Multivariate Analysis Lecture 17: Factor Analysis

Zhaoxia Yu
Professor, Department of Statistics

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Section 1

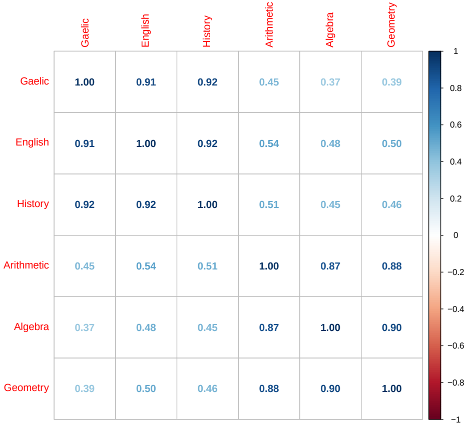
Motivating Example

A Motivating Example: Exam Score

```r
set.seed(6)
mu=rnorm(60, sd=1)
mu1=mu+rnorm(60)
mu2=mu+rnorm(60)
exam=cbind(mu1+rnorm(60, sd=0.5), mu1+rnorm(60, sd=0.5), mu1+rnorm(60, sd=0.5),
          mu2+rnorm(60, sd=0.5), mu2+rnorm(60, sd=0.5), mu2+rnorm(60, sd=0.5))
colnames(exam)=c("Gaelic","English","History",
                 "Arithmetic","Algebra","Geometry")
exam=data.frame(exam)
```

Pairwise Correlation

```r
corrplot(cor(exam), method="number")
```



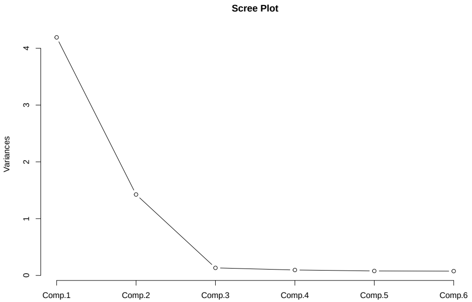
PCA

```r
obj=princomp(exam, cor=TRUE)
obj$loadings
```

```
##
## Loadings:
##            Comp.1 Comp.2 Comp.3 Comp.4 Comp.5 Comp.6
## Gaelic      0.394  0.456  0.176  0.149  0.756  0.107
## English     0.426  0.360 -0.127 -0.187 -0.465  0.650
## History     0.417  0.393 -0.139        -0.323 -0.740
## Arithmetic  0.414 -0.368  0.806        -0.193
## Algebra     0.396 -0.437 -0.456  0.659
## Geometry    0.402 -0.427 -0.275 -0.708  0.265
##
##                Comp.1 Comp.2 Comp.3 Comp.4 Comp.5 Comp.6
## SS loadings     1.000  1.000  1.000  1.000  1.000  1.000
## Proportion Var  0.167  0.167  0.167  0.167  0.167  0.167
## Cumulative Var  0.167  0.333  0.500  0.667  0.833  1.000
```

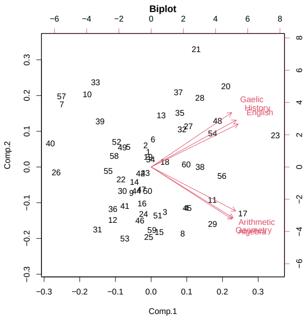
Visualize Eigenvalues

```
plot(obj, type="lines", main="Scree Plot")
```



Visualize First and Second PC (biplot)

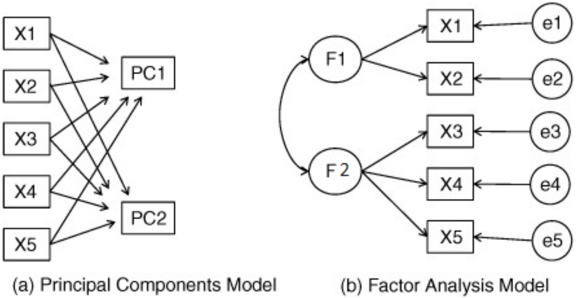
```
biplot(obj, main="Biplot")
```



- How to interpret the results?

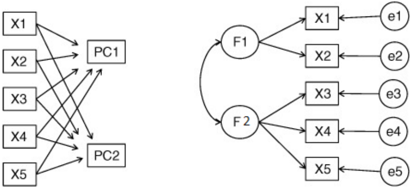
Introduction to Factor Analysis

```
knitr::include_graphics("img/PCA_FA.png")
```



https://livebook.manning.com/book/r-in-action-second-edition/chapter-14/6

PCA vs FA



(a) Principal Components Model (b) Factor Analysis Model

- Both reduce dimensionality
- Both use linear combinations
- PCA leads to principal components, which are linear combinations of functions
- FA leads to factors (latent and unobserved)

Section 2

The FA Model

The Factor Model

- Consider a random vector $\mathbf{X} \in \mathbb{R}^p$
- Let $\boldsymbol{\mu} \in \mathbb{R}^p$ denote the population mean
- Let $F \in \mathbb{R}^m$ denote m factors

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{pmatrix}, \boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{pmatrix}, \mathbf{F} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix}$$

The Factor model

- X_j, which is observable, is assumed to be a linear function of the unobservable common factors $f_1, \cdots, f_m$ plus specific errors.

$$
\begin{aligned}
X_1 &= \mu_1 + l_{11} f_1 + l_{12} f_2 + \cdots + l_{1m} f_m + \epsilon_1 \\
X_2 &= \mu_2 + l_{21} f_1 + l_{22} f_2 + \cdots + l_{2m} f_m + \epsilon_2 \\
&\vdots \\
X_p &= \mu_p + l_{p1} f_1 + l_{p2} f_2 + \cdots + l_{pm} f_m + \epsilon_p
\end{aligned}
$$

  where ϵ_j is called the specific factor for feature j.
- The means $\mu_1, \cdots, \mu_p$ are parameters
- The coefficients in the factor loading matrix are also parameters

The Factor model

- Let $\mathbf{L}$ denote the $p \times m$ matrix of factor loadings

$$\mathbf{L} = \begin{pmatrix} l_{11} & l_{12} & \cdots & l_{1m} \\ l_{21} & l_{22} & \cdots & l_{2m} \\ \vdots & \vdots & & \vdots \\ l_{p1} & l_{p2} & \cdots & l_{pm} \end{pmatrix}$$

- A compact expression of the factor model is

$$\mathbf{X}_{p\times 1} = \boldsymbol{\mu}_{p\times 1} + \mathbf{L}_{p\times m}\mathbf{F}_{m\times 1} + \boldsymbol{\epsilon}_{p\times 1}$$

Assumptions of FA

- $\mathbf{F}$ and ϵ are uncorrelated
- The common factors are uncorrelated

$$\mathbb{E}(\mathbf{F}) = 0, Cov(\mathbf{F}) = \mathbf{I}$$

- The specific factors are uncorrelated

$$\mathbb{E}(\epsilon) = 0, Cov(\epsilon) = \Psi$$

where Ψ is a diagonal matrix with non-negative values, i.e.,

$$\boldsymbol{\Psi} = \begin{pmatrix} \psi_1 & 0 & \dots & 0 \\ 0 & \psi_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \psi_p \end{pmatrix}$$

The Covariance

- By the factor model and its assumptions, we have

$$\begin{aligned}\Sigma &= cov(\mathbf{X}) \\ &= cov(\mathbf{LF} + \epsilon) \\ &= \mathbf{L}\,Cov(\mathbf{F})\mathbf{L}^T + \Psi \\ &= \mathbf{LL}^T + \Psi\end{aligned}$$

The last step is due to our assumption that $cov(\mathbf{F}) = \mathbf{I}$

The Covariance

- For $i \neq j$, the covariance between X_i (feature i) and X_j (feature j) is

$$\sigma_{ij} = cov(X_i, X_j) = \sum_{k=1}^{m} l_{ik} l_{jk}$$

- The variance of X_i is

$$\sigma_{ii} = \sum_{k=1}^{m} l_{ik}^2 + \psi_i$$

Section 3

Communality

Communality and Specific Variance

- From last slide

$$\sigma_{ii} = \sum_{k=1}^{m} l_{ik}^2 + \psi_i$$

- We say that the variance of X_i is partitioned into communality and specific variance where
  - communality is defined as $h_i^2 = \sum_{k=1}^{m} l_{ik}^2$, which is the proportion of variance contributed by common factors
  - specific variance ψ_i, which is the specific variance of X_i

Example of Communality

```r
obj=factanal(exam, factors=2)
L=obj$loadings[,1:2]
Psi=diag(obj$uniquenesses)
#communality
1-obj$uniquenesses
```

```
##     Gaelic   English   History Arithmetic   Algebra  Geometry
## 0.9156248 0.9222264 0.9241880  0.8546526 0.8948649 0.9160432
```

- For Geometry, the communality is _____ and the uniqueness (specific variance) is _____.

Section 4

Non-uniqueness

Non-uniqueness of Factor Loadings

- The factor loading coefficient is NOT unique.
- Suppose $\mathbf{X} = \boldsymbol{\mu} + \mathbf{LF} + \boldsymbol{\epsilon}$
- Consider any $m \times m$ orthogonal matrix Γ, which satisfies $\Gamma\Gamma^T = \Gamma^T\Gamma = \mathbf{I}$.
- Let $\tilde{\mathbf{L}} = \mathbf{L}\Gamma$ The model $\mathbf{X} = \boldsymbol{\mu} + \tilde{\mathbf{L}}\mathbf{F} + \boldsymbol{\epsilon}$

give the same Σ because

$$cov(\tilde{\mathbf{L}}\mathbf{F}) = \tilde{\mathbf{L}}cov(\mathbf{F})\tilde{\mathbf{L}}^T = \tilde{\mathbf{L}}\tilde{\mathbf{L}}^T = \mathbf{L}\Gamma\Gamma^T\mathbf{L}^T = \mathbf{L}\mathbf{L}^T = cov(\mathbf{LF})$$

Non-uniqueness of Factor Loadings

```
#Estimated Sigma
L%*%t(L) + Psi
```

```
##               Gaelic   English   History Arithmetic   Algebra  Geometry
## Gaelic     1.0000002 0.9104285 0.9161363  0.4431684 0.3743267 0.3881987
## English    0.9104285 1.0000000 0.9222470  0.5448653 0.4843854 0.4989042
## History    0.9161363 0.9222470 1.0000001  0.5128914 0.4493253 0.4636887
## Arithmetic 0.4431684 0.5448653 0.5128914  1.0000007 0.8702984 0.8814692
## Algebra    0.3743267 0.4843854 0.4493253  0.8702984 0.9999997 0.9053334
## Geometry   0.3881987 0.4989042 0.4636887  0.8814692 0.9053334 1.0000002
```

Non-uniqueness of Factor Loadings

- Consider a rotation matrix R and define $\tilde{\mathbf{L}} = LR$

```
theta=pi/6
R=matrix(c(cos(theta), sin(theta), -sin(theta), cos(theta)), 2,2)
L.tilde=L%*%R

L.tilde %*% t(L.tilde) + Psi
```

```
##               Gaelic   English   History Arithmetic   Algebra  Geometry
## Gaelic     1.0000002 0.9104285 0.9161363  0.4431684 0.3743267 0.3881987
## English    0.9104285 1.0000000 0.9222470  0.5448653 0.4843854 0.4989042
## History    0.9161363 0.9222470 1.0000001  0.5128914 0.4493253 0.4636887
## Arithmetic 0.4431684 0.5448653 0.5128914  1.0000007 0.8702984 0.8814692
## Algebra    0.3743267 0.4843854 0.4493253  0.8702984 0.9999997 0.9053334
## Geometry   0.3881987 0.4989042 0.4636887  0.8814692 0.9053334 1.0000002
```

Section 5

Factor Rotation

Subsection 1

Rotation for Better Interpretation

Rotation for Better Interpretation

- Interpretation of final results are easier for some choices of **L** than others.
- We often rotate the factors to gain insights or for better interpretation
- This is one advantage of factor analysis
- In practice,
  - Step 1: fit a factor model by imposing conditions that lead to a unique solution
  - Step 2: the loading matrix L is rotated (multiplied by an orthogonal matrix) in a way that gives a good interpretation of the data. Trial and error
- Well know criteria of rotation exist

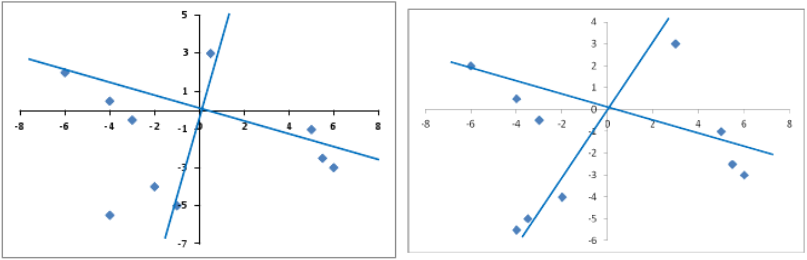
Subsection 2

Two Major Types of Rotation

Two Major Types of Rotation

- An orthogonal rotation
  - maintains the perpendicularity between factors after rotation.
  - assumes that factors are unrelated or independent of each other.
  - Varimax is the most commonly used method of orthogonal rotation.
- An oblique rotation
  - allows factors to be correlated and does not maintain a 90 degrees angle.
  - assumes that factors are related or dependent on each other.
  - One popular method is Promax
- Orthogonal rotations are mathematically appealing/convenient
- There is no reason that factors have to be uncorrelated

Two Major Types of Rotation



Varimax and Promax

- https://onlinelibrary.wiley.com/doi/pdf/10.1111/j.1745-3984.2006.00003.x
- Consider a rotation matrix with angle ψ

$$\begin{pmatrix} cos\psi & -sin\psi \\ sin\psi & cos\psi \end{pmatrix}$$

- The Varimax method looks for ψ that maximizes the Varimax criterion

$$\frac{1}{p}\sum_i \left[\sum_j l_{ij}^4/h_i - (\sum_j (l_{ij}/h_i)^2)^2/p\right]$$

- The Promax is based on Varimax. It basically shrinks small loadings by using a powerful function

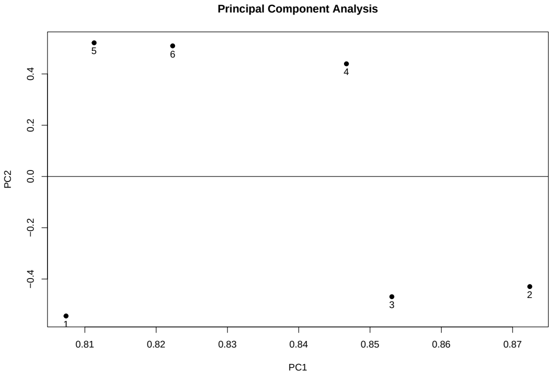
Factor Rotation

```r
principal(exam,nfactors=2, rotate="none")
```

```
## Principal Components Analysis
## Call: principal(r = exam, nfactors = 2, rotate = "none")
## Standardized loadings (pattern matrix) based upon correlation matrix
##             PC1   PC2   h2    u2 com
## Gaelic     0.81 -0.54 0.95 0.052 1.8
## English    0.87 -0.43 0.95 0.054 1.5
## History    0.85 -0.47 0.95 0.052 1.6
## Arithmetic 0.85  0.44 0.91 0.090 1.5
## Algebra    0.81  0.52 0.93 0.070 1.7
## Geometry   0.82  0.51 0.94 0.064 1.7
##
##                        PC1  PC2
## SS loadings           4.19 1.43
## Proportion Var        0.70 0.24
## Cumulative Var        0.70 0.94
## Proportion Explained  0.75 0.25
## Cumulative Proportion 0.75 1.00
##
## Mean item complexity =  1.6
## Test of the hypothesis that 2 components are sufficient.
##
## The root mean square of the residuals (RMSR) is  0.02
##  with the empirical chi square  0.79  with prob <  0.94
##
## Fit based upon off diagonal values = 1
```

Factor Rotation

```
plot(principal(exam,nfactors=2, rotate="none"))
```



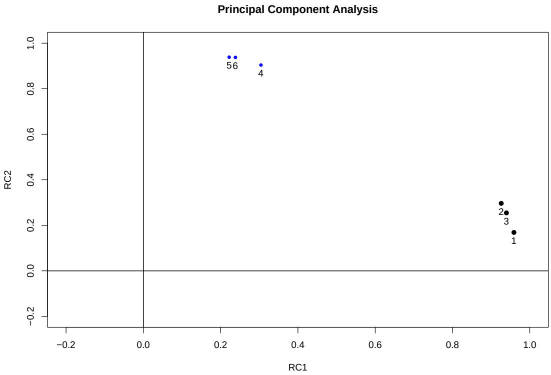
Factor Rotation: varimax (an orthogonal rotation)

```r
principal(exam, nfactors=2)
```

```
## Principal Components Analysis
## Call: principal(r = exam, nfactors = 2)
## Standardized loadings (pattern matrix) based upon correlation matrix
##             RC1  RC2   h2    u2 com
## Gaelic     0.96 0.17 0.95 0.052 1.1
## English    0.93 0.30 0.95 0.054 1.2
## History    0.94 0.25 0.95 0.052 1.1
## Arithmetic 0.30 0.90 0.91 0.090 1.2
## Algebra    0.22 0.94 0.93 0.070 1.1
## Geometry   0.24 0.94 0.94 0.064 1.1
##
##                        RC1  RC2
## SS loadings           2.86 2.76
## Proportion Var        0.48 0.46
## Cumulative Var        0.48 0.94
## Proportion Explained  0.51 0.49
## Cumulative Proportion 0.51 1.00
##
## Mean item complexity =  1.1
## Test of the hypothesis that 2 components are sufficient.
##
## The root mean square of the residuals (RMSR) is  0.02
##  with the empirical chi square  0.79  with prob <  0.94
##
## Fit based upon off diagonal values = 1
```

Factor Rotation: varimax (an orthogonal rotation)

```
plot(principal(exam, nfactors=2),xlim=c(-0.2,1),ylim=c(-0.2,1))
abline(h=0, v=0)
```



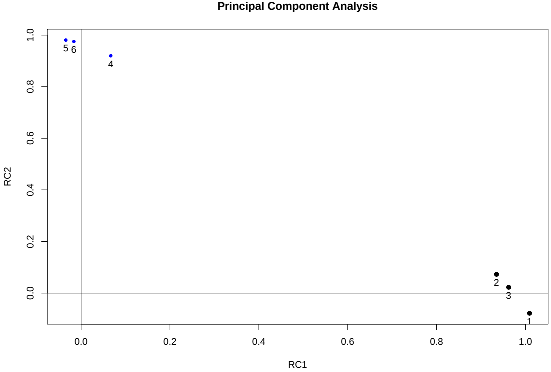
Factor Rotation: an oblique (non-orthogonal) rotation

```r
principal(exam,nfactors=2, rotate="promax")
```

```
## Principal Components Analysis
## Call: principal(r = exam, nfactors = 2, rotate = "promax")
## Standardized loadings (pattern matrix) based upon correlation matrix
##              RC1   RC2   h2    u2 com
## Gaelic      1.01 -0.08 0.95 0.052   1
## English     0.93  0.07 0.95 0.054   1
## History     0.96  0.02 0.95 0.052   1
## Arithmetic  0.07  0.92 0.91 0.090   1
## Algebra    -0.03  0.98 0.93 0.070   1
## Geometry   -0.02  0.98 0.94 0.064   1
##
##                        RC1  RC2
## SS loadings           2.84 2.78
## Proportion Var        0.47 0.46
## Cumulative Var        0.47 0.94
## Proportion Explained  0.50 0.50
## Cumulative Proportion 0.50 1.00
##
##  With component correlations of
##      RC1  RC2
## RC1 1.00 0.49
## RC2 0.49 1.00
##
## Mean item complexity =  1
## Test of the hypothesis that 2 components are sufficient.
##
```

Factor Rotation: an oblique (non-orthogonal) rotation

```
plot(principal(exam,nfactors=2, rotate="promax"))
```



Factor Rotation

- The following articles provide nice descriptions of the two major types of rotations:

https://scholarworks.umass.edu/cgi/viewcontent.cgi?article=1251&context=pare

https://www.theanalysisfactor.com/rotations-factor-analysis/

Section 6

Computation

Subsection 1

Method 1: Use PCA

Method 1: Use PCA

- By the spectral decomposition of Σ we have

$$\Sigma = \Gamma \Lambda \Gamma^T$$

where $\Gamma = (\gamma_1, \cdots, \gamma_p)$ is an orthogonal matrix and $\Lambda = diag(\lambda_1, \cdots, \lambda_p)$ be the diagonal matrix of eigenvalues.

- The spectral decomposition can be rewritten to

$$\boldsymbol{\Sigma} = \sum_{i=1}^{p} \lambda_i \gamma_i \gamma_i^T = \sum_{i=1}^{p} (\sqrt{\lambda_i} \gamma_i)(\sqrt{\lambda_i} \gamma_i)^T$$

Method 1: Use PCA

- Suppose that $\lambda_m, \lambda_{m+1}, \cdots, \lambda_p$ are small. Then

$$\mathbf{\Sigma} \approx \sum_{i=1}^{m} (\sqrt{\lambda_i}\gamma_i)(\sqrt{\lambda_i}\gamma_i)^T$$

- Let $\mathbf{L} = (\sqrt{\lambda_1}\gamma_1, \cdots, \sqrt{\lambda_m}\gamma_m)$
- Let $\mathbf{\Psi} = \mathbf{\Sigma} - \mathbf{L}\mathbf{L}^T$

Subsection 2

Method 2: MLE

Method 2: MLE

- We impose multivariate normality on the common and specific factors

$$\mathbf{F} \sim N(\mathbf{0}, \mathbf{I}), \epsilon \sim N(\mathbf{0}, \Psi)$$

- The log-likelihood is

$$l(\mu, \mathbf{L}, \Psi) = -\frac{np}{2}\log 2\pi - \frac{n}{2}\log|\mathbf{L}\mathbf{L}^T + \Psi| - \frac{1}{2}\sum_{i=1}^{n}(\mathbf{X}_i - \mu)^T(\mathbf{L}\mathbf{L}^T + \Psi)^{-1}(\mathbf{X}_i - \mu)$$

where $\mathbf{X}_i \in \mathbb{R}^p$ denotes the i observation (not the ith feature).

Subsection 3

The Number of Common Factors

The Number of Common Factors

- The $p \times p$ covariance matrix Σ is symmetric. As a result, there are $\frac{p(p+1)}{2}$ parameters.
- A factor mode imposes a structure on Σ
- For a FA model with m common factors
- A FA model a small number common factors, i.e., when m is small, the model uses fewer parameters
  - the model is more parsimonious
  - the model might not be adequate is m is too small
- One can test whether an m is large enough

Choose m of Factors Computed using PCA

- Cumulatative variance explained is should be reasonably large, such as >80%
- Look for elbow from the scree plot

A Goodness of Fit Test for the Adequacy of the Number of Common Factors

```r
knitr::include_graphics("img/TestNumberCommonFactors.png")
```

$$H_0: \underset{(p\times p)}{\boldsymbol{\Sigma}} = \underset{(p\times m)}{\mathbf{L}} \underset{(m\times p)}{\mathbf{L}'} + \underset{(p\times p)}{\boldsymbol{\Psi}}$$

$$-2\ln\Lambda = -2\ln\left[\frac{\text{maximized likelihood under } H_0}{\text{maximized likelihood}}\right]$$

$$= -2\ln\left(\frac{|\hat{\boldsymbol{\Sigma}}|}{|\mathbf{S}_n|}\right)^{-n/2} + n\left[\operatorname{tr}(\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{S}_n) - p\right]$$

It can be shown that $\operatorname{tr}(\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{S}_n) - p = 0$

$$= n\ln\left(\frac{|\hat{\boldsymbol{\Sigma}}|}{|\mathbf{S}_n|}\right)$$

Test for the Adequacy of the Number of Common Factors

- The number of parameters for covariance in the full model is

$$\frac{p(p+1)}{2}$$

- The number of parameters for covariance in the reduced model is

$$mp + p - \frac{m(m-1)}{2}$$

Note: $-\frac{m(m-1)}{2}$ is due to the nonuniqueness of $\mathbf{L}$.

- The difference is

$$\begin{aligned} df &= \frac{p(p+1)}{2} - [mp + p - \frac{m(m-1)}{2}] \\ &= \frac{1}{2}[(p-m)^2 - p - m] \end{aligned}$$

Test for the Adequacy of the Number of Common Factors

- The result indicates that 1 factor is not adequate as the p-value is small.
- The p-value is about whether the correlation structure specified in the proposed model is significantly different from that of the full model

```
factanal(exam, factors=1)$PVAL
```

```
##    objective
## 2.264759e-30
```

Test for the Adequacy of the Number of Common Factors

- The result indicates that 2 factors is adequate because the fit is not substantially from the full model.

```
factanal(exam, factors=2)$PVAL
```

```
## objective
##  0.930055
```

Section 7

CFA vs EFA

Exploratory Factor Analysis

- The FA approach we have discussed is exploratory in nature.
- In fact, we can perform EFA and identify latent factors by using only correlations, not the data
- The purpose of EFA is to explore the possible underlying structure that can explain the observed pattern of correlations
- EFA is used when researchers do not have a specific idea about the underlying structure of data
- EFA tries to identify the factor configuration (model)
- EFA is hypothesis-generating

Exploratory or Confirmatory Factor Analysis

- Confirmatory Factory Analysis (CFA) is used when a researcher has specific hypotheses or theories about the factor structure of their data.
- It is a “theory-driven” approach.
- In CFA, the researcher specifies the number of factors and which variables load onto which factors.
- CFA is typically used in later stages of research to test or confirm the factor structure suggested by EFA
- CFA is hypothesis testing. A pre-specified model is required

EFA vs CFA

- EFA tries to
- CFA
    - a model or several candidate models have been determined beforehand
    - the number of factors

Exploratory or Confirmatory Factor Analysis

- Use EFA when:
  - You are unsure about the underlying structure.
  - You aim to uncover complex patterns.
  - You need to form hypotheses and develop theory.
- Use CFA when:
  - You have a predetermined theory or model.
  - You aim to test the hypothesis about the factor structure.
  - You need to confirm or disconfirm theories.

Section 8

CFA: Example

Subsection 1

R Packages

R Packages

- Conduct CFA in R
- R packages:
  - sem
  - OpenMx
  - lavaan

Subsection 2

An Example using "lavaan"

An Example using “lavaan”

- CFA can be performed using the latent variable analysis (“lavaan”) package in r

```
model1 <- '
verbal =~ Gaelic + English + History
math =~ Arithmetic + Algebra + Geometry'

obj=cfa(model1, data=data.frame(exam))
```

Visualize the Model

```
#lavaanPlot(model=obj)
#lavaanPlot(model=obj, coefs=TRUE)
#semPaths(obj, what="est")
#Why set the factor loading for the fest feature to one? Id
```

Understand the Output

```r
#Std.lv: estimates when only latent variables are standardized
#Std.all: estimates when all variables are standardized.
summary(obj, fit.measures = TRUE)
```

Understand the Baseline model

```
model0 <- '
Gaelic ~~ Gaelic
English ~~ English
History ~~ History
Arithmetic ~~ Arithmetic
Algebra ~~ Algebra
Geometry ~~ Geometry
'
obj0=cfa(model0, data=data.frame(exam))
#https://stats.oarc.ucla.edu/wp-content/uploads/2020/02/fit3.png
#semPaths(obj0, what="est")
```

Understanding the Model Fit Statistics

- We often need to compare a FA (reduced) model to the full model
$$H_0 : \mathbf{\Sigma}(\mathbf{L}, \Psi) = \mathbf{\Sigma}$$
- Chi-square test. Derived from the likelihood ratio test. Depends on sample sizes.
- RMSEA: root mean square error of approximation compares the sample correlation matrix and the model correlation matrix. $<$0.06 is good.

$$RMSEA = \sqrt{\frac{\delta}{df(N-1)}}$$

where $\delta = \chi^2 - df$.

Understanding the Model Fit Statistics

- CFI: comparative fit index that measures the relative difference between two models; is not affected by sample size too much; between 0 and 1; the larger the better. $>$0.9 is good

$$CFI = \frac{\delta(\text{Baseline}) - \delta(\text{User})}{\delta(\text{Baseline})}$$

- More can be found in wikipedia and
  - https://doi.org/10.1016/B978-0-444-53737-9.50010-4
  - Bentler PM. Comparative fit indexes in structural models. Psychological bulletin. 1990 Mar;107(2):238.
  - http://www.davidakenny.net/cm/fit.htm

An Example using "lavaan"

```r
fitMeasures(obj0, c("chisq", "cfi", "rmsea"))
```

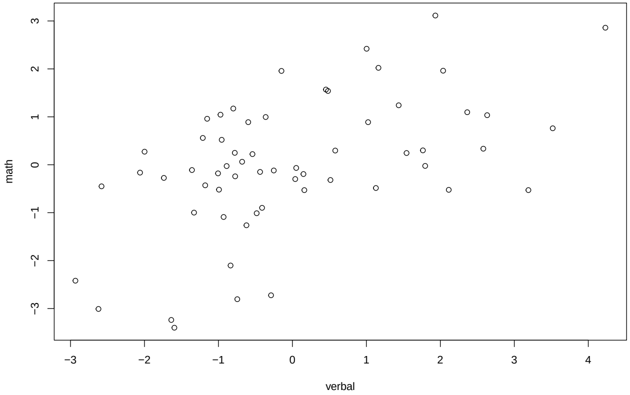
```
##   chisq     cfi   rmsea
## 462.072   0.000   0.705
```

```r
fitMeasures(obj, c("chisq", "cfi", "rmsea"))
```

```
## chisq   cfi rmsea
## 8.586 0.999 0.035
```

Compute Factor scores

```
objscores <- lavPredict(obj)
plot(objscores)
```



Helpful Resources

- https://quantdev.ssri.psu.edu/tutorials/intro-basic-confirmatory-factor-analysis
- https://lavaan.ugent.be/tutorial/tutorial.pdf
- https://stats.oarc.ucla.edu/r/seminars/rcfa/