$\boxed{2}$ 確率分布と母関数

周辺確率関数

$\downarrow$

累積分布関数 $F(x) = P(X \leq x)$

同時確率関数 $F(x, y) = \sum_{\substack{x' \leq x \\ y' \leq y}} p(x', y')$

$p_X(x) = \sum_y p(x, y)$

$\downarrow$

周辺分布

条件付き確率密度関数

$$f_{Y|X}(y|x) = \frac{f(x, y)}{f_X(x)}$$

例1

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\, dy = \underline{\frac{2}{\pi}\sqrt{1-x^2}}$$

$$f_{Y|X}(y|x) = \underline{\frac{1}{2\sqrt{1-x^2}}}$$

確率関数

確率密度関数 …… { 性質を知りたい!

モーメント母関数 , 確率母関数

$S = e^{\theta}$

$m(\theta) = E[e^{\theta x}]$

$= G(e^{\theta})$

$m'(0) = E[x]$

$m''(0) = E[x^2]$

原点まわりのモーメント

$E[X^k]$

$\phi(t) = m(it)$

特性関数

$G(s) = E[S^x] = \sum_x S^x p(x)$

$G'(1) = E[x]$

$G''(1) = E[x(x-1)]$

$V[x] = E[x^2] - E[x]^2$

$= G''(1) + G'(1) - (G'(1))^2$

★ {
- 確率分布と1対1対応
- 独立な変数の和が母関数の積に対応

例2

(1) $G(s) = \sum_x s^x p(x) = s^1 \cdot p + s^0 \cdot (1-p) = 1 + p(s-1)$

ベルヌーイ $\Rightarrow$ 二項分布 $\therefore$ $\underline{(1+p(s-1))^n}$

独立なn個の

ベルヌーイの和

例題 問2.1

(1) $\int_0^1\int_0^1 c(x+y)\,dx\,dy = \int_0^1 \frac{1}{2}c + cy\,dy = \left[\frac{1}{2}cy + \frac{1}{2}cy^2\right]_0^1 = \underline{c = 1}$

(2) $f_X(x) = \int_0^1 (x+y)\,dy = \left[xy + \frac{1}{2}y^2\right]_0^1 = \underline{x + \frac{1}{2}}$

(3) $f_{Y|X}(y|x) = \frac{f(x,y)}{f(x)} = \underline{\frac{x+y}{x+\frac{1}{2}}}$

問2.2

$$= E[s^x]$$

$$p+$$

$$G(s) = \sum_x s^x p(x) = s\,p(1-p) + s^2 p\cdot(1-p)^2 + s^3 p\cdot(1-p)^3 + \cdots$$

$$= p\cdot\frac{(\cancel{s(1-p)}-1)}{s(1-p)-1} = \frac{p}{1-s+sp}$$

$$E[x] = G'(1) = \qquad G''(1) = E[x^2] - E[x]$$

$$V[x] = E[x^2] - E[x]^2 = G''(s) + G'(s) - (G'(s))^2$$

$$G'(s) = \frac{p\cdot -1\cdot(p-1)}{(1-s+sp)^2} = \frac{1-p}{p}$$

$$G''(s) = \frac{+p(p-1)\cdot 2(\cancel{1-s+sp})(p-1)}{(1-s+sp)^{3}}$$

$s=1$を代入

$s=1$を代入 $G''(s) = \frac{2(1-p)^2}{p^2}$

$$E[x] = \frac{1-p}{p}$$

$$V[x] = \frac{2(1-p)^2}{p^2} + \frac{p(1-p)}{p^2} - \frac{(1-p)^2}{p^2}$$

$$= \frac{1-p}{p^2}$$

期待値・分散が求められる

問2.3

$\lambda > \theta$ ← 収束するように

$$m(\theta) = G[e^{\theta}] = \int e^{\theta x} \cdot f(x)\,dx = \lambda \int_0^{\infty} e^{(\theta-\lambda)x}\,dx$$

$$= \frac{\lambda}{\theta-\lambda}\left[e^{(\theta-\lambda)x} \right]_0^{\infty} = \frac{\lambda}{\lambda-\theta}$$

$E[x] = m'(0)$

$$m'(\theta) = \frac{\lambda}{(\lambda-\theta)^2} \qquad m'(0) = \frac{1}{\lambda}$$

$V[x] = E[x^2] - E[x]^2$

$= m''(0) - m'(0)^2$

$$m''(\theta) = \frac{+\lambda \cdot 2(\lambda-\theta)}{(\lambda-\theta)^4} \qquad m''(0) = \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2}$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$