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Worksheet 8
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Algorithms for Scientific Computing (Algorithmen des Wissenschaftlichen Rechnens)

Haar Wavelets

The wavelet families we look at (e.g. Haar wavelets) are constructed around a *mulitresolution analysis*, a nested sequence V_n of function spaces some of which properties are

$$\begin{aligned}
V_j &\subset V_{j+1},\ j \in \mathbb{Z} && (1)\\
\bigcap_{j=-\infty}^{\infty} V_j &= \{0\} && (2)\\
f(t) \in V_l &\Leftrightarrow f(2^{-l}t) \in V_0 && (3)\\
V_l &= V_{l-1} \oplus W_{l-1} \\
&= V_{l-2} \oplus W_{l-2} \oplus W_{l-1} \\
&= V_0 \oplus W_0 \oplus W_1 \oplus \cdots \oplus W_{l-1}, && (4)
\end{aligned}$$

with *orthogonal* functions $f \in V_j$ and $g \in W_j$, i.e. $< f, g >= 0$.
The theory of multiresolution analysis further states the existence of a unique function ϕ which satisfies a so-called *dilation equation* of the form

$$\phi(t) = \sum_{k \in \mathbb{Z}} c_k \cdot \phi(2t - k) \tag{5}$$

for coefficients c_k with $c_k \neq 0$ for $k \in [0, N]$ and $c_k = 0$ for every $k \notin [0, N]$.

Define another function, known as the **mother wavelet** or the **wavelet function** of the form

$$\psi(t) := \sum_{k \in \mathbb{Z}} (-1)^k c_{1-k} \cdot \phi(2t - k). \tag{6}$$

In case N is odd, i.e. we have an even number of coefficients that are not zero, the c_{1-k} changes to c_{N-k}!

With the help of ϕ and ψ, we can define *orthonormal nodal bases* $\{ \phi_{l,k} \}$ for V_l with

$$\begin{aligned}
\phi_{l,k}(t) &= \phi(2^l\, t - k) \\
span\{ \phi_{l,k} \} &= V_l,\ < \phi_{l,k}, \phi_{l,m} >= \delta_{k,m}\ \ k, m \in \mathbb{Z}. && (7)
\end{aligned}$$

The function ϕ is called **father wavelet** or the **scaling function**, and together with a **mother wavelet** ψ, they define the wavelet family. It is not necessary to know a specific formula for ϕ, the dilation equation (5) with its coefficients c_k together with the theory of multiresolution analysis provide enough information to derive the mother wavelet ψ as well as *orthonormal wavelet bases* $\{\ \psi_{l,m}\ \}$ for the W_l with

$$\begin{aligned} \psi_{l,k}(t) &= \psi(2^l\, t-k) \\ span\{\ \psi_{l,k}\ \} &= W_l,\ <\psi_{l,k},\psi_{l,m}>=\delta_{k,m}\ \ k,m\in\mathbb{Z}. \end{aligned} \tag{8}$$

The following two exercises will be done by hand. Feel free to verify you results via a Python implementation.

Exercise 1: Discrete Wavelet Transform

Compute the DWT for the Haar wavelets for the signal $s=[8,4,-1,1,0,4,1,7,-\frac{5}{2},-\frac{3}{2},0,-4,-2,-2,1,-5]$ using the Pyramidal Algorithm. Discuss the computation complexity of this method.

Exercise 2: Discrete Wavelet Transform 2D

Compute the DWT for the Haar wavelets for the 2D signal $s=\begin{bmatrix} 4 & 2 & 3 & 5 \\ 1 & -7 & 0 & 8 \\ -1 & -3 & 9 & -3 \\ 6 & -2 & -1 & 1 \end{bmatrix}$.