Manifold Learning and Sparse Representation: Assignment 13

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Exercise 250.

Proof. An easier proof than ref. [312] is offered as follows:

Obviously, the objective function is strictly convex with regard to Z. Thus, there exists a unique minimizer and we only have to verify that Eq. (11.2) is a minimizer.

We know that $\hat{Z} = U\Sigma_\lambda U^T$ is a minimizer iff.

$$0 \in \partial(\frac{1}{2}\|\phi(D) - \phi(D)\hat{Z}\|_F^2 + \lambda\|\hat{Z}\|_*) \Leftrightarrow \frac{1}{\lambda}K(I - \hat{Z}) \in \partial\|\hat{Z}\|_*,$$

where $K = \phi(D)\phi(D)^T = U\Sigma U^T$. This proposition holds due to the following fact

$$\begin{aligned}
\frac{1}{\lambda}K(I - \hat{Z}) &= \frac{1}{\lambda}U\Sigma U^T(I - U\Sigma_\lambda U^T) \\
&= \frac{1}{\lambda}U\Sigma(I - \Sigma_\lambda)U^T \\
&= \frac{1}{\lambda}U\Sigma \mathrm{diag}(\{\frac{\lambda}{\sigma_i}|\sigma_i > \lambda\}, I)U^T \\
&= U\mathrm{diag}(I, \{\frac{\sigma_i}{\lambda}|\sigma_i \leq \lambda\})U^T \\
&= U_0U_0^T + U_1WU_1^T,
\end{aligned}$$

where $W = \mathrm{diag}(\{\frac{\sigma_i}{\lambda}|\sigma_i \leq \lambda\}) = \mathrm{diag}(\{\frac{\sigma_i}{\lambda}|\frac{\sigma_i}{\lambda} \leq 1\})$ and U_0 $(U1)$ consists of eigenvectors corresponding to those eigen-values that are larger (less) than λ. We can verify that $\|W\|_2 \leq 1$ and hence $\frac{1}{\lambda}K(I - \hat{Z}) \in \partial\|\hat{Z}\|_*$ thanks to Theorem 126. □