Q1

4 Points

The inter-arrival time of market shocks is modeled by an exponential distribution with pdf $f(x) = \frac{1}{\theta} e^{-\frac{1}{\theta} x}, x > 0$. Consider a random sample $\{X_1, \cdots, X_n\}$ from this distribution.

Q1.4

1 Point

Let $\hat{\sigma}_X^2$ be the sample variance of the random sample. What is $\mathbb{E}[\hat{\sigma}_X^2]$? Is the sample standard deviation $\hat{\sigma}_X$ an unbiased estimator for θ?

$$\hat{\sigma}_X^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2$$

$$= \frac{1}{n-1} \sum_{i=1}^{n} (X_i^2 + \bar{X}_n^2 - 2X_i \bar{X}_n)$$

$$= \frac{1}{n-1} \left(\sum_{i=1}^{n} X_i^2 + n\theta^2 - 2\theta \sum_{i=1}^{n} X_i \right)$$

$$E(\hat{\sigma}_X^2) = \frac{1}{n-1} \sum_{i=1}^{n} E(X_i^2) + \frac{n}{n-1}\theta^2 - 2\theta \frac{1}{n-1} \sum_{i=1}^{n} E(X_i)$$

$$= \frac{n}{n-1} 2\theta^2 + \frac{n}{n-1}\theta^2 - 2\theta \frac{n}{n-1}\theta$$

$$= \frac{n}{n-1}\theta^2$$

$$\theta^2 = \frac{n-1}{n} E(\hat{\sigma}_X^2), \quad \theta = \sqrt{\frac{n-1}{n} E(\hat{\sigma}_X^2)}$$

$$\hat{\sigma}_X = \sqrt{\frac{1}{n-1} \Sigma (X_i - \bar{X}_n)^2} \Rightarrow E(\hat{\sigma}_X) = E\left(\sqrt{\frac{1}{n-1} \Sigma (X - \bar{X}_n)^2}\right)$$

$$\rightarrow \sqrt{\frac{1}{n-1}} E\left(\sqrt{\Sigma (X - \bar{X}_n)^2}\right) = \sqrt{\frac{1}{n-1}} E\left(\sqrt{n-1}\,(\hat{\sigma}_X^2)\right)$$

$$= E(\hat{\sigma}_X^2) \neq \sqrt{\frac{n-1}{n} E(\hat{\sigma}_X^2)} \Rightarrow \text{biased.}$$