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SS 2022
Worksheet 1
29.04.2022

Algorithms for Scientific Computing

1 Project: Interpolation of the Trajectory of the Asteroid Pallas – Sample Solution

Python Demo

see Jupyter Notebook solution `Worksheet_1.ipynb`.

Exercise 1

$$
\begin{aligned}
X_l &= \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} c_k e^{i2\pi kl/N} = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} c_k \cdot \left(\cos\left(\frac{2\pi kl}{N}\right) + i\sin\left(\frac{2\pi kl}{N}\right)\right) \\
&= \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \mathrm{Re}\{c_k\} \cdot \left(\cos\left(\frac{2\pi kl}{N}\right) + i\sin\left(\frac{2\pi kl}{N}\right)\right) + i\,\mathrm{Im}\{c_k\} \cdot \left(\cos\left(\frac{2\pi kl}{N}\right) + i\sin\left(\frac{2\pi kl}{N}\right)\right)
\end{aligned}
$$

Now we sort for real and imaginery part of X_l:

$$
\begin{aligned}
\mathrm{Re}\{X_l\} &= \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \mathrm{Re}\{c_k\} \cdot \cos\left(\frac{2\pi kl}{N}\right) - \mathrm{Im}\{c_k\} \cdot \sin\left(\frac{2\pi kl}{N}\right) \\
\mathrm{Im}\{X_l\} &= \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \mathrm{Im}\{c_k\} \cdot \cos\left(\frac{2\pi kl}{N}\right) + \mathrm{Re}\{c_k\} \cdot \sin\left(\frac{2\pi kl}{N}\right)
\end{aligned}
$$

Since X_l is real $\mathrm{Im}\{X_l\}$ must be zero. Thus, the remaining part is

$$
X_l = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \mathrm{Re}\{c_k\} \cdot \cos\left(\frac{2\pi kl}{N}\right) - \mathrm{Im}\{c_k\} \cdot \sin\left(\frac{2\pi kl}{N}\right)
$$

By reducing the sum to the range $k = 1, \ldots, \frac{N}{2} - 1$, we get

$$
\begin{aligned}
X_l = \operatorname{Re}\{c_0\} + \operatorname{Re}\{c_{N/2}\} \cdot \cos(\pi l) + \sum_{k=1}^{\frac{N}{2}-1} & \operatorname{Re}\{c_k\} \cdot \cos\left(\frac{2\pi kl}{N}\right) + \operatorname{Re}\{c_{-k}\} \cdot \cos\left(\frac{2\pi(-k)l}{N}\right) \\
& - \operatorname{Im}\{c_k\} \cdot \sin\left(\frac{2\pi kl}{N}\right) - \operatorname{Im}\{c_{-k}\} \cdot \sin\left(\frac{2\pi(-k)l}{N}\right)
\end{aligned}
$$

Using the symmetry of the sine and cosine functions, this can be derived to

$$
\begin{aligned}
X_l = \operatorname{Re}\{c_0\} + \operatorname{Re}\{c_{N/2}\} \cdot \cos(\pi l) + \sum_{k=1}^{\frac{N}{2}-1} & (\operatorname{Re}\{c_k\} + \operatorname{Re}\{c_{-k}\}) \cdot \cos\left(\frac{2\pi kl}{N}\right) \\
& + (\operatorname{Im}\{c_{-k}\} - \operatorname{Im}\{c_k\}) \cdot \sin\left(\frac{2\pi kl}{N}\right)
\end{aligned}
$$

Since $c_{-k} = c_k^*$ it is $\operatorname{Re}\{c_{-k}\} = \operatorname{Re}\{c_k\}$ and $\operatorname{Im}\{c_{-k}\} = -\operatorname{Im}\{c_k\}$. So, we get

$$
X_l = \operatorname{Re}\{c_0\} + \operatorname{Re}\{c_{N/2}\} \cdot \cos(\pi l) + \sum_{k=1}^{\frac{N}{2}-1} 2\operatorname{Re}\{c_k\} \cdot \cos\left(\frac{2\pi kl}{N}\right) - 2\operatorname{Im}\{c_k\} \cdot \sin\left(\frac{2\pi kl}{N}\right)
$$

For $N = 12$, $a_k = 2\operatorname{Re}\{c_k\}, b_k = -2\operatorname{Im}\{c_k\}$ for all $k = 1, \ldots, \frac{N}{2}$, $a_0 = c_0$ and $a_{\frac{N}{2}} = c_{\frac{N}{2}}$ we get equation (2):

$$
X_l = a_0 + \sum_{k=1}^{5} \left(a_k \cos\left(\frac{\pi kl}{6}\right) + b_k \sin\left(\frac{\pi kl}{6}\right) \right) + a_6 \cos(\pi l)
$$

Python Demo

see Jupyter Notebook solution `Worksheet_1.ipynb`.

Exercise 2

According the Hint:

If $X_l = -X_{12-l}$, then $X_6 = -X_{12-6}$ holds. This is possible only if also $X_6 = 0$ holds. Analog we get $X_0 = -X_{12-0} = -X_{12}$. Here the value X_{12} is the declination for the ascension of 360°, which is the same as the declination value for 0° due to the periodicity of the data. Thus, it must also apply that $X_0 = X_{12}$. So we can conclude that $X_0 = 0$ must hold.

With these considerations we can compute the values a_k and b_k, for example with the IPython-notebook `Worksheet_1.ipynb`.

According the coefficients:

We guess that the interpolation of the axis symmetrical data should only need the axis symmetrical basis functions (i.e. all $\cos$ functions), while the interpolation of the point symmetrical data will only depends on the point symmetric basis functions (i.e. all $\sin$ functions). Hence, in one case all coefficients $b_k = 0$, while in the other all $a_k = 0$.

To show this we can insert the symmetrical constraints into the equation for the interpolation

$$X_l = a_0 + \sum_{k=1}^{5} \left(a_k \cos\left(\frac{\pi k l}{6}\right) + b_k \sin\left(\frac{\pi k l}{6}\right)\right) + a_6 \cos\left(\pi l\right). \tag{1}$$

For X_{12-l} it must hold:

$$\begin{aligned} X_{12-l} &= a_0 + \sum_{k=1}^{5} \left(a_k \cos\left(\frac{\pi k(12-l)}{6}\right) + b_k \sin\left(\frac{\pi k(12-l)}{6}\right)\right) + a_6 \cos\left(\pi(12-l)\right) \\ &= a_0 + \sum_{k=1}^{5} \left(a_k \cos\left(2\pi k - \frac{\pi k l}{6}\right) + b_k \sin\left(2\pi k - \frac{\pi k l}{6}\right)\right) + a_6 \cos\left(12\pi - l\pi\right) \\ &= a_0 + \sum_{k=1}^{5} \left(a_k \cos\left(\frac{\pi k l}{6}\right) - b_k \sin\left(\frac{\pi k l}{6}\right)\right) + a_6 \cos\left(\pi l\right) \end{aligned}$$

If $X_l = X_{12-l}$, then holds $X_l - X_{12-l} = 0$, i.e.:

$$\begin{aligned} & a_0 + \sum_{k=1}^{5} \left(a_k \cos\left(\frac{\pi k l}{6}\right) + b_k \sin\left(\frac{\pi k l}{6}\right)\right) + a_6 \cos\left(\pi l\right) \\ - & \left(a_0 + \sum_{k=1}^{5} \left(a_k \cos\left(\frac{\pi k l}{6}\right) - b_k \sin\left(\frac{\pi k l}{6}\right)\right) + a_6 \cos\left(\pi l\right) \right) = 0. \end{aligned}$$

Simplified this results in (for all l):

$$\begin{aligned} & 2\sum_{k=1}^{5} b_k \sin\left(\frac{\pi k l}{6}\right) = 0 \\ \Rightarrow \quad & b_k = 0 \quad \text{for all } k. \end{aligned}$$

Remark: To be precise we would need to show that the solution $b_k = 0$ for all k is the only solution (uniqueness). We will skip this step here and note further that we would generally need to show the uniqueness of the trigonometric interpolation, since otherwise the Exercise would not have been properly stated at all. The uniqueness follows from the fact that the matrix is invertible (See the matrix from the lecture, which is, however, for the complex DFT).

Analog we get, if $X_l = -X_{12-l}$ which means $X_l + X_{12-l} = 0$:

$$2a_0 + 2\sum_{k=1}^{5} a_k \cos\left(\frac{\pi k l}{6}\right) + 2a_6 \cos\left(\pi l\right) = 0$$

$$\Rightarrow \quad a_k = 0 \quad \text{for all } k.$$