Worksheet 7: Haar Wavelets I

(1) Father wavelet or scaling $\phi(x)$

$$\phi_{\ell,k}(x) = \underset{\substack{\uparrow \\ \text{orthonormal} \\ \text{basis}}}{2^{\ell/2}} \phi(2^\ell x - k) \qquad V_\ell = \text{span}\{\phi_{\ell,k}(x)\}$$

(2) Mother wavelet $\psi(x)$, which will construct our surplus basis:

$$\psi_{\ell,k}(x) = 2^{\ell/2} \psi(\underbrace{2^\ell x - k}_{\frac{x - x_{\ell,k}}{h_\ell}}) \qquad \underset{\text{surplus space}}{W_\ell = \text{span}\{\psi_{\ell,k}(x)\}}$$

$$x_{\ell,k} = k \cdot 2^{-\ell}$$

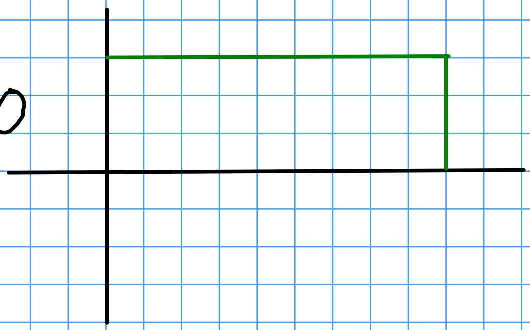
$$h_\ell = 2^{-\ell}$$

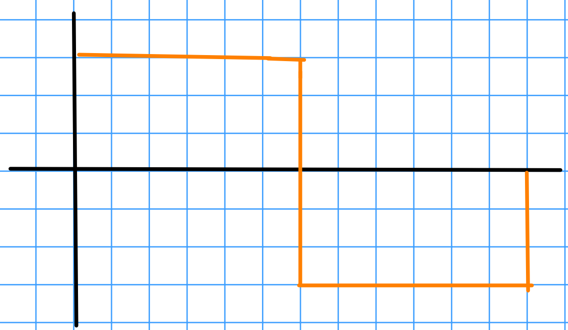
$$\phi(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{other} \end{cases}$$

$$\psi(x) = \begin{cases} 1 & \text{if } 0 < x < \frac{1}{2} \\ -1 & \text{if } \frac{1}{2} < x < 1 \\ 0 & \text{other} \end{cases}$$

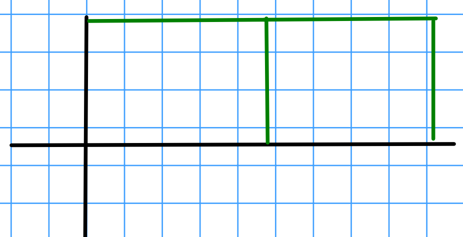
Haar wavelets

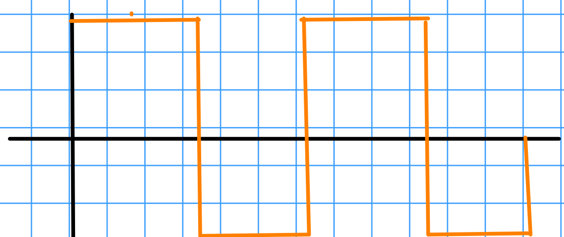
$\ell = 0$





$\ell = 1$





Properties:

- Functions in V_2 can be obtained by suitable additions of functions from V_1 and W_1

  $\longrightarrow V_\ell = V_{\ell-1} \oplus W_{\ell-1}$

- The Haar wavelets are orthogonal

  $\rightarrow \langle \psi_{\ell,i}, \psi_{m,j} \rangle = \int \psi_{\ell,i}(x)\, \psi_{m,j}(x) = \begin{cases} 1 & \text{if } \ell = m, i = j \\ 0 & \text{other} \end{cases}$

- Any function that can be written in terms of V_ℓ, can also be written in terms of $V_{\ell+1}$

  $\rightarrow V_\ell \subset V_{\ell+1}$

$$\phi_{\ell-1,k}(x) = \sum_i p_{i-2k}\, \phi_{\ell,i}(x)$$

$$\psi_{\ell-1,k}(x) = \sum_i q_{i-2k}\, \phi_{\ell,i}(x)$$

Scaling / Dilation Equations

$p_0 = \frac{1}{\sqrt{2}}, \; p_1 = \frac{1}{\sqrt{2}} \qquad q_0 = \frac{1}{\sqrt{2}}, \; q_1 = -\frac{1}{\sqrt{2}}$ for Haar

Filtering

Coarse to fine	Fine to coarse
$c_i^{\ell+1} = \sum_j \left(p_{i-2j} c_j^\ell + q_{i-2j} d_j^\ell \right)$ Reconstruction	$c_j^\ell = \sum_i p_{i-2j} c_i^{\ell+1}$ Low pass $d_j^\ell = \sum_i q_{i-2j} c_i^{\ell+1}$ High pass

Notation

- $c_i^{\ell+1}$: Nodal basis coefficients to represent arbitrary functions on level $\ell+1$
- c_i^{ℓ} : " on level ℓ
- d_i^{ℓ} : Wavelet coefficients to represent arbitrary functions on level
- p_{i-2j} : Nodal basis coefficients to represent nodal functions on level ℓ in terms of nodal basis of level $\ell+1$
- q_{i-2j} : Nodal basis coefficients to represent wavelets on level ℓ in terms of nodal basis of level $\ell+1$

Exercise 1:

Approximate scaling function ϕ using the cascade algorithm

Basic idea: The father wavelet can be written as

$$\phi(t) = \sum_k c_k \phi(2t-k)$$

$$\downarrow \text{finer}$$

$$\phi(4t-k')$$

$\vdots$ $\Rightarrow$ recursion

On the finest grid we choose the hat function and build up recursively

Here: $$\gamma_{n+1}(t) = \sum_k c_k \gamma_n(2t-k)$$

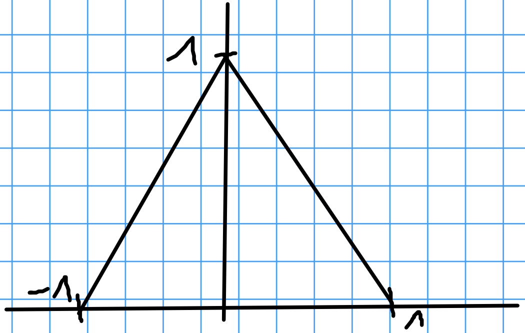
$$= \sum_k c_k \sum_m c_m \gamma_{n-1}(2(2t-k)-m)$$

$$= \sum_k c_k \sum_m c_m \gamma_{n-1}(\underbrace{2^2 t}_{\text{Refine}} - \underbrace{(2k+m)}_{\text{Shift}})$$

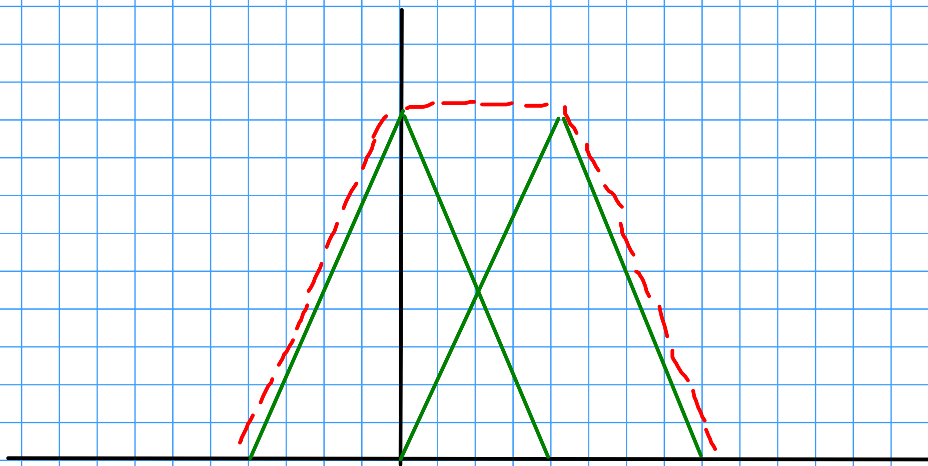
(i) Haar wavelet

Starting point $\gamma_0 = \max\{1-|x|, 0\}$

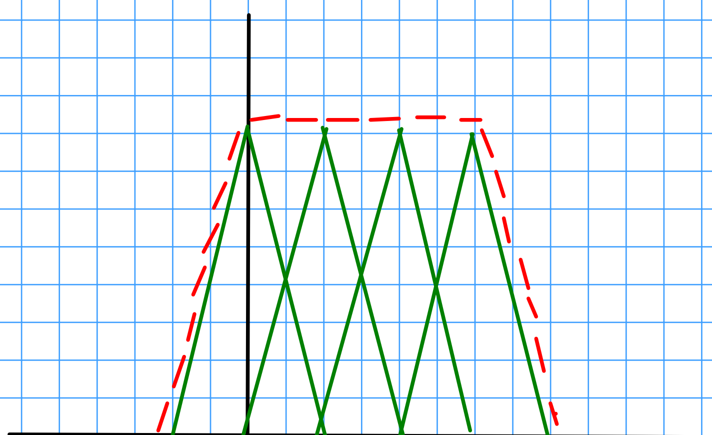
$c_0 = c = 1$



$n=1: \quad \gamma_1(x) = c_0\,\gamma_0(2x) + c_1\,\gamma_0(2x-1)$



$n=2: \quad \gamma_2(x) = c_0\,\gamma_1(2x) + c_1\,\gamma_1(2x-1)$

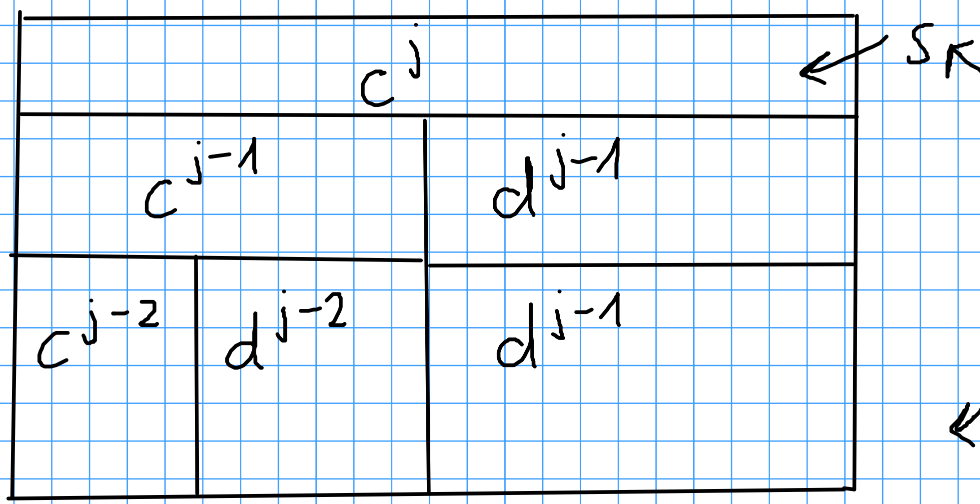


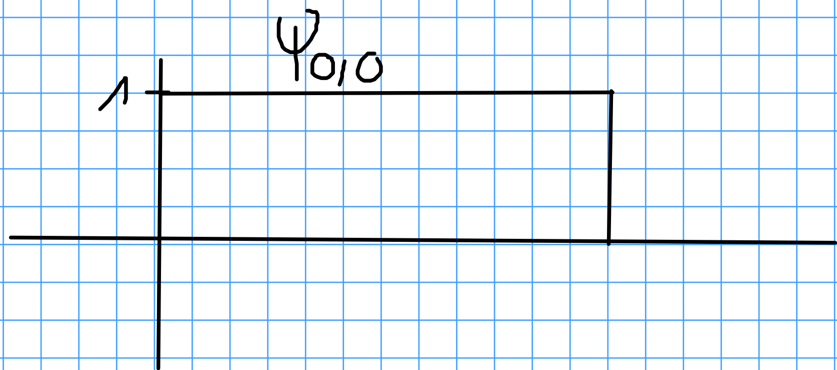
For mother wavelet:

- Alternation signs
- Reverse order of coefficients

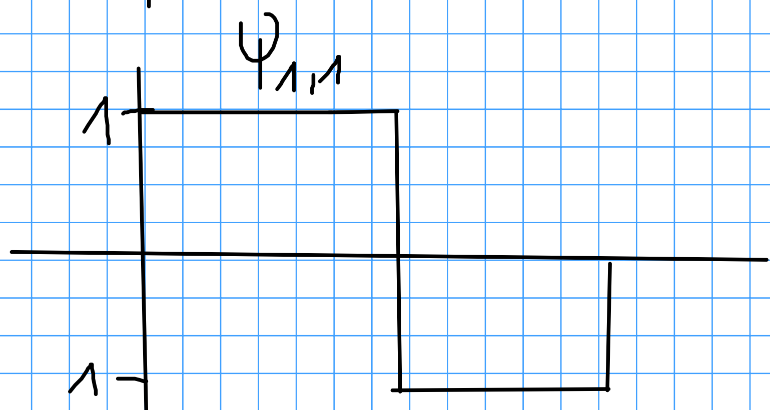
$$\gamma_{n+1} = \sum_k (-1)^k c_{N-k}\,\gamma_n(2t-k)$$

Ex2) Transformation for filtering

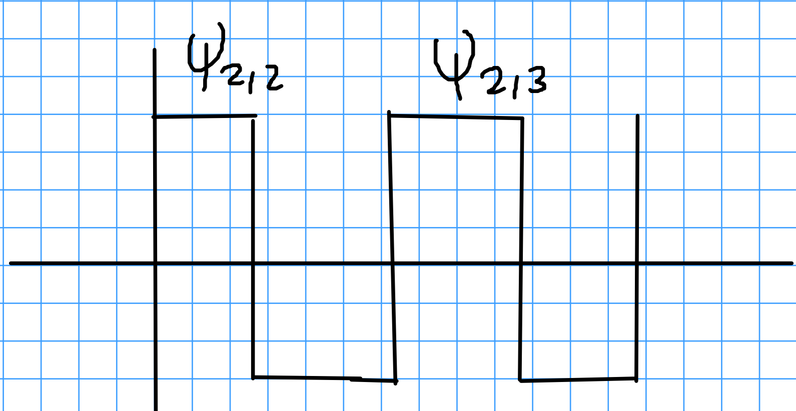




$\ell = 0$



$\ell = 1$



$\ell = 2$

$$
M = \begin{array}{cccccccc}
\psi_{0,0} & \psi_{1,1} & \psi_{2,2} & \psi_{2,3} & \psi_{3,4} & \psi_{3,5} & \psi_{3,6} & \psi_{3,7} \\
\end{array}
$$

$$
\begin{bmatrix}
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\
1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\
1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\
1 & -1 & 0 & -1 & 0 & 0 & 0 & -1
\end{bmatrix}
$$