Dynamics of two populations of phase oscillators with different frequency distributions

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A large variety of rhythms are observed in nature. Rhythms such as electroencephalogram signals in the brain can often be regarded as interacting. In this study, we investigate the dynamical properties of rhythmic systems in two populations of phase oscillators with different frequency distributions. We assume that the average frequency ratio between two populations closely approximates some small integer. Most importantly, we adopt a specific coupling function derived from phase reduction theory. Under some additional assumptions, the system of two populations of coupled phase oscillators reduces to a low-dimensional system in the continuum limit. Consequently, we find chimera states in which clustering and incoherent states coexist. Finally, we confirm consistent behaviors of the derived low-dimensional model and the original model.

I. INTRODUCTION

Rhythm plays a crucial role in many aspects of physics, biology, and chemistry [1,2]. To study the rhythmic phenomena quantitatively, the phase oscillator model has been widely used since the prominent studies had been done [3–5]. The Kuramoto model undergoes a transition from nonsynchronous to synchronous as the coupling strength increases. This model can also be solved exactly with the Lorentzian natural frequency distribution [6]. Although this discovery has triggered many subsequent works, many unsolved, related models are considered important in real-world situations [7]. One such important model is a system composed of multiple heterogeneous populations of phase oscillators. In fact, some real systems appear to have a hierarchical structure of multiple populations. For example, multiple neuronal modules in the brain seem to organize into structural networks [8,9]. To understand the dynamical properties of such systems, we need to theoretically investigate a model with multiple populations of oscillators. As a first step, we here investigate a two-population system of phase oscillators.

It is plausible that the characteristics of two populations are generally different and in several situations this property seems to play an important functional role. For example, the synchrony of different neuronal populations in the brain is positively correlated with the success of human tasking [9]. In real systems such as electroencephalograms, the average frequencies often largely differ across populations. Considering such a coupled system of two populations with different average frequency distributions, we can theoretically derive the correct form of the coupling function by the averaging method [10,11] in the phase reduction method. This derivation is essential if average frequencies between two populations are related by an integer ratio such as $k : 1$ (where k is an integer). This condition is called the resonant condition [12,13]. However, resonance has not been considered in most previous studies. Therefore, we will examine multi-frequency oscillator systems by applying a specific coupling function.

Moreover, multiple populations of phase oscillators exhibit interesting properties [13–18]. One of the most remarkable phenomena is the formation of chimera states in which synchronous and asynchronous states coexist [16,19–25]. This phenomenon was theoretically discovered by Kuramoto *et al.* [26], the solvable model was found later by Abrams *et al.* [27]. Later, the properties of chimera states were experimentally investigated [28–30]. Recent theoretical works have explored chimera states in more general situations [31–35]. However, in most studies on chimera states in multiple phase oscillator populations, the natural frequencies of the oscillators are assumed to be evenly distributed across the populations. As the next stage, we should theoretically examine heterogeneous frequency distributions across the populations. To this end, we treat a general situation in which the oscillator populations have different average frequencies. We focus on the resonant case with an integer frequency ratio. Here, we must derive the appropriate type of phase coupling function under the corresponding resonance condition. We show that such a system develops chimera states under some conditions and investigate the properties of these states.

Recently, Ott and Antonsen proposed a remarkable ansatz that reduces an infinite system of coupled phase oscillators in the continuum limit to a low-dimensional system [36–38]. This ansatz has been applied to a wide range of applications and has yielded many fruitful results [39–45]. Komarov and Pikovsky considered the resonant interactions among more than two oscillator communities and applied the Ott-Antonsen ansatz to a simple resonant case [13]. In this study, we consider two populations of phase oscillators in the more general resonant case $k : 1$, where k is not ± 1. However, because the populations interact through the resonant-type coupling function, we cannot straightforwardly reduce the system to low-dimensional equations using the Ott-Antonsen ansatz. To proceed with the analysis, we augment the Ott-Antonsen ansatz with the additional assumptions of Skardal *et al.* [46]. Consequently, our system reduces to a three-dimensional system of ordinary differential equations.

The remainder of this paper is structured as follows. Section II describes a coupled system of two populations of phase oscillators. In Sec. III, we reduce this system to a low-dimensional system. The emergent dynamical states such as the clustered chimera states are investigated in Sec. IV. In

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Sec. V, we numerically confirm that our results hold without the additional solvability assumption. Our method is extended to more general resonant conditions in Sec. VI. The paper concludes with a summary in Sec. VII.

II. PHASE REDUCTION AND MODEL EQUATIONS

As a preliminary, we discuss two interacting oscillators. The oscillators evolve by the following equations:

$$\frac{d\boldsymbol{x}_1}{dt} = \boldsymbol{f}_1(\boldsymbol{x}_1) + \epsilon \boldsymbol{g}_1(\boldsymbol{x}_1,\boldsymbol{x}_2), \tag{1}$$

$$\frac{d\boldsymbol{x}_2}{dt} = \boldsymbol{f}_2(\boldsymbol{x}_2) + \epsilon \boldsymbol{g}_2(\boldsymbol{x}_2,\boldsymbol{x}_1), \tag{2}$$

where $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are n-dimensional state vectors, $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ represent the intrinsic dynamics of the oscillators, and $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ are the interaction terms between the oscillators. We further suppose $|\epsilon| \ll 1$, and that the oscillators have nonperturbed limit cycles (when $\epsilon = 0$). The periods of oscillators 1 and 2 are $2\pi/\omega_1$ and $2\pi/\omega_2$, respectively, where ω_1 and ω_2 are the respective natural frequencies of the oscillators. The frequencies almost satisfy the resonant relation $k : 1$, where the natural frequency of the fast oscillator ω_1 is approximately k times the frequency of the slow oscillator ω_2:

$$\omega_1 \simeq k\omega_2. \tag{3}$$

If the frequencies satisfy Eq. (3), the resonant coupling function can be derived by phase reduction. For simplicity, ω_2 is denoted as ω in the following.

We first introduce the original phase variables θ_1 and θ_2 such that $d\theta_1/dt = k\omega$, $d\theta_2/dt = \omega$ near the limit cycle orbits $\boldsymbol{x}_{1,0}(t)$ and $\boldsymbol{x}_{2,0}(t)$ in the absence of the perturbations. Applying phase reduction, the dynamics of the phase variables θ_1 and θ_2 are determined from Eqs. (1) and (2) as

$$\frac{d\theta_1}{dt} = k\omega + \epsilon \boldsymbol{Z}_1(\theta_1) \cdot \boldsymbol{g}_{12}(\theta_1,\theta_2), \tag{4}$$

$$\frac{d\theta_2}{dt} = \omega + \epsilon \boldsymbol{Z}_2(\theta_2) \cdot \boldsymbol{g}_{21}(\theta_2,\theta_1), \tag{5}$$

where

$$\boldsymbol{Z}_1(\theta_1) = \nabla_{\boldsymbol{x}_1}\theta_1(\boldsymbol{x}_1)|_{\boldsymbol{x}_1=\boldsymbol{x}_{1,0}(\theta_1)},$$

$$\boldsymbol{Z}_2(\theta_2) = \nabla_{\boldsymbol{x}_2}\theta_2(\boldsymbol{x}_2)|_{\boldsymbol{x}_2=\boldsymbol{x}_{2,0}(\theta_1)}.$$

To separate the slow dynamics from Eqs. (4) and (5), we define slow phase variables ψ_1 and ψ_2 as $\theta_1 = k\omega t + \psi_1$ and $\theta_2 = \omega t + \psi_2$, respectively. The dynamics of the slow phase variables are described by

$$\frac{d\psi_1}{dt} = \epsilon \boldsymbol{Z}_1(\psi_1 + k\omega t) \cdot \boldsymbol{g}_{12}(\psi_1 + k\omega t, \psi_2 + \omega t), \tag{6}$$

$$\frac{d\psi_2}{dt} = \epsilon \boldsymbol{Z}_2(\psi_2 + \omega t) \cdot \boldsymbol{g}_{21}(\psi_2 + \omega t, \psi_1 + k\omega t). \tag{7}$$

Averaging the right-hand side in Eq. (6) over the period of the slow oscillator $2\pi/\omega$, we obtain

$$\frac{d\psi_1}{dt} = \epsilon \frac{\omega}{2\pi} \int_0^{2\pi/\omega} dt \, \boldsymbol{Z}_1(\psi_1 + k\omega t)$$

$$\cdot \boldsymbol{g}_{12}(\psi_1 + k\omega t, \psi_2 + \omega t)$$

$$= \frac{\epsilon}{2\pi} \int_0^{2\pi} d\Theta \, \boldsymbol{Z}_1(\psi_1 + k\Theta) \cdot \boldsymbol{g}_{12}(\psi_1 + k\Theta, \psi_2 + \Theta)$$

$$= \frac{\epsilon}{2\pi} \int_0^{2\pi} d\Theta \, \boldsymbol{Z}_1(\psi_1 - k\psi_2 + k\Theta)$$

$$\cdot \boldsymbol{g}_{12}(\psi_1 - k\psi_2 + k\Theta, \Theta), \tag{8}$$

where ψ_1 and ψ_2 are constants independent of t during one period. When evaluating the integral, we express the right-hand side of Eq. (8) as a function of $\psi_1 - k\psi_2$; namely $\Gamma_{12}(\psi_1 - k\psi_2)$. Performing similar operations the right-hand side of Eq. (7), we obtain the pair of averaged equations:

$$\frac{d\psi_1}{dt} = \epsilon \Gamma_{12}(\psi_1 - k\psi_2), \tag{9}$$

$$\frac{d\psi_2}{dt} = \epsilon \Gamma_{21}(k\psi_2 - \psi_1), \tag{10}$$

where

$$\Gamma_{12}(\psi_1 - k\psi_2) = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} dt \, \boldsymbol{Z}_1(\psi_1 + k\omega t)$$

$$\cdot \boldsymbol{g}_{12}(\psi_1 + k\omega t, \psi_2 + \omega t),$$

$$\Gamma_{21}(k\psi_2 - \psi_1) = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} dt \, \boldsymbol{Z}_2(\psi_2 + \omega t)$$

$$\cdot \boldsymbol{g}_{21}(\psi_2 + \omega t, \psi_1 + k\omega t).$$

This approximation is valid to order ϵ. Consequently, we obtain the evolutionary equations of the averaged phase variables θ_1 and θ_2:

$$\frac{d\theta_1}{dt} = k\omega + \epsilon \Gamma_{12}(\theta_1 - k\theta_2), \tag{11}$$

$$\frac{d\theta_2}{dt} = \omega + \epsilon \Gamma_{21}(k\theta_2 - \theta_1). \tag{12}$$

We emphasize that the coupling functions Γ_i ($i=1,2$) depend on $\theta_1 - k\theta_2$ (or $k\theta_2 - \theta_1$) when the natural frequencies of the oscillators satisfy the resonant relation $k : 1$. Generalizing this to the $m : n$ case, we can state that Γ_i ($i = 1, 2$) depends on $n\theta_1 - m\theta_2$ (or $m\theta_2 - n\theta_1$). In the case of the irrational ratio, the interaction functions become constants independent of the phase variables after the averaging [11] and we do not consider this trivial case in the present paper.

Our discussion of resonant interactions is now extended to populations of phase oscillators. We consider two populations of phase oscillators with different average frequencies. We assume that both populations have inherent Lorentzian (Cauchy) distributions of natural frequencies. The mean frequency ratio between the two populations is 2 : 1 (i.e., $k = 2$ in Fig. 1):

$$g_{\text{fast}}(\omega) = \frac{D}{\pi} \frac{1}{(\omega - 2\Omega)^2 + D^2}, \tag{13}$$

$$g_{\text{slow}}(\omega) = \frac{D}{\pi} \frac{1}{(\omega - \Omega)^2 + D^2}, \tag{14}$$

where D is the common width of the distributions and Ω is the mean of the distribution in the slow population. As usual, we assume dominance of the first term in the Fourier series of the coupling function and take $H(\theta) = \sin(\theta + \alpha)$ as in the Sakaguchi-Kuramoto model [47], where α is the phase lag
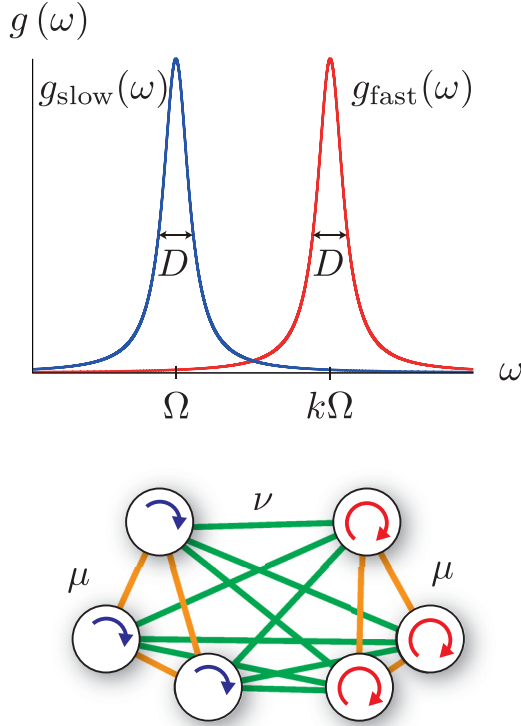
FIG. 1. Top: Natural frequency distributions of the phase oscillators in the two populations whose mean frequencies satisfy the resonant relation $k : 1$. Bottom: The coupling strengths is $\mu = (1 + A)/2$ within the same population and $\nu = (1 - A)/2$ between the two populations.

parameter. This parameter can be estimated directly if Eqs. (1) and (2) are explicitly given, but in this work we will consider general situations and regard α as the model parameter. We thus consider the following model:

$$
\frac{d\theta_i^{\text{fast}}}{dt} = \omega_i^{\text{fast}} + \frac{\mu}{N} \sum_{j=1}^{N} \sin\left(\theta_j^{\text{fast}} - \theta_i^{\text{fast}} - \alpha\right)
$$

$$
+ \frac{\nu}{N} \sum_{j=1}^{N} \sin\left(2\theta_j^{\text{slow}} - \theta_i^{\text{fast}} - \alpha\right), \quad (15)
$$

$$
\frac{d\theta_i^{\text{slow}}}{dt} = \omega_i^{\text{slow}} + \frac{\mu}{N} \sum_{j=1}^{N} \sin\left(\theta_j^{\text{slow}} - \theta_i^{\text{slow}} - \alpha\right)
$$

$$
+ \frac{\nu}{N} \sum_{j=1}^{N} \sin\left(\theta_j^{\text{fast}} - 2\theta_i^{\text{slow}} - \alpha\right), \quad (16)
$$

where $\theta_i^{\text{fast/slow}}$ is the phase of oscillator i ($i = 1, \ldots, N$) in the fast/slow population. Incorporating the phase reduction concept, we now consider the interactions between the oscillators. For this purpose, we introduce the parameter A and define $\mu = (1 + A)/2, \nu = (1 - A)/2$ as in Ref. [16]. In this setting, $\mu + \nu = 1$. The parameter A controls the ratio of the strengths of the interactions within each population and between the populations. When $0 < A < 1$, the interactions are stronger within the populations than across the populations; conversely,

when $-1 < A < 0$ the interactions across the populations dominate. In this study we fix the phase lag parameter $\alpha = \pi/2 - 0.05$ as in Ref. [21]. Under this condition, chimera states will appear, and a variety of dynamics with a reasonably general coverage are expected [16]. The general k case will be straightforwardly extended from the $k = 2$ case in Sec. VI.

III. REDUCTION TO LOW-DIMENSIONAL DYNAMICS

Using the Ott-Antonsen ansatz [36–38], we will attempt to reduce the system represented by Eqs. (13)–(16) to a low-dimensional system in the limit $N \to \infty$. However, our model includes the specific interactions such as $\sin\left(2\theta_j^{\text{slow}} - \theta_i^{\text{fast}} - \alpha\right)$ in Eqs. (15) and (16) so that we cannot use the Ott-Antonsen ansatz straightforwardly as the original case in Ref. [36]. To handle the high-order Kuramoto model, we use a modified version of the Ott-Antonsen ansatz used by Skardal *et al.* [46]. However, the modified ansatz alone does not reduce the original dynamics to a low-dimensional system. To overcome this difficulty, we employ one additional assumption in which we replace $\sin\left(\theta_j^{\text{slow}} - \theta_i^{\text{slow}} - \alpha\right)$ with $\sin\left(2\theta_j^{\text{slow}} - 2\theta_i^{\text{slow}} - \alpha\right)$ in Eq. (16). Equation (16) becomes

$$
\frac{d\theta_i^{\text{slow}}}{dt} = \omega_i^{\text{slow}} + \frac{\mu}{N} \sum_{j=1}^{N} \sin\left(2\theta_j^{\text{slow}} - 2\theta_i^{\text{slow}} - \alpha\right)
$$

$$
+ \frac{\nu}{N} \sum_{j=1}^{N} \sin\left(\theta_j^{\text{fast}} - 2\theta_i^{\text{slow}} - \alpha\right). \quad (17)
$$

The dynamics can be reduced by applying the Ott-Antonsen ansatz to Eq. (15) and the modified model Eq. (17). In general, however, it is unlikely that the interaction terms among the slow oscillators contain no first Fourier mode. In Sec. V, we will check the validity of the modified model under general conditions.

We now consider the continuum limit $N \to \infty$ in our modified model. The probability density functions (PDFs) of the fast and slow populations in the continuum limit are denoted as $f_{\text{fast}}(\theta, \omega, t)$ and $f_{\text{slow}}(\theta, \omega, t)$, respectively, where $f_j(\theta, \omega, t)d\theta d\omega$ is the fraction of oscillators with phase between θ and $\theta + d\theta$ and natural frequency between ω and $\omega + d\omega$ at time t in population $j(= \text{fast}, \text{slow})$. The common order parameter for the fast population is given by

$$
z_{\text{fast}}(t) = \lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j^{\text{fast}}}
$$

$$
= \int_{-\infty}^{\infty} d\omega \int_0^{2\pi} d\theta f_{\text{fast}}(\theta, \omega, t) e^{i\theta}. \quad (18)
$$

For the slow population, we define the Daido order parameter [48] as

$$
z_{\text{slow}}(t) = \lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} e^{2i\theta_j^{\text{slow}}}
$$

$$
= \int_{-\infty}^{\infty} d\omega \int_0^{2\pi} d\theta f_{\text{slow}}(\theta, \omega, t) e^{2i\theta}. \quad (19)
$$

Following the Ott-Antonsen ansatz [36–38] and its variant [46], we expand the PDFs as two Fourier series:

$$f_{\text{fast}}(\theta,\omega,t) = \frac{g_{\text{fast}}(\omega)}{2\pi}\left[1 + \sum_{n=1}^{\infty}(a(\omega,t)^n e^{in\theta} + \text{c.c.})\right], \quad (20)$$

$$f_{\text{slow}}(\theta,\omega,t) = \frac{g_{\text{slow}}(\omega)}{2\pi}\left[1 + \sum_{m=1}^{\infty}(b(\omega,t)^m e^{2im\theta} + \text{c.c.})\right],, \quad (21)$$

where c.c. stands for complex conjugate. The ansatz requires the conditions $|a(\omega,t)| < 1$ and $|b(\omega,t)| < 1$ for the convergence of Eqs. (20) and (21). To conserve the total number of oscillators in each population, the following continuity equations should be satisfied:

$$\frac{\partial f_j}{\partial t} + \frac{\partial}{\partial \theta_j}(f_j \dot{\theta}_j) = 0 \quad (j = \text{fast, slow}),$$

Substituting Eqs. (20) and (21) into these continuity equations, we obtain the ordinal differential equations for a and b:

$$\frac{\partial a}{\partial t} + i\omega a + \frac{\mu}{2}(z_{\text{fast}}a^2 e^{-i\alpha} - \bar{z}_{\text{fast}}e^{i\alpha})$$
$$+ \frac{\nu}{2}(z_{\text{slow}}a^2 e^{-i\alpha} - \bar{z}_{\text{slow}}e^{i\alpha}) = 0, \quad (22)$$

$$\frac{1}{2}\frac{\partial b}{\partial t} + i\omega b + \frac{\mu}{2}(z_{\text{slow}}b^2 e^{-i\alpha} - \bar{z}_{\text{slow}}e^{i\alpha})$$
$$+ \frac{\nu}{2}(z_{\text{fast}}b^2 e^{-i\alpha} - \bar{z}_{\text{fast}}e^{i\alpha}) = 0. \quad (23)$$

From Eqs. (18) and (20), we immediately find that

$$z_{\text{fast}}(t) = \int_{-\infty}^{\infty}d\omega g_{\text{fast}}(\omega)\bar{a}(\omega,t). \quad (24)$$

Given that the natural frequency distribution is Lorentzian Eq. (13), we can write

$$z_{\text{fast}}(t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}d\omega\left(\frac{1}{\omega - 2\Omega - iD} - \frac{1}{\omega - 2\Omega + iD}\right)$$
$$\times \bar{a}(\omega,t). \quad (25)$$

Following Ref. [36] we assume that $\bar{a}(\omega,t)$ is analytic in $\text{Im}\,\omega > 0$. From the complex conjugate of Eq. (22), we know that $\partial \bar{a}/\partial t \sim -(\text{Im}\,\omega)\bar{a}$ as $\text{Im}\,\omega \to \infty$. As $|\bar{a}(\omega,t)| < 1$, we also know that $\bar{a}(\omega,t) \to 0$ as $\text{Im}\,\omega \to \infty$. Integrating the right-hand side of Eq. (25) over the upper semicircular contour in the complex plain, we obtain

$$z_{\text{fast}}(t) = \bar{a}(2\Omega + iD,t).$$

A similar analysis gives

$$z_{\text{slow}}(t) = \bar{b}(\Omega + iD,t).$$

Substituting $\omega = 2\Omega + iD$ and $\omega = \Omega + iD$ in the complex conjugate of Eqs. (22) and (23), respectively, we obtain the dynamics of the complex order parameters of the two populations:

$$\frac{dz_{\text{fast}}}{dt} = (-D + 2\Omega i)z_{\text{fast}} + \frac{e^{-i\alpha}}{2}(\mu z_{\text{fast}} + \nu z_{\text{slow}})$$
$$- \frac{e^{i\alpha}}{2}(\mu \bar{z}_{\text{fast}} + \nu \bar{z}_{\text{slow}})z_{\text{fast}}^2, \quad (26)$$

$$\frac{dz_{\text{slow}}}{dt} = (-2D + 2\Omega i)z_{\text{slow}} + e^{-i\alpha}(\mu z_{\text{slow}} + \nu z_{\text{fast}})$$
$$- e^{i\alpha}(\mu \bar{z}_{\text{slow}} + \nu \bar{z}_{\text{fast}})z_{\text{slow}}^2. \quad (27)$$

We now rewrite Eqs. (26) and (27) in terms of the polar coordinates $z_{\text{fast}} = r_{\text{fast}}e^{-i\phi_{\text{fast}}}$ and $z_{\text{slow}} = r_{\text{slow}}e^{-i\phi_{\text{slow}}}$ and their phase difference $\phi = \phi_{\text{fast}} - \phi_{\text{slow}}$. The resulting system comprises three ordinary differential equations with three degrees of freedom $(r_{\text{fast}}, r_{\text{slow}}, \phi)$:

$$\frac{dr_{\text{fast}}}{dt} = -Dr_{\text{fast}} + \frac{1 - r_{\text{fast}}^2}{2}[\mu r_{\text{fast}}\cos\alpha$$
$$+ \nu r_{\text{slow}}\cos(\phi - \alpha)], \quad (28)$$

$$\frac{dr_{\text{slow}}}{dt} = -2Dr_{\text{slow}} + \frac{1 - r_{\text{slow}}^2}{2}[\mu r_{\text{slow}}\cos\alpha$$
$$+ \nu r_{\text{fast}}\cos(\phi + \alpha)], \quad (29)$$

$$\frac{d\phi}{dt} = \frac{1 + r_{\text{fast}}^2}{2}\left[\mu\sin\alpha - \nu\frac{r_{\text{slow}}}{r_{\text{fast}}}\sin(\phi - \alpha)\right]$$
$$- (1 + r_{\text{slow}}^2)\left[\mu\sin\alpha + \nu\frac{r_{\text{fast}}}{r_{\text{slow}}}\sin(\phi + \alpha)\right]. \quad (30)$$

From these equations, we found that the reduced system evolves independently of Ω, the mean of natural frequency distribution of the oscillators in the slow population. Later, we will show that this independency holds in general k cases.

IV. RESULTS: CLUSTERED CHIMERA STATE

In the previous section, we showed that if we make a slight modification to Eq. (16) and by replacing $\sin(\theta_j^{\text{slow}} - \theta_i^{\text{slow}} - \alpha)$ with $\sin(2\theta_j^{\text{slow}} - 2\theta_i^{\text{slow}} - \alpha)$, this modified system with Eq. (15) in the continuum limit can be simplified to the reduced system Eqs. (28)–(30) by applying the Ott-Antonsen ansatz. In this section, we numerically simulate the detailed dynamics of the modified and reduced systems. We set $D = 1.0 \times 10^{-3}$ and imposed the initial condition $r_{\text{fast}}, r_{\text{slow}} \simeq 1$ (Each θ_i was chosen from a uniform distribution in $[0, \pi/30]$), and each population comprised $N = 10^4$ oscillators. The initial order parameters in the reduced systems were set by substituting the conditions of the corresponding $N = 10^4$ modified systems.

Figure 2 plots the time evolutions of the order parameters of the reduced system for (a) $A = 0.9$, (b) $A = 0.1$, (c) $A = -0.1$, and (d) $A = -0.9$. In all cases, we can see the order parameters of the populations reach a steady state. Figures 2(a)–2(c) reveal three different types of dynamics; coherent, breathing chimera, and stable chimera. In the coherent state [Fig. 2(a)], the order parameters of both populations approach almost 1, indicating that the fast and slow oscillators form synchronous clusters. In the chimera state, the order parameter of one population becomes partially coherent while
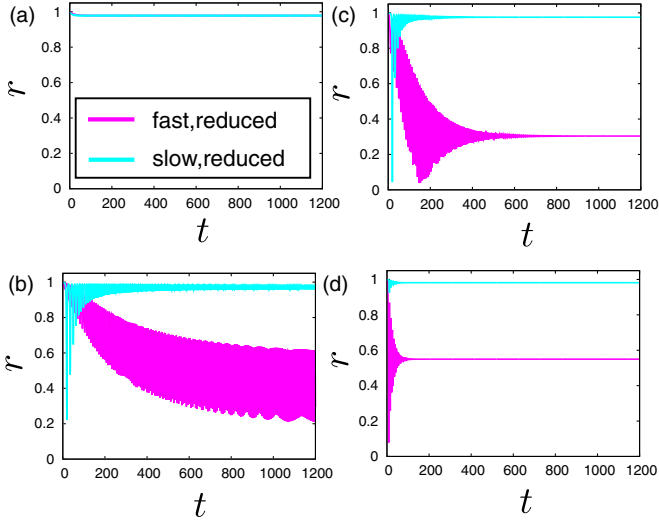
FIG. 2. Time evolutions of the order parameters of the populations in the reduced system for (a) $A = 0.9$, (b) $A = 0.1$, (c) $A = -0.1$, and (d) $A = -0.9$.
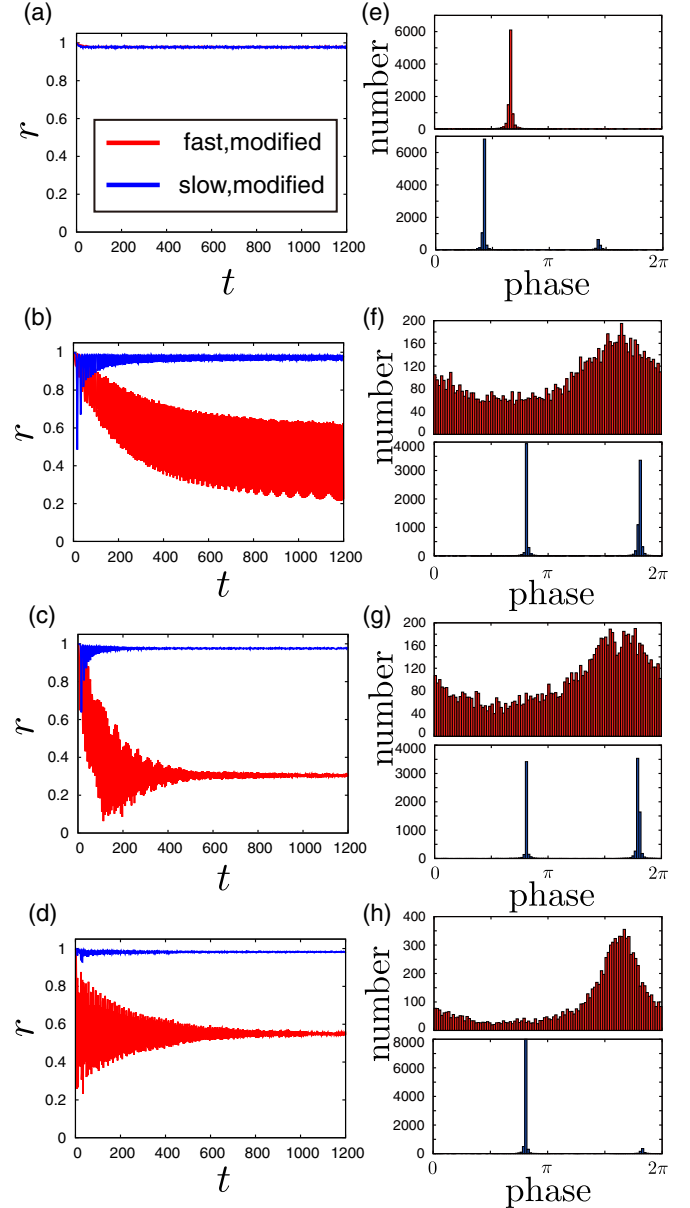

FIG. 3. Time evolutions of the order parameters of the populations in the modified systems; (a) $A = 0.9$, (b) $A = 0.1$, (c) $A = -0.1$, and (d) $A = -0.9$. Right panels show corresponding snapshots of the steady-state phase distributions (Up, fast population; Down, slow population).

that of the other population is coherent [27]. In this case, the slow oscillators are mutually synchronized while the fast oscillators are not. In the breathing chimera state [Fig. 2(b)], the order parameters oscillate. Conversely, in Fig. 2(c), the order parameters go finally to the fixed point, which is called the stable chimera state [16]. Figures 2(c) and 2(d) exhibit similar order parameter behaviors but different phase distributions. These differences will be thoroughly explored in later simulations of the modified system.

Figure 3 plots the evolving order parameters and instantaneous phase distributions of the modified system with $N = 10^4$. In Figs. 3(a)–3(d), the modified systems exhibit the same steady-state behavior as their counterpart reduced systems, but the transient behavior in Fig. 3(d) differs from that Fig. 2(d). In Fig. 2(d) at $A = -0.9$, the reduced system goes to the steady state faster than the modified system with $N = 10^4$ does in Fig. 3(d). This difference appears when the coupling between the populations is dominant.

Figure 3 also represents snapshots of the steady-state phase distributions of the two populations in the modified system. For $A = 0.1, -0.1,$ and 0.9 (Figs. 3(a)–3(c), respectively), the fast population is partially coherent, whereas the slow population splits into two clusters with a phase difference π. Similar states, known as clustered chimera states, have been reported in a delay-coupled system [49].

Let us check the result correspondence in the modified and reduced systems. Figures 4(a) and 4(b) plot the averages and standard deviations, respectively, of the steady-state order parameters in the system reduced by the ansatz and the modified system with $N = 10^4$. When the intrapopulation coupling strengths are sufficiently strong, both populations settle into coherent states. Weakened coupling leads to stable or breathing chimera states.

These results were obtained under the initial condition $r_{\text{fast}}, r_{\text{slow}} \simeq 1$. The steady-state order parameters behaved similarly under other starting conditions (data not shown), suggesting that the clustered chimera states in our system

are robust. However, the height ratio of two clusters in the asymptotic phase distributions of the slow oscillators does depend on the initial conditions, because the height difference have no effect on the phase dynamics in Eq. (17). Moreover, the transient behaviors crucially depend on the initial conditions in some parameter ranges.

V. VALIDITY OF SPECIFIC ASSUMPTION

For compatibility with the Ott-Antonsen ansatz, the above analysis imposed an unrealistic assumption on the interactions between the slow oscillators in Eq. (17). In real situations, the interaction term between the slow oscillators in Eq. (16) should
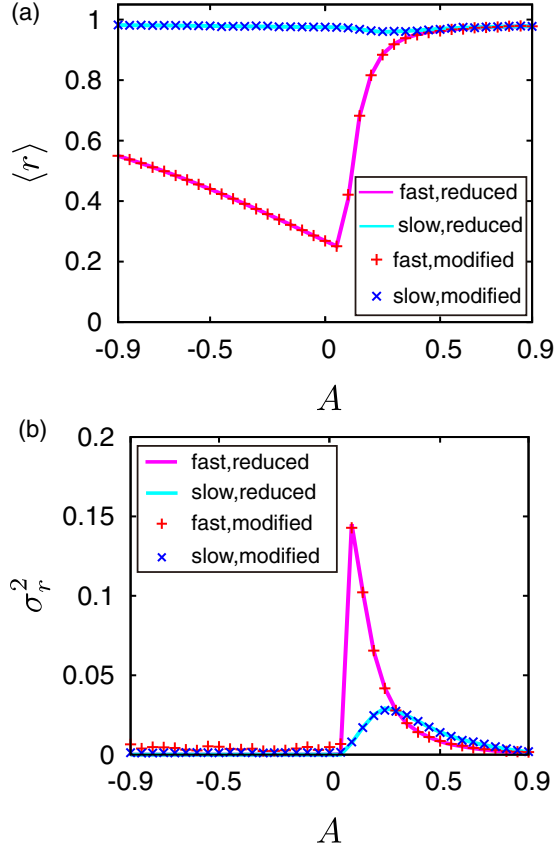
FIG. 4. Comparisons between the average (a) and the standard deviation (b) of the steady-state order parameters in the modified system (with $N = 10^4$) and the reduced system.
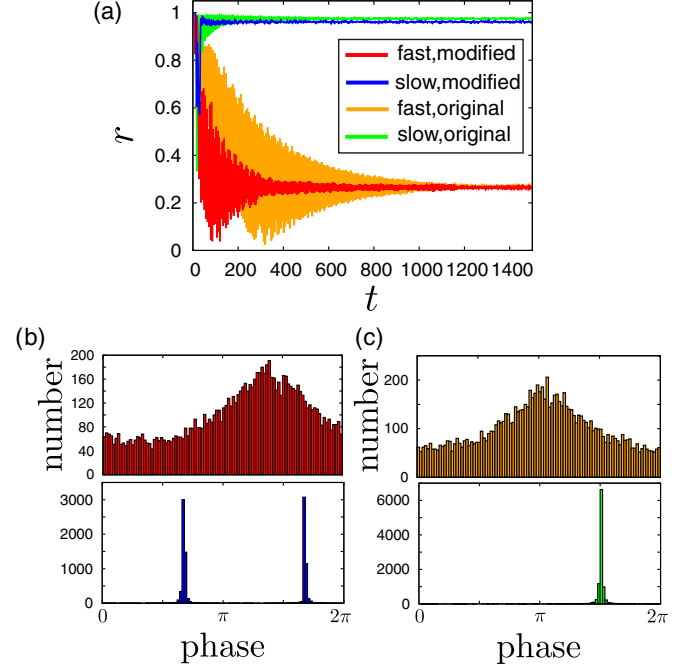

FIG. 5. (a) Time evolutions of the order parameters in the modified and original systems with $N = 10^4$ for $A = 0$. Bottom panels are snapshots of the steady-state phase distributions in the modified (b) and original (c) systems for $A = 0$.

contain the first mode in the Fourier series; consequently we need to reassess this assumption. Unfortunately, without replacing $\sin(\theta_j^{slow} - \theta_i^{slow} - \alpha)$ with $\sin(2\theta_j^{slow} - 2\theta_i^{slow} - \alpha)$, we cannot reduce the original system to a low-dimensional system through using the Ott-Antonsen ansatz. In this section, we investigate whether imposing the above assumption affects the dynamics of two oscillator populations.

After the above modification, the model Eqs. (15) and (17) can be rewritten as

$$
\frac{d\theta_i^{fast}}{dt} = \omega_i^{fast} + \mu \text{Im}\left[z_{fast}^{(N)} e^{-i(\theta_i^{fast} + \alpha)}\right]
$$
$$
+ \nu \text{Im}\left[z_{slow}^{(N)} e^{-i(\theta_i^{fast} + \alpha)}\right], \quad (31)
$$

$$
\frac{d\theta_i^{slow}}{dt} = \omega_i^{slow} + \mu \text{Im}\left[z_{slow}^{(N)} e^{-i(2\theta_i^{slow} + \alpha)}\right]
$$
$$
+ \nu \text{Im}\left[z_{fast}^{(N)} e^{-i(\theta_i^{slow} + \alpha)}\right], \quad (32)
$$

where $z_{fast}^{(N)}$ and $z_{slow}^{(N)}$ are the Kuramoto order parameter for the fast N oscillators and the Daido order parameter for the slow N oscillators, respectively. From these representations we see that all the interactions from the slow oscillators to the other can be described only through the Daido order parameter. As in Eq. (19), the Daido order parameter for $k = 2$ do not distinguish θ and $\theta + \pi$ in the phase variables of the slow oscillators. Therefore, with our modification the

difference between θ and $\theta + \pi$ in the phase variables of the slow oscillators does not matter and it is sufficient to use the Daido order parameter to examine the synchrony of the slow oscillators. However, without our modification, the model Eqs. (15) and (16) include the interactions from the slow oscillators that cannot be represented only by the Daido order parameters, and thus we have to be careful for how such condition affects the dynamics.

To scrutinize the effect of our assumption, we numerically examine the dynamics of the original system Eqs. (15) and (16). The natural frequency distributions of the populations are those of the modified system in Secs. II–IV. As the original system cannot be reduced to a low-dimensional system, we numerically simulate both systems with $N = 10^4$. As in Sec. IV, we set $D = 1.0 \times 10^{-3}$ and $\alpha = \pi/2 - 0.05$. The initial conditions were set to $r_{fast}, r_{slow} \simeq 1$ (each θ_i was chosen from uniform distribution in $[0, \pi/30]$).

The results for $A = 0$ (homogeneous coupling strengths in the system) are plotted in Fig. 5. Figure 5(a) compares the time evolutions of the order parameters in the modified and original systems with $N = 10^4$. Figures 5(b) and 5(c) are snapshots of the steady-state phase distributions in the modified and original systems, respectively. In Fig. 5(a), the order parameters of both systems reach similar steady states. However, closer inspection reveals that the asymptotic phase distributions of the slow oscillators differ between the two systems. Specifically, the slow population in the modified system splits into two clusters [Fig. 5(b)] but is unimodal in the original system [Fig. 5(c)]. Therefore, the original system settles into a normal rather than a clustered chimera state. The same result emerged under all tested conditions.
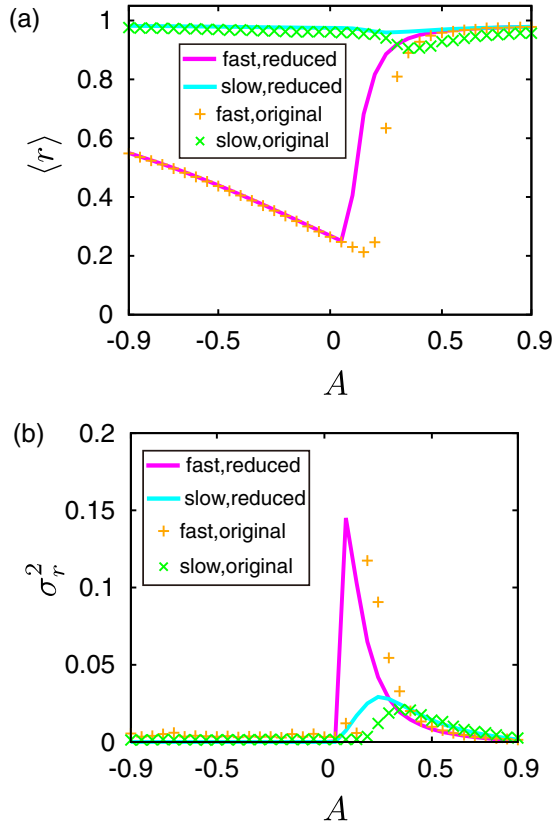
FIG. 6. Comparison between averages (a) and standard deviations (b) of the steady-state order parameters in the reduced and original systems ($N = 10^4$).

FIG. 7. (a) Time evolutions of the order parameters in the modified and original models with $N = 10^4$ for $A = 0$. Bottom panels are snapshots of the steady-state phase distributions in (b) the modified system and (c) the original system for $A = 0.2$.

The averages and standard deviations of the steady-state order parameters in the reduced and original systems are compared in Fig. 6 (Note that the results of the reduced system are replicated from Fig. 4). The behaviors of the reduced and original systems are qualitatively similar. We checked that this similarity remains if D is sufficiently small.

If we use not the Daido order parameter but the Kuramoto order parameter, the results might apparently be changed. However, considering the coupling functions of the slow oscillators in the modified system, it is more natural to use the Daido order parameter for $k = 2$ rather than the Kuramoto order parameter. In terms of the synchrony with the Daido order parameter, the modified and original systems show similar properties.

Finally, we remark on the different outcomes of the two systems. In Fig. 6, the asymptotic behaviors of the order parameters differ in certain ranges of the parameter A. The two models differ when A is small and positive; that is, when the intrapopulation interactions are slightly stronger than the interpopulation interactions.

To clarify this results, Fig. 7(a) plots the evolved order parameter dynamics of the modified and original systems with $N = 10^4$ for $A = 0.2$. Figures 7(b) and 7(c) represent snapshots of the steady-state phase distributions. In this case, the specific assumption alters the steady-state distribution of the modified system in two ways. First, it amplifies the peak of the fast oscillators relative to the original system. Second,
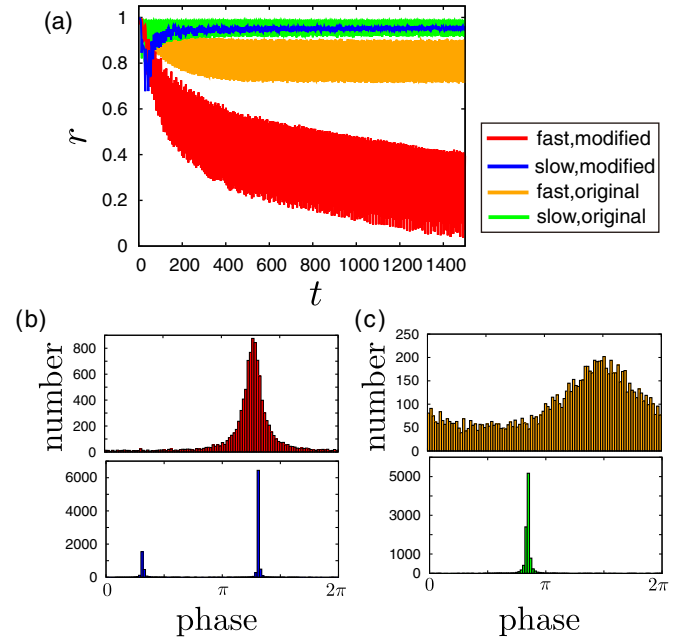
the slow oscillators separate into two clusters in the modified system, but they form a single cluster in the original system.

VI. GENERAL RESONANT CASE

As mentioned above, our method is readily extensible from the resonant condition $2 : 1$ to the general integer resonant condition $k : 1$. Thus, we consider a model of oscillators of two populations under the resonant condition $k : 1$:

$$\frac{d\theta_i^{\text{fast}}}{dt} = \omega_i^{\text{fast}} + \frac{\mu}{N} \sum_{j=1}^{N} \sin\left(\theta_j^{\text{fast}} - \theta_i^{\text{fast}} - \alpha\right)$$

$$+ \frac{\nu}{N} \sum_{j=1}^{N} \sin\left(k\theta_j^{\text{slow}} - \theta_i^{\text{fast}} - \alpha\right), \quad (33)$$

$$\frac{d\theta_i^{\text{slow}}}{dt} = \omega_i^{\text{slow}} + \frac{\mu}{N} \sum_{j=1}^{N} \sin\left(\theta_j^{\text{slow}} - \theta_i^{\text{slow}} - \alpha\right)$$

$$+ \frac{\nu}{N} \sum_{j=1}^{N} \sin\left(\theta_j^{\text{fast}} - k\theta_i^{\text{slow}} - \alpha\right), \quad (34)$$

where the corresponding natural frequency distributions are assumed to obey the Lorentz distributions of Fig. 1. Note that the strength parameter A satisfies $\mu = (1 + A)/2$ and $\nu = (1 - A)/2$. Thus, by replacing the interaction term $\sin(\theta_j^{\text{slow}} - \theta_i^{\text{slow}} - \alpha)$ in Eq. (34) with $\sin(k\theta_j^{\text{slow}} - k\theta_i^{\text{slow}} - \alpha)$, we can reduce the original system of Eqs. (33) and (34) in the continuum limit $N \to \infty$. After this modification, the PDF of the slow oscillator population

can be expressed as a Fourier series:

$$f_{\text{slow}}(\theta,\omega,t) = \frac{g_{\text{slow}}(\omega)}{2\pi}\left[1 + \sum_{m=1}^{\infty}(b(\omega,t)^m e^{ikm\theta} + \text{c.c.})\right],$$

where we assume $|b(\omega,t)| < 1$ under the Ott-Antonsen ansatz. We also define the complex order parameter for the slow oscillator population z_{slow}:

$$z_{\text{slow}}(t) = \int_{-\infty}^{\infty} d\omega \int_{0}^{2\pi} d\theta f_{\text{slow}}(\theta,\omega,t) e^{ik\theta}. \tag{35}$$

Similar to Sec III, the population dynamics are finally described by a set of ordinary differential equations with three degrees of freedom:

$$\frac{dr_{\text{fast}}}{dt} = -Dr_{\text{fast}} + \frac{1-r_{\text{fast}}^2}{2}[\mu r_{\text{fast}}\cos\alpha + \nu r_{\text{slow}}\cos(\phi-\alpha)], \tag{36}$$

$$\frac{dr_{\text{slow}}}{dt} = -kDr_{\text{slow}} + \frac{k(1-r_{\text{slow}}^2)}{2}[\mu r_{\text{slow}}\cos\alpha + \nu r_{\text{fast}}\cos(\phi+\alpha)], \tag{37}$$

$$\frac{d\phi}{dt} = \frac{1+r_{\text{fast}}^2}{2}\left[\mu\sin\alpha - \nu\frac{r_{\text{slow}}}{r_{\text{fast}}}\sin(\phi-\alpha)\right] - \frac{k(1+r_{\text{slow}}^2)}{2}\left[\mu\sin\alpha + \nu\frac{r_{\text{fast}}}{r_{\text{slow}}}\sin(\phi+\alpha)\right]. \tag{38}$$

Here, we used the polar coordinates $z_{\text{fast}} = r_{\text{fast}}e^{-i\phi_{\text{fast}}}$, $z_{\text{slow}} = r_{\text{slow}}e^{-i\phi_{\text{slow}}}$ and denoted the phase difference $\phi = \phi_{\text{fast}} - \phi_{\text{slow}}$ as in Sec. III. We remark that the system dynamics depend only on the ratio of the means of the natural frequencies, k. In other words, the absolute value of the mean frequencies Ω does not influence the collective behavior of the system.

Figure 8 shows the averages and standard deviations of the steady-state order parameters in the reduced and original systems for various k (3, 5, 7, 11). As in Sec. V, we set $N = 10^4$ and the initial conditions $r_{\text{fast}}, r_{\text{slow}} \simeq 1$. We also set $\alpha = \pi/2 - 0.05$ and $D = 1.0 \times 10^{-3}$. Qualitatively, the system Eqs. (36)–(38) exhibits the same dynamics as the system Eqs. (28)–(30), even at higher values of k.

However, the asymptotic behaviors differ between the reduced and original systems as k increases. The widening difference is especially apparent in the standard deviation. For large k, the order parameter of the slow oscillators tends to fluctuate with larger amplitudes in the original system, probably because of the imposed assumption. In typical Fourier series, the magnitudes of the low-frequency modes are more similar to the first Fourier mode amplitude than those of high-frequency modes.

Under the general condition $k : 1$, the modified system develops k-clustered chimera states, with k clusters of coherent slow oscillators. To our knowledge these states have not been previously reported. The slow oscillators in the k-clustered chimera states form k synchronous oscillator groups. The phase difference between the nearest clusters is approximately $2\pi/k$. The k-clustered chimera states can be considered as generalized versions of the two-cluster ones. Like the standard
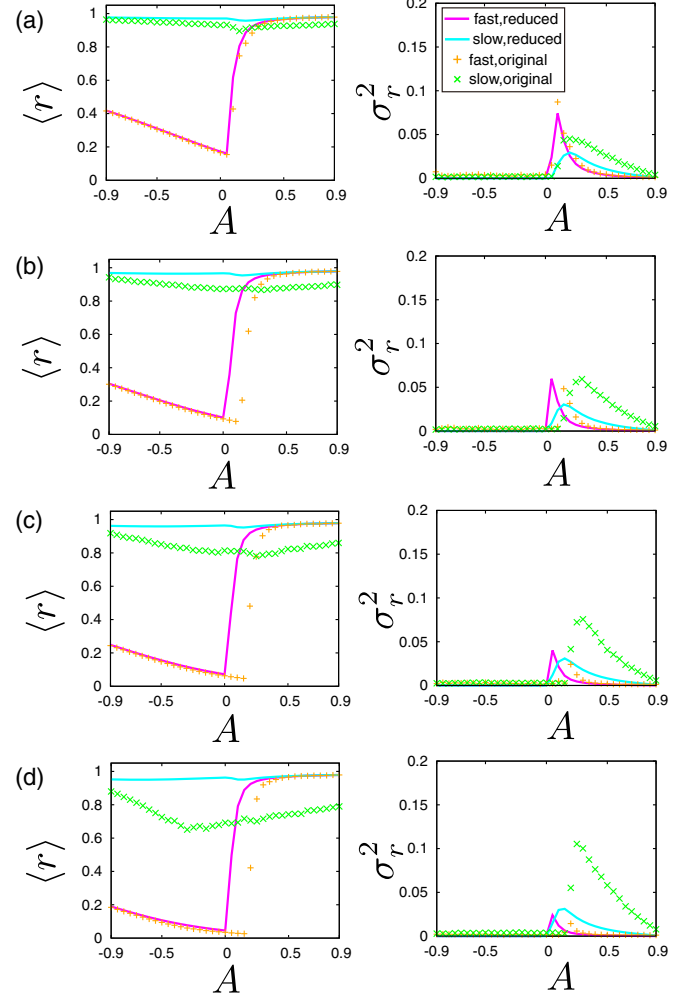

FIG. 8. (Left) Averages and (Right) standard deviations of the steady-state order parameters in the reduced and original systems with $N = 10^4$. Top to bottom: (a) $k = 3$, (b) $k = 5$, (c) $k = 7$, and (d) $k = 11$.

chimera state, the k-clustered chimera states are classifiable as stable or breathing.

We numerically checked the result correspondence between the reduced and modified systems with large N and general k. Therefore, our proposed reduction is valid in the general $k : 1$ case.

VII. CONCLUSION

Referring to phase reduction theory, we investigated the dynamics of multiple populations of phase oscillators. First, we assumed that the mean of frequency distribution in one population was twice faster than that in the other population. Applying the Ott-Antonsen ansatz [36,46] and imposing an additional assumption, we reduced the original system to a low-dimensional system of ordinary differential equations describing the time evolution of the order parameters. The population of slow oscillators was treated by the Daido order parameter [48]. Clustered chimera states emerged when the interpopulation coupling strength was relatively large.

We also investigated the general resonant condition. Our analysis was extensible from the simple resonant case $2:1$ to the general case $k:1$, where k is any integer. We confirmed that the results for the case $2:1$ were qualitatively replicated in the general case. However, for large k, our additional assumption significantly altered the dynamics of the original system.

As a future work, we will investigate multifrequency systems completely. In other words, we can think about dynamics of systems with the more general resonant case $m:n$. We can apply our approach to the resonant case $m:n$, although it requests us one more assumption on the interactions within populations of fast oscillators.

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