$\boxed{14}$ マルコフ連鎖

マルコフ性

$X \leftarrow$ マルコフ連鎖

$$P_m^{(n)}(x, B) = P(X_{n+m} \in B \mid X_n = x)$$

mステップ推移確率

斉時的有限マルコフ連鎖

確率行列

$$P_m(i, j) = P(X_{n+m} = j \mid X_n = i)$$

$$Q(m) = \begin{pmatrix} P_m(1,1) & \cdots & P_m(1,N) \\ \vdots & & \vdots \\ P_m(N,1) & \cdots & P_m(N,N) \end{pmatrix}$$

和が1

mステップ推移確率行列

S

1	2	...	n-1	n

図 推移確率の性質　　$P_n(k) = P(X_n = k)$

$$\pi_n = \underbrace{\left(P_n(1),\ P_n(2),\ \cdots\ P_n(N) \right)}$$

← N次元ベクトル

状態確率ベクトル

- π_0 ← 初期分布
- 和は1. ← 確率ベクトル

図 定常分布

$\pi := \lim_{n \to \infty} \pi_n$

$\pi = \pi Q$

Xの定常分布

図 (1) $Q(m+l) = Q(m) \cdot Q(l) = Q^m \cdot Q^l$

(2) $\pi_n = \pi_0 \cdot Q^n$

例1

$$P(i,j) = \begin{cases} q & (j=1) \\ 1-q & (j=2) \end{cases} \qquad Q = \begin{pmatrix} q & 1-q \\ q & 1-q \end{pmatrix}$$

例2

$$(a,b,c) = (a,b,c)\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

$$= \left(\frac{1}{2}b+\frac{1}{2}c,\ \frac{1}{2}a+\frac{1}{2}c,\ \frac{1}{2}a+\frac{1}{2}b\right)$$

$2a = b+c$ $\quad a-b=b-a$ $\quad \underbrace{a+b+c=1}_{\text{確率分布だから!}}$ $\quad \underbrace{(a,b,c)=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)}_{\text{離散一様分布}}$

$2b = a+c$ $\quad a=b$

$2c = a+b$ $\quad b=c$

$c=a$

$$(a,b,c) = (a,b,c)\begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix}$$

吸収状態

$$= \left(\frac{1}{3}a+\frac{1}{3}b,\ \frac{2}{3}a+\frac{1}{3}b,\ \frac{1}{3}b+c\right)$$

$b=0 \ - \ a=0 \qquad c=1$

④有限マルコフ連鎖のパラメータ推定.

尤度方程式の解.

$$l_n(\theta) = \sum_{j=1}^{n} \log P_\theta(x_{j-1}, x_j)$$

$$\frac{\partial}{\partial \theta} l_n(\hat{\theta}) = 0$$

$\boxed{\text{例3}}$

$A \to B$

25人　5人

$$\theta = \begin{pmatrix} 1-\theta & \theta & 0 \\ 0.1 & (0.9-\theta)^{49} & \theta \\ 0 & 0.1 & 0.9 \end{pmatrix}$$

1人

10人

$$(1-\theta)^{25} \cdot \theta^{5} \cdot (0.9-\theta)^{49} \cdot \theta \cdot (0.9)^{10}$$

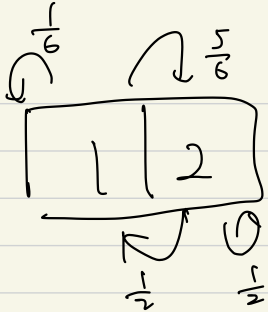
$$\log(\theta) = 25\log(1-\theta) + 6\log\theta + 49\log(0.9-\theta)$$

$$\frac{\log(\theta)}{d\theta} = \frac{-25}{1-\theta} + \frac{6}{\theta} + \frac{-49}{0.9-\theta}$$

$$\Rightarrow \underline{\theta = 0.07}$$

$\boxed{\text{例題}}$

問14.1



(1)

$$Q = \begin{pmatrix} \frac{1}{6}, & \frac{5}{6} \\ \frac{1}{2}, & \frac{1}{2} \end{pmatrix}$$

[2]

$$\pi = \pi Q$$

$$\pi \begin{pmatrix} -\frac{5}{6} & \frac{5}{6} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = 0$$

$$\begin{cases} a+b=1 \\ -\frac{5}{6}a+\frac{1}{2}b=0 \end{cases}$$

$$a+b=1$$
$$-\frac{5}{3}a+b=0$$
$$\frac{8}{3}a=1 \qquad a=\frac{3}{8},\ b=\frac{5}{8}$$

$$\lim_{n\to\infty} \pi_n = \left(\frac{3}{8}, \frac{5}{8}\right)$$

問14.2

a_1 a_2 a_3

状態

1	2	3

(1) $$Q = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{9} & \frac{2}{9} & \frac{2}{3} \end{pmatrix}$$

[2] $$\pi_1 = \pi_0 Q = (0.0.1)Q = \left(\frac{1}{9}, \frac{2}{9}, \frac{2}{3}\right)$$

$$\pi_2 = \pi_0 Q^2 = \left(\frac{1}{9}, \frac{2}{9}, \frac{2}{3}\right)Q = \left(\frac{4}{27}, \frac{8}{27}, \frac{5}{9}\right)$$

[3] $\pi = \pi Q$ となる

$$\pi \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & -\frac{1}{2} & \frac{1}{3} \\ \frac{1}{9} & \frac{2}{9} & -\frac{1}{3} \end{pmatrix}$$

a.b.c

$$-\frac{2}{3}a + \frac{1}{6}b + \frac{1}{9}c = 0$$

$$\frac{1}{3}a - \frac{1}{2}b + \frac{2}{9}c = 0$$

$$\frac{1}{3}a + \frac{1}{3}b - \frac{1}{3}c = 0$$

$$a+b+c=1$$
$$a+b-c=0$$
$$-12a+3b+2c=0$$

$$2a+2b=1$$
$$-10a+5b=0$$
$$-2a+b=0$$

$$3b=1$$

$$\underbrace{b=\frac{1}{3},\quad a=\frac{1}{6},\quad c=\frac{1}{2}}$$

$$\pi = \left(\frac{1}{6}, \frac{1}{3}, \frac{1}{2}\right)$$

$\boxed{\text{問14.3}}$

家　　1本

職　　1本

状態空間 $S = \{0, 1, 2\}$

$\pi_0 = (0, 1, 0)$

$$P_{ij} = P(X_{n+1} = j-1 \mid X_n = i-1)$$

$$Q = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1-\theta & \theta \\ 1-\theta & \theta & 0 \end{pmatrix}$$

$$1 \cdot (1-\theta) \cdot \theta \cdot (1-\theta) \cdot 1 \quad \theta \cdot \underline{(1-\theta) \cdot (1-\theta)}$$

$$\theta = \frac{1}{3}$$

[3] $\pi = \pi Q$ $\quad a = \dfrac{(1-\theta)}{(3-\theta)} = \dfrac{\frac{2}{3}}{\frac{8}{3}} = \dfrac{1}{4}$