Brownian Information

13. Compute $\mathbb{P}\{W_5 > 6 | W_3\}$.
    (a) Write the answer in terms of

$$F(t) \stackrel{\text{def}}{=} \int_{s=-\infty}^{t} \frac{1}{\sqrt{2\pi}} \exp\left[-\tfrac{1}{2}s^2\right] ds \qquad t \in \mathbb{R}$$

    (b) Write the answer in terms of erf.

14. Compute $\mathbb{E}[W_5^3 | W_3]$.

13.

(a) $P(W_5 > 6 \mid W_3)$

$= P(W_5 - W_3 > 6 - W_3)$

$= P\left(\frac{W_5 - W_3}{\sqrt{2}} > \frac{6 - W_3}{\sqrt{2}}\right)$

$Z \sim N(0,1)$

$= P\left(Z > \frac{6 - W_3}{\sqrt{2}}\right) = P\left(Z < -\frac{6 - W_3}{\sqrt{2}}\right)$

normal CDF

$= F\left(-\frac{6 - W_3}{\sqrt{2}}\right)$ ※1

(b) $F(t) = \frac{1}{2}\left[1 + \text{erf}\left(\frac{t}{\sqrt{2}}\right)\right]$

$\Rightarrow F\left(-\frac{6 - W_3}{\sqrt{2}}\right) = \frac{1}{2} + \left[1 + \text{erf}\left(-\frac{6 - W_3}{2}\right)\right]$

※

test via Monte-Carlo.

```
W3 = 7
W5 = np.random.normal(W3, sqrt(2), 10000)
print("Computational Value:", sum(W5>6)/10000)

Z = -(6-W3)/sqrt(2)
print("Theoratical Normal CDF Value:",N.cdf(Z))
print("Theoratical ERF Value:",0.5*(1+Erf(Z/sqrt(2))))
```

executed in 27ms, finished 19:03:20 2023-02-20

```
Computational Value: 0.7621
Theoratical Normal CDF Value: 0.7602499389065233
Theoratical ERF Value: 0.7602499389065233
```

14.

$$E[W_5^3 \mid W_3] = E[[(W_5 - W_3) + W_3]^3 \mid W_3]$$

$$= E[(W_5 - W_3)^3 + 3(W_5 - W_3)^2 W_3 + 3(W_5 - W_3)W_3^2 + W_3^3 \mid W_3]$$

$$= E[(W_5 - W_3)^3 \mid W_3] + 3W_3 E[(W_5 - W_3)^2 \mid W_3] + 3W_3^2 E[(W_5 - W_3) \mid W_3] + W_3^3$$

($E[(W_5 - W_3)^3 \mid W_3] = 0$, $E[(W_5 - W_3) \mid W_3] = 0$)

$$= 0 + 6W_3 + 0 + W_3^3$$

$$= 6W_3 + W_3^3 \ \text{※}$$

test via Monte-Carlo:

```
W3 = 7
a = np.random.normal(W3, sqrt(2), 10000)
print("Computational Value:", np.mean(a**3))
print("Theoratical Value:",W3**3+6*W3)
```

executed in 3ms, finished 19:08:01 2023-02-20

```
Computational Value: 385.92588318126843
Theoratical Value: 385
```