IE525B: Numerical Method in Finance

Homework 01

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Problem 01

Write in Python a basic Black Schole's pricer assuming risk neutral interest rate. Your pricer can take in spot, interest rate, a constant volatility number, strike, expiration, call/put type, and return its Black Scholes' price.

In [1]:
```
import math
import scipy.stats as stats


def black_scholes(spot, interest_rate, volatility, strike, expiration,
                  option_type):
    d1 = (math.log(spot / strike) +
          (interest_rate + 0.5 * volatility**2) * expiration) / (
              volatility * math.sqrt(expiration))
    d2 = d1 - volatility * math.sqrt(expiration)

    if option_type.lower() == "call":
        price = spot * stats.norm.cdf(d1) - strike * math.exp(
            -interest_rate * expiration) * stats.norm.cdf(d2)
    elif option_type.lower() == "put":
        price = strike * math.exp(
            -interest_rate *
            expiration) * stats.norm.cdf(-d2) - spot * stats.norm.cdf(-d1)
    else:
        raise ValueError("Invalid option type. Please enter 'call' or 'put'.")

    return price


# Example usage:
spot = 100 #spot price of an asset
interest_rate = 0.05 #interest rate
volatility = 0.2 #volatility
strike = 110 #strike price
expiration = 1 #expire in 1 year
option_type = "call"#option type

price = black_scholes(spot, interest_rate, volatility, strike, expiration,
```

```
                         option_type)
print(f"The Black-Scholes price for the {option_type} option is: {price:.2f}")
```

```
The Black-Scholes price for the call option is: 6.04
```

Problem 02

Try to Finish the first integral in the Black Scholes derivation either via the change of variable method, or the change of measure method(using S as the Numeraire).

Ans: Below is the derivation of both 2nd and 1st part of BS Model. I apologized that using hand writing since my iPad is not functioning.
I accomplished this by slightly different way compared to the one in the lecture note that I gained both d2 and d1 terms rather than just simply using one "I" term.
However, the result is same as the one in the lecture note, and have correctly derived both terms of BS Model.

$$\begin{cases} dS_t = rS_t\,dt + \sigma S_t\,d\tilde{W}_t, \quad S_t = S_0 e^{(r-\frac{1}{2}\sigma^2)T + \sigma\tilde{W}_t} \\ C_0 = e^{-rT}E^Q[(S_T - k)\mathbb{1}_{S_T > k}] = e^{-rT}E^Q[\max(S_T - k, 0)] \end{cases}$$

$$= \underbrace{e^{-rT}E^Q[S_T\mathbb{1}_{S_T > k}]}_{*\ \text{1st part}} - \underbrace{e^{-rT}E^Q[k\mathbb{1}_{S_T > k}]}_{\text{2nd part}}$$

(where Q is the prob - distribution)

$\Downarrow$

(2nd part)

$$e^{-rT}E^Q[k\mathbb{1}_{S_T > k}] = ke^{-rT}Q[S_T > k] = ke^{-rT}Q[S_0 e^{(r-\frac{1}{2}\sigma^2)T + \sigma\tilde{W}_t} > k]$$

$$= ke^{-rT}Q[\ln S_0 + (r - \tfrac{1}{2}\sigma^2)T + \sigma\tilde{W}_t > \ln k]$$

$$= ke^{-rT}Q\left[\tilde{W}_t > \frac{\ln k - \ln S_0 - (r - \frac{1}{2}\sigma^2)T}{\sigma}\right]$$

now, substitute.

$Q = \text{Norm - dist}$

$\Rightarrow \tilde{W}_t \sim N(0, t)$

$= \frac{\tilde{W}_t}{\sqrt{t}} \sim N(0,1)$

$$= ke^{-rT}N\left[\frac{-\ln k + \ln S_0 + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right]$$

$\downarrow$

$$= ke^{-rT}N[d_2]$$

$$= k\frac{e^{-rT}}{\sqrt{2\pi}}\int_{-\infty}^{d_2} \exp\left(-\frac{x^2}{2}\right)dx$$

(1st part)

$$e^{-rT} E^{Q}[S_T \mathbb{1}_{S_T>K}] = e^{-rT} E^{Q}[S_0 e^{(r-\frac{1}{2}\sigma^2)T + \sigma \tilde{W}_t} \mathbb{1}_{S_T>K}]$$

$$= e^{-rT} E^{Q}[S_0 e^{rT} e^{-\frac{1}{2}\sigma^2 T + \sigma \tilde{W}_t} \mathbb{1}_{S_T>K}]$$

$$= S_0 E^{Q}[e^{-\frac{1}{2}\sigma^2 T + \sigma \tilde{W}_t} \mathbb{1}_{S_T>K}]$$

Girsanov's Theorem: if $\tilde{W}_t$ is process with prob-dist Q, and if $Y(t) = \sigma t$

and if $W'_t = \tilde{W}_t - \sigma t$ with Q' (new process)

$$\Rightarrow \frac{dQ'}{dQ} = e^{\sigma \tilde{W}_t - \frac{1}{2}\sigma^2 t}.$$

and

$$S_T = S_0 e^{(r-\frac{1}{2}\sigma^2)T + \sigma(W'_t + \sigma T)}$$

$$= S_0 e^{(r+\frac{1}{2}\sigma^2)T + \sigma W'_t}$$

$$= S_0 E^{Q}[\frac{dQ'}{dQ} \mathbb{1}_{S_T>K}]$$

$$= S_0 E^{Q'}[\mathbb{1}_{S_T>K}]$$

$$= S_0 Q'[S_T > K] = S_0 Q'[S_0 e^{(r+\frac{1}{2}\sigma^2)T + \sigma W'_t} > K]$$

$$= S_0 Q'[\frac{B'_T}{\sqrt{T}} > \frac{\ln K - \ln S_0 - (r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}]$$

$$= S_0 N[\frac{-\ln K + \ln S_0 + (r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}] = S_0 N[d_1]$$

$$= \frac{S_0}{\sqrt{2\pi}} \int_{-\infty}^{d_1} \exp(-\frac{x^2}{2})\, dx.$$