{2} X_{4l} e^{-i2\pi kl/3} + e^{i2\pi k/6} \sum_{l=0}^{2} X_{4l+2} e^{-i2\pi kl/3} \\ \widetilde{c}_{k+3} &= \sum_{l=0}^{2} X_{4l} e^{-i2\pi kl/3} - e^{i2\pi k/6} \sum_{l=0}^{2} X_{4l+2} e^{-i2\pi kl/3} \end{aligned}$$

and

$$\begin{aligned} \widehat{c}_k &= \sum_{l=0}^{2} X_{4l+1} e^{-i2\pi kl/3} + e^{i2\pi k/6} \sum_{l=0}^{2} X_{4l+3} e^{-i2\pi kl/3} \\ \widehat{c}_{k+3} &= \sum_{l=0}^{2} X_{4l+1} e^{-i2\pi kl/3} - e^{i2\pi k/6} \sum_{l=0}^{2} X_{4l+3} e^{-i2\pi kl/3}, \end{aligned}$$

for each $k = 0, 1, 2$.

since all X_l are real, we can use the symmetry and write

$$\widehat{c}_{6-k} = \widehat{c}_k^* \qquad \text{and} \qquad \widetilde{c}_{6-k} = \widetilde{c}_k^*.$$

Since $\widehat{c}$ and $\widetilde{c}$ are each 6-periodic, the same would apply for $\widehat{c}_{-k} = \widehat{c}_k^*$ respectively, $\widetilde{c}_{-k} = \widetilde{c}_k^*$ but the index k for this case is $k = 0, \ldots, 6$.

We define the above required four DFTs of length 3 as

$$\begin{aligned} \mathcal{F}_k^{(0,4,8)} &:= \sum_{l=0}^{2} X_{4l} e^{-i2\pi kl/3} \qquad & \mathcal{F}_k^{(1,5,9)} &:= \sum_{l=0}^{2} X_{4l+1} e^{-i2\pi kl/3} \\ \mathcal{F}_k^{(2,6,10)} &:= \sum_{l=0}^{2} X_{4l+2} e^{-i2\pi kl/3} \qquad & \mathcal{F}_k^{(3,7,11)} &:= \sum_{l=0}^{2} X_{4l+3} e^{-i2\pi kl/3}. \end{aligned}$$

Then the $\widetilde{c}_k$s are computed from the following Butterflies:

$$
\begin{aligned}
\widetilde{c}_k &= \mathcal{F}_k^{(0,4,8)} + e^{i\pi k/3}\mathcal{F}_k^{(2,6,10)} \qquad \text{for} \quad k = 0,1,2 \\
\widetilde{c}_3 &= \mathcal{F}_0^{(0,4,8)} - \mathcal{F}_0^{(2,6,10)} \\
\widetilde{c}_k &= \widetilde{c}^*_{N-k} \qquad \text{for} \quad k = 4,5
\end{aligned}
$$

and the $\widehat{c}_k$ as well.

$$
\begin{aligned}
\widehat{c}_k &= \mathcal{F}_k^{(1,5,9)} + e^{i\pi k/3}\mathcal{F}_k^{(3,7,11)} \qquad \text{for} \quad k = 0,1,2 \\
\widehat{c}_3 &= \mathcal{F}_0^{(1,5,9)} - \mathcal{F}_0^{(3,7,11)} \\
\widehat{c}_k &= \widehat{c}^*_{N-k} \qquad \text{for} \quad k = 4,5
\end{aligned}
$$

Computation of the 3-DFTs

We can easily compute the 3-DFTs relatively easy, e.g.:

$$
\begin{aligned}
\mathcal{F}_0^{(0,4,8)} &= \sum_{l=0}^{2} X_{4l} \\
\mathcal{F}_1^{(0,4,8)} &= \sum_{l=0}^{2} X_{4l} e^{-i2\pi l/3} \\
\mathcal{F}_2^{(0,4,8)} &= \left\{ \mathcal{F}_1^{(0,4,8)} \right\}^*
\end{aligned}
$$

Exercise 2: DFT of Mirrored data

Steps:

① Index shift. Note that we do not interpret the sum as an empty sum, but as a short-hand notation for writing out all the terms separately. Following this idea, we switch the summation bounds in the next step.

② $\omega_N^{-kN} = e^{\frac{-2\pi ikN}{N}} = 1$

③ Change the summation bounds from $1, \ldots, N$ to $0, \ldots, N-1$. We can do this because it is $f_0 = f_N$ by definition. Furthermore, it is $\omega_N^{kn}\Big|_{n=0} = e^0 = 1 = e^{\frac{2\pi ikN}{N}} = \omega_N^{kn}\Big|_{n=N}$.

$$
\begin{aligned}
\widetilde{F}_k &= \frac{1}{N}\sum_{n=0}^{N-1}\widetilde{f}_n\omega_N^{-kn} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} f_{N-n}\omega_N^{-kn} \\
&\overset{①}{=} \frac{1}{N}\sum_{n=N}^{1} f_n\omega_N^{-k(N-n)} \\
&= \frac{1}{N}\sum_{n=1}^{N} f_n\omega_N^{-kN}\omega_N^{kn} \\
&\overset{②}{=} \frac{1}{N}\sum_{n=1}^{N} f_n\omega_N^{kn} \\
&\overset{③}{=} \frac{1}{N}\sum_{n=0}^{N-1} f_n\omega_N^{kn} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} f_n\omega_N^{-(-k)n} \\
&= F_{-k}
\end{aligned}
$$

Thus, the Fourier coefficients are also mirrored. The coefficients F_{-k} belong to the Ansatzfunctions $e^{i2\pi(-k)x}$, which unlike the original Ansatzfunctions, $e^{i2\pi kx}$, move in the "mirrored"' direction. Since the coefficients remain the same, the result is the "mirrored"' signal.

Exercise 3: DFT and "'Padding"'

For the classic Fast Fourier Transform the number of discrete data must be a power of two. If this is not the case, one could try to fill up the dataset by "zero" entries like this:

$$
\hat{f}_n := \begin{cases} f_n & \text{if} \quad n \le N-1 \\ 0 & \text{if} \quad N \le n \le M-1 \end{cases}
$$

The Fourier coefficients $\widehat{F}_k$ of the extended dataset then add up to

$$
\widehat{F}_k = \frac{1}{M}\sum_{n=0}^{M-1}\hat{f}_n\omega_M^{-kn} = \frac{1}{M}\sum_{n=0}^{N-1} f_n\omega_M^{-kn}.
$$

This looks like if the $\widehat{F}_k$ are just the $\frac{N}{M}$ multiple of the original coefficients from the transform of length N:

$$F_k = \frac{1}{N} \sum_{n=0}^{N-1} f_n \omega_N^{-kn}.$$

However, this is not the case, since

$$\omega_N^{-kn} \neq \omega_M^{-kn}.$$

So, the frequencies of the base functions do change.

If we take the Fourier transform as an interpolation problem, then the extension of the dataset is equal to an increment of the number of supporting points. Since the observed interval stays the same ($[0, 2\pi]$), the distance between the supporting points must decrease. By padding the dataset with ”zeros” we actually compressed the signal and therefore the signal must be assembled from higher-frequency oscillations.

We go on with the equation from above. First we show that

$$\omega_M^{-kn} = e^{-i2\pi kn/M} = e^{-i2\pi kn(N/M)/N} = \left(\omega_N^{-kn}\right)^{N/M}$$

holds and therefore

$$\widehat{F}_k = \frac{1}{M} \sum_{n=0}^{N-1} f_n \left(\omega_N^{-kn}\right)^{N/M} = \frac{N}{M} \cdot \frac{1}{N} \sum_{n=0}^{N-1} f_n \omega_N^{-k(N/M)n}$$

In general we cannot express this by the F_k. But if kN/M is an integer number, we get

$$\widehat{F}_k = \frac{N}{M} \cdot \frac{1}{N} \sum_{n=0}^{N-1} f_n \omega_N^{-k(N/M)n} = \frac{N}{M} F_{kN/M}$$

Explanation: The Fourier components $\widehat{F}_k$ of the compressed signal belong to the wave number k. In the original signal the same component would belong to the oscillation with wave number kN/M. If kN/M is an integer number, this Fourier component is also computed in the ”short” transformation and can be taken from the ”long” transformation directly without being changed. If kN/M is not an integer number, then there is no according component in the ”short” transformation.

Exercise 4: Circular Convolution Theorem

$$
\begin{aligned}
NF_kG_k &= N\left(\frac{1}{N}\sum_{n=0}^{N-1} f_n\omega_N^{-kn}\right)\left(\frac{1}{N}\sum_{m=0}^{N-1} g_m\omega_N^{-km}\right) \\
&= \frac{1}{N}\sum_{n=0}^{N-1} f_n\omega_N^{-kn}\left(\sum_{m=0}^{N-1} g_m\omega_N^{-km}\right) \\
&\overset{①}{=} \frac{1}{N}\sum_{n=0}^{N-1} f_n\omega_N^{-kn}\left(\sum_{l=0}^{N-1} g_{l-n}\omega_N^{-k(l-n)}\right) \\
&= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{l=0}^{N-1} f_n g_{l-n}\omega_N^{-k(n+l-n)} \\
&\overset{②}{=} \frac{1}{N}\sum_{l=0}^{N-1}\sum_{n=0}^{N-1} f_n g_{l-n}\omega_N^{-kl} \\
&= \frac{1}{N}\sum_{l=0}^{N-1}\left(\sum_{n=0}^{N-1} f_n g_{l-n}\right)\omega_N^{-kl} \\
&= \frac{1}{N}\sum_{l=0}^{N-1} h_l\omega_N^{-kl} \\
&= H_k
\end{aligned}
$$

① Index shift in the second sum using the periodicity

② Swap the sums

The full convolution cost $N\mathcal{O}(N) = \mathcal{O}(N^2)$

A FFT/IFFT cost $\mathcal{O}(N\log N)$ and the entrywise product of F and G $\mathcal{O}(N)$ so the cost of doing FFT $\rightarrow$ entrywise product $\rightarrow$ IFFT is $2\mathcal{O}(N\log N) + \mathcal{O}(N) + \mathcal{O}(N\log N) = \mathcal{O}(N\log N)$