Manifold Learning and Sparse Representation: Assignment 1

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Exercise 1.

The trace of A is

$$tr(A) = tr\begin{pmatrix}1 & 2 & 3\\ 2 & 5 & 5\\ 3 & 6 & 9\end{pmatrix} = 15.$$

Exercise 2.

The eigenvalue decomposition of A is

$$\begin{aligned}
A &= \begin{pmatrix}1 & 2 & 3\\ 2 & 5 & 5\\ 3 & 6 & 9\end{pmatrix}\\
&= \begin{pmatrix}-0.2615 & 0.1778 & -0.9487\\ -0.5623 & -0.8269 & 0.0000\\ -0.7845 & 0.5335 & 0.3162\end{pmatrix} \cdot diag(14.3007, 0.6993, 0) \cdot \begin{pmatrix}-0.2615 & 0.1778 & -0.9487\\ -0.5623 & -0.8269 & 0.0000\\ -0.7845 & 0.5335 & 0.3162\end{pmatrix}^T.
\end{aligned}$$

Exercise 3.

The full SVD of A is

$$\begin{aligned}
A &= \begin{pmatrix}1 & 2 & 3\\ 2 & 5 & 5\\ 3 & 6 & 9\\ 10 & 11 & 12\end{pmatrix}\\
&= \begin{pmatrix}-0.1409 & 0.8247 & 0.5477 & -0.0037\\ -0.3439 & 0.4263 & -0.7276 & 0.4131\\ -0.5470 & 0.0278 & -0.1880 & -0.8153\\ -0.7501 & -0.3706 & 0.3679 & 0.4058\end{pmatrix} \cdot \begin{pmatrix}25.4624 & 0 & 0\\ 0 & 1.2907 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \cdot \begin{pmatrix}-0.5045 & -0.7608 & -0.4082\\ -0.5745 & -0.0571 & 0.8165\\ -0.6445 & 0.6465 & -0.4082\end{pmatrix}^T.
\end{aligned}$$

The skinny SVD of A is

$$
\begin{aligned}
A &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 6 & 9 \\ 10 & 11 & 12 \end{pmatrix} \\
&= \begin{pmatrix} -0.1409 & 0.8247 & 0.5477 \\ -0.3439 & 0.4263 & -0.7276 \\ -0.5470 & 0.0278 & -0.1880 \\ -0.7501 & -0.3706 & 0.3679 \end{pmatrix} \cdot \begin{pmatrix} 25.4624 & 0 & 0 \\ 0 & 1.2907 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} -0.5045 & -0.7608 & -0.4082 \\ -0.5745 & -0.0571 & 0.8165 \\ -0.6445 & 0.6465 & -0.4082 \end{pmatrix}^T .
\end{aligned}
$$

Exercise 4.

Thanks to $P_X = X(X^T X)^{-1} X^T$, we can obtain the projection matrix associated with X as

$$
P_X = \begin{pmatrix} 0.4794 & -0.1850 & -0.2398 & 0.3973 \\ -0.1850 & 0.9343 & -0.0852 & 0.1412 \\ -0.2398 & -0.0852 & 0.8896 & 0.1830 \\ 0.3973 & 0.1412 & 0.1830 & 0.6968 \end{pmatrix}.
$$

Exercise 5.

Proof. We denote AXB by C. We have

$$
C_{i,j} = \sum_{k=1}^{p} \sum_{w=1}^{n} A_{i,k} X_{k,w} B_{w,j}.
$$

Thus, we can obtain

$$
LHS = \begin{pmatrix} C_{:,1} \\ \vdots \\ C_{:,q} \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{p} \sum_{w=1}^{n} A_{:,k} X_{k,w} B_{w,1} \\ \vdots \\ \sum_{k=1}^{p} \sum_{w=1}^{n} A_{:,k} X_{k,w} B_{w,q} \end{pmatrix} = \begin{pmatrix} \sum_{w=1}^{n} B_{w,1} A X_{:,w} \\ \vdots \\ \sum_{w=1}^{n} B_{w,q} A X_{:,w} \end{pmatrix},
$$

where we make use of the fact that $\sum_{k=1}^{p} A_{:,k} X_{k,w} = AX_{:,w}$ and $B_{w,1}, ..., B_{w,q}$ are commutative constants.

Also, by definition of the Kronecker product,

$$
B^T \otimes A = \begin{pmatrix} B_{1,1}A & \cdots & B_{n,1}A \\ \vdots & \ddots & \vdots \\ B_{1,q}A & \cdots & B_{n,q}A \end{pmatrix}.
$$

Then, we conclude by

$$
RHS = B^T \otimes A \cdot vec(X) = \begin{pmatrix} B_{1,1}A & \cdots & B_{n,1}A \\ \vdots & \ddots & \vdots \\ B_{1,q}A & \cdots & B_{n,q}A \end{pmatrix} \cdot \begin{pmatrix} X_{:,1} \\ \vdots \\ X_{:,n} \end{pmatrix} = \begin{pmatrix} \sum_{w=1}^{n} B_{w,1} A X_{:,w} \\ \vdots \\ \sum_{w=1}^{n} B_{w,q} A X_{:,w} \end{pmatrix} = LHS.
$$

□

Exercise 8.

Proof. We first note that

$$\begin{aligned} & A^{-1}U(C^{-1}+V^TA^{-1}U)^{-1}V^TA^{-1} \\ =& A^{-1}[AV^{-T}(C^{-1}+V^TA^{-1}U)U^{-1}]^{-1} \\ =& A^{-1}[(AV^{-T}C^{-1}+U)U^{-1}]^{-1} \\ =& A^{-1}[(A+UCV^T)V^{-T}C^{-1}U^{-1}]^{-1} \\ =& A^{-1}UCV^T(A+UCV^T)^{-1} \end{aligned} \tag{1}$$

By substituting Eq.(1), we can verify that

$$\begin{aligned} & [A^{-1}-A^{-1}U(C^{-1}+V^TA^{-1}U)^{-1}V^TA^{-1}]\cdot(A+UCV^T) \\ =& A^{-1}(A+UCV^T)-A^{-1}UCV^T(A+UCV^T)^{-1}(A+UCV^T) \\ =& I+A^{-1}UCV^T-A^{-1}UCV^T=I \end{aligned}$$

Thus, $(A+UCV^T)^{-1}=A^{-1}-A^{-1}U(C^{-1}+V^TA^{-1}U)^{-1}V^TA^{-1}$. □

Exercise 21.

The singular values of A denoted by σ_1,σ_2 and σ_3 are 25.4624, 1.2907 and 0, respectively.

- The 1-norm is $\max\{1+4+7+10, 2+5+8+11, 3+6+9+12\}=30$.
- The 2-norm is $\sigma_1(A)\approx 25.4624$.
- The Frobenious norm is $\sqrt{\sum_{i=1}^{3}\sigma_i^2}\approx\sqrt{25.4624^2+1.2907^2+0^2}\approx 25.4951$.
- The inf-norm is $\max\{1+2+3, 4+5+6, 7+8+9, 10+11+12\}=33$.
- The nuclear norm is $\sum_{i=1}^{3}\sigma_i\approx 25.4624+1.2907+0=26.7531$.
- The (2,1)-norm is $\sum_{i=1}^{n}\|A_i\|_2\approx 43.9445$.