9. Consider the event space $\Omega \stackrel{\text{def}}{=} [0,1)$ with probability measure dx. Consider the partition

$$\mathcal{P} \stackrel{\text{def}}{=} \{[0, 1/3), [1/3, 2/3), [2/3, 1)\}.$$

Assume that $f : \Omega \to \mathbb{R}$ is such that

$$\mathbb{E}\left[f|\mathcal{P}\right](x) = 5\mathbf{1}_{[0,1/3)}(x) - 2\mathbf{1}_{[1/3,2/3)}(x) + 6\mathbf{1}_{[2/3,1)}(x)$$

Define

$$g(x) = 4\mathbf{1}_{[0,1/3)}(x) - 2\mathbf{1}_{[1/3,2/3)}(x).$$

and compute $\mathbb{E}[fg]$.

let $g(x) = 4 \times 1_{[0,\frac{1}{3})}(x) - 2 \times 1_{[\frac{1}{3},\frac{2}{3})}(x)$

$= \sum_{n=1}^{3} \gamma_n 1_{A_n}(x)$

where $\begin{cases} \gamma_1 = 4 \\ \gamma_2 = -2 \\ \gamma_3 = 0 \end{cases}$ and $\begin{cases} A_1 = [0, \frac{1}{3}) \\ A_2 = [\frac{1}{3}, \frac{2}{3}) \\ A_3 = [\frac{2}{3}, 1) \end{cases}$

let $E[f|P](x) = 5 \times 1_{A_1}(x) - 2 \times 1_{A_2}(x) + 6 \times 1_{A_3}(x)$

$= \sum_{n=1}^{3} \bar{f}_{A_n} 1_{A_n}(x)$

where $\begin{cases} \bar{f}_{A_1} = 5 \\ \bar{f}_{A_2} = -2 \\ \bar{f}_{A_3} = 6 \end{cases}$

means that.

$\bar{f}_{A_n}$

* if x on $A_1 = [0, \frac{1}{3})$, we estimate f by 5.
  x on $A_2 = [\frac{1}{3}, \frac{2}{3})$, we estimate f by -2.
  x on $A_3 = [\frac{2}{3}, 1)$, we estimate f by 6.

$$\Rightarrow \mathbb{E}[fg] = \sum \gamma_n \times \mathbb{E}[f 1_{A_n}]$$

$$= \gamma_1 \mathbb{E}[\bar{f}_{A_1} 1_{A_1}] + \gamma_2 \mathbb{E}[\bar{f}_{A_2} 1_{A_2}] + \gamma_3 \mathbb{E}[\bar{f}_{A_3} 1_{A_3}]$$

$$= 4 \times 5 + (-2) \times (-2) + 0 \times 6$$

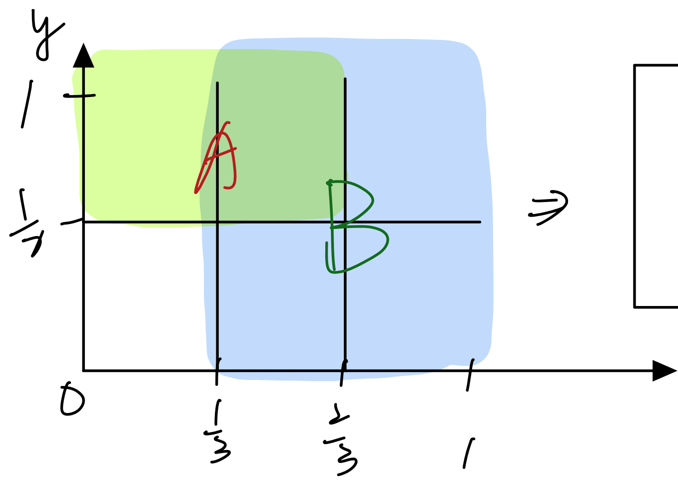
$$= 20 + 4 = 24 \text{ ※}$$

---

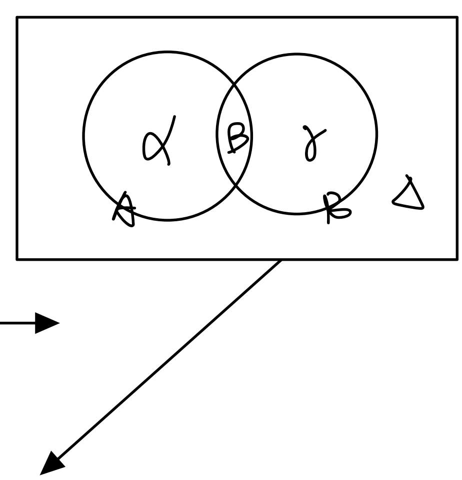
10. Define

$$\Omega \stackrel{\text{def}}{=} [0,1)^2$$
$$A \stackrel{\text{def}}{=} [0, 2/3) \times (1/2, 1)$$
$$B \stackrel{\text{def}}{=} [1/3, 1) \times [0, 1).$$

Enumerate and sketch all sets in $\sigma(\{A, B\})$.

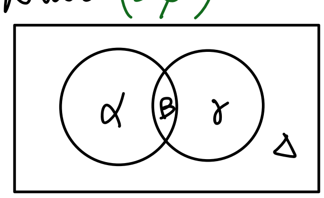


$\Rightarrow$

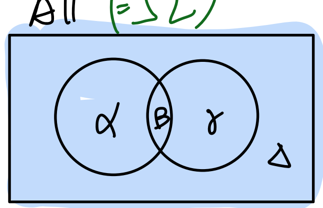


to get all $\sigma(\{A,B\})$s, we can list out all combinations of $\alpha, \beta, \gamma, \Delta$ (totally $2^4 = 16$)

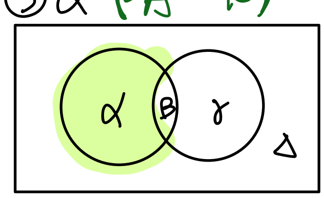
① Null $(=\emptyset)$



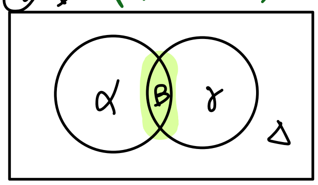
② All $(=\Omega)$



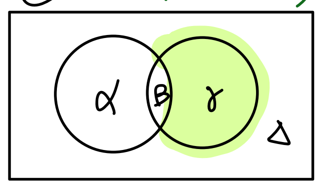
③ α $(=A-B)$



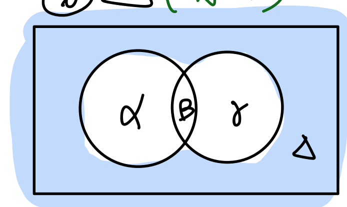
④ β $(= A \cap B)$



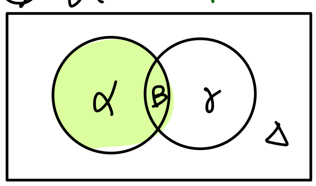
⑤ γ $(= B - A)$



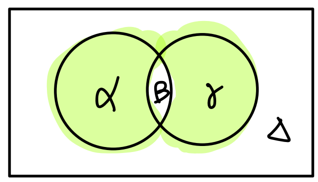
⑥ Δ $(= A \cup B)^c$



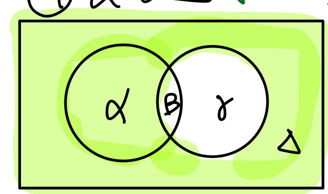
⑦ $\alpha \cup \beta$ $(= A)$



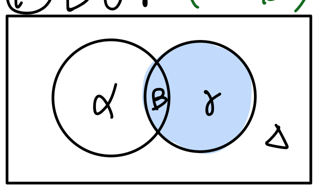
⑧ $\alpha \cup \gamma$ $(= A \ominus B)$



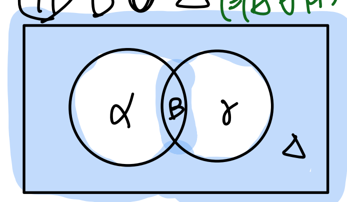
⑨ $\alpha \cup \Delta$ $(= B^c)$



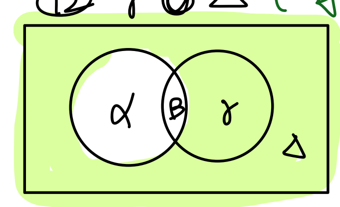
⑩ $\beta \cup \gamma$ $(= B)$



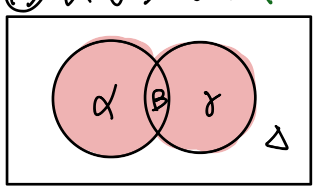
⑪ $\beta \cup \Delta$ $(= (A \ominus B)^c)$



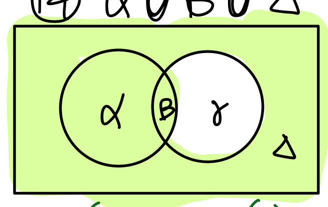
⑫ $\gamma \cup \Delta$ $(= A^c)$



⑬ $\alpha \cup \beta \cup \gamma$ $(= A \cup B)$

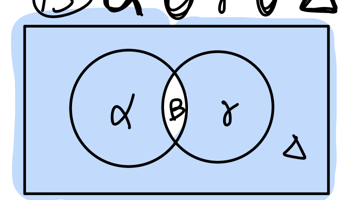


⑭ $\alpha \cup \beta \cup \Delta$



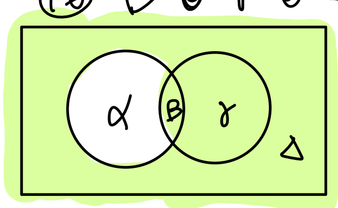
$(= A \cup B')$

⑮ $\alpha \cup \gamma \cup \Delta$



$(= A' \cup B')$

⑯ $\beta \cup \gamma \cup \Delta$



$(= A^c \cup B)$

$\Rightarrow$ We substitute all A with $[0, \frac{2}{3}) \times [\frac{1}{2}, 1)$ and B with $[\frac{1}{3}, 1) \times [0, 1)$, then we can get all 16 $\sigma(\{A, B\})$s.

11. Define $\Omega \overset{\text{def}}{=} [0,1)$ and

$$f(x) \overset{\text{def}}{=} x^3. \qquad x \in [0,1)$$

Compute the conditional expectation of f given the partition

$$P \overset{\text{def}}{=} \{[0,1/2], (1/2,3/4], (3/4,1)\}.$$

let $A_1 = [0, \frac{1}{2}]$, $A_2 = (\frac{1}{2}, \frac{3}{4}]$, $A_3 = (\frac{3}{4}, 1)$

then

$$\bar{f}_{A_1} = \frac{1}{\frac{1}{2} - 0} \int_0^{\frac{1}{2}} x^3 \, dx$$

$$= 2 \times \frac{1}{4} x^4 \Big|_0^{\frac{1}{2}} = 2 \times \frac{1}{4} \times \frac{1}{16}$$

$$= \frac{1}{32}$$

$$\bar{f}_{A_2} = \frac{1}{\frac{3}{4} - \frac{1}{2}} \int_{\frac{1}{2}}^{\frac{3}{4}} x^3 \, dx$$

$$= 4 \times \frac{1}{4} x^4 \Big|_{1/2}^{3/4} = x^4 \Big|_{1/2}^{3/4}$$

$$= 65/256$$

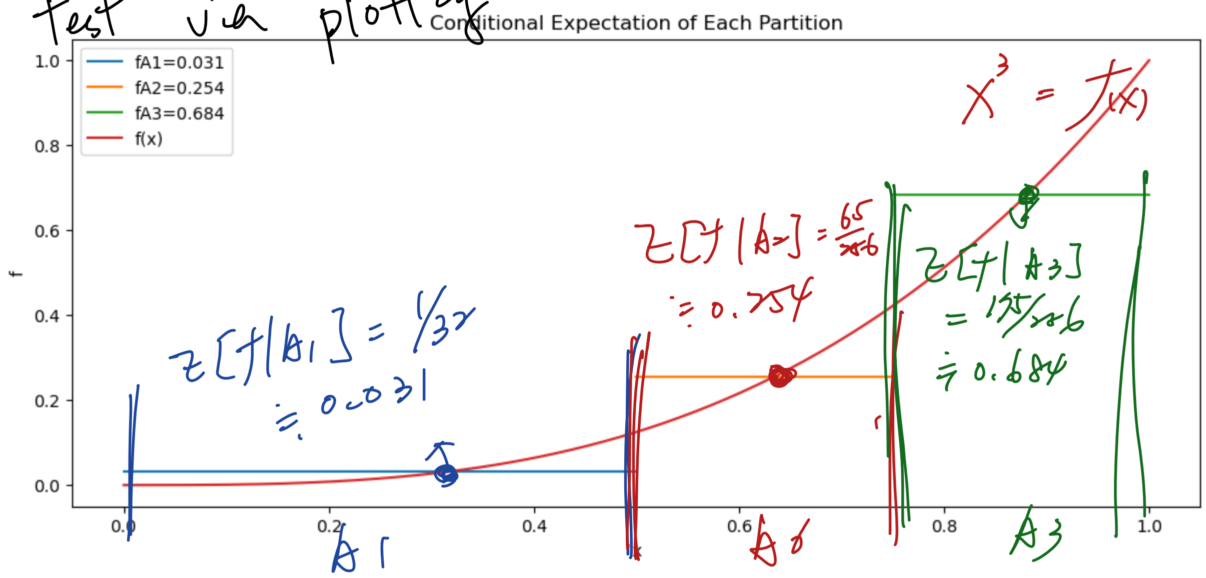
$$\bar{f}_{A_3} = \frac{1}{1 - 3/4} \int_{3/4}^{1} x^3 \, dx$$

$$= x^4 \Big|_{3/4}^{1} = 1 - \frac{81}{256} = \frac{175}{256}$$

$$\Rightarrow \begin{cases} \bar{f}_{A_1} = 1/32, & \text{if } x \in A_1, \ E[f \mid A_1] = 1/32 \\ \bar{f}_{A_2} = 65/256, & \text{if } x \in A_2, \ E[f \mid A_2] = 65/256 \\ \bar{f}_{A_3} = 175/256, & \text{if } x \in A_3, \ E[f \mid A_3] = 175/256 \end{cases}$$

\#

test via plotting:



12. Let X and Y be independent Gaussian random variables; let X have mean 1 and variance 4, and let Y have mean -1 and variance 2. Define

$$S_0 \stackrel{\text{def}}{=} 3, \qquad S_1 \stackrel{\text{def}}{=} S_0 \exp[X], \qquad \text{and} \qquad S_2 \stackrel{\text{def}}{=} S_1 \exp[Y].$$

$X \sim N(1,4)$

$Y \sim N(-1,2)$

Compute

(a) $\mathbb{E}[S_1]$

(b) $\mathbb{E}[S_2]$

(c) $\mathbb{E}[S_2|S_1]$.

(a) $E(S_1) = E(S_0 \exp X)$

$= 3E[e^X]$

let $y = \frac{X-1}{2}$, then $y \sim N(0,1)$

$\underbrace{2y+1 = X}$

$3E[\exp(2y+1)] = 3(E[e^{2y} \times e^1])$

$= 3e^1 E[e^{2y}] = 3e^1 \times e^2 = \underline{3e^3}$

(b) $E[S_2] = E[S_1 e^{Y}] = E[S_0 e^{X+Y}]$

$= 3\, E[e^{X+Y}]$

* let $Z = X+Y$. $Z \sim N(0, 6)$
* let $y = \frac{Z}{\sqrt{6}}$, $y \sim N(0,1)$, $\sqrt{6}\, y = Z$

$3E[e^{X+Y}] = 3 \times e^{\frac{6}{2}} = 3e^{3}$.

(c) $E[S_2 \mid S_1]$

$= E[S_2 \mid S_1 = s] = \sum_x x\, p(S_2 = x \mid S_1 = s)$

$= \sum_x x \frac{P(S_1 = s \cap S_2 = x)}{P(S_1 = s)} = \sum_x x \frac{\cancel{P(S_1 = s)}\, P(S_2 = x)}{\cancel{P(S_1 = s)}}$

∵ independent

$= \sum_x x\, p(S_2 = x) = E[S_2]$

$= E[S_1 \times \exp(Y)] = S_1\, E[\exp(Y)]$

$= S_1 \times \left[\exp\left(\frac{(\sqrt{2})^2}{2}\right) \div \exp(1)\right] = S_1$ #

S_1 here is constant.