

Algorithms for Scientific Computing

From Quadtrees to Space-Filling Orders

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TUM Uhrenturm

Part I

Quadtrees

Overview: Modelling of Geometric Objects

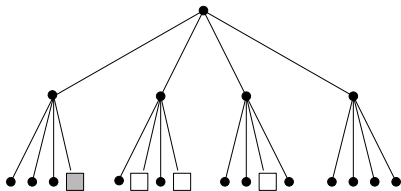
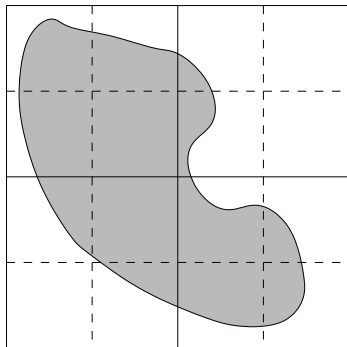
Surface-oriented models:

- wire-frame models
- augmented models using Bezier curves and planes
- typically described by graphs on nodes, edges, and faces

Volume-oriented models:

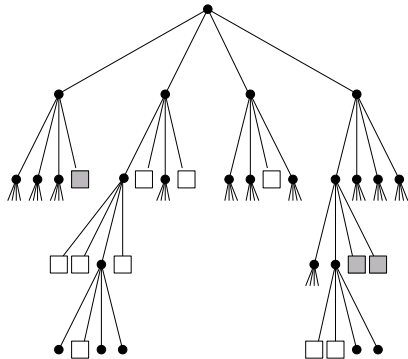
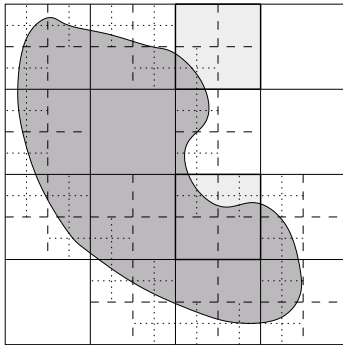
- Constructive Solid Geometry (boolean operations on primitives)
- voxel models: place object in a grid
- octrees: recursive refinement of voxel grids
- quadtrees: 2D analogon (voxel \rightsquigarrow pixel)

Quadrees to Describe Geometric Objects



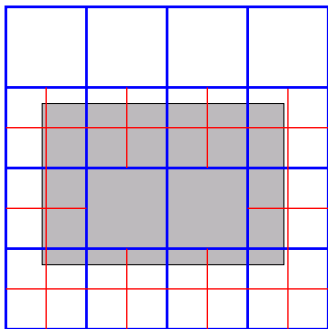
- start with an initial square (covering the entire domain)
- recursive substructuring in four subsquares

Quadrees to Describe Geometric Objects

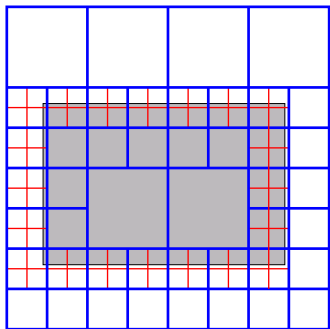


- start with an initial square (covering the entire domain)
- recursive substructuring in four subsquares
- adaptive refinement possible
- terminate, if squares entirely within or outside domain

Number of Quadtree Cells to Store a Rectangle



k=2

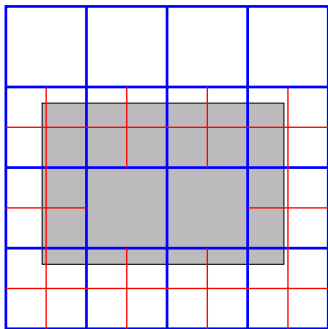


k=3

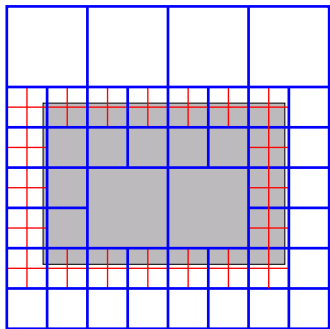
Terminal (t_k) and boundary (b_k) cells after k refinement steps (for $k > 2$):

$$\begin{aligned}
 b_k &= 2 \cdot b_{k-1} \\
 t_k &= t_{k-1} + 2 \cdot b_{k-1} \quad \Rightarrow \quad b_k = 2^{k-2} \cdot b_2 = \frac{5}{2} \cdot 2^k \\
 & \quad \quad \quad t_k = \dots = 5 \cdot 2^k - 14
 \end{aligned}$$

Number of Quadtree Cells to Store a Rectangle



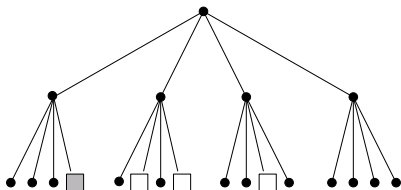
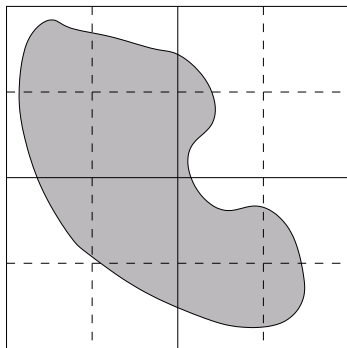
k=2



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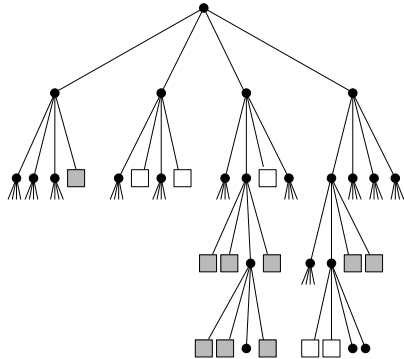
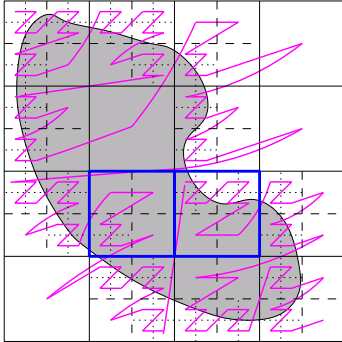
- uniformly ref. voxel-grid (level k): $(2^{d=2})^k = (2^k)^2 =: \mathcal{O}(N^2)$ cells
- quadtree-refined grid (level k): $\frac{15}{2} \cdot 2^k - 14 =: \mathcal{O}(N)$ cells
 \Rightarrow number of cells proportional to length of boundary ($N := 2^k$)

Storing a Quadtree – Sequentialisation



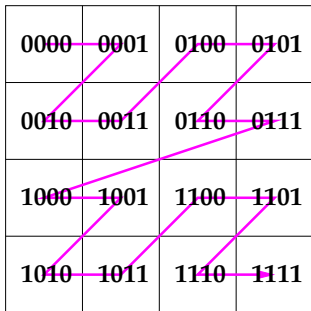
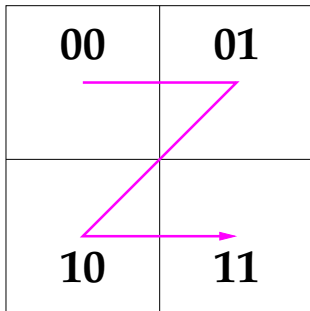
- sequentialise cell information according to **depth-first traversal**
- relative numbering of the child nodes determines sequential order

Storing a Quadtree – Sequentialisation



- sequentialise cell information according to **depth-first traversal**
- relative numbering of the child nodes determines sequential order
- here: leads to so-called **Morton order**

Morton Order (“Z curve”)



Relation to bit arithmetics:

- odd digits: position in vertical direction
- even digits: position in horizontal direction

Morton Order and Cantor's Mapping

Georg Cantor (1877):

$$0.\textcolor{red}{0}\textcolor{blue}{1}\textcolor{red}{1}\textcolor{blue}{1}\textcolor{red}{1}\textcolor{blue}{0}\textcolor{red}{0}\textcolor{blue}{1}\dots \rightarrow \begin{pmatrix} 0.\textcolor{red}{0}\textcolor{red}{1}\textcolor{red}{1}\textcolor{red}{0}\dots \\ 0.\textcolor{blue}{1}\textcolor{blue}{1}\textcolor{blue}{0}\textcolor{blue}{1}\dots \end{pmatrix}$$

- **bijective** mapping $[0, 1] \rightarrow [0, 1]^2$
- proved identical cardinality of $[0, 1]$ and $[0, 1]^2$
- provoked the question: is there a **continuous** mapping? (i.e. a curve)

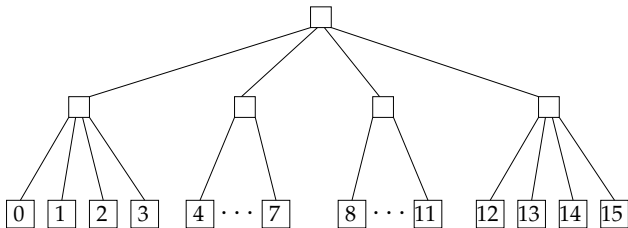
Similar: is there a contiguous order on the quadtree cells?

Preserving Neighbourship for a 2D Octree

Requirements:

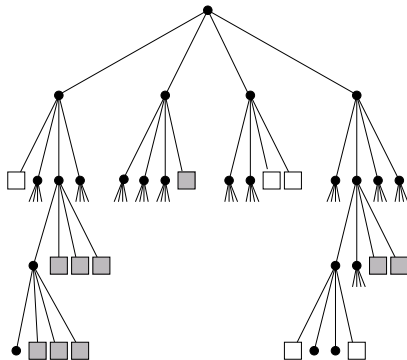
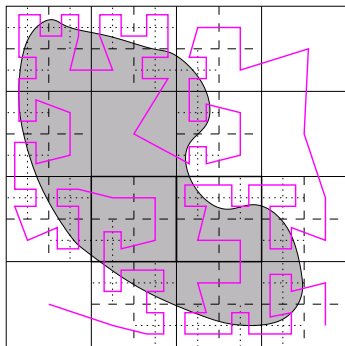
- consider a simple 4×4 -grid
- uniformly refined
- subsequently numbered cells should be neighbours in 2D

Leads to (more or less unique) numbering of children:



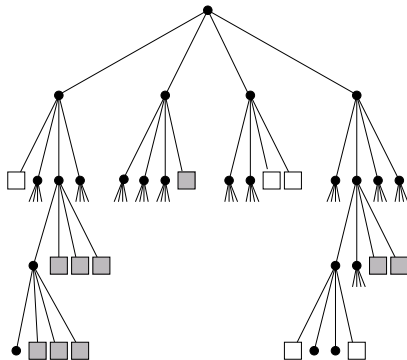
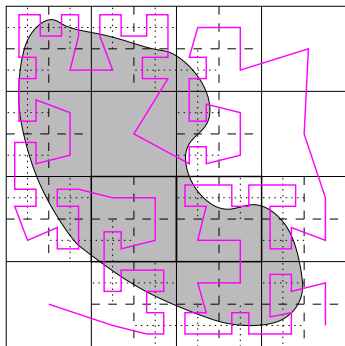
5	6	9	10
4	7	8	11
3	2	13	12
0	1	14	15

Preserving Neighbourship for a 2D Octree (2)



- adaptive refinement possible
- neighbours in sequential order remain neighbours in 2D

Preserving Neighbourship for a 2D Octree (2)



- adaptive refinement possible
- neighbours in sequential order remain neighbours in 2D
- here: similar to the concept of **Hilbert curves**

Open Questions

Algorithmics:

- How do we describe the sequential order algorithmically?
- What kind of operations are possible?
- Are there further “orderings” with the same or similar properties?

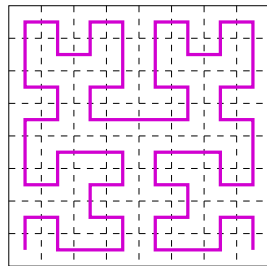
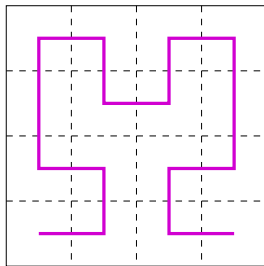
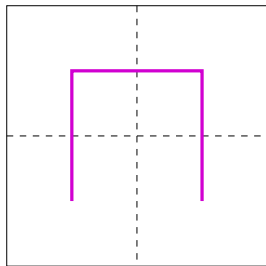
Applications:

- Can we quantify the “neighbour” property?
- In what applications can this property be useful?
- Which other properties and/or operations can be useful?

Part II

Hilbert Orders

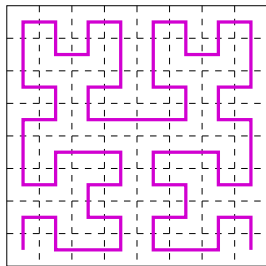
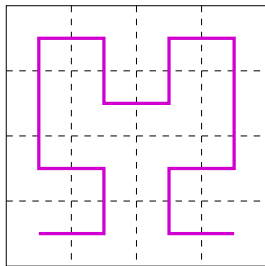
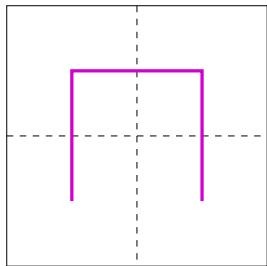
Construction of the Hilbert Order



Incremental construction of the Hilbert order:

- start with the basic pattern on 4 sub-squares
- combine four numbering patterns to obtain a twice-as-large pattern
- proceed with further iterations

Construction of the Hilbert Order

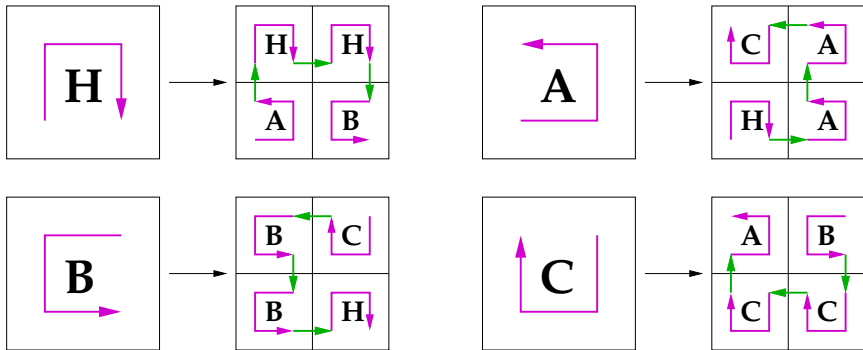


Recursive construction of the Hilbert order:

- start with the basic pattern on 4 subsquares
- for an existing grid and Hilbert order:
split each cell into 4 congruent subsquares
- order 4 subsquares following the rotated basic pattern,
such that a contiguous order is obtained

A Grammar for Describing the Hilbert Order

Examine pattern during the construction of the Hilbert order:



→ motivates a **Grammar** to generate the iterations

A Grammar for Describing the Hilbert Order

- Non-terminal symbols: $\{H, A, B, C\}$, start symbol H
- terminal characters: $\{\uparrow, \downarrow, \leftarrow, \rightarrow\}$
- productions:

$$\begin{array}{lcl}
 H & \leftarrow & A \uparrow H \rightarrow H \downarrow B \\
 A & \leftarrow & H \rightarrow A \uparrow A \leftarrow C \\
 B & \leftarrow & C \leftarrow B \downarrow B \rightarrow H \\
 C & \leftarrow & B \downarrow C \leftarrow C \uparrow A
 \end{array}$$

- replacement rule: in any word,
all non-terminals have to be replaced at the same time
 \rightarrow L-System (Lindenmayer)
- \Rightarrow the arrows describe the **iterations of the Hilbert curve** in “turtle graphics”

Using the Grammar to Describe the Hilbert Curve

The grammar for the Hilbert order prepares the mathematical definition of the curve (and proof of continuity):

- there are only four basic patterns that occur
(corresp. to the symbols $\{H, A, B, C\}$ of the grammar)
→ **closed recursive system!**
- two subsequent subsquares of the Hilbert-curve construction share a common edge(!)
→ follows from the fact that the move operators $\{\uparrow, \downarrow, \leftarrow, \rightarrow\}$
are sufficient to describe the operators
→ **contiguous order!** (leads to continuity of the curve)
- last but not least:
we have formalised the construction of the iterations
(towards formal definition of a mapping)

Part III

Applications of Space-Filling Orders

Sequentialising Multi-dimensional Data

Examples of multi-dimensional data structures:

- Matrices
- Image data (images, tomographic data, movies, ...)
- discretisation meshes (to discretise mathematical models in physics/...; PDE, etc.)
- Coordinates (often used in connection with graphs)
- tables (also in data bases)
- in computational finance and financial mathematics: “baskets” of stocks/options/...

Sequentialising Multi-dimensional Data (2)

Typical algorithms and operations:

- traversal (update/processing of all data; simulation meshes, e.g.)
- matrix operations (linear algebra, etc.)
- sequentialisation (e.g. to store data on discs or in main memory)
- partitioning of data (for parallelisation or in divide-and-conquer algorithms)
- sorting of data (to simplify further operations)
- in general: nested loops

```
for i from 1 to n do  
    for j from 1 to m do        ...
```


Demands on Efficient Sequentialisation

Effective Sequentialisation:

- unique numbering \Rightarrow requires bijective mapping
- sequentialisation without “holes” (for data structures, e.g.)

Efficient Sequentialisation:

- preserve neighbourhood properties \Rightarrow data locality
- fast, simple index computation
- “smoothness”, stability vs. small changes
- dimensional symmetry (no fast or slow dimensions)
- “clustering” of data

Application Examples

- **range queries** in image and raster data bases
- **image browsing** and **image search** in image collections
- heuristical approaches for graph-based algorithms (nearest neighbour, traveling salesman)
- collision detection
- **parallelisation** of data
- efficient use of **cache memory** (in simulations, e.g.)

Also: everything that involves Quadtrees/Octrees:

- Searching (collision detection, access to surface representations, etc.)
- Fast access to level-of-detail infos (geodata, graphics, games, ...)
- Dynamically adaptive meshes in scientific computing
- many more ...