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1 Verification of 8NodeBrick elements

1.1 Verification of 8NodeBrick cantilever beams

Problem description: Length=6m, Width=1m, Height=1m, Force=100N, E=1E8Pa, $\nu = 0.0$. The force direction was shown in Figure (1).

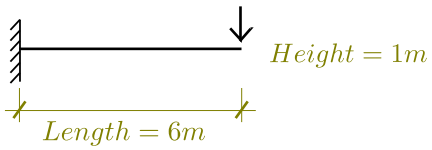
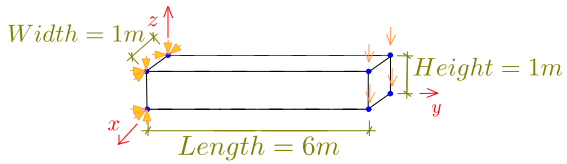


Figure 1: Problem description for cantilever beams

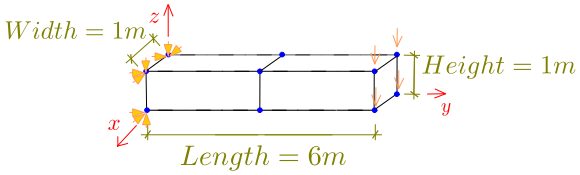
Theoretical displacement (bending and shear deformation):

$$\begin{aligned} d &= \frac{FL^3}{3EI} + \frac{FL}{GA} \\ &= \frac{100N \times 6^3 m^3}{3 \times 10^8 N/m^2 \times \frac{1}{12} m^4} + \frac{100N \times 6m}{5 \times 10^7 N/m^2 \times 1m^2} \\ &= 8.64 \times 10^{-4} m + 0.12 \times 10^{-4} m \\ &= 8.76 \times 10^{-4}\ m \end{aligned} \tag{1}$$

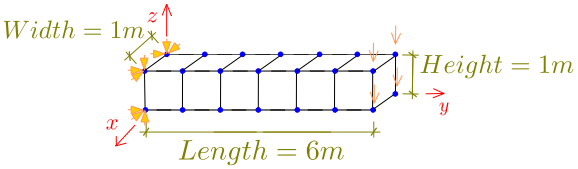
Numerical model:
The 8NodeBrick elements were shown in Figure (2).



(a) One 8NodeBrick element



(b) Two 8NodeBrick elements



(c) Six 8NodeBrick elements

Figure 2: 8NodeBrick elements for cantilever beams

All the ESSI results were listed in Table (1). The theoretical solution is 8.760E-04 m.

Table 1: Results for 8NodeBrick cantilever beams of different element numbers

Element number	1	2	6
8NodeBrick	4.61E-05 m	1.59E-04 m	5.84E-04 m
Error	94.74%	81.82%	33.33%

The errors were plotted in Figure (3).

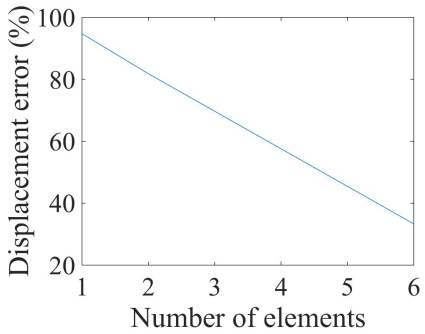


Figure 3: 8NodeBrick cantilever beam for different element number
Displacement error versus Number of elements

The ESSI model fei files for the table above are here

- ***Cantilever: different geometry***

In the figures above, only the model with geometry $6m \times 1m \times 1m$ was drawed. In the ESSI models, the geometry $10m \times 1m \times 1m$ and the geometry $20m \times 1m \times 1m$ were also calculated. In three different geometry models, all the element sizes were $1m \times 1m \times 1m$. Therefore, the number of elements used in each model were 6, 10 *and* 20 respectively.

The ESSI results for different geometry were listed in Table (2).

Table 2: Results for 8NodeBrick cantilever beams of different geometry

Geometry	8NodeBrick	Theoretical(bending)	Theoretical(shear)	Theoretical(all)	Error
1:6	5.84E-04 m	8.64E-04 m	1.20E-05 m	8.76E-04 m	33.33%
1:10	2.68E-03 m	4.00E-03 m	2.00E-05 m	4.02E-03 m	33.33%
1:20	2.14E-02 m	3.20E-02 m	4.00E-05 m	3.20E-02 m	33.33%

The ESSI model fei files for the table above are here

1.2 Verification of 8NodeBrick cantilever beam for different Poisson's ratio

Problem description: Length=6m, Width=1m, Height=1m, Force=100N, E=1E8Pa, $\nu = 0.0 - 0.49$. The force direction was shown in Figure (4).

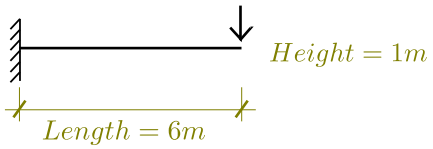


Figure 4: Problem description for cantilever beams of different Poisson's ratios

The theoretical solution for $\nu = 0.0$ was calculated below, while the solution for other Poisson's ratio were calculated by the similar process.

Theoretical displacement (bending and shear deformation):

$$\begin{aligned} d &= \frac{FL^3}{3EI} + \frac{FL}{GA} \\ &= \frac{100N \times 6^3 m^3}{3 \times 10^8 N/m^2 \times \frac{1}{12} m^4} + \frac{100N \times 6m}{5 \times 10^7 N/m^2 \times 1m^2} \\ &= 8.64 \times 10^{-4} m + 0.12 \times 10^{-4} m \\ &= 8.76 \times 10^{-4}\ m \end{aligned} \tag{2}$$

The rotation angle at the end:

$$\theta = \frac{FL^2}{2EI} = \frac{100N \times 6^2 m^2}{2 \times 10^8 N/m^2 \times \frac{1}{12} m^4} = 2.16 \times 10^{-4}\ rad = 0.0124^\circ \tag{3}$$

The 8NodeBrick elements for cantilever beams of different Poisson's ratios were shown in Figure (5):

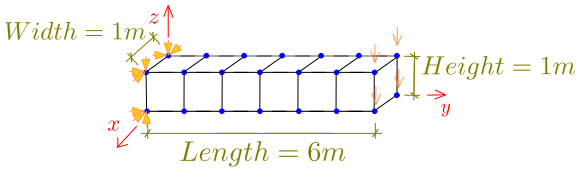


Figure 5: 8NodeBrick elements for cantilever beams of different Poisson's ratios

All the displacement results were listed in Table (3) - (5).

Table 3: ***Displacement*** results for 8NodeBrick cantilever beams with element side length 1 m

Poisson's ratio	8NodeBrick displacement	Theory displacement (bending)	Theory displacement (shear)	Theory displacement(all)	Error
0.00	5.840E-04 m	8.640E-04 m	1.200E-05 m	8.760E-04 m	33.33%
0.05	5.924E-04 m	8.640E-04 m	1.260E-05 m	8.766E-04 m	32.42%
0.10	5.969E-04 m	8.640E-04 m	1.320E-05 m	8.772E-04 m	31.95%
0.15	5.971E-04 m	8.640E-04 m	1.380E-05 m	8.778E-04 m	31.98%
0.20	5.922E-04 m	8.640E-04 m	1.440E-05 m	8.784E-04 m	32.58%
0.25	5.814E-04 m	8.640E-04 m	1.500E-05 m	8.790E-04 m	33.86%
0.30	5.634E-04 m	8.640E-04 m	1.560E-05 m	8.796E-04 m	35.95%
0.35	5.364E-04 m	8.640E-04 m	1.620E-05 m	8.802E-04 m	39.06%
0.40	4.970E-04 m	8.640E-04 m	1.680E-05 m	8.808E-04 m	43.57%
0.45	4.353E-04 m	8.640E-04 m	1.740E-05 m	8.814E-04 m	50.61%
0.49	3.142E-04 m	8.640E-04 m	1.788E-05 m	8.819E-04 m	64.37%

Then, in the same geometry, the element side length was cut into 0.5m.

Table 4: ***Displacement*** results for 8NodeBrick cantilever beams with element side length 0.5 m

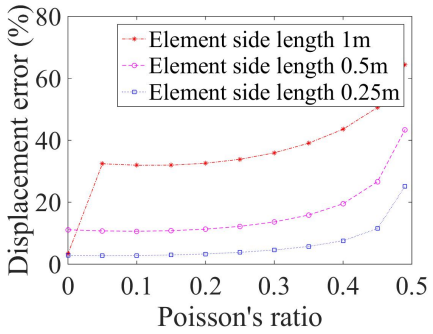
Poisson's ratio	8NodeBrick displacement	Theory displacement (bending)	Theory displacement (shear)	Theory displacement(all)	Error
0.00	7.787E-04 m	8.640E-04 m	1.200E-05 m	8.760E-04 m	11.11%
0.05	7.824E-04 m	8.640E-04 m	1.260E-05 m	8.766E-04 m	10.74%
0.10	7.839E-04 m	8.640E-04 m	1.320E-05 m	8.772E-04 m	10.63%
0.15	7.829E-04 m	8.640E-04 m	1.380E-05 m	8.778E-04 m	10.81%
0.20	7.790E-04 m	8.640E-04 m	1.440E-05 m	8.784E-04 m	11.31%
0.25	7.717E-04 m	8.640E-04 m	1.500E-05 m	8.790E-04 m	12.21%
0.30	7.597E-04 m	8.640E-04 m	1.560E-05 m	8.796E-04 m	13.63%
0.35	7.406E-04 m	8.640E-04 m	1.620E-05 m	8.802E-04 m	15.86%
0.40	7.089E-04 m	8.640E-04 m	1.680E-05 m	8.808E-04 m	19.52%
0.45	6.466E-04 m	8.640E-04 m	1.740E-05 m	8.814E-04 m	26.64%
0.49	4.990E-04 m	8.640E-04 m	1.788E-05 m	8.819E-04 m	43.42%

Finally, in the same geometry, the element side length was cut into 0.25m.

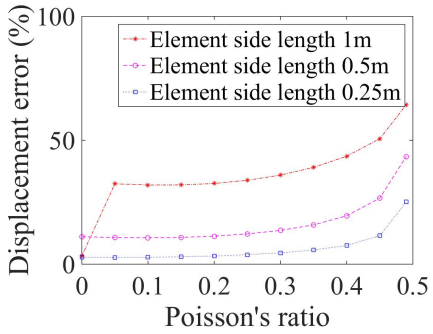
Table 5: ***Displacement*** results for 8NodeBrick cantilever beams with element side length 0.25 m

Poisson's ratio	8NodeBrick displacement	Theory displacement (bending)	Theory displacement (shear)	Theory displacement(all)	Error
0.00	8.511E-04 m	8.640E-04 m	1.200E-05 m	8.760E-04 m	2.84%
0.05	8.525E-04 m	8.640E-04 m	1.260E-05 m	8.766E-04 m	2.75%
0.10	8.527E-04 m	8.640E-04 m	1.320E-05 m	8.772E-04 m	2.79%
0.15	8.518E-04 m	8.640E-04 m	1.380E-05 m	8.778E-04 m	2.97%
0.20	8.494E-04 m	8.640E-04 m	1.440E-05 m	8.784E-04 m	3.30%
0.25	8.455E-04 m	8.640E-04 m	1.500E-05 m	8.790E-04 m	3.82%
0.30	8.393E-04 m	8.640E-04 m	1.560E-05 m	8.796E-04 m	4.58%
0.35	8.299E-04 m	8.640E-04 m	1.620E-05 m	8.802E-04 m	5.72%
0.40	8.141E-04 m	8.640E-04 m	1.680E-05 m	8.808E-04 m	7.57%
0.45	7.801E-04 m	8.640E-04 m	1.740E-05 m	8.814E-04 m	11.50%
0.49	6.603E-04 m	8.640E-04 m	1.788E-05 m	8.819E-04 m	25.12%

The errors were plotted in Figure (6).



(a) Error scale 0% - 80%



(b) Error scale 0% - 100%

Figure 6: 8NodeBrick cantilever beam for different Poisson's ratio ***Displacement error*** versus Poisson's ratio

The angle results were listed in Table (6).

Table 6: ***Rotation angle*** results for 8NodeBrick cantilever beams with element side length 1 m

Poisson's ratio	8NodeBrick angle (unit:°)	Theory angle (unit:°)	Error
0.00	8.25E-03	1.24E-02	33.46%
0.05	8.36E-03	1.24E-02	32.55%
0.10	8.42E-03	1.24E-02	32.08%
0.15	8.42E-03	1.24E-02	32.10%
0.20	8.35E-03	1.24E-02	32.67%
0.25	8.20E-03	1.24E-02	33.90%
0.30	7.95E-03	1.24E-02	35.89%
0.35	7.59E-03	1.24E-02	38.83%
0.40	7.07E-03	1.24E-02	43.00%
0.45	6.30E-03	1.24E-02	49.21%
0.49	4.93E-03	1.24E-02	60.20%

Then, in the same geometry, element side length was cut into 0.5m. The angle results were listed in Table (7).

Table 7: ***Rotation angle*** results for 8NodeBrick cantilever beams with with element side length 0.5 m

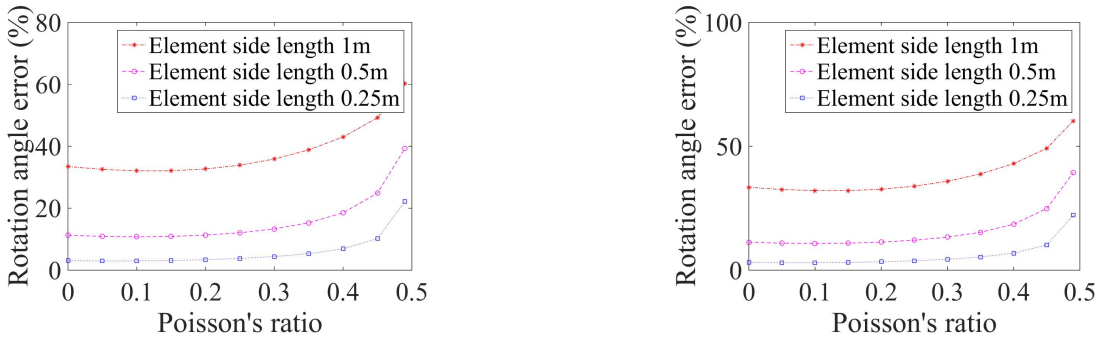
Poisson's ratio	8NodeBrick angle (unit:°)	Theory angle (unit:°)	Error
0.00	1.10E-02	1.24E-02	11.28%
0.05	1.10E-02	1.24E-02	10.91%
0.10	1.11E-02	1.24E-02	10.78%
0.15	1.10E-02	1.24E-02	10.90%
0.20	1.10E-02	1.24E-02	11.32%
0.25	1.09E-02	1.24E-02	12.09%
0.30	1.07E-02	1.24E-02	13.33%
0.35	1.05E-02	1.24E-02	15.29%
0.40	1.01E-02	1.24E-02	18.53%
0.45	9.32E-03	1.24E-02	24.87%
0.49	7.52E-03	1.24E-02	39.35%

Finally, in the same geometry, element side length was cut into 0.25m. The angle results were listed in Table (8).

Table 8: ***Rotation angle*** results for 8NodeBrick cantilever beams with with element side length 0.25 m

Poisson's ratio	8NodeBrick angle (unit:°)	Theory angle (unit:°)	Error
0.00	1.20E-02	1.24E-02	3.06%
0.05	1.20E-02	1.24E-02	2.97%
0.10	1.20E-02	1.24E-02	2.99%
0.15	1.20E-02	1.24E-02	3.12%
0.20	1.20E-02	1.24E-02	3.38%
0.25	1.19E-02	1.24E-02	3.79%
0.30	1.19E-02	1.24E-02	4.40%
0.35	1.17E-02	1.24E-02	5.33%
0.40	1.15E-02	1.24E-02	6.87%
0.45	1.11E-02	1.24E-02	10.22%
0.49	9.64E-03	1.24E-02	22.23%

The errors were plotted in Figure (7).



(a) Error scale 30% - 70%

(b) Error scale 0% - 100%

Figure 7: 8NodeBrick cantilever beam for different Poisson's ratio ***Rotation angle error*** versus Poisson's ratio

The ESSI model fei files for the table above are here

1.3 Test of irregular shaped 8NodeBrick cantilever beams

Cantilever model was used as an example. Three different shapes were tested.

In the first test, the upper two nodes of each element were moved one half element size along the $y-axis$, while the lower two nodes were kept at the same location. The element shape was shown in Figure (8).

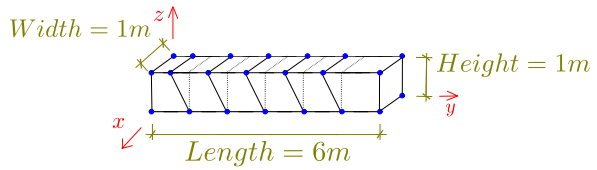


Figure 8: 8NodeBrick cantilever beams for irregular ***Shape 1***

In the second test, the upper two nodes of each element were moved 90% element size along the $y-axis$, while the lower two nodes were moved 90% element size in the other direction along the $y-axis$. The element shape was shown in Figure (9).

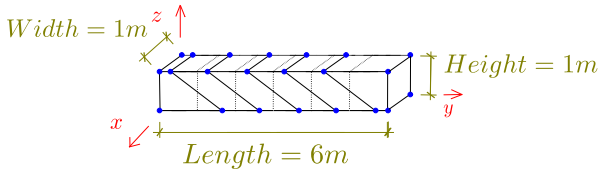


Figure 9: 8NodeBrick cantilever beams for irregular ***Shape 2***

In the third test, the upper two nodes of each element were moved one half element size with different directions along the $y-axis$, while the lower two nodes were kept at the same location. The element shape was shown in Figure (10).

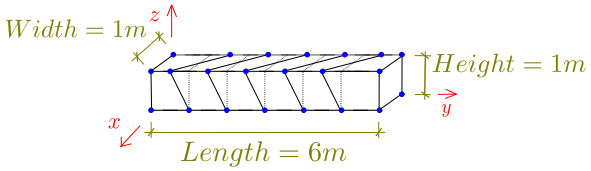
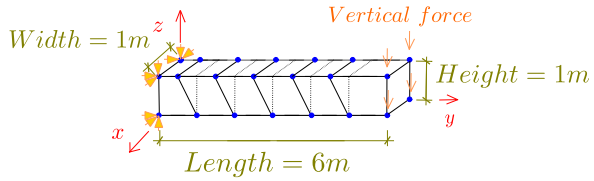
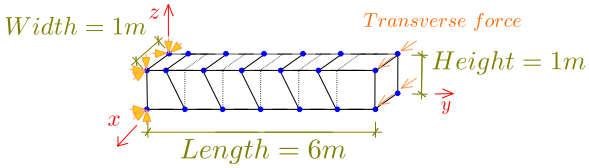


Figure 10: 8NodeBrick cantilever beams for irregular ***Shape 3***

The boundary conditions were shown in Figure (11), (12) and (13) .

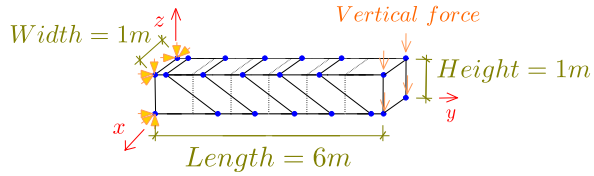


(a) Veritical force

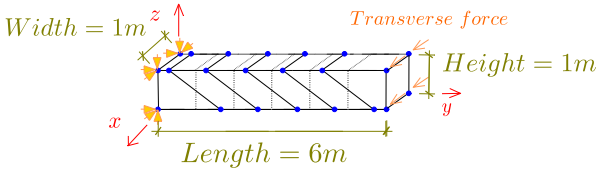


(b) Horizontal force

Figure 11: 8NodeBrick cantilever beam boundary conditions for irregular ***Shape 1***

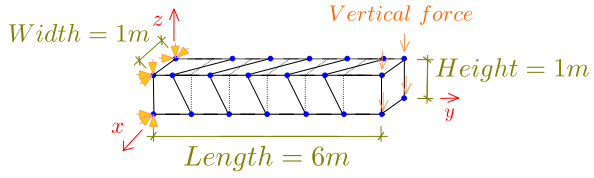


(a) Veritical force

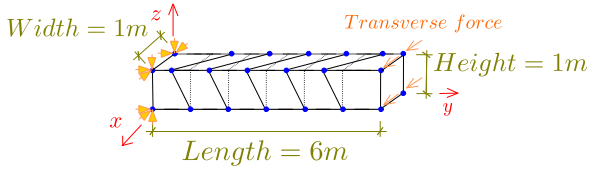


(b) Horizontal force

Figure 12: 8NodeBrick cantilever beam boundary conditions for irregular ***Shape 2***



(a) Veritical force



(b) Horizontal force

Figure 13: 8NodeBrick cantilever beam boundary conditions for irregular ***Shape 3***

The ESSI results were listed in Table (9).

Table 9: Results for 8NodeBrick cantilever beams of irregular shapes

Element Type	Force direction	Normal shape	Shape 1	Shape 2	Shape 3
8NodeBrick	Vertical (z)	5.840E-04 m	5.751E-04 m	2.959E-04 m	3.883E-04 m
8NodeBrick	Transverse (y)	5.840E-04 m	4.529E-04 m	1.390E-04 m	4.744E-04 m
Theoretical	-	8.760E-04 m	8.760E-04 m	8.760E-04 m	8.760E-04 m

The errors were listed in Table (10) and (11).

Table 10: Errors for irregular shaped 8NodeBrick compared to theoretical solution

Element Type	Force direction	Normal shape	Shape 1	Shape 2	Shape 3
8NodeBrick	Vertical (z)	33.33%	34.35%	66.22%	55.67%
8NodeBrick	Transverse (y)	33.33%	48.30%	84.13%	45.84%

Table 11: Errors for irregular shaped 8NodeBrick compared to normal shape

Element Type	Force direction	Normal shape	Shape 1	Shape 2	Shape 3
8NodeBrick	Vertical (z)	0.00%	1.52%	49.33%	33.51%
8NodeBrick	Transverse (y)	0.00%	22.45%	76.20%	18.77%

The ESSI model fei files for the table above are here

Then, the irregular beam was divided into small elements.

Problem description: Length=6m, Width=1m, Height=1m, Force=100N, E=1E8Pa, $\nu = 0.0$. The force direction was shown in Figure (14).

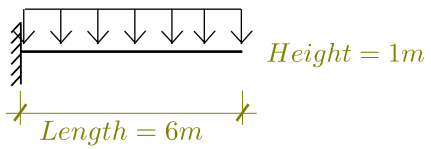


Figure 14: Problem description for cantilever beams under uniform load

Theoretical displacement (bending and shear deformation):

$$\begin{aligned} d &= \frac{qL^4}{8EI} + \frac{q\frac{L^2}{2}}{GA} \\ &= \frac{400N/m \times 12^4 m^4}{8 \times 10^8 N/m^2 \times \frac{2^4}{12} m^4} + \frac{400N/m \times \frac{12^2}{2} m^2}{\frac{10^8}{2} N/m^2 \times 2m \times 2m} \\ &= 7.776 \times 10^{-3} m + 1.44 \times 10^{-4} m \\ &= 7.920 \times 10^{-3} m \end{aligned} \tag{4}$$

The ESSI displacement results were listed in Table (12).

Table 12: Results for 8NodeBrick cantilever beams of irregular shapes with more elements

Element Type	Shape	Force direction	Number of division		
			1	2	4
8NodeBrick	shape1	Vertical (z)	5.37E-03 m	7.08E-03 m	7.71E-03 m
8NodeBrick	shape1	Transverse (y)	4.60E-03 m	6.66E-03 m	7.58E-03 m
8NodeBrick	shape2	Vertical (z)	2.74E-03 m	4.75E-03 m	6.43E-03 m
8NodeBrick	shape2	Transverse (y)	1.46E-03 m	2.72E-03 m	4.63E-03 m
8NodeBrick	shape3	Vertical (z)	9.21E-04 m	6.60E-03 m	7.56E-03 m
8NodeBrick	shape3	Transverse (y)	1.09E-03 m	6.09E-03 m	7.37E-03 m
Theoretical solution			7.92E-03 m	7.92E-03 m	7.92E-03 m

The error were listed in Table (13).

Table 13: Errors for 8NodeBrick cantilever beams of irregular shapes with more elements

Element Type	Shape	Force direction	Number of division		
			1	2	4
8NodeBrick	shape1	Vertical (z)	32.18%	10.63%	2.65%
8NodeBrick	shape1	Transverse (y)	41.95%	15.86%	4.34%
8NodeBrick	shape2	Vertical (z)	65.46%	40.01%	18.75%
8NodeBrick	shape2	Transverse (y)	81.51%	65.64%	41.60%
8NodeBrick	shape3	Vertical (z)	88.37%	16.67%	4.55%
8NodeBrick	shape3	Transverse (y)	86.19%	23.08%	6.94%

The errors were shown in Figure (15), (16) and (17).

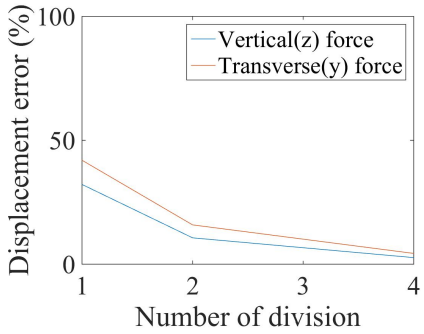


Figure 15: 8NodeBrick cantilever beam for irregular ***Shape 1***
Displacement error versus Number of division

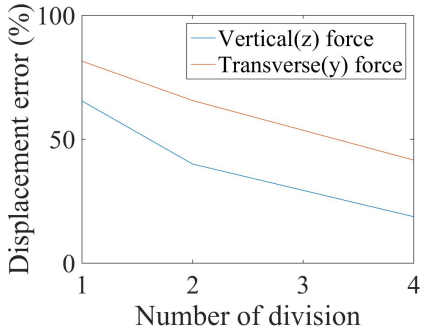


Figure 16: 8NodeBrick cantilever beam for irregular ***Shape 2***
Displacement error versus Number of division

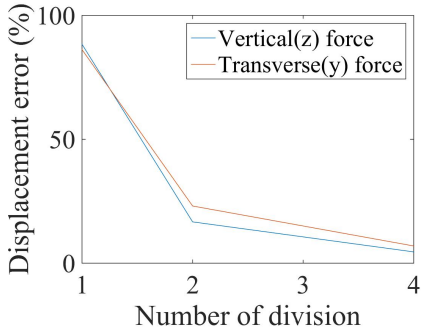


Figure 17: 8NodeBrick cantilever beam for irregular ***Shape 3***
Displacement error versus Number of division

The ESSI model fei files for the table above are here

1.4 Verification of 8NodeBrick edge clamped beams

Problem description: Length=6m, Width=1m, Height=1m, Force=100N, E=1E8Pa, $\nu = 0.0$. The force direction was shown in Figure (18).

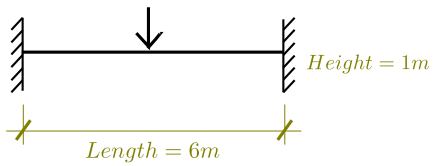


Figure 18: Problem description for clamped beams

The elment types and element sizes were same to the cantilever model. Only the boundary conditions and external force locations were changed.

Numerical model:

The 8NodeBrick elements were shown in Figure (19).

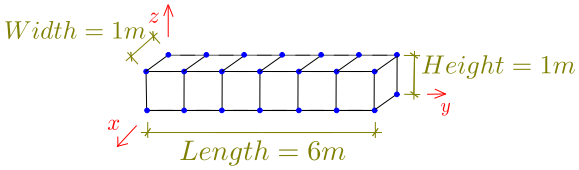


Figure 19: 8NodeBrick elements for clamped beams

Theoretical displacement (bending and shear deformation):

$$\begin{aligned} d &= \frac{FL^3}{192EI} + \frac{\frac{F}{2}\frac{L}{2}}{GA} \\ &= \frac{100N \times 6m^3}{192 \times 10^8 N/m^2 \times \frac{1}{12}m^4} + \frac{100N \times 6m}{4 \times 5 \times 10^7 N/m^2 \times 1m^2} \\ &= 1.35 \times 10^{-5}m + 0.3 \times 10^{-5}m \\ &= 1.65 \times 10^{-5}\ m \end{aligned} \tag{5}$$

The theoretical solution for $L = 6\ m$ was calculated above, while the solutions for other length were calculated by the similar process.

In the figures above, only the model with geometry $6m \times 1m \times 1m$ was drawed. In the ESSI models, the geometry $10m \times 1m \times 1m$ and the geometry $20m \times 1m \times 1m$ were also calculated. In three different geometry models, all the element sizes were $1m \times 1m \times 1m$. Therefore, the number of elements used in each model were 6, 10 *and* 20 respectively.

The results were listed in Table (14).

Table 14: Results for 8NodeBrick clamped beams of different geometry

Geometry	8NodeBrick	Theory(bending)	Theory(shear)	Theory(all)	Error
1:6	1.100E-05 m	1.35E-05 m	3.00E-06 m	1.65E-05 m	33.33%
1:10	4.500E-05 m	6.25E-05 m	5.00E-06 m	6.75E-05 m	33.33%
1:20	3.400E-04 m	5.00E-04 m	1.00E-05 m	5.10E-04 m	33.33%

The ESSI model fei files for the table above are here

In this section, the beam was cut into smaller elements with element side length 0.5m and 0.25m respectively. And the element side length of the original models is 1.0m. The numerical models were shown in Figure (20), (21) and (22).

Number of division 1:

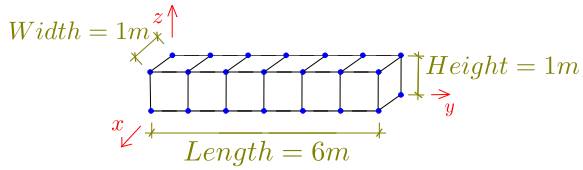


Figure 20: 8NodeBrick clamped beams with element side length 1.0m

Number of division 2:

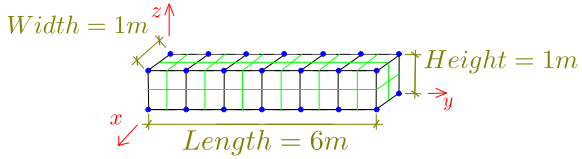


Figure 21: 8NodeBrick clamped beams with element side length 0.5m

Number of division 4:

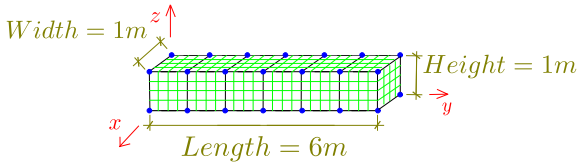


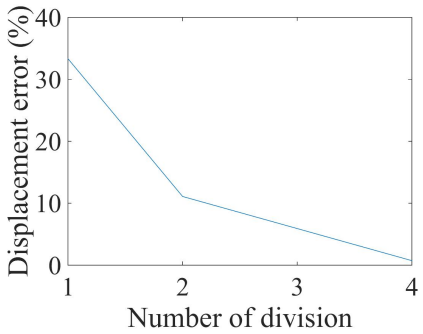
Figure 22: 8NodeBrick clamped beams with element side length 0.25m

The ESSI results were listed in Table (15). The theoretical solution is 1.65E-5 m.

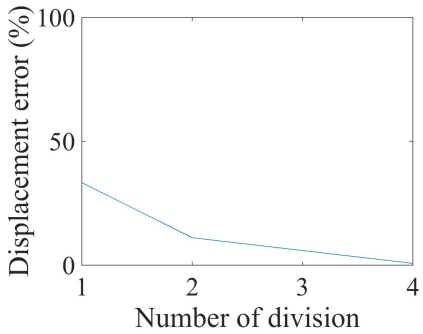
Table 15: Results for 8NodeBrick clamped beams with more elements

Element Type	Element side length		
	1 m	0.5 m	0.25 m
8NodeBrick	1.10E-05 m	1.47E-05 m	1.64E-05 m
Error	33.33%	11.09%	0.73%

The errors were plotted in Figure (23).



(a) Error scale 0% - 40%



(b) Error scale 0% - 100%

Figure 23: 8NodeBrick clamped beam for different element number
Displacement error versus Number of division

The ESSI model fei files for the table above are here

1.5 Verification of 8NodeBrick stress in cantilever beams

Problem description: Length=6m, Width=1m, Height=1m, Force=100N, E=1E8Pa, $\nu = 0.0$. The force direction was shown in Figure (24).

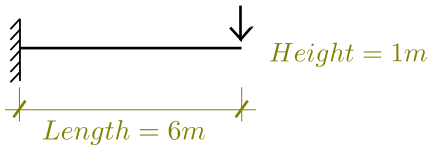


Figure 24: Problem description for cantilever beams of stress verification

The theoretical solution for the stress was calculated below.
The 8NodeBrick elements were shown in Figure (25).

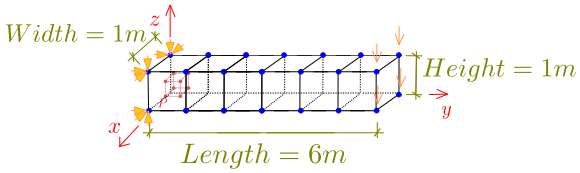


Figure 25: 8NodeBrick for cantilever beams of stress verification

The bending moment at the Gassian Point is

$$M = F(L - P_y) = 100N \times (6 - 0.2113)m = 578.87N \cdot m \tag{6}$$

The bending modulus is

$$I = \frac{bh^3}{12} = \frac{1}{12}m^4 \tag{7}$$

Therefore, the theoretical stress is

$$\sigma = \frac{M \cdot z}{I} = \frac{578.87N \cdot m \times (0.5 - 0.2113)m}{\frac{1}{12}m^4} = 2005Pa \tag{8}$$

To get a better result, the same geometry beam was also cut into small elements. When more elements were used, the theoretical stress was calculated again with the new coordinates. The calculation process is similar to the process above.
The numerical models were shown in Figure (26), (27) and (28).
Number of division 1:

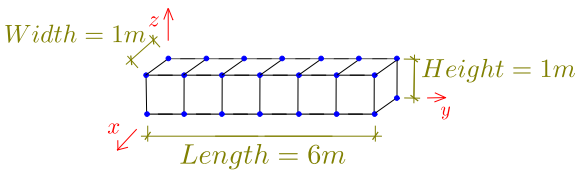


Figure 26: 8NodeBrick stress with element side length 1.0m

Number of division 2:

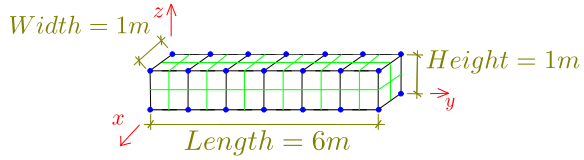


Figure 27: 8NodeBrick stress with element side length 0.5m

Number of division 4:

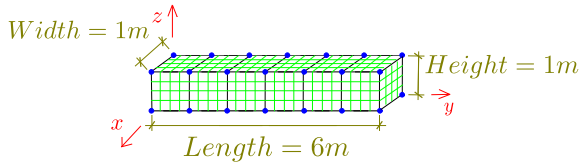


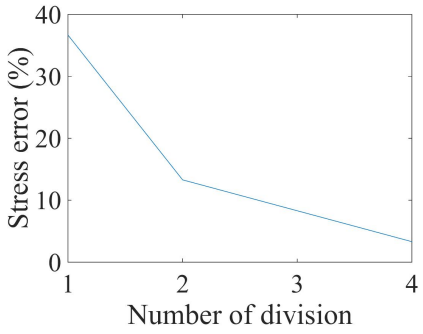
Figure 28: 8NodeBrick stress with element side length 0.25m

All the stress results were listed in Table (16).

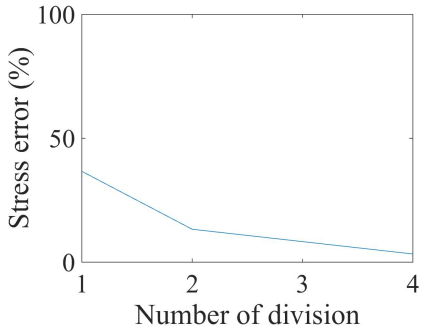
Table 16: Results for 8NodeBrick stress with more elements

Element Type	Element side length		
	1 m	0.5 m	0.25 m
8NodeBrick	1270.17 Pa	2418.60 Pa	3085.48 Pa
Theoretical	2005.26 Pa	2789.23 Pa	3191.27 Pa
Error	36.66%	13.29%	3.31%

The errors were plotted in Figure (29).



(a) Error scale 0% - 40%



(b) Error scale 0% - 100%

Figure 29: 8NodeBrick cantilever beams for stress verification
Stress error versus Number of division

The ESSI model fei files for the table above are here

1.6 Verification of 8NodeBrick square plate with four edges clamped

Problem description: Length=20m, Width=20m, Height=1m, Force=100N, E=1E8Pa, $\nu = 0.3$.

The four edges are clamped.

The load is the uniform normal pressure on the whole plate.

The plate flexural rigidity is

$$D = \frac{Eh^3}{12(1-\nu^2)} = \frac{10^8 N/m^2 \times 1^3 m^3}{12 \times (1-0.3^2)} = 9.1575 \times 10^6 \ N \cdot m \tag{9}$$

The theoretical solution is

$$d = \alpha_c \frac{qa^4}{D} = 0.00406 \times \frac{100 N/m^2 \times 20^4 m^4}{9.1575 \times 10^6 \ N \cdot m} = 2.2015 \times 10^{-3} m \tag{10}$$

where α_c is a coefficient, which depends on the ratio of plate length to width. In this problem, the coefficient[1] α_c is 0.00406.

The 8NodeBrick were shown in Figure (30) - (35).

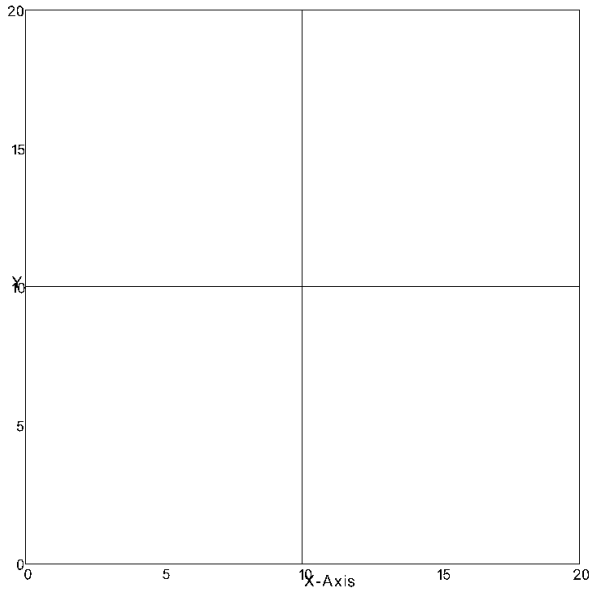


Figure 30: 8NodeBrick edge clamped square plate with element side length 10m

[1] Stephen Timoshenko, Theory of plates and shells (2nd edition). MrGRAW-Hill Inc, page120, 1959.

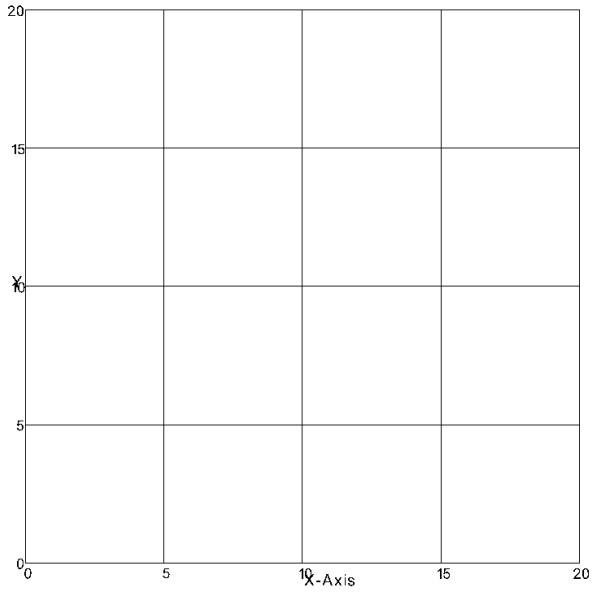


Figure 31: 8NodeBrick edge clamped square plate with element side length 5m

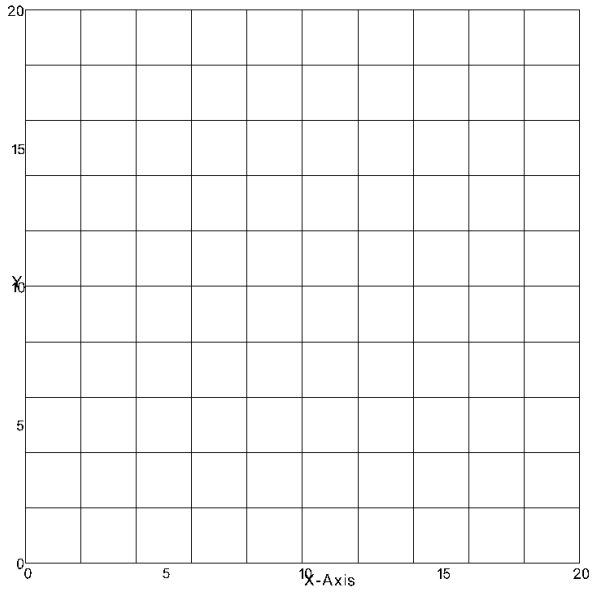


Figure 32: 8NodeBrick edge clamped square plate with element side length 2m

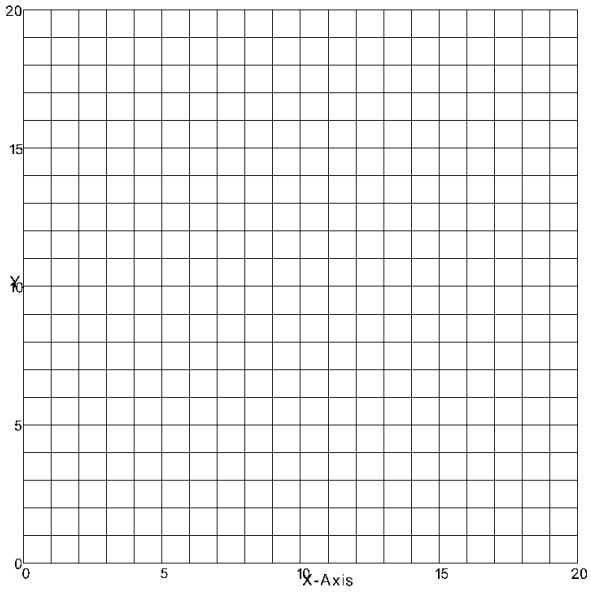


Figure 33: 8NodeBrick edge clamped square plate with element side length 1m

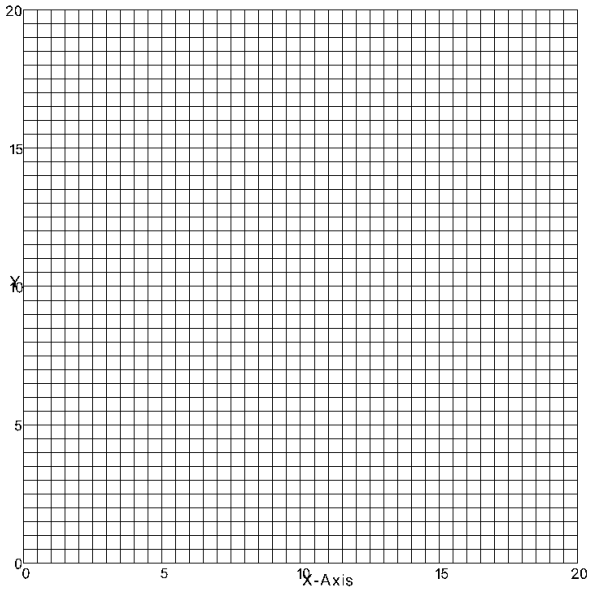


Figure 34: 8NodeBrick edge clamped square plate with element side length 0.5m

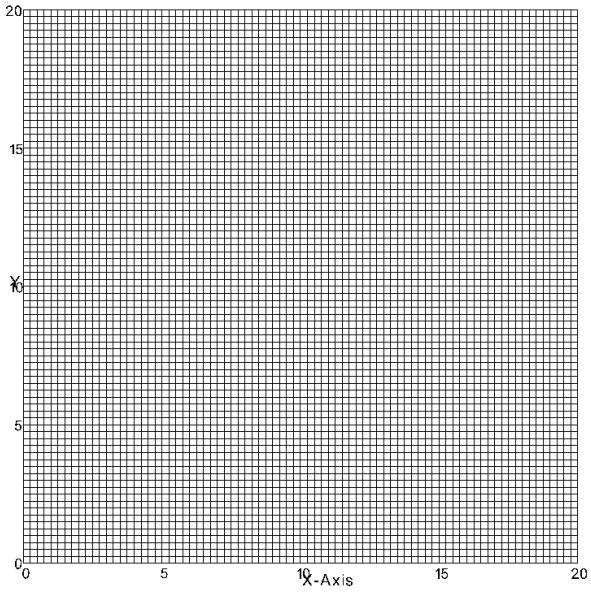


Figure 35: 8NodeBrick edge clamped square plate with element side length 0.25m

The results were listed in Table (17).

Table 17: Results for 8NodeBrick square plate with four edges clamped

Element type	8NodeBrick	8NodeBrick	8NodeBrick	Theoretical displacement
Number of layers	1layer	2layers	4layers	
Element side length	Height:1.00m	Height:0.50m	Height:0.25m	
10m	9.75E-05 m	9.75E-05 m	9.75E-05 m	2.20E-03 m
5m	3.28E-04 m	3.32E-04 m	3.32E-04 m	2.20E-03 m
2m	1.04E-03 m	1.10E-03 m	1.12E-03 m	2.20E-03 m
1m	1.56E-03 m	1.74E-03 m	1.79E-03 m	2.20E-03 m
0.5m	1.80E-03 m	2.30E-03 m	2.12E-03 m	2.20E-03 m
0.25m	1.87E-03 m	2.14E-03 m	2.23E-03 m	2.20E-03 m

The errors were listed in Table (18).

Table 18: Errors for 8NodeBrick square plate with four edges clamped

Element type	8NodeBrick	8NodeBrick	8NodeBrick
Number of layers	1layer	2layers	4layers
Element side length	Height:1.00m	Height:0.50m	Height:0.25m
10m	95.57%	95.57%	95.57%
5m	85.09%	84.94%	84.91%
2m	52.98%	50.09%	49.25%
1m	28.93%	21.17%	18.72%
0.5m	18.26%	4.58%	3.56%
0.25m	15.05%	2.70%	1.37%

The errors were plotted in Figure (36).

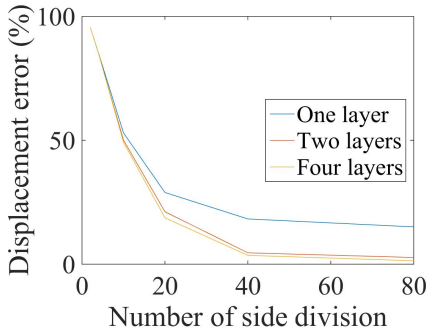


Figure 36: 8NodeBrick square plate with edge clamped
Displacement error versus Number of side division

The ESSI model fei files for the table above are here

- ***Square plate with edges clamped:*** $100m \times 100m \times 1m$

The same verification procedures above were did for the square plate of $100m \times 100m \times 1m$. The

1.7 Verification of 8NodeBrick square plate with four edges simply supported

Problem description: Length=20m, Width=20m, Height=1m, Force=100N, E=1E8Pa, $\nu = 0.3$.

The four edges are simply supported.

The load is the uniform normal pressure on the whole plate.

The plate flexural rigidity is

$$D = \frac{Eh^3}{12(1-\nu^2)} = \frac{10^8 N/m^2 \times 1^3 m^3}{12 \times (1-0.3^2)} = 9.1575 \times 10^6 \ N \cdot m \tag{11}$$

The theoretical solution is

$$d = \alpha_s \frac{qa^4}{D} = 0.00126 \times \frac{100 N/m^2 \times 20^4 m^4}{9.1575 \times 10^6 \ N \cdot m} = 7.0936 \times 10^{-3} m \tag{12}$$

where α_s is a coefficient, which depends on the ratio of plate length to width. In this problem, the coefficient[2] α_s is 0.00126.

The 8NodeBrick were shown in Figure (37) - (42).

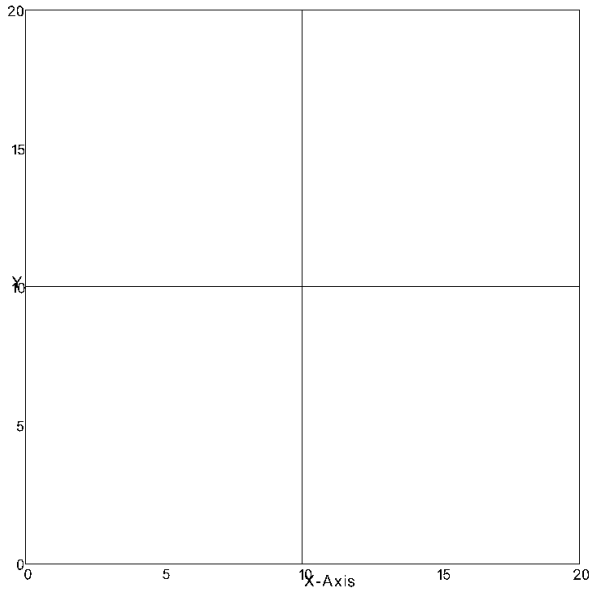


Figure 37: 8NodeBrick edge simply supported square plate with element side length 10m

[2]Stephen Timoshenko, Theory of plates and shells (2nd edition). MrGRAW-Hill Inc, page202, 1959.

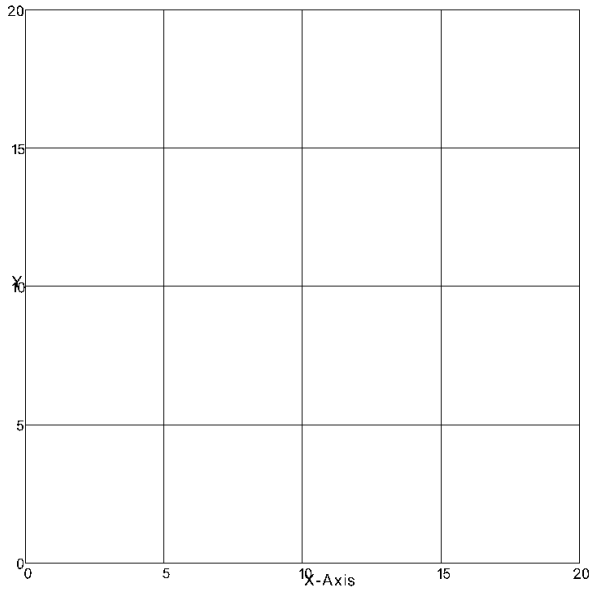


Figure 38: 8NodeBrick edge simply supported square plate with element side length 5m

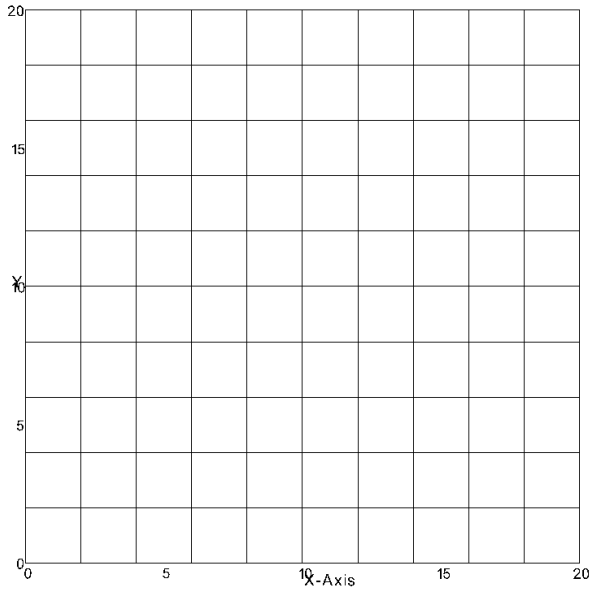


Figure 39: 8NodeBrick edge simply supported square plate with element side length 2m

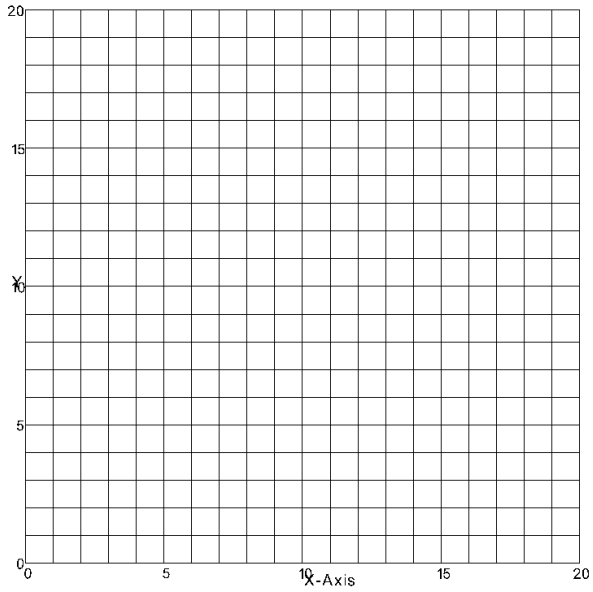


Figure 40: 8NodeBrick edge simply supported square plate with element side length 1m

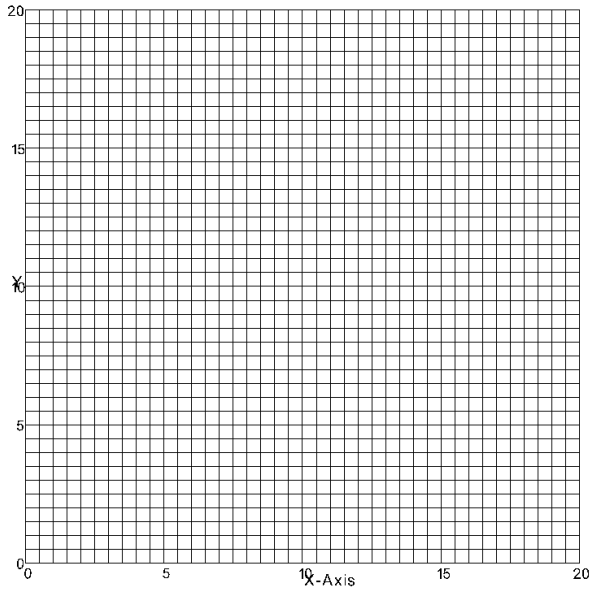


Figure 41: 8NodeBrick edge simply supported square plate with element side length 0.5m

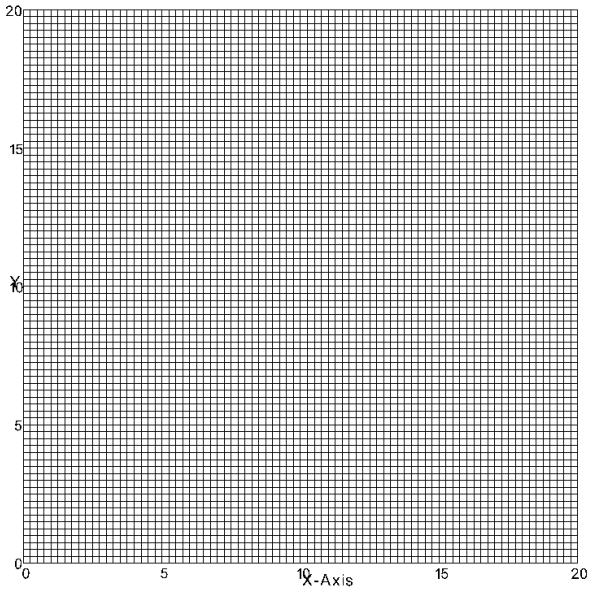


Figure 42: 8NodeBrick edge simply supported square plate with element side length 0.25m

The results were listed in Table (19).

Table 19: Results for 8NodeBrick square plate with four edges simply supported

Element type	8NodeBrick	8NodeBrick	Theoretical displacement
Number of layers	2layers	4layers	
Element side length	Height:0.50m	Height:0.25m	
10m	3.75E-004 m	3.76E-004 m	7.09E-03 m
5m	1.34E-003 m	1.35E-003 m	7.09E-03 m
2m	4.16E-003 m	4.27E-003 m	7.09E-03 m
1m	5.98E-003 m	6.22E-003 m	7.09E-03 m
0.5m	6.75E-003 m	7.04E-003 m	7.09E-03 m
0.25m	8.07E-003 m	7.30E-003 m	7.09E-03 m

The errors were listed in Table (20).

Table 20: Errors for 8NodeBrick square plate with four edges simply supported

Element type	8NodeBrick	8NodeBrick
Number of layers	2layers	4layers
Element side length	Height:0.50m	Height:0.25m
10m	94.72%	94.71%
5m	81.05%	80.91%
2m	41.31%	39.79%
1m	15.64%	12.38%
0.5m	4.88%	0.70%
0.25m	13.74%	2.86%

The errors were plotted in Figure (43).

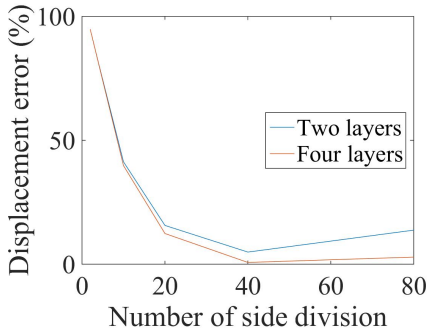


Figure 43: 8NodeBrick square plate with four edges simply supported
Displacement error versus Number of side division

The ESSI model fei files for the table above are here

1.8 Verification of 8NodeBrick circular plate with all edges clamped

Problem description: Diameter=20m, Height=1m, Force=100N, E=1E8Pa, $\nu = 0.3$.

The four edges are clamped.

The load is the uniform normal pressure on the whole plate.

The plate flexural rigidity is

$$D = \frac{Eh^3}{12(1-\nu^2)} = \frac{10^8 N/m^2 \times 1^3 m^3}{12 \times (1-0.3^2)} = 9.1575 \times 10^6 \ N \cdot m \tag{13}$$

The theoretical solution[3] is

$$d = \frac{qa^4}{64D} = \frac{100 N/m^2 \times 10^4 m^4}{64 \times 9.1575 \times 10^6 \ N \cdot m} = 1.7106 \times 10^{-3} m \tag{14}$$

The 8NodeBrick were shown in Figure (44) - (49).

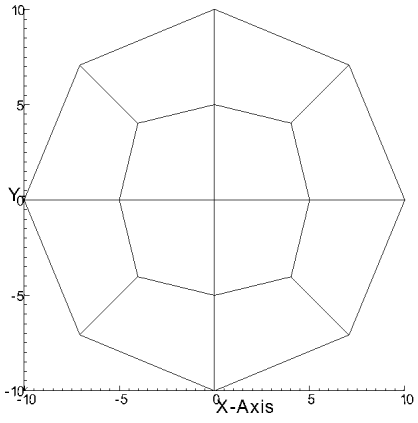


Figure 44: 8NodeBrick edge clamped circular plate with element side length 10m

[3]Stephen Timoshenko, Theory of plates and shells (2nd edition). MrGRAW-Hill Inc, page55, 1959.

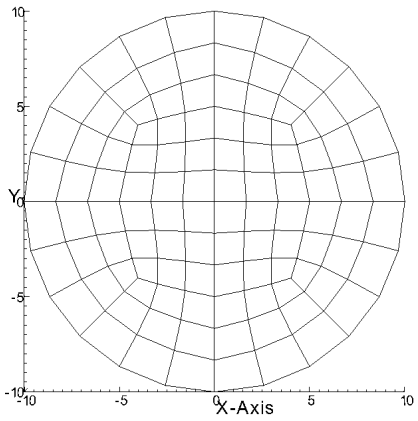


Figure 45: 8NodeBrick edge clamped circular plate with element side length 5m

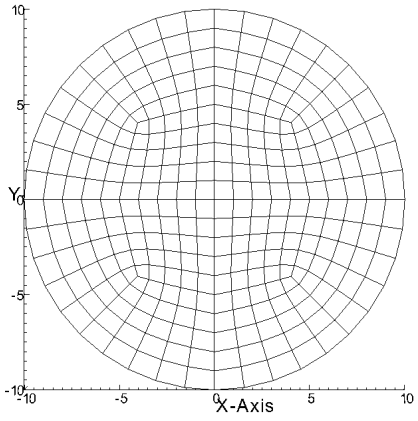


Figure 46: 8NodeBrick edge clamped circular plate with element side length 2m

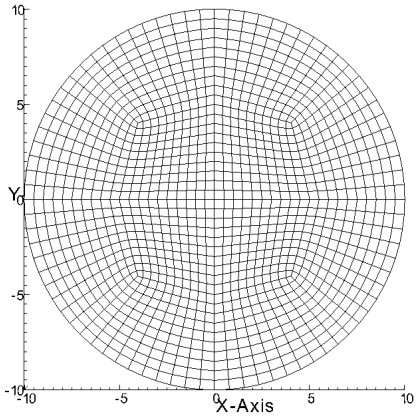


Figure 47: 8NodeBrick edge clamped circular plate with element side length 1m

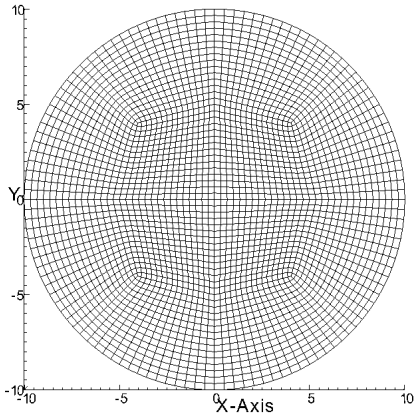


Figure 48: 8NodeBrick edge clamped circular plate with element side length 0.5m

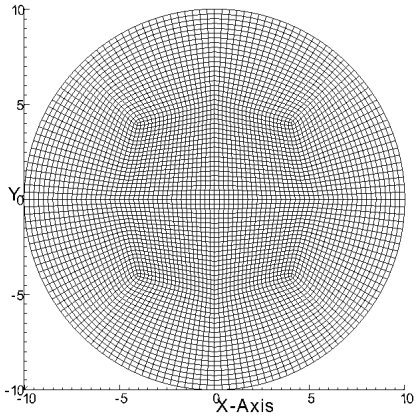


Figure 49: 8NodeBrick edge clamped circular plate with element side length 0.25m

The results were listed in Table (21).

Table 21: Results for 8NodeBrick circular plate with four edges clamped

Element type	8NodeBrick	8NodeBrick	8NodeBrick	Theoretical displacement
Number of layers	1layer	2layers	4layers	
Number of diameter divisions	Height:1.00m	Height:0.50m	Height:0.25m	
4	1.97E-04 m	1.99E-04 m	2.00E-04 m	1.71E-03 m
12	7.95E-04 m	8.47E-04 m	8.62E-04 m	1.71E-03 m
20	1.13E-03 m	1.25E-03 m	1.28E-03 m	1.71E-03 m
40	1.36E-03 m	1.54E-03 m	1.60E-03 m	1.71E-03 m
60	1.41E-03 m	1.62E-03 m	1.68E-03 m	1.71E-03 m
80	1.43E-03 m	1.64E-03 m	1.71E-03 m	1.71E-03 m

The errors were listed in Table (22).

Table 22: Errors for 8NodeBrick circular plate with four edges clamped

Element type	8NodeBrick	8NodeBrick	8NodeBrick
Number of layers	1layer	2layers	4layers
Number of diameter divisions	Height:1.00m	Height:0.50m	Height:0.25m
4	88.43%	88.32%	88.30%
12	53.43%	50.35%	49.47%
20	33.79%	27.00%	24.93%
40	20.14%	9.47%	6.03%
60	17.11%	5.34%	1.51%
80	16.01%	3.80%	0.19%

The errors were shown in Figure (50).

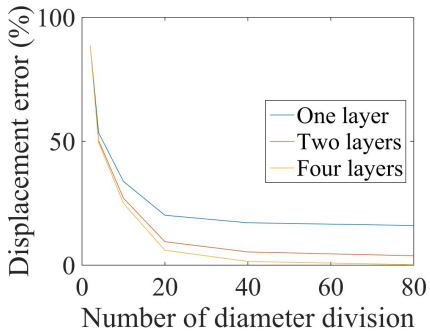


Figure 50: 8NodeBrick circular plate with edge clamped
Displacement error versus Number of side division

The ESSI model fei files for the table above are here

1.9 Verification of 8NodeBrick circular plate with all edges simply supported

Problem description: Diameter=20m, Height=1m, Force=100N, E=1E8Pa, $\nu = 0.3$.

The four edges are simply supported.

The load is the uniform normal pressure on the whole plate.

The plate flexural rigidity is

$$D = \frac{Eh^3}{12(1-\nu^2)} = \frac{10^8 N/m^2 \times 1^3 m^3}{12 \times (1-0.3^2)} = 9.1575 \times 10^6 \ N \cdot m \tag{15}$$

The theoretical solution[4] is

$$d = \frac{(5+\nu)qa^4}{64(1+\nu)D} = \frac{(5+0.3) \times 100 N/m^2 \times 10^4 m^4}{64 \times (1+0.3) \times 9.1575 \times 10^6 \ N \cdot m} = 6.956 \times 10^{-3} m \tag{16}$$

The 8NodeBrick were shown in Figure (51) - (56).

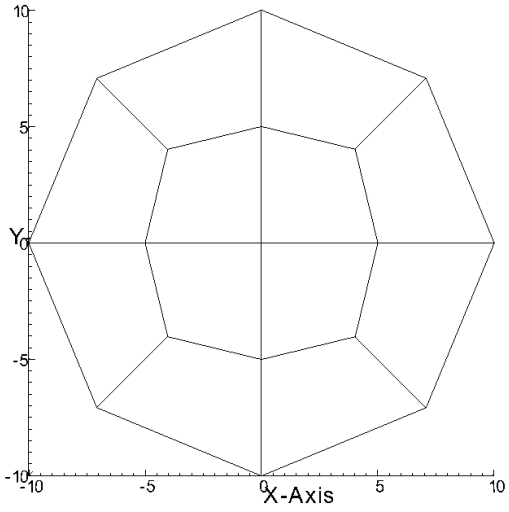


Figure 51: 8NodeBrick edge simply supported circular plate with element side length 10m

[4]Stephen Timoshenko, Theory of plates and shells (2nd edition). MrGRAW-Hill Inc, page55, 1959.

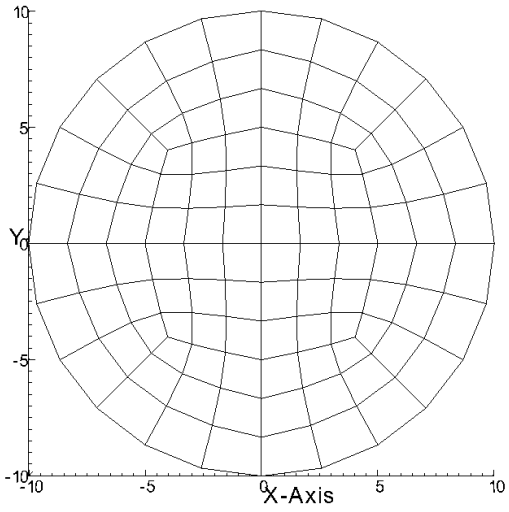


Figure 52: 8NodeBrick edge simply supported circular plate with element side length 5m

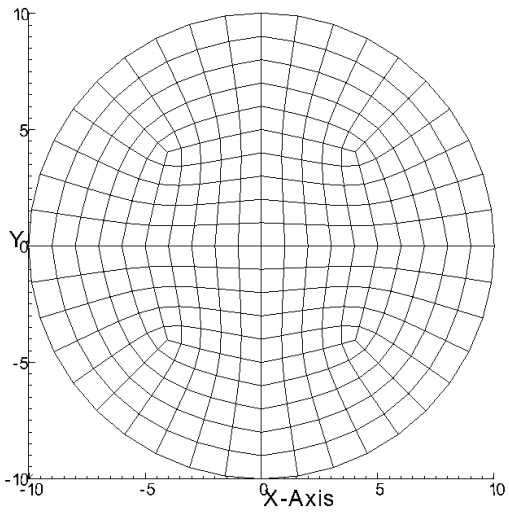


Figure 53: 8NodeBrick edge simply supported circular plate with element side length 2m

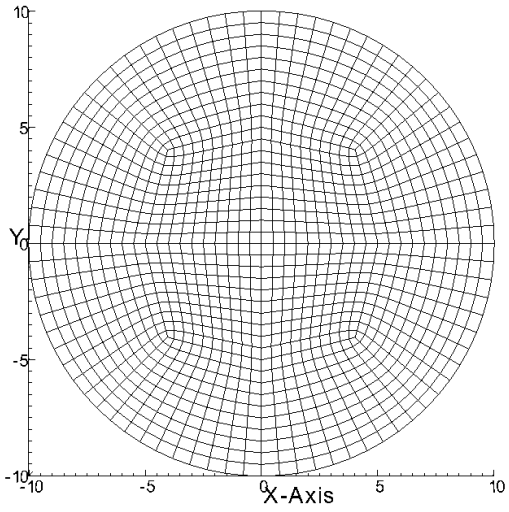


Figure 54: 8NodeBrick edge simply supported circular plate with element side length 1m

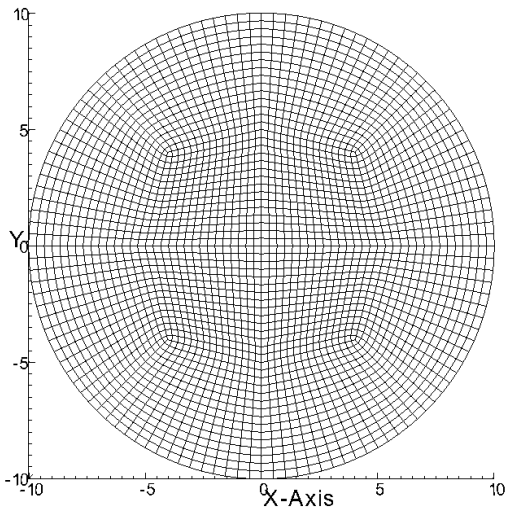


Figure 55: 8NodeBrick edge simply supported circular plate with element side length 0.5m

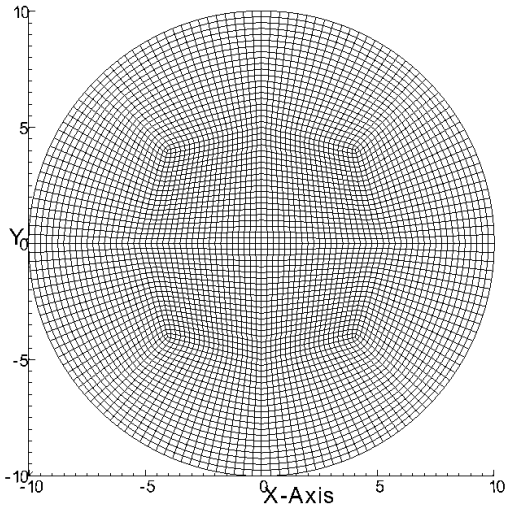


Figure 56: 8NodeBrick edge simply supported circular plate with element side length 0.25m

The results were listed in Table (23).

Table 23: Results for 8NodeBrick cicular plate with four edges simply supported

Element type	8NodeBrick	8NodeBrick	Theoretical displacement
Number of layers	2layers	4layers	
Number of diameter divisions	Height:0.50m	Height:0.25m	
4	6.35E-04 m	6.39E-04 m	6.96E-03 m
12	3.46E-03 m	3.57E-03 m	6.96E-03 m
20	4.96E-03 m	5.18E-03 m	6.96E-03 m
40	6.05E-03 m	6.37E-03 m	6.96E-03 m
60	6.30E-03 m	6.65E-03 m	6.96E-03 m
80	6.39E-03 m	6.76E-03 m	6.96E-03 m

The errors were listed in Table (24).

Table 24: Errors for 8NodeBrick cicular plate with four edges simply supported

Element type	8NodeBrick	8NodeBrick
Number of layers	2layers	4layers
Number of diameter divisions	Height:0.50m	Height:0.25m
4	90.87%	90.82%
12	50.19%	48.65%
20	28.64%	25.47%
40	13.09%	8.40%
60	9.45%	4.36%
80	8.10%	2.85%

The errors were plotted in Figure (57).

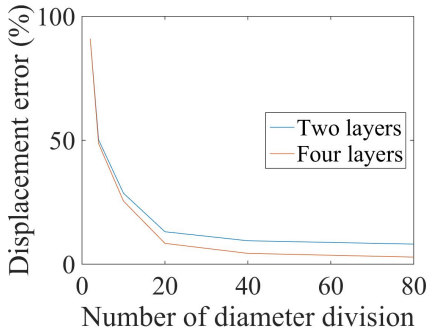


Figure 57: 8NodeBrick circular plate with edge simply supported
Displacement error versus Number of side division

The ESSI model fei files for the table above are here.

2 Verification of 27NodeBrick elements

2.1 Verification of 27NodeBrick cantilever beams

Problem description: Length=6m, Width=1m, Height=1m, Force=100N, E=1E8Pa, $\nu = 0.0$. The force direction was shown in Figure (58).

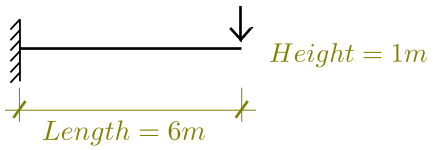
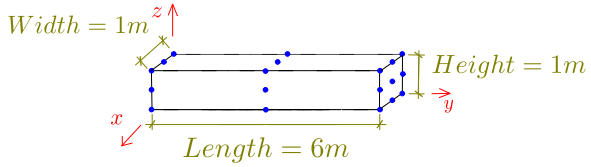


Figure 58: Problem description for cantilever beams

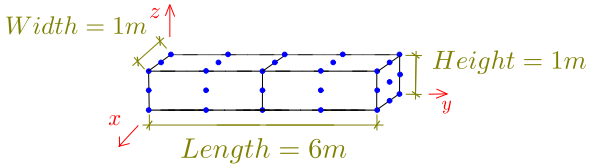
Theoretical displacement (bending and shear deformation):

$$\begin{aligned} d &= \frac{FL^3}{3EI} + \frac{FL}{GA} \\ &= \frac{100N \times 6^3 m^3}{3 \times 10^8 N/m^2 \times \frac{1}{12} m^4} + \frac{100N \times 6m}{5 \times 10^7 N/m^2 \times 1m^2} \\ &= 8.64 \times 10^{-4} m + 0.12 \times 10^{-4} m \\ &= 8.76 \times 10^{-4}\ m \end{aligned} \tag{17}$$

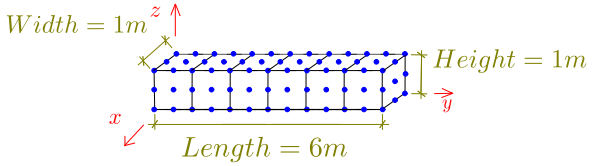
Numerical model:
The 27NodeBrick elements were shown in Figure (59).



(a) One 27NodeBrick element



(b) Two 27NodeBrick elements



(c) Six 27NodeBrick elements

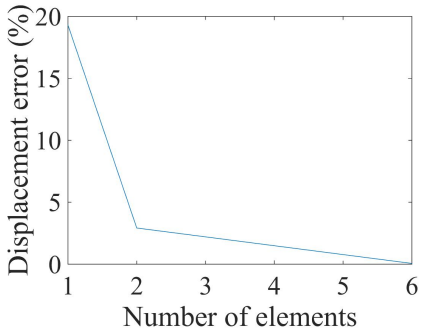
Figure 59: 27NodeBrick elements for cantilever beams

All the ESSI results were listed in Table (25).

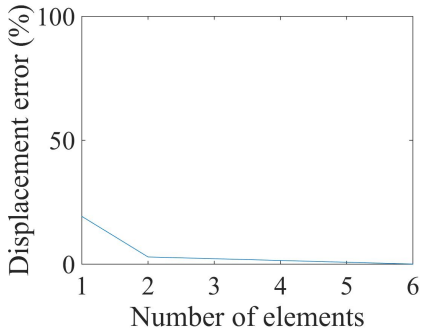
Table 25: Results for 27NodeBrick cantilever beams of different element numbers

Element number	1	2	6
27NodeBrick	7.07E-04 m	8.50E-04 m	8.75E-04 m
Error	19.30%	2.92%	0.06%

The errors were plotted in Figure (60).



(a) Error scale 0% - 20%



(b) Error scale 0% - 100%

Figure 60: 27NodeBrick cantilever beam for different element number
Displacement error versus Number of elements

The ESSI model fei files for the table above are here

- ***Cantilever: different geometry***

In the figures above, only the model with geometry $6m \times 1m \times 1m$ was drawed. In the ESSI models, the geometry $10m \times 1m \times 1m$ and the geometry $20m \times 1m \times 1m$ were also calculated. In three different geometry models, all the element sizes were $1m \times 1m \times 1m$. Therefore, the number of elements used in each model were 6, 10 *and* 20 respectively.

The ESSI results for different geometry were listed in Table (26).

Table 26: Results for 27NodeBrick cantilever beams of different geometry

Geometry	27NodeBrick	Theoretical(bending)	Theoretical(shear)	Theoretical(all)	Error
1:6	8.75E-04 m	8.64E-04 m	1.20E-05 m	8.76E-04 m	0.06%
1:10	4.02E-03 m	4.00E-03 m	2.00E-05 m	4.02E-03 m	0.02%
1:20	3.20E-02 m	3.20E-02 m	4.00E-05 m	3.20E-02 m	0.01%

The ESSI model fei files for the table above are here

2.2 Verification of 27NodeBrick cantilever beam for different Poisson's ratio

Problem description: Length=6m, Width=1m, Height=1m, Force=100N, E=1E8Pa, $\nu = 0.0 - 0.49$. The force direction was shown in Figure (61).

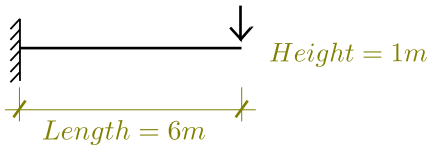


Figure 61: Problem description for cantilever beams of different Poisson's ratios

The theoretical solution for $\nu = 0.0$ was calculated below, while the solution for other Poisson's ratio were calculated by the similar process.

Theoretical displacement (bending and shear deformation):

$$\begin{aligned} d &= \frac{FL^3}{3EI} + \frac{FL}{GA} \\ &= \frac{100N \times 6^3 m^3}{3 \times 10^8 N/m^2 \times \frac{1}{12} m^4} + \frac{100N \times 6m}{5 \times 10^7 N/m^2 \times 1m^2} \\ &= 8.64 \times 10^{-4} m + 0.12 \times 10^{-4} m \\ &= 8.76 \times 10^{-4}\ m \end{aligned} \tag{18}$$

The rotation angle at the end:

$$\theta = \frac{FL^2}{2EI} = \frac{100N \times 6^2 m^2}{2 \times 10^8 N/m^2 \times \frac{1}{12} m^4} = 2.16 \times 10^{-4}\ rad = 0.0124^\circ \tag{19}$$

The 27NodeBrick elements for cantilever beams of different Poisson's ratios were shown in Figure (62):

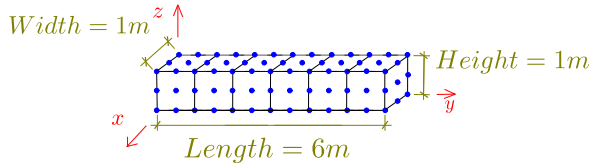


Figure 62: 27NodeBrick elements for cantilever beams of different Poisson's ratios

All the displacement results were listed in Table (27).

Table 27: ***Displacement*** results for 27NodeBrick cantilever beams with element side length 1 m

Poisson's ratio	27NodeBrick displacement	Theory displacement (bending)	Theory displacement (shear)	Theory displacement(all)	Error
0.00	8.804E-04 m	8.640E-04 m	1.200E-05 m	8.760E-04 m	0.06%
0.05	8.808E-04 m	8.640E-04 m	1.260E-05 m	8.766E-04 m	0.10%
0.10	8.805E-04 m	8.640E-04 m	1.320E-05 m	8.772E-04 m	0.24%
0.15	8.796E-04 m	8.640E-04 m	1.380E-05 m	8.778E-04 m	0.49%
0.20	8.778E-04 m	8.640E-04 m	1.440E-05 m	8.784E-04 m	0.86%
0.25	8.752E-04 m	8.640E-04 m	1.500E-05 m	8.790E-04 m	1.40%
0.30	8.715E-04 m	8.640E-04 m	1.560E-05 m	8.796E-04 m	2.14%
0.35	8.663E-04 m	8.640E-04 m	1.620E-05 m	8.802E-04 m	3.20%
0.40	8.588E-04 m	8.640E-04 m	1.680E-05 m	8.808E-04 m	4.80%
0.45	8.465E-04 m	8.640E-04 m	1.740E-05 m	8.814E-04 m	7.57%
0.49	8.248E-04 m	8.640E-04 m	1.788E-05 m	8.819E-04 m	12.56%

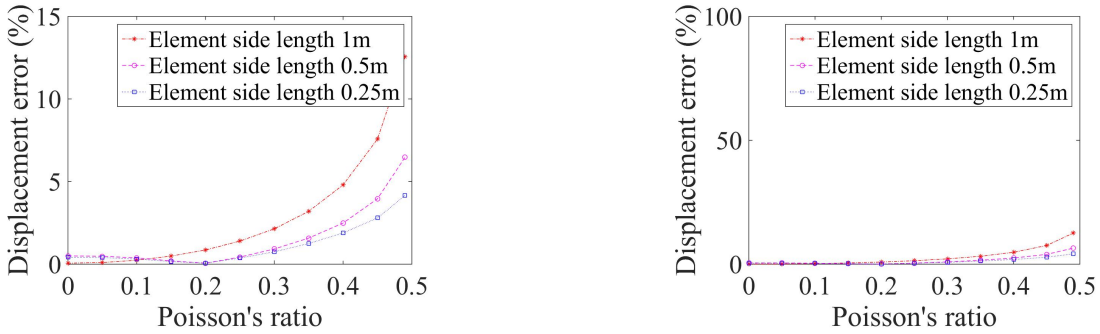
Table 28: ***Displacement*** results for 27NodeBrick cantilever beams with element side length 0.5 m

Poisson's ratio	27NodeBrick displacement	Theory displacement (bending)	Theory displacement (shear)	Theory displacement(all)	Error
0.00	8.804E-04 m	8.640E-04 m	1.200E-05 m	8.760E-04 m	0.50%
0.05	8.808E-04 m	8.640E-04 m	1.260E-05 m	8.766E-04 m	0.48%
0.10	8.805E-04 m	8.640E-04 m	1.320E-05 m	8.772E-04 m	0.38%
0.15	8.796E-04 m	8.640E-04 m	1.380E-05 m	8.778E-04 m	0.20%
0.20	8.778E-04 m	8.640E-04 m	1.440E-05 m	8.784E-04 m	0.06%
0.25	8.752E-04 m	8.640E-04 m	1.500E-05 m	8.790E-04 m	0.43%
0.30	8.715E-04 m	8.640E-04 m	1.560E-05 m	8.796E-04 m	0.92%
0.35	8.663E-04 m	8.640E-04 m	1.620E-05 m	8.802E-04 m	1.58%
0.40	8.588E-04 m	8.640E-04 m	1.680E-05 m	8.808E-04 m	2.49%
0.45	8.465E-04 m	8.640E-04 m	1.740E-05 m	8.814E-04 m	3.96%
0.49	8.248E-04 m	8.640E-04 m	1.788E-05 m	8.819E-04 m	6.47%

Table 29: ***Displacement*** results for 27NodeBrick cantilever beams with element side length 0.25 m

Poisson's ratio	27NodeBrick displacement	Theory displacement (bending)	Theory displacement (shear)	Theory displacement(all)	Error
0.00	8.797E-04 m	8.640E-04 m	1.200E-05 m	8.760E-04 m	0.42%
0.05	8.801E-04 m	8.640E-04 m	1.260E-05 m	8.766E-04 m	0.40%
0.10	8.799E-04 m	8.640E-04 m	1.320E-05 m	8.772E-04 m	0.31%
0.15	8.792E-04 m	8.640E-04 m	1.380E-05 m	8.778E-04 m	0.16%
0.20	8.778E-04 m	8.640E-04 m	1.440E-05 m	8.784E-04 m	0.06%
0.25	8.758E-04 m	8.640E-04 m	1.500E-05 m	8.790E-04 m	0.37%
0.30	8.730E-04 m	8.640E-04 m	1.560E-05 m	8.796E-04 m	0.75%
0.35	8.692E-04 m	8.640E-04 m	1.620E-05 m	8.802E-04 m	1.25%
0.40	8.641E-04 m	8.640E-04 m	1.680E-05 m	8.808E-04 m	1.89%
0.45	8.567E-04 m	8.640E-04 m	1.740E-05 m	8.814E-04 m	2.80%
0.49	8.452E-04 m	8.640E-04 m	1.788E-05 m	8.819E-04 m	4.16%

The errors were plotted in Figure (63).



(a) Error scale 0% - 15%

(b) Error scale 0% - 100%

Figure 63: 27NodeBrick cantilever beam for different Poisson's ratio ***Displacement error*** versus Poisson's ratio

The angle results were listed in Table (30).

Table 30: ***Rotation angle*** results for 27NodeBrick cantilever beams with element side length 1 m

Poisson's ratio	27NodeBrick angle (unit:°)	Theory angle (unit:°)	Error
0.00	1.238E-02	1.24E-02	0.19%
0.05	1.237E-02	1.24E-02	0.24%
0.10	1.236E-02	1.24E-02	0.34%
0.15	1.233E-02	1.24E-02	0.53%
0.20	1.230E-02	1.24E-02	0.80%
0.25	1.225E-02	1.24E-02	1.18%
0.30	1.219E-02	1.24E-02	1.70%
0.35	1.210E-02	1.24E-02	2.45%
0.40	1.196E-02	1.24E-02	3.55%
0.45	1.172E-02	1.24E-02	5.47%
0.49	1.130E-02	1.24E-02	8.89%

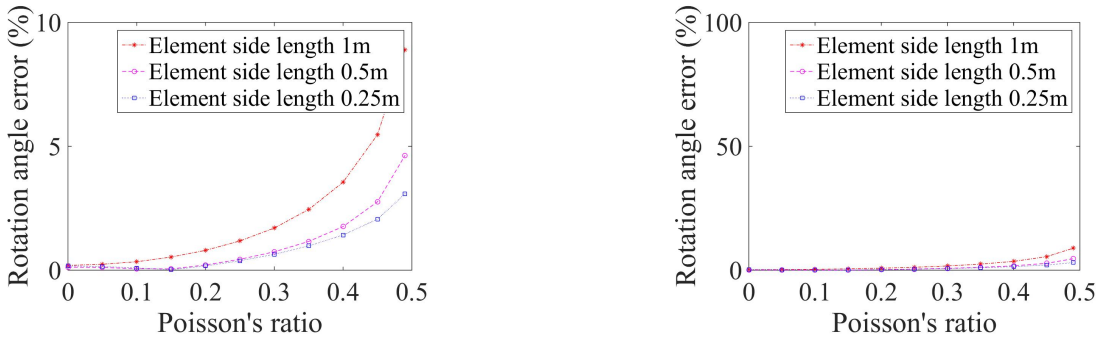
Table 31: ***Rotation angle*** results for 27NodeBrick cantilever beams with element side length 0.5 m

Poisson's ratio	27NodeBrick angle (unit:°)	Theory angle (unit:°)	Error
0.00	1.242E-02	1.24E-02	0.12%
0.05	1.241E-02	1.24E-02	0.11%
0.10	1.241E-02	1.24E-02	0.06%
0.15	1.239E-02	1.24E-02	0.05%
0.20	1.237E-02	1.24E-02	0.21%
0.25	1.235E-02	1.24E-02	0.44%
0.30	1.231E-02	1.24E-02	0.74%
0.35	1.226E-02	1.24E-02	1.16%
0.40	1.218E-02	1.24E-02	1.76%
0.45	1.206E-02	1.24E-02	2.76%
0.49	1.183E-02	1.24E-02	4.63%

Table 32: ***Rotation angle*** results for 27NodeBrick cantilever beams with element side length 0.25 m

Poisson's ratio	27NodeBrick angle (unit:°)	Theory angle (unit:°)	Error
0.00	1.242E-02	1.24E-02	0.17%
0.05	1.242E-02	1.24E-02	0.15%
0.10	1.241E-02	1.24E-02	0.09%
0.15	1.240E-02	1.24E-02	0.02%
0.20	1.238E-02	1.24E-02	0.17%
0.25	1.235E-02	1.24E-02	0.38%
0.30	1.232E-02	1.24E-02	0.64%
0.35	1.228E-02	1.24E-02	0.98%
0.40	1.222E-02	1.24E-02	1.42%
0.45	1.214E-02	1.24E-02	2.06%
0.49	1.202E-02	1.24E-02	3.08%

The errors were plotted in Figure (64).



(a) Error scale 0% - 10%

(b) Error scale 0% - 100%

Figure 64: 27NodeBrick cantilever beam for different Poisson's ratio ***Rotation angle error*** versus Poisson's ratio

The ESSI model fei files for the table above are here

2.3 Test of irregular shaped 27NodeBrick cantilever beams

Cantilever model was used as an example. Three different shapes were tested.

In the first test, the upper two nodes of each element were moved one half element size along the $y-axis$, while the lower two nodes were kept at the same location. The element shape was shown in Figure (65).

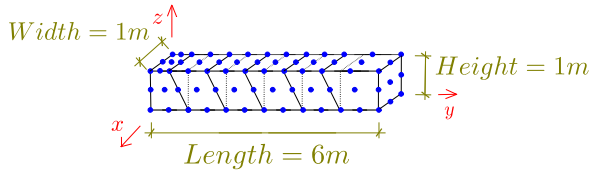


Figure 65: 27NodeBrick cantilever beams for irregular ***Shape 1***

In the second test, the upper two nodes of each element were moved 90% element size along the $y-axis$, while the lower two nodes were moved 90% element size in the other direction along the $y-axis$. The element shape was shown in Figure (66).

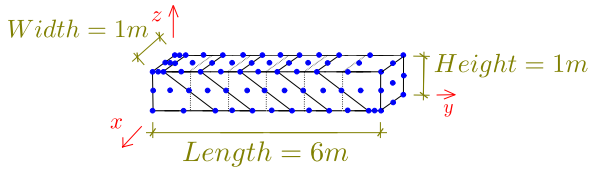


Figure 66: 27NodeBrick cantilever beams for irregular ***Shape 2***

In the third test, the upper two nodes of each element were moved one half element size with different directions along the $y-axis$, while the lower two nodes were kept at the same location. The element shape was shown in Figure (67).

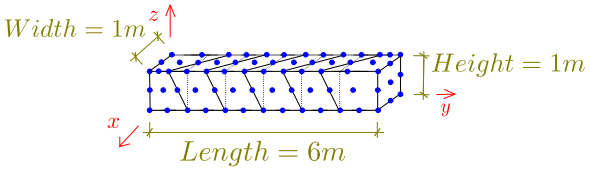
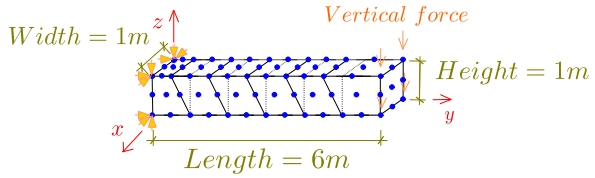
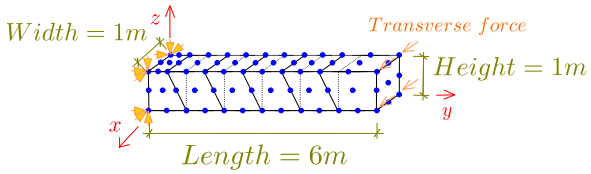


Figure 67: 27NodeBrick cantilever beams for irregular ***Shape 3***

The boundary conditions were shown in Figure (68), (69) and (70) .

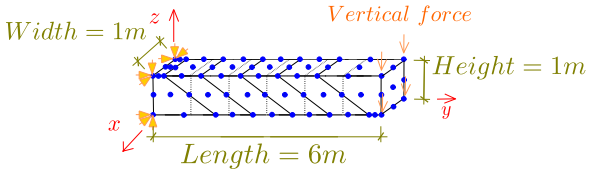


(a) Veritical force

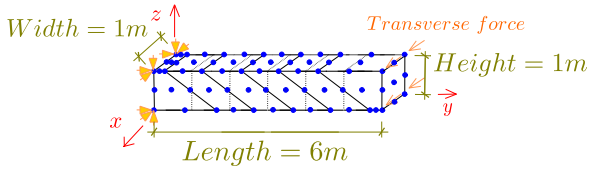


(b) Horizontal force

Figure 68: 27NodeBrick cantilever beam boundary conditions for irregular ***Shape 1***

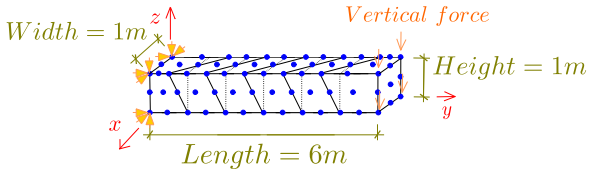


(a) Veritical force

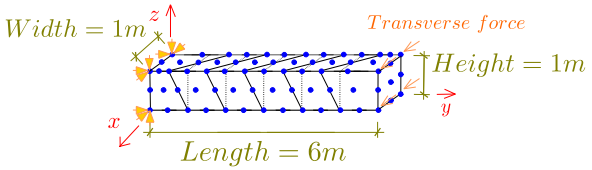


(b) Horizontal force

Figure 69: 27NodeBrick cantilever beam boundary conditions for irregular ***Shape 2***



(a) Veritical force



(b) Horizontal force

Figure 70: 27NodeBrick cantilever beam boundary conditions for irregular ***Shape 3***

The ESSI results were listed in Table (33).

Table 33: Results for 27NodeBrick cantilever beams of irregular shapes

Displacements for irregular shaped element					
Element Type	Force direction	Normal shape	Shape 1	Shape 2	Shape 3
27NodeBrick	Vertical (z)	8.755E-04 m	8.819E-04 m	8.709E-04 m	8.837E-04 m
27NodeBrick	Transverse (y)	8.755E-04 m	8.831E-04 m	8.462E-04 m	8.824E-04 m
Theoretical	-	8.760E-04 m	8.760E-04 m	8.760E-04 m	8.760E-04 m

The errors were listed in Table (34) and (35).

Table 34: Errors for irregular shaped 27NodeBrick compared to theoretical solution

Errors for irregular shaped element, compared to theoretical solutions					
Element Type	Force direction	Normal shape	Shape 1	Shape 2	Shape 3
27NodeBrick	Vertical (z)	0.06%	0.67%	0.58%	0.88%
27NodeBrick	Transverse (y)	0.06%	0.81%	3.40%	0.73%

Table 35: Errors for irregular shaped 27NodeBrick compared to normal shape

Errors for irregular shaped element, compared to normal shape					
Element Type	Force direction	Normal shape	Shape 1	Shape 2	Shape 3
27NodeBrick	Vertical (z)	0.00%	0.74%	0.52%	0.94%
27NodeBrick	Transverse (y)	0.00%	0.87%	3.34%	0.79%

The ESSI model fei files for the table above are here

Then, the beam was divided into small elements.

Problem description: Length=6m, Width=1m, Height=1m, Force=100N, E=1E8Pa, $\nu = 0.0$. The force direction was shown in Figure (71).

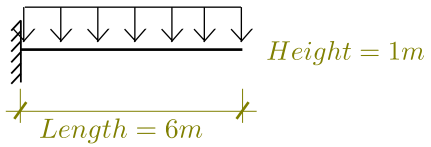


Figure 71: Problem description for cantilever beams under uniform pressure

Theoretical displacement (bending and shear deformation):

$$\begin{aligned} d &= \frac{qL^4}{8EI} + \frac{q\frac{L^2}{2}}{GA} \\ &= \frac{400N/m \times 12^4 m^4}{8 \times 10^8 N/m^2 \times \frac{2^4}{12} m^4} + \frac{400N/m \times \frac{12^2}{2} m^2}{\frac{10^8}{2} N/m^2 \times 2m \times 2m} \\ &= 7.776 \times 10^{-3} m + 1.44 \times 10^{-4} m \\ &= 7.920 \times 10^{-3} m \end{aligned} \tag{20}$$

The ESSI displacement results were listed in Table (36).

Table 36: Results for 27NodeBrick cantilever beams of irregular shapes with more elements

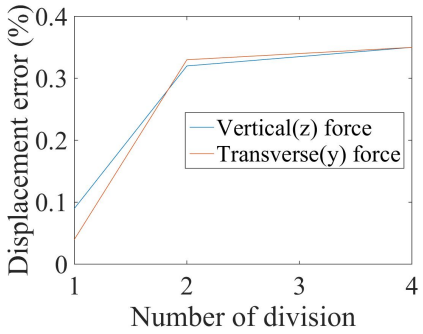
Element Type	Shape	Force direction	Number of division		
			1	2	4
27NodeBrick	shape1	Vertical (z)	7.913E-03 m	7.946E-03 m	7.948E-03 m
27NodeBrick	shape1	Transverse (y)	7.923E-03 m	7.946E-03 m	7.948E-03 m
27NodeBrick	shape2	Vertical (z)	7.741E-03 m	7.930E-03 m	7.947E-03 m
27NodeBrick	shape2	Transverse (y)	7.371E-03 m	7.894E-03 m	7.944E-03 m
27NodeBrick	shape3	Vertical (z)	1.982E-03 m	7.946E-03 m	7.948E-03 m
27NodeBrick	shape3	Transverse (y)	1.979E-03 m	7.947E-03 m	7.948E-03 m
Theoretical solution			7.920E-03 m	7.920E-03 m	7.920E-03 m

The error were listed in Table (37).

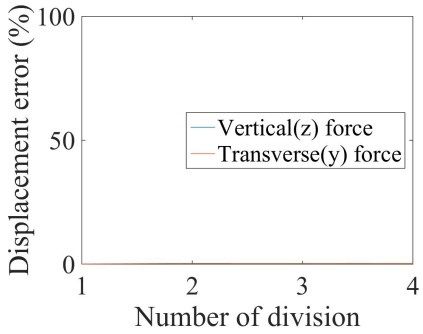
Table 37: Errors for 27NodeBrick cantilever beams of irregular shapes with more elements

Element Type	Shape	Force direction	Number of division		
			1	2	4
27NodeBrick	shape1	Vertical (z)	0.09%	0.32%	0.35%
27NodeBrick	shape1	Transverse (y)	0.04%	0.33%	0.35%
27NodeBrick	shape2	Vertical (z)	2.25%	0.13%	0.34%
27NodeBrick	shape2	Transverse (y)	6.93%	0.33%	0.31%
27NodeBrick	shape3	Vertical (z)	74.97%	0.32%	0.35%
27NodeBrick	shape3	Transverse (y)	75.02%	0.34%	0.35%

The errors were shown in Figure (72), (73) and (74).

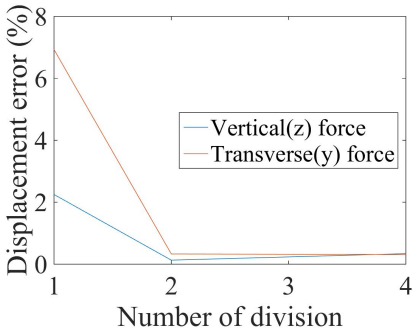


(a) Error scale 0% - 0.4%

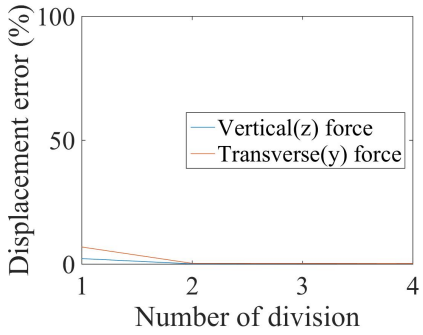


(b) Error scale 0% - 100%

Figure 72: 27NodeBrick cantilever beam for irregular ***Shape 1***
Displacement error versus Number of division

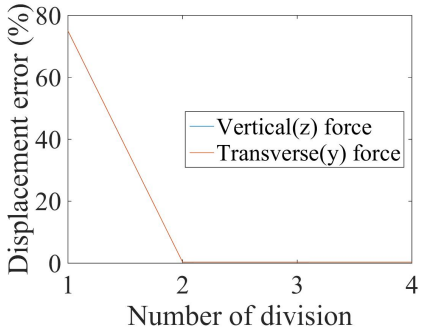


(a) Error scale 0% - 8%

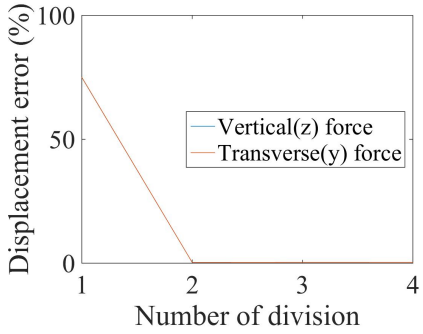


(b) Error scale 0% - 100%

Figure 73: 27NodeBrick cantilever beam for irregular ***Shape 2***
Displacement error versus Number of division



(a) Error scale 0% - 80%



(b) Error scale 0% - 100%

Figure 74: 27NodeBrick cantilever beam for irregular ***Shape 3***
Displacement error versus Number of division

The ESSI model fei files for the table above are here

2.4 Verification of 27NodeBrick edge clamped beams

Problem description: Length=6m, Width=1m, Height=1m, Force=100N, E=1E8Pa, $\nu = 0.0$. The force direction was shown in Figure (75).

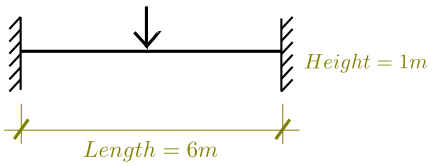


Figure 75: Problem description for clamped beams

The elment types and element sizes were same to the cantilever model. Only the boundary conditions and external force locations were changed.

The 27NodeBrick elements were shown in Figure (76).

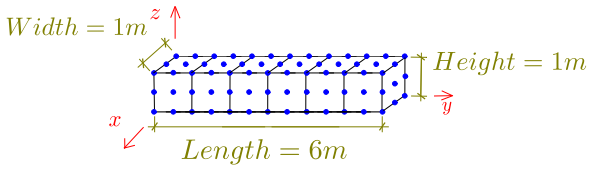


Figure 76: 27NodeBrick elements for clamped beams

Theoretical displacement (bending and shear deformation):

$$\begin{aligned} d &= \frac{FL^3}{192EI} + \frac{\frac{F}{2}\frac{L}{2}}{GA} \\ &= \frac{100N \times 6m^3}{192 \times 10^8 N/m^2 \times \frac{1}{12}m^4} + \frac{100N \times 6m}{4 \times 5 \times 10^7 N/m^2 \times 1m^2} \\ &= 1.35 \times 10^{-5}m + 0.3 \times 10^{-5}m \\ &= 1.65 \times 10^{-5}\ m \end{aligned} \tag{21}$$

The theoretical solution for $L = 6\ m$ was calculated above, while the solutions for other length were calculated by the similar process.

In the figures above, only the model with geometry $6m \times 1m \times 1m$ was drawed. In the ESSI models, the geometry $10m \times 1m \times 1m$ and the geometry $20m \times 1m \times 1m$ were also calculated. In three different geometry models, all the element sizes were $1m \times 1m \times 1m$. Therefore, the number of elements used in each model were 6, 10 *and* 20 respectively.

The results were listed in Table (38).

Table 38: Results for 27NodeBrick clamped beams of different geometry

Geometry	27NodeBrick	Theory(bending)	Theory(shear)	Theory(all)	Error
1:6	1.636E-05 m	1.35E-05 m	3.00E-06 m	1.65E-05 m	0.83%
1:10	6.727E-05 m	6.25E-05 m	5.00E-06 m	6.75E-05 m	0.34%
1:20	5.095E-04 m	5.00E-04 m	1.00E-05 m	5.10E-04 m	0.09%

The ESSI model fei files for the table above are here

In this section, the beam was cut into smaller elements with element side length 0.5m and 0.25m respectively. And the element side length of the original models is 1.0m. The numerical models were shown in Figure (77), (78) and (79).

Number of division 1:

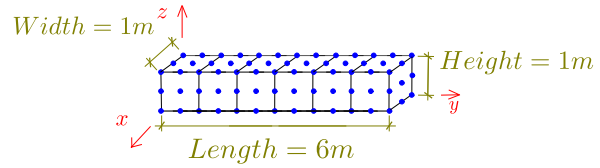


Figure 77: 27NodeBrick clamped beams with element side length 1.0m

Number of division 2:

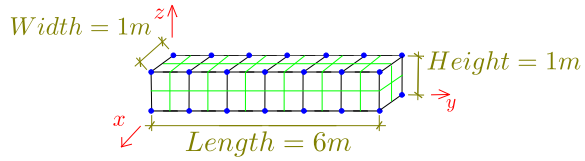


Figure 78: 27NodeBrick clamped beams with element side length 0.5m

Number of division 4:

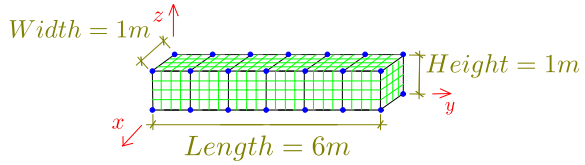


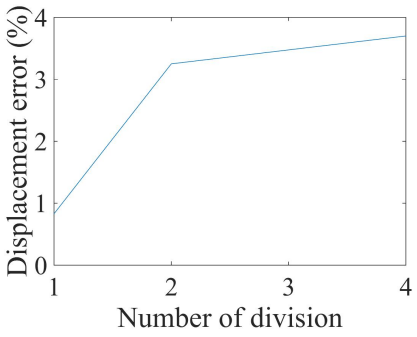
Figure 79: 27NodeBrick clamped beams with element side length 0.25m

The ESSI results were listed in Table (39). The theoretical solution is 1.65E-5 m.

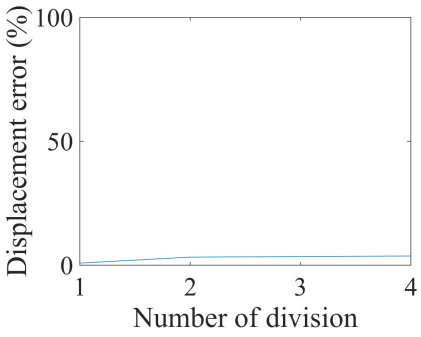
Table 39: Results for 27NodeBrick clamped beams with more elements

Element Type	Element side length		
	1 m	0.5 m	0.25 m
27NodeBrick	1.64E-05 m	1.70E-05 m	1.71E-05 m
Error	0.83%	3.25%	3.70%

The errors were plotted in Figure (80).



(a) Error scale 0% - 4%



(b) Error scale 0% - 100%

Figure 80: 27NodeBrick clamped beam for different element number
Displacement error versus Number of division

The ESSI model fei files for the table above are here

2.5 Verification of 27NodeBrick stress in cantilever beams

Problem description: Length=6m, Width=1m, Height=1m, Force=100N, E=1E8Pa, $\nu = 0.0$. The force direction was shown in Figure (81).

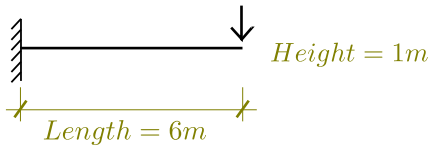


Figure 81: Problem description for cantilever beams of stress verification

The theoretical solution for the stress was calculated below.
The 27NodeBrick elements were shown in Figure (82).

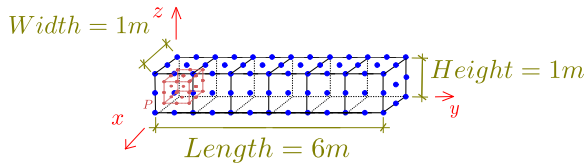


Figure 82: 27NodeBrick for cantilever beams of stress verification

The bending moment at the Gassian Point is

$$M = F(L - P_y) = 100N \times (6 - 0.1127)m = 588.73N \cdot m \tag{22}$$

The bending modulus is

$$I = \frac{bh^3}{12} = \frac{1}{12}m^4 \tag{23}$$

Therefore, the theoretical stress is

$$\sigma = \frac{M \cdot z}{I} = \frac{588.73N \cdot m \times (0.5 - 0.1127)m}{\frac{1}{12}m^4} = 2736Pa \tag{24}$$

To get a better result, the same geometry beam was also cut into small elements. When more elements were used, the theoretical stress was calculated again with the new coordinates. The calculation process is similar to the process above.
The numerical models were shown in Figure (83), (84) and (85).
Number of division 1:

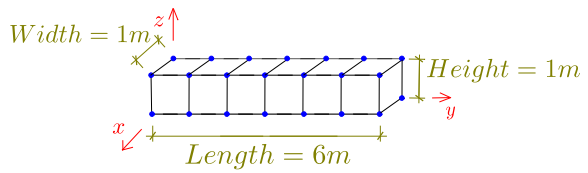


Figure 83: 27NodeBrick stress with element side length 1.0m

Number of division 2:

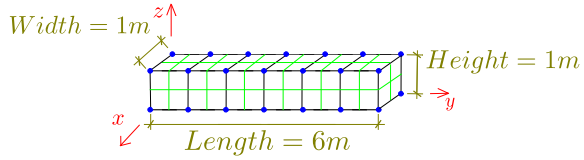


Figure 84: 27NodeBrick stress with element side length 0.5m

Number of division 4:

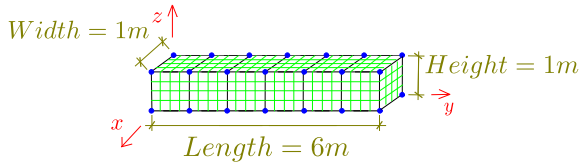
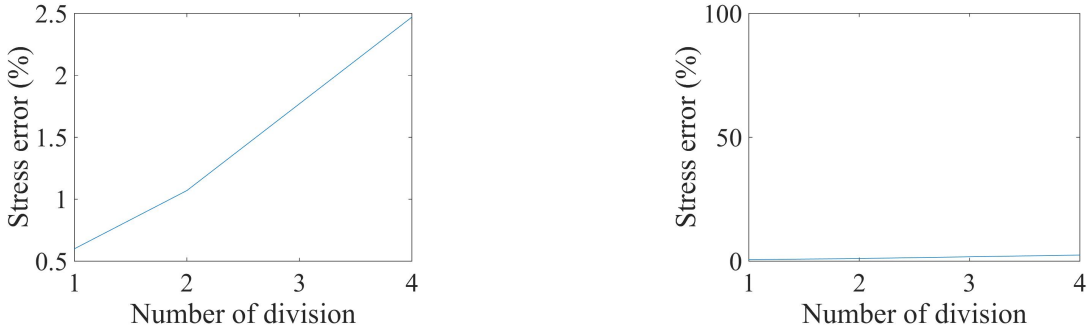


Figure 85: 27NodeBrick stress with element side length 0.25m

All the stress results were listed in Table (40).

Table 40: Results for 27NodeBrick stress with more elements

Element Type	Element side length		
	1 m	0.5 m	0.25 m
27NodeBrick	2719.81 Pa	3198.19 Pa	3464.76 Pa
Theoretical	2736.17 Pa	3164.27 Pa	3381.18 Pa
Error	0.60%	1.07%	2.47%



(a) Error scale 0% - 2.5%

(b) Error scale 0% - 100%

Figure 86: 27NodeBrick cantilever beams for stress verification
Stress error versus Number of division

The ESSI model fei files for the table above are here

2.6 Verification of 27NodeBrick square plate with four edges clamped

Problem description: Length=20m, Width=20m, Height=1m, Force=100N, E=1E8Pa, $\nu = 0.3$.

The four edges are clamped.

The load is the uniform normal pressure on the whole plate.

The plate flexural rigidity is

$$D = \frac{Eh^3}{12(1-\nu^2)} = \frac{10^8 N/m^2 \times 1^3 m^3}{12 \times (1-0.3^2)} = 9.1575 \times 10^6 \ N \cdot m \tag{25}$$

The theoretical solution is

$$d = \alpha_c \frac{qa^4}{D} = 0.00406 \times \frac{100 N/m^2 \times 20^4 m^4}{9.1575 \times 10^6 \ N \cdot m} = 2.2015 \times 10^{-3} m \tag{26}$$

where α_c is a coefficient, which depends on the ratio of plate length to width. In this problem, the coefficient[5] α_c is 0.00406.

The 27NodeBrick were shown in Figure (87) - (92).

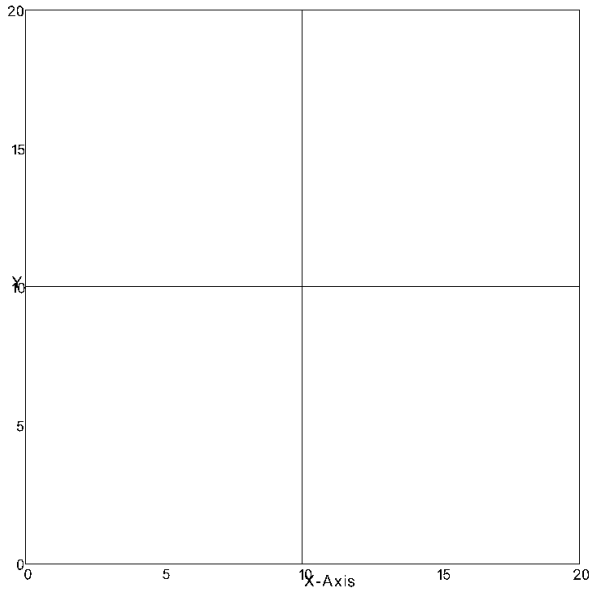


Figure 87: 27NodeBrick edge clamped square plate with element side length 10m

[5]Stephen Timoshenko, Theory of plates and shells (2nd edition). MrGRAW-Hill Inc, page120, 1959.

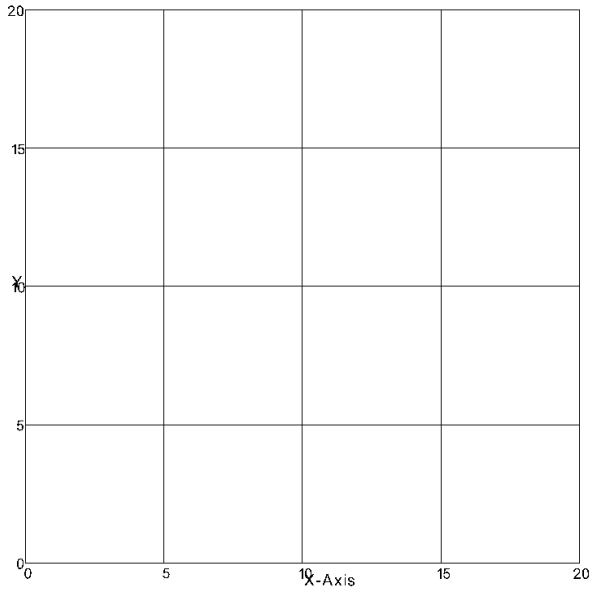


Figure 88: 27NodeBrick edge clamped square plate with element side length 5m

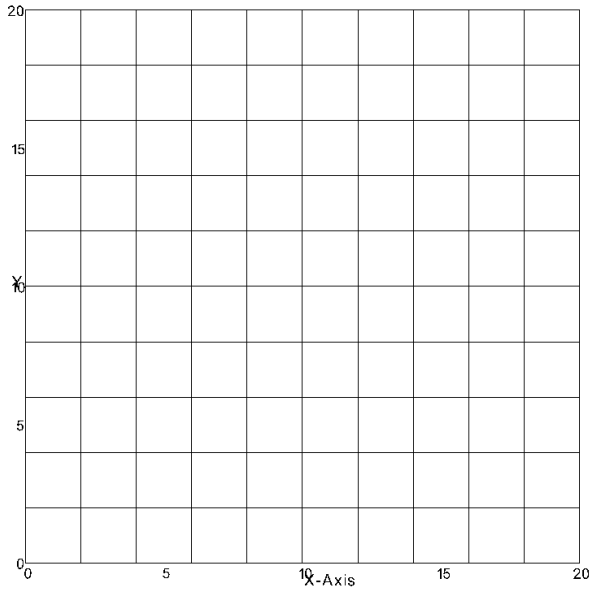


Figure 89: 27NodeBrick edge clamped square plate with element side length 2m

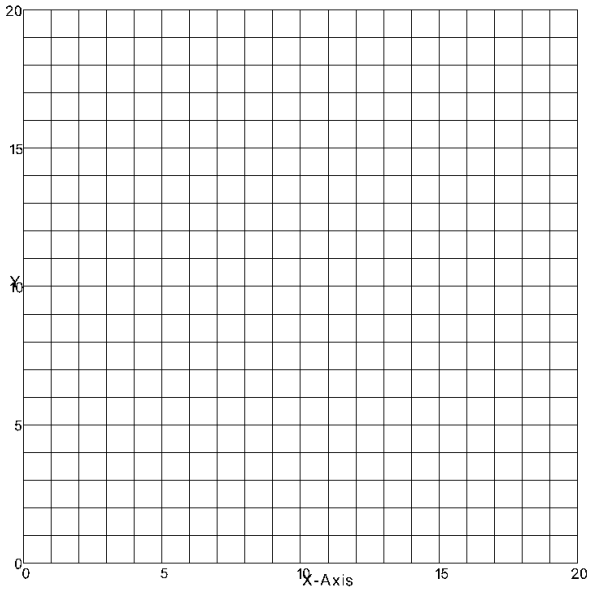


Figure 90: 27NodeBrick edge clamped square plate with element side length 1m

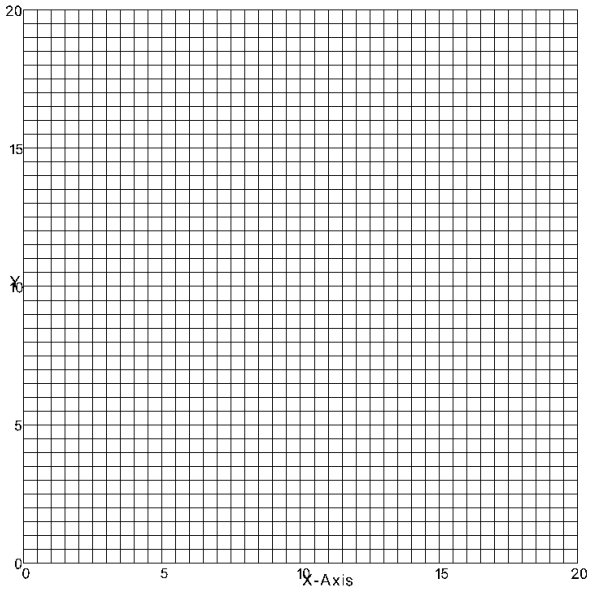


Figure 91: 27NodeBrick edge clamped square plate with element side length 0.5m

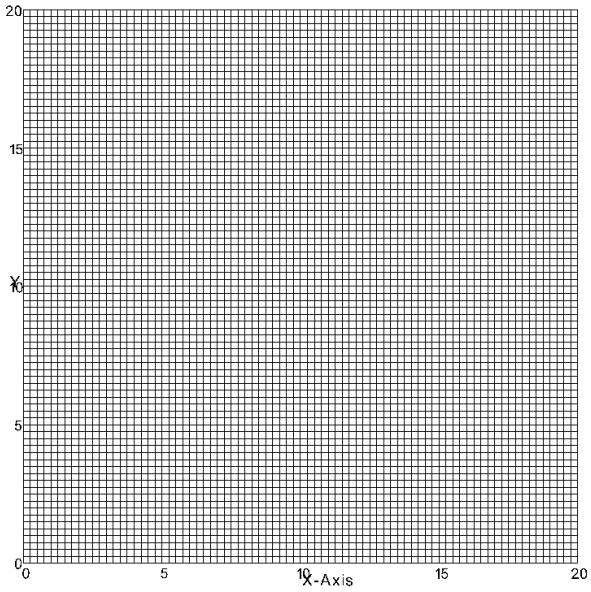


Figure 92: 27NodeBrick edge clamped square plate with element side length 0.25m

The results were listed in Table (41).

Table 41: Results for 27NodeBrick square plate with four edges clamped

Element type	27NodeBrick	27NodeBrick	27NodeBrick	Theoretical displacement
Number of layers	1layer	2layers	4layers	
Element side length	Height:1.00m	Height:0.50m	Height:0.25m	
10m	4.82E-004 m	4.82E-004 m	4.82E-004 m	2.20E-03 m
5m	1.97E-003 m	1.98E-003 m	1.98E-003 m	2.20E-03 m
2m	2.25E-003 m	2.26E-003 m	2.26E-003 m	2.20E-03 m
1m	2.28E-003 m	2.29E-003 m	2.29E-003 m	2.20E-03 m
0.5m	2.29E-003 m	2.30E-003 m	2.30E-003 m	2.20E-03 m
0.25m	2.29E-003 m	2.30E-003 m	-[6]	2.20E-03 m

The errors were listed in Table (42).

[6]This model run out of memory on machine cml01 (memory: 23.5GB). This model has 233,289 nodes with 3 dofs, which may require 40GB memory.

Table 42: Errors for 27NodeBrick square plate with four edges clamped

Element type	27NodeBrick	27NodeBrick	27NodeBrick
Number of layers	1layer	2layers	4layers
Element side length	Height:1.00m	Height:0.50m	Height:0.25m
10m	78.11%	78.10%	78.10%
5m	10.67%	10.19%	10.16%
2m	2.23%	2.79%	2.83%
1m	3.56%	4.16%	4.22%
0.5m	3.96%	4.58%	4.65%
0.25m	4.08%	4.70%	-

The errors were plotted in Figure (93).

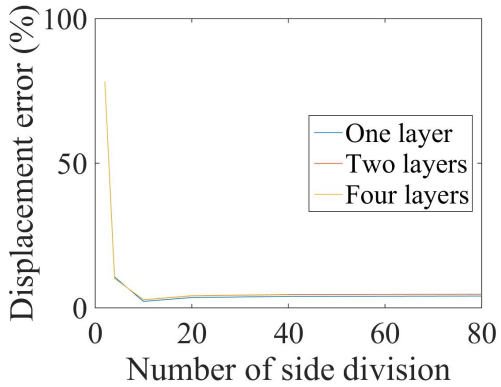


Figure 93: 27NodeBrick square plate with edge clamped
Displacement error versus Number of side division

The ESSI model fei files for the table above are here

2.7 Verification of 27NodeBrick square plate with four edges simply supported

Problem description: Length=20m, Width=20m, Height=1m, Force=100N, E=1E8Pa, $\nu = 0.3$.

The four edges are simply supported.

The load is the uniform normal pressure on the whole plate.

The plate flexural rigidity is

$$D = \frac{Eh^3}{12(1-\nu^2)} = \frac{10^8 N/m^2 \times 1^3 m^3}{12 \times (1-0.3^2)} = 9.1575 \times 10^6 \ N \cdot m \tag{27}$$

The theoretical solution is

$$d = \alpha_s \frac{qa^4}{D} = 0.00126 \times \frac{100 N/m^2 \times 20^4 m^4}{9.1575 \times 10^6 \ N \cdot m} = 7.0936 \times 10^{-3} m \tag{28}$$

where α_s is a coefficient, which depends on the ratio of plate length to width. In this problem, the coefficient[7] α_s is 0.00126.

The 27NodeBrick were shown in Figure (94) - (99).

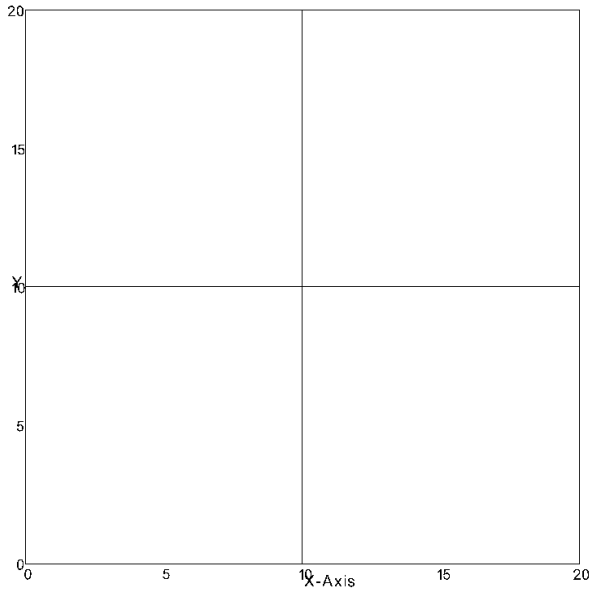


Figure 94: 27NodeBrick edge simply supported square plate with element side length 10m

[7]Stephen Timoshenko, Theory of plates and shells (2nd edition). MrGRAW-Hill Inc, page202, 1959.

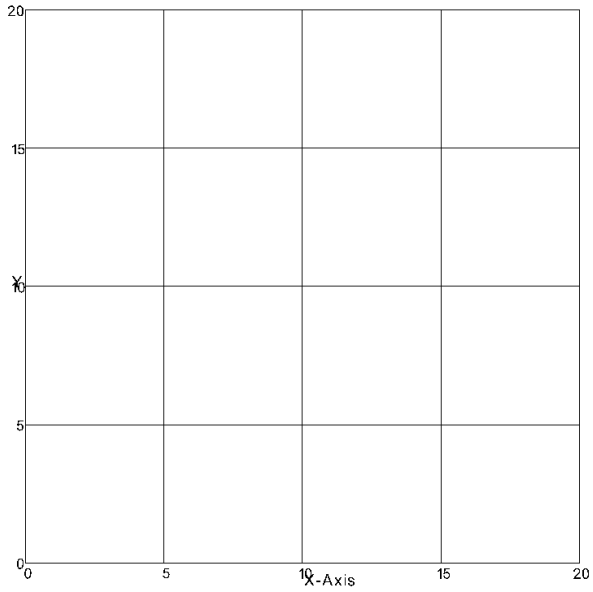


Figure 95: 27NodeBrick edge simply supported square plate with element side length 5m

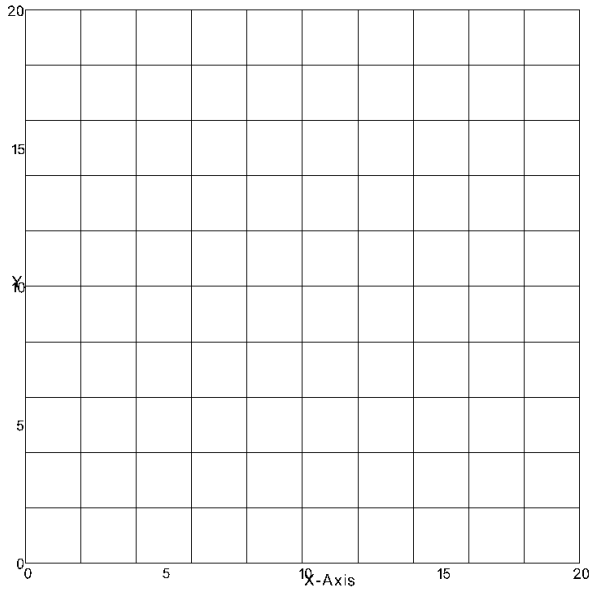


Figure 96: 27NodeBrick edge simply supported square plate with element side length 2m

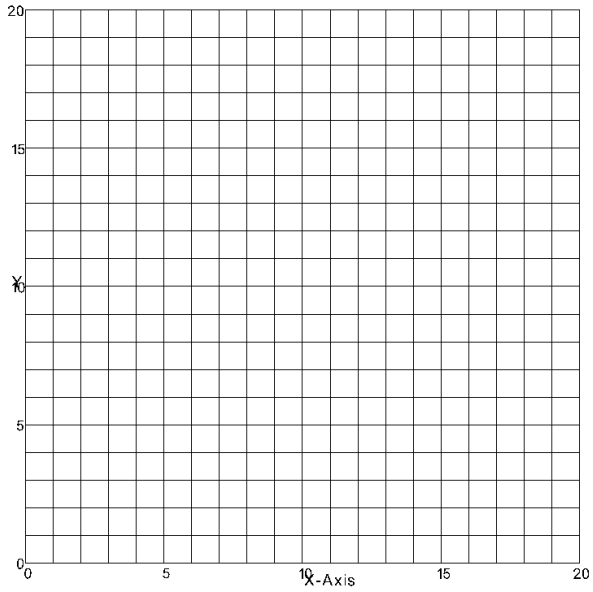


Figure 97: 27NodeBrick edge simply supported square plate with element side length 1m

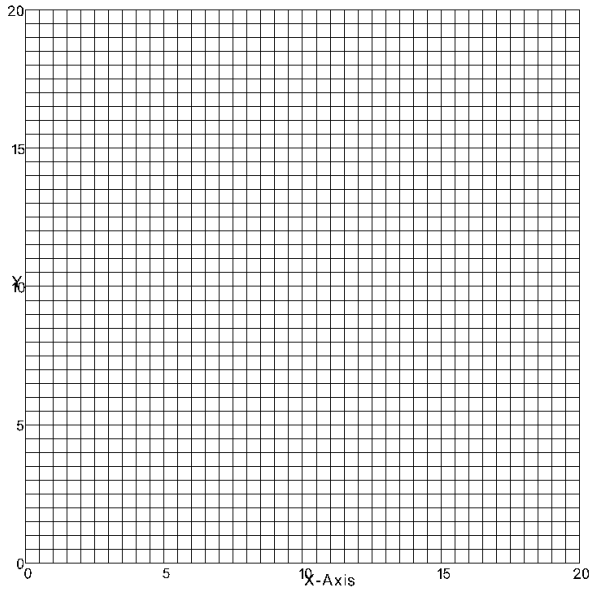


Figure 98: 27NodeBrick edge simply supported square plate with element side length 0.5m

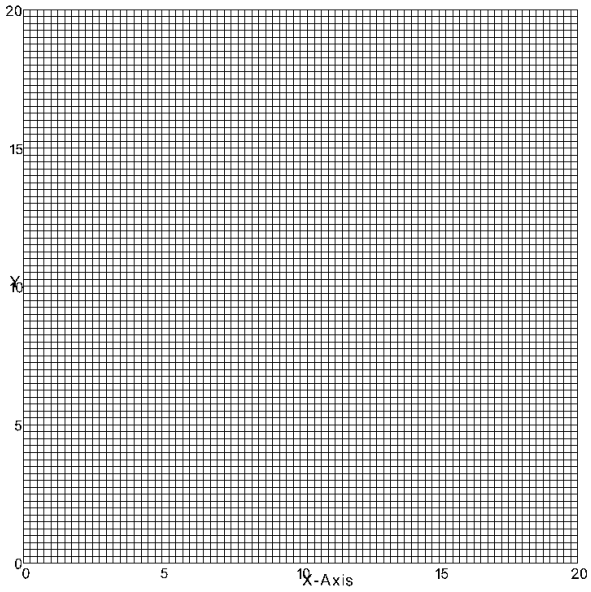


Figure 99: 27NodeBrick edge simply supported square plate with element side length 0.25m

The results were listed in Table (43).

Table 43: Results for 27NodeBrick square plate with four edges simply supported

Element type	27NodeBrick	27NodeBrick	Theoretical displacement
Number of layers	2layers	4layers	
Element side length	Height:0.50m	Height:0.25m	
10m	6.54E-003 m	6.54E-003 m	7.09E-03 m
5m	7.24E-003 m	7.24E-003 m	7.09E-03 m
2m	7.44E-003 m	7.44E-003 m	7.09E-03 m
1m	7.49E-003 m	7.49E-003 m	7.09E-03 m
0.5m	7.50E-003 m	7.50E-003 m	7.09E-03 m
0.25m	7.51E-003 m	-[8]	7.09E-03 m

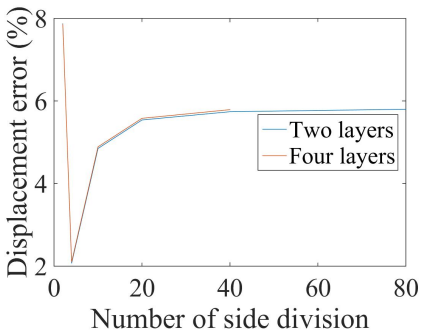
The errors were listed in Table (44).

[8]This model run out of memory on machine cml01 (memory: 23.5GB). This model has 233,289 nodes with 3 dofs, which may require 40GB memory.

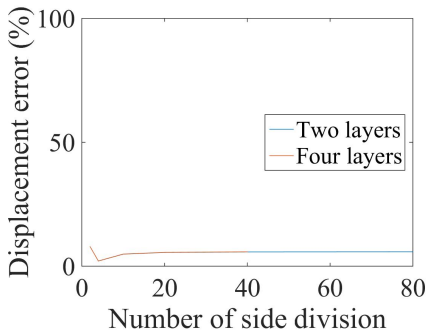
Table 44: Errors for 27NodeBrick square plate with four edges simply supported

Element type	27NodeBrick	27NodeBrick
Number of layers	2layers	4layers
Element side length	Height:0.50m	Height:0.25m
10m	7.87%	7.85%
5m	2.07%	2.10%
2m	4.85%	4.89%
1m	5.54%	5.58%
0.5m	5.74%	5.79%
0.25m	5.80%	-

The errors were plotted in Figure (100).



(a) Error scale 0% - 8%



(b) Error scale 0% - 100%

Figure 100: 27NodeBrick square plate with edge simply supported
Displacement error versus Number of side division

The ESSI model fei files for the table above are here

2.8 Verification of 27NodeBrick circular plate with all edges clamped

Problem description: Diameter=20m, Height=1m, Force=100N, E=1E8Pa, $\nu = 0.3$.

The four edges are clamped.

The load is the uniform normal pressure on the whole plate.

The plate flexural rigidity is

$$D = \frac{Eh^3}{12(1-\nu^2)} = \frac{10^8 N/m^2 \times 1^3 m^3}{12 \times (1-0.3^2)} = 9.1575 \times 10^6 \ N \cdot m \tag{29}$$

The theoretical solution[9] is

$$d = \frac{qa^4}{64D} = \frac{100 N/m^2 \times 10^4 m^4}{64 \times 9.1575 \times 10^6 \ N \cdot m} = 1.7106 \times 10^{-3} m \tag{30}$$

The 27NodeBrick were shown in Figure (101) - (106).

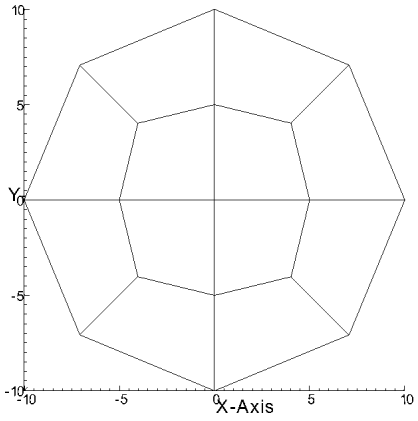


Figure 101: 27NodeBrick edge clamped circular plate with element side length 10m

[9] Stephen Timoshenko, Theory of plates and shells (2nd edition). MrGRAW-Hill Inc, page55, 1959.

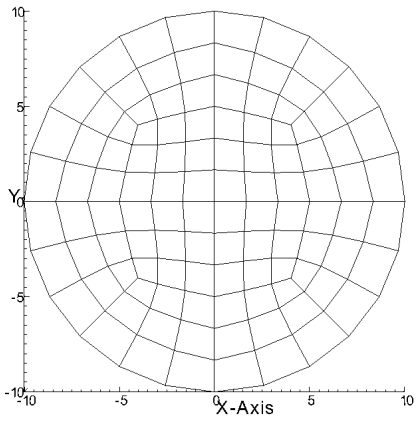


Figure 102: 27NodeBrick edge clamped circular plate with element side length 5m

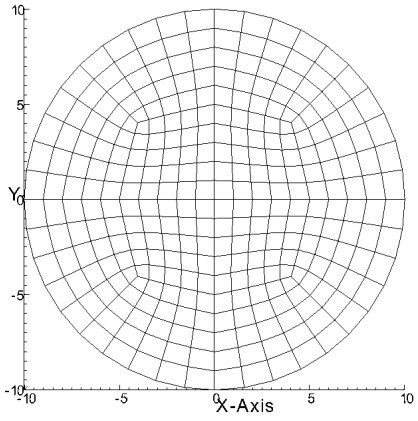


Figure 103: 27NodeBrick edge clamped circular plate with element side length 2m

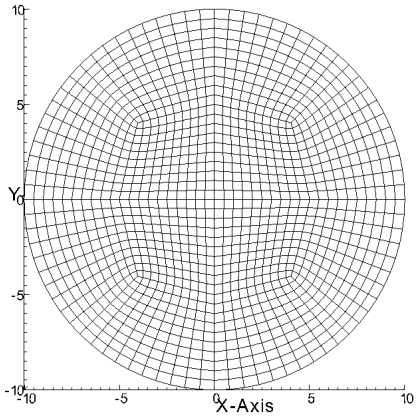


Figure 104: 27NodeBrick edge clamped circular plate with element side length 1m

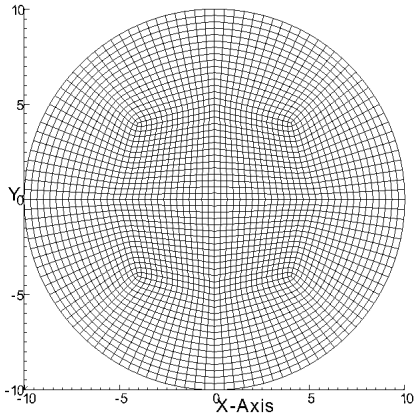


Figure 105: 27NodeBrick edge clamped circular plate with element side length 0.5m

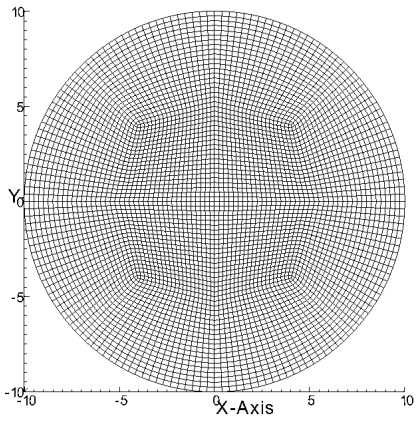


Figure 106: 27NodeBrick edge clamped circular plate with element side length 0.25m

The results were listed in Table (45).

Table 45: Results for 27NodeBrick circular plate with four edges clamped

Element type	27NodeBrick	27NodeBrick	27NodeBrick	Theoretical displacement
Number of layers	1layer	2layers	4layers	
Number of diameter divisions	Height:1.00m	Height:0.50m	Height:0.25m	
4	2.777E-03 m	2.788E-03 m	2.789E-03 m	1.706E-03 m
12	2.772E-03 m	2.786E-03 m	2.787E-03 m	1.706E-03 m
20	2.545E-03 m	2.556E-03 m	2.558E-03 m	1.706E-03 m
40	1.758E-03 m	1.768E-03 m	1.769E-03 m	1.706E-03 m
60	1.762E-03 m	1.772E-03 m	1.773E-03 m	1.706E-03 m
80	1.763E-03 m	1.773E-03 m	1.774E-03 m	1.706E-03 m

The errors were listed in Table (46).

Table 46: Errors for 27NodeBrick circular plate with four edges clamped

Element type	27NodeBrick	27NodeBrick	27NodeBrick
Number of layers	1layer	2layers	4layers
Number of diameter divisions	Height:1.00m	Height:0.50m	Height:0.25m
4	62.75%	63.42%	63.47%
12	62.46%	63.27%	63.34%
20	49.14%	49.82%	49.91%
40	3.03%	3.62%	3.68%
60	3.25%	3.83%	3.91%
80	3.32%	3.91%	3.99%

The errors were shown in Figure (107).

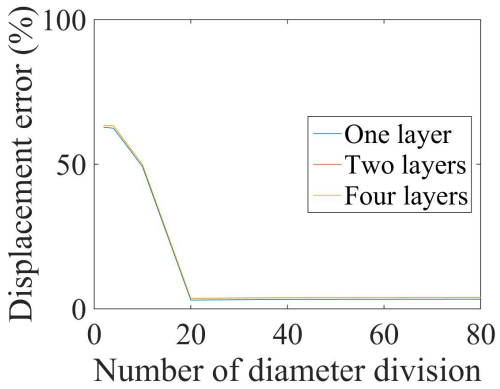


Figure 107: 27NodeBrick circular plate with edge clamped
Displacement error versus Number of side division

The ESSI model fei files for the table above are here

2.9 Verification of 27NodeBrick circular plate with all edges simply supported

Problem description: Diameter=20m, Height=1m, Force=100N, E=1E8Pa, $\nu = 0.3$.

The four edges are simply supported.

The load is the uniform normal pressure on the whole plate.

The plate flexural rigidity is

$$D = \frac{Eh^3}{12(1-\nu^2)} = \frac{10^8 N/m^2 \times 1^3 m^3}{12 \times (1-0.3^2)} = 9.1575 \times 10^6 \ N \cdot m \tag{31}$$

The theoretical solution[10] is

$$d = \frac{(5+\nu)qa^4}{64(1+\nu)D} = \frac{(5+0.3) \times 100 N/m^2 \times 10^4 m^4}{64 \times (1+0.3) \times 9.1575 \times 10^6 \ N \cdot m} = 6.956 \times 10^{-3} m \tag{32}$$

The 27NodeBrick were shown in Figure (108) - (113).

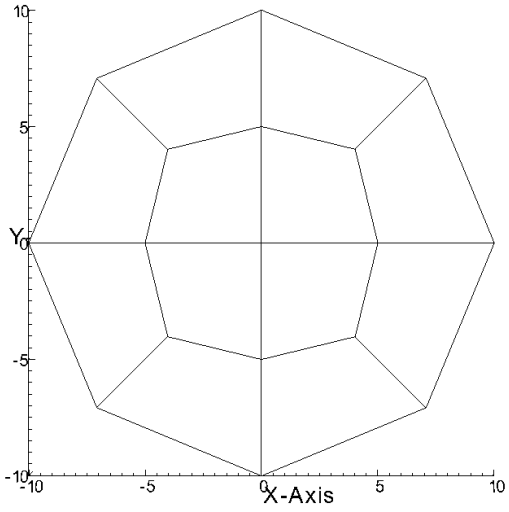


Figure 108: 27NodeBrick edge simply supported circular plate with element side length 10m

[10] Stephen Timoshenko, Theory of plates and shells (2nd edition). MrGRAW-Hill Inc, page55, 1959.

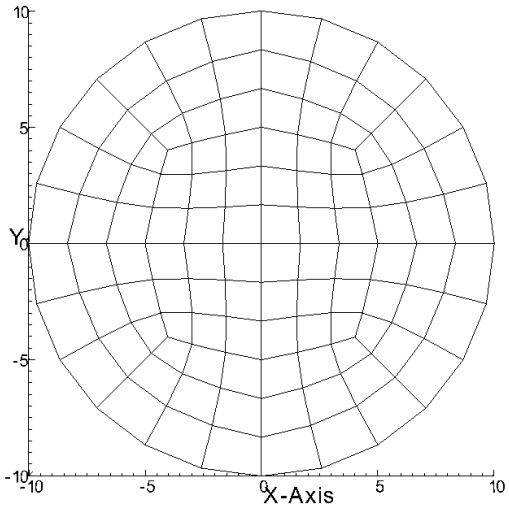


Figure 109: 27NodeBrick edge simply supported circular plate with element side length 5m

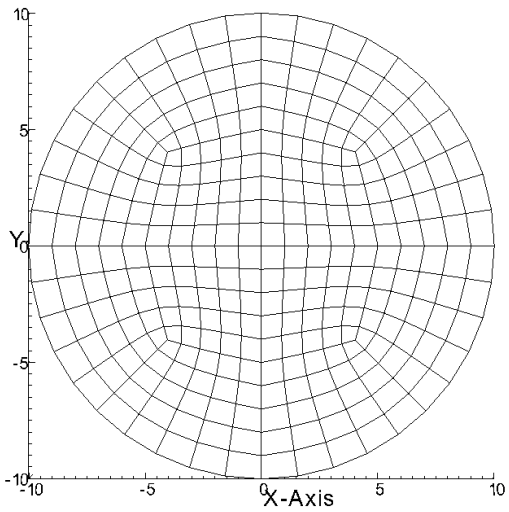


Figure 110: 27NodeBrick edge simply supported circular plate with element side length 2m

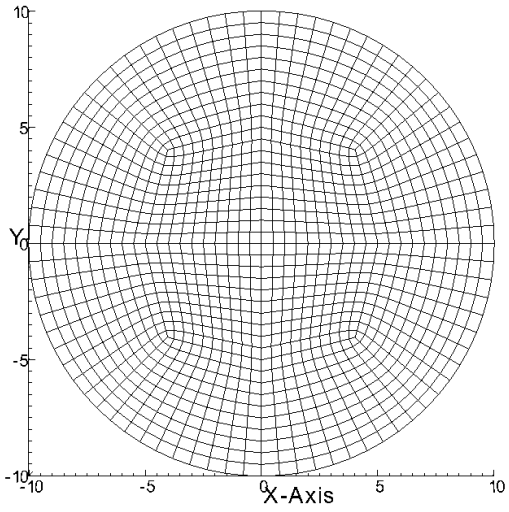


Figure 111: 27NodeBrick edge simply supported circular plate with element side length 1m

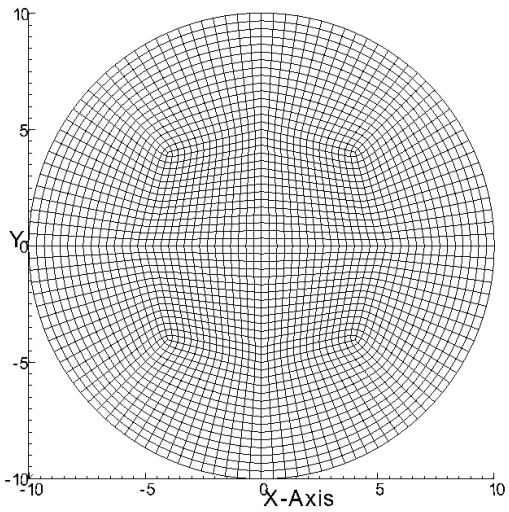


Figure 112: 27NodeBrick edge simply supported circular plate with element side length 0.5m

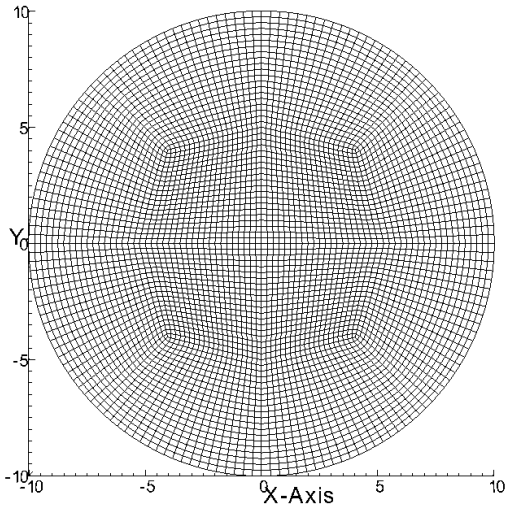


Figure 113: 27NodeBrick edge simply supported circular plate with element side length 0.25m

The results were listed in Table (47).

Table 47: Results for 27NodeBrick cicular plate with four edges simply supported

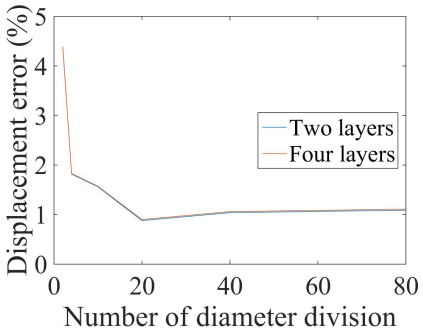
Element type	27NodeBrick	27NodeBrick	Theoretical displacement
Number of layers	2layers	4layers	
Number of diameter divisions	Height:0.50m	Height:0.25m	
4	7.259E-03 m	7.261E-03 m	6.956E-03 m
12	7.083E-03 m	7.084E-03 m	6.956E-03 m
20	7.064E-03 m	7.065E-03 m	6.956E-03 m
40	7.018E-03 m	7.019E-03 m	6.956E-03 m
60	7.029E-03 m	7.030E-03 m	6.956E-03 m
80	7.032E-03 m	7.034E-03 m	6.956E-03 m

The errors were listed in Table (48).

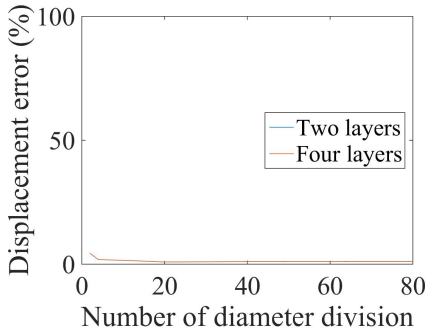
Table 48: Errors for 27NodeBrick cicular plate with four edges simply supported

Element type	27NodeBrick	27NodeBrick
Number of layers	2layers	4layers
Number of diameter divisions	Height:0.50m	Height:0.25m
4	4.36%	4.38%
12	1.82%	1.83%
20	1.56%	1.57%
40	0.88%	0.90%
60	1.04%	1.06%
80	1.09%	1.11%

The errors were plotted in Figure (114).



(a) Error scale 0% - 5%



(b) Error scale 0% - 100%

Figure 114: 27NodeBrick circular plate with edge simply supported
Displacement error versus Number of side division

The ESSI model fei files for the table above are here

Verification for 4NodeANDES

3 Verification of 4NodeANDES elements

3.1 Verification of 4NodeANDES cantilever beams

Problem description: Length=6m, Width=1m, Height=1m, Force=100N, E=1E8Pa, $\nu = 0.0$. The force direction was shown in Figure (115).

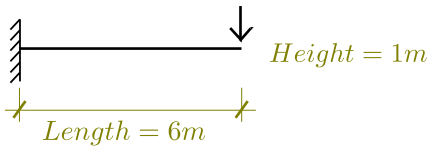


Figure 115: Problem description for cantilever beams

Theoretical displacement (bending and shear deformation):

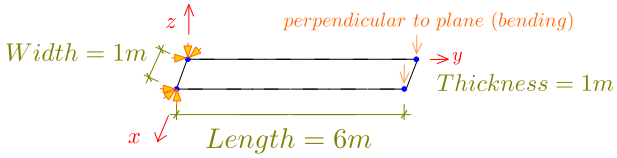
$$\begin{aligned} d &= \frac{FL^3}{3EI} + \frac{FL}{GA} \\ &= \frac{100N \times 6^3 m^3}{3 \times 10^8 N/m^2 \times \frac{1}{12} m^4} + \frac{100N \times 6m}{5 \times 10^7 N/m^2 \times 1m^2} \\ &= 8.64 \times 10^{-4} m + 0.12 \times 10^{-4} m \\ &= 8.76 \times 10^{-4}\ m \end{aligned} \tag{33}$$

4NodeANDES element model:

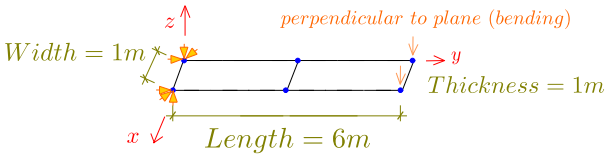
- ***Force direction: perpendicular to plane (bending)***

When the force direction is perpendicular to the plane, only the bending deformation is calculated in 4NodeANDES elements.

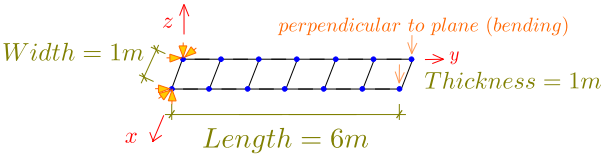
The 4NodeANDES elements were shown in Figure (116).



(a) One 4NodeANDES element



(b) Two 4NodeANDES elements



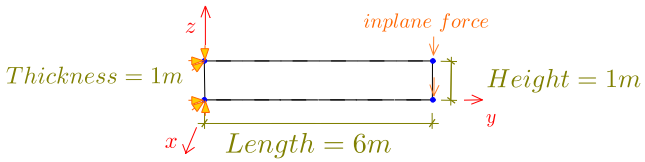
(c) Six 4NodeANDES elements

Figure 116: 4NodeANDES elements for cantilever beams under force perpendicular to plane

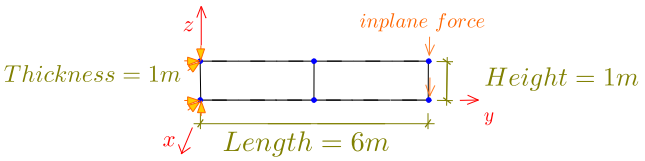
- ***Force direction: inplane force***

When the force direction is inplane, both the bending and shear deformation are calculated in 4NodeANDES elements.

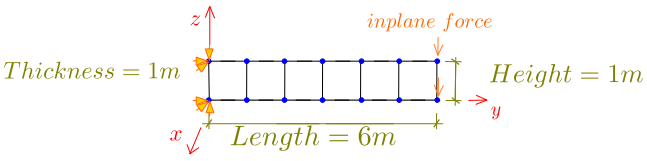
The 4NodeANDES elements under inplane force were shown in Figure (117).



(a) One 4NodeANDES element



(b) Two 4NodeANDES elements



(c) Six 4NodeANDES elements

Figure 117: 4NodeANDES elements for cantilever beams under inplane force

The ESSI results for the force ***perpendicular to plane (bending)*** were listed in Table (49). The theoretical solution is 8.760E-04 m.

Table 49: Results for 4NodeANDES cantilever beams under the force perpendicular to plane (bending)

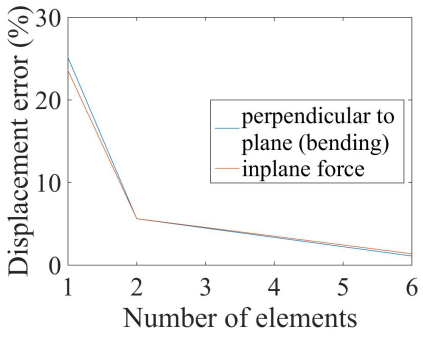
Element number	1	2	6
4NodeANDES	6.56E-04 m	8.27E-04 m	8.86E-04 m
Error	25.14%	5.62%	1.11%

The ESSI results for the ***inplane force*** were listed in Table (50).
The theoretical solution is 8.760E-04 m.

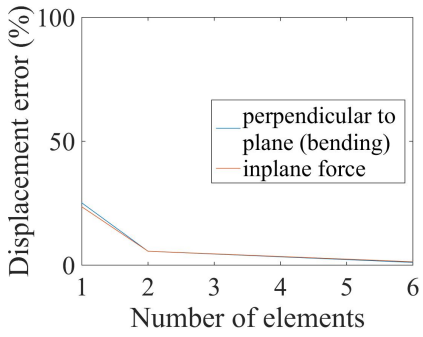
Table 50: Results for 4NodeANDES cantilever beams under the inplane force

Element number	1	2	6
4NodeANDES	6.70E-04 m	8.27E-04 m	8.64E-04 m
Error	23.56%	5.63%	1.38%

The errors were plotted in Figure (118).



(a) Error scale 0% - 30%



(b) Error scale 0% - 100%

Figure 118: 4NodeANDES cantilever beam for different element number Displacement error versus Number of elements

The ESSI model fei files for the table above are here

- ***Beam: different geometry***

In the figures above, only the model with geometry $6m \times 1m \times 1m$ was drawed. In the ESSI models, the geometry $10m \times 1m \times 1m$ and the geometry $20m \times 1m \times 1m$ were also calculated. In three different geometry models, all the element sizes were $1m \times 1m \times 1m$. Therefore, the number of elements used in each model were 6, 10 *and* 20 respectively.

The ESSI results for the force ***perpendicular to plane (bending)*** were listed in Table (51).

Table 51: Results for 4NodeANDES cantilever for the force perpendicular to plane (bending)

Geometry	4NodeANDES	Theoretical(bending)	Theoretical(shear)	Theoretical(all)	Error
1:6	8.64E-04 m	8.64E-04 m	1.20E-05 m	8.76E-04 m	1.38%
1:10	4.00E-03 m	4.00E-03 m	2.00E-05 m	4.02E-03 m	0.50%
1:20	3.20E-02 m	3.20E-02 m	4.00E-05 m	3.20E-02 m	0.13%

The ESSI results for the ***inplane force*** were listed in Table (52).

Table 52: Results for 4NodeANDES cantilever beams for the inplane force

Geometry	4NodeANDES	Theoretical(bending)	Theoretical(shear)	Theoretical(all)	Error
1:6	8.86E-04 m	8.64E-04 m	1.20E-05 m	8.76E-04 m	1.11%
1:10	4.04E-03 m	4.00E-03 m	2.00E-05 m	4.02E-03 m	0.42%
1:20	3.21E-02 m	3.20E-02 m	4.00E-05 m	3.20E-02 m	0.04%

The ESSI model fei files for the table above are here

3.2 Verification of 4NodeANDES cantilever beam for different Poisson's ratio

Problem description: Length=6m, Width=1m, Height=1m, Force=100N, E=1E8Pa, $\nu = 0.0 - 0.49$. The force direction was shown in Figure (119).

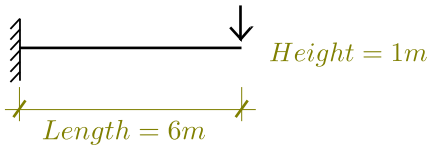


Figure 119: Problem description for cantilever beams of different Poisson's ratios

The theoretical solution for $\nu = 0.0$ was calculated below, while the solution for other Poisson's ratio were calculated by the similar process.

Theoretical displacement (bending and shear deformation):

$$\begin{aligned} d &= \frac{FL^3}{3EI} + \frac{FL}{GA} \\ &= \frac{100N \times 6^3 m^3}{3 \times 10^8 N/m^2 \times \frac{1}{12} m^4} + \frac{100N \times 6m}{5 \times 10^7 N/m^2 \times 1m^2} \\ &= 8.64 \times 10^{-4} m + 0.12 \times 10^{-4} m \\ &= 8.76 \times 10^{-4}\ m \end{aligned} \tag{34}$$

The rotation angle at the end:

$$\theta = \frac{FL^2}{2EI} = \frac{100N \times 6^2 m^2}{2 \times 10^8 N/m^2 \times \frac{1}{12} m^4} = 2.16 \times 10^{-4}\ rad = 0.0124^\circ \tag{35}$$

The 4NodeANDES elements for cantilever beams of different Poisson's ratios were shown in Figure (120) and (121):

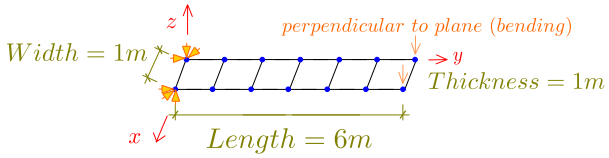


Figure 120: 4NodeANDES elements for different Poisson's ratios under the force perpendicular to plane (bending)

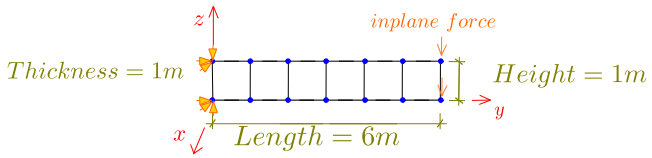


Figure 121: 4NodeANDES elements for different Poisson's ratios under the inplane force

The ESSI results for the force ***perpendicular to plane (bending)*** were listed in Table (53) - (55).

Table 53: ***Displacement error*** results for 4NodeANDES with element side length 1 m under the force perpendicular to plane (bending)

Poisson's ratio	4NodeANDES displacement	Theory displacement (bending)	Theory displacement (shear)	Theory displacement(all)	Error
0.00	8.639E-04 m	8.640E-04 m	1.200E-05 m	8.760E-04 m	1.38%
0.05	8.635E-04 m	8.640E-04 m	1.260E-05 m	8.766E-04 m	1.49%
0.10	8.622E-04 m	8.640E-04 m	1.320E-05 m	8.772E-04 m	1.71%
0.15	8.599E-04 m	8.640E-04 m	1.380E-05 m	8.778E-04 m	2.04%
0.20	8.566E-04 m	8.640E-04 m	1.440E-05 m	8.784E-04 m	2.48%
0.25	8.522E-04 m	8.640E-04 m	1.500E-05 m	8.790E-04 m	3.05%
0.30	8.466E-04 m	8.640E-04 m	1.560E-05 m	8.796E-04 m	3.75%
0.35	8.398E-04 m	8.640E-04 m	1.620E-05 m	8.802E-04 m	4.59%
0.40	8.315E-04 m	8.640E-04 m	1.680E-05 m	8.808E-04 m	5.60%
0.45	8.216E-04 m	8.640E-04 m	1.740E-05 m	8.814E-04 m	6.78%
0.49	8.124E-04 m	8.640E-04 m	1.788E-05 m	8.819E-04 m	7.88%

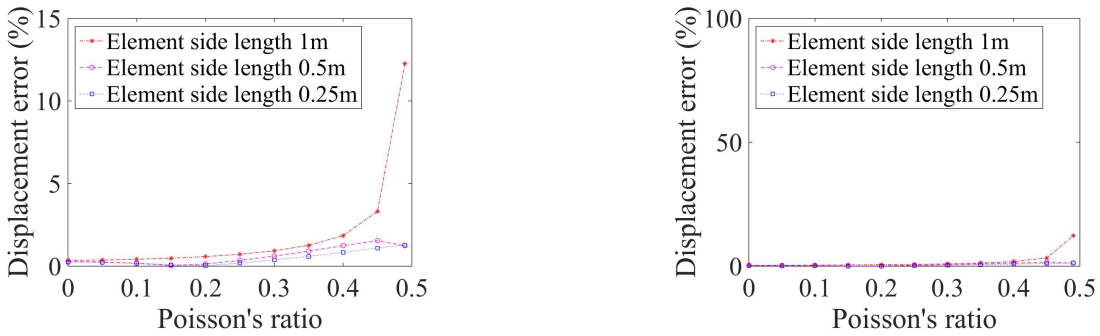
Table 54: ***Displacement error*** results for 4NodeANDES with element side length 0.5 m under the force perpendicular to plane (bending)

Poisson's ratio	4NodeANDES displacement	Theory displacement (bending)	Theory displacement (shear)	Theory displacement(all)	Error
0.00	8.724E-04 m	8.640E-04 m	1.200E-05 m	8.760E-04 m	0.41%
0.05	8.724E-04 m	8.640E-04 m	1.260E-05 m	8.766E-04 m	0.48%
0.10	8.717E-04 m	8.640E-04 m	1.320E-05 m	8.772E-04 m	0.62%
0.15	8.703E-04 m	8.640E-04 m	1.380E-05 m	8.778E-04 m	0.85%
0.20	8.682E-04 m	8.640E-04 m	1.440E-05 m	8.784E-04 m	1.17%
0.25	8.652E-04 m	8.640E-04 m	1.500E-05 m	8.790E-04 m	1.57%
0.30	8.615E-04 m	8.640E-04 m	1.560E-05 m	8.796E-04 m	2.06%
0.35	8.569E-04 m	8.640E-04 m	1.620E-05 m	8.802E-04 m	2.65%
0.40	8.514E-04 m	8.640E-04 m	1.680E-05 m	8.808E-04 m	3.34%
0.45	8.449E-04 m	8.640E-04 m	1.740E-05 m	8.814E-04 m	4.14%
0.49	8.388E-04 m	8.640E-04 m	1.788E-05 m	8.819E-04 m	4.88%

Table 55: ***Displacement error*** results for 4NodeANDES with element side length 0.25 m under the force perpendicular to plane (bending)

Poisson's ratio	4NodeANDES displacement	Theory displacement (bending)	Theory displacement (shear)	Theory displacement(all)	Error
0.00	8.640E-04 m	8.640E-04 m	1.200E-05 m	8.760E-04 m	1.37%
0.05	8.637E-04 m	8.640E-04 m	1.260E-05 m	8.766E-04 m	1.47%
0.10	8.627E-04 m	8.640E-04 m	1.320E-05 m	8.772E-04 m	1.65%
0.15	8.611E-04 m	8.640E-04 m	1.380E-05 m	8.778E-04 m	1.90%
0.20	8.588E-04 m	8.640E-04 m	1.440E-05 m	8.784E-04 m	2.23%
0.25	8.559E-04 m	8.640E-04 m	1.500E-05 m	8.790E-04 m	2.63%
0.30	8.523E-04 m	8.640E-04 m	1.560E-05 m	8.796E-04 m	3.10%
0.35	8.480E-04 m	8.640E-04 m	1.620E-05 m	8.802E-04 m	3.66%
0.40	8.429E-04 m	8.640E-04 m	1.680E-05 m	8.808E-04 m	4.31%
0.45	8.370E-04 m	8.640E-04 m	1.740E-05 m	8.814E-04 m	5.04%
0.49	8.316E-04 m	8.640E-04 m	1.788E-05 m	8.819E-04 m	5.71%

The errors were plotted in Figure (122).



(a) Error scale 0% - 15%

(b) Error scale 0% - 100%

Figure 122: 4NodeANDES cantilever beam for force perpendicular to the plane(bending)
Displacement error versus Poisson's ratio

The ESSI results for the ***inplane force*** were listed in Table (56) - (58).

Table 56: ***Displacement error*** results for 4NodeANDES with element side length 1 m under the inplane force

Poisson's ratio	4NodeANDES displacement	Theory displacement (bending)	Theory displacement (shear)	Theory displacement(all)	Error
0.00	8.790E-04 m	8.640E-04 m	1.200E-05 m	8.760E-04 m	0.35%
0.05	8.799E-04 m	8.640E-04 m	1.260E-05 m	8.766E-04 m	0.37%
0.10	8.809E-04 m	8.640E-04 m	1.320E-05 m	8.772E-04 m	0.42%
0.15	8.821E-04 m	8.640E-04 m	1.380E-05 m	8.778E-04 m	0.48%
0.20	8.835E-04 m	8.640E-04 m	1.440E-05 m	8.784E-04 m	0.58%
0.25	8.853E-04 m	8.640E-04 m	1.500E-05 m	8.790E-04 m	0.72%
0.30	8.878E-04 m	8.640E-04 m	1.560E-05 m	8.796E-04 m	0.93%
0.35	8.913E-04 m	8.640E-04 m	1.620E-05 m	8.802E-04 m	1.26%
0.40	8.971E-04 m	8.640E-04 m	1.680E-05 m	8.808E-04 m	1.85%
0.45	9.107E-04 m	8.640E-04 m	1.740E-05 m	8.814E-04 m	3.32%
0.49	9.901E-04 m	8.640E-04 m	1.788E-05 m	8.819E-04 m	12.27%

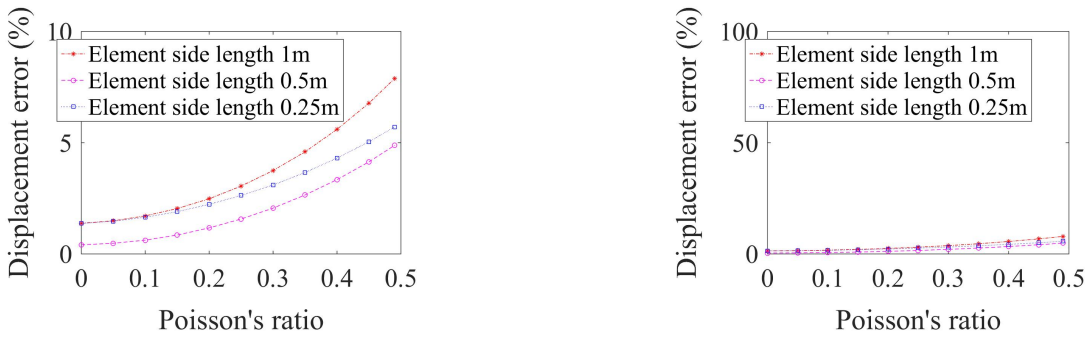
Table 57: ***Displacement error*** results for 4NodeANDES with element side length 0.5 m under the inplane force

Poisson's ratio	4NodeANDES displacement	Theory displacement (bending)	Theory displacement (shear)	Theory displacement(all)	Error
0.00	8.784E-04 m	8.640E-04 m	1.200E-05 m	8.760E-04 m	0.28%
0.05	8.788E-04 m	8.640E-04 m	1.260E-05 m	8.766E-04 m	0.25%
0.10	8.787E-04 m	8.640E-04 m	1.320E-05 m	8.772E-04 m	0.17%
0.15	8.782E-04 m	8.640E-04 m	1.380E-05 m	8.778E-04 m	0.04%
0.20	8.772E-04 m	8.640E-04 m	1.440E-05 m	8.784E-04 m	0.13%
0.25	8.759E-04 m	8.640E-04 m	1.500E-05 m	8.790E-04 m	0.35%
0.30	8.742E-04 m	8.640E-04 m	1.560E-05 m	8.796E-04 m	0.61%
0.35	8.722E-04 m	8.640E-04 m	1.620E-05 m	8.802E-04 m	0.91%
0.40	8.699E-04 m	8.640E-04 m	1.680E-05 m	8.808E-04 m	1.24%
0.45	8.679E-04 m	8.640E-04 m	1.740E-05 m	8.814E-04 m	1.54%
0.49	8.709E-04 m	8.640E-04 m	1.788E-05 m	8.819E-04 m	1.24%

Table 58: ***Displacement error*** results for 4NodeANDES with element side length 0.25 m under the inplane force

Poisson's ratio	4NodeANDES displacement	Theory displacement (bending)	Theory displacement (shear)	Theory displacement(all)	Error
0.00	8.782E-04 m	8.640E-04 m	1.200E-05 m	8.760E-04 m	0.25%
0.05	8.786E-04 m	8.640E-04 m	1.260E-05 m	8.766E-04 m	0.23%
0.10	8.788E-04 m	8.640E-04 m	1.320E-05 m	8.772E-04 m	0.18%
0.15	8.786E-04 m	8.640E-04 m	1.380E-05 m	8.778E-04 m	0.09%
0.20	8.781E-04 m	8.640E-04 m	1.440E-05 m	8.784E-04 m	0.03%
0.25	8.774E-04 m	8.640E-04 m	1.500E-05 m	8.790E-04 m	0.19%
0.30	8.763E-04 m	8.640E-04 m	1.560E-05 m	8.796E-04 m	0.38%
0.35	8.750E-04 m	8.640E-04 m	1.620E-05 m	8.802E-04 m	0.59%
0.40	8.734E-04 m	8.640E-04 m	1.680E-05 m	8.808E-04 m	0.84%
0.45	8.717E-04 m	8.640E-04 m	1.740E-05 m	8.814E-04 m	1.10%
0.49	8.706E-04 m	8.640E-04 m	1.788E-05 m	8.819E-04 m	1.28%

The errors were plotted in Figure (122).



(a) Error scale 0% - 10%

(b) Error scale 0% - 100%

Figure 123: 4NodeANDES cantilever beam for inplane force ***Displacement error*** versus Poisson's ratio

The angle results for the force ***perpendicular to plane (bending)*** were listed in Table (59).

Table 59: ***Rotation angle*** results for element side length 1 m under the force perpendicular to plane (bending)

Poisson's ratio	4NodeANDES angle (unit:°)	Theory angle (unit:°)	Error
0.00	1.238E-02	1.240E-02	0.19%
0.05	1.237E-02	1.240E-02	0.23%
0.10	1.236E-02	1.240E-02	0.34%
0.15	1.234E-02	1.240E-02	0.52%
0.20	1.230E-02	1.240E-02	0.78%
0.25	1.226E-02	1.240E-02	1.12%
0.30	1.221E-02	1.240E-02	1.54%
0.35	1.214E-02	1.240E-02	2.07%
0.40	1.206E-02	1.240E-02	2.70%
0.45	1.197E-02	1.240E-02	3.46%
0.49	1.188E-02	1.240E-02	4.16%

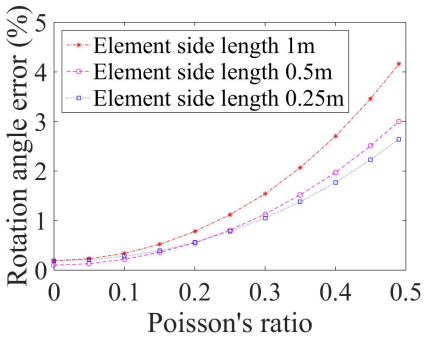
Table 60: ***Rotation angle*** results for element side length 0.5 m the force perpendicular to plane (bending)

Poisson's ratio	4NodeANDES angle (unit:°)	Theory angle (unit:°)	Error
0.00	1.239E-02	1.240E-02	0.10%
0.05	1.238E-02	1.240E-02	0.13%
0.10	1.237E-02	1.240E-02	0.22%
0.15	1.236E-02	1.240E-02	0.36%
0.20	1.233E-02	1.240E-02	0.55%
0.25	1.230E-02	1.240E-02	0.81%
0.30	1.226E-02	1.240E-02	1.13%
0.35	1.221E-02	1.240E-02	1.52%
0.40	1.216E-02	1.240E-02	1.97%
0.45	1.209E-02	1.240E-02	2.51%
0.49	1.203E-02	1.240E-02	3.00%

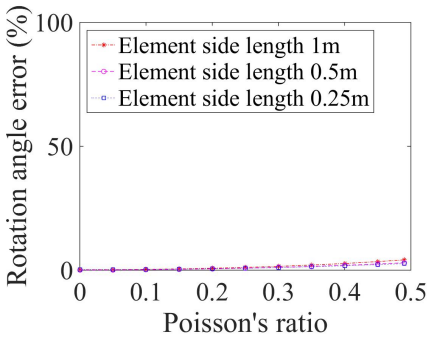
Table 61: ***Rotation angle*** results for element side length 0.25 m under the force perpendicular to plane (bending)

Poisson's ratio	4NodeANDES angle (unit:°)	Theory angle (unit:°)	Error
0.00	1.238E-02	1.240E-02	0.19%
0.05	1.237E-02	1.240E-02	0.21%
0.10	1.237E-02	1.240E-02	0.28%
0.15	1.235E-02	1.240E-02	0.39%
0.20	1.233E-02	1.240E-02	0.56%
0.25	1.230E-02	1.240E-02	0.78%
0.30	1.227E-02	1.240E-02	1.05%
0.35	1.223E-02	1.240E-02	1.38%
0.40	1.218E-02	1.240E-02	1.77%
0.45	1.212E-02	1.240E-02	2.23%
0.49	1.207E-02	1.240E-02	2.64%

The errors were plotted in Figure (124).



(a) Error scale 0% - 5%



(b) Error scale 0% - 100%

Figure 124: 4NodeANDES cantilever beam for force perpendicular to the plane(bending)
Rotation angle error versus Poisson's ratio

The ESSI results for the ***inplane force*** were listed in Table (62 - (64).

Table 62: ***Rotation angle*** results for element side length 1 m under the inplane force

Poisson's ratio	4NodeANDES angle (unit:°)	Theory angle (unit:°)	Error
0.00	1.254E-02	1.240E-02	1.14%
0.05	1.255E-02	1.240E-02	1.19%
0.10	1.256E-02	1.240E-02	1.26%
0.15	1.257E-02	1.240E-02	1.35%
0.20	1.258E-02	1.240E-02	1.47%
0.25	1.260E-02	1.240E-02	1.64%
0.30	1.263E-02	1.240E-02	1.89%
0.35	1.269E-02	1.240E-02	2.30%
0.40	1.278E-02	1.240E-02	3.08%
0.45	1.305E-02	1.240E-02	5.28%
0.49	1.506E-02	1.240E-02	21.43%

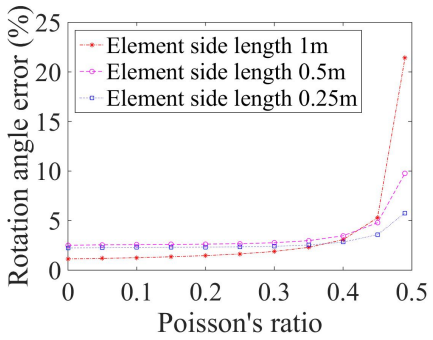
Table 63: ***Rotation angle*** results for element side length 0.5 m under the inplane force

Poisson's ratio	4NodeANDES angle (unit:°)	Theory angle (unit:°)	Error
0.00	1.271E-02	1.240E-02	2.51%
0.05	1.272E-02	1.240E-02	2.56%
0.10	1.272E-02	1.240E-02	2.58%
0.15	1.272E-02	1.240E-02	2.60%
0.20	1.273E-02	1.240E-02	2.63%
0.25	1.273E-02	1.240E-02	2.67%
0.30	1.274E-02	1.240E-02	2.77%
0.35	1.277E-02	1.240E-02	2.98%
0.40	1.283E-02	1.240E-02	3.47%
0.45	1.299E-02	1.240E-02	4.79%
0.49	1.361E-02	1.240E-02	9.78%

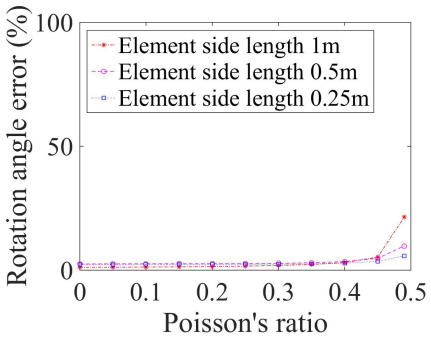
Table 64: ***Rotation angle*** results for element side length 0.25 m under the inplane force

Poisson's ratio	4NodeANDES angle (unit:°)	Theory angle (unit:°)	Error
0.00	1.268E-02	1.240E-02	2.24%
0.05	1.268E-02	1.240E-02	2.27%
0.10	1.268E-02	1.240E-02	2.30%
0.15	1.269E-02	1.240E-02	2.31%
0.20	1.269E-02	1.240E-02	2.33%
0.25	1.269E-02	1.240E-02	2.35%
0.30	1.270E-02	1.240E-02	2.41%
0.35	1.271E-02	1.240E-02	2.53%
0.40	1.275E-02	1.240E-02	2.83%
0.45	1.284E-02	1.240E-02	3.58%
0.49	1.312E-02	1.240E-02	5.77%

The errors were plotted in Figure (124).



(a) Error scale 0% - 25%



(b) Error scale 0% - 100%

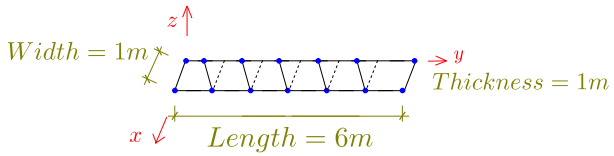
Figure 125: 4NodeANDES cantilever beam for inplane force ***Rotation angle error*** versus Poisson's ratio

The ESSI model fei files for the table above are here

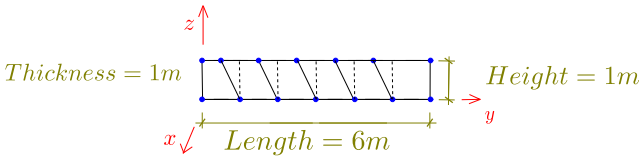
3.3 Test of irregular shaped 4NodeANDES cantilever beams

Cantilever model was used as an example. Three different shapes were tested.

In the ***first*** test, the upper two nodes of each element were moved one half element size along the $y-axis$, while the lower two nodes were kept at the same location. The element shape was shown in Figure (126).



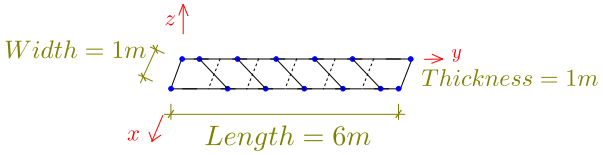
(a) Horizontal plane



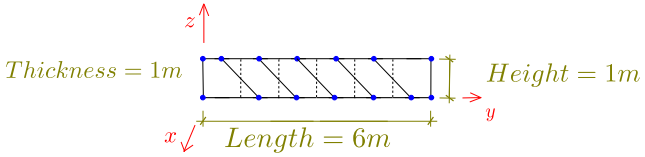
(b) Veritical plane

Figure 126: 4NodeANDES cantilever beam for irregular ***Shape 1***

In the ***second*** test, the upper nodes of each element were moved 50% element size along the $y-axis$, while the lower nodes were moved 50% element size in the other direction along the $y-axis$. The element shape was shown in Figure (127).



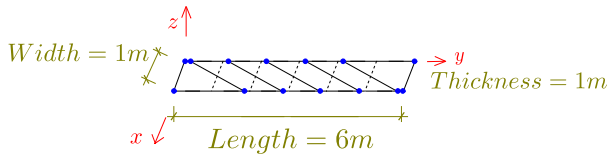
(a) Horizontal plane



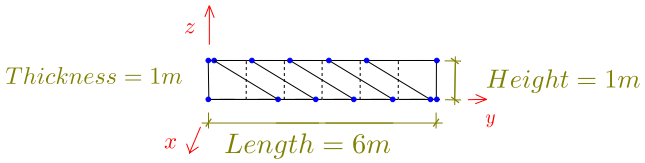
(b) Veritical plane

Figure 127: 4NodeANDES cantilever beam for irregular ***Shape 2***

In the ***third*** test, the upper two nodes of each element were moved 90% element size with different directions along the $y-axis$, while the lower nodes were moved 90% element size in the other direction along the $y-axis$. The element shape was shown in Figure (128).



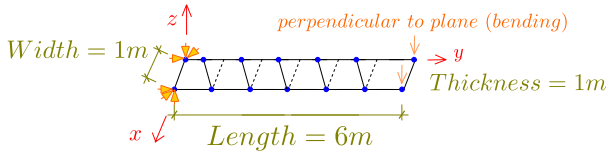
(a) Horizontal plane



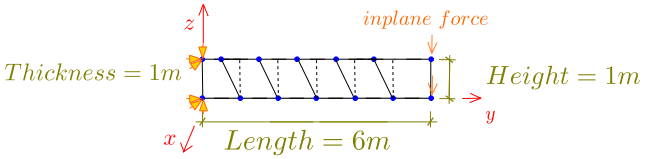
(b) Veritical plane

Figure 128: 4NodeANDES cantilever beam for irregular ***Shape 3***

The boundary conditions were shown in Figure (129), (130) and (131).

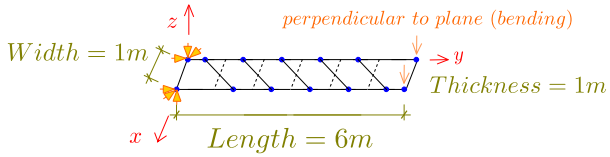


(a) Horizontal plane

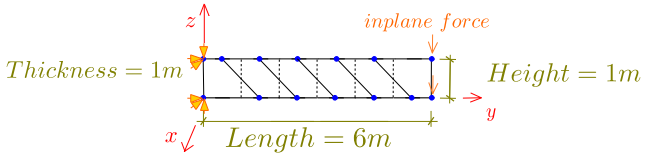


(b) Veritical plane

Figure 129: 4NodeANDES cantilever beam boundary conditions for irregular ***Shape 1***

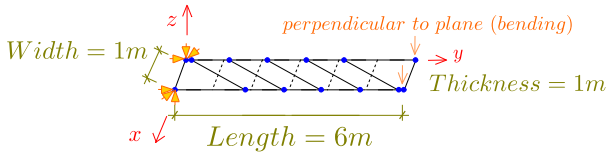


(a) Horizontal plane

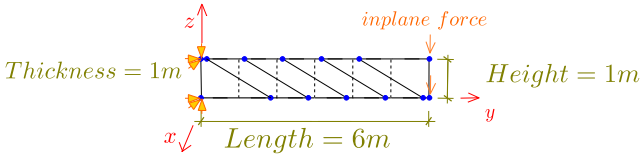


(b) Veritical plane

Figure 130: 4NodeANDES cantilever beam boundary conditions for irregular ***Shape 2***



(a) Horizontal plane



(b) Veritical plane

Figure 131: 4NodeANDES cantilever beam boundary conditions for irregular ***Shape 3***

The ESSI results were listed in Table (65).

Table 65: Results for 4NodeANDES cantilever beams of irregular shapes

Displacements for irregular shaped element					
Element Type	Force direction	Normal shape	Shape 1	Shape 2	Shape 3
4NodeANDES	perpendicular to plane (bending)	8.639E-04 m	8.602E-04 m	8.534E-04 m	7.851E-04 m
4NodeANDES	inplane force	8.857E-04 m	7.036E-04 m	4.263E-04 m	1.909E-04 m
Theoretical	-	8.760E-04 m	8.760E-04 m	8.760E-04 m	8.760E-04 m

The errors were listed in Table (66) and (67).

Table 66: Errors for irregular shaped 4NodeANDES compared to theoretical solution

Errors for irregular shaped element, compared to theoretical solutions					
Element Type	Force direction	Normal shape	Shape 1	Shape 2	Shape 3
4NodeANDES	perpendicular to plane (bending)	1.38%	1.80%	2.58%	10.38%
4NodeANDES	inplane force	1.11%	19.68%	51.34%	78.21%

Table 67: Errors for irregular shaped 4NodeANDES compared to normal shape

Errors for irregular shaped element, compared to normal shape					
Element Type	Force direction	Normal shape	Shape 1	Shape 2	Shape 3
4NodeANDES	perpendicular to plane (bending)	0.00%	0.42%	1.22%	9.12%
4NodeANDES	inplane force	0.00%	20.56%	51.87%	78.45%

The ESSI model fei files for the table above are here

Then, the beam was divided into small elements.

Problem description: Length=6m, Width=1m, Height=1m, Force=100N, E=1E8Pa, $\nu = 0.0$. The force direction was shown in Figure (132).

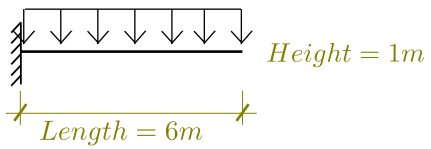


Figure 132: Problem description for cantilever beams under uniform pressure

Theoretical displacement (bending and shear deformation):

$$\begin{aligned} d &= \frac{qL^4}{8EI} + \frac{q\frac{L^2}{2}}{GA} \\ &= \frac{400N/m \times 12^4 m^4}{8 \times 10^8 N/m^2 \times \frac{2^4}{12} m^4} + \frac{400N/m \times \frac{12^2}{2} m^2}{\frac{10^8}{2} N/m^2 \times 2m \times 2m} \\ &= 7.776 \times 10^{-3} m + 1.44 \times 10^{-4} m \\ &= 7.920 \times 10^{-3} m \end{aligned} \tag{36}$$

The ESSI displacement results were listed in Table (68).

Table 68: Results for 4NodeANDES cantilever beams of irregular shapes with more elements

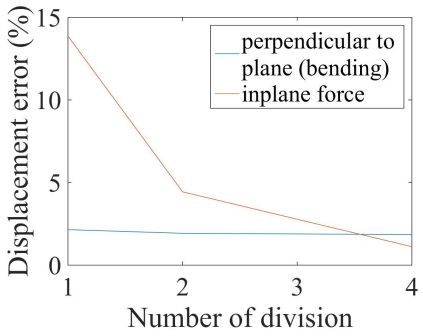
Element Type	Shape	Force direction	Number of division		
			1	2	4
4NodeANDES	shape1	perpendicular to plane (bending)	7.750E-03 m	7.768E-03 m	7.774E-03 m
4NodeANDES	shape1	inplane force	6.822E-03 m	7.569E-03 m	7.832E-03 m
4NodeANDES	shape2	perpendicular to plane (bending)	7.656E-03 m	7.734E-03 m	7.765E-03 m
4NodeANDES	shape2	inplane force	3.875E-03 m	5.855E-03 m	7.074E-03 m
4NodeANDES	shape3	perpendicular to plane (bending)	6.637E-03 m	7.139E-03 m	7.521E-03 m
4NodeANDES	shape3	inplane force	1.555E-03 m	2.424E-03 m	3.896E-03 m
Theoretical solution			7.920E-03 m	7.920E-03 m	7.920E-03 m

The error were listed in Table (69).

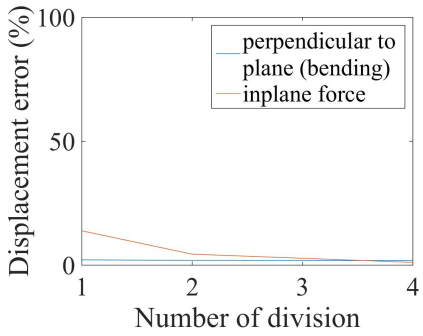
Table 69: Errors for 4NodeANDES cantilever beams of irregular shapes with more elements

Element Type	Shape	Force direction	Number of division		
			1	2	4
4NodeANDES	shape1	perpendicular to plane (bending)	2.15%	1.93%	1.85%
4NodeANDES	shape1	inplane force	13.87%	4.44%	1.12%
4NodeANDES	shape2	perpendicular to plane (bending)	3.33%	2.35%	1.96%
4NodeANDES	shape2	inplane force	51.07%	26.07%	10.68%
4NodeANDES	shape3	perpendicular to plane (bending)	16.21%	9.86%	5.04%
4NodeANDES	shape3	inplane force	80.37%	69.40%	50.80%

The errors were shown in Figure (133), (134) and (135).

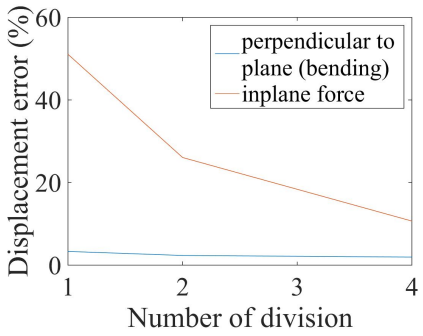


(a) Error scale 0% - 15%

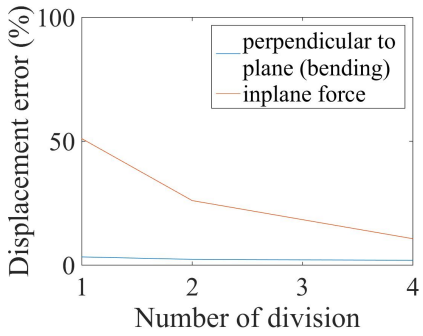


(b) Error scale 0% - 100%

Figure 133: 4NodeANDES cantilever beam for irregular ***Shape 1***
Displacement error versus Number of division

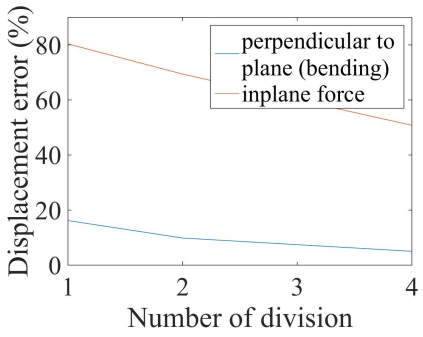


(a) Error scale 0% - 60%

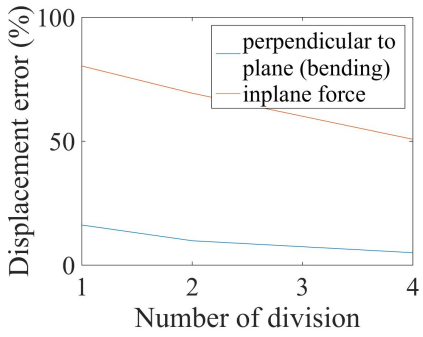


(b) Error scale 0% - 100%

Figure 134: 4NodeANDES cantilever beam for irregular ***Shape 2***
Displacement error versus Number of division



(a) Error scale 0% - 80%



(b) Error scale 0% - 100%

Figure 135: 4NodeANDES cantilever beam for irregular ***Shape 3***
Displacement error versus Number of division

The ESSI model fei files for the table above are here

3.4 Verification of 4NodeANDES edge clamped beams

Problem description: Length=6m, Width=1m, Height=1m, Force=100N, E=1E8Pa, $\nu = 0.0$. The force direction was shown in Figure (136).

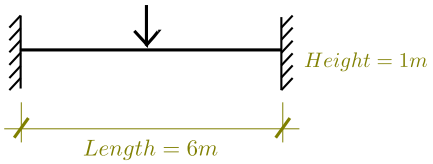


Figure 136: Problem description for clamped beams

The elment types and element sizes were same to the cantilever model. Only the boundary conditions and external force locations were changed.

The 4NodeANDES elements were shown in Figure (137) and (138).

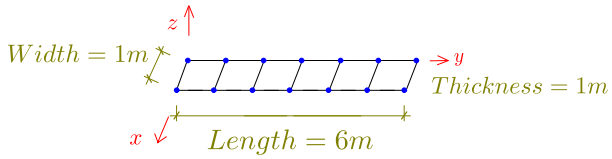


Figure 137: 4NodeANDES elements for clamped beams horizontal plane

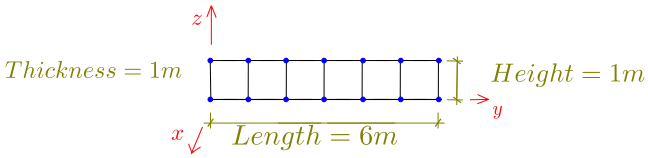


Figure 138: 4NodeANDES elements for clamped beams vertical plane

Theoretical displacement (bending and shear deformation):

$$\begin{aligned} d &= \frac{FL^3}{192EI} + \frac{\frac{F}{2}\frac{L}{2}}{GA} \\ &= \frac{100N \times 6m^3}{192 \times 10^8 N/m^2 \times \frac{1}{12}m^4} + \frac{100N \times 6m}{4 \times 5 \times 10^7 N/m^2 \times 1m^2} \\ &= 1.35 \times 10^{-5}m + 0.3 \times 10^{-5}m \\ &= 1.65 \times 10^{-5}\ m \end{aligned} \tag{37}$$

The theoretical solution for $L = 6\ m$ was calculated above, while the solutions for other length were calculated by the similar process.

In the figures above, only the model with geometry $6m \times 1m \times 1m$ was drawed. In the ESSI models, the geometry $10m \times 1m \times 1m$ and the geometry $20m \times 1m \times 1m$ were also calculated. In three different geometry models, all the element sizes were $1m \times 1m \times 1m$. Therefore, the number of elements used in each model were 6, 10 *and* 20 respectively.

The ESSI results for the force ***perpendicular to plane (bending)*** were listed in Table (70).

Table 70: Results for 4NodeANDES clamped beams under the force perpendicular to plane (bending)

Geometry	4NodeANDES	Theory(bending)	Theory(shear)	Theory(all)	Error
1:6	1.347E-05 m	1.35E-05 m	3.00E-06 m	1.65E-05 m	18.36%
1:10	6.245E-05 m	6.25E-05 m	5.00E-06 m	6.75E-05 m	7.48%
1:20	4.999E-04 m	5.00E-04 m	1.00E-05 m	5.10E-04 m	1.98%

The ESSI results for the ***inplane force*** were listed in Table (71).

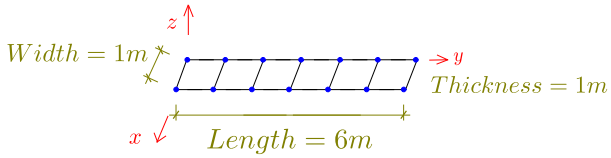
Table 71: Results for 4NodeANDES clamped beams under the inplane force

Geometry	4NodeANDES	Theory(bending)	Theory(shear)	Theory(all)	Error
1:6	1.622E-05 m	1.35E-05 m	3.00E-06 m	1.65E-05 m	1.70%
1:10	6.796E-05 m	6.25E-05 m	5.00E-06 m	6.75E-05 m	0.68%
1:20	5.123E-04 m	5.00E-04 m	1.00E-05 m	5.10E-04 m	0.45%

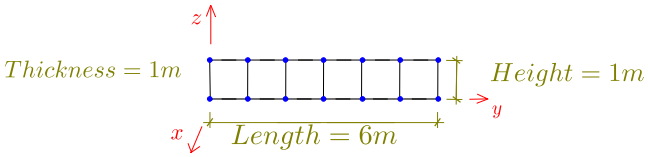
The ESSI model fei files for the table above are here

In this section, the beam was cut into smaller elements with element side length 0.5m and 0.25m respectively. And the element side length of the original models is 1.0m. The numerical models were shown in Figure (139), (140) and (141).

Number of division 1:



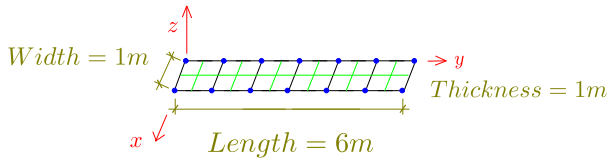
(a) Horizontal plane



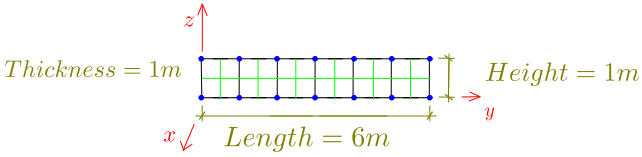
(b) Veritical plane

Figure 139: 4NodeANDES clamped beam with element side length 1.0m

Number of division 2:



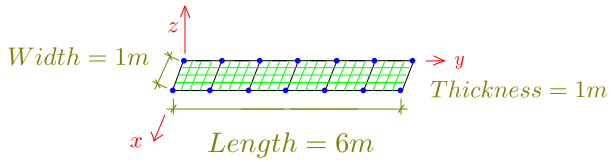
(a) Horizontal plane



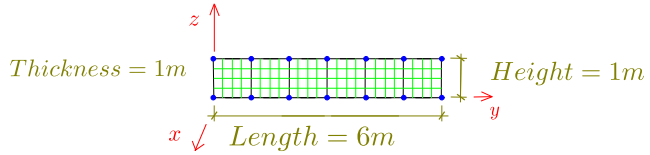
(b) Veritical plane

Figure 140: 4NodeANDES clamped beam with element side length 0.5m

Number of division 4:



(a) Horizontal plane



(b) Veritical plane

Figure 141: 4NodeANDES clamped beam with element side length 0.25m

The ESSI results for the force ***perpendicular to plane (bending)*** were listed in Table (72). The theoretical solution is 1.65E-5 m.

Table 72: Results for 4NodeANDES clamped beams under the force perpendicular to plane (bending)

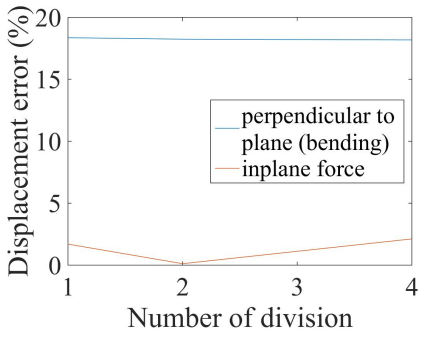
Element Type	Element side length		
	1 m	0.5 m	0.25 m
4NodeANDES	1.347E-05 m	1.35E-05 m	1.35E-05 m
Error	18.36%	18.24%	18.18%

The ESSI results for the ***inplane force*** were listed in Table (73). The theoretical solution is 1.65E-5 m.

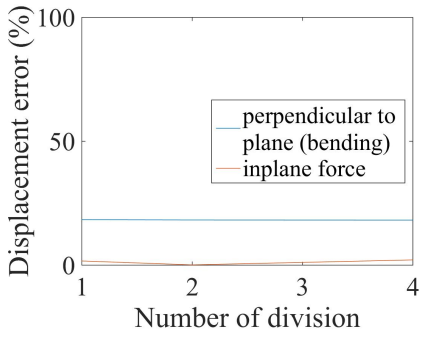
Table 73: Results for 4NodeANDES clamped beams under the inplane force

Element Type	Element side length		
	1 m	0.5 m	0.25 m
4NodeANDES	1.62E-05 m	1.65E-05 m	1.69E-05 m
Error	1.70%	0.12%	2.12%

The errors were plotted in Figure (142).



(a) Error scale 0% - 20%



(b) Error scale 0% - 100%

Figure 142: 4NodeANDES clamped beam for different element number
Displacement error versus Number of division

The ESSI model fei files for the table above are here

3.5 Verification of 4NodeANDES square plate with four edges clamped

Problem description: Length=20m, Width=20m, Height=1m, Force=100N, E=1E8Pa, $\nu = 0.3$.

The four edges are clamped.

The load is the uniform normal pressure on the whole plate.

The plate flexural rigidity is

$$D = \frac{Eh^3}{12(1-\nu^2)} = \frac{10^8 N/m^2 \times 1^3 m^3}{12 \times (1-0.3^2)} = 9.1575 \times 10^6 \ N \cdot m \tag{38}$$

The theoretical solution is

$$d = \alpha_c \frac{qa^4}{D} = 0.00406 \times \frac{100 N/m^2 \times 20^4 m^4}{9.1575 \times 10^6 \ N \cdot m} = 2.2015 \times 10^{-3} m \tag{39}$$

where α_c is a coefficient, which depends on the ratio of plate length to width. In this problem, the coefficient[11] α_c is 0.00406.

The 4NodeANDES were shown in Figure (143) - (148).

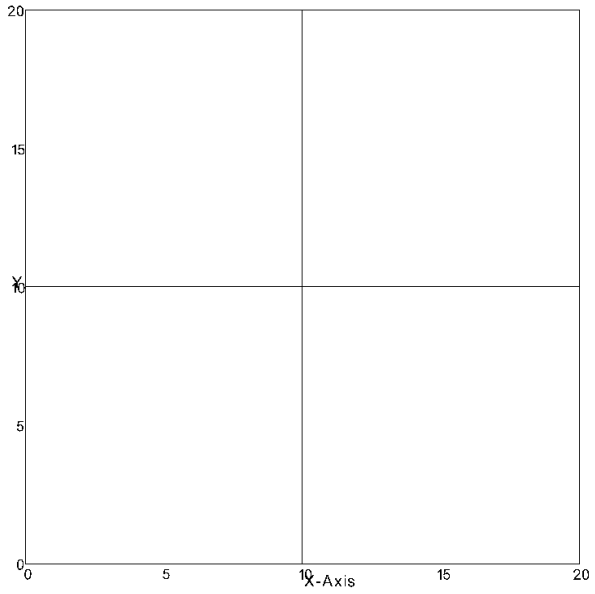


Figure 143: 4NodeANDES edge clamped square plate with element side length 10m

[11] Stephen Timoshenko, Theory of plates and shells (2nd edition). MrGRAW-Hill Inc, page120, 1959.

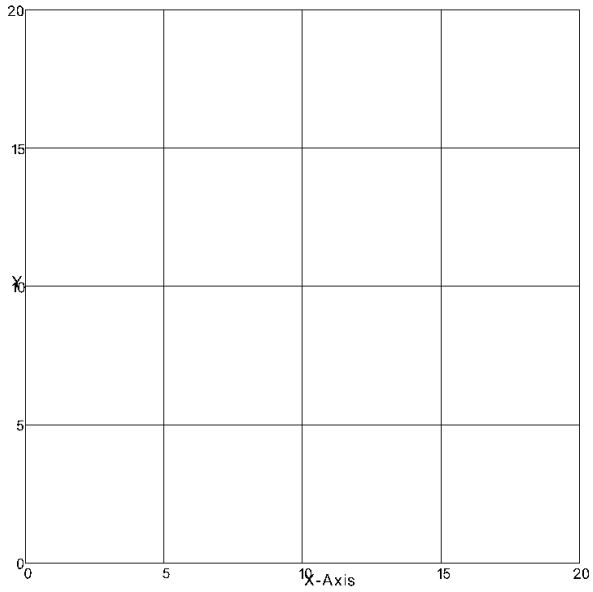


Figure 144: 4NodeANDES edge clamped square plate with element side length 5m

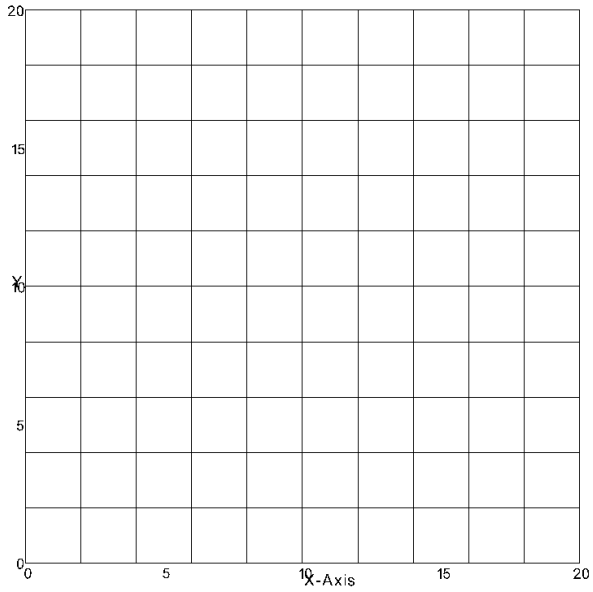


Figure 145: 4NodeANDES edge clamped square plate with element side length 2m

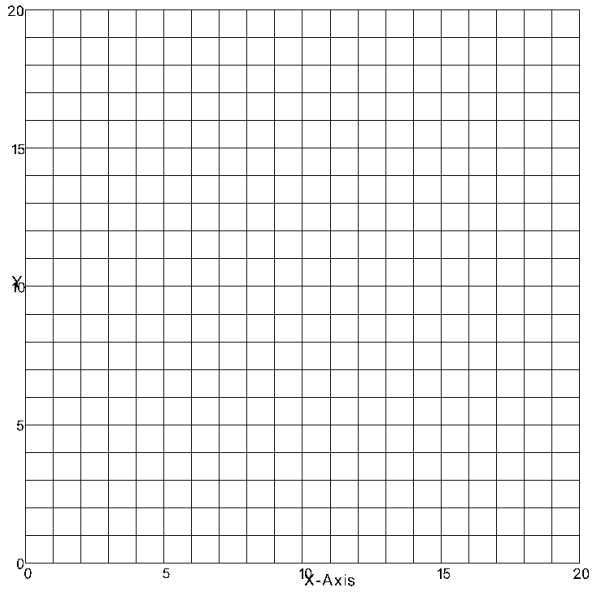


Figure 146: 4NodeANDES edge clamped square plate with element side length 1m

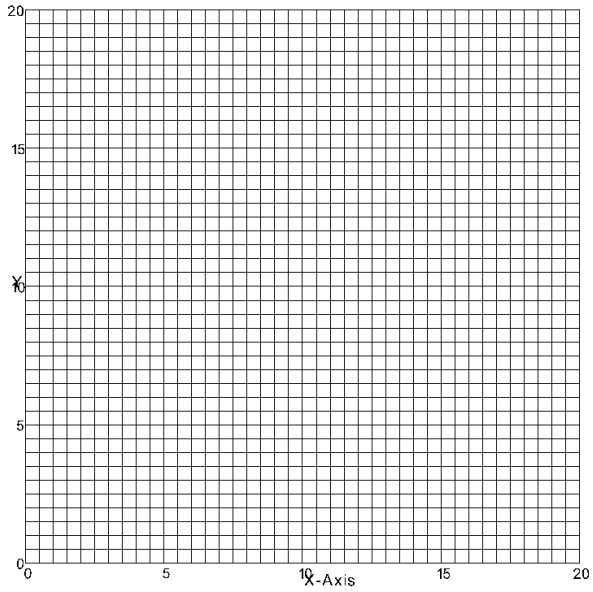


Figure 147: 4NodeANDES edge clamped square plate with element side length 0.5m

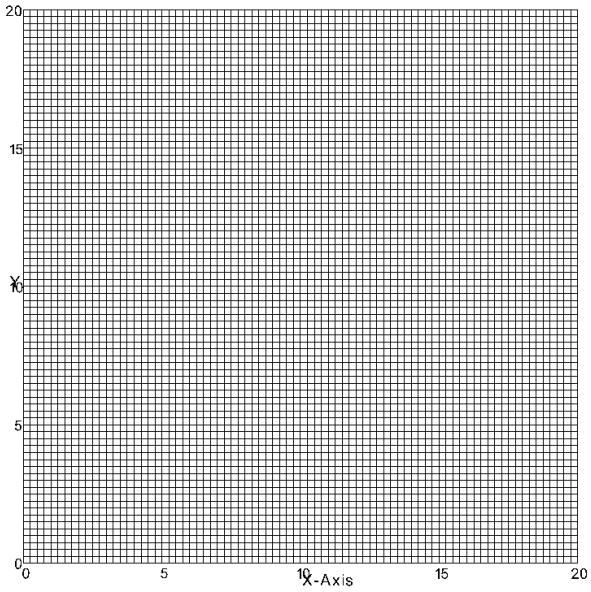


Figure 148: 4NodeANDES edge clamped square plate with element side length 0.25m

The results were listed in Table (74).

Table 74: Results for 4NodeANDES square plate with four edges clamped

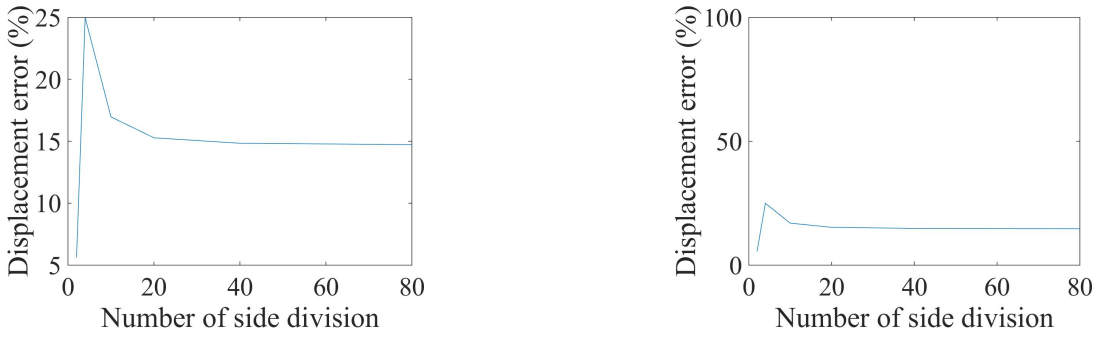
Element type	4NodeANDES	Theoretical
Element side length	Height:1.00m	displacement
10m	2.33E-003 m	2.20E-03 m
5m	2.75E-003 m	2.20E-03 m
2m	2.58E-003 m	2.20E-03 m
1m	2.54E-003 m	2.20E-03 m
0.5m	2.53E-003 m	2.20E-03 m
0.25m	2.53E-003 m	2.20E-03 m

The errors were listed in Table (75).

Table 75: Errors for 4NodeANDES square plate with four edges clamped

Element type	4NodeANDES
Element side length	Height:1.00m
10m	5.65%
5m	24.98%
2m	16.97%
1m	15.28%
0.5m	14.84%
0.25m	14.73%

The errors were plotted in Figure (149).



(a) Error scale 0% - 25%

(b) Error scale 0% - 100%

Figure 149: 4NodeANDES square plate with edge clamped
Displacement error versus Number of side division

The ESSI model fei files for the table above are here

3.6 Verification of 4NodeANDES square plate with four edges simply supported

Problem description: Length=20m, Width=20m, Height=1m, Force=100N, E=1E8Pa, $\nu = 0.3$.

The four edges are simply supported.

The load is the uniform normal pressure on the whole plate.

The plate flexural rigidity is

$$D = \frac{Eh^3}{12(1-\nu^2)} = \frac{10^8 N/m^2 \times 1^3 m^3}{12 \times (1-0.3^2)} = 9.1575 \times 10^6 \ N \cdot m \tag{40}$$

The theoretical solution is

$$d = \alpha_s \frac{qa^4}{D} = 0.00126 \times \frac{100 N/m^2 \times 20^4 m^4}{9.1575 \times 10^6 \ N \cdot m} = 7.0936 \times 10^{-3} m \tag{41}$$

where α_s is a coefficient, which depends on the ratio of plate length to width. In this problem, the coefficient[12] α_s is 0.00126.

The 4NodeANDES were shown in Figure (150) - (155).

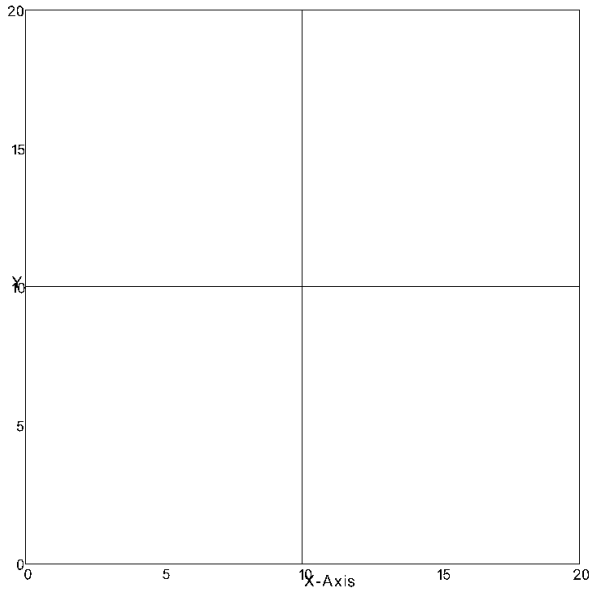


Figure 150: 4NodeANDES edge simply supported square plate with element side length 10m

[12]Stephen Timoshenko, Theory of plates and shells (2nd edition). MrGRAW-Hill Inc, page202, 1959.

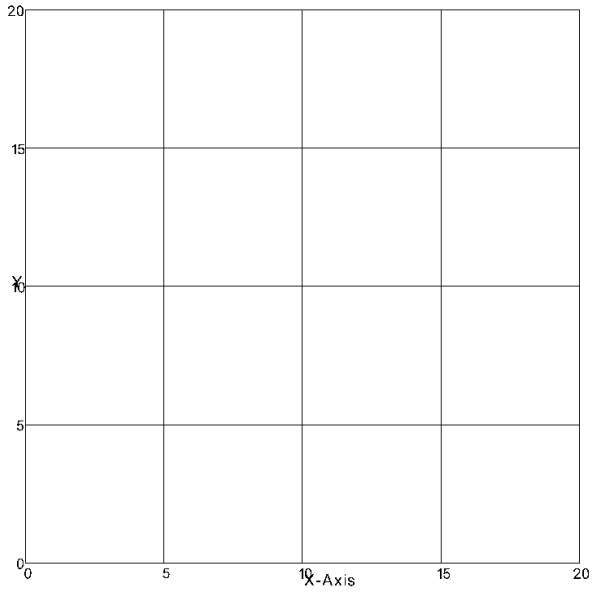


Figure 151: 4NodeANDES edge simply supported square plate with element side length 5m

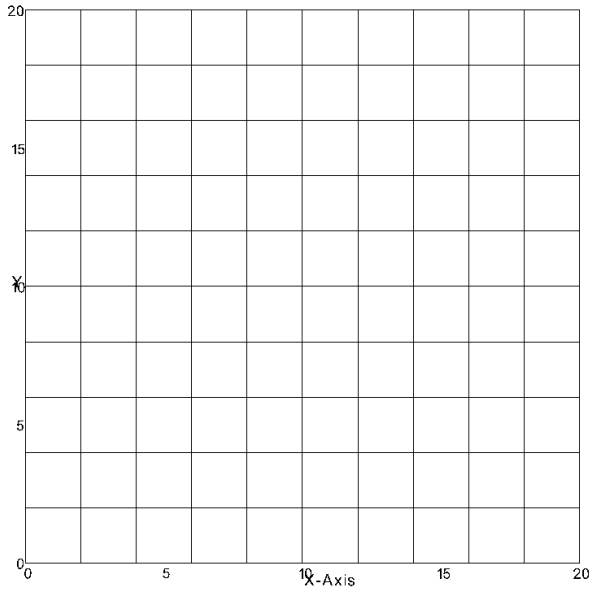


Figure 152: 4NodeANDES edge simply supported square plate with element side length 2m

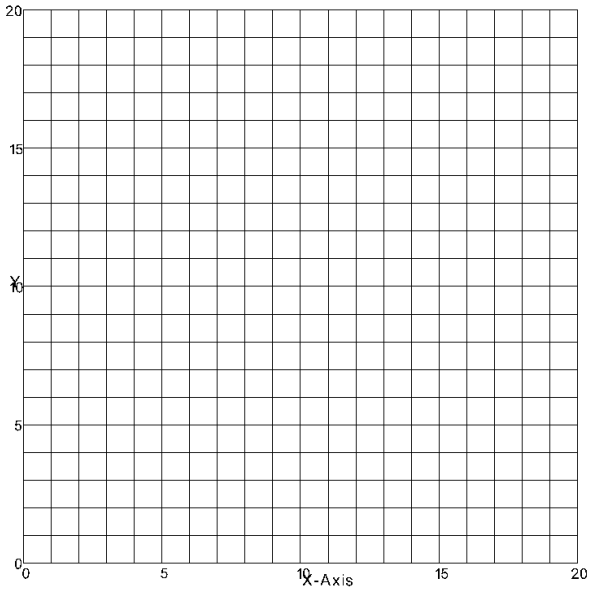


Figure 153: 4NodeANDES edge simply supported square plate with element side length 1m

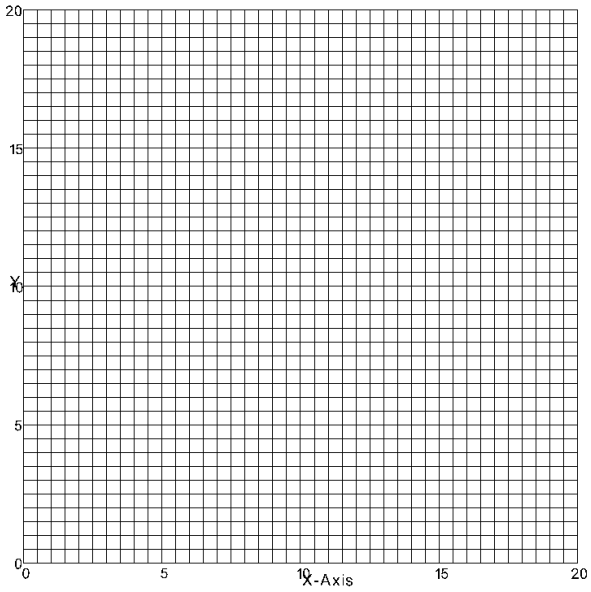


Figure 154: 4NodeANDES edge simply supported square plate with element side length 0.5m

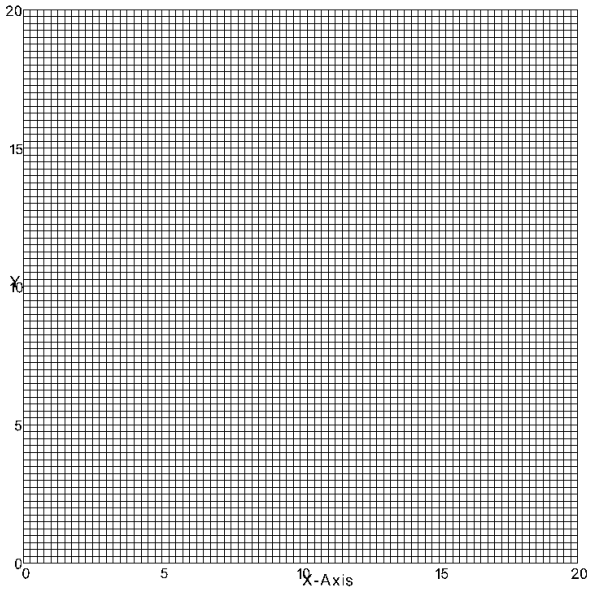


Figure 155: 4NodeANDES edge simply supported square plate with element side length 0.25m

The results were listed in Table (76).

Table 76: Results for 4NodeANDES square plate with four edges simply supported

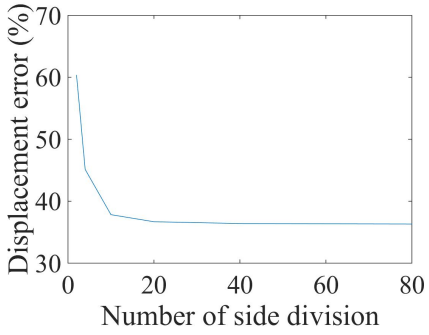
Element type	4NodeANDES	Theoretical
Element side length	Height:1.00m	displacement
10m	1.14E-002 m	7.09E-03 m
5m	1.03E-002 m	7.09E-03 m
2m	9.78E-003 m	7.09E-03 m
1m	9.70E-003 m	7.09E-03 m
0.5m	9.68E-003 m	7.09E-03 m
0.25m	9.67E-003 m	7.09E-03 m

The errors were listed in Table (77).

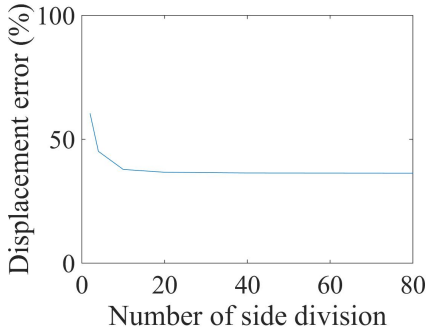
Table 77: Errors for 4NodeANDES square plate with four edges simply supported

Element type	4NodeANDES
Element side length	Height:1.00m
10m	60.34%
5m	45.14%
2m	37.83%
1m	36.69%
0.5m	36.40%
0.25m	36.32%

The errors were plotted in Figure (156).



(a) Error scale 0% - 70%



(b) Error scale 0% - 100%

Figure 156: 4NodeANDES square plate with edge simply supported
Displacement error versus Number of side division

The ESSI model fei files for the table above are here

3.7 Verification of 4NodeANDES circular plate with all edges clamped

Problem description: Diameter=20m, Height=1m, Force=100N, E=1E8Pa, $\nu = 0.3$.

The four edges are clamped.

The load is the uniform normal pressure on the whole plate.

The plate flexural rigidity is

$$D = \frac{Eh^3}{12(1-\nu^2)} = \frac{10^8 N/m^2 \times 1^3 m^3}{12 \times (1-0.3^2)} = 9.1575 \times 10^6 \ N \cdot m \tag{42}$$

The theoretical solution[13] is

$$d = \frac{qa^4}{64D} = \frac{100 N/m^2 \times 10^4 m^4}{64 \times 9.1575 \times 10^6 \ N \cdot m} = 1.7106 \times 10^{-3} m \tag{43}$$

The 4NodeANDES were shown in Figure (157) - (162).

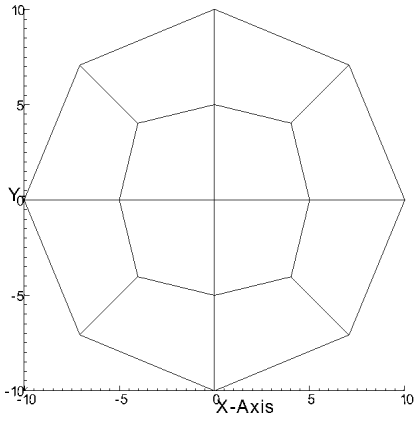


Figure 157: 4NodeANDES edge clamped circular plate with element side length 10m

[13]Stephen Timoshenko, Theory of plates and shells (2nd edition). MrGRAW-Hill Inc, page55, 1959.

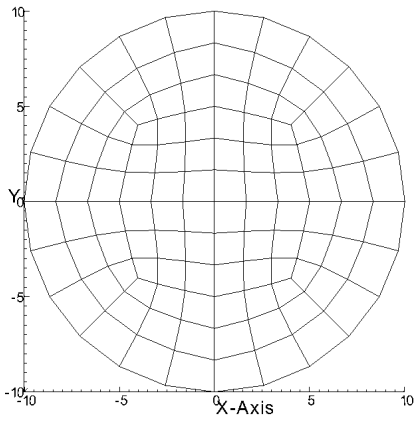


Figure 158: 4NodeANDES edge clamped circular plate with element side length 5m

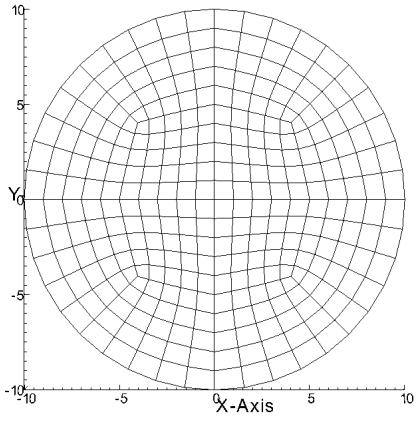


Figure 159: 4NodeANDES edge clamped circular plate with element side length 2m

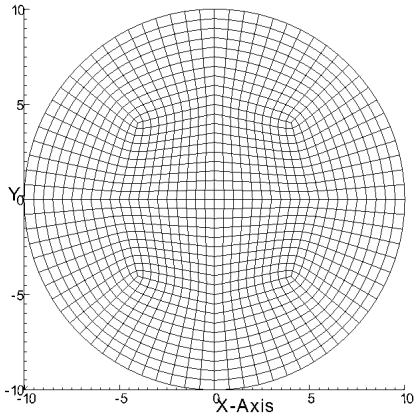


Figure 160: 4NodeANDES edge clamped circular plate with element side length 1m

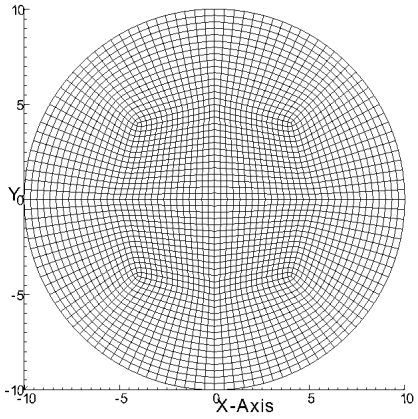


Figure 161: 4NodeANDES edge clamped circular plate with element side length 0.5m

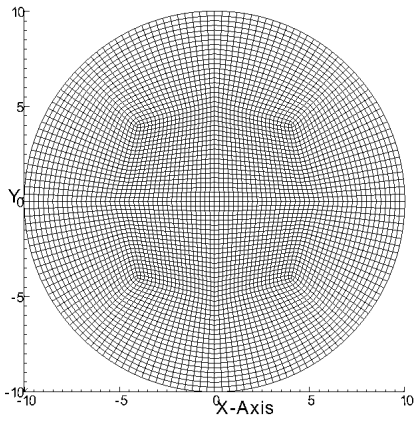


Figure 162: 4NodeANDES edge clamped circular plate with element side length 0.25m

The results were listed in Table (78).

Table 78: Results for 4NodeANDES circular plate with four edges clamped

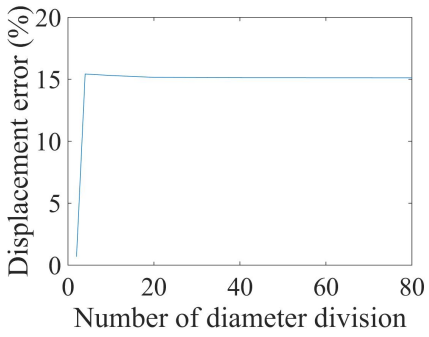
Element type	4NodeANDES	Theoretical
Element side length	Height:1.00m	displacement
10m	1.69E-003 m	1.706E-03 m
5m	1.97E-003 m	1.706E-03 m
2m	1.97E-003 m	1.706E-03 m
1m	1.96E-003 m	1.706E-03 m
0.5m	1.96E-003 m	1.706E-03 m
0.25m	1.96E-003 m	1.706E-03 m

The errors were listed in Table (79).

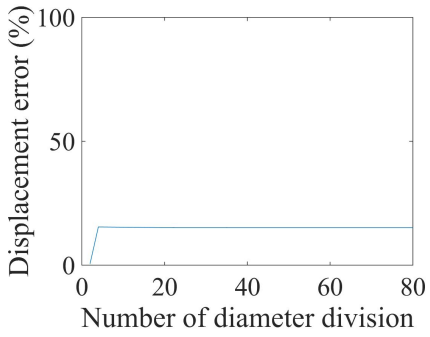
Table 79: Errors for 4NodeANDES circular plate with four edges clamped

Element type	4NodeANDES
Element side length	Height:1.00m
10m	0.71%
5m	15.43%
2m	15.31%
1m	15.16%
0.5m	15.13%
0.25m	15.12%

The errors were shown in Figure (163).



(a) Error scale 0% - 20%



(b) Error scale 0% - 100%

Figure 163: 4NodeANDES circular plate with edge clamped
Displacement error versus Number of side division

The ESSI model fei files for the table above are here

3.8 Verification of 4NodeANDES circular plate with all edges simply supported

Problem description: Diameter=20m, Height=1m, Force=100N, E=1E8Pa, $\nu = 0.3$.

The four edges are simply supported.

The load is the uniform normal pressure on the whole plate.

The plate flexural rigidity is

$$D = \frac{Eh^3}{12(1-\nu^2)} = \frac{10^8 N/m^2 \times 1^3 m^3}{12 \times (1-0.3^2)} = 9.1575 \times 10^6 \ N \cdot m \tag{44}$$

The theoretical solution[14] is

$$d = \frac{(5+\nu)qa^4}{64(1+\nu)D} = \frac{(5+0.3) \times 100 N/m^2 \times 10^4 m^4}{64 \times (1+0.3) \times 9.1575 \times 10^6 \ N \cdot m} = 6.956 \times 10^{-3} m \tag{45}$$

The 4NodeANDES were shown in Figure (164) - (169).

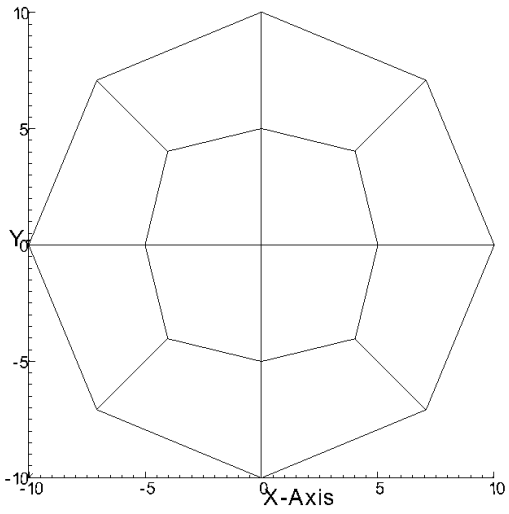


Figure 164: 4NodeANDES edge simply supported circular plate with element side length 10m

[14] Stephen Timoshenko, Theory of plates and shells (2nd edition). MrGRAW-Hill Inc, page55, 1959.

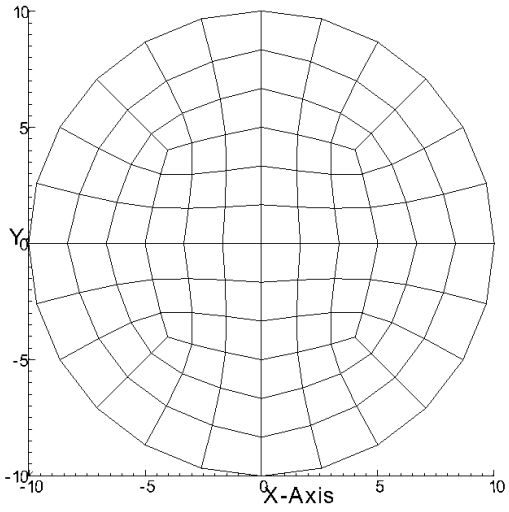


Figure 165: 4NodeANDES edge simply supported circular plate with element side length 5m

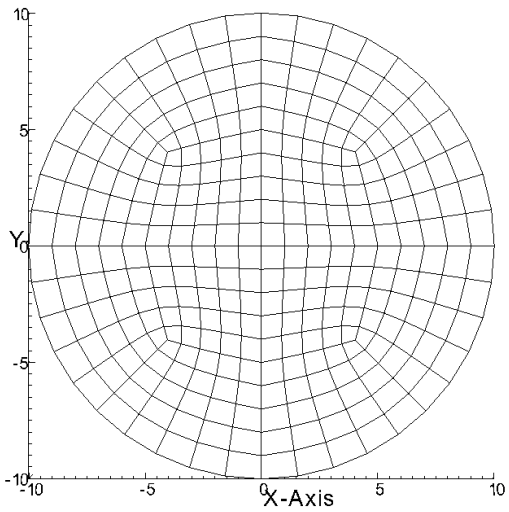


Figure 166: 4NodeANDES edge simply supported circular plate with element side length 2m

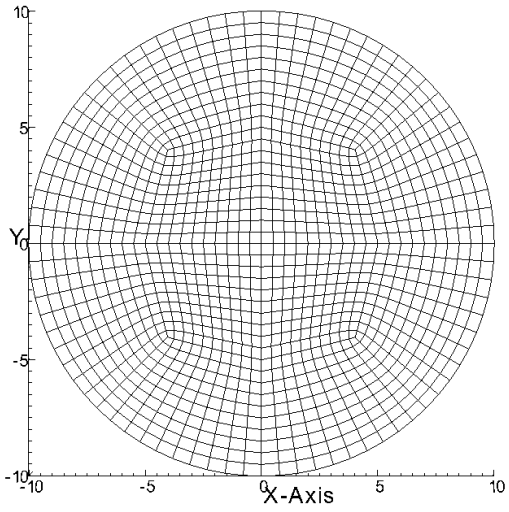


Figure 167: 4NodeANDES edge simply supported circular plate with element side length 1m

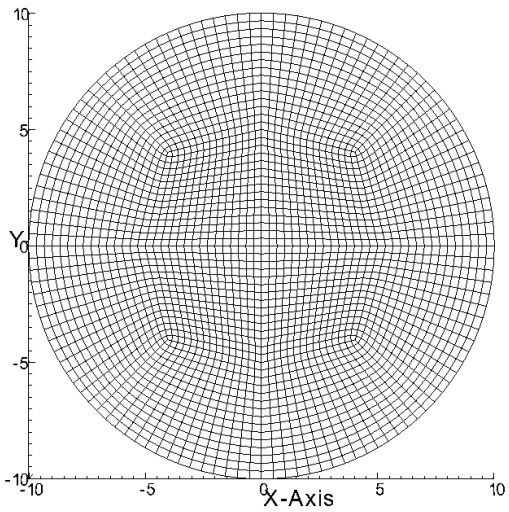


Figure 168: 4NodeANDES edge simply supported circular plate with element side length 0.5m

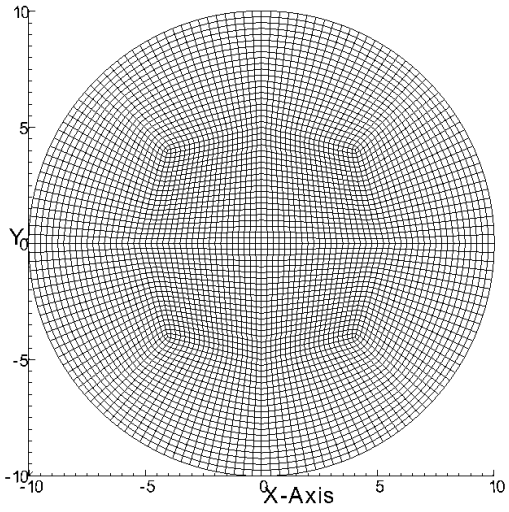


Figure 169: 4NodeANDES edge simply supported circular plate with element side length 0.25m

The results were listed in Table (80).

Table 80: Results for 4NodeANDES cicular plate with four edges simply supported

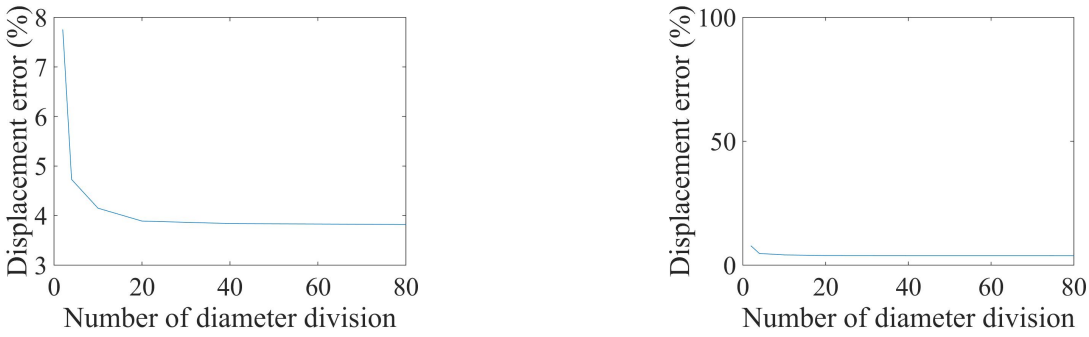
Element type	4NodeANDES	Theoretical
Element side length	Height:1.00m	displacement
10m	7.50E-003 m	6.956E-03 m
5m	7.29E-003 m	6.956E-03 m
2m	7.25E-003 m	6.956E-03 m
1m	7.23E-003 m	6.956E-03 m
0.5m	7.22E-003 m	6.956E-03 m
0.25m	7.22E-003 m	6.956E-03 m

The errors were listed in Table (81).

Table 81: Errors for 4NodeANDES cicular plate with four edges simply supported

Element type	4NodeANDES
Element side length	Height:1.00m
10m	7.75%
5m	4.73%
2m	4.15%
1m	3.89%
0.5m	3.84%
0.25m	3.82%

The errors were plotted in Figure (170).



(a) Error scale 0% - 8%

(b) Error scale 0% - 100%

Figure 170: 4NodeANDES circular plate with edge simply supported
Displacement error versus Number of side division

The ESSI model fei files for the table above are here