Unless otherwise specified, W is a Brownian motion with filtration $\mathscr{W}$
Define

$$\operatorname{erf}(\ell) \stackrel{\text{def}}{=} \frac{1}{\sqrt{\pi}} \int_{t=-\ell}^{\ell} \exp\left[-t^2\right] dt; \qquad \ell \in \mathbb{R}$$

Technology

1. Compute $1 + 1$.

   Solution: 2

Gaussian

2. Suppose that X is Gaussian with mean 4 and variance 1. Explicitly compute $\mathbb{E}[X^3]$ (hint: perhaps you might want to write out the binomial expansion of $(a+b)^3$)

3. If η is a standard Gaussian random variable, compute

$$\mathbb{E}\left[\exp\left[2+3\eta\right]\right]$$

4. Let η be a standard Gaussian random variable. Compute $\mathbb{P}\{\eta \leq 3\}$ in terms of erf.

Central Limit Theorem

5. Suppose that $\{X_n\}_n$ are an independent and identically distributed collection of random variables with common law

$$\mathbb{P}\{X = -1\} = \mathbb{P}\{X = 0\} = \mathbb{P}\{X = 1\} = \frac{1}{3}.$$

(a) Compute the common mean μ of the X_n's.

(b) Compute the common variance σ^2 of the X_n's.

(c) Compute the characteristic function $\phi_X(\theta) \stackrel{\text{def}}{=} \mathbb{E}\left[e^{\imath\theta X}\right]$ for $\theta \in \mathbb{R}$. (I want you to work through the definition of the characteristic function)

(d) Define

$$S_N \stackrel{\text{def}}{=} \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \{X_n - \mu\}.$$

Using the statement of the central limit theorem (you don't need to reprove anything) find the limit of S_N as $N \nearrow \infty$.