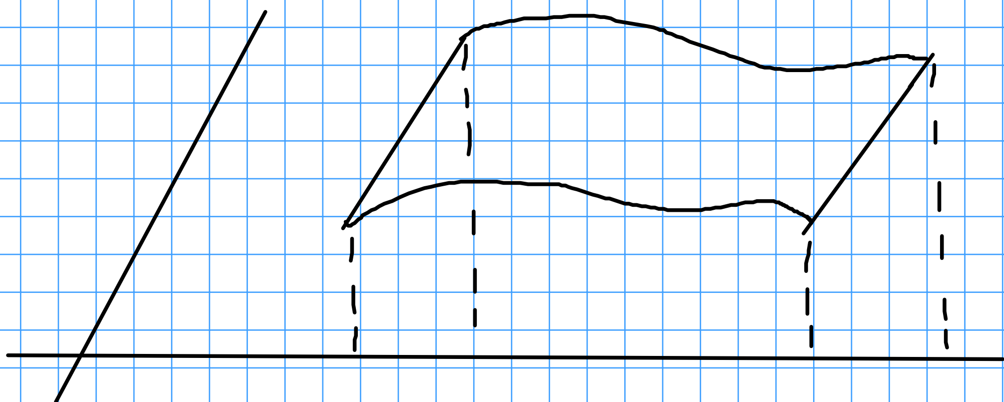
Worksheet 9: Multi-dimensional Quadrature

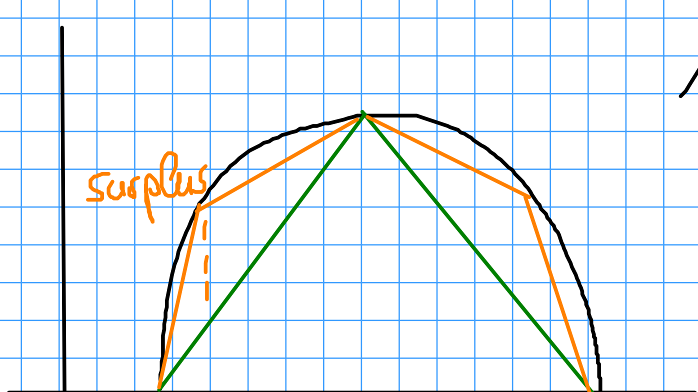
Approximate integrals in higher dimensions



calculate volume

Methods like CT or CS do not perform in high dimensions
In particular the ε-complexity (# of function evaluations to reach a certain error margin) looks quite bad
Hierarchical methods help to mitigate the curse of dimensionality

Archimedes n-d

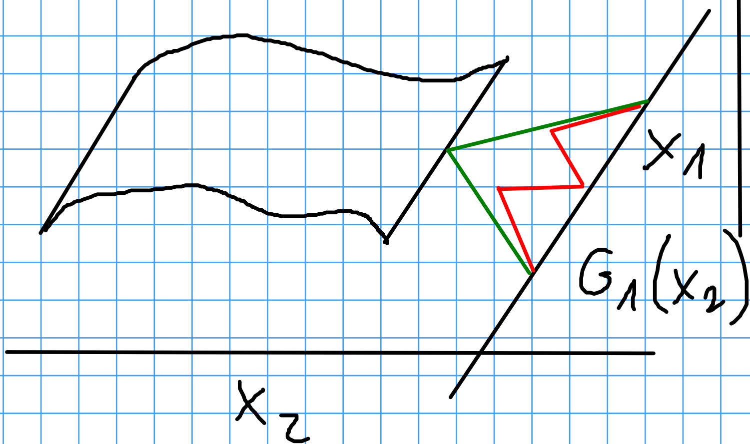


Apply same principle for higher dimensions

Naive Approach

Theorem of Fubini

$$\int_{\Omega} f(x_1, x_2)\, dx = \int_{\Omega_1}\int_{\Omega_2} f(x_1, x_2)\, dx_1\, dx_2$$

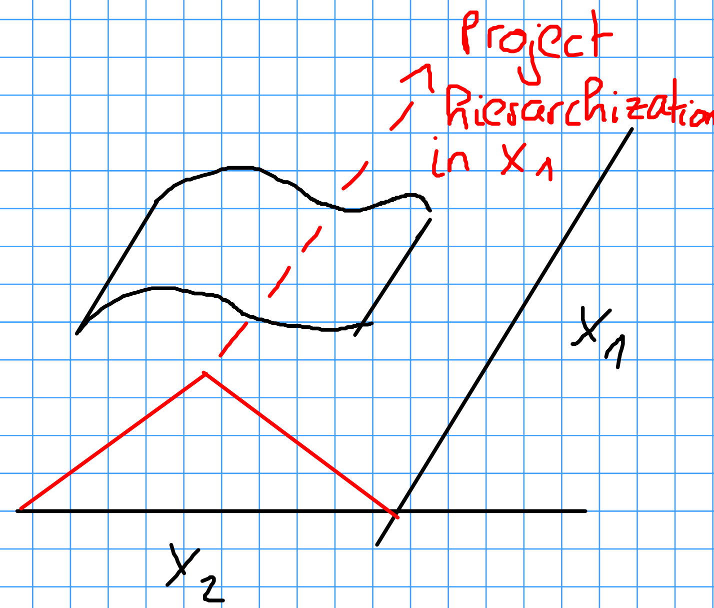


Do the same in x_2 based on result from x_1: $G_1(x_2)$

- First compute integrals in x_1 direction
  $\Rightarrow$ involves hierarchization in x_1
- Now repeat for x_2 direction

$\rightarrow$ We end up with d nested loops, and have no real advantages besides adaptivity

Improved Approach

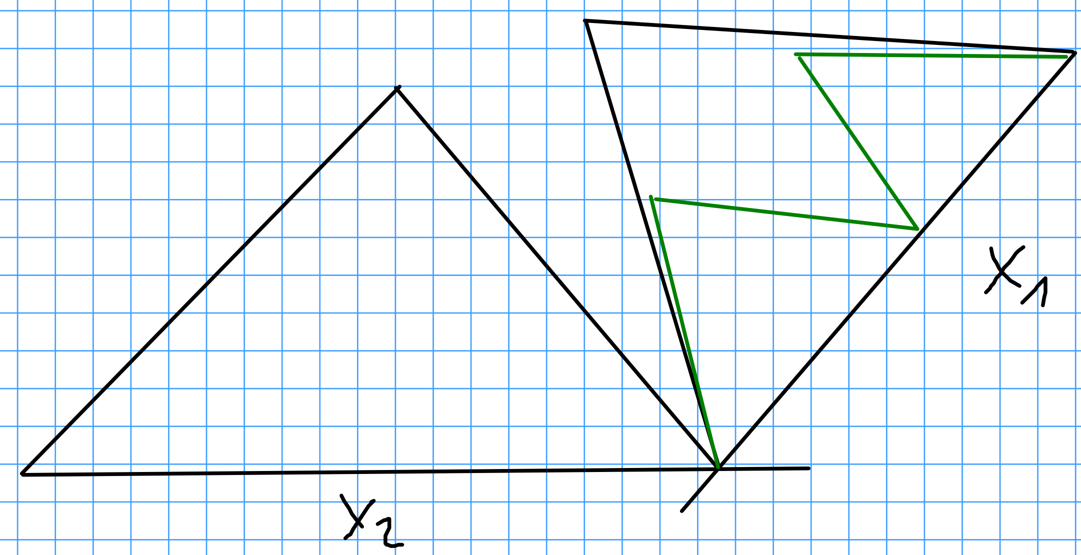


- First write hierarchical are elements in x_2-dir as a function of x_1
- Integrate then in x_1-dir (also using hierarchization)

$\rightarrow$ Involves reduction of number of points where we need to calculate the integral

Exercise 1

Pagoda



- Scale and shif mother of all hat functions $\Rightarrow f_x, f_y$
- Represent 2D piecewise linear function as multiplication of 1D linear functions $\Rightarrow f_x \cdot f_y$
- For each level : 2^{n-1} intervals , $x_{e,i} = i \cdot h_e$
- Meshgrid samples domain

Optional:

Height of triangles changes

$\rightarrow$ Getting smaller deeper into levels

Same happens with Pagodas

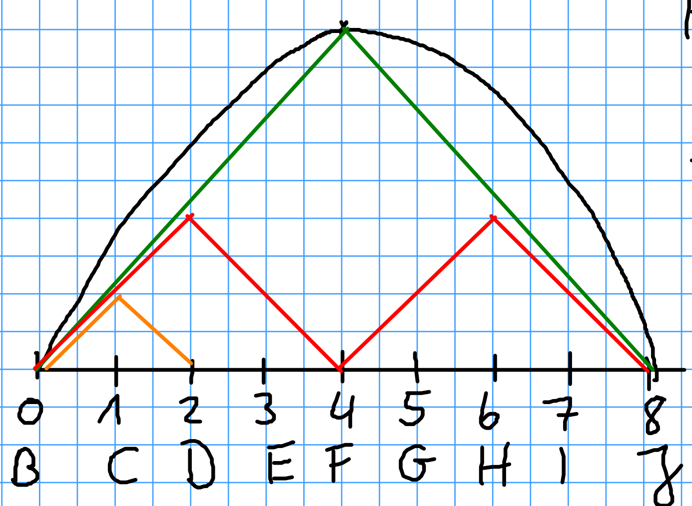
Exercise 2

Approximate integral of

$$f(x_1, x_2) = 16x_1(1-x_1)x_2(1-x_2)$$

$$2^3 = 8 \Rightarrow 3 \text{ levels}$$

We will first hierarchize and find the surpluse on the x_2 direction



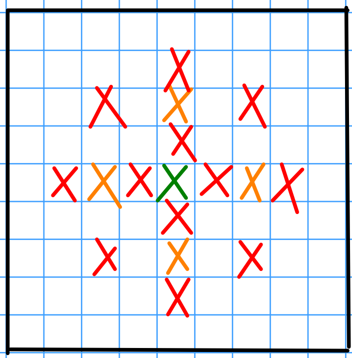
Height of triangle

$$f\left(\frac{a+b}{2}\right) - \frac{f(a)+f(b)}{2}$$

- Hierarchize in x_1-direction
- We are not integrating f anymore, but the surpluses
- Now we have the height of pagodas
- Before: 1 column represented one triangle
  Now: 1 row represents one triangle
- Multiply height of pagoda with base
- Base area of pagoda is $2^{-|\vec{e}|}$, $|\vec{e}| = |e_1| + |e_2|$

$e = 1: \; h_1 = 2^{-1} = 0.5$

$$2^{-|\vec{e}|} = 2^{-(|e_1| + |e_2|)} = 2^{-|e_1|} \cdot 2^{-|e_1|}$$



Volumes

X = 0.250

X = 0.031

X = 0.004