Manifold Learning and Sparse Representation: Assignment 3

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Exercise 46.

The results are listed in Table 1. (The code in Matlab is attached.)

Euclidean distance	0.734042
l1 distance	0.668856
Angular distance	0.773333
Cosine distance	**0.645510**

Thus, *Cosine distance* offers the best dissimilarity metrics in terms of classification.

Exercise 48.

Prove $\frac{1}{2}\sum_{i,j} W_{ij}\|\mathbf{f_i}-\mathbf{f_j}\|^2 = tr(\mathbf{F}\mathbf{L}_W\mathbf{F}^T)$, where $\mathbf{F}=(\mathbf{f_1},...,\mathbf{f_n})$.

Proof.

$$
\begin{aligned}
\frac{1}{2}\sum_{i,j} W_{ij}\|\mathbf{f_i}-\mathbf{f_j}\|^2 &= \frac{1}{2}\sum_{i,j} W_{ij}(\mathbf{f_i}-\mathbf{f_j})^T(\mathbf{f_i}-\mathbf{f_j}) \\
&= \frac{1}{2}\sum_{i,j} W_{ij}(\mathbf{f_i^T f_i}+\mathbf{f_j^T f_j}-\mathbf{f_i^T f_j}-\mathbf{f_j^T f_i}) \\
&= \sum_i(\sum_j W_{ij})\mathbf{f_i}^T\mathbf{f_i} - \sum_{i,j} W_{ij}\mathbf{f_j}^T\mathbf{f_i} \\
&= \sum_i d_i\mathbf{f_i}^T\mathbf{f_i} - \sum_{i,j} W_{ij}\mathbf{f_j}^T\mathbf{f_i} \\
&= tr(\sum_i d_i\mathbf{f_i}^T\mathbf{f_i}) - tr(\sum_{i,j} W_{ij}\mathbf{f_j}^T\mathbf{f_i}) \\
&= tr(\sum_i \mathbf{f_i}d_i\mathbf{f_i}^T) - tr(\sum_{i,j} \mathbf{f_i}W_{ij}\mathbf{f_j}^T) \\
&= tr(\mathbf{F}\mathbf{D}_W\mathbf{F}^T) - tr(\mathbf{F}\mathbf{W}\mathbf{F}^T) \\
&= tr(\mathbf{F}(\mathbf{D}_W-\mathbf{W})\mathbf{F}^T) \\
&= tr(\mathbf{F}\mathbf{L}_W\mathbf{F}^T)
\end{aligned}
$$

□

Derive the Laplacian matrix of $\frac{1}{2}\sum_{i,j} W_{ij}(f_i - f_j)^2$ under the constrain $\sum_i W_{ii} f_i^2 = 1$.

Let $\mathbf{g} \triangleq \mathbf{D}_W^{\frac{1}{2}}\mathbf{f}$ so that the constrain $\sum_i W_{ii} f_i^2 = 1$ deduces to $\|\mathbf{g}\| = 1$. Then, we have

$$\begin{aligned}
\frac{1}{2}\sum_{i,j} W_{ij}(f_i - f_j)^2 &= \sum_i W_{ii} f_i^2 - \sum_{i,j} W_{ij} f_i f_j \\
&= 1 - \mathbf{f}^T\mathbf{W}\mathbf{f} = \mathbf{g}^T\mathbf{g} - \mathbf{g}^T\mathbf{D}^{-\frac{1}{2}}\mathbf{W}\mathbf{D}^{-\frac{1}{2}}\mathbf{g} \\
&= \mathbf{g}^T(\mathbf{I} - \mathbf{D}^{-\frac{1}{2}}\mathbf{W}\mathbf{D}^{-\frac{1}{2}})\mathbf{g}^T \\
&\triangleq \mathbf{g}^T\tilde{\mathbf{L}}_W\mathbf{g}^T
\end{aligned}$$

where the Laplaceian matrix $\tilde{\mathbf{L}}_W = \mathbf{I} - \mathbf{D}^{-\frac{1}{2}}\mathbf{W}\mathbf{D}^{-\frac{1}{2}}$.