5. Suppose that $\{X_n\}_n$ are an independent and identically distributed collection of random variables with common law

$$\mathbb{P}\{X=-1\} = \mathbb{P}\{X=0\} = \mathbb{P}\{X=1\} = \frac{1}{3}.$$

(a) Compute the common mean μ of the X_n's.

(b) Compute the common variance σ^2 of the X_n's.

(c) Compute the characteristic function $\phi_X(\theta) \overset{\text{def}}{=} \mathbb{E}\left[e^{\imath\theta X}\right]$ for $\theta \in \mathbb{R}$. (I want you to work through the definition of the characteristic function)

(d) Define

$$S_N \overset{\text{def}}{=} \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \{X_n - \mu\}.$$

Using the statement of the central limit theorem (you don't need to reprove anything) find the limit of S_N as $N \nearrow \infty$.

(a) $\mu = E(X) = \frac{1}{3}(-1+0+1) = 0$

(b) $\sigma^2 = E(X^2) = \frac{1}{3}(1+0+1) = \frac{2}{3}$

(c) $\phi_X(\theta) = E(\exp(i\theta X))$

$$= \frac{1}{3}e^{\theta i} + \frac{1}{3}e^{-\theta i} + \frac{1}{3}$$

$$= \frac{e^{\theta i} + e^{-\theta i} + 1}{3}$$

$$= \frac{(e^{\theta i} + e^{-\theta i})\frac{1}{2} \times 2 + 1}{3} = \frac{1 + 2\cos(\theta)}{3}$$

(d) CLT : $\frac{1}{\sqrt{N}} \sum_{n=1}^{N} \frac{X_n - \mu}{\sigma}$ will $\to N(0,1)$

if $N \to \infty$.

$$\Rightarrow \frac{1}{\sqrt{N}} \sum_{1}^{N} \frac{X_n - \mu}{\sigma} \sim N(0,1)$$

$$\Rightarrow \frac{1}{\sigma} \frac{1}{\sqrt{N}} \sum_{1}^{N} (X_n - \mu) \sim N(0,1)$$

$$= \frac{1}{\sqrt{N}} \sum_{n=1}^{N} (X_n - \mu) \sim \mathcal{N}\left(0, \frac{2}{3}\right)$$

$$\boxed{\begin{aligned} &Y = \sigma X \\ &\Rightarrow E(Y) = \sigma E(X) = 0 \\ &Var(Y) = Var(\sigma X) \\ &\quad = \sigma^2 Var(X) = \sigma^2 = \frac{2}{3} \end{aligned}}$$

$$\Rightarrow S_N \xrightarrow[N \to \infty]{} \mathcal{N}\left(0, \frac{2}{3}\right)$$