Worksheet 6 : Numerical Quadrature for 1D Functions

Exercise 1)

$$\int_a^b x^n \, dx = \frac{x^{n+1}}{n+1}\Big|_a^b = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}$$

(1) $$\int_0^1 -4x(x-1)\, dx = -\int_0^1 (x^2-x)\, dx$$
$$= -4\left(\frac{x^3}{3}\Big|_0^1 - \frac{x^2}{2}\Big|_0^1\right)$$
$$= -4\left(\frac{1}{3}-\frac{1}{2}\right) = -4\left(\frac{2-3}{6}\right) = \frac{4}{6} = \frac{2}{3}$$

(2) $$\int_0^1 \frac{8}{9}\left(-16x^4+40x^3-35x^2+11x\right) dx$$
$$= \frac{8}{9}\left(-\frac{16}{5}+10-\frac{35}{5}+\frac{11}{2}\right) = \frac{76}{135}$$

Finding the integral analytically is not always possible

- We don't know $f(x)$ and only have some data points
- We know $f(x)$ but there is no analytical integral known

We want to find a numerical approximation of integral

Approach: ① approximate/interpolate
② integrate

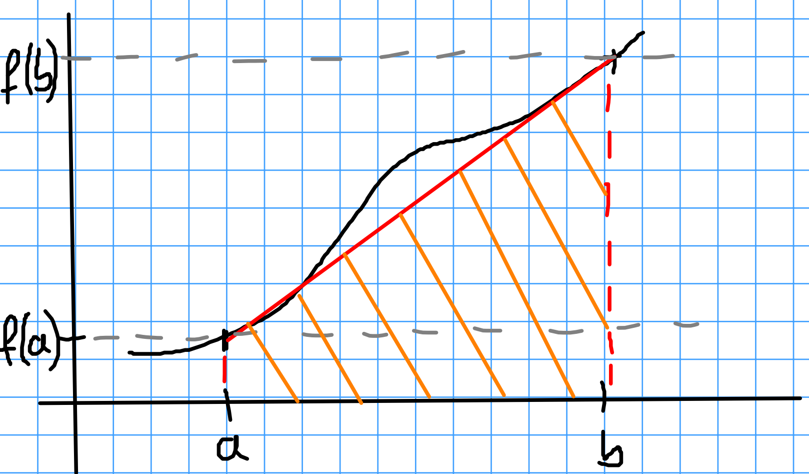
Quadrature

$$\int_a^b f(x)\,dx \approx \int_a^b p_n(x)\,dx = \sum_{i=0}^{n} w_i f(x_i) \quad \text{Newton-Cotes}$$

Approximation

$$x_i = a + ih$$

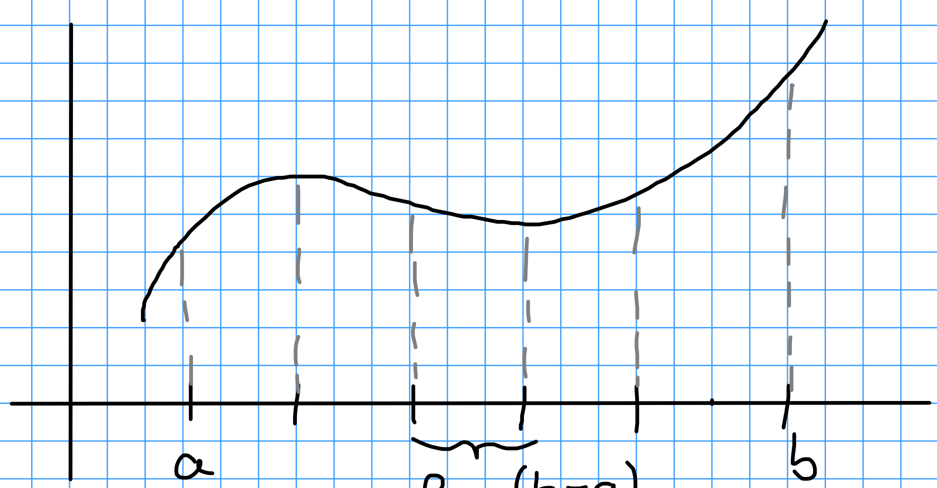
Exercise: Trapezoidal rule: $n=1$



$$T = (b-a)\frac{f(a)+f(b)}{2}$$

The error bound

$$\Theta\left((b-a)^3\right)$$

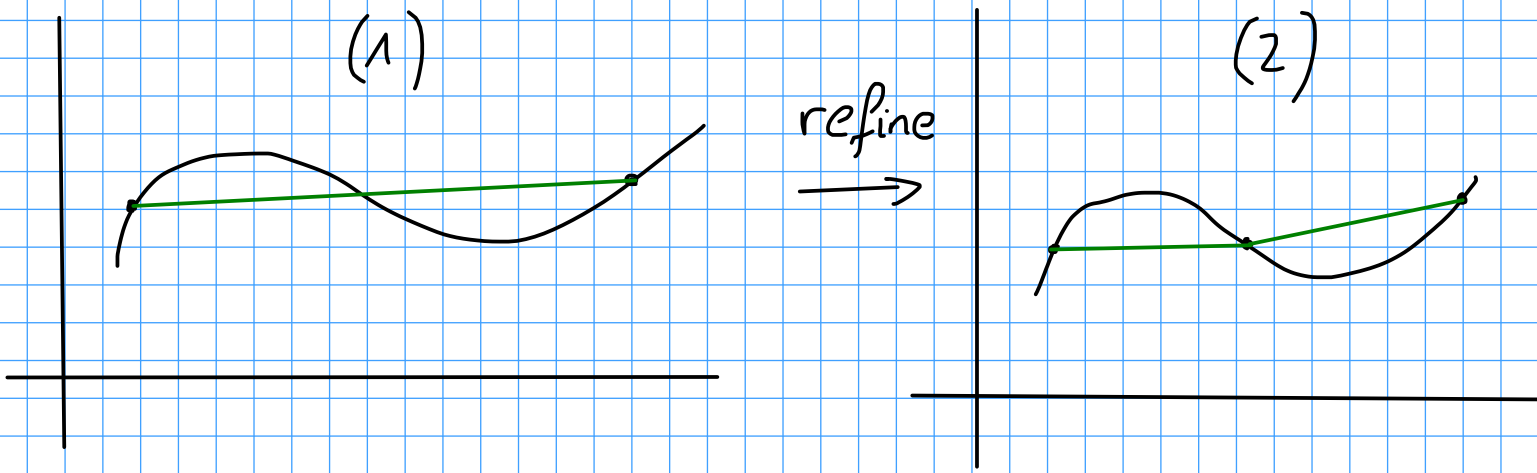


$$h = \frac{(b-a)}{n}$$

$$CT = h \cdot \left[\frac{f(a)}{2} + \sum_{i=1}^{n-1} f(a+ih) + \frac{f(b)}{2}\right]$$

Error: $\Theta(h^2)$

Hierarchical idea:

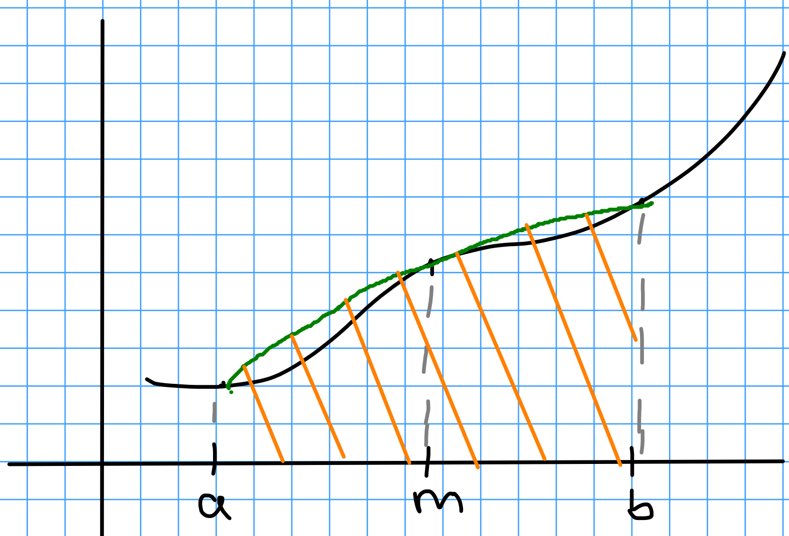


Problem: The calculation at (1) will not help us for (2).

$\longrightarrow$ We have to discard our old result from (1) to increase accuracy

$\Rightarrow$ With a hierarchical approach we can refine and use the calculations we already have

Simpson's rule: $n=2$



$$S = (b-a)\frac{f(a)+4f\left(\frac{a+b}{2}\right)+f(b)}{6}$$

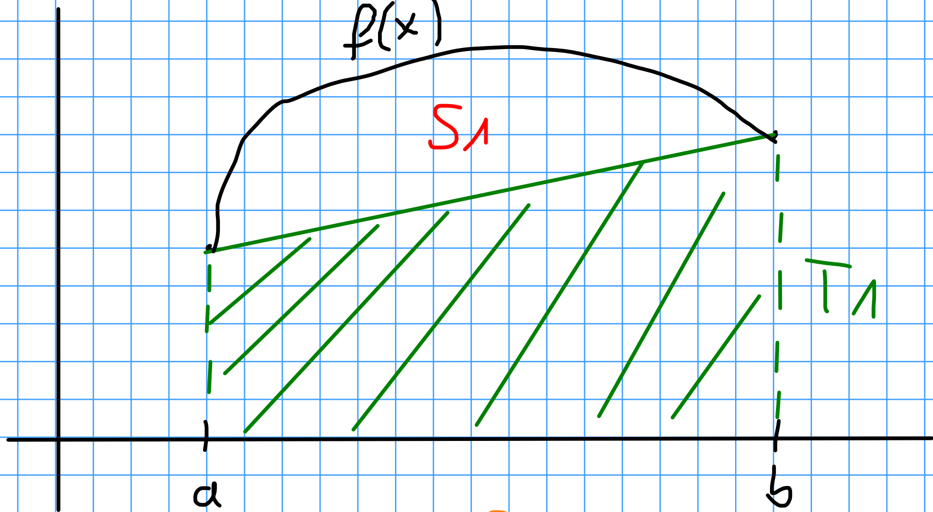
Error: $\mathcal{O}((b-a)^5)$

CS = see lecture

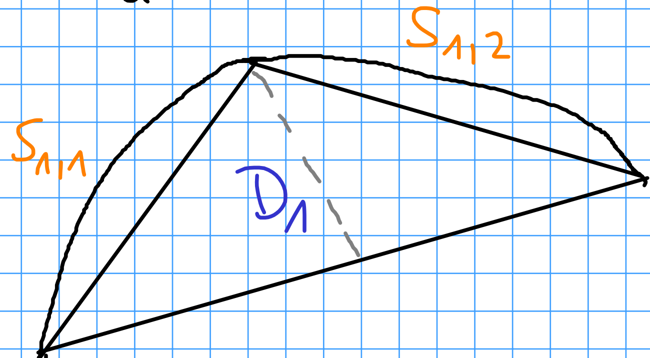
Error: $\mathcal{O}(h^4)$

We are similar problems as for the Trapezoidal rule regarding hierarchical idea

Exercise 3 : Archimedes' Approach



$$F_1 = T_1 + S_1$$



$$S_1 = D_1 + S_{1,2} + S_{1,2}$$

Hierarchical approach

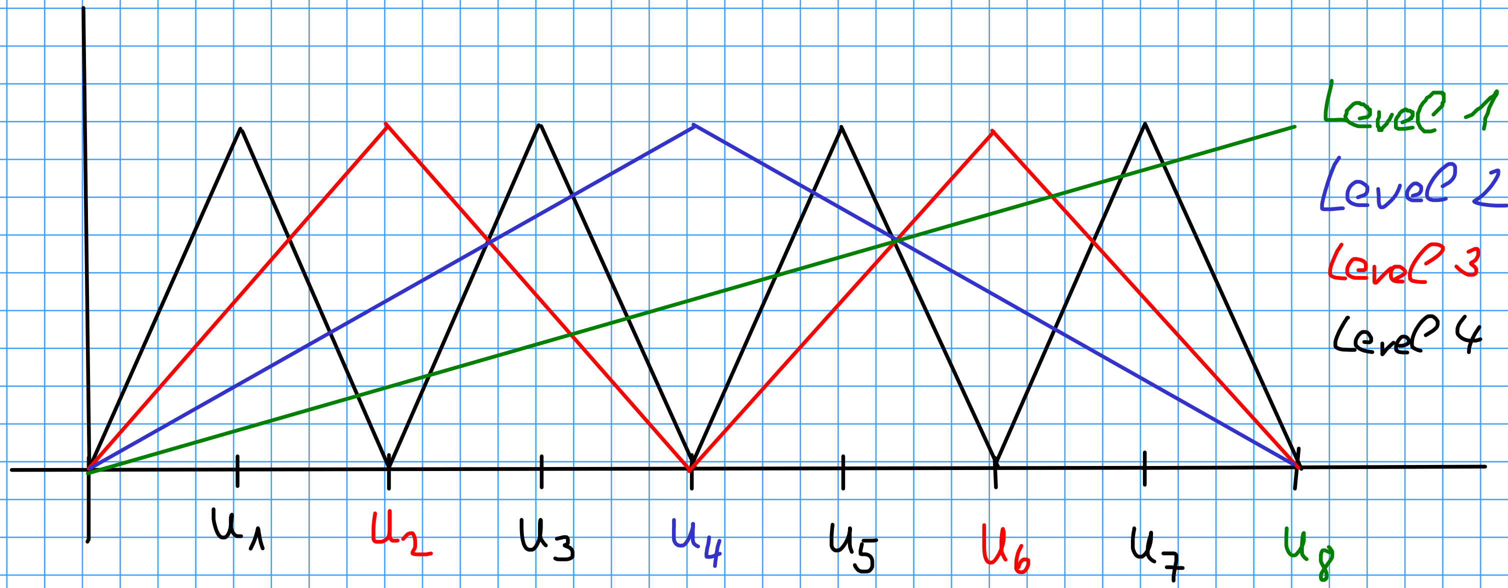
$$T_1 = \frac{b-a}{2}\left(f(a) + f(b)\right)$$

$$D_1 = \frac{b-a}{2}\underbrace{\left(f\left(\frac{a+b}{2}\right) - \frac{f(a)+f(b)}{2}\right)}_{\text{height of triangle}}$$

$\Rightarrow$ leads to recursion

a) vector u: representing function evaluations of a parabola

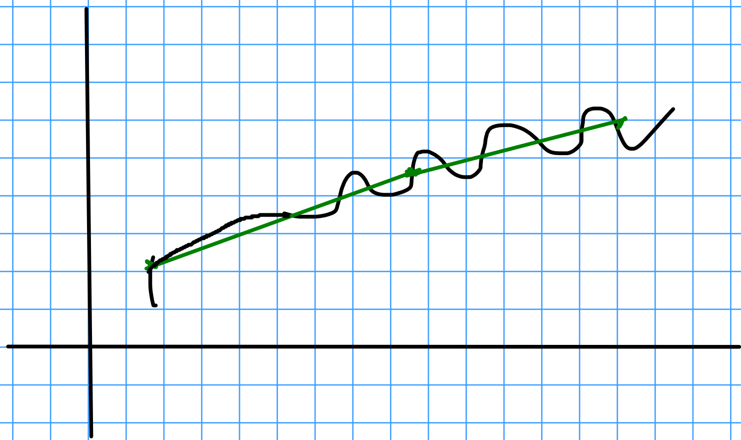
vector v: heights of triangles



Coefficient v: $\quad f(u_3) - \dfrac{f(u_2) + f(u_4)}{2} \qquad f(u_6) - \dfrac{f(u_4) + f(u_8)}{2}$

$\xleftarrow{-1} u_3 \xrightarrow{+1} \qquad \xleftarrow{-2} u_6 \xrightarrow{+2}$

Exercise 4: Adaptivity



- Properties of function (if known)
- Error is bound
- Hierarchical methods