**10/10** Questions Answered
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# IE 522 HW12

## Q1

5 Points

Consider the time series model given by the following:

$$Equation\ 1: \quad Y_t - \mu = \phi_1(Y_{t-1} - \mu) + \phi_2(Y_{t-2} - \mu) + \epsilon_t,$$

where $\phi_1 = -0.3, \phi_2 = 0.1, \mu = 1, \epsilon_t \sim N(0, \sigma_\epsilon^2), \sigma_\epsilon^2 = 1.$

### Q1.1

1 Point

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What are the roots of $\phi(z) = 1 - \phi_1 z - \phi_2 z^2$?

$Z_1 = 5, Z_2 = -2.$

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## Q1

5 Points

Consider the time series model given by the following:

$$Equation\ 1: \quad Y_t - \mu = \phi_1(Y_{t-1} - \mu) + \phi_2(Y_{t-2} - \mu) + \epsilon_t,$$

where $\phi_1 = -0.3, \phi_2 = 0.1, \mu = 1, \epsilon_t \sim N(0, \sigma_\epsilon^2), \sigma_\epsilon^2 = 1.$

### Q1.1

1 Point

What are the roots of $\phi(z) = 1 - \phi_1 z - \phi_2 z^2$?

$\phi(z) = 1 - \phi_1 z - \phi_2 z^2$ , $\begin{cases} \phi_1 = -0.3 \\ \phi_2 = 0.1 \end{cases}$

$\Rightarrow \phi(z) = 1 + 0.3z - 0.1z^2$

$-0.1(z^2 - 3z - 10) = 0$

$-0.1(z-5)(z+2) = 0$

$\Rightarrow z_1 = 5, \quad z_2 = -2$

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$\phi(z) = (1 - \frac{1}{5}z)(1 + \frac{1}{2}z)$

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## Q1.2

1 Point

Is $\{Y_t\}$ stationary? Why?

Yes. Since $|1/2|$ and $|-1/5| < 1$, they are $absolutely\ summable$, which make $Y_t$ $stationary$.

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**Q1.2**

1 Point

Is $\{Y_t\}$ stationary? Why?

$$\phi(z) = (1 - \tfrac{1}{5}z)(1 + \tfrac{1}{2}z)$$

$$\downarrow$$

$$Y_t - M = -0.3(Y_{t-1} - M) + 0.1(Y_{t-2} - M) + \varepsilon_t$$

$$\downarrow$$

$$(1 + 0.3L - 0.1L^2)(Y_t - M) = \varepsilon_t$$

$$\downarrow$$

$$(1 + \tfrac{1}{2}L)(1 - \tfrac{1}{5}L)(Y_t - M) = \varepsilon_t$$

$$Y_t = M + (1 + \tfrac{1}{2}L)^{-1}(1 - \tfrac{1}{5}L)^{-1}\varepsilon_t$$

$$= 1 + (1 + \tfrac{1}{2}L)^{-1}(1 - \tfrac{1}{5}L)^{-1}\varepsilon_t$$

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## Q1.3

1 Point

Using the Yule-Walker equation, derive an equation relating $\gamma(0)$ and $\gamma(1)$.

$$-3\gamma(1) = \gamma(0).$$

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**Q1.3**

1 Point

Using the Yule-Walker equation, derive an equation relating $\gamma(0)$ and $\gamma(1)$.

$$Y_t - \mu = -0.3(Y_{t-1} - \mu) + 0.1(Y_{t-2} - \mu) + \varepsilon_t$$

$$\to \gamma(h) = -0.3\,\gamma(h-1) + 0.1\,\gamma(h-2)$$

$$\to \rho(h) = -0.3\,\rho(h-1) + 0.1\,\rho(h-2)$$

$$\rho(1) = -0.3\,\rho(0) + 0.1\,\rho(-1) \qquad \rho(-1) = \rho(1$$

$$0.9\,\rho(1) = -0.3\,\rho(0) = -0.3$$

$$\downarrow$$

$$\rho(1) = -\frac{1}{3} = \frac{\gamma(1)}{\gamma(0)}$$

$$-3\gamma(1) = \gamma(0) \quad ※$$

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## Q1.4

1 Point

Compute the variance of the two sides of Equation 1 to get another relationship for $\gamma(0)$ and $\gamma(1)$. Using this and the relationship in question 1.3, compute $\gamma(0)$ and $\gamma(1)$.

$(1)\ 0.9\gamma(0) = -0.06\gamma(1) + 1$
$(2)\ \gamma(0) = 3/2.64, \gamma(1) = 1/-2.64.$

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**Q1.4**

1 Point

Compute the variance of the two sides of Equation 1 to get another relationship for $\gamma(0)$ and $\gamma(1)$. Using this and the relationship in question 1.3, compute $\gamma(0)$ and $\gamma(1)$.

① $Y_t - \mu = \phi_1 (Y_{t-1} - \mu) + \phi_2 (Y_{t-2} - \mu) + \varepsilon_t$

$$\gamma(0) = Var(\underline{\phi_1(Y_{t-1} - \mu)} + \phi_2 (Y_{t-2} - \mu) + \underline{\varepsilon_t})$$

$$= \phi_1^2 \gamma(0) + \phi_2^2 \gamma(0) + \sigma_\varepsilon^2$$

$$+ 2\phi_1\phi_2 \gamma(1)$$

$$+ 2\phi_1 \, cov(Y_{t-1} - \mu, \varepsilon_t)$$

$$+ 2\phi_2 \, cov(Y_{t-2} - \mu, \varepsilon_t)$$

$$\gamma(0) = \phi_1^2 \gamma(0) + \phi_2^2 \gamma(0) + 2\phi_1\phi_2 \gamma(1) + \sigma_\varepsilon^2$$

$$\Rightarrow \gamma(0) = 0.09 \gamma(0) + 0.01 \gamma(0) - 0.06 \gamma(1) + 1$$

( var(aX - bY) = var[aX + (-b)Y] = a^2*var(X) + (-b)^2*var(Y) + 2*a*(-b)*cov(X,Y) = a^2*var(X) + b^2*var(Y) - 2*a*b*cov(X,Y) )

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$$0.9 \gamma(0) = -0.06 \gamma(1) + 1 \quad ※$$

②
$$\begin{cases} -3\gamma(1) = \gamma(0) \\ 0.9\gamma(0) = -0.06\gamma(1) + 1 \end{cases}$$

$$= -2.7 \gamma(1) = -0.06 \gamma(1) + 1$$

$$= -2.64 \gamma(1) = 1, \quad \gamma(1) = 1/-2.64$$

$$\gamma(0) = 3/2.64$$

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## Q1.5

1 Point

Suppose the model in Equation 1 is expanded to an ARMA(2,1) model:

$$Y_t - \mu = \phi_1(Y_{t-1} - \mu) + \phi_2(Y_{t-2} - \mu) + \epsilon_t + \theta_1 \epsilon_{t-1},$$

where $\theta_1 = 2$, $\phi_1, \phi_2, \sigma_\epsilon^2$ stay the same. Is $\{Y_t\}$ stationary? Why?

Choose Files No file chosen $|1/2|, |-1/5| < 1$, it is $stationary$.

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**Q1.5**
1 Point

Suppose the model in Equation 1 is expanded to an ARMA(2,1) model:

$$Y_t - \mu = \phi_1(Y_{t-1} - \mu) + \phi_2(Y_{t-2} - \mu) + \epsilon_t + \theta_1\epsilon_{t-1},$$

where $\theta_1 = 2, \phi_1, \phi_2, \sigma_\epsilon^2$ stay the same. Is $\{Y_t\}$ stationary? Why?

$$Y_t - \mu = \phi_1(Y_{t-1} - \mu) + \phi_2(Y_{t-2} - \mu) + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

$$\Rightarrow Y_t - \mu = -0.3(Y_{t-1} - \mu) + 0.1(Y_{t-2} - \mu) + \varepsilon_t + 2\varepsilon_{t-1}$$

$$\downarrow$$

$$(1 + 0.3L - 0.1L^2)(Y_t - \mu) = \varepsilon_t + 2\varepsilon_{t-1}$$

$$\downarrow$$

$$(1 + \tfrac{1}{2}L)(1 - \tfrac{1}{5}L)(Y_t - \mu) = \varepsilon_t + 2\varepsilon_{t-1}$$

$$\Rightarrow (1 + \tfrac{1}{2}L)(1 - \tfrac{1}{5}L)(Y_t - \mu) = (1 + 2L)\varepsilon_t$$

$$\phi(L)(Y_t - \mu) = \theta(L)\varepsilon_t$$

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## Q2

5 Points

From R library Ecdat, get Monthly risk free rate from Jan 1960 to Dec 2002. Compute monthly changes of the risk free rate.

```
library(Ecdat)
data(Capm)
dr=diff(r)
```

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## Q2.1

1 Point

Suppose Conditional MLE is used to estimate the parameters of an AR(2) model for $\{Y_t = dr_t, 1 \leq t \leq m\}$:

$$Y_t = a + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma_\epsilon^2).$$

Conditional on the first two observations $y_1, y_2$, write down the expression of the log-likelihood of observing $Y_3 = y_3, \cdots, Y_m = y_m$.

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**Q2.1**

1 Point

Suppose Conditional MLE is used to estimate the parameters of an AR(2) model for $\{Y_t = dr_t, 1 \le t \le m\}$:

$$Y_t = a + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma_\epsilon^2).$$

Conditional on the first two observations $y_1, y_2$, write down the expression of the log-likelihood of observing $Y_3 = y_3, \cdots, Y_m = y_m$.

$$Y_3 \sim N(a + \phi_1 y_1 + \phi_2 y_2 , \sigma_\epsilon^2)$$

$$f_{Y_3}(y_3) = \frac{1}{\sqrt{2\pi\sigma_\epsilon^2}} e^{-\frac{(y_3 - a - \phi_1 y_1 - \phi_1 y_2)^2}{2\sigma_\epsilon^2}}$$

Log-likelihood func

$$= \log\left( \frac{1}{\sqrt{2\pi\sigma_\epsilon^2}} \times e^{-\frac{(y_3 - a - \phi_1 y_1 - \phi_2 y_2)^2}{2\sigma_\epsilon^2}} \right)$$

$\log(ab)$
$= \log a + \log b$

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$$= \log \frac{1}{\sqrt{2\pi\sigma_\epsilon^2}} - \frac{(y_3 - a - \phi_1 y_1 - \phi_2 y_2)^2}{2\sigma_\epsilon^2}$$

$$= -\frac{1}{2} \log 2\pi\sigma_\epsilon^2 - \frac{(y_3 - a - \phi_1 y_1 - \phi_2 y_2)^2}{2\sigma_\epsilon^2}$$

$$f_{Y_4|Y_3}(y_4 | y_3) = \frac{1}{\sqrt{2\pi\sigma_\epsilon^2}} e^{-\frac{(y_4 - a - \phi_1 y_2 - \phi_2 y_3)^2}{2\sigma_\epsilon^2}}$$

log-likelihood func:

$\log [ f_{Y_3}(y_3) \times f_{Y_4|Y_3}(y_4 | y_3) ]$

$$= \log \left[ \left( \frac{1}{\sqrt{2\pi\sigma_i^2}} \right)^2 \right.$$

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$$\times\, e^{\left[-\frac{(y_4 - a - \phi_1 y_2 - \phi_2 y_3)^2}{2\sigma_\varepsilon^2} - \frac{(y_3 - a - \phi_1 y_1 - \phi_2 y_2)^2}{2\sigma_\varepsilon^2}\right]}$$

$$= -\log 2\pi\sigma_\varepsilon^2 - \sum_{i=3}^{4} \frac{(y_i - a - \phi_1 y_{i-2} - \phi_2 y_{i-1})^2}{2\sigma_\varepsilon^2}$$

$\downarrow$

log - likelihood func for $y_m$

$$\Rightarrow \frac{-(n-2)}{2} \log 2\pi\sigma_\varepsilon^2 - \sum_{i=3}^{n} \frac{(y_i - a - \phi_1 y_{i-2} - \phi_2 y_{i-1})^2}{2\sigma_\varepsilon^2}$$

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## Q2.2

1 Point

Suppose Exact MLE is used to estimate the parameters of a MA(2) model for $\{Y_t = dr_t, 1 \leq t \leq m\}$:

$$Y_t = \mu + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}, \quad \epsilon_t \sim N(0, \sigma_\epsilon^2).$$

What's the mean vector and covariance matrix of the multivariate normal distribution for

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$(Y_1, \ldots, Y_m)^T$ in terms of $\mu, \theta_1, \theta_2, \sigma_\epsilon^2$?

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**Q2.2**

1 Point

Suppose Exact MLE is used to estimate the parameters of a MA(2) model for $\{Y_t = dr_t, 1 \le t \le m\}$:

$$Y_t = \mu + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}, \quad \epsilon_t \sim N(0, \sigma_\epsilon^2).$$

What's the mean vector and covariance matrix of the multivariate normal distribution for $(Y_1, \cdots, Y_m)^\top$ in terms of $\mu, \theta_1, \theta_2, \sigma_\epsilon^2$?

① mean vector $= (\underbrace{\mu, \mu \dots, \mu}_{m})^T$

② $\gamma(0) = Var(Y_t) = Var(\mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2})$

$= \sigma_\varepsilon^2 + \theta_1^2 \sigma_\varepsilon^2 + \theta_2^2 \sigma_\varepsilon^2 = (1 + \theta_1^2 + \theta_2^2)\sigma_\varepsilon^2$

$\gamma(1) = Cov(Y_t, Y_{t+1})$

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$= E[(\varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2}) \times (\varepsilon_{t+1} + \theta_1 \varepsilon_t + \theta_2 \varepsilon_{t-1})]$

$= E(\theta_1 \varepsilon_t^2) + E(\theta_1 \theta_2 \varepsilon_{t-1}^2)$

$= \theta_1 \sigma_\varepsilon^2 + \theta_1 \theta_2 \sigma_\varepsilon^2 = (\theta_1 + \theta_1 \theta_2)\sigma_\varepsilon^2$

$\gamma(2) = Cov(Y_t, Y_{t+2})$

$= E[(\varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2}) \times (\varepsilon_{t+2} + \theta_1 \varepsilon_{t+1} + \theta_2 \varepsilon_t)]$

$= \theta_2 \sigma_\varepsilon^2$

$\gamma(h)$ for $h \geq 3 = 0$

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$$\Rightarrow \Sigma =$$

$$\begin{bmatrix}
(1+\theta_1^2+\theta_2^2)\sigma_\varepsilon^2 & (\theta_1+\theta_1\theta_2)\sigma_\varepsilon^2 & \theta_2\sigma_\varepsilon^2 & \cdots & \cdots \\
(\theta_1+\theta_1\theta_2)\sigma_\varepsilon^2 & (1+\theta_1^2+\theta_2^2)\sigma_\varepsilon^2 & \ddots & & \vdots \\
\theta_2\sigma_\varepsilon^2 & & \ddots & & \theta_2\sigma_\varepsilon^2 \\
\vdots & & & \ddots & (\theta_1+\theta_1\theta_2)\sigma_\varepsilon^2 \\
\cdots & & \theta_2\sigma_\varepsilon^2 & (\theta_1+\theta_1\theta_2)\sigma_\varepsilon^2 & (1+\theta_1^2+\theta_2^2)\sigma_\varepsilon^2
\end{bmatrix}$$

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## Q2.3

1 Point

Use auto.arima to fit dr. Using the fit you obtain, forecast the monthly change from Dec 2002 to Jan 2003. What's the corresponding forecast for the interest rate in Jan 2003?

Series: dr

ARIMA(1,0,1) with zero mean

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Coefficients:

| | ar1 | ma1 |
|---|---|---|
| | 0.4551 | -0.6761 |
| s.e. | 0.1104 | 0.0903 |

sigma^2 = 0.004751: log likelihood = 647.65

AIC=-1289.29 AICc=-1289.25 BIC=-1276.56

-------------------------------------------------------------------

$$Y_t - \mu = \phi_1(Y_{t-1} - \mu) + \epsilon_t + \theta_1\epsilon_{t-1}$$

$$= Y_t - \mu = 0.4551(Y_{t-1} - \mu) + \epsilon_t - 0.6761\epsilon_{t-1}$$

---

$$Forcast\ model = \hat{Y_{t+1}} = \mu + 0.4551(Y_t - \mu) - 0.6761\epsilon_t$$

---

Monthly Change = Y_hat = -0.01154477

---

The IRR of January = J2003 = 0.09845523

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```
#Q2-3
#Data Preparation
library(Ecdat)
data(Capm)
r = Capm$rf
dr = diff(r)

#Fit ARIMA model
library(forecast)
auto.arima(dr)

#Get December 2002 to January 2003 monthly change
Choose Files No file chosen
Y_t = dr[length(dr)]
Y_t1 = dr[length(dr)-1]
Y_hat = mu + 0.4551*(Y_t - mu) - 0.6761*(Y_t - Y_t1)
Y_hat

#Get January 2003
D2002 = r[length(r)]
J2003 = D2002 + Y_hat
J2003
```

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## Q2.4

1 Point

What's the 90% prediction interval for the monthly change from Dec 2002 to Jan 2003? What's the corresponding 90% prediction interval for the interest rate in Jan 2003?

The 90% prediction interval of montely change from December 2022 to January 2023 = [-0.1079146,  0.1188449]

---------------------------------------------------------------

The 90% prediction interval of January 2023 = [0.00208540143421547, 0.228844940056894]

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```
#Data Preparation
library(Ecdat)
data(Capm)
r = Capm$rf
dr = diff(r)

#Fit ARIMA model
library(forecast)
fit = auto.arima(dr)
fit

```

```
#Get December 2002 to January 2003 monthly change
mu = mean(dr)
Y_t = dr[length(dr)]
Y_t1 = dr[length(dr)-1]
Y_hat = mu + 0.4551*(Y_t - mu) - 0.6761*(Y_t - Y_t1)
Y_hat

#Get Prediction Interval for monthly change
alpha = 0.1
f = forecast(fit, 1, level = 90)
f
#Get Prediction Interval for January 2003
D2002 = r[length(r)]
J2003ub = D2002 + f$upper
J2003lb = D2002 + f$lower
print(paste(J2003lb, J2003ub))
```

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## Q2.5

1 Point

If you try to produce a $k$-step-ahead forecast, when $k \to +\infty$, what'll be your forecast for the monthly change? What'll be the variance of the forecasting error?

$(1)$ multi-step-ahead forcast converges to $\mu = -0.0004271845$.
$(2)$ The varience converges to $0.005044025$.

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Q2-5.R

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```r
#Q2-5
#Data Preparation
library(Ecdat)
data(Capm)
r = Capm$rf
dr = diff(r)

#Fit ARIMA model
library(forecast)
fit = auto.arima(dr)
fit

#Get December 2002 to January 2003 monthly change
mu = mean(dr)
Y_t = dr[length(dr)]
Y_t1 = dr[length(dr)-1]
Y_hat = mu + 0.4551*(Y_t - mu) - 0.6761*(Y_t - Y_t1)
mu

#Get the gamma(0) after infinity periods
gamma = fit$sigma2 * (1 + (-0.6761)^2 + 2*0.4551*(-0.6761)) / (1 - 0.4551^2)
gamma

```

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