Ito Integration

15. Define

$$M_t \stackrel{\text{def}}{=} \int_{s=0}^t e^{-s} dW_s.$$

For each $t > 0$, compute

(a) $\mathbb{E}[M_t]$.

(b) $\mathbb{E}[M_t^2]$

16. Define

$$A_t \stackrel{\text{def}}{=} \int_{s=0}^t W_s^3 dW_s$$

Compute

(a) $\mathbb{E}[A_t]$

(b) $\langle A \rangle_t$.

(c) $\mathbb{E}[A_t^2]$.

※ Martingale Property check!

1. $M_t = \int_0^t e^{-s} dW_s$

$E[M_t] = \int_0^t e^{-s} E[W_s - W_{s+\delta t}] = 0$

⇒ It's Martingale Process!

2. $A_t = \int_0^t W_s^3 dW_s$

$E[A_t] = \int_0^t \underbrace{\cancel{E[W_s^3]}}_{0} E[W_s - W_{s+\delta t}] = 0$

⇒ It's Martingale Process!

15.

(a) $E[M_t] = E\left[\int_0^t e^{-s} dW_s\right]$

$= E[\,E[M_t \mid \mathcal{F}_0]] = E[M_0] = 0$

(b) $E[M_t^2] = E\left[\left(\int_0^t e^{-s} dW_s\right)^2\right]$

$= E[\langle M \rangle_t]$

$\circledast$ $\langle M \rangle_t = \int_0^t (e^{-s})^2 ds = \int e^{-2s} ds = -\frac{1}{2} e^{-2s} \Big|_0^t$

$= -\frac{1}{2}(e^{-2t} - 1)$

$\Rightarrow E\left[-\frac{1}{2} e^{-2t} + \frac{1}{2}\right] = -\frac{1}{2} E[e^{-2t}] + \frac{1}{2}$

$= -\frac{1}{2} e^{-2t} + \frac{1}{2}$ #

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$$A_t = \int_0^t W_s^3 \, dW_s$$

(a)

$$E[A_t] = E[\,E[A_t \mid \mathcal{F}_0]\,] = 0$$

(b) $\langle A \rangle_t = A_t^2 = \int_0^t (W_s^3)^2 \, ds$

$$= \int_0^t W_s^6 \, ds$$

(c) $E[A_t^2] = E[\langle A \rangle_t] = E\left[\int_0^t W_s^6 \, ds\right]$

$$= \int_0^t E[W_s^6] \, ds = \frac{6!}{2^3 \, 3!} \int_0^t s^3 \, ds$$

$$= \frac{\cancel{6}^{3} \; 5 \; \cancel{4} \; \cancel{3}\cancel{2}\cancel{1}}{\cancel{8} \; \cancel{2} \; \cancel{3}\cancel{2}\cancel{1}} \times \frac{s^4}{4}\Big|_0^t$$

$$= 15 \times \frac{t^4}{4} = \frac{15 t^4}{4}.$$