Algorithms of Scientific Computing

Discrete Sine Transform (DST)

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DFT and Symmetry

	INPUT		TRANSFORM
real symmetry	$f_n \in \mathbb{R}$	$\rightarrow$	Real DFT (RDFT)
even symmetry	$f_n = f_{-n}$	$\rightarrow$	Discrete Cosine Transform (DCT)
odd symmetry	$f_n = -f_{-n}$	$\rightarrow$	Discrete Sine Transform (DST)

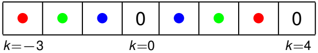
“QUARTER-WAVE”	INPUT		TRANSFORM
even symmetry	$f_n = f_{-n-1}$	$\rightarrow$	QW-DCT
odd symmetry	$f_n = -f_{-n-1}$	$\rightarrow$	QW-DST

Real-valued Input Data with “Odd” Symmetry

Given: $2N$ input data $f_{-N+1}, \ldots, f_N$, all $f_n \in \mathbb{R}$, with

$$f_{-n} = -f_n, \qquad \text{in particular} \quad f_0 = f_N = f_{-N} = 0$$

For example, for $N = 4$:



The DFT then has the following form:

$$\begin{aligned}
F_k &= \frac{1}{2N} \sum_{n=-N+1}^{N} f_n \omega_{2N}^{-nk} \\
&= \frac{1}{2N} \left(\underbrace{f_0}_{=0} + \sum_{n=1}^{N-1} \left(f_n \omega_{2N}^{-nk} + f_{-n} \omega_{2N}^{nk} \right) + \underbrace{f_N}_{=0} \omega_{2N}^{-Nk} \right) \\
&= \frac{1}{2N} \sum_{n=1}^{N-1} f_n \left(\omega_{2N}^{-nk} - \omega_{2N}^{nk} \right) = \frac{-i}{N} \sum_{n=1}^{N-1} f_n \sin\left(\frac{\pi nk}{N} \right).
\end{aligned}$$

Symmetry in the Coefficients

Transform to f_n with symmetry $f_{-n} = -f_n$ gives:

$$F_k = \frac{-i}{N} \sum_{n=1}^{N-1} f_n \sin\left(\frac{\pi n k}{N}\right) \qquad \text{for } \; k = -N+1, \ldots, N.$$

Same symmetry in the coefficients F_k:

$$F_{-k} = \frac{-i}{N} \sum_{n=1}^{N-1} f_n \sin\left(\frac{\pi n(-k)}{N}\right) = \frac{-i}{N} \sum_{n=1}^{N-1} f_n \left(-\sin\frac{\pi n k}{N}\right) = -F_k$$

$\Rightarrow$ leads to the same (up to scaling) **"discrete sine transform"**

Discrete Sine Transform (DST)

From DFT of real-valued, odd symmetric data:

$$F_k = -\frac{i}{N} \sum_{n=1}^{N-1} f_n \sin\left(\frac{\pi nk}{N}\right), \quad k = 1, \dots N-1 \, .$$

Analogue calculation for IDFT gives:

$$f_n = 2i \sum_{k=1}^{N-1} F_k \sin\left(\frac{\pi nk}{N}\right), \quad n = 1, \dots N-1 \, .$$

$\Rightarrow$ definition of the discrete sine transform ($\widehat{F}_k := iF_k$):

$$\widehat{F}_k = \frac{1}{N} \sum_{n=1}^{N-1} f_n \sin\left(\frac{\pi nk}{N}\right), \qquad f_n = 2 \sum_{k=1}^{N-1} \widehat{F}_k \sin\left(\frac{\pi nk}{N}\right),$$

Computation of the Discrete Sine Transform

Via pre-/postprocessing:

(1) generate $2N$ vector with odd symmetry

$$x_{-k} = -x_k \qquad \text{for } k = 1, \ldots N-1$$
$$x_0 = x_N = 0$$

(2) coefficients X_k via fast, real-valued FFT on vector x

(3) postprocessing: $\widehat{X}_k = -\operatorname{Im}\{X_k\}$ for $k = 1, \ldots, N-1$.

(4) if necessary: scaling

Recall: New Transforms and Symmetries

We obtain a new pair of transforms:

$$F_k = -\frac{i}{N}\sum_{n=1}^{N-1} f_n \sin\left(\frac{\pi nk}{N}\right) \qquad f_n = 2i\sum_{k=1}^{N-1} F_k \sin\left(\frac{\pi nk}{N}\right)$$

- both transforms work on **data sets** that **are neither symmetric nor periodic**
- for the particular case of the DST, we have in mind that $f_0 = f_N = 0$
- if we extend the data sets according to the symmetry rules, then the reflected (and thus symmetric) sets become periodic
- the two transforms are connected to the DFT and iDFT via a 3-step procedure:
  1. extend/duplicate the data set in a symmetric way
  2. apply the DFT/iDFT
  3. extract the symmetric half of the transformed data set
- this equivalence has two important consequences:
  1. we may compute the sine transforms ($N-1$ numbers that require sums over $N-1$ terms $\Rightarrow \mathcal{O}(N^2)$ operations) by using an FFT in step 2 $\Rightarrow$ reduces work to $\mathcal{O}(N \log N)$
  2. we prove that DST and iDST are inverse operations to each other (as the DFT and iDFT are inverse to each other on the respective symmetric data)

Summary: Survey on DCT/DST Variants

Symmetry properties $\Leftrightarrow$ how is data continued at boundaries:

beg. \ end	even	odd
even	x	x
odd	x	x

$\Rightarrow$ 4 possibilities

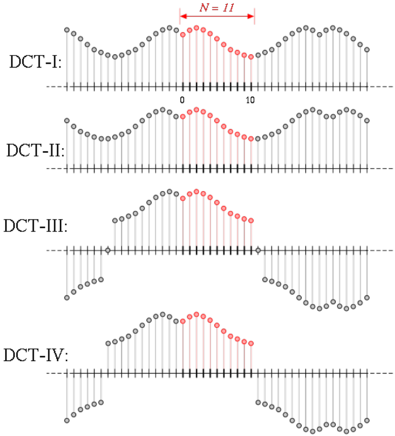
beg. \ end	mirror	copy
mirror	x	x
copy	x	x

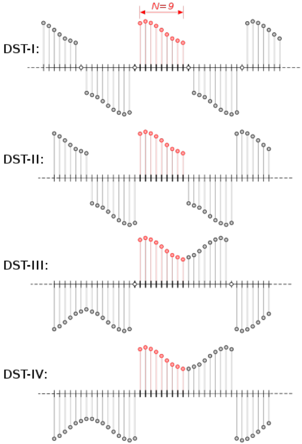
$\Rightarrow$ 4 possibilities

$\Rightarrow$ in total: 16 possibilities (8 DCT, 8 DST)

Summary: Survey on DCT/DST Variants (2)

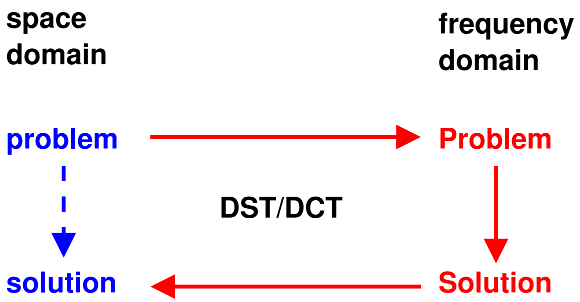
Common schemes of DCT (left) and DST (right) (images: Wikipedia):





Application: DCT/DST for PDE (Spectral Methods)

nice application: DST for Fast Poisson Solver



Attention: limits/problems for using DFT with PDE include

- irregular (i.e. non-rectangular) domains
- variable coefficients in problem

$\Rightarrow$ other methods: FVM, FEM (fast linear solvers, multigrid, etc.)