A Note on Tensors

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§. Definition and Transformation

(V^* is the dual space of V)

A tensor of type (r, s) on a vector space V is a C-valued function T on

$$V \times \cdots \times V \times V^* \times \cdots \times V^*,\ (V : r; V^* : s) \tag{1}$$

which is linear in each arguments. This property is called **multilinearity**. The set of tensors of the type (r, s) on a vector space V, denoted as $\mathcal{T}_s^r(V)$, form a vector space. Let $\{e_i\}_{i=1\ldots n}$ be a basis for V and $\{e^i\}_{i=1\ldots n}$ the corresponding dual basis. Then, denoting the ith component of v_p as v_p^i and the jth components of f_q as f_{qj}, we have

$$\begin{aligned} T(v_1, \ldots, v_r, f_1, \ldots, f_s) &= v_1^{i_1} \ldots v_r^{i_r} f_{1j_1} \ldots f_{sj_s} T(e_{i_1}, \ldots, e_{i_r}, e^{j_1}, \ldots, e^{j_s}) \\ &= v_1^{i_1} \ldots v_r^{i_r} f_{1j_1} \ldots f_{sj_s} T_{i_1,\ldots,i_r}^{j_1,\ldots,j_s} \end{aligned} \tag{2}$$

where the numbers

$$\boxed{T_{i_1,\ldots,i_r}^{j_1,\ldots,j_s} = T(e_{i_1}, \ldots, e_{i_r}, e^{j_1}, \ldots, e^{j_s})} \tag{3}$$

are called the components of T in basis of $\{e_i\}_{i=1\ldots n}$.

Example 1. The Levi-Civita tensor ϵ on $\mathbb{R}^3$ is defined by

$$\epsilon(u, v, w) = (u \times v) \cdot w \tag{4}$$

. If $\{e_1, e_2, e_3\}$ is the standard basis for $\mathbb{R}^3$, then (4) yields

$$\begin{aligned} \epsilon_{ijk} &= \epsilon(e_i, e_j, e_k) \\ &= (e_i \times e_j) \cdot e_k \\ &= \hat{\epsilon}_{ijk} \end{aligned} \tag{5}$$

Example 2. Change of basis

Suppose we have two bases for V, $\mathcal{B} = \{e_i\}$ and $\mathcal{B}' = \{e_{i'}\}$. Each of the $e_{i'}$ can be expressed as

$$e_{i'} = A_{i'}^j e_j \tag{6}$$

for some $A_{i'}^j$. Likewise, there exists number $A_i^{j'}$ such that

$$e_i = A_i^{j'} e_{j'}. \tag{7}$$

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We then have

$$e_i = A_i^{j'} e_{j'} = A_i^{j'} A_{j'}^k e_k \tag{8}$$

and can then conclude that

$$A_i^{j'} A_{j'}^k = \delta_i^k,\ A_{i'}^j A_{k'}^j = \delta_{i'}^{k'}. \tag{9}$$

Correspondingly, one can prove that

$$e^i = A_{j'}^i e^{j'},\ e^{i'} = A_j^{i'} e^j \tag{10}$$

[Hint: $e^{i'}(e_j) = e^{i'}(A_j^{k'} e_{k'}) = A_j^{k'} \delta_{k'}^{i'} = A_j^{i'}$]. Now we are ready to derive the transformation of a general (r, s) tensor T,

$$\boxed{\begin{aligned} T^{j_1',\dots,j_s'}_{i_1',\dots,i_r'} &= A^{k_1}_{i_1'} \dots A^{k_r}_{i_r'} A^{j_1'}_{l_1} \dots A^{j_s'}_{l_s} T(e_{k_1}, \dots, e_{k_r}, e^{l_1}, \dots, e^{l_s}) && (11) \\ &= A^{k_1}_{i_1'} \dots A^{k_r}_{i_r'} A^{j_1'}_{l_1} \dots A^{j_s'}_{l_s} T^{l_1,\dots,l_s}_{k_1,\dots,k_r} && (12) \end{aligned}}$$

To the end, it is useful to introduce the matrices

$$\boxed{A = \begin{pmatrix} A_1^{1'} & \dots & A_n^{1'} \\ \vdots & \vdots & \vdots \\ A_1^{n'} & \dots & A_n^{n'} \end{pmatrix},\ A^{-1} = \begin{pmatrix} A_{1'}^{1} & \dots & A_{1'}^{n} \\ \vdots & \vdots & \vdots \\ A_{n'}^{1} & \dots & A_{n'}^{n} \end{pmatrix}} \tag{13}$$

The matrices satisfy $AA^{-1} = A^{-1}A = I$.

§. The Tensor Product

Given two finite-dimensional vector spaces V and W, we would construct a product vector space, which we denote $V \otimes W$, whose elements are in the same sense "product" of vectors $v \in V$ and $w \in W$. We denote these product by $v \otimes w$. This product is bilinear such that

$$\boxed{(c_1 v_1 + v_2) \otimes (c_2 w_1 + w_2) = c_1 c_2 v_1 \otimes w_1 + c_1 v_1 \otimes w_2 + c_2 v_2 \otimes w_1 + v_2 \otimes w_2,\ c_{1,2} \in \mathbb{C}} \tag{14}$$

Two vector can be expanded in terms of $\{e_i\}$ and $\{f_i\}$ for V and W as

$$v \otimes w = v^i w^j e_i \otimes f_j. \tag{15}$$

Now we define the tensor product $v \otimes w$ to be the element of $V \otimes W$ as below:

$$\boxed{(v \otimes w)(h, g) = v(h)w(g),\ \forall h \in V^*, g \in W^*, (g, h) \in V^* \times W^*} \tag{16}$$

Properties of tensor product:

$$\begin{aligned} (V \otimes W)^* &= (V^* \otimes W^*) \\ (V_1 \otimes V_2) \otimes V_3 &= V_1 \otimes (V_2 \otimes V_3) \end{aligned} \tag{17}$$

Now consider the tensor product of a vector space and its dual, i.e.

$$V^* \otimes \cdots \otimes V^* \otimes V \otimes \cdots \otimes V,\ (V : s; V^* : r) \tag{18}$$

Actually it is identical to $\mathcal{T}_s^r$. Since the space in (18) has the basis $\mathcal{B}_s^r = \{e^{i_1} \otimes \cdots \otimes e^{i_r} \otimes e_{j_1} \otimes \cdots \otimes e_{j_s}\}$, we can conclude that $\mathcal{B}_s^r$ is the basis of T_s^r. In fact, we claim that if $T \in \mathcal{T}_s^r$ has the component $T_{i_1,\ldots,i_r}^{j_1,\ldots,j_s}$, then

$$\boxed{T = T_{i_1,\ldots,i_r}^{j_1,\ldots,j_s} e^{i_1} \otimes \cdots \otimes e^{i_r} \otimes e_{j_1} \otimes \cdots \otimes e_{j_s}} \tag{19}$$

To prove this, (3) and (16) are required. A tensor product like $f \otimes g = f_i g_j e^i \otimes e^j \in \mathcal{T}_0^2$ thus has the components $(f \otimes g)_{ij} = f_i g_j$.

Example 3. Contraction, which is the generalization of trace function to a tensor of arbitrary rank: Given $T \in \mathcal{T}_s^r$, we can define a contraction of T to be any $(r-1, s-1)$ tensor by substituting e^i into one of the argument, e_i into another and then summing over i. For instance, we can get the $(r-1, s-1)$ tensor $\widetilde{T}$ defined as

$$\widetilde{T}(v_1, \ldots, v_{r-1}, f_1, \ldots, f_{s-1}) = T(v_1, \ldots, v_{r-1}, e_i, f_1, \ldots, f_{s-1}, e^i) \tag{20}$$

The components of $\widetilde{T}$ are

$$T_{i_1,\ldots,i_{r-1}}^{j_1,\ldots,j_{s-1}} = T_{i_1,\ldots,i_{r-1},l}^{j_1,\ldots,j_{s-1},l} \tag{21}$$

If we have two linear operators A and B, then their tensor product $A \otimes B \in \mathcal{T}_2^2$ has components

$$(A \otimes B)_{ik}^{jl} = A_i^j B_k^l. \tag{22}$$

and contracting on the first and the last index gives a $(1,1)$ tensor AB whose components are

$$(AB)_k^j = A_l^j B_k^l. \tag{23}$$

Example 4. $V^* \otimes V$

Given $f \otimes v \in V^* \otimes V$, we can define a linear operator by $(f \otimes v)(w) = f(w)v$. More generally, given

$$T_i^j e^i \otimes e_j \in V^* \otimes V \tag{24}$$

we can define a linear operator T by

$$T(v) = T_i^j e^i(v) e_j = v^i T_i^j e_j,\ v \in V \tag{25}$$

Here, I provide one application to the quantum mechanics. Let $\mathcal{H}$ be the Hilbert space and let $\phi, \psi \in \mathcal{H}$ so that $L(\phi) \in \mathcal{H}^*$. The tensor product of $L(\phi)$ and ψ, which we write as $L(\phi) \otimes \psi$, is written in Dirac notation as $|\psi\rangle \langle\phi|$

Example 5. Maxwell Stress Tensor is defined in $(2,0)$ form as

$$T_{(2,0)} = \boldsymbol{E} \otimes \boldsymbol{E} + \boldsymbol{B} \otimes \boldsymbol{B} - \frac{1}{2}(\boldsymbol{E} \cdot \boldsymbol{E} + \boldsymbol{B} \cdot \boldsymbol{B})g. \tag{26}$$

Field tensor $F^{\mu\nu}$ of electrodynamics is also a $(2,0)$ tensor. The Lorentz force law

$$\frac{dp^\mu}{dt} = qF^u{}_\nu v^\nu. \tag{27}$$

§. Symmetric and Antisymmetric Tensors

A symmetric tensor is an $(r,0)$ tensor, those value us unaffected by the interchanges of any two of its arguments, that is

$$T(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_r) = T(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_r) \tag{28}$$

for any i, j. Symmetric $(0,r)$ tensor is defined similarly. Here, symmetric $(r,0)$ and $(0,r)$ tensors each form vector spaces, denoted $S^r(V)$ and $S^r(V^*)$ respectively.

An antisymmetric tensor is one whose value changes sign under transposition of any two of its arguments, i.e.,

$$T(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_r) = -T(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_r) \tag{29}$$

The space formed by the $(r,0)$ and $(0,r)$ antisymmetric tensors are denoted as $\Lambda^r V$ and $\Lambda^r V^*$.

Some properties of antisymmetric tensors are listed below,

1. $T(v_1, \ldots, v_r) = 0$ if $v_i = v_j$ for any $i \neq j$
2. $T(v_1, \ldots, v_r) = 0$ if $\{v_1, \ldots, v_r\}$ is linearly dependent.
3. If dim $V = n$, then the only tensor in $\Lambda^r V^*$ and $\Lambda^r V$ for $r > n$ is the 0 tensor.

An important operation on antisymmetric tensors is the **wedge product**: Given $f, g \in V^*$, we define the wedge product of f and g, denoted $f \wedge g$, to be the antisymmetric $(2,0)$ tensor defined by

$$\boxed{f \wedge g = f \otimes g - g \otimes f} \tag{30}$$

Expanding in terms of e^i gives

$$f \wedge g = f_i g_j e^i \wedge e^j \tag{31}$$

We define our wedge product $f_1 \wedge \cdots \wedge f_r$ to be the sum of all tensor products of terms $f_{i_1} \otimes \cdots \otimes f_{i_r}$ where each term get $+$ or $-$ sign depending on whether an odd or even number of transpositions of the factors are necessary to obtain it from $f_1 \otimes \cdots \otimes f_r$. The number is odd, the term is signed to be -1 while even a $+1$. In fact, that $\{e^{i_1} \wedge \cdots \wedge e^{i_r}\}_{i_1 < \cdots < i_r}$ is a basis for $\Lambda^r V^*$ 5t Considering $\mathbb{R}^n$ with the standard inner product. Let $\{e_i\}_{i=1,\ldots,n}$ be an orthonormal basis for $\mathbb{R}^n$ and consider the tensor

$$\boxed{\epsilon = e^1 \wedge \cdots \wedge e^n \ \in \Lambda^r \mathbb{R}^{n*}} \tag{32}$$

For $n = 2$, consider two vector u and v on (x, y) plane, we have

$$\epsilon(u, v) = \epsilon_{i,j} u^i v^j = u^x v^y - u^y v^x = (u \times v)^z \tag{33}$$

Example 6. Determinant

The determinant of A, denoted $|A|$ or det$A$, is calculated

$$\boxed{\begin{aligned} |A| &= \epsilon(A_1, \ldots, A_n) && (34) \\ &= \sum_{i1,\ldots,i_n} \epsilon_{i_1,\ldots,i_n} A_{i_1,1} \ldots A_{i_n,n} && (35) \end{aligned}}$$

Pseudovector in $\mathbb{R}^3$

Pseudovector is a tensor on $\mathbb{R}^3$ whose components transform like vectors under rotations but do not change sign under inversion, such as magnetic field vector $\boldsymbol{B}$, angular velocity $\boldsymbol{\omega}$. It turns out that pseudovectors like these are actually elements of $\Lambda^2\mathbb{R}^3$, which is known as bivector. In term of matrices , this corresponds to the identification

$$\boxed{\begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \to \begin{pmatrix} x \\ y \\ z \end{pmatrix}.} \tag{36}$$

This identification can be embodied in a on-on-one and onto map J from $\Lambda^2\mathbb{R}^3$ to $\mathbb{R}^3$. If $\alpha \in \Lambda^2\mathbb{R}^3$, we can expand it as

$$\alpha = \alpha^{23} e_2 \wedge e_3 + \alpha^{31} e_3 \wedge e_1 + \alpha^{12} e_1 \wedge e_2 \tag{37}$$

and we then define J in components to be

$$\boxed{\begin{aligned} & J : \Lambda^2\mathbb{R}^3 \to \mathbb{R}^3 && (38) \\ & \qquad \alpha^{ij} \to (J(\alpha))^i = \frac{1}{2} \epsilon^i_{jk} \alpha^{jk} && (39) \end{aligned}}$$

Textbook:

An Introduction to Tensors and Group Theory for Physicists, Nadir Jeevanjee