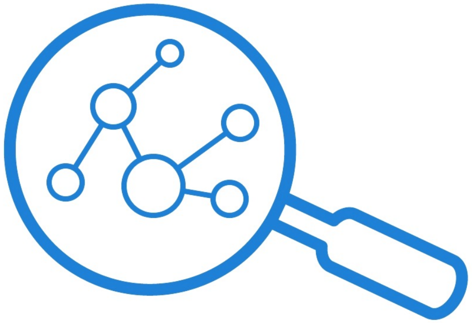
数据科学与大数据技术的数学基础

第十四讲



计算机学院
余皓然

2024/6/13

课程内容

Part1 随机化方法
一致性哈希 布隆过滤器 CM Sketch方法 最小哈希 欧氏距离下的相似搜索 Jaccard相似度下的相似搜索
Part2 谱分析方法
主成分分析 奇异值分解 谱图论
Part3 最优化方法
压缩感知

压缩感知

基于mutual coherence的
信号恢复L0最小化问题分析

压缩感知

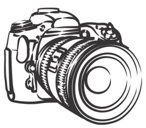
压缩感知中的信号恢复问题（Signal Recovery Problem in Compressive Sensing）

给定$p\times n$测量矩阵$\mathbf{C}$、$n\times n$矩阵$\mathbf{V}$、$p\times 1$测量向量$\boldsymbol{y}$，求解：

$$\min \|\boldsymbol{s}\|_0$$
$$\text{s.t. } \mathbf{CV}\boldsymbol{s}=\boldsymbol{y}.$$

$\|\boldsymbol{s}\|_0$为$n\times 1$稀疏向量$\boldsymbol{s}$的非零元素个数。

感知设备（如相机）
测量矩阵$\mathbf{C}$



原始信号 $\boldsymbol{x}=\mathbf{V}\boldsymbol{s}$ → 测量向量 $\boldsymbol{y}=\mathbf{C}\boldsymbol{x}=\mathbf{CV}\boldsymbol{s}$

信号感知

解决上述
信号恢复问题，得到$\tilde{\boldsymbol{s}}$

测量向量 → 恢复信号 $\tilde{\boldsymbol{x}}=\mathbf{V}\tilde{\boldsymbol{s}}$

信号恢复

（1）恢复信号$\tilde{\boldsymbol{x}}$能否有效地还原原始信号$\boldsymbol{x}$?

在$p\times n$测量矩阵$\mathbf{C}$满足什么条件时，$\tilde{\boldsymbol{x}}$可以有效地还原$\boldsymbol{x}$?

压缩感知

压缩感知中的信号恢复问题（Signal Recovery Problem in Compressive Sensing）

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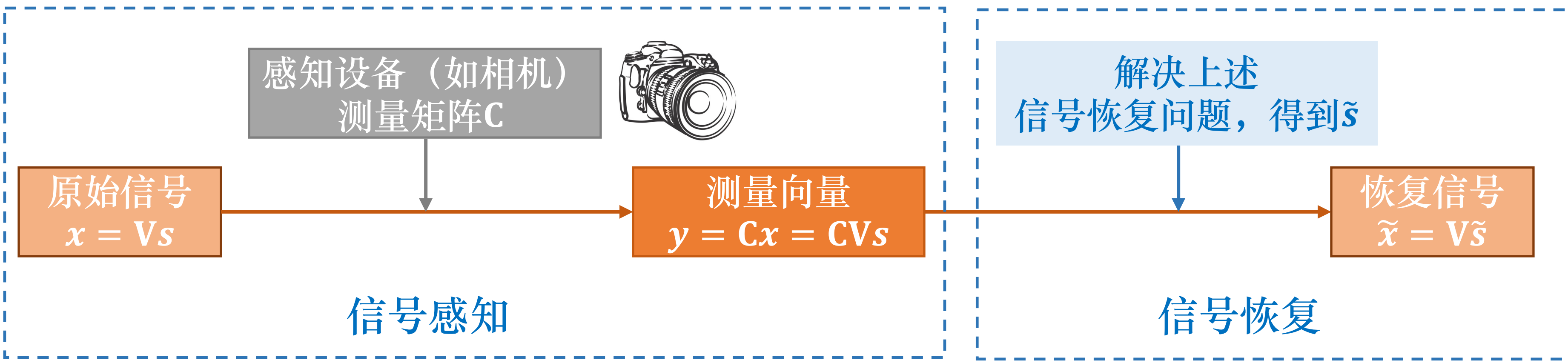

（1）恢复信号$\tilde{\boldsymbol{x}}$能否有效地还原原始信号$\boldsymbol{x}$?

在$p×n$测量矩阵C满足什么条件时，$\tilde{\boldsymbol{x}}$可以有效地还原$\boldsymbol{x}$?

在RIP之前，主要的分析工具是矩阵的mutual coherence

基本工具：spark

对$n\times 1$稀疏向量$\boldsymbol{s}$，现收到$p\times 1$测量向量$\boldsymbol{y}$。能否通过求如下问题恢复$\boldsymbol{s}$

$$\min \|\boldsymbol{s}\|_0$$
$$\text{s.t. } \mathbf{CV}\boldsymbol{s}=\boldsymbol{y}.$$

其中矩阵**CV**维数为$p\times n$。

Definition (Donoho and Elad [2003])

The *spark* of a given matrix $\mathbf{A}$ is the smallest number of columns from $\mathbf{A}$ that are linearly dependent, written as $\mathrm{spark}(\mathbf{A})$.

Consider the following matrix A.

$$A=\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & 0 & 2 \\ 1 & 2 & 0 & 3 \\ 1 & 0 & -3 & 4 \end{bmatrix}$$

The spark of this matrix equals 3 because:

- There is no set of 1 column of A which are linearly dependent.
- There is no set of 2 columns of A which are linearly dependent.
- But there is a set of 3 columns of A which are linearly dependent. The first three columns are linearly dependent because

$$\begin{pmatrix}1\\1\\1\\1\end{pmatrix}-\frac{1}{2}\begin{pmatrix}2\\2\\2\\0\end{pmatrix}+\frac{1}{3}\begin{pmatrix}0\\0\\0\\-3\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}.$$

内容取自Wotao Yin <Sparse Optimization Lecture: Sparse Recovery Guarantees>及Wikipedia <Spark>

基本工具：spark

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$$\min \|\boldsymbol{s}\|_0$$
$$\text{s.t. } \mathbf{CV}\boldsymbol{s} = \boldsymbol{y}.$$

其中矩阵**CV**维数为$p\times n$。

Theorem (Gorodnitsky and Rao [1997])

即$\mathbf{CV}\boldsymbol{s} = \boldsymbol{y}$

If $\boxed{\mathbf{A}\mathbf{x} = \mathbf{b}}$ *has a solution* $\mathbf{x}$ *obeying* $\|\mathbf{x}\|_0 < \operatorname{spark}(\mathbf{A})/2$, *then* $\mathbf{x}$ *is the sparsest solution.*

- **Proof idea**: if there is a solution $\mathbf{y}$ to $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{x} - \mathbf{y} \neq 0$, then $\mathbf{A}(\mathbf{x} - \mathbf{y}) = 0$ and thus

$$\|\mathbf{x}\|_0 + \|\mathbf{y}\|_0 \geq \boxed{\|\mathbf{x} - \mathbf{y}\|_0 \geq \operatorname{spark}(\mathbf{A})}$$

根据$\mathbf{A}(\boldsymbol{x}-\boldsymbol{y}) = \mathbf{0}$和spark的定义易得

or $\|\mathbf{y}\|_0 \geq \operatorname{spark}(\mathbf{A}) - \|\mathbf{x}\|_0 > \operatorname{spark}(\mathbf{A})/2 > \|\mathbf{x}\|_0$.

由条件中$\|\boldsymbol{x}\|_0$不等式可得

即在此条件下，一定可以恢复原信号

内容取自Wotao Yin <Sparse Optimization Lecture: Sparse Recovery Guarantees>

基本工具：spark

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For example, if matrix $\mathbf{A} \in \mathbb{R}^{m\times n} (m < n)$ has entries $A_{ij} \sim \mathcal{N}(0, 1)$, then $\mathrm{rank}(\mathbf{A}) = m = \mathrm{spark}(\mathbf{A}) - 1$ with probability 1.

基本工具：coherence

对$n\times 1$稀疏向量$\boldsymbol{s}$，现收到$p\times 1$测量向量$\boldsymbol{y}$。能否通过求如下问题恢复$\boldsymbol{s}$

$$\min \|\boldsymbol{s}\|_0$$
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其中矩阵CV维数为$p\times n$。

Definition (Mallat and Zhang [1993])

The (mutual) coherence of a given matrix $\mathbf{A}$ is the largest absolute normalized inner product between different columns from $\mathbf{A}$. Suppose $\mathbf{A} = [\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n]$. The mutual coherence of $\mathbf{A}$ is given by

$$\mu(\mathbf{A}) = \max_{k,j,k\neq j} \frac{|\mathbf{a}_k^\top \mathbf{a}_j|}{\|\mathbf{a}_k\|_2 \cdot \|\mathbf{a}_j\|_2}.$$

- It characterizes the dependence between columns of $\mathbf{A}$
- For unitary matrices, $\mu(\mathbf{A}) = 0$
- For matrices with more columns than rows, $\mu(\mathbf{A}) > 0$
- For recovery problems, we desire a small $\mu(\mathbf{A})$ as it is similar to unitary matrices.

内容取自Wotao Yin <Sparse Optimization Lecture: Sparse Recovery Guarantees>

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Theorem (Donoho and Elad [2003])

$$\operatorname{spark}(\mathbf{A}) \geq 1 + \mu^{-1}(\mathbf{A}).$$

证明略

Corollary

If $\mathbf{Ax} = \mathbf{b}$ has a solution $\mathbf{x}$ obeying $\|\mathbf{x}\|_0 < (1+\mu^{-1}(\mathbf{A}))/2$, then $\mathbf{x}$ is the unique sparsest solution.

说明CV各列越接近正交，越容易恢复信号

内容取自Wotao Yin <Sparse Optimization Lecture: Sparse Recovery Guarantees>

基本工具：coherence

对$n\times1$稀疏向量$\boldsymbol{s}$，现收到$p\times1$测量向量$\boldsymbol{y}$。能否通过求如下问题恢复$\boldsymbol{s}$

$$\min \|\boldsymbol{s}\|_0 \quad \text{s.t.}\ \mathbf{CV}\boldsymbol{s}=\boldsymbol{y}.$$

其中矩阵**CV**维数为$p\times n$。

Theorem (Gorodnitsky and Rao [1997])　基于spark的信号可恢复的充分条件

If $\mathbf{Ax}=\mathbf{b}$ *has a solution* $\mathbf{x}$ *obeying* $\|\mathbf{x}\|_0<\operatorname{spark}(\mathbf{A})/2$, *then* $\mathbf{x}$ *is the sparsest solution.*

Corollary　基于coherence的信号可恢复的充分条件

If $\mathbf{Ax}=\mathbf{b}$ *has a solution* $\mathbf{x}$ *obeying* $\|\mathbf{x}\|_0<(1+\mu^{-1}(\mathbf{A}))/2$, *then* $\mathbf{x}$ *is the unique sparsest solution.*

哪一个更实用?

内容取自Wotao Yin <Sparse Optimization Lecture: Sparse Recovery Guarantees>

基本工具：coherence

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Corollary 基于coherence的信号可恢复的充分条件

If $\mathbf{Ax} = \mathbf{b}$ *has a solution* $\mathbf{x}$ *obeying* $\|\mathbf{x}\|_0 < (1 + \mu^{-1}(\mathbf{A}))/2$*, then* $\mathbf{x}$ *is the unique sparsest solution.*

For $\mathbf{A} \in \mathbb{R}^{m\times n}$ where $m < n$, $(1 + \mu^{-1}(\mathbf{A}))$ is at most $1 + \sqrt{m}$ but spark can be $1 + m$. spark is more useful.

Assume $\mathbf{Ax} = \mathbf{b}$ has a solution with $\|\mathbf{x}\|_0 = k < \text{spark}(\mathbf{A})/2$. It will be the unique ℓ_0 minimizer. Will it be the ℓ_1 minimizer as well? Not necessarily.

However, $\|\mathbf{x}\|_0 < (1 + \mu^{-1}(\mathbf{A}))/2$ is a sufficient condition.

内容取自Wotao Yin <Sparse Optimization Lecture: Sparse Recovery Guarantees>

压缩感知

基于mutual coherence的信号恢复L1最小化问题分析

优化问题

问题
$$\min \|\boldsymbol{s}\|_0 \quad \text{s.t.}\ \mathbf{CV}\boldsymbol{s} = \boldsymbol{y}.$$
是非凸优化问题，是NP难问题

解决方案：用新目标函数近似原目标函数$\|\boldsymbol{s}\|_0$

改为求解
$$\min \|\boldsymbol{s}\|_1 \quad \text{s.t.}\ \mathbf{CV}\boldsymbol{s} = \boldsymbol{y}.$$

在什么条件下，L1最小化问题的解也是L0最小化问题的解?

基于coherence的分析

在什么条件下，求解L1问题可得L0问题最优解：

$$\min \|\boldsymbol{s}\|_1 \quad \text{s.t. } \mathbf{CV}\boldsymbol{s} = \boldsymbol{y}.$$

$$\min \|\boldsymbol{s}\|_0 \quad \text{s.t. } \mathbf{CV}\boldsymbol{s} = \boldsymbol{y}.$$

Theorem (Donoho and Elad [2003], Gribonval and Nielsen [2003])

If $\mathbf{A}$ *has normalized columns and* $\mathbf{Ax} = \mathbf{b}$ *has a solution* $\mathbf{x}$ *satisfying*

$$\|\mathbf{x}\|_0 < \frac{1}{2}\left(1 + \mu^{-1}(\mathbf{A})\right),$$

条件和L0问题分析中条件相同

then this $\mathbf{x}$ *is the unique minimizer with respect to both* ℓ_0 *and* ℓ_1.

基于coherence的分析

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then this $\mathbf{x}$ *is the unique minimizer with respect to both* ℓ_0 *and* ℓ_1.

Proof sketch:

$\boldsymbol{x}$非零元素位置的集合

- Previously we know $\mathbf{x}$ is the unique ℓ_0 minimizer; let $S := \mathrm{supp}(\mathbf{x})$
- Suppose $\mathbf{y}$ is the ℓ_1 minimizer but not $\mathbf{x}$; we study $\mathbf{e} := \mathbf{y} - \mathbf{x}$
- $\mathbf{e}$ must satisfy $\mathbf{Ae} = 0$ and $\|\mathbf{e}\|_1 \le 2\|\mathbf{e}_S\|_1$
- $\mathbf{A}^\top \mathbf{Ae} = 0 \Rightarrow |e_j| \le (1+\mu(\mathbf{A}))^{-1}\mu(\mathbf{A})\|\mathbf{e}\|_1, \forall j$
- the last two points together contradict the assumption

（1）先证明$\|\boldsymbol{e}_{S^c}\|_1 \le \|\boldsymbol{e}_S\|_1$
（2）再结合$\|\boldsymbol{e}\|_1 = \|\boldsymbol{e}_S\|_1 + \|\boldsymbol{e}_{S^c}\|_1$

Result bottom line: allow $\|\mathbf{x}\|_0$ up to $O(\sqrt{m})$ for exact recovery

内容取自Wotao Yin <Sparse Optimization Lecture: Sparse Recovery Guarantees>

基于coherence的分析

在什么条件下，求解L1问题可得L0问题最优解：

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（1）先证明$\|\boldsymbol{e}_{S^c}\|_1 \le \|\boldsymbol{e}_S\|_1$

Proof sketch:

一：利用$\|\boldsymbol{y}\|_1 \le \|\boldsymbol{x}\|_1$得$\|\boldsymbol{x} + \boldsymbol{e}_S + \boldsymbol{e}_{S^c}\|_1 \le \|\boldsymbol{x}\|_1$
二：根据S^c定义得$\|\boldsymbol{x} + \boldsymbol{e}_S + \boldsymbol{e}_{S^c}\|_1 = \|\boldsymbol{x}_S + \boldsymbol{e}_S\|_1 + \|\boldsymbol{e}_{S^c}\|_1$
三：由一和二得到$\|\boldsymbol{x}_S + \boldsymbol{e}_S\|_1 + \|\boldsymbol{e}_{S^c}\|_1 \le \|\boldsymbol{x}\|_1 = \|\boldsymbol{x}_S\|_1$
四：利用三角不等式$\|\boldsymbol{x}_S\|_1 - \|\boldsymbol{e}_S\|_1 \le \|\boldsymbol{x}_S + \boldsymbol{e}_S\|_1$化简上式

- Previously we know $\mathbf{x}$ is t
- Suppose $\mathbf{y}$ is the ℓ_1 minin
- $\mathbf{e}$ must satisfy $\mathbf{Ae} = 0$ and $\|\mathbf{e}\|_1 \le 2\|\mathbf{e}_S\|_1$
- $\mathbf{A}^\top \mathbf{Ae} = 0 \Rightarrow |e_j| \le (1 + \mu(\mathbf{A}))^{-1}\mu(\mathbf{A})\|\mathbf{e}\|_1, \forall j$
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Result bottom line: allow $\|\mathbf{x}\|_0$ up to $O(\sqrt{m})$ for exact recovery

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基于coherence的分析

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Lemma 5. *For any $\boldsymbol{\delta} \in \mathbb{R}^n$ and $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ such that $\boldsymbol{A}\boldsymbol{\delta} = \mathbf{0}$ and all columns of $\boldsymbol{A}$ are unit vectors, the bound*

$$|\delta_i| \le \frac{\mu}{\mu+1}\|\boldsymbol{\delta}\|_1,$$

where μ is the mutual coherence of $\boldsymbol{A}$, holds.

Proof: If $\boldsymbol{A}\boldsymbol{\delta} = \mathbf{0}$, then $\left(\boldsymbol{A}^\top\boldsymbol{A} - \boldsymbol{I}_n\right)\boldsymbol{\delta} = -\boldsymbol{\delta}$. Thus,

$$|\delta_i| \le \sum_{j=1}^n |\delta_j| \left|\left(\boldsymbol{A}^\top\boldsymbol{A} - \boldsymbol{I}_n\right)_{i,j}\right| \le \sum_{j=1}^n \mu|\delta_j| - \mu|\delta_i|.$$

Rearranging the terms yields the sought statement. ■

Theorem (Donoho and Elad [20

If $\mathbf{A}$ *has normalized columns* a

$\|\mathbf{x}\|$

then this $\mathbf{x}$ *is the unique minin*

对右侧左侧向量每个元素展开

Proof sketch:

- Previously we know $\mathbf{x}$ is the unique ℓ_0 minimizer; let $S := \mathrm{supp}(\mathbf{x})$
- Suppose $\mathbf{y}$ is the ℓ_1 minimizer but not $\mathbf{x}$; we study $\mathbf{e} := \mathbf{y} - \mathbf{x}$
- $\mathbf{e}$ must satisfy $\mathbf{A}\mathbf{e} = 0$ and $\|\mathbf{e}\|_1 \le 2\|\mathbf{e}_S\|_1$
- $\boxed{\mathbf{A}^\top\mathbf{A}\mathbf{e} = 0 \Rightarrow |e_j| \le (1+\mu(\mathbf{A}))^{-1}\mu(\mathbf{A})\|\mathbf{e}\|_1, \forall j}$
- the last two points together contradict the assumption

利用coherence定义及$\mathbf{A}^T\mathbf{A}$对角线元素为1的性质

Result bottom line: allow $\|\mathbf{x}\|_0$ up to $O(\sqrt{m})$ for exact recovery

内容取自Wotao Yin <Sparse Optimization Lecture: Sparse Recovery Guarantees>
引理证明取自Van Luong Le et al. <Selective l1 minimization for sparse recovery>

基于coherence的分析

在什么条件下，求解L1问题可得L0问题最优解：

$$\min \|\boldsymbol{s}\|_1 \quad \text{s.t. } \boldsymbol{CVs} = \boldsymbol{y}. \qquad\qquad \min \|\boldsymbol{s}\|_0 \quad \text{s.t. } \boldsymbol{CVs} = \boldsymbol{y}.$$

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- $\mathbf{A}^\top\mathbf{Ae} = 0 \Rightarrow |e_j| \le (1 + \mu(\mathbf{A}))^{-1}\mu(\mathbf{A})\|\mathbf{e}\|_1, \forall$
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代入$\|\boldsymbol{e}\|_1 \le 2\|\boldsymbol{e}_S\|_1 \le 2\|\boldsymbol{e}\|_\infty\|\boldsymbol{x}\|_0$
以及定理关于$\|\boldsymbol{x}\|_0$的条件和$\|\boldsymbol{e}\|_\infty \ne 0$
推得$|e_j| < \|\boldsymbol{e}\|_\infty, \forall j$，显然不成立

Result bottom line: allow $\|\mathbf{x}\|_0$ up to $O(\sqrt{m})$ for exact recovery

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基于coherence的分析

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allow $\|\mathbf{x}\|_0$ up to $O(\sqrt{m})$ for exact recovery　　m是**A**的行数

太小了，如果该界为紧的，那么为了无损压缩稀疏度为K的信号，就需要准备行数为K^2的矩阵**C**

内容取自Wotao Yin <Sparse Optimization Lecture: Sparse Recovery Guarantees>

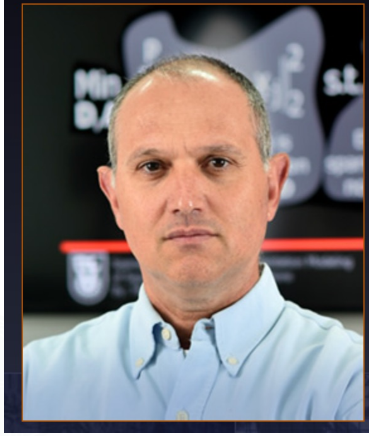
基于coherence的分析

David Donoho

Anne T. and Robert M. Bass Professor of
Humanities and Sciences
Professor of Statistics

David Donoho has studied the exploitation of sparse signals in signal recovery, including for denoising, superresolution, and solution of underdetermined equations. His research with collaborators showed that ell-1 penalization was an effective and even optimal way to exploit sparsity of the object to be recovered. He coined the notion of *compressed sensing* which has impacted many scientific and technical fields, including magnetic resonance imaging in medicine, where it has been implemented in FDA-approved medical imaging protocols and is already used in millions of actual patient MRIs.

In recent years David and his postdocs and students have been studying large-scale covariance matrix estimation, large-scale matrix denoising, detection of rare and weak signals among many pure noise non-signals, compressed sensing and related scientific imaging problems, and most recently, empirical deep learning.



{ Short Bio & Research Interests }

more info

Michael Elad holds a B.Sc. (1986), M.Sc. (1988) and D.Sc. (1997) in Electrical Engineering from the Technion, Israel Institute of Technology. After several years of industrial research, Michael served as a research associate at Stanford University during the years 2001-2003. Since 2003 he holds a permanent faculty position in the Computer-Science department at the Technion. Prof. Elad also holds a courtesy appointment in the Technion's Electrical & Computer Engineering (ECE) Departement.

Michael Elad works in the fields of signal and image processing and machine learning, specializing in particular on inverse problems, sparse representations, deep learning, and generative models. Prof. Elad has authored hundreds of technical publications in leading venues, many of which have led to exceptionally high impact.

resume

压缩感知

基于Restricted Isometry Property的信号恢复L0最小化问题分析

有限等距性质

对$n\times 1$稀疏向量$\boldsymbol{s}$，现收到$p\times 1$测量向量$\boldsymbol{y}$。能否通过求如下问题恢复$\boldsymbol{s}$

$$\min \|\boldsymbol{s}\|_0$$
$$\text{s.t. } \mathbf{CV}\boldsymbol{s} = \boldsymbol{y}.$$

其中矩阵**CV**维数为$p\times n$。

有限等距性质（Restricted Isometry Property (RIP)）

对于任意稀疏度为K的向量$\boldsymbol{s}$及给定$\delta_K > 0$，矩阵**CV**满足：

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对$n\times 1$稀疏向量$\boldsymbol{s}$，现收到$p\times 1$测量向量$\boldsymbol{y}$。能否通过求如下问题恢复$\boldsymbol{s}$

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Theorem 1 (ℓ_0-minimization). *If A is $(2k,\epsilon)$-RIP for any $\epsilon < 1$ and $\|x\|_0 = k$ then $z^\star = x$ is the unique minimizer of:*

$$\min \|z\|_0 \qquad \textit{subject to} \qquad Az = Ax.$$

内容取自Christopher Musco <Princeton COS 521 Advanced Algorithm Design: Compressed Sensing>

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Proof. Suppose there is some optimal solution $z\neq x$. So $\|x-z\|\neq0$. Since it is optimal for the ℓ_0 minimization problem, it must be that $\|z\|_0\le\|x\|_0=k$ So $x-z$ is at most $2k$-sparse. It follows from the $(2k,\epsilon)$-RIP property of A that:

$$(1-\epsilon)\|x-z\|_2^2\le\|Ax-Az\|_2^2.$$

However, if $Az=Ax$, $\|Ax-Az\|_2=0$, so we have a contradiction if $\epsilon<1$: the right hand side is positive but the left side is 0. So it must be that any optimum z is equal to x. □

内容取自Christopher Musco <Princeton COS 521 Advanced Algorithm Design: Compressed Sensing>

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条件是否容易达到?

行数为$O\left(\frac{2k\log\frac{n}{2k}}{\epsilon^2}\right)$的随机高斯矩阵以很大概率满足

与第十二讲结论一致

内容取自Christopher Musco <Princeton COS 521 Advanced Algorithm Design: Compressed Sensing>

压缩感知

基于Restricted Isometry Property的信号恢复L1最小化问题分析

有限等距性质

在什么条件下，求解L1问题可得L0问题最优解：

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Theorem 4. [Performance of BP via RIP, Candès, Tao, Romberg, 2006]
If $\delta_{2k} < \sqrt{2} - 1$, then for any vector $\boldsymbol{x}$, the solution to basis pursuit satisfies

L1最小化问题

$$\|\hat{\boldsymbol{x}} - \boldsymbol{x}\|_2 \leq C_0 k^{-1/2} \|\boldsymbol{x} - \boldsymbol{x}_k\|_1.$$

where $\boldsymbol{x}_k$ is the best k-term approximation of $\boldsymbol{x}$ for some constant C_0.

内容取自Yuejie Chi <ECE 8201: Low-dimensional Signal Models for High-dimensional Data Analysis>

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行数为$\Theta\left(k \log \frac{n}{k}\right)$的随机高斯矩阵以很大概率满足

相比于Coherence刻画的充分条件，对矩阵行数的要求从稀疏度的平方降了下来

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Proof of Theorem 4: Set $\hat{\boldsymbol{x}} = \boldsymbol{x} + \boldsymbol{h}$. We already show $\boldsymbol{A}\boldsymbol{h} = 0$. The goal is to establish that $\boldsymbol{h} = 0$ when $\boldsymbol{A}$ satisfies the desired RIP.

The first step is to decompose $\boldsymbol{h}$ into a sum of vectors $\boldsymbol{h}_{T_0}$, $\boldsymbol{h}_{T_1}$, $\boldsymbol{h}_{T_2}$, $\ldots$, each of sparsity at most k. Here, T_0 corresponds to the locations of the k largest coefficients of $\boldsymbol{x}$; T_1 to the locations of the k largest coefficients of $\boldsymbol{h}_{T_0^c}$, T_2 to the locations of the next k largest coefficients of $\boldsymbol{h}_{T_0^c}$, and so on.

The proof proceeds in two steps:

1. the first step shows that the size of $\boldsymbol{h}$ outside of $T_0 \cup T_1$ is essentially bounded by that of $\boldsymbol{h}$ on $T_0 \cup T_1$.

2. the second step shows that $\|\boldsymbol{h}_{T_0 \cup T_1}\|_2$ is appropriately small.

有限等距性质

在什么条件下，求解L1问题可得L0问题最优解：

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Step 1: Note that for each $j \geq 2$,

利用∞范数定义放缩　利用T_{j-1}定义放缩

$$\|\boldsymbol{h}_{T_j}\|_2 \leq \sqrt{k}\|\boldsymbol{h}_{T_j}\|_\infty \leq \frac{1}{\sqrt{k}}\|\boldsymbol{h}_{T_{j-1}}\|_1$$

therefore

对每个j分别用上式

$$\sum_{j\geq 2}\|\boldsymbol{h}_{T_j}\|_2 \leq \frac{1}{\sqrt{k}}\sum_{j\geq 1}\|\boldsymbol{h}_{T_j}\|_1 = \frac{1}{\sqrt{k}}\|\boldsymbol{h}_{T_0^c}\|_1.$$

This allows us to bound

三角不等式

$$\|\boldsymbol{h}_{(T_0\cup T_1)^c}\|_2 \overset{=}{\leq} \|\sum_{j\geq 2}\boldsymbol{h}_{T_j}\|_2 \leq \sum_{j\geq 2}\|\boldsymbol{h}_{T_j}\|_2 \leq \frac{1}{\sqrt{k}}\|\boldsymbol{h}_{T_0^c}\|_1. \qquad (*)$$

Given $\hat{\boldsymbol{x}} = \boldsymbol{x} + \boldsymbol{h}$ is the optimal solution, we have

$$\|\boldsymbol{x}\|_1 \geq \|\boldsymbol{x}+\boldsymbol{h}\|_1 = \sum_{i\in T_0}|x_i + h_i| + \sum_{i\in T_0^c}|x_i + h_i|$$

$$\geq \|\boldsymbol{x}_{T_0}\|_1 - \|\boldsymbol{h}_{T_0}\|_1 + \|\boldsymbol{h}_{T_0^c}\|_1 - \|\boldsymbol{x}_{T_0^c}\|_1,$$

有限等距性质

在什么条件下，求解L1问题可得L0问题最优解：

$$\min \|\boldsymbol{s}\|_1 \quad \text{s.t. } \mathbf{CV}\boldsymbol{s} = \boldsymbol{y}. \qquad\qquad \min \|\boldsymbol{s}\|_0 \quad \text{s.t. } \mathbf{CV}\boldsymbol{s} = \boldsymbol{y}.$$

which gives

$$\begin{aligned}\|\boldsymbol{h}_{T_0^c}\|_1 &\le \|\boldsymbol{h}_{T_0}\|_1 + \|\boldsymbol{x}\|_1 - \|\boldsymbol{x}_{T_0}\|_1 + \|\boldsymbol{x}_{T_0^c}\|_1 \\ &\le \|\boldsymbol{h}_{T_0}\|_1 + 2\|\boldsymbol{x}_{T_0^c}\|_1 := \|\boldsymbol{h}_{T_0}\|_1 + 2\|\boldsymbol{x} - \boldsymbol{x}_k\|_1.\end{aligned}$$

三角不等式

Combining with (*), we have

$$\|\boldsymbol{h}_{(T_0\cup T_1)^c}\|_2 \le \frac{1}{\sqrt{k}}\|\boldsymbol{h}_{T_0^c}\|_1 \le \frac{1}{\sqrt{k}}\|\boldsymbol{h}_{T_0}\|_1 + \frac{2}{\sqrt{k}}\|\boldsymbol{x} - \boldsymbol{x}_k\|_1.$$

Step 2: We next bound $\|\boldsymbol{h}_{T_0\cup T_1}\|_2$. Note that

$$0 = \boldsymbol{A}\boldsymbol{h} = \boldsymbol{A}\boldsymbol{h}_{T_0\cup T_1} + \sum_{j\ge 2} \boldsymbol{A}\boldsymbol{h}_{T_j},$$

we have by RIP

$$(1-\delta_{2k})\|\boldsymbol{h}_{T_0\cup T_1}\|_2^2 \le \|\boldsymbol{A}\boldsymbol{h}_{T_0\cup T_1}\|_2^2 = |\langle \boldsymbol{A}\boldsymbol{h}_{T_0\cup T_1}, \sum_{j\ge 2} \boldsymbol{A}\boldsymbol{h}_{T_j}\rangle|.$$

内容取自Yuejie Chi <ECE 8201: Low-dimensional Signal Models for High-dimensional Data Analysis>

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Using Proposition 2, we have for $j \geq 2$

$$\begin{aligned}|\langle \boldsymbol{A}\boldsymbol{h}_{T_0\cup T_1}, \boldsymbol{A}\boldsymbol{h}_{T_j}\rangle| &\leq |\langle \boldsymbol{A}\boldsymbol{h}_{T_0}, \boldsymbol{A}\boldsymbol{h}_{T_j}\rangle| + |\langle \boldsymbol{A}\boldsymbol{h}_{T_1}, \boldsymbol{A}\boldsymbol{h}_{T_j}\rangle| \\ &\leq \delta_{2k}(\|\boldsymbol{h}_{T_0}\|_2 + \|\boldsymbol{h}_{T_1}\|_2)\|\boldsymbol{h}_{T_j}\|_2 \\ &\leq \delta_{2k}\sqrt{2}\|\boldsymbol{h}_{T_0\cup T_1}\|_2\|\boldsymbol{h}_{T_j}\|_2,\end{aligned}$$

Proposition 2.

$$|\langle \boldsymbol{A}\boldsymbol{x}_1, \boldsymbol{A}\boldsymbol{x}_2\rangle| \leq \delta_{s_1+s_2}\|\boldsymbol{x}_1\|_2\|\boldsymbol{x}_2\|_2$$

for all $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ *that are supported on disjoint subsets* $T_1, T_2 \subset [n]$ *with* $|T_1| \leq s_1$ *and* $|T_2| \leq s_2$.

Proof: Without loss of generality assume $\|\boldsymbol{x}_1\|_2 = \|\boldsymbol{x}_2\|_2 = 1$. Applying the parallelogram identity, which says

$$\begin{aligned}|\langle \boldsymbol{A}\boldsymbol{x}_1, \boldsymbol{A}\boldsymbol{x}_2\rangle| &= \frac{1}{4}|\|\boldsymbol{A}\boldsymbol{x}_1 + \boldsymbol{A}\boldsymbol{x}_2\|_2^2 - \|\boldsymbol{A}\boldsymbol{x}_1 + \boldsymbol{A}\boldsymbol{x}_2\|_2^2| \\ &\leq \frac{1}{4}|2(1+\delta_{s_1+s_2}) - 2(1-\delta_{s_1+s_2})| \leq \delta_{s_1+s_2}.\end{aligned}$$

有限等距性质

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$$\begin{aligned}
|\langle \boldsymbol{A}\boldsymbol{h}_{T_0\cup T_1}, \boldsymbol{A}\boldsymbol{h}_{T_j}\rangle| &\leq |\langle \boldsymbol{A}\boldsymbol{h}_{T_0}, \boldsymbol{A}\boldsymbol{h}_{T_j}\rangle| + |\langle \boldsymbol{A}\boldsymbol{h}_{T_1}, \boldsymbol{A}\boldsymbol{h}_{T_j}\rangle| \\
&\leq \delta_{2k}(\|\boldsymbol{h}_{T_0}\|_2 + \|\boldsymbol{h}_{T_1}\|_2)\|\boldsymbol{h}_{T_j}\|_2 \\
&\leq \delta_{2k}\sqrt{2}\|\boldsymbol{h}_{T_0\cup T_1}\|_2\|\boldsymbol{h}_{T_j}\|_2,
\end{aligned}$$

which gives

$$\begin{aligned}
(1-\delta_{2k})\|\boldsymbol{h}_{T_0\cup T_1}\|_2^2 &\leq \sum_{j\geq 2} |\langle \boldsymbol{A}\boldsymbol{h}_{T_0\cup T_1}, \boldsymbol{A}\boldsymbol{h}_{T_j}\rangle| \\
&\leq \sqrt{2}\delta_{2k}\|\boldsymbol{h}_{T_0\cup T_1}\|_2 \sum_{j\geq 2}\|\boldsymbol{h}_{T_j}\|_2 \\
&\leq \sqrt{2}\delta_{2k}\|\boldsymbol{h}_{T_0\cup T_1}\|_2 \frac{1}{\sqrt{k}}\|\boldsymbol{h}_{T_0^c}\|_1,
\end{aligned}$$

therefore

$$\|\boldsymbol{h}_{T_0\cup T_1}\|_2 \leq \frac{\sqrt{2}\delta_{2k}}{(1-\delta_{2k})}\frac{1}{\sqrt{k}}\|\boldsymbol{h}_{T_0^c}\|_1 \leq \rho\frac{1}{\sqrt{k}}(\|\boldsymbol{h}_{T_0}\|_1 + 2\|\boldsymbol{x}-\boldsymbol{x}_k\|_1)$$

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where $\rho := \frac{\sqrt{2}\delta_{2k}}{(1-\delta_{2k})}$. Since $\|\boldsymbol{h}_{T_0}\|_1 \le \sqrt{k}\|\boldsymbol{h}_{T_0}\|_2 \le \sqrt{k}\|\boldsymbol{h}_{T_0\cup T_1}\|_2$, we can bound

$$\|\boldsymbol{h}_{T_0\cup T_1}\|_2 \le \frac{2\rho}{1-\rho}\frac{\|\boldsymbol{x}-\boldsymbol{x}_k\|_1}{\sqrt{k}}.$$

Finally,

$$\begin{aligned}
\|\hat{\boldsymbol{x}}-\boldsymbol{x}\|_2 = \|\boldsymbol{h}\|_2 &\le \|\boldsymbol{h}_{T_0\cup T_1}\|_2 + \|\boldsymbol{h}_{(T_0\cup T_1)^c}\|_2 \\
&\le \|\boldsymbol{h}_{T_0\cup T_1}\|_2 + \frac{1}{\sqrt{k}}\|\boldsymbol{h}_{T_0}\|_1 + \frac{2}{\sqrt{k}}\|\boldsymbol{x}-\boldsymbol{x}_k\|_1 \\
&\le 2\|\boldsymbol{h}_{T_0\cup T_1}\|_2 + \frac{2}{\sqrt{k}}\|\boldsymbol{x}-\boldsymbol{x}_k\|_1 \\
&\le \frac{2(1+\rho)}{1-\rho}\frac{\|\boldsymbol{x}-\boldsymbol{x}_k\|_1}{\sqrt{k}}.
\end{aligned}$$

Therefore, $C_0 := \frac{2(1+\rho)}{1-\rho}$. The requirement on δ_{2k} comes from the fact that we need $1-\rho > 0$ to avoid the bound to blow up.

内容取自Yuejie Chi <ECE 8201: Low-dimensional Signal Models for High-dimensional Data Analysis>

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- (Foucart-Lai) If $\delta_{2k+2} < 1$, then $\exists$ a sufficiently small p so that ℓ_p minimization is guaranteed to recovery any k-sparse x
- (Candes) $\delta_{2k} < \sqrt{2} - 1$ is sufficient
- (Foucart-Lai) $\delta_{2k} < 2(3 - \sqrt{2})/7 \approx 0.4531$ is sufficient
- RIP gives $\kappa(\mathbf{A}_S) \le \sqrt{(1+\delta_k)/(1-\delta_k)}$, $\forall |S| \le k$; so $\delta_{2k} < 2(3-\sqrt{2})/7$ gives $\kappa(\mathbf{A}_S) \le 1.7$, $\forall |S| \le 2m$, very well-conditioned.
- (Mo-Li) $\delta_{2k} < 0.493$ is sufficient
- (Cai-Wang-Xu) $\delta_k < 0.307$ is sufficient
- (Cai-Zhang) $\delta_k < 1/3$ is sufficient and necessary for universal ℓ_1 recovery

内容取自Wotao Yin <Sparse Optimization Lecture: Sparse Recovery Guarantees>

本讲小结

- 信号恢复L0最小化问题的理论分析
- 信号恢复L1最小化问题的理论分析

主要参考资料

清华大学张颢 <现代数字信号处理2> 课堂视频

Wotao Yin <Sparse Optimization Lecture: Sparse Recovery Guarantees> Slides

Wikipedia <Spark> Webpage

Van Luong Le et al. <Selective l1 minimization for sparse recovery> Paper

Christopher Musco <Princeton COS 521 Advanced Algorithm Design: Compressed Sensing> Lecture Notes

Yuejie Chi <ECE 8201: Low-dimensional Signal Models for High-dimensional Data Analysis> Slides

谢谢！