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Worksheet 10
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Algorithms for Scientific Computing

Hierarchization in Higher Dimensions, Spatial Adaptivity

Exercise 1: Hierarchization in Higher Dimensions

In this exercise we will implement the multi-recursive algorithm for hierarchization of a multi-dimensional regular sparse grid. The structure of the code resembles strongly the one-dimensional case. We have a class (**PagodaFunction**) representing our grid points.

(i) Implement the refinement criterion **MinLevelCriterion** that adds all points up to a specified level to a given grid.
Hint: In your grid traversal, try to avoid multiple visits to the same grid points.

(ii) Implement the function **hierarchize** efficiently using a recursive approach.
Hint: The underlying traversal algorithm can be implemented similar to the one in (i).

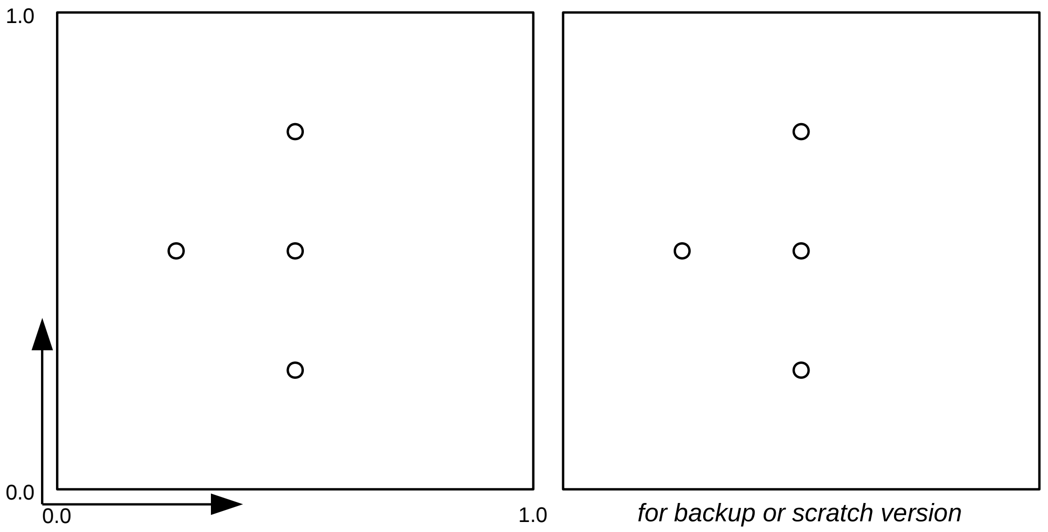
(iii) Implement a function to compute the volume of the sparse grid interpolant.

Exercise 2: Adaptive Sparse Grids

Here, the exercise is to adaptively refine a 2-dimensional sparse grid without boundary. We follow the notation introduced in the lecture and choose our domain accordingly with $\Omega = [0.0, 1.0]^2$.

1. In the following image you see an incomplete regular sparse grid V_2^1. Insert the missing grid points using small **squares**. What are the level-index-vector pairs $\vec{l}, \vec{i}$ for each of them?
2. Use the (modified) picture from the previous task to perform two steps of adaptive refinement:
   (a) Refine grid point $\vec{l}, \vec{i} = (1,2),(1,3)$: create all hierarchical children. Draw its children as small **triangles**. Make sure that you also insert all missing hierarchical parents (and parents of parents, . . .) of these children to make the grid suitable for typical algorithms on sparse grids.

   (b) Now refine grid point $(2,2),(3,3)$. Again, do not forget to create all missing parents. Draw all new points as small **crosses**.



Exercise 3: The Combination Technique – A Different View on Sparse Grids

Dealing with hierarchical bases often turns out to be sophisticated. On this worksheet we will therefore see how the so-called *combination technique* finds a sparse grid interpolant, that approximates a function on a number of full grids, each consisting only of a "relatively small" number of grid points.

Let $u_{\underline{l}}$ ($\underline{l} \in \mathbb{N}^2$) for a $u : [0,1]^2 \to \mathbb{R}$ the interpolant in $V_{\underline{l}}$ (interpolating piecewise bilinearly at the inner grid points, at the boundary u is assumed to be zero again).

(i) $V_{\underline{l}}$ can be decomposed into a set of subspaces $W_{\underline{l}}$. Accordingly, the interpolant $u_{\underline{l}} \in V_{\underline{l}}$ can be written as a sum of $w_{\underline{l}} \in W_{\underline{l}}$.

Spot the grid associated with $u_{(3,2)}$ in the right part of Figure 1. Identify those subspaces in the left part that are needed to reconstruct $u_{(3,2)}$.

(ii) Use the result from (i) to rewrite

$$\sum_{|\underline{l}|_1 = n+1} u_{\underline{l}}\ , \qquad n \in \mathbb{N}$$

for the two-dimensional case as a weighted sum of $w_{\underline{l}}$.

Hint: Look at the subspace scheme in Figure 1 and count the occurrences of each subspace in the sum. What do you notice when comparing $w_{\underline{l}}$ with common level $n = |\underline{l}|_1 + dim - 1$?

(iii) In the final step use the previous results to give a representation of the sparse grid interpolant

$$u_n^D := \sum_{|\underline{l}|_1 \le n+1} w_{\underline{l}}$$

as a weighted sum of $u_{\underline{l}}$. Again, count the occurrences of the $w_{\underline{l}}$.

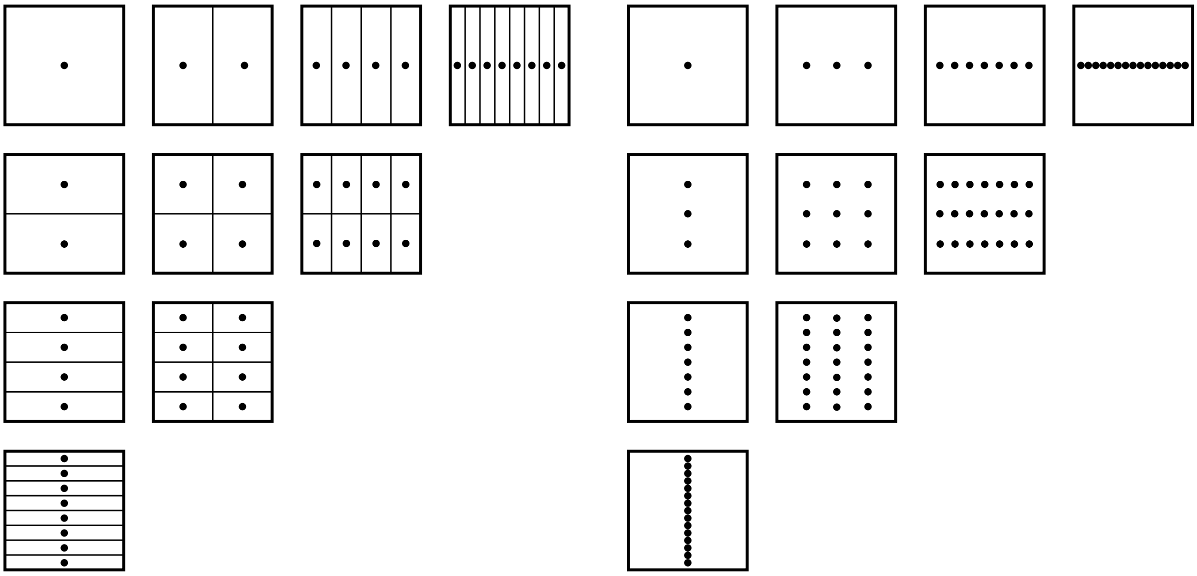


Figure 1: The two parts in the picture show the grid points and supports associated with interpolants $w_{\underline{l}}$ (left) and $u_{\underline{l}}$ (right) up to level 4 for the 2d case.

(iv) Assume you are talking to a person who knows how to approximate the volume $F_2(u)$ through the trapezoidal rule (in 2d) with respect to $u_{\underline{l}}$. Give instructions on how to write a program that implements a sparse grid approximation of $F_2(u)$. Remember Archimedes quadrature.

(v) Compare this method with Archimedes quadrature — what are the (dis-)advantages?

Optional: One-dimensional Sparse Grids—An Adaptive Implementation

Last week we introduced Archimedes' approach to approximate the integral $F(f, a, b) = \int_a^b f(x)\, dx$ of a function $f : \mathbb{R} \to \mathbb{R}$, respectively to approximate the function f itself.

For the one-dimensional case we want to formalize this approach and generalize it in the following ways:

- Let $\phi(x)$ be the "mother of all hat functions" with

$$\phi(x) = \begin{cases} x+1 & \text{for } -1 \le x < 0 \\ 1-x & \text{for } 0 \le x < 1 \\ 0 & \text{else} \end{cases} \tag{1}$$

- The data structure used to store the hierarchical coefficients is now called *Sparse Grid*.
- A sparse grid is defined by a particular set of interpolation points $x_{l,i}$ and associated ansatz functions $\phi_{l,i}(x)$ with

$$\phi_{l,i}(x) = \phi(\ 2^l \cdot \left(x - i \cdot \frac{1}{2^l}\right)\ ) = \phi(2^l \cdot x - i)\ , \qquad l \in \mathbb{N}^+, i \in \{1, 3, \ldots, 2^l - 1\} \tag{2}$$

- Archimedes' approach from the lecture corresponds to a *regular* sparse grid.
- To improve the quality of approximation for arbitrary functions f we introduce spatial adaptivity.

Your task is to implement the missing parts in the members of the *SparseGrid1d* class and turn it into a fully working adaptive implementation of a one-dimensional sparse grid. Import and use the class *GridPoint* and look at the comments in the provided code snippets for some more details.

a) The constructor *__init__* creates a grid containing all grid points on levels $l \leq minLevel$. A given function f is then evaluated at those points before *hierarchization* is performed eventually to obtain the hierarchical coefficients.
Implement this behavior.

b) Implement the member function *computeVolume* that computes an approximation for $F(f, 0, 1)$ using the current sparse grid interpolant.

c) Implement the member function *refineAdaptively* that takes a certain refinement criterion (see source code) and inserts new grid points accordingly.