(a) Compute $\mathbb{E}[W_1^2 W_s^2]$ for $0 \le s \le 1$.

(b) Compute

$$\mathbb{E}\left[W_1^2 \int_{s=0}^{1} W_s^2 \, ds\right].$$

(a) $E(W_1^2 W_s^2) = E[[(W_1^2 + W_s^2) - W_s^2] W_s^2]$

$= E[(W_1^2 + W_s^2) W_s^2 - W_s^4]$

$= E[[(W_1 - W_s)^2 + 2W_1 W_s] W_s^2 - W_s^4]$

$= E[(W_1 - W_s)^2 W_s^2 + 2W_1 W_s^3 - W_s^4]$

$= E[W_{1-s}^2] E[W_s^2] + E[2W_1 W_s^3] - E[W_s^4]$

$= (1-s)s + E[2[(W_1 - W_s) + W_s] W_s^3] - E[W_s^4]$

$= (1-s)s + E[2[W_s^3(W_1 - W_s) + W_s^4]] - E[W_s^4]$

$= (1-s)s + E[2W_s^3(W_1 - W_s)] + 2E[W_s^4] - E[W_s^4]$

$= (1-s)s + 2E[W_s^3] E[W_{1-s}]$ $\quad + E[W_s^4]$

$= s - s^2 + 0 + 3s^2 = s + 2s^2$ ※

(b) $E[W_1^2 \int_0^1 W_s^2 \, ds] = E[\int_0^1 W_1^2 W_s^2 \, ds]$

$= E[\int_0^1 (2s^2 + s) ds] = E[(\frac{2}{3}s^3 + \frac{1}{2}s^2)\Big|_0^1]$

$= E[\frac{2}{3} + \frac{1}{2}] = E[\frac{4+3}{6}] = 7/6$ ※

7. Compute

(a) $\mathbb{E}[(W_{10} - W_7)^2]$.

(b) $\mathbb{E}[(W_5 - W_1)^2]$.

(c) $\mathbb{E}[(W_{10} - W_7)(W_5 - W_1)]$.

(d) $\mathbb{E}\left[(3\{W_{10} - W_7\} + 4\{W_5 - W_1\})^2\right]$.

(a) $\mathbb{E}[(W_{10} - W_7)^2] = \mathbb{E}[(W_3 - W_0)^2]$

$= \mathbb{E}[W_3^2] = 3$

(b) $\mathbb{E}[(W_5 - W_1)^2] = \mathbb{E}[(W_4 - W_0)^2]$

$= \mathbb{E}[W_4^2] = 4$

(c) $\mathbb{E}[(W_{10} - W_7)(W_5 - W_1)]$

$= \mathbb{E}[(W_{10} - W_7)]\,\mathbb{E}[(W_5 - W_1)] = \mathbb{E}[W_3]\,\mathbb{E}[W_4]$

$= 0$

(d) $\mathbb{E}[[3(W_{10} - W_7) + 4(W_5 - W_1)]^2]$

$= \mathbb{E}[9(W_{10} - W_7)^2 + 16(W_5 - W_1)^2$

$\quad + 24(W_{10} - W_7)(W_5 - W_1)]$

$= 9\,\mathbb{E}[(W_{10} - W_7)^2] + 16\,\mathbb{E}[(W_5 - W_1)^2]$

$\quad + 24\,\mathbb{E}(W_{10} - W_7)\,\mathbb{E}(W_5 - W_1)$

$= 9\,\mathbb{E}[W_3^2] + 16\,\mathbb{E}(W_4^2) + 24\,\mathbb{E}(W_3)\,\mathbb{E}(W_4)$

$= 9 \times 3 + 16 \times 4 = 91$

8. This is a key calculation for *exponential martingales* (useful in *numéraire* calculations). Compute

(a) $\mathbb{E}[W_{10} - W_6]$.

(b) $\mathbb{E}[(W_{10} - W_6)^2]$.

(c) $\mathbb{E}\left[\exp\left[3(W_{10} - W_6)\right]\right]$.

(d)

$$\mathbb{E}\left[\exp\left[3(W_{10} - W_6) - 18\right]\right]$$

(a) $\mathbb{E}[W_{10} - W_6] = \mathbb{E}[W_4 - W_0]$

$= \mathbb{E}(W_4) = 0$

(b) $\mathbb{E}[(W_{10} - W_6)^2] = \mathbb{E}[W_4^2] = 4$

(c) $\mathbb{E}[\exp[3(W_{10} - W_6)]]$

$= \mathbb{E}[\exp(3W_{10}) \div \exp(3W_6)]$

$= \mathbb{E}[\exp(3W_{10})] \div \mathbb{E}[\exp 3(W_6)]$

$\downarrow$

$*$ $W_{10} \sim N(0, 10)$ $*$ $W_6 \sim N(0, 6)$

$y = \dfrac{W_{10}}{\sqrt{10}} \sim N(0, 1)$ $y = \dfrac{W_6}{\sqrt{6}} \sim N(0, 1)$

$\downarrow$

$= \mathbb{E}[\exp(3\sqrt{10}\, y)] \div \mathbb{E}[\exp(3\sqrt{6}\, y)]$

$= \exp(45) \div \exp(27) = \exp(45 - 27)$

$= \exp(18)$

(d) $\mathbb{E}(\exp[3(W_{10} - W_6)) - 18])$

$= \mathbb{E}[\exp(3(W_{10} - W_6))] \div \mathbb{E}(\exp(18))$

$= e^{18} \div e^{18} = 1$ $*$