2. Suppose that X is Gaussian with mean 4 and variance 1. Explicitly compute $\mathbb{E}[X^3]$ (hint: perhaps you might want to write out the binomial expansion of $(a+b)^3$)

let $Y = X - 4$,

$$\Rightarrow \begin{rcases} E(Y) = E(X) - 4 \\ Var(Y) = Var(X) \end{rcases} Y \sim N(0,1)$$

$$\Rightarrow E(X^3) = E[(Y+4)^3]$$
$$= E[Y^3 + 12Y^2 + 12Y + 64]$$
$$= E[Y^3] + 12E[Y^2] + 12E(Y] + 64$$

$$\boxed{\mathbb{E}[X^{2k}] = \frac{(2k)!}{2^k k!} \quad \text{and} \quad \mathbb{E}[X^{2k+1}] = 0}$$

$$= 0 + 12 + 0 + 64$$
$$= 76$$

* Validation via Monte-Carlo.

```
μ, σ = 4, 1 # mean and standard deviation
μs =[]
for i in range(1000):
    X = np.random.normal(μ, σ, 10000)
    X3 = s**3
    μs.append(X3.mean())
μs = np.array(μs)
print(μs.mean())
```

executed in 452ms, finished 14:52:03 2023-01-26

76.04779568847707

3. If η is a standard Gaussian random variable, compute

$$\mathbb{E}\left[\exp\left[2+3\eta\right]\right]$$

$\eta \sim N(0,1)$

$$E(\exp(2+3\eta)) = E(e^{2} \times e^{3\eta})$$
$$= e^{2} \times \underbrace{E(\exp(3\eta))}$$
$$= e^{2} \times M_X(3)$$
$$= e^{2} \times e^{9/2}$$
$$= e^{6.5} = 665.14163\ldots$$

✲ Validation via Monte-Carlo

```python
l = []
for i in range(1000000):
    η = np.random.normal(0, 1, 1000)
    a = exp(2+3*η)
    l.append(a.mean())
l = np.array(l)
print(l.mean())
```

executed in 48.9s, finished 15:07:32 2023-01-26

```
664.2383962895137
```

4. Let η be a standard Gaussian random variable. Compute $\mathbb{P}\{\eta \leq 3\}$ in terms of erf.

$$P(\eta \leq 3) = 1 - \underbrace{P(\eta > 3)}$$

$$P(\eta > 3) = \frac{1}{2}(|\eta| > 3)$$
$$= \frac{1}{2}\left[1 - \operatorname{erf}(3/\sqrt{2})\right]$$

$$1 - \frac{1}{2}\left[1 - \operatorname{erf}(3/\sqrt{2})\right]$$
$$= \frac{1}{2} + \frac{1}{2}\operatorname{erf}(3/\sqrt{2})$$
$$= \frac{1}{2}\left(1 + \operatorname{erf}(3/\sqrt{2})\right)$$
$$= 0.99865010,$$

✡ Validation via Gaussian CDF

```
norm.cdf(3)
```

executed in 4ms, finished 15:22:01 2023-01-26

```
0.9986501019683699
```