Numerical Method in Finance

Homework 02

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Problem 01

Observe SPX option prices such as below. Pick an expiration date. Assume a reasonable interest rate(say 4%). Index level can be observed at the top(say 3970). Invert implied vols from your Black Scholes pricers for all the strikes in that expiration. Hence you need to now code a root finding algorithm in Python which takes as inputs option prices from the market, and your BS pricer, find the implied volatility for the given strike and expiration.
S&P 500 Index Options Prices - Barchart.com

Basic Import

In [20]:
```
import math
import scipy.stats as si
import numpy as np
import pandas as pd
```

Function Defination

In [21]:
```
def black_scholes(S, K, T, r, sigma, option_type='call'):
    d1 = (np.log(S / K) + (r + 0.5 * sigma ** 2) * T) / (sigma * np.sqrt(T))
    d2 = d1 - sigma * np.sqrt(T)

    if option_type == 'call':
        price = S * si.norm.cdf(d1) - K * np.exp(-r * T) * si.norm.cdf(d2)
    elif option_type == 'put':
        price = K * np.exp(-r * T) * si.norm.cdf(-d2) - S * si.norm.cdf(-d1)
```

```python
    else:
        raise ValueError("Invalid option type. Choose either 'call' or 'put'

    return price

def implied_volatility(S, K, T, r, market_price, option_type='call', initial
    sigma = initial_guess
    for _ in range(max_iter):
        option_price = black_scholes(S, K, T, r, sigma, option_type)
        if abs(option_price - market_price) < tol:
            return sigma

        # Calculate option price derivative with respect to sigma (vega)
        d1 = (np.log(S / K) + (r + 0.5 * sigma ** 2) * T) / (sigma * np.sqrt
        vega = S * si.norm.pdf(d1) * np.sqrt(T)

        if vega == 0:
            return sigma

        # Newton-Raphson method
        sigma -= (option_price - market_price) / vega

    return sigma
```

Read Data

In [25]:

```python
# set up date time
now = np.datetime64('2023-04-08')
expiration_date = np.datetime64('2023-08-18','D')

print("Observation stock Price: 3970")
print("Risk free rate: 4%")
print(f"T: {(expiration_date - now).astype(float) / 365.0}")
print("Observed at 2023-04-08")
print("Expiration date: 2023-08-18")

# read from csv
df = pd.read_csv('/Users/yu-chingliao/Library/CloudStorage/GoogleDrive-josep
df.head()
```

```
Observation stock Price: 3970
Risk free rate: 4%
T: 0.36164383561643837
Observed at 2023-04-08
Expiration date: 2023-08-18
```

Out[25]:

	Strike	Moneyness	Bid	Midpoint	Ask	Last	Change	%Chg	Volume	Open Int	C
0	4,025.00	+1.95%	260.2	262.35	264.5	217.50	0.0	unch	0.0	5.0	ui
1	4,030.00	+1.83%	256.8	258.90	261.0	241.07	0.0	unch	0.0	20.0	ui
2	4,040.00	+1.58%	249.9	252.00	254.1	272.40	0.0	unch	0.0	1.0	ui
3	4,050.00	+1.34%	243.1	245.20	247.3	224.43	0.0	unch	0.0	82.0	ui
4	4,060.00	+1.10%	236.4	238.45	240.5	248.80	0.0	unch	0.0	5.0	ui

In [26]:

```
# our observation
df['Observed Stock Price'] = 3970
df['Rf Rate'] = 0.04
df['T'] = (expiration_date - now).astype(float) / 365.0

# select only using col from the dataframe
df=df[['Observed Stock Price','Strike','T','Rf Rate','Midpoint','IV']]

# convert type to lower
df['IV']= df['IV'].str.lower()
df.dropna(inplace = True)

# Check datatype
df['Strike'] = df['Strike'].apply(lambda x: float(x.replace(',', ''))).astyp

market_data_lst = df.values.tolist()

implied_vols = []

for data in market_data_lst:
    iv = implied_volatility(*data)
    implied_vols.append(iv)
    S, K, T, r, market_price, option_type = data
    print(f"Option Type: {option_type} Strike Price: {K}, Option Price: {mar
```

```
Option Type: call Strike Price: 4025.0, Option Price: 262.35, Implied Volat
ility: 0.27438068453380005
Option Type: call Strike Price: 4030.0, Option Price: 258.9, Implied Volati
lity: 0.2731727514755915
Option Type: call Strike Price: 4040.0, Option Price: 252.0, Implied Volati
lity: 0.2707144332669891
Option Type: call Strike Price: 4050.0, Option Price: 245.2, Implied Volati
lity: 0.2683035305038347
Option Type: call Strike Price: 4060.0, Option Price: 238.45, Implied Volat
ility: 0.2658859104829015
Option Type: call Strike Price: 4070.0, Option Price: 231.75, Implied Volat
ility: 0.26346005595126704
Option Type: call Strike Price: 4075.0, Option Price: 228.45, Implied Volat
ility: 0.26227636378580904
Option Type: call Strike Price: 4080.0, Option Price: 225.1, Implied Volati
lity: 0.26102439993362264
Option Type: call Strike Price: 4090.0, Option Price: 218.6, Implied Volati
lity: 0.2586823349762912
Option Type: call Strike Price: 4100.0, Option Price: 212.15, Implied Volat
ility: 0.2563272694334568
Option Type: call Strike Price: 4110.0, Option Price: 205.7, Implied Volati
lity: 0.2539049725416949
Option Type: call Strike Price: 4120.0, Option Price: 199.4, Implied Volati
lity: 0.25157145640349765
Option Type: call Strike Price: 4125.0, Option Price: 196.2, Implied Volati
lity: 0.2503256144478668
Option Type: call Strike Price: 4130.0, Option Price: 193.1, Implied Volati
lity: 0.2491671828915923
Option Type: call Strike Price: 4140.0, Option Price: 186.95, Implied Volat
ility: 0.24684848232642895
Option Type: call Strike Price: 4150.0, Option Price: 180.85, Implied Volat
ility: 0.24450820801975676
Option Type: call Strike Price: 4160.0, Option Price: 174.85, Implied Volat
ility: 0.24219747465504668
Option Type: call Strike Price: 4170.0, Option Price: 168.9, Implied Volati
lity: 0.23986145135927098
Option Type: call Strike Price: 4175.0, Option Price: 166.0, Implied Volati
lity: 0.23874329817570558
Option Type: call Strike Price: 4180.0, Option Price: 163.1, Implied Volati
lity: 0.23760487326092863
Option Type: put Strike Price: 4025.0, Option Price: 133.8, Implied Volatil
ity: 0.1420414621170814
Option Type: put Strike Price: 4030.0, Option Price: 135.3, Implied Volatil
ity: 0.14094245355007545
Option Type: put Strike Price: 4040.0, Option Price: 138.2, Implied Volatil
ity: 0.1385517608670967
Option Type: put Strike Price: 4050.0, Option Price: 141.25, Implied Volati
lity: 0.1361970690123368
Option Type: put Strike Price: 4060.0, Option Price: 144.3, Implied Volatil
ity: 0.13371459224696386
Option Type: put Strike Price: 4070.0, Option Price: 147.4, Implied Volatil
ity: 0.13114970943560286
Option Type: put Strike Price: 4075.0, Option Price: 149.0, Implied Volatil
ity: 0.12986688036381433
Option Type: put Strike Price: 4080.0, Option Price: 150.6, Implied Volatil
ity: 0.1285472211292378
```

```
Option Type: put Strike Price: 4090.0, Option Price: 153.9, Implied Volatil
ity: 0.1258986185209881
Option Type: put Strike Price: 4100.0, Option Price: 157.25, Implied Volati
lity: 0.1231408077914357
Option Type: put Strike Price: 4110.0, Option Price: 160.65, Implied Volati
lity: 0.12026158186078888
Option Type: put Strike Price: 4120.0, Option Price: 164.15, Implied Volati
lity: 0.11730106260703817
Option Type: put Strike Price: 4125.0, Option Price: 165.9, Implied Volatil
ity: 0.11574435212111864
Option Type: put Strike Price: 4130.0, Option Price: 167.7, Implied Volatil
ity: 0.11418847498225378
Option Type: put Strike Price: 4140.0, Option Price: 171.35, Implied Volati
lity: 0.11095907174586575
Option Type: put Strike Price: 4150.0, Option Price: 175.1, Implied Volatil
ity: 0.1075898855754572
Option Type: put Strike Price: 4160.0, Option Price: 178.9, Implied Volatil
ity: 0.103993162042896
Option Type: put Strike Price: 4170.0, Option Price: 182.8, Implied Volatil
ity: 0.10018729943821145
Option Type: put Strike Price: 4175.0, Option Price: 184.75, Implied Volati
lity: 0.0981442246832659
Option Type: put Strike Price: 4180.0, Option Price: 186.75, Implied Volati
lity: 0.09605827357761818
```

Problem 02

Finish the derivation of Black-Scholes formula for pricing a compo option.

Let S be the foreign stock price, X be the strike price in domestic currency, r be the domestic risk-free rate,q be the foreign risk-free rate, σ_s be the volatility of the stock, and σx be the volatility of the exchange rate. Let the correlation between the stock and the exchange rate be ρ. We'll use the Black-Scholes-Merton model to price this option.

Let $C(S, X, T)$ be the price of a compo call option with time to expiration T. To derive the Black-Scholes formula for a compo option, we can follow these steps:

1. Define the risk-neutral measure.
2. Write down the stochastic differential equations (SDEs) for the stock price and exchange rate.
3. Derive the risk-neutral dynamics of the stock price and exchange rate.
4. Determine the dynamics of the stock price in domestic currency.
5. Apply Ito's Lemma to find the dynamics of the option price.
6. Solve the resulting partial differential equation (PDE) using risk-neutral pricing.

Describe:

1. Risk-neutral measure:

In the risk-neutral world, the domestic and foreign risk-free rates are used to discount the expected payoffs. Therefore, the domestic risk-free rate, r, and the foreign risk-free rate, q, are the appropriate discount rates.

2. SDEs for the stock price and exchange rate:

Let S_t be the foreign stock price, and let X_t be the exchange rate (units of domestic currency per unit of foreign currency). Then, the SDEs for S_t and X_t are:

$$dS_t = S_t(qdt + \sigma_s dW_{s,t})$$

$$dX_t = X_t(rdt + \sigma_x dW_{x,t})$$

where $W_{s,t}$ and $W_{x,t}$ are Wiener processes representing the stochastic components of the stock price and exchange rate, respectively, with correlation ρ: $dW_{s,t}dW_{x,t} = \rho dt$.

3. Risk-neutral dynamics:

Under the risk-neutral measure, the drift terms change. The risk-neutral SDEs for S_t and X_t become:

$$dS_t = S_t((r-q)dt + \sigma_s dW_{s,t})$$

$$dX_t = X_t(rdt + \sigma_x dW_{x,t})$$

4. Dynamics of the stock price in domestic currency:

Now, we need to find the dynamics of the stock price in domestic currency, $S_t X_t$. Using Ito's product rule:

$$d(S_t X_t) = S_t dX_t + X_t dS_t + dS_t dX_t$$

Substitute the risk-neutral SDEs of S_t and X_t into the equation above:

$$d(S_t X_t) = S_t X_t(rdt + \sigma_x dW_{x,t}) + X_t S_t((r-q)dt + \sigma_s dW_s$$
$$\rho\sigma_s\sigma_x S_t X_t dt$$

5. Apply Ito's Lemma to the option price:

Using Ito's Lemma, we can find the dynamics of the option price, $C(S_t X_t, t)$:

$$dC = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial(S_t X_t)}d(S_t X_t) + \frac{1}{2}\frac{\partial^2 C}{\partial(S_t X_t)^2}(d(S_t X_t))^2$$

Substitute the dynamics of $S_t Xt$ we derived in step 4 into the equation above:

$dC = \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial (S_t X_t)}$
$(S_t X_t(rdt + \sigma_x dW_{x,t}) + X_t S_t((r-q)dt + \sigma_s dW_{s,t}) + \rho\sigma_s\sigma_x S_t X_t dt) + \frac{1}{2} \frac{\partial^2 C}{\partial (S_t X_t)^2}$
$(d(S_t X_t))^2$

Now we have the dynamics of the option price under the risk-neutral measure.

6. Solve the PDE using risk-neutral pricing:

The option price can be obtained by solving the following PDE:

$$\frac{\partial C}{\partial t} + rS_t X_t \frac{\partial C}{\partial (S_t X_t)} + \frac{1}{2} (S_t X_t)^2 (\sigma_s^2 + \sigma_x^2 + 2\rho\sigma_s\sigma_x) \frac{\partial^2 C}{\partial (S_t X_t)^2} - rC = 0$$

with the boundary condition at expiration T:

$$C(S_T X_T, T) = max(S_T X_T - K, 0)$$

where K is the strike price in domestic currency. This PDE is similar to the standard Black-Scholes PDE, but the volatility term is now the combined volatility of the stock and the exchange rate.

To solve this PDE, we can use a similar approach to the standard Black-Scholes formula. First, apply the following transformations:

$$y = ln(S_t X_t), v(y, t) = e^{-\alpha y - \beta t} C(S_t X_t, t)$$

where α and β are constants to be determined later. After transforming the PDE and applying the chain rule, we obtain a heat equation, which can be solved using standard methods. Finally, we can find the Black-Scholes formula for the compo option by applying the inverse transformations.

The resulting Black-Scholes formula for a compo call option is:

$$C(S_t X_t, t) = S_t X_t e^{-q(T-t)} N(d1) - Ke^{-r(T-t)} N(d2)$$

where

$$d1 = \frac{ln(\frac{S_t X_t}{} K) + (r - q + \frac{1}{2}(\sigma_s^2 + \sigma_x^2 + 2\rho\sigma_s\sigma_x)(T-t))}{\sqrt{(\sigma_s^2 + \sigma_x^2 + 2\rho\sigma_s\sigma_x)(T-t)}}$$

$$d2 = d1 - \sqrt{(\sigma_s^2 + \sigma_x^2 + 2\rho\sigma_s\,\sigma_x\,)(T-t)}$$

and $N(\cdot)$ is the cumulative distribution function of the standard normal distribution.

A similar derivation can be done for a compo put option, which will yield the corresponding Black-Scholes formula for that case.