Manifold Learning and Sparse Representation: Assignment 4

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Exercise 50.

Proof.

$$
\begin{aligned}
\frac{1}{2}\sum_{i,j}\pi_i p_{ij}\left(\frac{g_i}{\sqrt{\pi_i}}-\frac{g_j}{\sqrt{\pi_j}}\right)^2 &= \frac{1}{2}\sum_{i,j}\left(\pi_i p_{ij}\frac{g_i^2}{\pi_i}+\frac{g_j^2}{\pi_j}-2\frac{g_i g_j}{\sqrt{\pi_i}\sqrt{\pi_j}}\right) \\
&= \frac{1}{2}\sum_{i,j} p_{ij} g_i^2 + \frac{1}{2}\sum_{i,j}\pi_i p_{ij}\frac{g_j^2}{\pi_j} - \sum_{i,j} p_{ij}\sqrt{\pi_i}\frac{g_i g_j}{\sqrt{\pi_j}} \\
&= \frac{1}{2}\sum_{i} g_i^2 + \frac{1}{2}\sum_{j}\pi_j\frac{g_j^2}{\pi_j} - \frac{1}{2}\sum_{i,j} p_{ij}\sqrt{\pi_i}\frac{g_i g_j}{\sqrt{\pi_j}} - \frac{1}{2}\sum_{i,j} p_{ji}\sqrt{\pi_j}\frac{g_i g_j}{\sqrt{\pi_i}} \\
&= \sum_{i} g_i^2 - \frac{1}{2}\sum_{i,j} p_{ij}\sqrt{\pi_i}\frac{g_i g_j}{\sqrt{\pi_j}} - \frac{1}{2}\sum_{i,j} p_{ji}\sqrt{\pi_j}\frac{g_i g_j}{\sqrt{\pi_i}} \\
&= g^T g - \frac{1}{2}(g^T \Pi^{1/2} P \Pi^{-1/2} g + g^T \Pi^{-1/2} P^T \Pi^{1/2} g) \\
&= g^T (I - \frac{1}{2}(\Pi^{1/2} P \Pi^{-1/2} + \Pi^{-1/2} P^T \Pi^{1/2}))g \\
&= g^T (I - \Theta) g
\end{aligned}
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$\square$