1.

From the note of Notes_knn.pdf, we get:

new - $X_0, Y_0 \in \{0, 1\}$, and suppose

$f(X_0) < 1/2$

$P(Y_0 = 1) = f(X_0) \Rightarrow \hat{Y}_B = 0$

From Baysian model

$\Rightarrow P(Y_0 \neq \hat{Y}_B) = P(Y_0 \neq 0) = f(X_0) = p$

$1-NN: p(\hat{Y}_{1nn} = 1) = f(X_{1nn}) \xrightarrow[n\to\infty]{} f(X_0) = p$

As $n \to \infty$

$\Rightarrow p(\hat{Y}_{3NN} = 1) \leq \binom{3}{2} p^2(1-p) + \binom{3}{3} p^3$

$= 3p^2(1-p) + p^3 = p^2(3-2p)$

$\Rightarrow p(\hat{Y}_{3NN} \neq Y_0) \leq p(\hat{Y}_{3NN} = 1)\, p(Y_0 = 0) +$

$p(\hat{Y}_{3NN} = 0)\, p(Y = 1)$

$= p^2(3-2p)(1-p) + (1 - p^2(3-2p))p$

$= 3p^2 - 2p^3 - 3p^3 + 2p^4 + p - 3p^3 + 2p^4$

$= p(1-p)(4p - 4p^2 + 1)$

We want to show $1.4p \geq 4p^4 - 8p^3 + 3p^2 + p$

$\Leftrightarrow 4p^4 - 8p^3 + 3p^2 - 0.4p \leq 0$

Since $0 < p < 1$, so $-4p^3 + 8p^2 - 3p + 0.4 \geq 0$

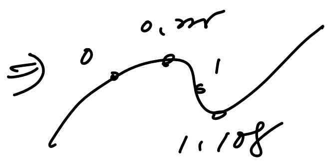
Let $f(p)\ -4p^3 + 8p^2 - 3p + 0.4$

$$\frac{d}{dp} f(x) = -12p^2 + 16p - 3 = 0$$

$$\Rightarrow p = \frac{16 \pm \sqrt{12}}{24} \approx 1.108 \ \& \ 0.225$$

$f'(1) > 0$, $f'(0) > 0$ $\Rightarrow$ when $0.225 < p < 1.108$

$f'(p) < 0$

$\Rightarrow$ 

For $p = 0 \Rightarrow f(0) \geq 0$

$p = 1 \Rightarrow f(1) \geq 0$

$\therefore$ when $0 \leq p \leq 1 \Rightarrow 1.4p \geq 4p^4 - 8p^3 + 3p^2 + p$

$\Rightarrow$ The upper bound can be weaken to $P_{3NN} \leq 1.4p$.

$\Rightarrow$ We have upper bound of the asymptotic error of $1-NN$ is $2p$ so we know after we weaken the upperbound of $3-NN$, it can be smaller than ub of the asymptotic error of $1-NN$.

2.

(a) Using $2^{|A|} \times 1$ matrix which is indexed by subset of A, and the i th entry is 1 if index is a subset of A_k, where A_k is a subset of A.

Denote that $\phi(A) = (\phi_{A_k}(A))_{A_k \subseteq A} = I_{A_k \subseteq A}$

Then,

$K(A_1, A_2) = \langle \phi(A_1), \phi(A_2) \rangle$

$= \sum_{A_k \subseteq A} \phi_{A_k}(A_1)\, \phi_{A_k}(A_2)$

$= \sum_{A_k \subseteq A} I_{A_k \subseteq A_1, A_k \subseteq A_2} = \sum_{A_k \subseteq A} I_{A_k \subseteq \{A_1 \cap A_2\}}$

$= 2^{|A_1 \cap A_2|}$

(b) Let $u^{\otimes 2} = u \cdot u^T$

$\phi(A) = (\phi_{A_k}(A))^{\otimes 2}_{A_k \subseteq A}$, where $\phi_{A_k}(A) = I_{A_k \subseteq A}$

then

$K(A_1, A_2) = \langle \phi(A_1), \phi(A_2) \rangle$

$= \left(\sum_{A_k \subseteq A} \phi_{A_k}(A_1)\, \phi_{A_k}(A_2)\right)^2$

$= \left(2^{|A_1 \cap A_2|}\right)^2 = 4^{|A_1 \cap A_2|}$

(c) $\phi(A) = (\phi_{A_k}(A))_{A_k \subseteq A}$

$\phi(A) = \mathbb{1}_{A_k \subseteq A} (\sqrt{a-1})^{|A_k|}, \ a > 1$

$K(A_1, A_2) = \langle \phi(A_1), \phi(A_2) \rangle$

$= \sum_{A_k \subseteq A} \phi_{A_k}(A_1) \phi_{A_k}(A_2) = \sum_{A_k \subseteq A} \sqrt{a-1}^{|A_k|}$

$A_k \in \{A_1 \cap A_2\}$

$= a^{|A_1 \cap A_2|}, \ a > 1$

3.

(a) when $m = 1$, (with only R_1) $\Rightarrow \hat{y} = \frac{1}{|R_1|} y$

$$Dof = \frac{\sum cov(y_i, \hat{y}_i)}{\sigma^2} = \sum_{R_1} \frac{1}{|R_1|} \frac{\sigma^2}{\sigma^2} = 1$$

$\Rightarrow$ when $m = 1$, $Dof = 1 = m$

Suppose $Dof = m$ is true, for $m = k$,

$k \in N$, then, when $m = k+1$

$$\hat{y} = \begin{pmatrix} \frac{1}{|R_1|} & 0 & \cdots & 0 \\ 0 & \frac{1}{|R_2|} & & \\ \vdots & & \ddots & \\ 0 & & & \frac{1}{|R_{k+1}|} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

$$\text{Dof} = \frac{\sum cov(y_i, \hat{y}_i)}{\sigma^2} = \left(\sum_{R_1} \frac{1}{|R_1|} \frac{\sigma^2}{\sigma^2} + \sum_{R_2} \frac{1}{|R_2|} \frac{\sigma^2}{\sigma^2} + \right.$$

$$\left. \cdots + \sum_{R_K} \frac{1}{|R_K|} \frac{\sigma^2}{\sigma^2} \right) + \sum_{R_{K+1}} \frac{1}{|R_{K+1}|} \frac{\sigma^2}{\sigma^2} = K+1$$

$\Rightarrow$ When $m = K+1$, $\text{Dof} = K+1 = m$

By Mathematical Induction, $\text{Dof} = m$, $\forall m = n$.

(b)

```r
set.seed(1)
n <- 100
p <- 10
X <- matrix(rnorm(n*p), ncol=p)
y <- rnorm(n)
```

(c)

```r
library(tree)
m <- c(1, 5, 10)
dof <- numeric(length(m))
for (i in seq_along(m)) {
  tree.fit <- tree(y ~ ., data=data.frame(X, y), mincut=m[i])
  yhat <- predict(tree.fit)
  covar <- cov(y, yhat) / var(y)
  dof[i] <- sum(diag(covar))
}
dof
```

```
[1]  1.154744  4.209729 11.457395
```

$\Rightarrow$ Confirming the result in (a).