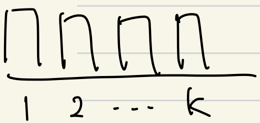
5 離散型分布

○ 一様分布



$$E[X] = \sum_x x p(x) = 1\cdot\frac{1}{k} + 2\cdot\frac{1}{k} \cdots + \frac{k}{k} = \frac{1}{k}\cdot\frac{k(k+1)}{2} = \frac{k+1}{2}$$

$$E[X^2] = \frac{1}{k}(1+4+9\cdots+k^2) = \frac{1}{k}\cdot\frac{k(k+1)(2k+1)}{6} = \frac{(k+1)(2k+1)}{6}$$

$$V[X] = \frac{2(k+1)(2k+1)}{12} - \frac{3(k+1)^2}{12} = \frac{(k+1)(4k+2-3k-3)}{12} = \frac{k^2-1}{12}$$

○ ベルヌーイ分布　成功確率 p　$Bin(1, p)$

$$P(X=x) = p^x(1-p)^{1-x}$$

$$E[X] = \sum_x x\, p(x) = 1\cdot p + 0\cdot(1-p) = p$$

$$E[X^2] = \sum_x x^2 p(x) = p$$

$$V[X] = p - p^2 = pq$$

○ 二項分布　$Bin(n, p)$　$Y = X_1 + X_2 + \cdots + X_n$

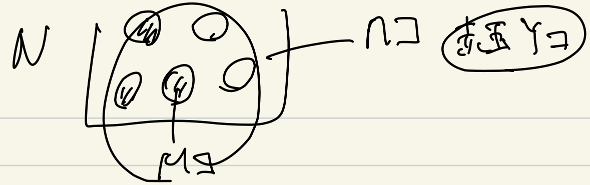
$$E[s^Y] = E[s^{X_1} s^{X_2} \cdots] = E[s^{X_1}] \times E[s^{X_2}] \cdots = (ps+q)^n$$

$$E[Y] = E[X_1] + E[X_2] + \cdots + E[X_n] = np$$

$$V[Y] = npq$$

再生性

○超幾何分布



○復元抽出 $\mathrm{Bin}\left(n, \frac{M}{N}\right)$

○非復元抽出 $\mathrm{HG}(N, M, n)$ $\quad P(Y=y) = \dfrac{{}_M C_y \times {}_{N-M} C_{n-y}}{{}_N C_n}$

○ポアソン分布 $\mathrm{Po}(\lambda)$: $P(Y=y) = \dfrac{\lambda^y}{y!} e^{-\lambda}$

$\mathrm{Bin}(n, p)$

$np = \lambda$ と固定して

$\begin{cases} n \to \infty \\ p \to 0 \end{cases}$

確率母関数

$$G(s) = E[s^Y] = \sum_{y=0}^{\infty} s^y \cdot \frac{\lambda^y}{y!} \cdot e^{-\lambda}$$

$$= e^{-\lambda} \cdot \sum_{y=0}^{\infty} \frac{(s\lambda)^y}{y!}$$

$$\boxed{e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}}$$

$$= e^{-\lambda} \cdot e^{s\lambda} = e^{\lambda(s-1)}$$

よって

$$E[Y] = G'(1)$$

$$E[Y(Y-1)] = G''(1)$$

$$V[Y] = E[Y^2] - E[Y]^2$$

$$= G''(1) + G'(1) - \{G'(1)\}^2$$

$$= \lambda^2 + \lambda - \lambda^2 = \lambda$$

$$G'(s) = \lambda \cdot e^{\lambda(s-1)}$$

$$G'(1) = \lambda$$

$$G''(s) = \lambda^2 \cdot e^{\lambda(s-1)}$$

$$G''(1) = \lambda^2$$

再生性 成り立つ

○ **幾何分布**　$Geo(p)$　$P(X=x)=(1-p)^x \cdot p$　q

$$G(s)=E[s^X]=\sum_x s^x \cdot p(x)=p+s(1-p)p+s^2(1-p)^2p\cdots$$

$$=p\cdot\frac{1}{1-sq}=\frac{p}{1-qs}$$

$$G'(s)=E[X\cdot s^{X-1}]=\frac{pq}{(1-qs)^2}\qquad G'(1)=E[X]=\frac{q}{p}$$

$$G''(s)=E[X(X-1)s^{X-2}]=\frac{+pq^2(1-qs)\cdot 2}{((1-qs)^{4})^{3}}=\frac{2pq^2}{(1-q)^3p^3}=\frac{2q^2}{p^2}$$

X^2-X

$$V[X]=\frac{2q^2}{p^2}+\frac{pq}{p^2}-\frac{q^2}{p^2}=\frac{q^2+pq}{p^2}=\frac{q}{p^2}$$

(1) 無記憶性

$X\sim Geo(p)$　$P(X\geq t_1+t_2 \mid X\geq t_1)=P(X\geq t_2)$

$t_1, t_2=0,1,2\cdots$

$P(X \geqq t) = q^t$　　$(\text{左辺}) = \dfrac{q^{t_1+t_2}}{q^{t_1}} = q^{t_2} = (\text{右辺})$

はじめて成功を W とすると　$W = X+1$

定義より $NB(1, P) = Geo(P)$

○負の二項分布　$NB(r, P)$

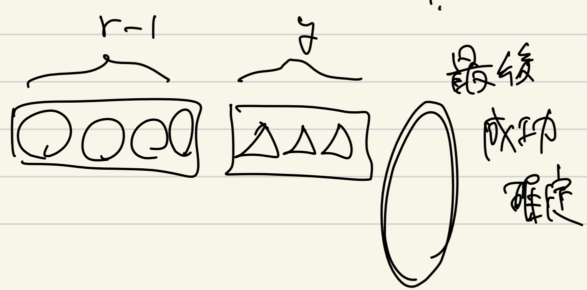
r回成功して、
成功確率pの時の失敗回数 = Y

幾何分布を独立にr個足した分布 → $\begin{cases} E[X] = r \times \dfrac{q}{p} \\ V[X] = r \times \dfrac{q^2}{p} \end{cases}$

$P(Y=y) = {}_rH_y \cdot p^r \cdot q^y$

$\downarrow$

${}_{r+y-1}C_y$



$X_1, X_2 \cdots X_r \sim Geo(p)$ の時. $X_1+X_2 \cdots + X_r \sim NB(r, p)$

$E[X_k] = \frac{q}{p}$ $\quad E[X_1 + \cdots X_r] = r \times \frac{q}{p} = \frac{qr}{p}$ 再生性あり

$V[X_k] = \frac{q}{p^2}$ $\quad V[X_1 + X_2 + \cdots + X_r] = r \times \frac{q}{p^2} = \frac{qr}{p^2}$

多項分布

独立にn回試行する時.

結果jが起こる回数 $Y^{(j)}$ とする.

$Y := (Y^{(1)}, \cdots, Y^{(k)})$ を多項分布

$M(n : p_1 \ldots p_k)$

図

① p_1

② p_2

⋮

ⓚ p_k

↓

$p_1 + p_2 + \cdots + p_k = 1$

$$P(Y^{(1)} = y^{(1)}, \cdots, Y^{(k)} = y^{(k)}) = \frac{n!}{y^{(1)}! \cdots y^{(k)}!} \cdot p_1^{y^{(1)}} \cdots p_k^{y^{(k)}}$$

$$y^{(1)} + \cdots + y^{(k)} = n$$

$n=1$のとき.

$$E[X^{(j)}] = p_j$$

$$V[X^{(j)}] = p_j(1-p_j)$$

$$Cov[X^{(j)}, X^{(j')}] = E[X^{(j)}, X^{(j')}] - E[X^{(j)}] \cdot E[X^{(j')}]$$

$$= 0 - p_j \cdot p_{j'}$$

$$G(s_1, \cdots s_k) = E[s_1^{X^{(1)}} \cdots s_k^{X^{(k)}}] = s_1^1 s_2^0 \cdots p_1 + \cdots = s_1 p_1 + \cdots + s_k p_k$$

$Y = X_1 + X_2 + \cdots + X_n$ と表せられる. よって $E[Y^{(j)}] = n p_j$

$V[Y^{(j)}] = n p_j (1-p_j)$

$Cov[\ ] = -n p_j \cdot p_{j'}$

$G(\ ) = (s_1 p_1 + \cdots s_k p_k)^n$

再生性

$$
\begin{tikzcd}
\boxed{\text{幾何分布}} \arrow[r] & \boxed{\text{負の二項分布}} \arrow[r, leftrightarrow, "\text{反転}"] & \boxed{\text{二項分布}} \arrow[r, "\text{多変量}"] \arrow[d, "\text{極限}\ np=\lambda,\ n\to\infty"] & \boxed{\text{多項分布}} \\
 & & \boxed{\text{ポアソン分布}} &
\end{tikzcd}
$$

例題

問5.1

$$\beta = 1-(1-p)^n$$

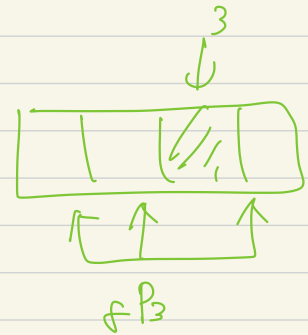
$$(1-p)^n = 0.02$$

$$-np = -3.9$$

$$Y \sim Bin(n, p)$$

$$n = 5000 \times 3.9$$

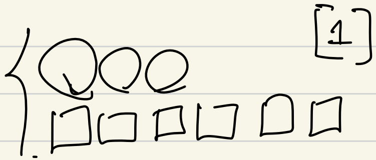
$$= \underline{19500}$$



問5.2

$$P(X=x) = \frac{{}_{40}C_x \cdot {}_{39}C_{25-x}}{{}_{79}C_{25}}$$

問5.3



$$E[X_i^2] = 0^2 \cdot P(X_i=0) + 1^2 \cdot P(X_i=1) = P(X_i=1)$$

$$= \boxed{\frac{{}_8P_3 \times 3}{{}_9P_4}} \quad ? \quad = \underline{\frac{1}{3}}$$

[2]

$$E[X_i X_j] = 1 \cdot \left(P(X_i = 1) \text{ かつ } P(X_j = 1) \right) + \underbrace{0 \cdots}_{0}$$

$$= \frac{{}_7P_2 \cdot {}_3P_2}{{}_9P_4} = \underline{\frac{1}{12}}$$

$$\frac{1}{12} - \frac{1}{9} = -\frac{1}{36}$$

$$E[X_i X_j] - E[X_i]E[X_j]$$

[3] $E[X_i] = \frac{1}{3}$ $\quad V[X_i] = \frac{1}{3} - \frac{1}{9} = \underline{\frac{2}{9}}$

$$V = \frac{1}{16} V[X_1 + X_2 + X_3 + X_4]$$

$$\frac{8}{9} - \frac{3}{9} = \frac{5}{9}$$

$$= \frac{1}{16} \left\{ V[X_1] + V[X_2] + V[X_3] + V[X_4] + \sum_{(12)} \mathrm{cov}[X_i, X_j] \right. = \underline{\frac{5}{144}}$$

問5.4

$X \sim Po(1.5)$

$Y \sim Po(3)$

$X \sim Po(\lambda)$

$P(x) = \dfrac{\lambda^{x}}{x!} e^{-\lambda}$

$E[X] = \lambda$

$V[X] = \lambda$

$\mathrm{Bin}\left(5, \frac{1}{3}\right)$

$\dfrac{a}{a+b}$

$\boxed{ア}$ $1.5 + 3 = \underline{4.5}$

$\boxed{イ}$

$X+Y \sim Po(4.5)$

$$P(X = x \mid X+Y = 5) = \frac{P(X=x,\ Y=5-x)}{P(X+Y=5)} = \frac{\dfrac{(1.5)^{x}}{x!} \cdot \dfrac{(3)^{5-x}}{(5-x)!}}{\dfrac{(4.5)^{5}}{5!} \cdot \cancel{e^{-4.5}}}$$

$$= \frac{5!}{x!(5-x)!} \cdot \frac{(1.5)^{x}(3)^{5-x}}{(4.5)^{x}}$$

${}_5C_x$

$$\left(\frac{a}{a+b}\right)^{x}\left(\frac{b}{a+b}\right)^{5-x}$$

$np = 5 \times \frac{1}{3} = \underline{1.67}$

問 5.5

[1] カード n 種類　k 種類　$k+1$ 種類目　$p_k = \frac{n-k}{n}$　$X \sim Geo(p)$

$\frac{1}{p_k} = \frac{n}{n-k}$

$$\sum_{k=0}^{n-1} \frac{n}{n-k} = \sum_{k=0}^{3} \left(\frac{4}{4-0} + \frac{4}{4-1} + \frac{4}{4-2} + \frac{4}{4-3} \right)$$

$$= 4 \left(\underbrace{\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}}_{\frac{12+6+4+3}{12}} \right) = \frac{25}{3}$$

$n = 5$
$k = 4$

$\frac{5}{1} = 5$

[2] $y = \frac{137}{12}$

$$x = \boxed{\frac{25}{3}} + 5 = \frac{40}{3}$$