Mercury's Perihelion Advance from the Solar Quadrupole Moment

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Abstract
Utilizing the method of General Relativity Theroy(GRT), Mercury's perihelion advance can be precisely interpreted. This interpretation is one of the most well-known tests of GRT. However, with the development of space probe technology, many new details of celestial bodies can be obtained than ever brfore, expecially the solar. According to new observational data obtained from the Solar Heliospheric Satellite(SoHO) and from the Global Oscillations Network Group(GONG), the internal rotational velocity distribution can be determined and, farther more, the solar quadrupole J_2. Due to J_2 the gravitational potential should be modified by adding a term of r^{-3}. The adjunction can conduce to Mercury's perihelion as well, which was once thought to be a challenge to Einstein's theory. The main goal of this paper is to calculate how many arcsec per century of Mercury's perihelion advance is contributed by the solar quadrupole moment with two different methods and compared it with the perihelion shift given from GRT.

Keywords: Mercury's perihelion, gravitation, Solar, gravitational potential, central force

1. Introduction

Urbain Jean Joseph Le Verriere[1] was the first astronomer to calculate Mercury's perihelion advance applying Newton's Law of Gravitation. In 1858, he started to check the observations towards the solar in detail, relying mainly on 14 very accurate solar transit times between 1697 and 1848. He found that Mercury's perihelion is processing slowly just as prospected, the observed precession was approximate 565 seconds of arc per century, however, on the accordance of his calculation it should be 527 seconds of arc per century, leaving 38 seconds of arc he couldn't explain. In his opinion, Mercury's perihelion advance was caused by the additional force exerted by one unknown planet.

This can be modeled precisely by an approximation that replace each one of the outer planet by a ring of uniform density and the same mass, as Price and Rush have showed in their paper[2]. Therefore, the mass density $\lambda_i = M_i/2\pi R_i$, where M_i stands for the mass of each planet and R_i the orbit radius. The force on the Mercury provided by the ring can be written as

$$\boldsymbol{F}_i = [mG\pi\lambda_i/({R_i}^2 - a^2)]\boldsymbol{e}_r, \tag{1}$$

where m is the mass of Mercury and a is its orbital semi-major axis. Then from Newton's second law, the differential equation relates the gravitational force and the motion:

$$\Phi(r) = F_o(r) + F(r) = m(\ddot{r} - \dot{\theta}^2 r), \tag{2}$$

where $\Phi(r)$ is the total applied forces, F_o and $F(r)$ are forces exerted by the solar and the external planetary rings respectively, and $F(r) = G\pi m \sum_{i=2}^{9} \lambda_i a/(R_i^2 - a^2)$. Utilizing a Taylor series and solving the differential equation for apsidal angle, which is defined as the angle between the perihelion and aphelion, we obtain:

$$\Psi = \frac{\pi}{\sqrt{3 + a\left[\frac{\Phi'(a)}{\Phi(a)}\right]}}, \tag{3}$$

where Ψ is the apsidal angle. After substituting $\Phi'(a)$ and $\Phi(a)$ into the Eq(3), it can be rewritten as:

$$\Psi = \frac{\pi}{\sqrt{3 + \frac{-2F_o + Gm\pi a \sum_{i=2}^{9} \lambda_i \frac{R_i^2 + a^2}{(R_i^2 - a^2)^2}}{F_o + F(a)}}}. \tag{4}$$

Now, applying a binomial expansion to the Eq(4):

$$\Psi = \pi \left[1 - \frac{F(a)}{F_o} - \frac{Gm\pi a \sum_{i=2}^{9} \lambda_i \frac{R_i^2 + a^2}{(R_i^2 - a^2)^2}}{2F_o}\right]. \tag{5}$$

The numerical value for Ψ can be calculated by substituting the data into the Eq(5), according to Price and Rush's paper, $\Psi = \pi(1 + 9.884 \times 10^{-7})$, and then the procession of Mercury's orbit per revolution is

$$\frac{2\Psi - 2\pi}{T}, \tag{6}$$

where T is the period of Mercury's orbit, 87.969 days on earth, and convert it to a more convenient form: $procession = 531.9 arcsec/century$. Still Newton's theory failed to explain the observed procession using the known masses in the solar system.

2. The interpretation from GRT

Schwarzschild metric is one of the most famous solution to the gravitational field equation, and it can be used to explain the advance of Mercury's perihelion. The metric for the spacetime around a spherically symmetric non-rotating mass m is[3]:

$$(\mathrm{d}s)^2 = \frac{(\mathrm{d}r)^2}{1 - \frac{2Gm}{c^2 r}} + r^2\left[(\mathrm{d}\theta)^2 + \sin^2\theta(\mathrm{d}\phi)^2\right] - c^2\left(1 - \frac{2Gm}{c^2 r}\right)(\mathrm{d}t)^2, \tag{7}$$

where G is the gravitational constant and c is the speed of light.

The PPN formalism is a frame work to parametrize various theory of gravity in a systematic way. β and γ are the most important of ten parameters. The metric in isotropic coordinate can be written as[4]:

$$\mathrm{d}s^2 = -\left[1 - 2\frac{GM}{c^2 r} - 2\beta\left(\frac{GM}{c^2 r}\right)^2\right](c\mathrm{d}t)^2 + \left(1 + 2\gamma\frac{GM}{c^2 r}\right)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2). \tag{8}$$

In the metric, γ is related to the amount of curvature produced by a unit rest mass, and the parameter β represents the degree of non-linearity in the superposition law for gravity. In GRT, β=γ=1.

The curves of Mercury's orbits do not close when the coefficient of ϕ is unequal to 1. A Keplerian orbit modified by the metric can be written (Misner et al. 1973)

$$r = \frac{a(1-e^2)}{1 + ecos[(1 - \delta\phi/2\pi)\phi]}, \tag{9}$$

where a is the semi-major axis, e is the eccentricity, and ϕ is the true anomaly. The procession of the perihelion in one orbital period is

$$\delta\phi = \frac{6\pi GM}{a(1-e^2)c^2}\frac{(2 - \beta + 2\gamma)}{3}. \tag{10}$$

According to the data from NASA's Solar system bodies: Mercury website, the perihelion shift is 42.9 seconds of arc per century, and this can be explained by Eq(10). The interpretation to Mercury's perihelion advance was one of the most successful examples to explore the universe utilizing the general relativity theory, and it has became one of the strongest verifications to Einstein's theory.

3. Solar quadrupole moment

Due to the rotation, the shape of solar is not a prefect sphere, this has been confirmed by observations, solar is slightly more flat at the poles than the equator. The gravitational quadrupole moment J_2 of the solar is corresponding to the second Legendre polynomial as a function of co-latitude in an expansion of the gravitational field on Legendre polynomials. The gravitational potential outside the solar[5]:

$$\Phi(r,\theta) = -\frac{GM}{r}\left[1 - \sum_{n=even}^{\infty}\left(\frac{R_s}{r}\right)J_n P_n(cos\theta)\right], \tag{11}$$

where Φ is the gravitational potential expressed in polar coordinates(r,θ,ϕ), M and R_s are mass and average radius of the solar respectively. And J_2 is related to the solar oblateness Δ_ϕ[6].

However, in this paper, the main method to obtain the quadrupole moment is to expand the gravitational potential as Taylor series. For the continuous distribution of mass, potential of the gravitational field at the point $\boldsymbol{r}$ due to a distribution $m(\boldsymbol{r}')$ can be written as the integral:

$$\Phi(\boldsymbol{r}) = -\int\frac{G\rho\mathrm{d}V}{|\boldsymbol{r} - \boldsymbol{r}'|} \tag{12}$$

where ρ represent the density of the solar, for simplicity, we choose the center of the solar as the original point. Expanding Eq(11) as Taylor series:

$$\Phi(\boldsymbol{r}) = -\int G\rho \mathrm{d}V \left\{ \frac{1}{r} + \frac{\boldsymbol{r}' \cdot \boldsymbol{r}}{r^3} + \frac{1}{2}\left[3\frac{(\boldsymbol{r}' \cdot \boldsymbol{r})^2}{r^5} - \frac{r'^2}{r^3}\right] + \cdots \right\}, \tag{13}$$

For convenient, $\boldsymbol{r}$,$\boldsymbol{r}'$ and $\mathrm{d}V$ can be denoted as[7]:

$$\begin{aligned} &\boldsymbol{r} = (x,y,z) = (x^1,x^2,x^3); \\ &\boldsymbol{r}' = (x',y',z') = (x'^1,x'^2,x'^3); \\ &\mathrm{d}V = \mathrm{d}x'^1 x'^2 x'^3 \end{aligned} \tag{14}$$

Then Φ can be written as:

$$\Phi = -G\left[\frac{M}{r} + \frac{1}{r^3}\sum_k x^k D^k + \frac{1}{2}\sum_{k,l} Q^{kl}\frac{x^k x^l}{r^5} + \cdots\right], (k,l = 1,2,3) \tag{15}$$

where $M = \int \rho \mathrm{d}x'^1 x'^2 x'^3$ is the mass of solar, $D^k = \int x'^k \rho \mathrm{d}x'^1 x'^2 x'^3$ is the dipole moment, in this coordinate $D^k = 0$, and the most important parameter Q^{kl} is a quadrupole tensor,

$$Q^{kl} = \int (3x'^k x'^l - r'^2 \delta_i^k)\rho \mathrm{d}x'^1 x'^2 x'^3, \tag{16}$$

to rewritten it in matrix form,

$$Q = \begin{bmatrix} Q^{11} & Q^{12} & Q^{13} \\ Q^{21} & Q^{22} & Q^{23} \\ Q^{31} & Q^{32} & Q^{33} \end{bmatrix}. \tag{17}$$

Now to simplify the problem, we can deal with ρ as constant and assume the shape of the sun is biaxial ellipsoid and the equation of the ellipsoid is $x^2/A^2 + y^2/A^2 + z^2/B^2 < 1$. Now according to the observation the average radius $R_s = 695500km$, and $A - B = 30km$, therefore A and B must conform to the constraint condition: $\frac{4\pi}{3}A^2B = \frac{4\pi}{3}R_s^3$. Solving these two equations yields $A = 695510km$ and $B = 695480km$. Then we can get the tensor Q^{kl}, integrate the Eq(16), we obtain

$$Q = \frac{1}{5}M(A^2 - B^2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \tag{18}$$

With no argument, we assume the orbit of Mercury is coincident with the equatorial plane of the sun(the orbit inclination of Mercury is just 3.38°). Thus we can write the potential of the gravitational field and neglect the higher order terms:

$$\Phi(\boldsymbol{r}) = -\frac{GM}{r} - \frac{GM(A^2 - B^2)}{10r^3}, \tag{19}$$

the term of r^{-3} was modified by the quadrupole moment of solar.

4. Mercury's perihelion advance

Now we already have obtained the modified $\boldsymbol{\Phi(r)}$, then what is supposed to do is to get the advance of the Mercury's perihelion utilizing the $\boldsymbol{\Phi(r)}$. The gravitational potential of Mercury is:

$$U = -\frac{k}{r} + \delta U, \delta U = \frac{s}{r^3} \tag{20}$$

where $k = GMm, s = -GMm(A^2 - B^2)/10$, and m is the mass of Mercury. Here two different methods are applied to find the advance $\delta\phi$ in one Mercury orbit period, one is integral method and another is using the regular perturbation theory.

4.1 The integral method

If U just contains the first term $-k/r$, the equation of the orbit can be written as:

$$r = \frac{p}{1 + e\cos\phi}, \tag{21}$$

where $2p$ is latus rectum of the orbit and e is the eccentricity, they are given as:

$$p = \frac{L^2}{mk}, e = \sqrt{1 + \frac{2EM}{mk^2}}, \tag{22}$$

where L is the angular momentum and E is total mechanic energy. Applying Newton's second law, the motion equation is

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr} + U(r). \tag{23}$$

Then we can get the orbit from the equation below, which can be obtained from Eq(23)£¬

$$\phi = \int_{r_{min}}^{r_{max}} \frac{Ldr}{r^2\sqrt{2m(E-U) - \frac{L^2}{r^2}}} \tag{24}$$

This can be rewritten as the partial derivative of L,

$$\phi = -2\frac{\partial}{\partial L}\int_{r_{min}}^{r_{max}} \sqrt{2m(E-U) - \frac{L^2}{r^2}}dr \tag{25}$$

Now, if a small modification δU is applied to U, we obtain

$$\delta\phi = \frac{\partial\phi}{\partial U}\delta U = \frac{\partial}{\partial L}\int_{r_{min}}^{r_{max}} \frac{2m\delta U dr}{r^2\sqrt{2m(E+k/r) - \frac{L^2}{r^2}}} = \frac{\partial}{\partial L}\left(\frac{2m}{L}\int_0^{\pi} r^2\delta U \mathrm{d}\phi\right) \tag{26}$$

Then substitute $\delta U = s/r^3$ into Eq(26), and replace r using the relation between r and ϕ in Eq(21), and the relation between p and L in Eq(22), we will get the final result of $\delta\phi$,

$$
\begin{aligned}
\delta\phi &= \frac{\partial}{\partial L}\left(\frac{2m}{L}\int_0^{\pi}\frac{s}{r}\mathrm{d}\phi\right) \\
&= \frac{\partial}{\partial L}\left[\frac{2m}{L}\int_0^{\pi}\frac{s}{p}(1+e\cos\phi)\mathrm{d}\phi\right] \\
&= \frac{\partial}{\partial L}\left(\frac{2\pi ms}{Lp}\right) \\
&= \frac{\partial}{\partial L}\left(\frac{2\pi m^2sk}{L^3}\right) \\
&= -\frac{6\pi m^2sk}{L^4} \\
&= -\frac{6\pi s}{kp^2}
\end{aligned}
\tag{27}
$$

After substituting s into the result we can find

$$\delta\phi = \frac{3}{5}\pi\frac{A^2-B^2}{p^2}. \tag{28}$$

To calculate the value of $\delta\phi$, the orbital elements[8] of Mercury is necessary, as shown in Table(1). From the geometrical relationship of ellipse, we have $p = b^2/a$,

Table 1: The orbital elements of Mercury

Mean distance from the sun	0.387AU(5.79×10^7km)
Maximum distance from the sun	0.467AU(6.98×10^7km)
Minimum distance from the sun	0.307AU(4.6×10^7km)
Siderial period	87.969 days

where a and b are the maximum and minimum distance from the sun respectively. Substituting these values into Eq(28), we have $\delta\phi = 8.56\times10^{-8}rad$, this is the perihelion shift in one Mercury year(87.969 days on earth), it can be converted to a more convenient form 0.0397seconds of arc per century.

4.2 The method of regular perturbation theory

The Hamiltonian function of Mercury without the modification(here it is treated as a perturbation) is given

$$H_0 = \frac{1}{2m}\left(p_r^2+\frac{p_\phi^2}{r^2}\right)-\frac{k}{r}, \tag{29}$$

and the Hamilton-Jacoby equation(HJE) is,

$$H\left(q,\frac{\partial S}{\partial q},t\right)+\frac{\partial S}{\partial t}=0, \tag{30}$$

where S, the generating function, is the solution of HJE, and q is the generalized coordinates of a mechanical system. Here, the HJE can be written as,

$$\frac{1}{2m}\left(\frac{\partial S}{\partial r}\right)^2+\frac{1}{2mr^2}\left(\frac{\partial S}{\partial \phi}\right)^2-\frac{k}{r}+\frac{\partial S}{\partial t}=0, \tag{31}$$

for the Hamiltonian function does not depend on time explicitly, separating the equation is feasible,

$$S(q,P,t) = W(q,P) + f(t), \tag{32}$$

where P is the generalized momentum. Then Eq(31) can be transformed to

$$\begin{aligned} \frac{1}{2m}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial S}{\partial \phi}\right)^2 - \frac{k}{r} = E, \\ \frac{\mathrm{d}f(t)}{\mathrm{d}t} = -E, \end{aligned} \tag{33}$$

where E is a constant, and on this occasion, it is the mechanical energy. From Eq(33), $f(t) = -Et$ can be obtained immediately, then to find the complete solution we must solve the first equation. The method is separating variables as before,

$$\left(\frac{\partial W}{\partial \phi}\right)^2 = -r^2\left(\frac{\partial W}{\partial r}\right)^2 + 2mkr + 2mr^2E = L^2, \tag{34}$$

here, L is a constant just like E and the physical meaning of L is angular momentum. From these equations, the solution of Eq(31) can be found ,

$$S = -Et + L\phi + \int \sqrt{2mE + \frac{2mk}{r} - \frac{L^2}{r^2}}\mathrm{d}r. \tag{35}$$

After the regular transformation, the Hamiltonian function becomes

$$K = H + \frac{\partial S}{\partial t} = 0, \tag{36}$$

and the generalized coordinates is

$$\begin{aligned} \beta_1 = \frac{\partial S}{\partial E} = -t + \int \frac{m\mathrm{d}r}{\sqrt{2mE + \frac{2mk}{r} - \frac{L^2}{r^2}}}; \\ \beta_2 = \frac{\partial S}{\partial L} = \phi - \int \frac{L\mathrm{d}r}{r^2\sqrt{2mE + \frac{2mk}{r} - \frac{L^2}{r^2}}}, \end{aligned} \tag{37}$$

notice that $E < 0$ for ellipse orbit. To do the first integral, substitution is necessary[9], then we will find the relation between t and r, it is

$$\begin{aligned} r = a - l\cos\xi; \\ t = \sqrt{\frac{m}{2|E|}}(a\xi - l\sin\xi) - \beta_1 = \sqrt{\frac{ma}{k}}(a\xi - l\sin\xi) - \beta_1, \end{aligned} \tag{38}$$

where ξ is a parameter, ξ varying from 0 to 2π corresponds Mercury completes an orbit around the sun and t varies from 0 to T, the period of revolution. And

$$\begin{aligned} a = \frac{k}{2|E|}, \\ l = \sqrt{\frac{k^2}{4E^2} - \frac{L^2}{2m|E|}} = ea. \end{aligned} \tag{39}$$

Then, from the second integral in (37) we will get the orbit,

$$r = \frac{p}{1+e\cos(\phi-\beta_2)}. \tag{40}$$

Now, if the modification is added to the Hamiltonian function, it becomes

$$H = H_0 + \varepsilon H', \tag{41}$$

where $\varepsilon H' = s/r^3$ and the Hamiltonian function after regular transformation K no longer equals to 0, it becomes

$$K = H + \frac{\partial S}{\partial t} = \varepsilon H'. \tag{42}$$

At that moment, according to Hamiltonian regular equation, β_1 and β_2 are given

$$\begin{aligned} \dot{\beta}_1 &= \frac{\partial K}{\partial E}; \\ \dot{\beta}_2 &= \frac{\partial K}{\partial L}, \end{aligned} \tag{43}$$

and β_1 and β_2 are not constants from (43), instead, it is β_1 and β_2 that lead to Mercury's perihelion shift. From the first equation in (43), we find $\beta_1 = \frac{1}{2}t$, then

$$t = \frac{2}{3}\sqrt{\frac{ma}{k}}(a\xi - l\sin\xi), \tag{44}$$

And the remaining task is to find β_2, to achieve this we need the result of Eq(44) and the first equation in (38). From the second equation in (43), we have

$$\begin{aligned} \beta_2 &= \int_0^T \frac{\partial K}{\partial L}\mathrm{d}t \\ &= -\int_0^T \frac{3s}{r^4}\frac{\partial r}{\partial L}\mathrm{d}t \\ &= -\frac{sL}{m|E|l}\sqrt{\frac{ma}{k}}\int_0^{2\pi}\frac{\cos\xi\,\mathrm{d}\xi}{(a-l\cos)^3} \\ &= -\frac{sL}{m|E|l}\sqrt{\frac{ma}{k}}\frac{3\pi al}{(a^2-l^2)^{\frac{5}{2}}}. \end{aligned} \tag{45}$$

Substitute Eq(22),Eq(39) and $p = b^2/a$ into Eq(45), we obtain

$$\beta_2 = -\frac{6\pi s}{kp^2}. \tag{46}$$

Here, β_2 is Mercury's perihelion shift in one period of revolution, it is the same with the result from integral method. Now, the orbit of Mercury is

$$r = \frac{p}{1+e\cos(\eta\phi+\phi_0)}, \tag{47}$$

where ϕ_0 is a constant related to the original position. The coefficient η is related to the procession, when η is unequal to 1, the procession will exist, as shown in Figure

1. The distance from the sun r go through a period correspond $\eta(\phi-\beta_2)=2\pi$, and β is the perihelion shift in one period.

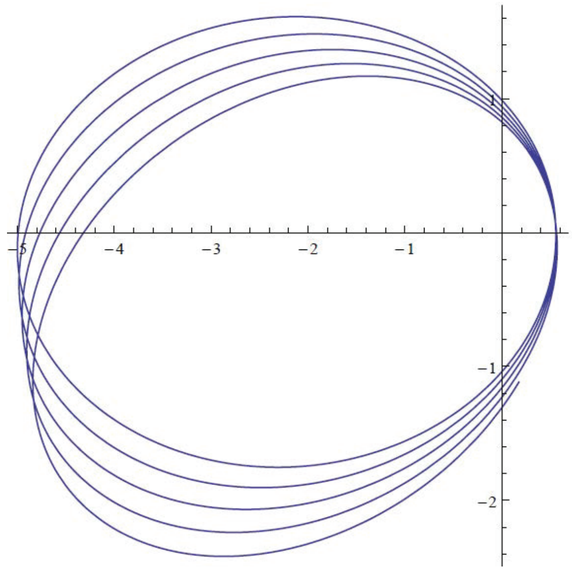

Figure 1: The perihelion shift with $p=1, e=0.8$ and $\eta=0.99$

5. Conclusions

Because of solar quadrupole moment, the total perihelion should contain an additional contribution,

$$\delta\phi=\frac{6\pi GM}{a(1-e^2)c^2}\frac{(2-\beta+2\gamma)}{3}-\frac{6\pi s}{kp^2}. \tag{48}$$

However, compared to the first term, perihelion caused by solar quadrupole moment is so small(about 0.92%), it is in the error range of the result obtained from GRT. Further exploration to the space will give a precise measurement of solar oblateness, then the general relativity theory will be tested to a new accuracy.

References

[1] Baum, Richard and Sheehan,William, In Search of Plante Vulcan: The Ghost in Newton's CLockwork Universe, Plenum Trade, New York. 1997

[2] Michael P. Price and William F. Rush, Nonrelativistic contribution to Mercury's perihelion precession, Am. J. Phys.47(6), June 1979

[3] D. X. Lu, University Physics, China Higher Education Press and Springer-Verlag(1999)

[4] Frank P. Pijpers, Helioseismic determination of the solar gravitational quadrupole moment,Mon. Not. R. Astron. Soc. 000,000-000(1994)

[5] Sophie Pireaux and Jean-Pierre Rozelot, Solar Quadrupole Moment and Purely Relativistic Gravitation Contributions to Mercury's Perihelion Advance, Astrophysics and Space Science 284:1159-1194, 2003

[6] L. Iorio, On the possibility of measuring the solar oblateness and some relativistic effects from planetary ranging, A&A, 433:385-393, April 2005.

[7] Hans C. Ohanian and Remo Ruffini, GRAVITATION AND SPACE-TIME(Second Edition), W.W. Norton & Co. 1994

[8] Kaufmann, Solar System Data: Mercury Data, http://hyperphysics.phy-astr.gsu.edu/hbase/solar/soldata2.html

[9] Landau L. D. and Lifshitz E. M. Course of Theoretical Physics by L.D.LANDAU, Volume 1, Butterworth-Heinemann