

# Multivariate Analysis Lecture 4: A Random Sample from A Multivariate Distribution

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## Section 1

### Review of Lecture 03

# Outline

- Definitions
- A random variable from a univariate distribution
- A random sample from a univariate distribution
- A random vector from a multivariate distribution
- A random sample from a multivariate distribution
- Linear combinations of a random vector

## Definitions: Mean and Variance

- Let  $X$  be a random variable. Its mean, denoted by  $E[X]$ , is defined as

$$\mu = E[X] = \begin{cases} \int_{-\infty}^{\infty} xf(x)dx & \text{if } X \text{ is continuous} \\ \sum_i x_i p_i & \text{if } X \text{ is discrete} \end{cases}$$

- Its variance, denoted by  $Var(X)$ , is defined as  $E[(X - \mu)^2]$ ,

$$\sigma^2 = Var[X] = E[(X - \mu)^2] = \begin{cases} \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx & \text{if } X \text{ is continuous} \\ \sum_i (x_i - \mu)^2 p_i & \text{if } X \text{ is discrete} \end{cases}$$

## Definitions: Mean Vector and Covariance Matrix

- Let  $\mathbf{X}_{p \times 1} = (X_1, \dots, X_p)^T$  be a random vector. Its mean is defined as

$$E[\mathbf{X}] = \begin{pmatrix} E[X_1] \\ \vdots \\ E[X_p] \end{pmatrix}$$

- Its covariance matrix is defined as

$$\boldsymbol{\Sigma}_{p \times p} = \text{Cov}(\mathbf{X}) = E[(\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})^T]$$

# Random Vectors: An Example

- Let  $X_1, \dots, X_n$  be iid random variables from a univariate distribution with mean  $\mu$  and variance  $\sigma^2$ . We say  $X_1, \dots, X_n$  form a random sample.
- We often stack the random variables vertically:

$$\mathbf{X}_{n \times 1} = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix},$$

- Remark:  $\mathbf{X}$  is a **random vector** with mean vector and covariance matrix

$$E[\mathbf{X}] = \mu \mathbf{1}_n = \begin{pmatrix} \mu \\ \vdots \\ \mu \end{pmatrix}, \text{Cov}(\mathbf{X}) = \sigma^2 \mathbf{I}_n = \begin{bmatrix} \sigma^2 & 0 & \dots & 0 \\ 0 & \sigma^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma^2 \end{bmatrix}$$

# A Random Vector From a Multivariate Distribution

- Let  $\mathbf{X}_{p \times 1}$  be a random vector from a multivariate distribution with mean vector  $\boldsymbol{\mu}_{p \times 1}$  and covariance matrix  $\boldsymbol{\Sigma}_{p \times p}$ .
- In other words,
  - $E(\mathbf{X}) = \boldsymbol{\mu}$
  - $Cov(\mathbf{X}) = \boldsymbol{\Sigma}$ .

# A Random Sample From a Multivariate Distribution

- Consider a random sample  $\mathbf{X}_1, \dots, \mathbf{X}_n$  from a multivariate distribution with mean vector  $\boldsymbol{\mu}_{p \times 1}$  and covariance matrix  $\boldsymbol{\Sigma}_{p \times p}$ .
- We often stack the random vectors to form an  $n \times p$  matrix:

$$\mathbf{X}_{n \times p} = \begin{pmatrix} \mathbf{X}_1^T \\ \vdots \\ \mathbf{X}_n^T \end{pmatrix}$$



# A Random Sample From a Multivariate Distribution: Sample Mean Vector

- Sample mean vector is

$$\bar{\mathbf{X}} = \frac{1}{n} \sum_{i=1}^n \mathbf{x}_i = \left( \frac{1}{n} \mathbf{1}_n^T \mathbf{X} \right)^T$$

It is a random vector with

- mean vector  $E[\bar{\mathbf{X}}] = \boldsymbol{\mu}$ , i.e., the sample mean vector is unbiased for the population mean vector.  $\bar{\mathbf{X}}$  can be used to estimate  $\boldsymbol{\mu}$ .
- covariance matrix  $\text{Cov}(\bar{\mathbf{X}}) = \frac{1}{n} \boldsymbol{\Sigma}$

# A Random Sample From a Multivariate Distribution: Sample Covariance Matrix

- The sample covariance matrix is

$$\mathbf{S} = \frac{1}{n-1} \sum_{i=1}^n (\mathbf{x}_i - \bar{\mathbf{x}})(\mathbf{x}_i - \bar{\mathbf{x}})^T$$

- It is unbiased for  $\mathbf{\Sigma}$ , i.e.,  $E[\mathbf{S}] = \mathbf{\Sigma}$ .
- We showed that

$$\mathbf{S} = \frac{1}{n-1} \mathbf{X}^T \mathbf{C} \mathbf{X}$$

where  $\mathbf{C}_{n \times n} = \mathbf{I} - \frac{1}{n} \mathbf{J} = \mathbf{I} - \frac{1}{n} \mathbf{1}\mathbf{1}^T$

- This expression is helpful when we derive the distribution of  $\mathbf{S}$ .

## Section 2

# Linear Combination of a Random Vector

## Definition of a Linear Combination of a Random Vector

- Let  $\mathbf{X}_{p \times 1} = (X_1, \dots, X_n)^T$  be a  $p$ -dimensional random vector with mean vector  $\boldsymbol{\mu}$  and covariance matrix  $\boldsymbol{\Sigma}$ .
- Consider a linear combination of the form:

$$Y = \mathbf{a}^T \mathbf{X} = \sum_{i=1}^p a_i X_i$$

where  $\mathbf{a}$  is a  $p$ -dimensional constant vector.

- Note  $Y = \mathbf{a}^T \mathbf{X} = \mathbf{X}^T \mathbf{a}$ , both gives the same univariate random variable  $Y$ .
- E.g.,  $\mathbf{X} = (X_1, X_2, X_3)^T$ ,  $\mathbf{a} = (1/3, 1/3, 1/3)^T$ . Then

$$Y = \mathbf{a}^T \mathbf{X} = \frac{1}{3}(X_1 + X_2 + X_3)$$

# Mean of $Y = \mathbf{a}^T \mathbf{X}$

- The mean of  $Y$  can be expressed as:

$$\begin{aligned} E(Y) &= E(\mathbf{a}^T \mathbf{X}) \\ &= \mathbf{a}^T E(\mathbf{X}) \\ &= \mathbf{a}^T \boldsymbol{\mu} \end{aligned}$$

- Intuitively, the mean of  $Y$  is a weighted average of the components of  $\mathbf{X}$ , with weights given by the corresponding components of  $\mathbf{a}$ .

# Variance of $Y$

- $Var(Y) = \mathbf{a}^T \mathbf{\Sigma} \mathbf{a}$

Proof: Because  $Y - EY$  is univariate,  $Y - EY = (Y - EY)^T$ .  
Therefore,

$$\begin{aligned} Var(Y) &= E[(Y - EY)^2] = E[(Y - EY)(Y - EY)^T] \\ &= E[(\mathbf{a}^T \mathbf{X} - \mathbf{a}^T \boldsymbol{\mu})(\mathbf{a}^T \mathbf{X} - \mathbf{a}^T \boldsymbol{\mu})^T] = E[\mathbf{a}^T (\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})^T \mathbf{a}] \\ &= \mathbf{a}^T E[(\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})^T] \mathbf{a} = \mathbf{a}^T Cov(\mathbf{X}) \mathbf{a} = \mathbf{a}^T \mathbf{\Sigma} \mathbf{a} \end{aligned}$$

The last step is due to the definition of covariance matrix.

- The variance of  $Y$  depends on the covariance structure of  $\mathbf{X}$ , as well as the weights given by  $\mathbf{a}$ .

# Linear Combinations of Iris Setosa Features

- Recall that for the iris setosa,  $\mathbf{X}$  is  $50 \times 4$ .
- Consider a linear combination of the features  $Y = \mathbf{X}b$ , where

$$b = \begin{pmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{pmatrix}$$

- $Yb$  is a  $50 \times 1$  vector, with the  $i$ th row be the average of the four features of the  $i$ th iris setosa flower. To see this

# Linear Combinations of Iris Setosa Features

$$\begin{aligned} Y = Xb &= \begin{pmatrix} X_1^T \\ \vdots \\ X_n^T \end{pmatrix} b = \begin{pmatrix} X_1^T b \\ \vdots \\ X_n^T b \end{pmatrix} = \begin{pmatrix} b^T X_1 \\ \vdots \\ b^T X_n \end{pmatrix} \\ &= \begin{pmatrix} \frac{x_{11} + x_{12} + x_{13} + x_{14}}{4} \\ \vdots \\ \frac{x_{n1} + x_{n2} + x_{n3} + x_{n4}}{4} \end{pmatrix} \end{aligned}$$



## Linear Combinations of Iris Setosa Features: sample mean

```
setosa=as.matrix(iris[iris$Species=="setosa", 1:4])
sample.meanvec=matrix(colMeans(setosa), 4, 1)
rownames(sample.meanvec)=colnames(setosa)
colnames(sample.meanvec)="mean"
b=matrix(1/4, 4, 1)
Y=setosa%*%b
#sample mean of Y: the following two results are the same
mean(Y)
```

```
## [1] 2.5355
```

```
t(b)%*%sample.meanvec
```

```
##          mean
## [1,] 2.5355
```

## Linear Combinations of Iris Setosa Features: sample variance

*#sample variance of Y: the following two results are the same*

```
var(Y)
```

```
##           [,1]  
## [1,] 0.03844617
```

```
t(b)%*%cov(setosa)%*%b
```

```
##           [,1]  
## [1,] 0.03844617
```

## Section 3

### A Simulated Study

# Daily Intake of Protein

- This is a simulated data set
- For adults, the recommended range of daily protein intake is between 0.8 g/kg and 1.8 g/kg of body weight
- 60 observations
- 4 sources of proteins
  - meat
  - dairy
  - vegetables / nuts / tofu
  - other

# Choose Mean Vector and Covariance Matrix

- The multivariate distribution has

- mean vector

$$\boldsymbol{\mu} = (24, 16, 8, 8)^T$$

- covariane matrix

$$\boldsymbol{\Sigma} = \begin{pmatrix} 1.3 & 0.3 & 0.3 & 0.3 \\ 0.3 & 1.3 & 0.3 & 0.3 \\ 0.3 & 0.3 & 1.3 & 0.3 \\ 0.3 & 0.3 & 0.3 & 1.3 \end{pmatrix}$$

## Define Mean Vector and Covariance Matrix in R

```
#the library "MASS" is required  
library(MASS)  
my.cov=4*(diag(4) + 0.3* rep(1,4)%o%rep(1,4))  
eigen(my.cov)#to check whether the cov matrix is p.d.
```

```
## eigen() decomposition  
## $values  
## [1] 8.8 4.0 4.0 4.0  
##  
## $vectors  
##      [,1]      [,2]      [,3]      [,4]  
## [1,] -0.5  0.8660254  0.0000000  0.0000000  
## [2,] -0.5 -0.2886751 -0.5773503 -0.5773503  
## [3,] -0.5 -0.2886751 -0.2113249  0.7886751  
## [4,] -0.5 -0.2886751  0.7886751 -0.2113249
```

# Simulate A Random Sample

```
set.seed(1)
x=mvrnorm(n, mu=my.mean, Sigma=my.cov)
dim(x)
```

```
## [1] 60  4
```

```
protein=as.matrix(data.frame(meat=x[,1],dairy=x[,2],
                             veg=x[,3], other=x[,4]))
```

# The simulated data

```
protein
```

##		meat	dairy	veg	other
##	[1,]	29.08891	17.54865	5.814221	7.264953
##	[2,]	23.65965	13.06336	8.734581	9.452868
##	[3,]	26.43410	16.83504	9.278807	8.409798
##	[4,]	21.68232	15.51922	3.379171	5.954558
##	[5,]	22.22387	15.45446	8.804571	7.562144
##	[6,]	25.54395	16.46835	8.556332	10.299174
##	[7,]	20.15075	14.71290	10.660378	7.584075
##	[8,]	25.44330	14.98680	4.866275	6.323171
##	[9,]	23.41142	16.34138	6.667006	6.164109
##	[10,]	28.21604	16.64242	5.874860	7.078538
##	[11,]	22.58127	13.61817	5.178349	5.652878
##	[12,]	22.19211	16.04745	8.714666	6.732854
##	[13,]	25.07026	16.80008	7.180086	9.716474



# Sample Mean and Sample Covariance

```
xbar=matrix(colMeans(protein), 4, 1)
t(xbar)
```

```
##           [,1]      [,2]      [,3]      [,4]
## [1,] 24.03403 15.92836 7.66049 7.738634
```

```
S=cov(protein)
S
```

```
##           meat      dairy      veg      other
## meat  4.2956426 0.8150757 1.1294478 0.5532420
## dairy 0.8150757 4.4052993 0.3497889 0.2337300
## veg   1.1294478 0.3497889 5.1705794 0.5897121
## other 0.5532420 0.2337300 0.5897121 4.5287293
```

# Estimation

- An unbiased estimator of  $\boldsymbol{\mu}$  is the sample mean vector, i.e.,  $\hat{\boldsymbol{\mu}} = \bar{\mathbf{X}}$ .
- An unbiased estimator of  $\boldsymbol{\Sigma}$  is the sample covariance matrix  $\mathbf{S}$ , i.e.,  $\hat{\boldsymbol{\Sigma}} = \mathbf{S}$
- We have shown that  $Var(\bar{\mathbf{X}}) = \frac{1}{n}\boldsymbol{\Sigma}$ , where  $n = 60$ .
- We can estimate it by

$$\hat{Var}(\bar{\mathbf{X}}) = \frac{1}{60}\mathbf{S}$$

# Linear Functions/Combinations: Three Questions

- Q1: Construct a large-sample (approximate) C.I. for protein from meat. In other words, the parameter of interest is  $\mu_1$ .
- Q2: Construct a large-sample C.I. for the total protein intake
- Q3: Construct a large-sample C.I. for the difference of protein intake between from meat and from vegetable

# Linear Functions/Combinations: Question 1

- Q1: Construct a large-sample (approximate) C.I. for protein from meat. In other words, the parameter of interest is  $\mu_1$ .
- Estimate  $\bar{X}_1 = 24.0$ .
- We need compute the standard error (s.e.) of  $\bar{X}_1$ , which is defined as  $se(\bar{X}_1) = \sqrt{\hat{var}(\bar{X}_1)}$
- Two ways to compute the s.e.,
  - 1  $se(\bar{X}_1) = \sqrt{4.2956/60} = 0.27$
  - 2 The calculation can also be done by noticing that  $\bar{X}_1$  is a linear combination of  $\bar{\mathbf{X}}$ :  $\bar{X}_1 = \mathbf{a}^T \bar{\mathbf{X}}$ , where  $\mathbf{a}^T = (1, 0, 0, 0)$ . Thus,

$$\hat{var}(\bar{X}_1) = \mathbf{a}^T \frac{\mathbf{S}}{60} \mathbf{a}$$

## Linear Functions/Combinations: Question 2

- Q2: Construct a large-sample C.I. for the total protein intake
- The parameter of interest is  $\mu_1 + \mu_2 + \mu_3 + \mu_4 = \mathbf{a}^T \boldsymbol{\mu}$ , where  $\mathbf{a} = (1, 1, 1, 1)^T$ .
- Estimate:  $\mathbf{a}^T \bar{\mathbf{X}}$
- Standard error:  $\sqrt{\mathbf{a}^T \frac{\mathbf{S}}{n} \mathbf{a}}$

## Linear Functions/Combinations: Question 3

- Q3: Construct a large-sample C.I. for the difference of protein intake between from meat and from vegetable
- The parameter of interest is  $\mu_1 - \mu_3 = \mathbf{a}^T \boldsymbol{\mu}$ , where  $\mathbf{a} = (1, 0, -1, 0)^T$ .
- Estimate:  $\mathbf{a}^T \bar{\mathbf{X}}$
- Standard error:  $\sqrt{\mathbf{a}^T \frac{\mathbf{S}}{n} \mathbf{a}}$

# Linear Functions/Combinations: Question 1 (continued)

- R code to compute using the above two ways

```
# Method 1
```

```
sqrt(S[1,1]/60)
```

```
## [1] 0.2675706
```

```
# Method 2
```

```
a=matrix(c(1,0,0,0),4,1)
```

```
sqrt(t(a)%*%S%*%a/60)
```

```
##           [,1]
```

```
## [1,] 0.2675706
```

- Both methods give  $s.e.(\bar{X}_1) = 0.27$
- An approximate 95% C.I. for  $\mu_1$  is  $24.0 \pm 1.96 * 0.27$

## Linear Functions/Combinations: Question 2 (continued)

```
a=matrix(1,4,1)
t(a)%*% xbar #estimate
```

```
##           [,1]
## [1,] 55.36152
```

```
sqrt(t(a)%*%S%*%a/60) #standard error
```

```
##           [,1]
## [1,] 0.6550095
```

```
#a large-sample 95% C.I.
```

```
c(t(a)%*% xbar- 1.96*sqrt(t(a)%*%S%*%a/60),
  t(a)%*% xbar+ 1.96*sqrt(t(a)%*%S%*%a/60))
```

```
## [1] 54.07770 56.64534
```



## Linear Functions/Combinations: Question 3 (continued)

```
a=matrix(c(1,0,-1,0),4,1)
t(a)%*% xbar #estimate
```

```
##           [,1]
## [1,] 16.37354
```

```
sqrt(t(a)%*%S%*%a/60) #standard error
```

```
##           [,1]
## [1,] 0.3465864
```

```
#a large-sample 95% C.I.
```

```
c(t(a)%*% xbar- 1.96*sqrt(t(a)%*%S%*%a/60),
  t(a)%*% xbar+ 1.96*sqrt(t(a)%*%S%*%a/60))
```

```
## [1] 15.69423 17.05285
```

## Section 4

# Generalized Variance

# Why Do We Need Generalized Variance?

- For a random variable (i.e., univariate), we quantify dispersion using variance and standard deviation.
- For a random vector (i.e., multivariate), we use its covariance matrix to quantify the dispersion as well as the relationships between different variables / features.
  - The dispersion information is represented by a matrix, which has  $p(p+1)/2$  unique parameters

# What is Generalized Variance?

- It is attempting to have a scalar summary (i.e., a single number) to quantify the “total” amount of dispersion for a multivariate distribution
- Generalized variance
  - Provides an overall measure of dispersion of the multivariate distribution
  - One choice is the determinant:  $|\Sigma|$ .
  - A larger determinant indicates a greater degree of dispersion

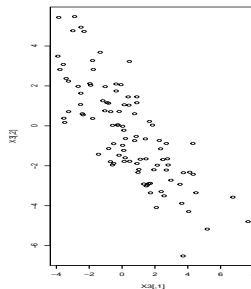
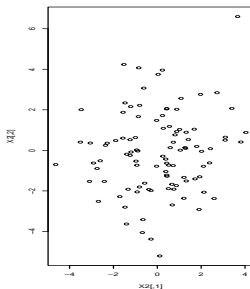
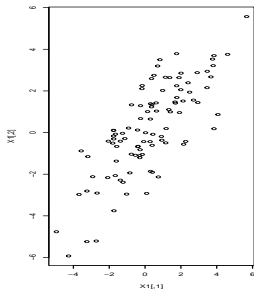
## Generalized Variance: An Example

$$\Sigma_1 = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}, \Sigma_2 = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \Sigma_3 = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}$$

```
Sigma1=matrix(c(5,4,4,5), 2,2)
X1=mvrnorm(100, mu=c(0,0), Sigma=Sigma1)
Sigma2=matrix(c(3,0,0,3), 2,2)
X2=mvrnorm(100, mu=c(0,0), Sigma=Sigma2)
Sigma3=matrix(c(5,-4,-4,5), 2,2)
X3=mvrnorm(100, mu=c(0,0), Sigma=Sigma3)
```

# Generalized Variance: An Example

```
par(mfrow=c(1,3))  
plot(X1); plot(X2); plot(X3)
```



## Generalized Variance might (NOT) be useful

- In the example above,  $|\Sigma_1| = |\Sigma_2| = |\Sigma_3| = 9!$
- $|\Sigma|$  does not tell the orientations.
- $|\Sigma|$  is useful to compare two patterns when they have nearly the same orientations.
- The generalized variance does not capture all the information contained in the covariance matrix.
- The eigenvalues provide more information than the determinant - Principal Component Analysis!

## How to Interpret A Covariance Matrix? - the 2D Situation

<https://datascienceplus.com/understanding-the-covariance-matrix/>



## Section 5

Normal Distributions: univariate, multivariate,  
matrix normal distributions

# The Big Picture: Univariate vs Multivariate

- **Review:** A random sample, denoted by  $X_1, \dots, X_n$ , from a (univariate) normal distribution  $N(\mu, \sigma^2)$ 
  - What are the distributions of  $\bar{X}, s^2$ ? What useful statistics can be constructed?
- **New material:** A random sample, denoted by  $\mathbf{X}_1, \dots, \mathbf{X}_n$ , from a multivariate normal distribution  $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ 
  - What are the distributions of  $\bar{\mathbf{X}}, \mathbf{S}$ ? What useful statistics can be constructed?

# The Big Picture: Univariate

- A random sample, denoted by  $X_1, \dots, X_n$ , from a (univariate) normal distribution  $N(\mu, \sigma^2)$
- Let  $\mathbf{X}_{n \times 1} = (X_1, \dots, X_n)^T$ . It is random vector with a multivariate normal distribution, i.e.,

$$\mathbf{X}_{n \times 1} = (X_1, \dots, X_n)^T \sim \mathbf{N}(\mu \mathbf{1}, \sigma^2 \mathbf{I})$$

- 1  $\bar{X} \sim N(\mu, \sigma^2/n)$
- 2  $\frac{(n-1)s^2}{\sigma^2} \sim \chi_{n-1}^2$
- 3 Independence between  $\bar{X}$  and  $s^2$ .
- 4 a t-statistic is

$$\frac{\frac{\bar{X} - \mu}{\sqrt{\sigma^2/n}}}{\sqrt{\frac{(n-1)s^2/\sigma^2}{n-1}}} = \frac{\sqrt{n}(\bar{X} - \mu)}{s}$$

It follows the t-distribution with  $n-1$  degrees of freedom, denoted by  $t_{n-1}$ .

# The Big Picture: Multivariate

- A random sample  $\mathbf{X}_1, \dots, \mathbf{X}_n$  from a multivariate normal distribution  $\mathbf{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ .
- Let

$$\mathbf{X}_{n \times p} = \begin{pmatrix} \mathbf{x}_1^T \\ \vdots \\ \mathbf{x}_n^T \end{pmatrix}$$

$\mathbf{X}$  follows a matrix normal distribution.

- 1 Sample mean vector follows a multivariate normal, i.e.,  $\bar{\mathbf{X}} \sim \mathbf{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}/n)$
- 2 Sample covariance matrix  $(n-1)\mathbf{S}$  follows a Wishart distribution, i.e.,  $(n-1)\mathbf{S} \sim \text{Wishart}_p(n-1, \boldsymbol{\Sigma})$
- 3 Independence between  $\bar{\mathbf{X}}$  and  $\mathbf{S}$ .
- 4 Hoetelling's  $T^2$ :  $T^2 = (\bar{\mathbf{X}} - \boldsymbol{\mu})^T \left(\frac{\mathbf{S}}{n}\right)^{-1} (\bar{\mathbf{X}} - \boldsymbol{\mu})$