Financial Derivatives Final Exam

Yu-Ching Liao

May 7, 2023

1 Question 1

Consider a two-factor stochastic volatility model for a stock:

$$
\begin{aligned}
\frac{dS}{S} &= \mu^S dt + \sigma_t^I dW1_t + \beta\sigma_t^M dW2_t \\
\frac{dS^M}{S^M} &= \mu^M dt + \sigma_t^M dW2_t \\
d\sigma_t^I &= \kappa^I(\bar{\sigma}^I - \sigma_t^I)dt + b^I\sigma_t^I dW3_t \\
d\sigma_t^M &= \kappa^M(\bar{\sigma}^M - \sigma_t^M)dt + b^M\sigma_t^M dW4_t
\end{aligned}
$$

where $dW1$, $dW2$, $dW3$, and $dW4$ are mutually independent Brownian motions. Here S_M is the market portfolio, β is a constant, and the volatility of the stock is $\sigma_t^S = \sqrt{(\sigma_t^I)^2 + (\beta\sigma_t^M)^2}$. The instantaneous correlation of dS and dS_M is then $\rho_t = \frac{\beta\sigma_t^M}{\sigma_t^S}$.

If we have a derivative security, f, whose payoffs are determined by ρ_t what partial differential equation would f's price have to satisfy in a complete market with no arbitrage? (Be sure to define all the notation that you use. Specify any extra assumptions you are making.)

Let's denote the price of the derivative security as $F(S, S_M, \sigma_I, \sigma_M, t)$. In a complete market with no arbitrage, F should satisfy a PDE that ensures the absence of arbitrage opportunities. We start by using Ito's lemma to find the dynamics of the price F:

$$
\begin{aligned}
dF = {} & \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial S}dS + \frac{\partial F}{\partial S_M}dS_M + \frac{\partial F}{\partial \sigma_I}d\sigma_I + \frac{\partial F}{\partial \sigma_M}d\sigma_M \\
& + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}(dS)^2 + \frac{1}{2}\frac{\partial^2 F}{\partial S_M^2}(dS_M)^2 + \frac{1}{2}\frac{\partial^2 F}{\partial \sigma_I^2}(d\sigma_I)^2 + \frac{1}{2}\frac{\partial^2 F}{\partial \sigma_M^2}(d\sigma_M)^2 \\
& + \frac{\partial^2 F}{\partial S \partial S_M}(dS)(dS_M) + \frac{\partial^2 F}{\partial S \partial \sigma_I}(dS)(d\sigma_I) + \frac{\partial^2 F}{\partial S \partial \sigma_M}(dS)(d\sigma_M) \\
& + \frac{\partial^2 F}{\partial S_M \partial \sigma_I}(dS_M)(d\sigma_I) + \frac{\partial^2 F}{\partial S_M \partial \sigma_M}(dS_M)(d\sigma_M) + \frac{\partial^2 F}{\partial \sigma_I \partial \sigma_M}(d\sigma_I)(d\sigma_M)
\end{aligned}
$$

Substitute the given dynamics of dS, dS_M, $d\sigma_I$, and $d\sigma_M$:

$$
\begin{aligned}
dF = {} & \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial S}(\mu^S dt + \sigma_t^I dW1_t + \beta\sigma_t^M dW2_t) + \frac{\partial F}{\partial S_M}(\mu^M dt + \sigma_t^M dW2_t) \\
& + \frac{\partial F}{\partial \sigma_I}(\kappa^I(\bar{\sigma}^I - \sigma_t^I)dt + b^I\sigma_t^I dW3_t) + \frac{\partial F}{\partial \sigma_M}(\kappa^M(\bar{\sigma}^M - \sigma_t^M)dt + b^M\sigma_t^M dW4_t) \\
& + \ldots
\end{aligned}
$$

The absence of arbitrage opportunities implies that F must be a martingale under the risk-neutral measure Q. Therefore, the drift term of dF under Q must be zero. We can then write the risk-neutral PDE for the price $F(S, S_M, \sigma_I, \sigma_M, t)$:

$$
\begin{aligned}
&\frac{\partial F}{\partial t} + \mu^S \frac{\partial F}{\partial S} + \mu^M \frac{\partial F}{\partial S_M} + \kappa^I(\bar{\sigma}^I - \sigma_t^I)\frac{\partial F}{\partial \sigma_I} + \kappa^M(\bar{\sigma}^M - \sigma_t^M)\frac{\partial F}{\partial \sigma_M} \\
&\quad + \frac{1}{2}(\sigma_t^I)^2\frac{\partial^2 F}{\partial S^2} + \frac{1}{2}(\sigma_t^M)^2\frac{\partial^2 F}{\partial S_M^2} + \frac{1}{2}(b^I\sigma_t^I)^2\frac{\partial^2 F}{\partial \sigma_I^2} + \frac{1}{2}(b^M\sigma_t^M)^2\frac{\partial^2 F}{\partial \sigma_M^2} \\
&\quad + \rho_t\sigma_t^S\sigma_t^M\frac{\partial^2 F}{\partial S \partial S_M} = rF
\end{aligned}
$$

where r is the risk-free rate. This PDE represents the no-arbitrage condition for the price $F(S, S_M, \sigma_I, \sigma_M, t)$ of the derivative security f, whose payoffs are determined by the instantaneous correlation ρ_t in a complete market.

2 Question 2

From the simulation, we get the output as shown below:

Breakeven correlation	0.46830141809585857
Profit or loss with 0.10 higher correlation	\$426783.12
Profit or loss with 0.10 lower correlation	\$-1333622.48

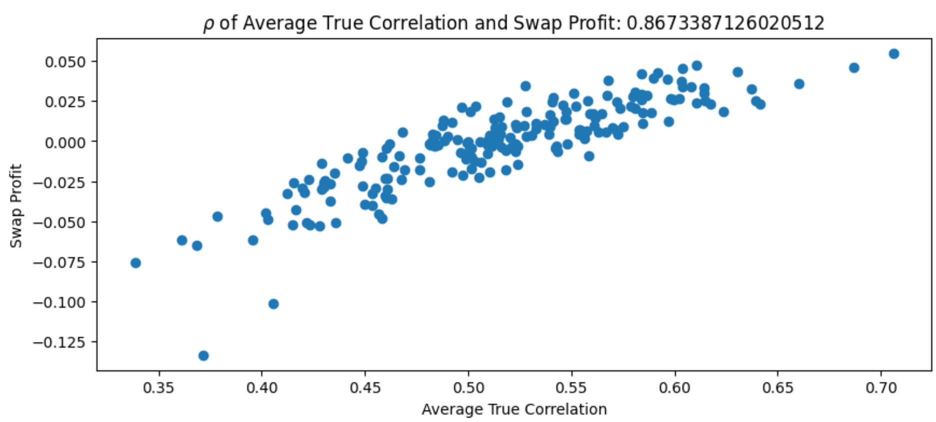

Figure 1: Average True Correlation vs Swap Profit

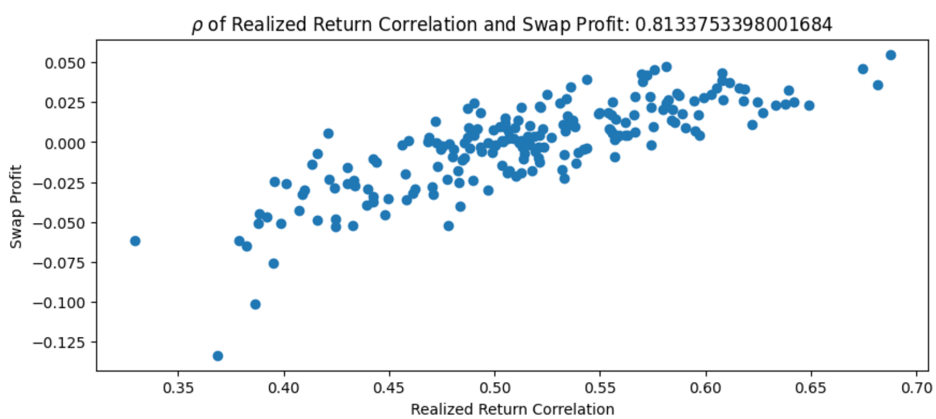

Figure 2: Realized Return Correlation vs Swap Profit

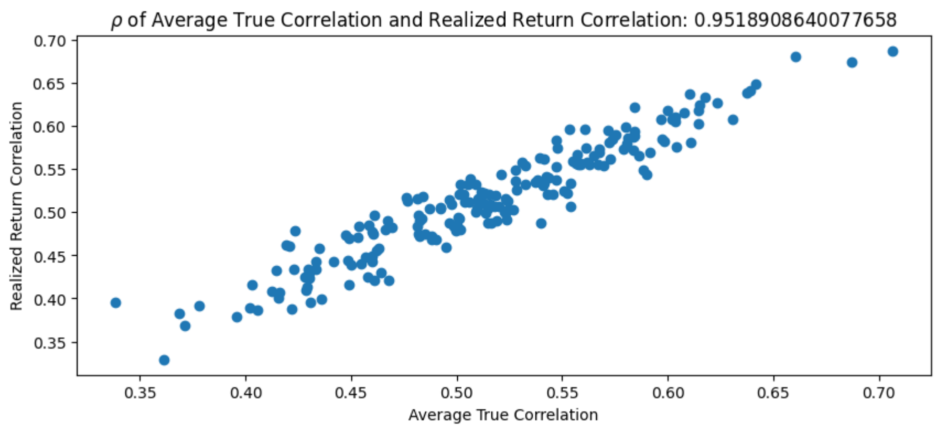

Figure 3: Average True Correlation vs Realized Return Correlation

3 Question 3

Below are the plots of first 5 sample paths:

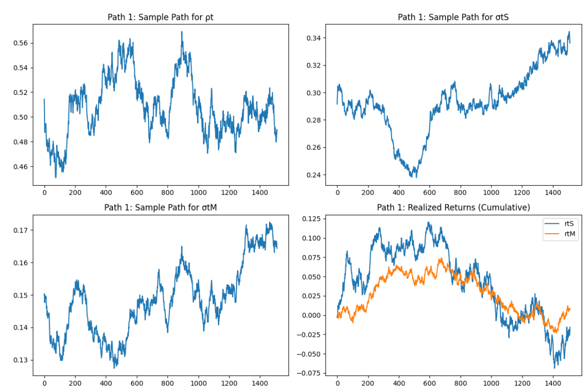

Figure 4: ρ_t, σ_t^S, σ_t^M, r_t^S and r_t^M for Path 1

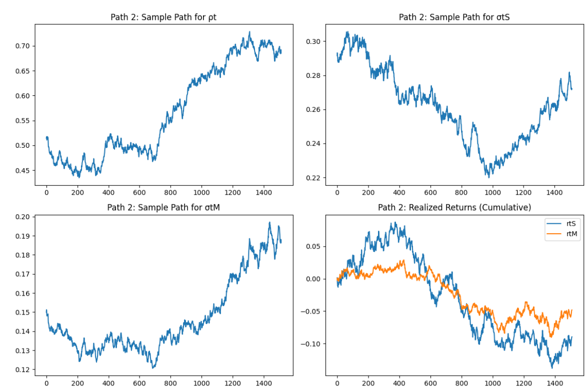

Figure 5: ρ_t, σ_t^S, σ_t^M, r_t^S and r_t^M for Path 2

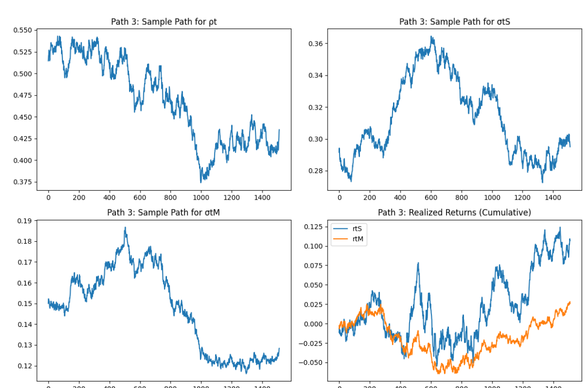

Figure 6: ρ_t, σ_t^S, σ_t^M, r_t^S and r_t^M for Path 3

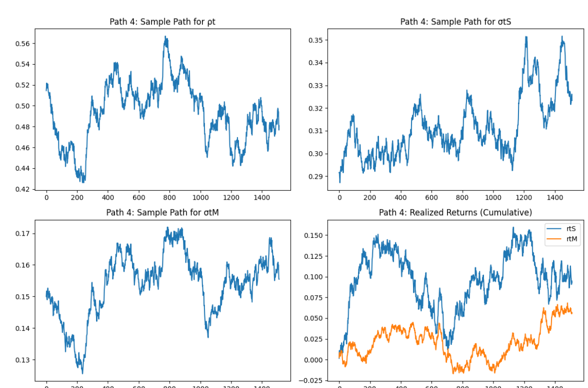

Figure 7: ρ_t, σ_t^S, σ_t^M, r_t^S and r_t^M for Path 4

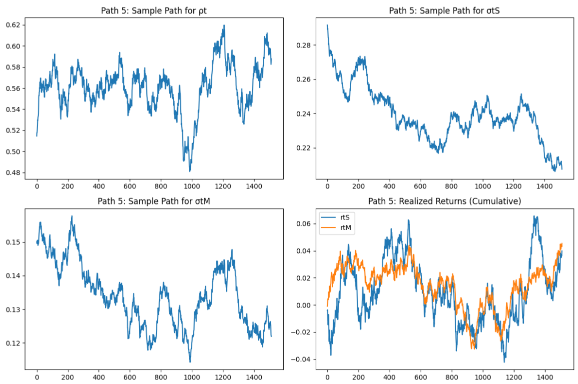

Figure 8: ρ_t, σ_t^S, σ_t^M, r_t^S and r_t^M for Path 5

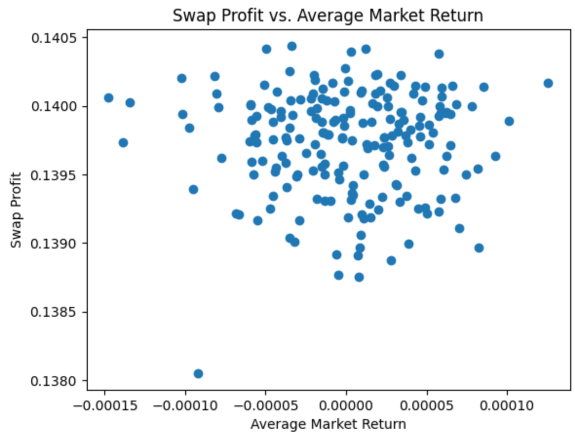

Figure 9: Average Market Return vs Swap Profit

(i) Is correlation directional in this model? Does your trade tend to make more money in down markets than up markets?

The model's correlation isn't inherently directional. In this model, there is no clear directional relationship between the correlation and the stock market. This is because the model assumes constant expected returns for both the stock and market portfolio ($\mu_S = \mu_M = 0.0$). The model also assumes that volatility is driven by independent Brownian motions, which does not create any dependency between correlation and the direction of the market. And we can as well see from the scatter plot above that return and swap profit do not have obvious relationship.

(ii) Do we tend to observe higher correlation when market volatility is higher?

In the given model, we can observe that market volatility and correlation are related due to the presence of the β parameter, which connects the stock's volatility to the market's volatility. As the market's volatility increases, it will affect the stock's volatility, which in turn affects the correlation. However, it's important to note that the model assumes a constant β, which may not accurately capture the empirical relationship between market volatility and correlation.

The assumptions in the model leading to each finding are:

1. The presence of the β parameter that connects the stock's volatility to the market's volatility.
2. The constant β assumption, which may not accurately capture the empirical relationship between market volatility and correlation.

4 Question 4

Calculate the weighted average of implied and realized volatilities for the component sector ETFs using equal weights:

$$\text{weighted\_avg\_implied\_vol} = \sum_{i=1}^{n} w_i \cdot \text{implied\_vols}_i$$

$$\text{weighted\_avg\_realized\_vol} = \sum_{i=1}^{n} w_i \cdot \text{realized\_vols}_i$$

Calculate the profit or loss for both the SPY ETF and the sector ETFs:

$$\text{SPY\_profit\_loss} = (\text{SPY\_realized\_vol} - \text{SPY\_implied\_vol}) \cdot \text{Notional}$$

$$\text{sector\_profit\_loss} = (\text{weighted\_avg\_realized\_vol} - \text{weighted\_avg\_implied\_vol}) \cdot \text{Notional}$$

Calculate the net profit or loss:

$$\text{net\_profit\_loss} = \text{SPY\_profit\_loss} - \text{sector\_profit\_loss}$$

From this calculation, we get the output as shown below:

Estimated Net Profit/Loss	$ -113333.33

Implications:

With the information from Question 2, the correlation coefficients show strong positive relationships between the average true correlation, realized return correlation, and swap profit. The breakeven correlation is now 0.4683, and the profit/loss figures for 0.10 higher/lower correlation are $426783.12 and $-1333622.48, respectively.

The net profit/loss calculation still shows a loss if the notional value of the market vol swap was $10 million. However, the strong positive correlations indicate that the trade would have been more profitable if the correlation was higher than the breakeven value. The strategy of betting on continued high correlation by going long on the index vol swap and short on the sector ETFs would have been more successful with higher correlation levels.

I pledge that I have not received assistance from anyone in preparing this exam.