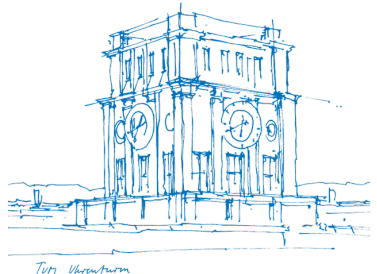


# Algorithms for Scientific Computing

## Wavelets

Michael Bader  
Technical University of Munich  
Summer 2022



## Part I

# Haar Wavelets as an Hierarchical Basis

## Remember the 1D Hierarchical Basis

- “mother of all hat functions”:  $\phi(x) := \max\{1 - |x|, 0\}$
- hat functions on level  $l \in \mathbb{N}$  with mesh width  $h_l = 2^{-l}$  at grid points  $x_{l,i} = i \cdot h_l$ :

$$\phi_{l,i}(x) := \phi\left(\frac{x - x_{l,i}}{h_l}\right)$$

- hierarchical basis functions on level  $l$ :

$$\phi_{l,i}(x) \quad \text{for all } i \in \mathcal{I}_l := \{i : 1 \leq i < 2^l, i \text{ odd}\}$$

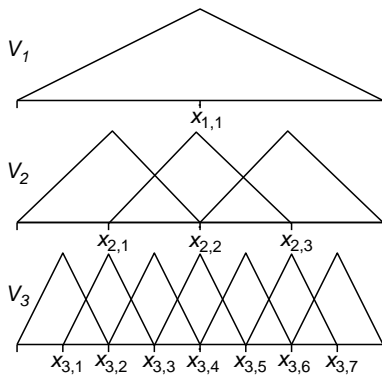
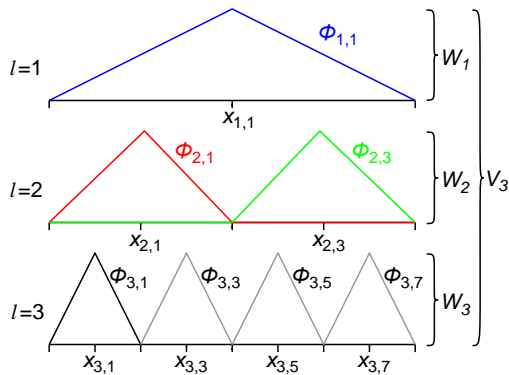
- resulting hierarchical basis

$$\Psi_n := \bigcup_{l=1}^n \{\phi_{l,i} : i \in \mathcal{I}_l\}.$$

- with corresponding function spaces:

$$W_l := \text{span} \{\phi_{l,i} : i \in \mathcal{I}_l\} \quad \text{and} \quad V_n = \bigoplus_{l=1}^n W_l$$

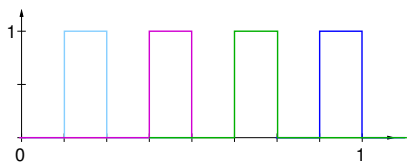
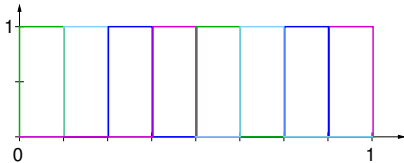
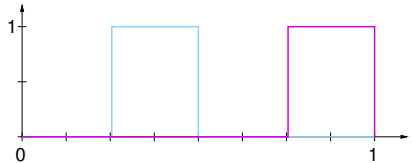
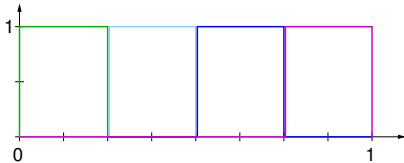
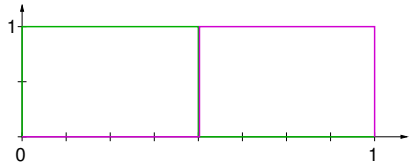
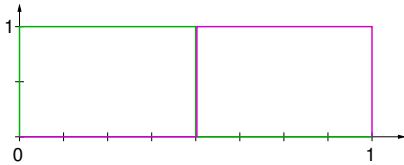
# Hierarchical vs. Nodal Basis



→ for piecewise *linear* (basis) functions

**Now: how to build a piecewise constant basis?**

# Piecewise Constant Basis – Attempt # 1



# Piecewise Constant Basis – Attempt # 1 (cont.)

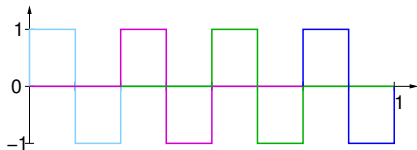
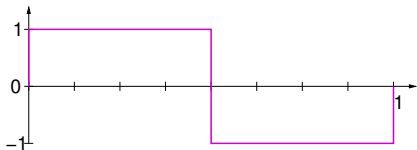
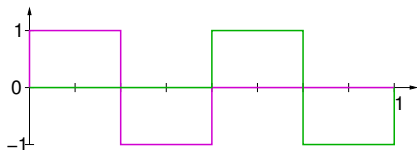
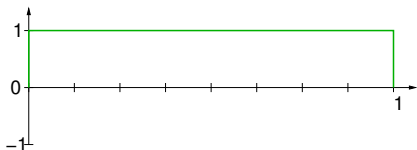
## Discussion:

- obviously qualifies as a “hierarchical basis” w.r.t. hierarchical levels and mesh widths
- built from a “mother of all step functions”, e.g.:

$$\phi(x) := \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

- hierarchical basis functions on level  $l$ :  $\phi_{l,i}(x) := \phi\left(\frac{x - x_{l,i}}{h_l}\right)$
- nodal basis on level  $l$ :  $V_l = \text{span}\{\phi_{l,i}(x) : i = 0, \dots, 2^l - 1\}$
- hierarchical surplus:  $W_l = \text{span}\{\phi_{l,i}(x) : 1 \leq i < 2^l, i \text{ odd}\}$
- would hierarchical surpluses be small in such a setting?
- are functions represented well by coarse-level basis functions?

## Attempt #2: “Hierarchical Haar Basis”



- for each *interval*, we obtain a contribution from each *level*
- course-level representations will consist of *average values*
- each “surplus” level add corrections to averages

# Hierarchical Haar Basis

- again a hierarchical basis with “mother Haar function”:

$$\psi(x) := \begin{cases} 1 & \text{if } 0 < x < 1 \\ -1 & \text{if } 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

- hierarchical Haar basis functions on level  $l$ :

$$\psi_{l,i}(x) := \psi\left(\frac{x - x_{l,i}}{h_l}\right) \quad \text{for all } i \in \mathcal{I}_l := \{i : 0 \leq i < 2^l, i \text{ even}\}$$

- hierarchical surplus space for each level:

$$W_l := \text{span} \{ \psi_{l,i} : i \in \mathcal{I}_l \}$$

- space of piecewise constant functions  $V_n = \bigoplus_{l=0}^n W_l$   
 $\rightarrow$  includes a step function on interval  $(0, 1)$  for  $l = 0$



# Hierarchical Haar Basis – Coefficients

- consider a piecewise constant function  $\in V_1$ :

$$s(x) := a\phi_{1,0}(x) + b\phi_{1,1}(x) \begin{cases} a & \text{if } 0 < x < \frac{1}{2} \\ b & \text{if } \frac{1}{2} < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

- condition in interval  $0 < x < \frac{1}{2}$ :

$$v_{0,0} \underbrace{\psi_{0,0}(x)}_{=\phi_{0,0}(x)} + v_{1,0}\psi_{1,0}(x) = v_{0,0} + v_{1,0} = a$$

- condition in interval  $\frac{1}{2} < x < 1$ :

$$v_{0,0}\psi_{0,0}(x) + v_{1,0}\psi_{1,0}(x) = v_{0,0} - v_{1,0} = b$$

- solve linear system of equations:

$$v_{0,0} = \frac{1}{2}(a + b) \quad v_{1,0} = \frac{1}{2}(a - b)$$

# Hierarchical Haar Basis – Transformation

- represent a piecewise constant function  $s(x) \in V_l$ :

$$s(x) = \sum_{i=0}^{2^l-1} c_{l,i} \phi_{l,i}(x)$$

- write as coarse function plus hierarchical surplus:

$$s(x) = \underbrace{\sum_i c_{l,i} \phi_{l,i}(x)}_{\in V_l} = \underbrace{\sum_i c_{l-1,i} \phi_{l-1,i}(x)}_{\in V_{l-1}} + \underbrace{\sum_{i \in \mathcal{I}_l} d_{l,i} \psi_{l,i}(x)}_{\in W_l}$$

- examine intervals  $x_{l,2i} < x < x_{l,2i+1}$  and  $x_{l,2i+1} < x < x_{l,2i+2}$ :

$$c_{l-1,i} + d_{l,2i} = c_{l,2i} \quad \text{and} \quad c_{l-1,i} - d_{l,2i} = c_{l,2i+1}$$

- leads to formula for  $c_{l-1,i}$  and  $d_{l,2i}$  (note the even index  $2i$ ):

$$c_{l-1,i} = \frac{1}{2}(c_{l,2i} + c_{l,2i+1}) \quad d_{l,2i} = \frac{1}{2}(c_{l,2i} - c_{l,2i+1})$$

## Part II

# Haar Wavelets as Wavelets

# Change of Notation – Scaling Function

- define **scaling function**:

$$\phi(x) := \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

- nodal basis functions on level  $l$ :

$$\phi_{l,k}(x) := 2^{l/2} \phi\left(\frac{x - x_{l,k}}{h_l}\right) = 2^{l/2} \phi\left(\frac{x - k \cdot 2^{-l}}{2^{-l}}\right) = 2^{l/2} \phi(2^l x - k)$$

(remember:  $x_{l,k} = k \cdot 2^{-l}$  and  $h_l = 2^{-l}$ )

- scaling with  $2^{l/2}$  to be discussed ...
- resulting nodal basis on level  $l$ :

$$V_l = \text{span}\{\phi_{l,k}(x) : k = 0, \dots, 2^l - 1\}$$

# Change of Notation – Wavelet Functions

- define **mother Haar wavelet**:

$$\psi(x) := \begin{cases} 1 & \text{if } 0 < x < \frac{1}{2} \\ -1 & \text{if } \frac{1}{2} < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

- Haar wavelet functions** on level  $l$ :

$$\psi_{l,k}(x) := 2^{l/2} \psi(2^l x - k) = 2^{l/2} \psi\left(\frac{x - 2^{-l}k}{2^{-l}}\right) = 2^{l/2} \psi\left(\frac{x - x_{l,k}}{h_l}\right)$$

for  $k = 0, \dots, 2^l - 1$ , (but no “stride two”)

- Important changes:
  - shifted numbering of levels:  $\psi(x)$  defined on  $[0, 1]$
  - thus: supports of  $\psi_{l,k}(x)$  and  $\psi_{l,k+1}(x)$  no longer overlap
  - index  $k = 0, \dots, 2^l - 1$  used with “stride 1”

## Change of Notation – Wavelet Functions (cont.)

- define **mother Haar wavelet**:

$$\psi(x) := \begin{cases} 1 & \text{if } 0 < x < \frac{1}{2} \\ -1 & \text{if } \frac{1}{2} < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

- Haar wavelet functions** on level  $l$ :

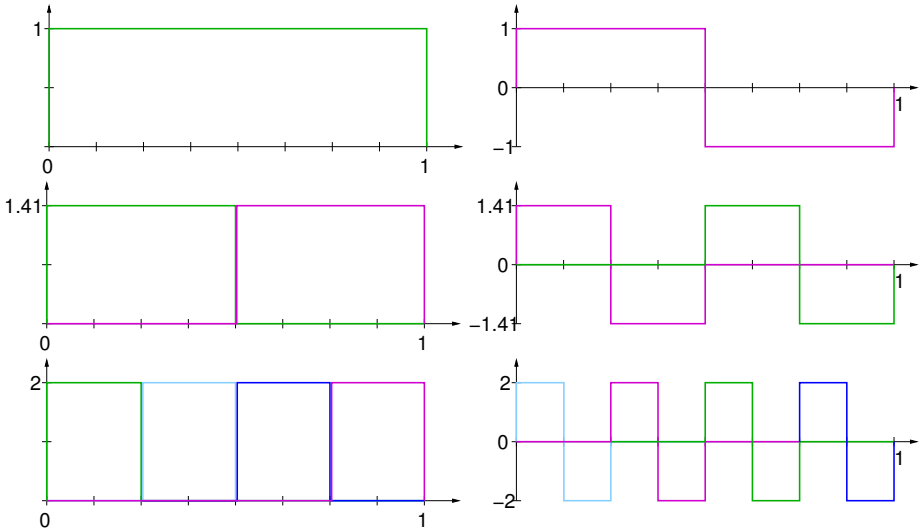
$$\psi_{l,k}(x) := 2^{l/2} \psi(2^l x - k) \quad \text{for } k = 0, \dots, 2^l - 1.$$

- wavelet space for each level:

$$W_l := \text{span} \left\{ \psi_{l,k} : k = 0, \dots, 2^l - 1 \right\}$$

- definition of function spaces:  $V_{l+1} = V_l \oplus W_l$

# Haar Wavelet Functions



# Haar Wavelets – Transform

- represent a piecewise constant function  $s(x) \in V_l$ :

$$s(x) = \sum_{k=0}^{2^l-1} c_{l,k} \phi_{l,k}(x)$$

- write as coarse function plus hierarchical surplus:

$$s(x) = \underbrace{\sum_k c_{l,k} \phi_{l,k}(x)}_{\in V_l} = \underbrace{\sum_k c_{l-1,k} \phi_{l-1,k}(x)}_{\in V_{l-1}} + \underbrace{\sum_k d_{l-1,k} \psi_{l-1,k}(x)}_{\in W_{l-1}}$$

- transform  $c_{l,2k}$  to  $c_{l-1,k}$  and  $d_{l-1,k}$ :

$$c_{l-1,k} = \frac{1}{\sqrt{2}}(c_{l,2k} + c_{l,2k+1}) \quad d_{l-1,k} = \frac{1}{\sqrt{2}}(c_{l,2k} - c_{l,2k+1})$$

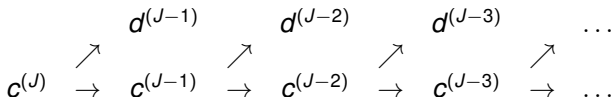
- backward transform  $c_{l-1,k}$  and  $d_{l-1,k}$  to  $c_{l,2k}$  and  $c_{l,2k+1}$ :

$$c_{l,2k} = \frac{1}{\sqrt{2}}(c_{l-1,k} + d_{l-1,k}) \quad c_{l,2k+1} = \frac{1}{\sqrt{2}}(c_{l-1,k} - d_{l-1,k})$$

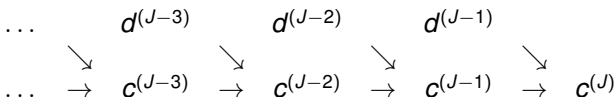


## Haar Wavelets – Transform (2)

- scheme for wavelet decomposition:



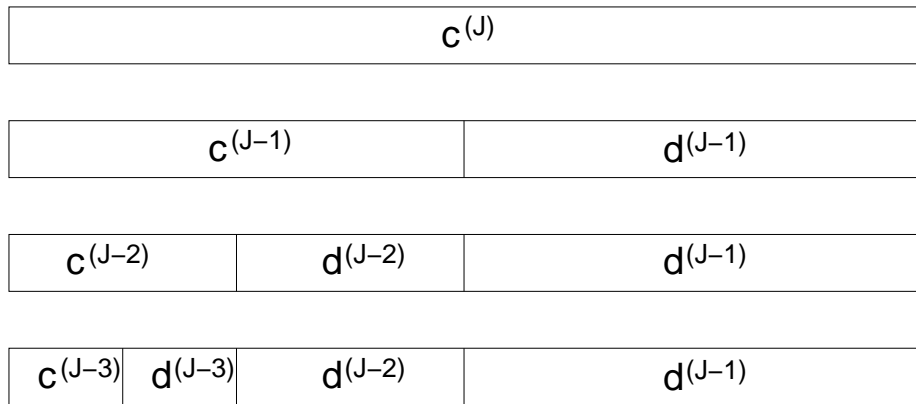
- scheme for assembly:



- Note: computational effort for transformations is only  $\mathcal{O}(N)$

## Haar Wavelets – Transform (3)

Scheme for data structures:



# Haar Wavelets – Orthogonality

- Haar wavelets are **orthogonal** functions:

$$\int \psi_{l,i}(x) \psi_{m,j}(x) dx := \begin{cases} 1 & \text{if } l = m \text{ and } i = j \\ 0 & \text{otherwise} \end{cases}$$

- two different wavelet functions  $\psi_{l,i} \neq \psi_{l,j}$  on the same level  $l$

$$\int \psi_{l,i}(x) \psi_{l,j}(x) dx = 0 \quad (\text{no overlap of functions!})$$

- two wavelet functions  $\psi_{l,i} \neq \psi_{m,j}$  on different levels  $l < m$

$$\int \psi_{l,i}(x) \psi_{m,j}(x) dx = \psi_{l,i}(x_{m,j}^+) \int \psi_{m,j}(x) dx = 0$$

- scalar product of a wavelet functions  $\psi_{l,i}$  with itself

$$\int (\psi_{l,i}(x))^2 dx = \int_{x_{l,i}}^{x_{l,i}+2^{-l}} (2^{l/2})^2 dx = 1$$

# Haar Wavelets – Summary and Next Steps

Haar wavelets:

- hierarchical basis of **piecewise constant** and ...
- ... **orthogonal** basis functions
- $\mathcal{O}(N)$  effort for hierarchical transformation (compare tutorial)

Next steps:

- applications in signal and image processing
- extension to 2D (and higher dimensions)
- is there a piecewise linear/polynomial/higher-order orthogonal(!) wavelet basis?

## Part III

# Wavelets in Signal and Image Processing

# Scaling Functions and Wavelet Functions in 2D

Use tensor product, as for hierarchical basis:

- 2D scaling functions on levels  $l_1, l_2$ :

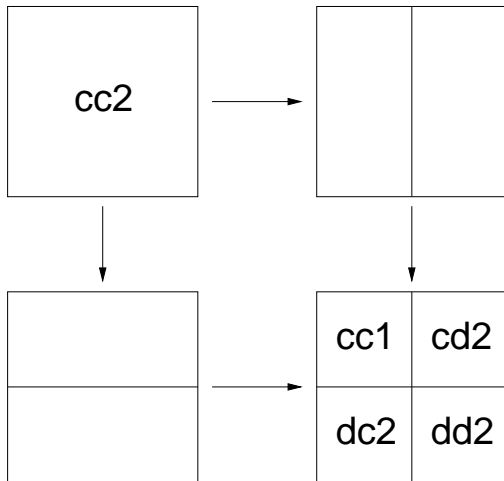
$$\phi_{\vec{l}, \vec{k}}(x_1, x_2) := \phi_{l_1, l_2, k_1, k_2}(x_1, x_2) := \phi_{l_1, k_1}(x_1) \cdot \phi_{l_2, k_2}(x_2)$$

- 2D wavelet functions on levels  $l_1, l_2$ :

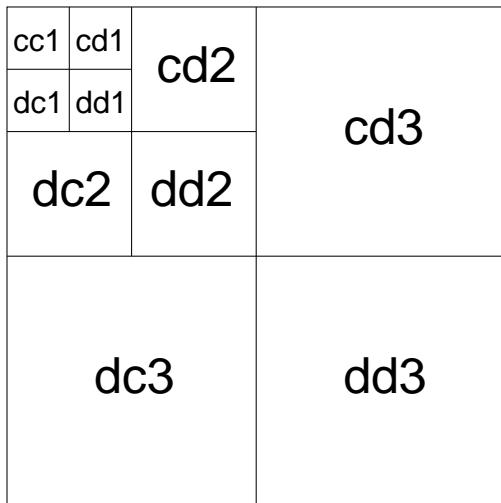
$$\psi_{\vec{l}, \vec{k}}(x_1, x_2) := \psi_{l_1, l_2, k_1, k_2}(x_1, x_2) := \psi_{l_1, k_1}(x_1) \cdot \psi_{l_2, k_2}(x_2)$$

- thus straightforward extension to 3D and higher dimensions
- construction of basis function equivalent to 2D/3D Fourier Transform (also for Hierarchical Basis)
- however: 2D/3D Wavelet transform typically not a straightforward sequence of 1D transforms  
→ instead: sequence of level-wise transforms

## 2D Wavelets – A Single Transformation Step



## 2D Wavelets – Storage Scheme





# Wavelet-Based Compression of Image Data

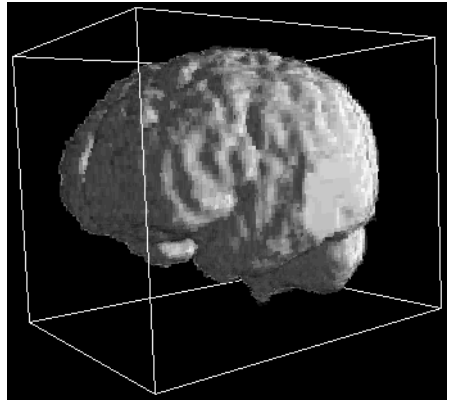
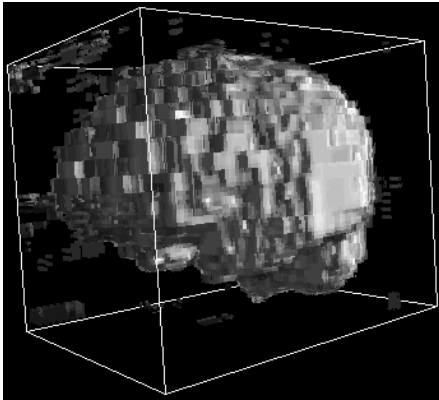
## Typical steps for image compression:

1. Conversion of colour model  
(separation of brightness and colour information)
2. **2D discrete Wavelet transform**
3. **Quantisation of the coefficients** ( $\rightarrow$  reduce information)
4. efficient encoding  
(loss-less compression of the quantised coefficients)

In practice:

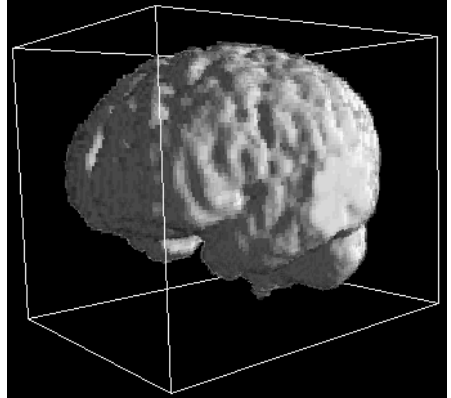
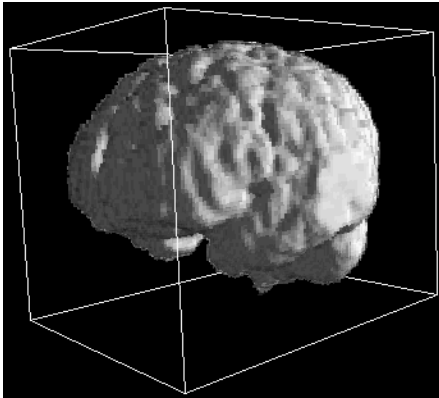
- different algorithms: EZF, SPIHT, ...
- similar to JPEG, but often much better quality
- see, e.g., Walker: “Wavelet-based Image Compression”  
for full details

## Example: 3D Image Compression



(wavelet-based compression of raster data, A. Dehmel)

## Example: 3D Image Compression (2)



(wavelet-based compression of raster data, A. Dehmel)

# From Fourier Transform to Wavelets

## (Discrete) Fourier Transform:

$$f(x) \sim \sum c_k e^{ikx} \quad \text{or} \quad f_n = \sum F_k e^{i\pi kn/N}$$

- $f$  contains only spatial information
- $c_k, F_k$  contain only frequency information
- no relation between frequency and location

## Windowed Fourier Transform:

$$f(x) = \frac{1}{2\pi} \int \int F(u, k) g(x - u) e^{ikx} dk du, \quad F(u, k) = \int f(x) g(x - u) e^{-ikx} dx$$

- $F(u, k)$ : frequency  $k$  at location  $u$
- $g(\xi)$  a window function
  - narrow windows do not allow to locate coarse frequencies
  - but wide windows decrease accuracy in location

# From Fourier Transform to Wavelets (2)

## Continuous Wavelet Transform:

$$W(a, b) = \int f(t) \psi_a^b(t) dt \quad \text{and} \quad f(x) = \frac{1}{C_\psi} \int \int W(a, b) \frac{\psi_a^b(x)}{a^2} da db$$

- continuous in  $a$  and  $b$
- $\psi_a^b(t) = |a|^{-1/2} \psi\left(\frac{t-b}{a}\right)$  with “mother wavelet”  $\psi$
- infinitely many (redundant) coefficients  $\rightarrow$  computationally not feasible

## Multiresolution Analysis/Discrete Wavelet Transform:

- restrict  $(a, b)$  to discrete values  $(a, b) := \left(\frac{1}{2^j}, \frac{k}{2^j}\right)$
- thus discrete wavelet functions:

$$\psi_{j,k} = \psi_k^{2^{-j}} = 2^{j/2} \psi(2^j t - k)$$

- combines frequency and location: higher spatial resolution for higher frequencies

## Part IV

# More Complicated Wavelets

*Reference/more details  $\rightsquigarrow$  Aboufadel & Schlicker: Discovering Wavelets*

# Mother and Father Wavelets – General Situation

- **mother wavelet**  $\psi(x)$
- **father wavelet**  $\phi(x)$ , also called **scaling function**
- basis built from scaling functions on each level  $l$ :

$$\phi_{l,k}(x) := 2^{l/2} \phi(2^l x - k) \quad V_l := \text{span} \{ \phi_{l,k}(x) \}$$

- surplus basis built from wavelet functions on each level  $l$ :

$$\psi_{l,k}(x) := 2^{l/2} \psi(2^l x - k) \quad W_l := \text{span} \{ \psi_{l,k}(x) \}$$

- definition of function spaces:  $V_{l+1} = V_l \oplus W_l$
- wavelet basis functions are **orthonormal**:

$$\langle \psi_{l,k}(x), \psi_{m,j}(x) \rangle = \int \psi_{l,k}(x) \psi_{m,j}(x) dx = \begin{cases} 1 & \text{if } l = m \text{ and } k = j \\ 0 & \text{otherwise} \end{cases}$$

- also: scaling basis functions are orthonormal on each level

# Scaling and Wavelet Functions

- note:  $\phi_{l-1,k} \in V_l \supset V_{l-1}$ , and also  $\psi_{l-1,k} \in V_l = V_{l-1} \oplus W_{l-1}$
- hence, all  $\phi_{l-1,k}$  and  $\psi_{l-1,k}$  can be uniquely represented via the basis functions of  $V_l$ , i.e., the  $\phi_{l,k}$ :

$$\phi_{l-1,0}(x) = \sum_i p_i \phi_{l,i}(x) = 2^{l/2} \sum_i p_i \phi(2^l x - i)$$

$$\psi_{l-1,0}(x) = \sum_i q_i \phi_{l,i}(x) = 2^{l/2} \sum_i q_i \phi(2^l x - i)$$

- for efficiency:  $p_i$  and  $q_i$  should be non-zero for only a few  $i$
- for Haar wavelets:

$$p_0 = \frac{1}{\sqrt{2}}, \quad p_1 = \frac{1}{\sqrt{2}}, \quad \text{all other } p_i = 0$$

$$q_0 = \frac{1}{\sqrt{2}}, \quad q_1 = -\frac{1}{\sqrt{2}}, \quad \text{all other } q_i = 0$$



## Scaling and Wavelet Functions (2)

- do for all scaling functions  $\phi_{l-1,k}$ :

$$\begin{aligned}\phi_{l-1,k}(x) &= 2^{l/2} \sum_i p_i \phi\left(2^l x - 2k - i\right) \\ &\stackrel{2k+i \rightarrow i}{=} 2^{l/2} \sum_i p_{i-2k} \phi\left(2^l x - i\right) = \sum_i p_{i-2k} \phi_{l,i}(x)\end{aligned}$$

Note:  $\phi_{l,k}(x) = 2^{l/2} \phi\left(2^l x - k\right) = 2^{l/2} \phi\left(2^l x - 2^l \frac{k}{2^l}\right) = 2^{l/2} \phi\left(2^l \left(x - \frac{k}{2^l}\right)\right) = \phi_{l,0}\left(x - \frac{k}{2^l}\right)$   
and thus:  $\phi_{l-1,k}(x) = \phi_{l-1,0}\left(x - \frac{k}{2^{l-1}}\right) = 2^{l/2} \sum_i q_i \phi\left(2^l \left(x - \frac{k}{2^{l-1}}\right) - i\right) = \dots$

- and similar for wavelet functions:  $\psi_{l-1,k}(x) = \sum_i q_{i-2k} \phi_{l,i}(x)$
- for Haar wavelets:  
 $p_{i-2k}$  and  $q_{i-2k}$  are non-zero only for  $i = 2k$  and  $i = 2k + 1$ :

$$\begin{aligned}\phi_{l-1,k}(x) &= \frac{1}{\sqrt{2}} \phi_{l,2k}(x) + \frac{1}{\sqrt{2}} \phi_{l,2k+1}(x) \\ \psi_{l-1,k}(x) &= \frac{1}{\sqrt{2}} \phi_{l,2k}(x) - \frac{1}{\sqrt{2}} \phi_{l,2k+1}(x)\end{aligned}$$

# Wavelet Transformations and Filtering

- consider a signal function represented on (fine) level  $l + 1$ :

$$f_{l+1}(x) = \sum_i c_i^{(l+1)} \phi_{l+1,i}(x)$$

- and a decomposition  $f_{l+1} = f_l + g_l$ , where  $f_l \in V_l$  and  $g_l \in W_l$ :

$$\begin{aligned} f_{l+1}(x) &= \sum_i c_i^{(l+1)} \phi_{l+1,i}(x) = \sum_j c_j^{(l)} \phi_{l,j}(x) + \sum_j d_j^{(l)} \psi_{l,j}(x) \\ &= \sum_j \left( c_j^{(l)} \sum_i p_{i-2j} \phi_{l+1,i}(x) \right) + \sum_j \left( d_j^{(l)} \sum_i q_{i-2j} \phi_{l+1,i}(x) \right) \\ &= \sum_i \phi_{l+1,i}(x) \sum_j \left( p_{i-2j} c_j^{(l)} + q_{i-2j} d_j^{(l)} \right) \end{aligned}$$

- two different representations of  $f_{l+1}(x)$ , but  $\{\phi_{l+1,k}(x)\}$  a basis:

$$\Rightarrow c_i^{(l+1)} = \sum_j \left( p_{i-2j} c_j^{(l)} + q_{i-2j} d_j^{(l)} \right)$$

## Wavelet Transformations and Filtering (2)

- $p_i$  and  $q_i$  determine transformation of coefficients:

$$c_i^{(l+1)} = \sum_j \left( p_{i-2j} c_j^{(l)} + q_{i-2j} d_j^{(l)} \right)$$

- solves assembly:  
for given  $f_l$  and  $g_l$  (i.e., given coefficients  $c_j^{(l)}$  and  $d_j^{(l)}$ ),  
find coefficients  $c_i^{(l+1)}$  for  $f_{l+1}$
- for Haar wavelets:

$$\begin{aligned} \text{even } i: \quad c_i^{(l+1)} &= \frac{1}{\sqrt{2}} c_{i/2}^{(l)} + \frac{1}{\sqrt{2}} d_{i/2}^{(l)} \\ \text{odd } i: \quad c_i^{(l+1)} &= \frac{1}{\sqrt{2}} c_{(i-1)/2}^{(l)} - \frac{1}{\sqrt{2}} d_{(i-1)/2}^{(l)} \end{aligned}$$

## Wavelet Transformations and Filtering (3)

- now: fine-level representation given as

$$f_{l+1}(x) = \sum_i c_i^{(l+1)} \phi_{l+1,i}(x)$$

- wanted: decomposition  $f_{l+1} = f_l + g_l$  with

$$f_l(x) + g_l(x) = \sum_j c_j^{(l)} \phi_{l,j}(x) + \sum_j d_j^{(l)} \psi_{l,j}(x)$$

- use that  $\{\phi_{l,k}(x)\}$  and  $\{\psi_{l,k}(x)\}$  are **orthonormal** basis for  $V_l$  and  $W_l$ , and  $V_l \perp W_l$ :

$$\begin{aligned} \Rightarrow c_j^{(l)} &= \langle f_{l+1}(x), \phi_{l,j}(x) \rangle = \left\langle \sum_i c_i^{(l+1)} \phi_{l+1,i}(x), \phi_{l,j}(x) \right\rangle \\ &= \sum_i c_i^{(l+1)} \langle \phi_{l+1,i}(x), \phi_{l,j}(x) \rangle \\ &= \dots \end{aligned}$$

## Wavelet Transformations and Filtering (4)

- continued:

$$\begin{aligned}
 c_j^{(l)} &= \langle f_{l+1}(x), \phi_{l,j}(x) \rangle = \dots = \sum_i c_i^{(l+1)} \langle \phi_{l+1,i}(x), \phi_{l,j}(x) \rangle \\
 &= \sum_i c_i^{(l+1)} \left\langle \phi_{l+1,i}(x), \sum_k p_{k-2j} \phi_{l+1,k}(x) \right\rangle \\
 &= \sum_i c_i^{(l+1)} \sum_k p_{k-2j} \left\langle \phi_{l+1,i}(x), \phi_{l+1,k}(x) \right\rangle = \sum_i c_i^{(l+1)} p_{i-2j}
 \end{aligned}$$

- similar computation for  $d_j^{(l)}$ , and therefore:

$$c_j^{(l)} = \sum_i p_{i-2j} c_i^{(l+1)} \quad d_j^{(l)} = \sum_i q_{i-2j} c_i^{(l+1)}$$

- again, for Haar wavelets:

$$c_j^{(l)} = \frac{1}{\sqrt{2}} c_{2j}^{(l+1)} + \frac{1}{\sqrt{2}} c_{2j+1}^{(l+1)} \quad d_j^{(l)} = \frac{1}{\sqrt{2}} c_{2j}^{(l+1)} - \frac{1}{\sqrt{2}} c_{2j+1}^{(l+1)}$$

# Wavelet Transformations and Filtering – Summary

Wanted: decomposition  $f_{l+1} = f_l + g_l$  with

- coarser representation  $f_l(x) = \sum c_j^{(l)} \phi_{l,j}(x)$  with

$$c_j^{(l)} = \sum_i p_{i-2j} c_i^{(l+1)}$$

corresponds to a **low-pass filter** (averaging)

- oscillatory surplus  $g_l(x) = \sum d_j^{(l)} \psi_{l,j}(x)$  with

$$d_j^{(l)} = \sum_i q_{i-2j} c_i^{(l+1)}$$

corresponds to a **high-pass filter** (difference computation)

- and reconstruction:  $c_i^{(l+1)} = \sum_j \left( p_{i-2j} c_j^{(l)} + q_{i-2j} d_j^{(l)} \right)$

# How to Determine the Filtering Coefficients?

- we need coefficients for low-pass and high-pass filter:

$$c_j^{(l)} = \sum_i p_{i-2j} c_i^{(l+1)} \quad d_j^{(l)} = \sum_i q_{i-2j} c_i^{(l+1)}$$

- reconstruction then:  $c_i^{(l+1)} = \sum_j (p_{i-2j} c_j^{(l)} + q_{i-2j} d_j^{(l)})$
- requires **scaling equation** for scaling and wavelet functions:

$$\phi_{l-1,k}(x) = \sum_i p_{i-2k} \phi_{l,i}(x) \quad \psi_{l-1,k}(x) = \sum_i q_{i-2k} \phi_{l,i}(x)$$

- requires **orthogonal** scaling and wavelet functions:
  - $\phi_{l,k} \perp \phi_{l,j}$  and  $\psi_{l,k} \perp \psi_{l,j}$  for  $k \neq j$
  - $\psi_{l,k} \perp \phi_{m,j}$  if  $m \leq l$  and arbitrary  $k, j$  (i.e.,  $W_l \perp V_m$ )

## How to Determine the Wavelet Functions? (2)

- **scaling equation** for mother and father wavelet:

$$\phi(x) = \sqrt{2} \sum_k p_k \phi(2x - k) \quad \psi(x) = \sqrt{2} \sum_k q_k \phi(2x - k)$$

also called **dilation equation**

- for Haar wavelet:

$$\phi(x) = \phi(2x) + \phi(2x - 1) \quad \psi(x) = \phi(2x) - \phi(2x - 1)$$

- for more complicated wavelets:
  - more than 2 non-zeros  $p_k$  (and  $q_k$ )
  - $p_k$  and  $q_k$  determined to satisfy orthogonality
  - often no analytical expression for  $\phi(x)$  and  $\psi(x)$  available
  - obtain  $\phi(x)$  and  $\psi(x)$  as solutions of the scaling equation  
→ see worksheet “cranking the machine”



# Towards More Complicated Wavelets

## “Wish List:”

- orthonormal basis of scaling functions on each level:

$$\langle \phi_{l,k}(x), \phi_{l,j}(x) \rangle = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{otherwise} \end{cases}$$

- scaling/wavelet functions obey top scaling equation:

$$\phi_{l-1,k}(x) = \sum_i p_{i-2k} \phi_{l,i}(x) \quad \psi_{l-1,k}(x) = \sum_i q_{i-2k} \phi_{l,i}(x)$$

- scaling/wavelet functions have **compact support**  
 $\leadsto p_i \neq 0$  only for few  $i$  (same for  $q_i$ )
- as additional criteria: “vanishing moments” of wavelet functions

$$\int \psi(t) dt = 0 \quad \int t\psi(t) dt = 0 \quad \text{etc.}$$

## Towards More Complicated Wavelets (2)

orthonormal basis of scaling functions:

- on each level:

$$\langle \phi_{l,k}(x), \phi_{l,j}(x) \rangle = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{otherwise} \end{cases}$$

- combine with scaling equation and compact support:

$$\phi_{l-1,k}(x) = \sum_i p_{i-2k} \phi_{l,i}(x) \quad \text{where } p_i \neq 0 \text{ only for few } i$$

and obtain:

$$\begin{aligned} \langle \phi_{l-1,k}(x), \phi_{l-1,m}(x) \rangle &= \left\langle \sum_i p_{i-2k} \phi_{l,i}(x), \sum_j p_{j-2m} \phi_{l,j}(x) \right\rangle \\ &= \sum_i p_{i-2k} \sum_j p_{j-2m} \langle \phi_{l,i}(x), \phi_{l,j}(x) \rangle = \sum_i p_{i-2k} p_{i-2m} \end{aligned}$$

- in particular (for  $k = m$ ):  $\sum_i (p_{i-2k})^2 = \sum_i p_i^2 = 1$

## Towards More Complicated Wavelets (3)

- in addition – for  $k = 0$  and arbitrary  $m \neq 0$ :

$$\langle \phi_{l-1,0}(x), \phi_{l-1,m}(x) \rangle = \sum_i p_i p_{i-2m} = 0$$

- similar argument: scaling and wavelet functions are orthogonal!

$$\begin{aligned} \langle \phi_{l-1,0}(x), \psi_{l-1,0}(x) \rangle &= \left\langle \sum_i p_i \phi_{l,i}(x), \sum_j q_j \phi_{l,j}(x) \right\rangle \\ &= \sum_i p_i \sum_j q_j \langle \phi_{l,i}(x), \phi_{l,j}(x) \rangle = \sum_i p_i q_i \stackrel{!}{=} 0 \end{aligned}$$

- and wavelet functions of one level are orthogonal:

$$\langle \psi_{l,k}(x), \psi_{l,m}(x) \rangle = 0 \quad \rightsquigarrow \quad \sum_i q_i q_{i-2(k-m)} = \begin{cases} 0 & \text{if } k \neq m \\ 1 & \text{if } k = m \end{cases}$$

- to satisfy these requirements:  $q_k = (-1)^k p_{1-k}$

## Daubechies Wavelets (D4)

- setting:  $\phi(x) = 0$  outside of interval  $[0, 3]$   
 $\rightarrow$  non-zero coefficients are  $p_0, p_1, p_2$ , and  $p_3$
- orthogonality requires  $\sum p_i^2 = 1$  and  $\sum p_i p_{i-2m} = 0$ :

$$p_0^2 + p_1^2 + p_2^2 + p_3^2 = 1 \quad \text{and} \quad p_0 p_2 + p_1 p_3 = 0$$

- plus vanishing moments  $\int \psi(t) dt = 0$  and  $\int t \psi(t) dt = 0$   
 together with  $q_k = (-1)^k p_{1-k}$  leads to

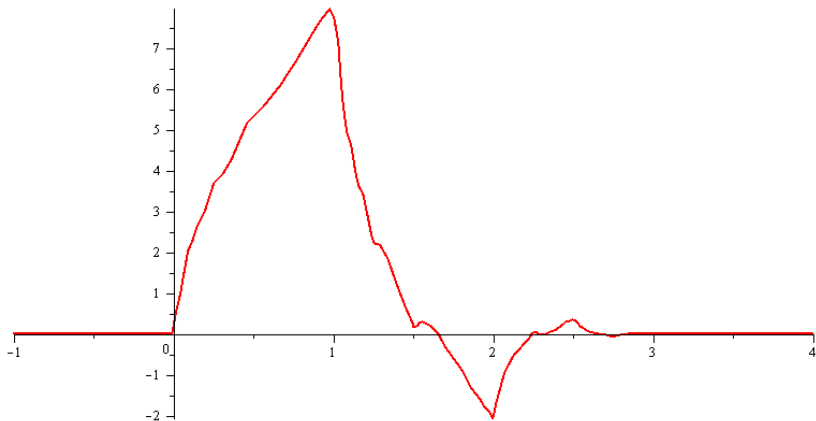
$$-p_0 + p_1 - p_2 + p_3 = 0 \quad \text{and} \quad -p_1 + 2p_2 - 3p_3 = 0$$

- one solution to this system:

$$p_0 = \frac{1 + \sqrt{3}}{4\sqrt{2}}, \quad p_1 = \frac{3 + \sqrt{3}}{4\sqrt{2}}, \quad p_2 = \frac{3 - \sqrt{3}}{4\sqrt{2}}, \quad p_3 = \frac{1 - \sqrt{3}}{4\sqrt{2}}$$

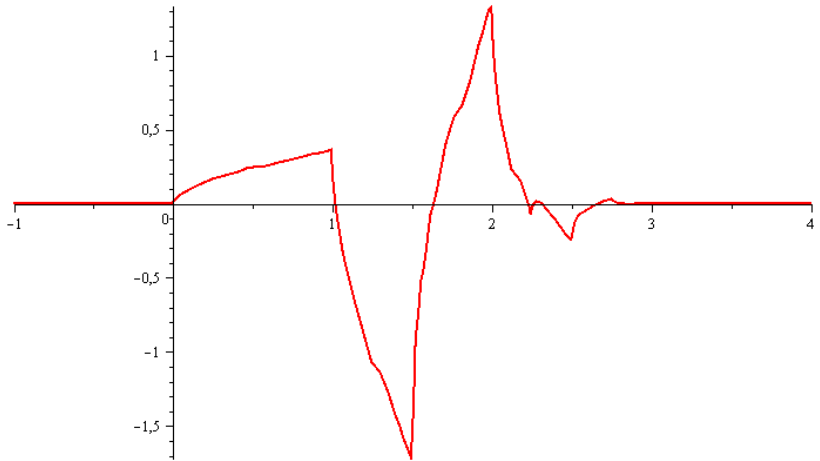
# Daubechies Wavelets (D4) – Scaling Function

no analytical expression available  $\rightarrow$  iterative approximation



see tutorials:  $\rightarrow$  “**cranking the machine**”

# Daubechies Wavelets (D4) – Wavelet Function



## Daubechies Wavelets (D4) – Transform

**Algorithm to compute  $c_j^{(l)} = \sum_i p_{i-2j} c_i^{(l+1)}$  and  $d_j^{(l)} = \sum_i q_{i-2j} c_i^{(l+1)}$  (1 level):**

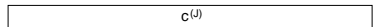
- Input: coefficients  $c_i^{(l+1)}$ ,  $i = 0, \dots, 2^{l+1} - 1$  stored in array  $c[:]$
- requires helper array  $cl[:]$  for  $c_j^{(l)}$ ,  $j = 0, \dots, 2^l - 1$
- requires helper array  $dl[:]$  for  $d_j^{(l)}$ ,  $j = 0, \dots, 2^l - 1$
- filter coefficients  $p_i$  and  $q_i$  stored in arrays  $p[0:3]$  and  $q[0:3]$  resp.
- main loop:

```
for j from 0 to 2l - 1 do
```

```
    cl[j] = p[0]*c[2*j] + p[1]*c[2*j+1] + p[2]*c[2*j+2] + p[3]*c[2*j+3];
```

```
    dl[j] = q[0]*c[2*j] + q[1]*c[2*j+1] + q[2]*c[2*j+2] + q[3]*c[2*j+3];
```

```
end do
```



- copy  $cl[:]$  and  $dl[:]$  into  $c[:]$  as in:



- missing: treat boundaries  $\rightsquigarrow$  e.g.: periodic wrap-around

# Finally: Multiresolution Analysis

## Definition: **Multiresolution Analysis**

- nested sequence of function spaces:

$$\dots \subset V_0 \subset V_1 \subset V_2 \subset V_3 \subset \dots$$

- with a scaling function  $\phi$   
such that  $\phi(2^l x - k)$  is an orthonormal Basis of  $V_l$   
(and  $V_l = \text{span}\{\phi_{l,k} : k \in \mathbb{Z}\}$ )
- $\bigcup V_l$  is **dense** in  $L^2(\mathbb{R})$
- $V_l$  satisfy **separation property**:  $\bigcap V_l = \{0\}$
- $f(t) \in V_l$  if and only if  $f(2^{-l}t) \in V_0$

Last but not least: find coefficients  $c_k$  such that  $s(x) \approx \sum c_k \phi_{l,k}(x)$ ?

- use orthogonality:  $c_k = \langle s(x), \phi_{l,k}(x) \rangle$   
(orthogonal projection to space  $V_l$ )



# References and Further Material

## Textbooks:

- E. Aboufadel, S. Schlicker: *Discovering Wavelets*. Wiley, New York, 1999
- I. Daubechies: *Ten Lectures on Wavelets*. CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM, 1992.
- J. S. Walker: *A Primer on Wavelets and their Scientific Applications, Second Edition*. Chapman and Hall/CRC, 2008.

## Articles:

- G. Strang: *Wavelet transforms versus Fourier transforms*. Bulletin of the American Mathematical Society 28 (1993), p. 288–305.
- J. S. Walker: *Wavelet-based Image Compression*

See the course webpage for URLs and online access.