Consider a point of mass M moving through a homogeneous medium along the z-axis at a constant velocity $\vec{V} = V_0\vec{k}$. To find the turbulence inside the medium, we employ the equations of hydrodynamics in the absence of viscous and thermal dissipations,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \tag{1}$$
$$\rho \left(\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} \right) = -\nabla p - \rho \nabla \Phi, \tag{2}$$
$$\nabla^2 \Phi = 4\pi G \left[\rho + M\delta(\vec{r} - \vec{V_0}t) \right], \tag{3}$$
$$p = c_s^2 \rho/\gamma \tag{4}$$

where $\rho, \vec{v}, p$ are density, velocity and pressure of the medium as functions of $\vec{r}$ and t. Let $\rho = \rho_0 + \delta\rho(\vec{r}, t)$ and to the lowest order (here $\vec{v}$ and $\delta\rho$ are small perturbations), equation (1)-(4) can be rewritten as

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot (\vec{v}) = 0, \tag{5}$$
$$\rho_0 \left(\frac{\partial \vec{v}}{\partial t} \right) = -\nabla p - \rho_0 \nabla \Phi, \tag{6}$$
$$\nabla^2 \Phi = 4\pi G \left[\rho_0 + M\delta(\vec{r} - \vec{V_0}t) \right], \tag{7}$$
$$p = c_s^2 \rho/\gamma. \tag{8}$$

Introducing $\psi = \delta\rho/\rho_0$, we obtain a wave equation

$$\frac{\partial^2 \psi}{\partial t^2} - \frac{c_s^2}{\gamma}\nabla^2 \psi - c_s^2 k_J^2 = 4\pi G M \delta(\vec{r} - V_0 t\vec{k}). \tag{9}$$

where $k_J^2 = 4\pi G\rho_0/c_s^2$. Since there is no fluctuation at $t = 0$, we have the initial conditions $\psi(\vec{r}, 0) = \frac{\partial \psi}{\partial t}|_{t=0} = 0$. If we ignore the self gravitational interaction, or equivalently omit $c_s^2 k_J^2$, Eq (9) is reduced to

$$\frac{\partial^2 \psi}{\partial t^2} - c_s'^2 \nabla^2 \psi = 4\pi G M \delta(\vec{r} - V_0 t\vec{k}), \tag{10}$$

where $c_s'^2 = c_s^2/\gamma$. Using the Kirchhoff formula, the solution can be written as

$$\psi(\vec{r}, t) = \int_0^t \left[\frac{GM}{c_s'^2 (t - \tau)} \iint_{S_{c_s'(t-\tau)}} \delta(\vec{r} + \vec{\xi} - V_0 \tau \vec{k}) dS \right] d\tau, \tag{11}$$

where the sphere $S_{c_s'(t-\tau)} = \{\xi \in \mathbb{R}^3 \mid |\xi| = c_s'(t - \tau)\}$. The integral of δ function vanishes unless

$$c_s'^2 (t - \tau)^2 = (z - V_0 \tau)^2 + x^2 + y^2, \tag{12}$$

here we use the connections between $\vec{r}, \vec{\xi}$ and τ: $x = -\xi_1, y = -\xi_2, z + \xi_3 = V_0\tau$ and $\xi_1^2 + \xi_2^2 + \xi_3^2 = c_s'^2 (t - \tau)^2$. If the τ that satisfies Eq (12) exists, one requires

$$c_s'^2 (z - V_0 t)^2 - r_z^2 (V_0^2 - c_s'^2) \geq 0, \tag{13}$$

where $r_z = \sqrt{x^2 + y^2}$. One case of interests is the hypersonic motion of the mass, say $V_0 > c_s'$, where the existence of non-zero solution requires

$$r_z \leq \frac{|z - V_0 t|}{\sqrt{\mathcal{M}^2 - 1}} \tag{14}$$

where $\mathcal{M} = V_0/c_s'$. As we can see, inequality (14) defines the down stream of a conic shock whose cone angle is $\Theta = \sin^{-1} \frac{c_s'}{V_0}$ and the density in this region is given by ψ. Now our task is to find ψ analytically in the spherical coordinates. In the spherical coordinate system $\vec{\xi} = (r_\xi, \theta_\xi, \phi_\xi)$, $\vec{r} = (r, \theta, \phi)$ and $\vec{r} - V_0\tau\vec{k} = (r', \theta', \phi')$, where

$$r' = \sqrt{r^2 \sin^2\theta + (r\cos\theta - V_0\tau)^2}, \tag{15}$$

$$r_\xi = c_s'(t-\tau) \tag{16}$$

$$\tag{17}$$

Note that

$$\delta(\vec{r} - \vec{r}_0) = \frac{1}{r^2 \sin\theta}\delta(r - r_0)\delta(\theta - \theta_0)\delta(\phi - \phi_0), \tag{18}$$

$\delta(\vec{r} + \vec{\xi} - V_0\tau\vec{k})$ can be written as

$$\delta(\vec{r} + \vec{\xi} - V_0\tau\vec{k}) = \frac{1}{c_s'^2(t-\tau)^2 \sin\theta_\xi}\delta[r' - c_s'(t-\tau)]\delta(\theta_\xi + \theta')\delta(\phi_\xi + \phi'). \tag{19}$$

The integral in Eq (11) becomes

$$\psi(\vec{r}, t) = \int_0^t \left[\frac{GM}{c_s'^2(t-\tau)} \iint_{S_{c_s'(t-\tau)}} \delta(\vec{r} + \vec{\xi} - V_0\tau\vec{k}) dS \right] d\tau \tag{20}$$

$$= \frac{GM}{c_s'^2} \int_0^t d\tau \int_0^\pi d\theta_\xi \int_0^{2\pi} d\phi_\xi \frac{1}{c_s'^2(t-\tau)^3 \sin\theta_\xi}\delta[r' - c_s'(t-\tau)]\delta(\theta_\xi + \theta')\delta(\phi_\xi + \phi')c_s'^2(t-\tau)^2 \sin\theta_\xi \tag{21}$$

$$= \frac{GM}{c_s'^2} \int_0^t d\tau \frac{\delta[\sqrt{r^2 \sin^2\theta + (r\cos\theta - V_0\tau)^2} - c_s'(t-\tau)]}{t-\tau} \tag{22}$$

$$= \frac{GM}{c_s'^2} \sum_{\tau_i \in [0,t]} \frac{1}{(t-\tau_i)\,|f'(\tau_i)|}, \tag{23}$$

where $f(\tau) = \sqrt{r^2 \sin^2\theta + (r\cos\theta - V_0\tau)^2} - c_s'(t-\tau)$ and τ_i is the ith root of $f(\tau) = 0$. Inside the cone defined by Eq. (14), we have

$$\tau_{1,2} = \frac{rV_0\cos\theta - c_s'^2 t \pm \sqrt{c_s'^2(r\cos\theta - V_0 t)^2 - r^2\sin^2\theta(V_0^2 - c_s'^2)}}{V_0^2 - c_s^2} \tag{24}$$

and

$$f'(\tau_i) = c_s' + \frac{V_0(V_0\tau_i - r\cos\theta)}{\sqrt{r^2\sin^2\theta + (r\cos\theta - V_0\tau_i)^2}} \tag{25}$$

$$= c_s' + \frac{V_0(V_0\tau_i - r\cos\theta)}{c_s'(t-\tau_i)} \tag{26}$$

$$= \frac{-c_s^2 t + V_0 r\cos\theta + \tau_i(c_s'^2 - V_0^2)}{c_s'(t-\tau_i)}. \tag{27}$$

Combining Eqs (23),(24) and (27), we obtain

$$\psi(\vec{r}, t) = \frac{GM}{c_s'^2} \sum_{\tau_{1,2}\in[0,t]} \frac{c_s'}{\sqrt{c_s'^2(r\cos\theta - V_0 t)^2 - r^2\sin^2\theta(V_0^2 - c_s'^2)}} \tag{28}$$

$$= \frac{GM}{c_s'^2} \sum_{\tau_{1,2}\in[0,t]} \frac{c_s'}{\sqrt{c_s'^2(r\cos\theta - V_0 t)^2 - r^2\sin^2\theta(V_0^2 - c_s'^2)}}. \tag{29}$$

Now, let's justify the physical meaning of the solution. In the reference frame that is adhered to the mass, (R, θ_m, ϕ_m), we have

$$R\cos\theta_m = r\cos\theta - V_0 t \tag{30}$$
$$R\sin\theta_m = r\sin\theta, \tag{31}$$

and

$$\psi(\vec{r}, t) = \frac{GM}{c_s'^2 R}\frac{g}{\sqrt{1-\mathcal{M}^2\sin^2\theta_m}}, \tag{32}$$

where $g = 0, 1, 2$ is determined by the distribution of τ_1 and τ_2. Since τ_i is solved from the equation

$$g(\tau) = (V_0^2 - c_s'^2)\tau^2 + 2(c_s'^2 t - rV_0\cos\theta)\tau + r^2 - c_s'^2 t^2 = 0, \tag{33}$$

or

$$h(\tau) = (V_0^2 - c_s'^2)\tau^2 + 2(c_s'^2 t - V_0 R\cos\theta_m - V_0^2 t)\tau + R^2 + (V_0^2 - c_s'^2)t^2 + 2RV_0 t\cos\theta_m = 0. \tag{34}$$

Obviously $h(t) = R^2 > 0$ (we only consider the points inside the cone). Hence we only need to consider the symmetric axis

$$s = \frac{-c_s'^2 t + V_0 R\cos\theta_m + V_0^2 t}{V_0^2 - c_s'^2} \tag{35}$$

and

$$h(0) = R^2 + (V_0^2 - c_s'^2)t^2 + 2RV_0 t\cos\theta_m. \tag{36}$$

Case 1: g=2, if $0 < s < t$ and $h(0) > 0$. Here $0 < s < t$ gives

$$\frac{c_s'^2 - V_0^2}{V_0}t < x_m = R\cos\theta_m < 0 \tag{37}$$

and $h(0) > 0$ gives

$$0 < R < -V_0 t\cos\theta_m - t\sqrt{c_s'^2 - V_0^2\sin^2\theta_m} \text{ or } R > -V_0 t\cos\theta_m + t\sqrt{c_s'^2 - V_0^2\sin^2\theta_m}. \tag{38}$$

Case 2, g=1, if $h(0) < 0$ which is determined by

$$V_0 t\cos\theta_m - t\sqrt{c_s'^2 - V_0^2\sin^2\theta_m} < R < -V_0 t\cos\theta_m + t\sqrt{c_s'^2 - V_0^2\sin^2\theta_m} \tag{39}$$

Case 3, g=0, if $h(0) > 0$ and $s <= 0$ which is determined by

$$x_m <= \frac{c_s'^2 - V_0^2}{V_0}t \tag{40}$$

and

$$0 < R < -V_0 t\cos\theta_m - t\sqrt{c_s'^2 - V_0^2\sin^2\theta_m} \text{ or } R > -V_0 t\cos\theta_m + t\sqrt{c_s'^2 - V_0^2\sin^2\theta_m}. \tag{41}$$

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Note: all the discussions are in the case $\pi - \Theta < \theta_m < \pi$.

These solutions are based on the assumption that the distribution of medium gas is initially homogeneous, hence we need to take the limit $t \to \infty$ inEq (32), which gives the continuous distribution

$$\psi(\vec{r}, t) = \frac{2GM}{c_s'^2 R}\frac{1}{\sqrt{1-\mathcal{M}^2\sin^2\theta_m}}. \tag{42}$$