General Instructions

Material:

You may only use one hand-written sheet of paper (size A4, on both pages).

Any other material including electronic devices of any kind is forbidden.

Use only the exam paper that was handed out to solve the exercises. If the space on a page is not enough, choose any empty space provided and state where you continue with your solution. For additional notes and sketches, you can obtain separate exam sheets.

Do not use pencil, or red or green ink.

Errors and ambiguities:

If you think that a question contains an error or ambiguity, then correct it or choose an interpretation that allows you to complete the exercise.

Make sure to point out your correction or disambiguation in your solution!

Note that questions, esp. concerning errors or ambiguities, will not be answered during the exam.

General hint:

Often, exercises b), c), etc. can be solved without the results from the previous exercise a); if you are stuck with exercise a), then don't immediately skip exercises b), c), etc.

Maximum score:

The total number of points of this collection of exam questions exceeds the number of points typically achievable in an exam. In the final exam, the achievable number of points will be specified and whether these include an overhead compared to the 100% mark used for the grading scheme.

Working time:

120 minutes.

Please switch off your cell phones!

Proposed Solution

1	2	3	4	5	$\sum$
/11	/12	/10	/10	/11	/54

Last name, first name, student ID:

1 Quarter-Wave Discrete Sine Transform (≈ 5 + 3 + 3 = 11 points)

Given is a real-valued input data-set $f_0, \ldots, f_{2N-1} \in \mathbb{R}$ (i.e., length $2N$) with the symmetry conditions $f_{2N-n-1} = -f_n$. Complete the tasks below.

a) Show that the corresponding Quarter-Wave Fourier coefficients

$$F_k = \frac{1}{2N} \sum_{n=0}^{2N-1} f_n \omega_{2N}^{-k\left(n+\frac{1}{2}\right)} \tag{1}$$

have only imaginary values and can be written as

$$F_k = -\frac{i}{N} \sum_{n=0}^{N-1} f_n \sin\left(\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right) \tag{2}$$

The proof is done in the following steps:

① Isolate the symmetry condition

② Insert the symmetry condition

③ Assemble terms to a sum over f_n

④ Make terms “imaginary”

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$$
\begin{aligned}
F_k &= \frac{1}{2N}\sum_{n=0}^{2N-1} f_n \omega_{2N}^{-k\left(n+\frac{1}{2}\right)} \\
&\overset{①}{=} \frac{1}{2N}\left(\sum_{n=0}^{N-1} f_n \omega_{2N}^{-k\left(n+\frac{1}{2}\right)} + \sum_{n=N}^{2N-1} f_n \omega_{2N}^{-k\left(n+\frac{1}{2}\right)}\right) \\
&\overset{②}{=} \frac{1}{2N}\left(\sum_{n=0}^{N-1} f_n \omega_{2N}^{-k\left(n+\frac{1}{2}\right)} - \sum_{n=N}^{2N-1} f_{2N-n-1} \omega_{2N}^{-k\left(n+\frac{1}{2}\right)}\right) \\
&= \frac{1}{2N}\left(\sum_{n=0}^{N-1} f_n \omega_{2N}^{-k\left(n+\frac{1}{2}\right)} - \sum_{n=0}^{N-1} f_n \omega_{2N}^{-k\left(2N-n-1+\frac{1}{2}\right)}\right) \\
&= \frac{1}{2N}\left(\sum_{n=0}^{N-1} f_n \omega_{2N}^{-k\left(n+\frac{1}{2}\right)} - \sum_{n=0}^{N-1} f_n \omega_{2N}^{k\left(n+\frac{1}{2}\right)}\right) \\
&\overset{③}{=} \frac{1}{2N}\sum_{n=0}^{N-1} f_n \underbrace{\left(\omega_{2N}^{-k\left(n+\frac{1}{2}\right)} - \omega_{2N}^{k\left(n+\frac{1}{2}\right)}\right)}_{=\omega_{2N}^{-k\left(n+\frac{1}{2}\right)} - \left(\omega_{2N}^{k\left(n+\frac{1}{2}\right)}\right)^{*} = -2i\operatorname{Im}\{\omega_{2N}^{-k\left(n+\frac{1}{2}\right)}\}} \\
&\overset{④}{=} \frac{-2i}{2N}\sum_{n=0}^{N-1} f_n \underbrace{\operatorname{Im}\{e^{i2\pi k\left(n+\frac{1}{2}\right)/2N}\}}_{=\operatorname{Im}\left\{\cos\left(\frac{\pi k\left(n+\frac{1}{2}\right)}{N}\right)+i\sin\left(\frac{\pi k\left(n+\frac{1}{2}\right)}{N}\right)\right\}} \\
&= -\frac{i}{N}\sum_{n=0}^{N-1} f_n \sin\left(\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right) \in \mathbb{C} \qquad \text{q.e.d.}
\end{aligned}
$$

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b) Show that the coefficients F_k of the QW-DST again justify a symmetry condition!

Since there are only sine terms, we assume (and get) an odd symmetry:

$$
\begin{aligned}
F_{2N-k} &= -\frac{i}{N}\sum_{n=0}^{N-1} f_n \sin\left(\frac{\pi}{N}(2N-k)\left(n+\frac{1}{2}\right)\right) \\
&= -\frac{i}{N}\sum_{n=0}^{N-1} f_n \sin\left(2\pi-\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right) \\
&= \frac{i}{N}\sum_{n=0}^{N-1} f_n \sin\left(\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right) \\
&= -F_k
\end{aligned}
$$

Since all $F_k = -F_{2N-k}$, we need the F_k only for $k = 0, \ldots, N$ for a Sine Transform.

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c) Let `real-FFT(g,N)` be a procedure that computes the Fourier coefficients G_k efficiently from a real data-set `g` that consists of $2N$ values g_n.

Describe a **procedure** `QW-DST(g,N)` that uses the given procedure `real-FFT` to compute the coefficients F_k for $k = 0, \ldots, N-1$ from equation (2) for the (non-symmetrical) real data $f_0, \ldots, f_{N-1}$, stored in the parameter field `g`.

Note: `real-FFT(g,N)` **does not** compute a QW-RDFT!

QW-DST algorithm reduced to real FFT:

(a) For $n = 0, \ldots, N-1$ set

$$g_n = f_n,$$
$$g_{2N-n-1} = -f_n.$$

(b) Use $2N$ `real-FFT` to compute G_k from g_n (for $k = 0, \ldots, N$).

(c) Calculate the QW-DST coefficients $F_k = G_k \omega_{2N}^{-\frac{k}{2}}$

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2 Wavelets Approximation (≈ 8 + 4 = 12 points)

The cubic Battle-Lemarié wavelet is given by the coefficients $\{c_k\}$:

$$c_0 = \frac{1}{8} \qquad c_1 = \frac{1}{2} \qquad c_2 = \frac{3}{4} \qquad c_3 = \frac{1}{2} \qquad c_4 = \frac{1}{8}\,. \tag{3}$$

Starting with the mother hat function

$$\phi_0(t) = \max\{1 - |t|,\, 0\}, \tag{4}$$

we can compute an approximation of the scaling and wavelet functions by iterating over their dilation equations.

Ignoring the scaling factor such as $\frac{1}{\sqrt{2}}$, the dilation equation for the scaling (father) function is given by

$$\phi_{n+1}(t) = \sum_{k=0}^{4} c_k \cdot \phi_n(2t - k), \tag{5}$$

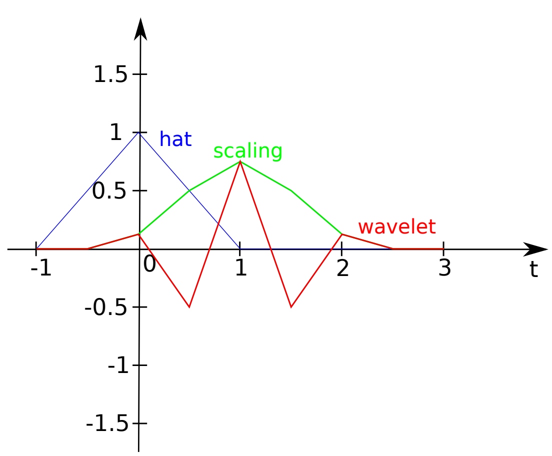
and the dilation equation for the wavelet (mother) function is given by

$$\psi_{n+1}(t) = \sum_{k=0}^{4} (-1)^k c_{K-k} \cdot \phi_n(2t - k), \quad \text{where } K = 4. \tag{6}$$

a) In the given figure below, sketch the first approximation of the scaling function $\phi_1(t)$ and the wavelet function $\psi_1(t)$ on the interval $[-1, 3]$.

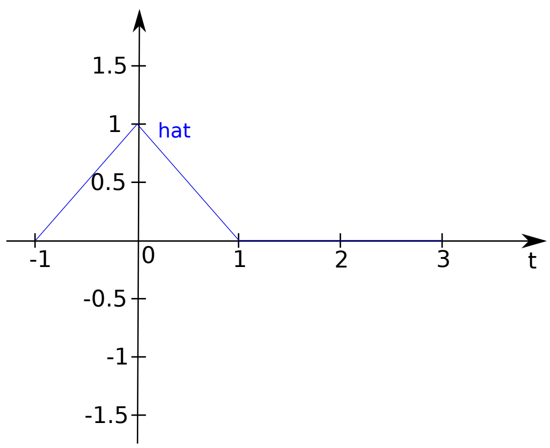
Hint 1: Use the given helper table on the right to compute the values of $\phi_1(t)$ and $\psi_1(t)$ at given points to help for your sketching.

Hint 2: Backup figure and helper table is provided in case you need to correct your solutions.



	$\phi_1(t)$	$\psi_1(t)$
$t = -1.0$	0.0	0.0
$t = -0.5$	0.0	0.0
$t = 0.0$	0.125	0.125
$t = 0.5$	0.5	-0.5
$t = 1.0$	0.75	0.75
$t = 1.5$	0.5	-0.5
$t = 2.0$	0.125	0.125
$t = 2.5$	0.0	0.0
$t = 3.0$	0.0	0.0

Backup figure and helper table



	$\phi_1(t)$	$\psi_1(t)$
$t = -1.0$		
$t = -0.5$		
$t = 0.0$		
$t = 0.5$		
$t = 1.0$		
$t = 1.5$		
$t = 2.0$		
$t = 2.5$		
$t = 3.0$		

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b) By observation, what conclusion can you draw for the integral of the obtained wavelet, i.e., $\int_{-\infty}^{+\infty} \psi_1(t)$? Explain, what this implies for wavelets in general.

Observation:

- Area above x-axis equals area below x-axis, i.e., $\rightarrow \int_{-\infty}^{+\infty} \psi_1(t) = 0$

What it implies:

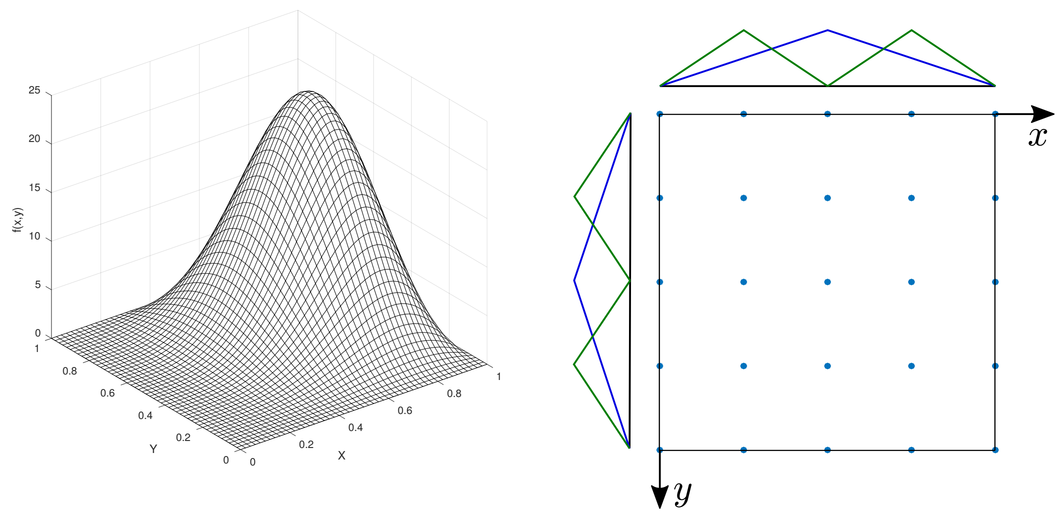
- The mean is zero $\rightarrow$ The wavelet basis functions will represent fluctuations to the average of the coarser level.
- (admissibility condition is met)

3 2D Hierarchization and Sparse Grids (≈ 3 + 2 + 5 = 10 points)

In this problem we want to interpolate the two-dimensional function $f(x, y)$ given by

$$f(x, y) = 128\ x^3 \cdot \sin(\pi x) \cdot y \cdot \sin(\pi y) \tag{7}$$

in the domain $\Omega = [0, 1]^2$ using the hierarchical hat function basis, as discussed in the lectures. As you can see in Figure 1, the function is 0 on the boundaries and the most important features are located in the area of $x \geq 0.5$ and $y \geq 0.5$.



Function to interpolate

Full grid of level (2,2)

Figure 1: Function and full grid

a) **2D Hierarchization:** We first use a hierarchical basis on a full grid to interpolate function f. Grid points (x_i, y_j) and hierarchical basis functions are illustrated in Figure 1. Table 1 shows the function evaluation $f(x, y)$ at each grid point (x_i, y_j). Perform 2D-hierarchization to transform the function values into hierarchical surpluses (round to **two digits** after the decimal point).

Use Tables 2 and 3 (extra tables are given on the right in case you want to correct an error) or redraw these tables on an extra sheet to compute your solution.

Table 1: Function values $f(x_i, y_j)$.

$y_j \backslash x_i$	0.00	0.25	0.50	0.75	1.00
0.00	0.00	0.00	0.00	0.00	0.00
0.25	0.00	0.25	2.83	6.75	0.00
0.50	0.00	0.71	8.00	19.09	0.00
0.75	0.00	0.75	8.49	20.25	0.00
1.00	0.00	0.00	0.00	0.00	0.00

Table 2: Hierarchization (intermediate results; use right table if you need to correct errors)

$y_j \backslash x_i$	0.00	0.25	0.50	0.75	1.00
0.00	0.00	0.00	0.00	0.00	0.00
0.25	0.00	-1.16	2.83	5.34	0.00
0.50	0.00	-3.29	8.00	15.09	0.00
0.75	0.00	-3.49	8.49	16.01	0.00
1.00	0.00	0.00	0.00	0.00	0.00

$y_j \backslash x_i$	0.00	0.25	0.50	0.75	1.00
0.00					
0.25					
0.50					
0.75					
1.00					

Table 3: Hierarchization (results; use right table if you need to correct errors)

$y_j \backslash x_i$	0.00	0.25	0.50	0.75	1.00
0.00	0.00	0.00	0.00	0.00	0.00
0.25	0.00	0.48	-1.17	-2.21	0.00
0.50	0.00	-3.29	8.00	15.09	0.00
0.75	0.00	-1.85	4.49	8.46	0.00
1.00	0.00	0.00	0.00	0.00	0.00

$y_j \backslash x_i$	0.00	0.25	0.50	0.75	1.00
0.00					
0.25					
0.50					
0.75					
1.00					

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b) **2D Hierarchical Surpluses:** Consider the hierarchical interpolant on $(N+1) \times (N+1)$ grids for increasing number of grid points $N = 2^L$. Characterize how the size of the surpluses decreases for sufficiently smooth functions f (state at least two properties).

- along horizontal or vertical grid lines, the hierarchical surpluses decrease by $\frac{1}{4}$ from level to level
- along "diagonals", the hierarchical surpluses decrease by $\frac{1}{16}$
- the absolute size of the surpluses depends on the (mixed) second derivatives of f

c) **Sparse Grid:** Now we use a sparse grid as shown in Figure 2 to interpolate function f. Grid points have been hierarchized. A grid point can be uniquely identified by a level-index-vector pair $\vec{l},\vec{i} := (l_x, l_y), (i_x, i_y)$.

Table 4: Level 2 regular sparse grid points

$\vec{l},\vec{i}$	$(1,1),(1,1)$	$(2,1),(1,1)$	$(2,1),(3,1)$	$(1,2),(1,1)$	$(1,2),(1,3)$
$\alpha_{\vec{l},\vec{i}}$	8.00	-3.29	15.09	-1.17	4.49

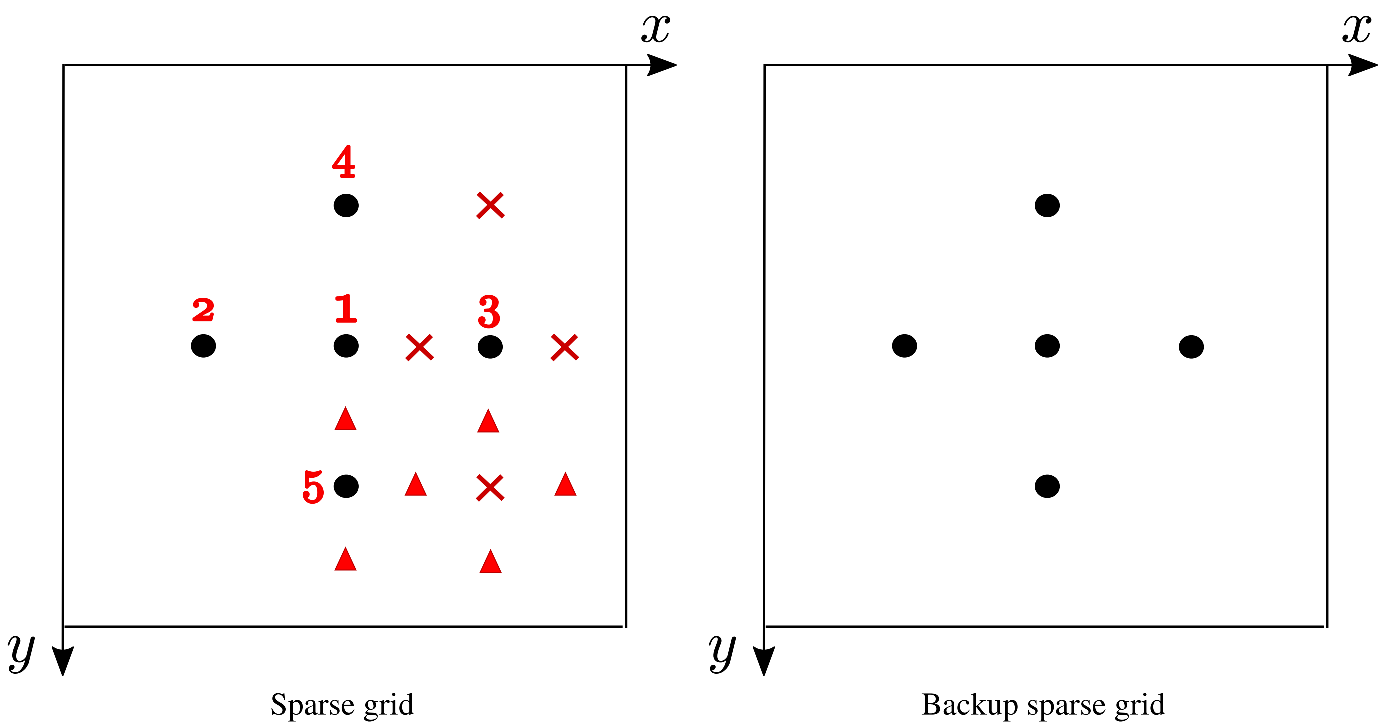


Figure 2: Sparse grid

i) Label the grid points in the picture with numbers $1, 2, \dots$ according to the first 2 lines in Table 4.

ii) Fill in the hierarchical surpluses $\alpha_{\vec{l},\vec{i}}$ in the third line. Use the function values provided in task a). Round to **two digits** after the decimal point.

iii) Refine the grid point with the highest hierarchical surplus value. Use the "$\times$" symbol to mark the added grid points in Figure 2.

iv) Refine grid point $(2,2),(3,3)$. Use the "$\triangle$" symbol to mark the added grid points in Figure 2.

Hint: Do not forget to check for missing parents in grid refinement.

4 Sparse Grids (≈ 6 + 4 = 10 points)

a) Name **two** different data structures for sparse grids. For each of them, discuss the following points:

(i) Hierarchization/dehierarchization (Hint: consider data access and traversal complexity)

(ii) Spatial adaptivity

(iii) Memory consumption

Discuss any two of these data structures:

Arrays

(i) Access by index $O(1)$, need mapping between hierarchical index $(\overrightarrow{l}, \overrightarrow{i})$ and "flat" (1D) index j. Hierarchical neighbors can be deduced from the current node's hierarchical index $(\overrightarrow{l}, \overrightarrow{i})$.

(ii) Cannot add or delete element $\rightarrow$ Does not support spatial adaptivity. However, dimensional adaptivity can be accommodated.

(iii) Store only data, i.e., v_j, at each grid point. No need to store $(\overrightarrow{l}, \overrightarrow{i})$ if correct mapping is provided. Cache efficient!

Trees

(i) One hierarchical neighbour is the parent node, the other one (ancestor) can be found from the root node. The neighbours can be passed as additional parameters in the algorithm.

(ii) Spatial Adaptivity is done straightforward by adding the nodes corresponding to the adaptive points.

(iii) On each node: $2d$ pointers to children, d optional pointers for parent (ancestors can be trace back through parent pointer) $\rightarrow$ relatively high memory consumption.

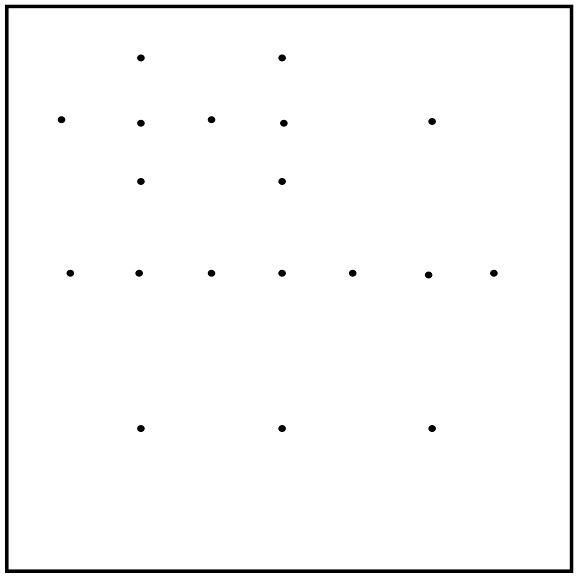
Hash tables

(i) Access by Key—hierarchical index $(\overrightarrow{l}, \overrightarrow{i})$ $rightarrow$ complexity $O(1)$, computation of indices (hashing).

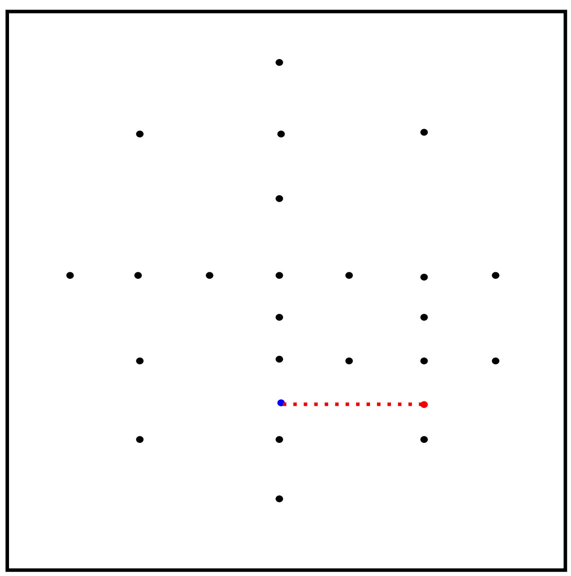
(ii) Spatial Adaptivity poses no problem, additional points can be inserted independently of existing points in the data structure.

(iii) Memory consumption is low, because for each node, we only need to store a Key $(\overrightarrow{l}, \overrightarrow{i})$ and a data v_j. Cache inefficient due to hashing.

b) Discuss whether the following two grids are valid sparse grids. Explain your reasons. You can annotate the grids directly.



Valid: adaptive refinement, no missing parent



Invalid: point in red missing a parent (blue)

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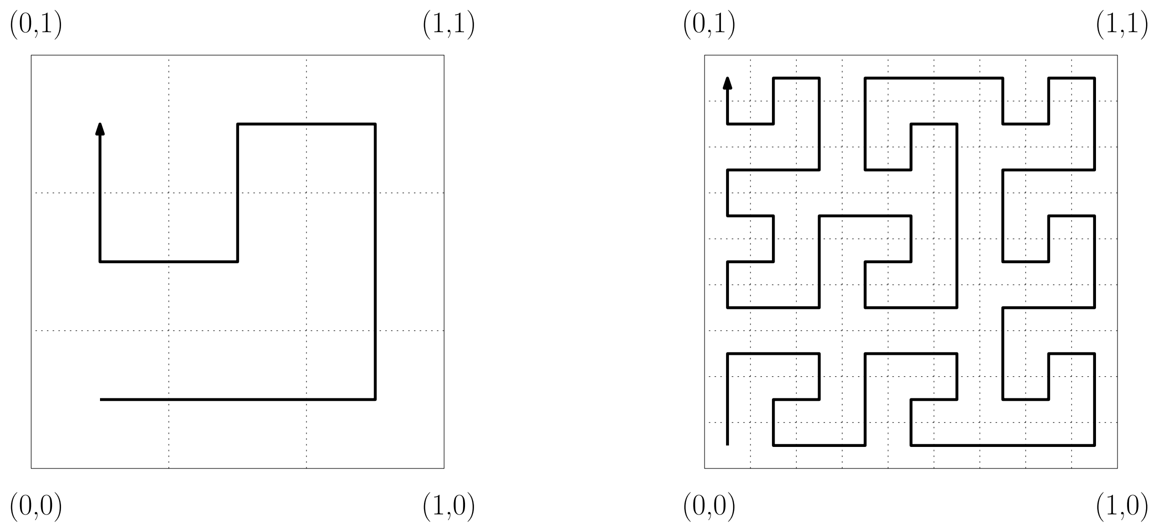


Figure 3: First two iterations of a Peano-Meander curve.

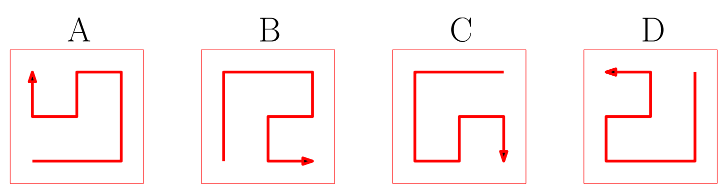
5 Space-filling Curves - Peano-Meander Curve ($\approx$ 5 + 3 + 3 = 11 points)

Figure 3 depicts the first two iterations of a Meander type Peano curve.

a) Find a grammar which constructs the Peano-Meander curve from figure 3.

Terminals: $\uparrow, \downarrow, \leftarrow, \rightarrow$

Nonterminals: A, B, C, D



Startsymbol: A

Productions:

- $A \longrightarrow B \rightarrow B \rightarrow A \uparrow A \uparrow A \leftarrow C \downarrow D \leftarrow D \uparrow A$
- $B \longrightarrow A \uparrow A \uparrow B \rightarrow B \rightarrow B \downarrow D \leftarrow C \downarrow C \rightarrow B$

- $C \longrightarrow D \leftarrow D \leftarrow C \downarrow C \downarrow C \rightarrow A \uparrow B \rightarrow B \downarrow C$
- $D \longrightarrow C \downarrow C \downarrow D \leftarrow D \leftarrow D \uparrow B \rightarrow A \uparrow A \leftarrow D$

b) Arithmetical representation of the Peano-Meander curve

1) Given is a parametrization $q(t)$ of the Peano-Meander curve

$$q(0_9.n_1n_2n_3n_4...) = Q_{n_1} \circ Q_{n_2} \circ Q_{n_3} \circ Q_{n_4} \circ ... \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{8}$$

where $t = 0_9.n_1n_2n_3n_4...$ is the representation of t in a base nine system. Determine the operators Q_1, Q_4 and Q_6.

$$Q_1 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix}$$
$$= \frac{1}{3} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix}$$

$$Q_4 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$Q_6 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$$
$$= \frac{1}{3} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$$

2) Compute the coordinates of $q(\frac{2}{3})$ and $q(\frac{1}{2})$.

$\frac{2}{3} = 0_9.6$

$q\left(\frac{2}{3}\right) = Q_6 \binom{0}{0} = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$

$\frac{1}{2} = 0_9.44444\ldots$

$$q\left(\tfrac{1}{2}\right) = Q_4 \circ Q_4 \circ \ldots \binom{0}{0} = \lim_{n\to\infty} Q_4^n \binom{0}{0}$$

Write Q_4 in the following form:

$$Q_4 = \underbrace{\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}}_{=:A} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{=:v} + \underbrace{\begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}}_{=:b} = Av + b.$$

From this, we get
$Q_4^n v = A^n v + A^{n-1} b + \ldots + Ab + b$
and
$(I - A)\left(A^{n-1} b + \ldots + Ab + b\right) = b - A^n b = (I - A^n)\, b$
Now we can compute
$\binom{\alpha}{\beta} =: \lim_{n\to\infty} Q_4^n \binom{0}{0} = \lim_{n\to\infty} \left(\underbrace{A^n v}_{\to 0} (I - A)^{-1} (I - \underbrace{A^n}_{\to 0}) b \right) =$
$(I - A)^{-1}\, b$

$$\Leftrightarrow (I - A)\binom{\alpha}{\beta} = b \Leftrightarrow \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix} \binom{\alpha}{\beta} = \binom{\frac{2}{3}}{\frac{2}{3}}$$

Solution: $\alpha = 1$, $\beta = 1$.