Q1

2 Points

Let $Z_0, Z_1, \cdots$, be independent standard normal random variables, and $X_t = Z_0 + Z_t, t \geq 1$.

Q1.1

1 Point

Compute the mean and autocovariances of the series $X = \{X_1, X_2, \cdots, X_t, \cdots\}$ for all possible lags.

$\gamma(0) = 2$, $\gamma(h) = 1$ for $h >= 1$.

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Q1

2 Points

Let $Z_0, Z_1, \cdots$, be independent standard normal random variables, and $X_t = Z_0 + Z_t, t \geq 1$.

$N(0,1)$

Q1.1

1 Point

Compute the mean and autocovariances of the series $X = \{X_1, X_2, \cdots, X_t, \cdots\}$ for all possible lags.

$Z_0, Z_1, \ldots \sim N(0,1)$

$X_1 = Z_0 + Z_1$

$X_2 = Z_0 + Z_2$

$\vdots$

$X_t = Z_0 + Z_t$

$E(X_t) = E(Z_0) + E(Z_t) = 0$

$\gamma(0) = Var(X_1) = Var(Z_0 + Z_1)$, since Z_0, Z_1 is iid.

$\Rightarrow Var(Z_0) + Var(Z_1) + \underbrace{Cov(Z_0, Z_1)}_{=0}$

$= 1 + 1 = 2$

$\gamma(1) = Cov(X_1, X_2) = E[(X_1 - \overset{0}{E(X)})(X_2 - \overset{0}{E(X)})]$

$= E(X_1 X_2) = E[(Z_0 + Z_1)(Z_0 + Z_2)]$

$= E[Z_0^2 + Z_0 Z_2 + Z_0 Z_1 + Z_1 Z_2)]$

$$= E(z_0^2) = 1$$

$$\text{Var}(z_0) = 1 = E(z_0^2) - E(z_0)^2 \Rightarrow E(z_0^2)$$

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$$\gamma(2) = Cov(X_1, X_3) = E[(X_1 X_3)]$$
$$= E[(Z_0 + Z_1)(Z_0 + Z_3)]$$
$$= E[(Z_0^2)] = 1$$

$$\Rightarrow \gamma(0) = 2, \quad \gamma(h) = 1, \; h \geq 1$$

A: mean = 0, Autocov $\begin{cases} \gamma(0) = 2 \\ \gamma(t) = 1, \; t \geq 1 \end{cases}$

Q1.2

1 Point

Is X weakly stationary? Why? Is X a white noise? Why?

Yes. It is weakly stationary process. As I have shown below:

(1) $E(X_t)$ regradless of time t, always is 0.

(2) Covarience depends on $lag - h$ only.

No. It is not white noise. Since $cov(X_s, X_t) = 1$, not 0.

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Q1.2
1 Point

Is X weakly stationary? Why? Is X a white noise? Why?

① $E(X_t) = E(Z_0 + Z_t) \underset{Z\text{'s iid}}{=} E(Z_0) + E(Z_t) = 0$

② $\gamma(0) = Var(X_t) = Var(Z_0 + Z_t)$
$= Var(Z_0) + Var(Z_t) + 2Cov(Z_0, Z_t)$ iid
$= Var(Z_0) + Var(Z_t)$
$= [E(Z_0^2) - E(Z_0)^2] + [E(Z_t^2) - E(Z_t)^2]$
$= 2$

$\gamma(h) = Cov(X_t, X_{t+h})$
$= E[(X_t - E(X_t))(X_{t+h} - E(X_{t+h}))]$
$= E[X_t \cdot X_{t+h}]$
$= E[(Z_0 + Z_t)(Z_0 + Z_{t+h})]$
$= E[Z_0^2 + Z_0 Z_{t+h} + Z_0 Z_t + Z_t Z_{t+h}]$
$= E[Z_0^2] = 1$

Q2

4 Points

In the following, $\{\epsilon_t\}$ is a white noise with variance $\sigma_\epsilon^2 = 1$. Consider a MA(3) process $Y_t = \mu + \epsilon_t + 2\epsilon_{t-1} + 3\epsilon_{t-2} + 4\epsilon_{t-3}$.

Q2.1

1 Point

Is Y_t stationary (here stationarity refers to weak stationarity)?

Yes, as I shown below,
(1) the mean of Y_t is μ, which is indepent with t,
(2) and the γ (covarience) will die out when $h >= 4$.
Thus, Y_t process is stationary. (The description itself confirms that Y_t process is $MA3$ model, which already confirms that it is stationary by itself.)

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$$Y_t = \mu + \varepsilon_t + 2\varepsilon_{t-1} + 3\varepsilon_{t-2} + 4\varepsilon_{t-3}$$

$$\Rightarrow E(Y_t) = E(\mu) + E(\varepsilon_t) + 2E(\varepsilon_{t-1}) + 3E(\varepsilon_{t-2}) + 4E(\varepsilon_{t-3})$$
$$= \mu$$

$$Cov(Y_t, Y_{t+4}) = \gamma(4)$$
$$= E[(\varepsilon_t + 2\varepsilon_{t-1} + 3\varepsilon_{t-2} + 4\varepsilon_{t-3}) \times (\varepsilon_{t+4} + 2\varepsilon_{t+3} + 3\varepsilon_{t+2} + 4\varepsilon_{t+1})]$$
$$= 0$$

2. If X and Y are independent, then $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$.

$$Cov(Y_t, Y_{t+3}) = \gamma(3)$$
$$= E[(\varepsilon_t + 2\varepsilon_{t-1} + 3\varepsilon_{t-2} + 4\varepsilon_{t-3})(\varepsilon_{t+3} + 2\varepsilon_{t+2} + 3\varepsilon_{t+1} + 4\varepsilon_t)] = 4E(\varepsilon_t^2)$$
$$= 4$$

$$Var(\varepsilon_t) = 1 = E(\varepsilon_t^2) - E(\varepsilon_t)^2 = E(\varepsilon_t^2)$$

$$\gamma(2) = Cov(Y_t, Y_{t+2})$$
$$= E[(\varepsilon_t + 2\varepsilon_{t-1} + 3\varepsilon_{t-2} + 4\varepsilon_{t-3})(\varepsilon_{t+2} + 2\varepsilon_{t+1} + 3\varepsilon_t + 4\varepsilon_{t-1})]$$
$$= 3E(\varepsilon_t^2) + 8E(\varepsilon_{t-1}^2) = 11$$

$$\gamma(1) = Cov(Y_t, Y_{t+1})$$
$$= E[(\varepsilon_t + 2\varepsilon_{t-1} + 3\varepsilon_{t-2} + 4\varepsilon_{t-3})$$

$$(\varepsilon_{t+1} + 2\varepsilon_t + 3\varepsilon_{t-1} + 4\varepsilon_{t-2})]$$

$$= 2E(\varepsilon_t^2) + 6E(\varepsilon_{t-1}^2) + 12E(\varepsilon_{t-2}^2) = 20$$

Q2.2

1 Point

What's the variance of Y_t?

30

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$$Var(Y_t) = Cov(Y_t, Y_t) = E((Y_t - \mu_Y)^2)$$

$$= E[(\varepsilon_t + 2\varepsilon_{t-1} + 3\varepsilon_{t-2} + 4\varepsilon_{t-3})^2]$$

$$= E[\varepsilon_t^2 + 2\varepsilon_{t-1}\varepsilon_t + 3\varepsilon_t\varepsilon_{t-2} + 4\varepsilon_t\varepsilon_{t-3}$$

$$+ 2\varepsilon_t\varepsilon_{t-1} + 4\varepsilon_{t-1}^2 + 6\varepsilon_{t-1}\varepsilon_{t-2} + 8\varepsilon_{t-1}\varepsilon_{t-3}$$

$$+ 3\varepsilon_t\varepsilon_{t-2} + 6\varepsilon_{t-1}\varepsilon_{t-2} + 9\varepsilon_{t-2}^2 + 12\varepsilon_{t-2}\varepsilon_{t-3}$$

$$+ 4\varepsilon_t\varepsilon_{t-3} + 8\varepsilon_{t-1}\varepsilon_{t-3} + 12\varepsilon_{t-2}\varepsilon_{t-3} + 16\varepsilon_{t-3}^2]$$

$$= E(\varepsilon_t^2) + 4\cancel{E[\varepsilon_t\varepsilon_{t-1}]} + 6\cancel{E[\varepsilon_t\varepsilon_{t-2}]} + 8\cancel{E[\varepsilon_t\varepsilon_{t-3}]}$$

$$+ 4(\varepsilon_{t-1}^2) + 12\cancel{E[\varepsilon_{t-1}\varepsilon_{t-2}]} + 16\cancel{E[\varepsilon_{t-1}\varepsilon_{t-3}]}$$

$$+ 9(\varepsilon_{t-2}^2) + 24\cancel{E(\varepsilon_{t-2}\varepsilon_{t-3})} + 16E(\varepsilon_{t-3}^2)$$

$$= E(\varepsilon_t^2) + 4(\varepsilon_{t-1}^2) + 9(\varepsilon_{t-2}^2) + 16E(\varepsilon_{t-3}^2)$$

$$= 1 + 4 + 9 + 16 = 30$$

$$* \quad Var(\varepsilon_t) = 1 = E(\varepsilon_t^2) - E(\varepsilon_t)^2 = E(\varepsilon_t^2)$$

Q2.3

1 Point

What's the lag-1 autocorrelation?

$2/3$

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$$\gamma(1) = Cov(Y_t, Y_{t+1})$$

$$= E[(\varepsilon_t + 2\varepsilon_{t-1} + 3\varepsilon_{t-2} + 4\varepsilon_{t-3})$$
$$(\varepsilon_{t+1} + 2\varepsilon_t + 3\varepsilon_{t-1} + 4\varepsilon_{t-2})]$$

$$= 2E(\varepsilon_t^2) + 6E(\varepsilon_{t-1}^2) + 12E(\varepsilon_{t-2}^2)$$

$$= 20$$

$$Var(Y_t) = Cov(Y_t, Y_t) = E((Y_t - \mu_Y)^2)$$

$$= E[(\varepsilon_t + 2\varepsilon_{t-1} + 3\varepsilon_{t-2} + 4\varepsilon_{t-3})^2]$$

$$= E[\varepsilon_t^2 + 2\varepsilon_{t-1}\varepsilon_t + 3\varepsilon_t\varepsilon_{t-2} + 4\varepsilon_t\varepsilon_{t-3}$$
$$+ 2\varepsilon_t\varepsilon_{t-1} + 4\varepsilon_{t-1}^2 + 6\varepsilon_{t-1}\varepsilon_{t-2} + 8\varepsilon_{t-1}\varepsilon_{t-3}$$
$$+ 3\varepsilon_t\varepsilon_{t-2} + 6\varepsilon_{t-1}\varepsilon_{t-2} + 9\varepsilon_{t-2}^2 + 12\varepsilon_{t-2}\varepsilon_{t-3}$$
$$+ 4\varepsilon_t\varepsilon_{t-3} + 8\varepsilon_{t-1}\varepsilon_{t-3} + 12\varepsilon_{t-2}\varepsilon_{t-3} + 16\varepsilon_{t-3}^2]$$

$$= E(\varepsilon_t^2) + 4E[\cancel{\varepsilon_t\varepsilon_{t-1}}] + 6E[\cancel{\varepsilon_t\varepsilon_{t-2}}] + 8E[\cancel{\varepsilon_t\varepsilon_{t-3}}]$$
$$+ 4(\varepsilon_{t-1}^2) + 12E[\cancel{\varepsilon_{t-1}\varepsilon_{t-2}}] + 16E[\cancel{\varepsilon_{t-1}\varepsilon_{t-3}}]$$
$$+ 9(\varepsilon_{t-2}^2) + 24E(\cancel{\varepsilon_{t-2}\varepsilon_{t-3}}) + 16E(\varepsilon_{t-3}^2)$$

$$= E(\varepsilon_t^2) + 4(\varepsilon_{t-1}^2) + 9(\varepsilon_{t-2}^2) + 16E(\varepsilon_{t-3}^2)$$

$$= 1 + 4 + 9 + 16 = 30 = \gamma(0)$$

$$\Rightarrow \gamma(0) = 30, \quad \gamma(1) = 20, \quad \rho(1) = \frac{\gamma(1)}{\gamma(0)} = 2/3.$$

Q2.4

1 Point

What are lag-h autocorrelations for $h \geq 4$?

0

▾ A11Q2-4.png

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$$Var(Y_t) = Cov(Y_t, Y_t) = E((Y_t - \mu_Y)^2)$$

$$= E[(\varepsilon_t + 2\varepsilon_{t-1} + 3\varepsilon_{t-2} + 4\varepsilon_{t-3})^2]$$

$$= E[\varepsilon_t^2 + 2\varepsilon_{t-1}\varepsilon_t + 3\varepsilon_t\varepsilon_{t-2} + 4\varepsilon_t\varepsilon_{t-3}$$

$$+ 2\varepsilon_t\varepsilon_{t-1} + 4\varepsilon_{t-1}^2 + 6\varepsilon_{t-1}\varepsilon_{t-2} + 8\varepsilon_{t-1}\varepsilon_{t-3}$$

$$+ 3\varepsilon_t\varepsilon_{t-2} + 6\varepsilon_{t-1}\varepsilon_{t-2} + 9\varepsilon_{t-2}^2 + 12\varepsilon_{t-2}\varepsilon_{t-3}$$

$$+ 4\varepsilon_t\varepsilon_{t-3} + 8\varepsilon_{t-1}\varepsilon_{t-3} + 12\varepsilon_{t-2}\varepsilon_{t-3} + 16\varepsilon_{t-3}^2]$$

$$= E(\varepsilon_t^2) + 4E[\varepsilon_t\varepsilon_{t-1}] + 6E[\varepsilon_t\varepsilon_{t-2}] + 8E[\varepsilon_t\varepsilon_{t-3}]$$

$$+ 4(\varepsilon_{t-1}^2) + 12E[\varepsilon_{t-1}\varepsilon_{t-2}] + 16E[\varepsilon_{t-1}\varepsilon_{t-3}]$$

$$+ 9(\varepsilon_{t-2}^2) + 24E(\varepsilon_{t-2}\varepsilon_{t-3}) + 16E(\varepsilon_{t-3}^2)$$

$$= E(\varepsilon_t^2) + 4(\varepsilon_{t-1}^2) + 9(\varepsilon_{t-2}^2) + 16E(\varepsilon_{t-3}^2)$$

$$= 1 + 4 + 9 + 16 = 30$$

$$Cov(Y_t, Y_{t+4}) = \gamma(4)$$

$$= E[(\varepsilon_t + 2\varepsilon_{t-1} + 3\varepsilon_{t-2} + 4\varepsilon_{t-3}) \times$$

$$(\varepsilon_{t+4} + 2\varepsilon_{t+3} + 3\varepsilon_{t+2} + 4\varepsilon_{t+1})]$$

$$= 0 \quad \Rightarrow \gamma(h) = 0 \text{ for } h \geq 4$$

$$\gamma(h) = 0, \quad \gamma(0) = 30. \quad \rho(h) = \frac{\gamma(h)}{\gamma(0)} = 0, \; h \geq 4$$

Q3

4 Points

Let $\phi(L) = 1 + \frac{1}{2}L$.

Q3.1

1 Point

What's $\phi(L)^{-1}$?

(shown below)

A11Q3-1.png

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Q3

4 Points

Let $\phi(L) = 1 + \frac{1}{2}L$.

Q3.1

1 Point

What's $\phi(L)^{-1}$?

$\phi(L) = 1 + \frac{1}{2}L, \Rightarrow \phi_1 = -\frac{1}{2}$

$\phi^{-1}(L) = 1 + \phi_1 L + \phi_1^2 L^2 + \phi_1^3 L^3 + \cdots$

$= 1 - \frac{1}{2}L + \frac{1}{4}L^2 - \frac{1}{8}L^3 + \cdots$

Q3.2

1 Point

Is $\phi(L)^{-1}$ absolutely summable? Is $\phi(L)^{-1}\epsilon_t$ stationary? Here ϵ_t is a white noise.

Yes. (reason shown below)

Yes. (reason shown below)

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Q3.2

1 Point

Is $\phi(L)^{-1}$ absolutely summable? Is $\phi(L)^{-1}\epsilon_t$ stationary? Here ϵ_t is a white noise.

①

$$\phi(L)^{-1} = 1 - \frac{1}{2}L + \frac{1}{4}L^2 - \frac{1}{8}L^3 + \cdots\cdots$$

$$\Rightarrow \psi(L) = \psi_0 + \psi_1 L + \psi_2 L^2 + \psi_3 L^3$$

$$\psi_0 = 1,\ \psi_1 = -\frac{1}{2},\ \psi_2 = \frac{1}{4},\ \psi_3 = -\frac{1}{8} \ \cdots\cdots$$

$$\Rightarrow \psi_t = \left(-\frac{1}{2}\right)^t$$

$$\sum_{j=0}^{\infty}|\psi_j| = \sum_{j=0}^{\infty}\left|\left(-\frac{1}{2}\right)^j\right| = \sum_{j=0}^{\infty}\left(\frac{1}{2}\right)^j = \frac{1}{1-\frac{1}{2}} = 2 < \infty$$

$\Rightarrow$ Absolutely summable (converging)

②

(if $\{X_t\}$ is stationary & ψ_j's is absolutely summable

$\rightarrow$ $Y_t = \psi(L)X_t$ is stationary)

$\Rightarrow$ ε_t is stationary (since it is weakly stationay process)

$\phi^{-1}(L)$ is absolutely summable (as shown above)

$\rightarrow$ $\phi^{-1}(L)\,\varepsilon_t$ is stationary ! *

Q3.3

1 Point

Consider an AR(1) process $Y_t = 3 - \frac{1}{2}Y_{t-1} + \epsilon_t$. Find μ so that $Y_t - \mu = -\frac{1}{2}(Y_{t-1} - \mu) + \epsilon_t$. Provide the MA($\infty$) representation for Y_t.

$\mu = 2$

(shown below)

▾ A11Q3-3.png

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Q3.3

1 Point

Consider an AR(1) process $Y_t = 3 - \frac{1}{2}Y_{t-1} + \epsilon_t$. Find μ so that $Y_t - \mu = -\frac{1}{2}(Y_{t-1} - \mu) + \epsilon_t$. Provide the MA($\infty$) representation for Y_t.

AR(1): $Y_t = a + \phi_1 Y_{t-1} + \varepsilon_t$

$\Rightarrow Y_t = 3 - \frac{1}{2} Y_{t-1} + \varepsilon_t$

$Y_t - \mu = -\frac{1}{2}(Y_{t-1} - \mu) + \varepsilon_t$

$Y_t - \mu = -\frac{1}{2} Y_{t-1} + \frac{1}{2}\mu + \varepsilon_t$

$Y_t = \frac{3}{2}\mu - \frac{1}{2}Y_{t-1} + \varepsilon_t$

$\frac{3}{2}\mu = 3$

$\mu = 2,$

A: $\mu = 2$

$Y_t - \mu = -\frac{1}{2}(Y_{t-1} - \mu) + \varepsilon_t$

$Y_{t-1} - \mu = -\frac{1}{2}(Y_{t-2} - \mu) + \varepsilon_{t-1}$

$\downarrow$

$$\begin{aligned} Y_t - \mu &= -\frac{1}{2}\left(-\frac{1}{2}(Y_{t-2} - \mu) + \varepsilon_{t-1}\right) + \varepsilon_t \\ &= \frac{1}{4}(Y_{t-2} - \mu) - \frac{1}{2}\varepsilon_{t-1} + \varepsilon_t \\ &= \frac{1}{4}\left(-\frac{1}{2}(Y_{t-3} - \mu) + \varepsilon_{t-2}\right) - \frac{1}{2}\varepsilon_{t-1} + \varepsilon_t \\ &= -\frac{1}{8}(Y_{t-3} - \mu) + \frac{1}{4}\varepsilon_{t-2} - \frac{1}{2}\varepsilon_{t-1} + \varepsilon_t \end{aligned}$$

$\downarrow$

$$\begin{aligned} Y_t &= \mu + \varepsilon_t - \frac{1}{2}\varepsilon_{t-1} + \frac{1}{4}\varepsilon_{t-2} - \frac{1}{8}\varepsilon_{t-3} + \cdots \\ &= \mu + \sum_{j=0}^{\infty} \phi_1^j \varepsilon_{t-j} \end{aligned}$$

Q3.4

1 Point

Is $\{Y_t\}$ stationary? Why? What's the mean of Y_t?

Yes. (reason shown below)

$mean = \mu = 2.$

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Q3.4
1 Point

Is $\{Y_t\}$ stationary? Why? What's the mean of Y_t?

$$Y_t = \mu + \varepsilon_t - \frac{1}{2}\varepsilon_{t-1} + \frac{1}{4}\varepsilon_{t-2} - \frac{1}{8}\varepsilon_{t-3} + \cdots$$

$$\Rightarrow Y_t = \mu + \sum_{j=0}^{\infty} \phi_1^j \varepsilon_{t-j},$$

$$\phi_1 = -\frac{1}{2} \Rightarrow |\phi_1| = |-\tfrac{1}{2}| < 1$$

$\Rightarrow \{Y_t\}$ is stationary, since it satisfied $|\phi_1| < 1$ for $Y_t = \mu + \sum_{j=0}^{\infty} \phi_1^j \varepsilon_{t-j}$.

$$E(Y_t) = E(\mu) + \underbrace{E(\varepsilon_t)}_{0} - \frac{1}{2}\underbrace{E(\varepsilon_{t-1})}_{0} + \underbrace{\cdots}_{0} = \mu$$

IE 522 HW11

UNGRADED

STUDENT
Yu-Ching Liao

TOTAL POINTS
- / 10 pts

QUESTION 1

(no title) 2 pts

1.1 (no title) 1 pt

1.2 (no title) 1 pt

QUESTION 2

(no title) 4 pts

2.1 (no title) 1 pt

2.2 (no title) 1 pt

2.3	(no title)	1 pt
2.4	(no title)	1 pt

QUESTION 3

(no title) 4 pts

3.1	(no title)	1 pt
3.2	(no title)	1 pt
3.3	(no title)	1 pt
3.4	(no title)	1 pt