Manifold Learning and Sparse Representation: Assignment 12

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Exercise 229.

Proof. By the method of Lagrange multipliers,

$$L = \|C\|_* + \frac{\alpha}{2}\|D - A\|_F^2 + Y^T(A - AC).$$

We assume that $A = U_r \Lambda_r V_r^T$, where r is arbitrary, while V_{-r} is a basis for the orthonormal to V_r so that $I = V_r V_r^T + V_{-r} V_{-r}^T$. Thanks to Eq. (10.56), we know C has a close-form solution, i.e., $V_r V_r^T$.

Then, by the first order conditions,

$$V_r V_r^T + W = A^T Y = V_r \Lambda_r U_r^T Y \Rightarrow W = V_r(\Lambda_r U_r^T Y - V_r^T) \Rightarrow V_{-r}^T W = 0,$$

$$Y(I - C^T) = \alpha(D - A) \Rightarrow D = A + \frac{1}{\alpha} Y V_{-r} V_{-r}^T.$$

Since we also have $V_r^T W = 0$, $W = 0$. Then $U_r^T Y = \Lambda_r^{-1} V_r^T$, which implies $Y = U_r \Lambda_r^{-1} V_r^T + U_{-r} B$ for some B. By substitution to the second equation above, we have

$$D = A + \frac{1}{\alpha}(U_r \Lambda_r^{-1} V_r^T + U_{-r} B) V_{-r} V_{-r}^T = A + \frac{1}{\alpha} U_{-r} B V_{-r} V_{-r}^T.$$

Thus, $\|D - A\|_F^2 = \frac{1}{\alpha^2}\|BV_{-r}\|_F^2$ which can be minimized when $\Lambda_{-r} = BV_{-r}$ is diagonal. This means that $D = [U_r, U_{-r}]\mathrm{diag}(\Lambda_r, \Lambda_{-r})[V_r, V_{-r}]^T = [U_r, U_{-r}]\mathrm{diag}(\Sigma_r, \Sigma_{-r})[V_r, V_{-r}]^T$ is a SVD for D. That is, V_r and U_r are top r left and right singular vectors of D, respectively. Further, the loss function can be written as the following form

$$\|C\|_* + \frac{\alpha}{2}\|D - A\|_F^2 = r + \frac{1}{\alpha}\sum_{k>r} \sigma_k^2.$$

□

Exercise 240.

Proof.

- It is trivial to verify that those three conditions hold for any subset of domain Ω if itself contains any permutation of its elements, which can be proved by mathematical induction.
- $\bigcap_{i=1}^m \Omega_i \subset \Omega_i$ for all i and then $\{f_i\}$ satisfies the EBD conditions on $\bigcap_{i=1}^m \Omega_i$, respectively. Therefore, by trivial verification, we can conclude then linear combinations of $\{f_i\}$ satisfies the EBD conditions as well.

□