HW1

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October 8, 2021

[13]:
```
library(ggplot2)
set.seed(1)
```

1 1.5

[25]:
```
m <- 10000
```

[42]:
```
# approach a.
x <- rcauchy(m, 0, 1)
theta <- sum(x > 2) / m
theta
```

0.1449

$X \sim \frac{1}{\pi(1+x^2)}$. The expectation of $\phi(X)$ is

$$\begin{aligned} E(\phi(X)) &= 1 \times P(\phi(X)=1) + 0 \times P(\phi(X)=0) \\ &= P(\phi(X)=1) \\ &= P(X>2) \\ &= \theta \end{aligned}$$

[46]:
```
# standard deviation
sd <- sqrt(1/(m-1) * sum(((x>2)-theta)^2))
sd
```

0.352029533532697

[44]:
```
# approach b.
theta <- 1 / 2 * sum(abs(x) > 2) / m
theta
```

0.1478

$X \sim \frac{1}{\pi(1+x^2)}$. The expectation of $\phi(X)$ is

$$
\begin{aligned}
E(\phi(X)) &= 1/2 \times P(\phi(X) = 1/2) + 0 \times P(\phi(X) = 0) \\
&= 1/2 \times (P(X > 2) + P(X < -2)) \\
&= P(X > 2) \\
&= \theta
\end{aligned}
$$

[51]:
```
# standard deviation
sd <- sqrt(1/(m-1) * sum((1/2*(abs(x)>2)-theta)^2))
sd
```

0.147642230833806

[50]:
```
# approach c.
x <- runif(m, 0, 1/2)
theta <- 1 / 2 * sum(dcauchy(x, 0, 1)) / m
theta
```

0.147634848537703

$X \sim U(0, 1/2)$. The expectation of $\phi(X)$ is

$$
\begin{aligned}
E(\phi(X)) &= 1/2 \times E(f(X)) \\
&= 1/2 \times \int_0^{1/2} \frac{2}{\pi(1+x^2)} dx
\end{aligned}
$$

Let $t = 1/x$, the above integration becomes

$$
\begin{aligned}
&= \int_{\infty}^{2} \frac{-1}{\pi(1+t^2)} dt \\
&= \int_{2}^{\infty} \frac{1}{\pi(1+t^2)} dt \\
&= P(Y > 2) \quad where\ Y \sim \frac{1}{\pi(1+y^2)} \\
&= \theta
\end{aligned}
$$

[55]:
```
# standard deviation
sd <- sqrt(1/(m-1) * sum((1/2*dcauchy(x, 0, 1)-theta)^2))
sd
```

0.00983345846102847

Approach c has the smallest standard deviation while approach a has the largest one.

2 2.4

The noninformative prior is $\pi(\mu,\sigma_x^2) \propto \frac{1}{\sigma_x^2}$, $\pi(\lambda,\sigma_y^2) \propto \frac{1}{\sigma_y^2}$. We integrate μ and λ to obtain the posterior of σ_x^2 and σ_y^2

$$\begin{aligned}
p(\sigma_x^2,\sigma_y^2|X,Y) &\propto \int\int \sigma_x^{-n-2}\exp(\frac{-\sum(x_i-\mu)^2}{2\sigma_x^2})\sigma_y^{-m-2}\exp(\frac{-\sum(y_i-\lambda)^2}{2\sigma_y^2})d\mu d\lambda \\
&= \int\int \sigma_x^{-n-2}\exp(\frac{-\sum(x_i-\bar{x})^2-n(\bar{x}-\mu)^2}{2\sigma_x^2})\sigma_y^{-m-2}\exp(\frac{-\sum(y_i-\bar{y})^2-m(\bar{y}-\lambda)^2}{2\sigma_y^2})d\mu d\lambda \\
&= \sigma_x^{-n-1}\exp(\frac{-\sum(x_i-\bar{x})^2}{2\sigma_x^2})\sigma_y^{-m-1}\exp(\frac{-\sum(y_i-\bar{y})^2}{2\sigma_y^2}) \\
&= \sigma_x^{-n-1}\exp(\frac{-(n-1)s_x^2}{2\sigma_x^2})\sigma_y^{-m-1}\exp(\frac{-(m-1)s_y^2}{2\sigma_y^2})
\end{aligned}$$

Since σ_x^2 and σ_y^2 are independent, they respectively have the inverse gamma posterior distribution as

$$\begin{aligned}
p(\sigma_x^2|X,Y) &\propto InverseGamma(\frac{n-1}{2},\frac{(n-1)s_x^2}{2}) \\
p(\sigma_y^2|X,Y) &\propto InverseGamma(\frac{m-1}{2},\frac{(m-1)s_y^2}{2})
\end{aligned}$$

Equivalently,

$$\begin{aligned}
p(\frac{1}{\sigma_x^2}|X,Y) &\propto \Gamma(\frac{n-1}{2},\frac{(n-1)s_x^2}{2}) \\
p(\frac{1}{\sigma_y^2}|X,Y) &\propto \Gamma(\frac{m-1}{2},\frac{(m-1)s_y^2}{2})
\end{aligned}$$

Therefore, $\frac{(n-1)s_x^2}{\sigma_x^2} \sim \chi^2(n-1)$, $\frac{(m-1)s_y^2}{\sigma_y^2} \sim \chi^2(m-1)$ and they are independent. So

$$F = \frac{\frac{(n-1)s_x^2}{\sigma_x^2(n-1)}}{\frac{(m-1)s_y^2}{\sigma_y^2(m-1)}} = \frac{\sigma_y^2/s_y^2}{\sigma_x^2/s_x^2} \sim F(n-1,m-1)$$

3 2.6

$\theta = (\mu,\sigma^2)$, the likelihood is

$$\begin{aligned}
L(Y|\theta) &= \frac{1}{(2\pi\sigma^2)^{n/2}}\exp(\frac{-\sum(y_i-\mu)^2}{2\sigma^2}) \\
l(Y|\theta) &= \frac{-n}{2}\log(2\pi\sigma^2) - \frac{\sum(y_i-\mu)^2}{2\sigma^2}
\end{aligned}$$

The first partial derivatives is

$$\begin{aligned}
\frac{\partial l}{\partial \mu} &= \frac{\sum(y_i-\mu)}{\sigma^2} \\
\frac{\partial l}{\partial \sigma^2} &= \frac{-n}{2\sigma^2} + \frac{\sum(y_i-\mu)^2}{2\sigma^4}
\end{aligned}$$

Since Fisher score has the property

$$E\left[\left(\frac{\partial l}{\partial \theta}\right)^2\right] = -E\left[\frac{\partial^2 l}{\partial \theta^2}\right]$$

We can calculate

$$\begin{pmatrix} \frac{\sum(y_i-\mu)}{\sigma^2} \\ \frac{-n}{2\sigma^2} + \frac{\sum(y_i-\mu)^2}{2\sigma^4} \end{pmatrix} \begin{pmatrix} \frac{\sum(y_i-\mu)}{\sigma^2} & \frac{-n}{2\sigma^2} + \frac{\sum(y_i-\mu)^2}{2\sigma^4} \end{pmatrix} = \begin{pmatrix} \frac{[\sum(y_i-\mu)]^2}{\sigma^4} & \frac{-n\sum(y_i-\mu)}{2\sigma^4} + \frac{\sum(y_i-\mu)\sum(y_i-\mu)^2}{2\sigma^6} \\ \frac{-n\sum(y_i-\mu)}{2\sigma^4} + \frac{\sum(y_i-\mu)\sum(y_i-\mu)^2}{2\sigma^6} & \frac{n^2}{4\sigma^4} - \frac{n\sum(y_i-\mu)^2}{2\sigma^6} + \frac{[\sum(y_i-\mu)^2]^2}{4\sigma^8} \end{pmatrix}$$

Then we calculate the expectation respectively

$$\begin{aligned} E\left(\frac{[\sum(y_i-\mu)]^2}{\sigma^4}\right) &= \frac{E\left((\sum y_i)^2 - 2n\mu\sum y_i + n^2\mu^2\right)}{\sigma^4} \\ &= \frac{E((\sum y_i)^2) - n^2\mu^2}{\sigma^4} \\ &= \frac{Var(\sum y_i) + (E(\sum y_i))^2 - n^2\mu^2}{\sigma^4} \\ &= \frac{n\sigma^2 + n^2\mu^2 - n^2\mu^2}{\sigma^4} \\ &= \frac{n}{\sigma^2} \end{aligned}$$

and

$$\begin{aligned} &E\left(\frac{-n\sum(y_i-\mu)}{2\sigma^4} + \frac{\sum(y_i-\mu)\sum(y_i-\mu)^2}{2\sigma^6}\right) = E\left(\frac{\sum(y_i-\mu)\sum(y_i-\mu)^2}{2\sigma^6}\right) \\ &= \frac{E\left((\sum y_i - n\mu)(\sum y_i^2 - 2\mu\sum y_i + n\mu^2)\right)}{2\sigma^6} \\ &= \frac{E(\sum y_i \sum y_i^2) - n\mu E(\sum y_i^2) - 2\mu E((\sum y_i)^2) + 2n\mu^2 E(\sum y_i) + n\mu^2 E(\sum y_i) - n^2\mu^3}{2\sigma^6} \\ &= \frac{E(\sum y_i^2 + \sum_{i\neq j} y_i y_j^2) - n\mu(n\mu^2 + n\sigma^2) - 2\mu(n^2\mu^2 + n^2\sigma^2) + 2n^2\mu^3 + n^2\mu^3 - n^2\mu^3}{2\sigma^6} \\ &= \frac{n\mu^3 + 3n\mu\sigma^2 + n(n-1)\mu(\mu^2+\sigma^2) - n\mu(n\mu^2 + n\sigma^2) - 2\mu n\sigma^2}{2\sigma^6} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} E(\frac{n^2}{4\sigma^4} - \frac{n\sum(y_i-\mu)^2}{2\sigma^6} + \frac{[\sum(y_i-\mu)^2]^2}{4\sigma^8}) &= \frac{n^2}{4\sigma^4} - \frac{n^2E(((y-\mu))^2)}{2\sigma^6} + \frac{E((\sum(y_i-\mu)^2)^2)}{4\sigma^8} \\ &= \frac{n^2}{4\sigma^4} - \frac{n^2\sigma^2}{2\sigma^6} + \frac{(E(\sum(y_i-\mu)^2))^2 + Var(\sum(y_i-mu)^2)}{4\sigma^8} \\ &= \frac{n^2}{4\sigma^4} - \frac{n^2\sigma^2}{2\sigma^6} + \frac{\left[nE((y-\mu)^2)\right]^2 + nVar((y_i-\mu)^2)}{4\sigma^8} \\ &= \frac{n^2}{4\sigma^4} - \frac{n^2\sigma^2}{2\sigma^6} + \frac{n^2\sigma^4 + n\left(E((y-\mu)^4) - [E((y-\mu)^2)]^2\right)}{4\sigma^8} \\ &= \frac{n^2}{4\sigma^4} - \frac{n^2\sigma^2}{2\sigma^6} + \frac{n^2\sigma^4 + n(3\sigma^4 - \sigma^4)}{4\sigma^8} \\ &= \frac{n}{2\sigma^4} \end{aligned}$$

The fisher information matrix is

$$\begin{pmatrix} \frac{n}{\sigma^2} & 0 \\ 0 & \frac{n}{2\sigma^4} \end{pmatrix}$$

so the invariant prior is

$$\pi(\mu, \sigma^2) \propto \frac{1}{\sigma^3}$$

4 2.12

The likelihood is

$$\begin{aligned} L(Y|\theta, \sigma^2) &= (\frac{1}{\sqrt{2\pi}})^n \sigma^{-n} \exp\{\frac{-1}{2\sigma^2}(Y - X\theta)'(Y - X\theta)\} \\ &= (\frac{1}{\sqrt{2\pi}})^n \sigma^{-n} \exp\{\frac{-1}{2\sigma^2}(vs^2 + (\theta - \hat{\theta})'X'X(\theta - \hat{\theta}))\} \end{aligned}$$

where $v = n - d$, $s^2 = (Y - \hat{Y})'(Y - \hat{Y})/v$, $vs^2 =$SSE.
Since we need to obtain the marginal posterior distribution of θ, we need to integrate out σ^2 as

$$p(\theta|Y) = \int p(\theta, \sigma^2|Y) d\sigma^2$$

The joint posterior is

$$\begin{aligned} P(\theta, \sigma^2|Y) &\propto L(Y|\theta, \sigma^2)\pi(\theta, \sigma^2) \\ &\propto \sigma^{-n-2} \exp\{\frac{-1}{2\sigma^2}(vs^2 + (\theta - \hat{\theta})'X'X(\theta - \hat{\theta}))\} \end{aligned}$$

The marginal posterior distribution of θ is

$$\begin{aligned} p(\theta|Y) &\propto \int_0^\infty \sigma^{-n-2} \exp\{\frac{-1}{2\sigma^2}(vs^2 + (\theta - \hat{\theta})'X'X(\theta - \hat{\theta}))\} d\sigma^2 \\ &= \int_0^\infty \phi^{\frac{n-2}{2}} \exp\{\frac{-1}{2}(vs^2 + (\theta - \hat{\theta})'X'X(\theta - \hat{\theta}))\phi\} d\phi \quad (\phi = \frac{1}{\sigma^2}) \\ &= (\frac{vs^2 + (\theta - \hat{\theta})'X'X(\theta - \hat{\theta})}{2})^{-\frac{n}{2}} \int_0^\infty t^{\frac{n-2}{2}} e^{-t} dt \quad (t = \frac{vs^2 + (\theta - \hat{\theta})'X'X(\theta - \hat{\theta})}{2}\phi) \\ &= \Gamma(\frac{n}{2})(\frac{vs^2 + (\theta - \hat{\theta})'X'X(\theta - \hat{\theta})}{2})^{-\frac{n}{2}} \\ &= \Gamma(\frac{n}{2})(\frac{vs^2}{2})^{-\frac{n}{2}}(1 + \frac{(\theta - \hat{\theta})'X'X(\theta - \hat{\theta})}{vs^2})^{-\frac{n}{2}} \end{aligned}$$

Since the density of the multivariate t distribution follows the form

$$\frac{\Gamma((v + d)/2)}{\Gamma(v/2)v^{d/2}\pi^{d/2}|\Sigma|^{1/2}}(1 + \frac{(x - \mu)'\Sigma^{-1}(x - \mu)}{v})^{-n/2}$$

Here, $\Sigma^{-1} = \frac{X'X}{s^2}$, $\sqrt{\pi} = \Gamma(\frac{1}{2})$, $x = \theta$ and $\mu = \hat{\theta}$.
Therefore, the above marginal distribution is

$$p(\theta|Y) = \frac{\Gamma(\frac{n}{2})|X'X|^{1/2}s^{-d}}{(\Gamma(1/2))^d\Gamma(v/2)v^{d/2}}(1 + \frac{(\theta - \hat{\theta})'X'X(\theta - \hat{\theta})}{vs^2})^{-\frac{n}{2}}$$

which is said to be the multivariate t distribution.

5 2.13

Solution 1

From result 2.1.1, $p(\theta,\sigma^2|Y) = p(\sigma^2|s^2) \times p(\theta|\hat{\theta},\sigma^2)$ where the marginal distribution of σ^2 is $vs^2\chi^{-2}(v)$ and the conditional distribution θ, given σ^2, is $N(\hat{\theta},\sigma^2(X'X)^{-1})$.
Therefore,

$$\begin{aligned}
\frac{(\theta-\hat{\theta})'X'X(\theta-\hat{\theta})}{ds^2} &= \frac{(\theta-\hat{\theta})'X'X(\theta-\hat{\theta})/\sigma^2 d}{s^2/\sigma^2} \\
&= \frac{\chi^2(d)/d}{s^2\chi^2(v)/s^2 v} \\
&= \frac{\chi^2(d)/d}{\chi^2(v)/v} \\
&= F(d,v)
\end{aligned}$$

Solution 2

(From wikipedia)The definition of multivariate t distribution, for the case of p dimensions is, based on the observation that if y and u are independent and distributed as $N(0,\Sigma)$ and $\chi^2(v)$ respectively, and $y/\sqrt{u/v} = x-\mu$, then x has the density

$$\frac{\Gamma((v+p)/2)}{\Gamma(v/2)v^{p/2}\pi^{p/2}|\Sigma|^{1/2}}(1+\frac{(x-\mu)'\Sigma^{-1}(x-\mu)}{v})^{-(v+p)/2}$$

and is said to be distributed as a multivariate t distribution with parameters Σ, μ, v.
So

$$\begin{aligned}
\frac{(x-\mu)'\Sigma^{-1}(x-\mu)}{p} &= \frac{v}{u}\frac{y'\Sigma^{-1}y}{p} \\
&= \frac{y'\Sigma^{-1}y/p}{u/v} \\
&= \frac{\chi^2(p)}{\chi^2(v)} \\
&= F(p,v)
\end{aligned}$$

Here, $\Sigma^{-1} = \frac{X'X}{s^2}$, $p = d$, $x = \theta$ and $\mu = \hat{\theta}$, hence

$$\frac{(\theta-\hat{\theta})'X'X(\theta-\hat{\theta})}{ds^2} \sim F(d,v)$$

(Actually the above two solutions almost have the same meaning.)

6 2.14

6.1 a

The pdf of π is

$$f(\pi) = \frac{\pi^{\alpha-1}(1-\pi)^{\beta-1}}{B(\alpha,\beta)}$$

Since $\pi = \frac{\lambda\alpha}{\lambda\alpha+\beta}$, the pdf of λ is

$$
\begin{aligned}
g(\lambda) &= \frac{(\frac{\lambda\alpha}{\lambda\alpha+\beta})^{\alpha-1}(\frac{\beta}{\lambda\alpha+\beta})^{\beta-1}}{B(\alpha,\beta)}\frac{\alpha\beta}{(\lambda\alpha+\beta)^2} \\
&= \frac{\lambda^{\alpha-1}\alpha^\alpha\beta^\beta}{(\lambda\alpha+\beta)^{\alpha+\beta}B(\alpha,\beta)} \sim F(2\alpha,2\beta)
\end{aligned}
$$

That is, λ has the F distribution.

6.2 b

Since $\lambda = e^{2\delta}$, the pdf of δ is

$$
\begin{aligned}
p(\delta) &= \frac{e^{2\delta(\alpha-1)}\alpha^\alpha\beta^\beta}{B(\alpha,\beta)(e^{2\delta}\alpha+\beta)^{\alpha+\beta}}2e^{2\delta} \\
&= \frac{2(2\alpha)^\alpha(2\beta)^\beta e^{2\delta\alpha}}{B(\alpha,\beta)(2\beta+2\alpha e^{2\delta})^{\alpha+\beta}}
\end{aligned}
$$

6.3 c

Let

$$
\begin{aligned}
P(\delta) &= p(\delta)d\delta \\
&= \frac{2(2\alpha)^\alpha(2\beta)^\beta e^{2\delta\alpha}}{B(\alpha,\beta)(2\beta+2\alpha e^{2\delta})^{\alpha+\beta}}d\delta
\end{aligned}
$$

when α and β are large, we can use Stirling's approximation of the Beta function

$$
\begin{aligned}
P(\delta) &= \frac{2(2\alpha)^\alpha(2\beta)^\beta(\alpha+\beta)^{\alpha+\beta-1/2}}{\sqrt{2\pi}(2\beta+2\alpha e^{2\delta})^{\alpha+\beta}\alpha^{\alpha-1/2}\beta^{\beta-1/2}}e^{2\alpha\delta}d\delta \\
&= \frac{(2\alpha+2\beta)^{\alpha+\beta}}{\sqrt{2\pi}(2\beta+2\alpha e^{2\delta})^{\alpha+\beta}\sqrt{\frac{1}{2}(\frac{1}{2\alpha}+\frac{1}{2\beta})}}e^{2\alpha\delta}d\delta
\end{aligned}
$$

let $\mu = \frac{1}{2}\log\left[(1-\frac{1}{2\alpha})/(1-\frac{1}{2\beta})\right]$, $\sigma^2 = \frac{1}{4}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)$, since we want to prove they are approximately the mean and variance of $p(\delta)$, we only need to prove that $t = \frac{\delta-\mu}{\sigma}$ has the standard normal distribution.
It follows

$$
\begin{aligned}
P(\delta) &= \frac{(2\alpha+2\beta)^{\alpha+\beta}}{\sqrt{2\pi}\sigma(2\beta+2\alpha e^{2\delta})^{\alpha+\beta}}e^{2\alpha\delta}d\delta \\
P(t) &= \frac{1}{\sqrt{2\pi}}\left(\frac{2\alpha+2\beta}{2\alpha e^{2(t\sigma+\mu)}+2\beta}\right)^{\alpha+\beta}e^{2\alpha(t\sigma+\mu)}dt \\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{2\alpha+2\beta}{2\alpha e^{2\beta(t\sigma+\mu)/(\alpha+\beta)}+2\beta e^{-2\alpha(t\sigma+\mu)/(\alpha+\beta)}}\right)^{\alpha+\beta}dt
\end{aligned}
$$

Use Taylor expansion for the denominator,

$$
\begin{aligned}
2\alpha e^{2\beta(t\sigma+\mu)/(\alpha+\beta)} + 2\beta e^{-2\alpha(t\sigma+\mu)/(\alpha+\beta)} &= 2\alpha(1+\frac{2\beta(t\sigma+\mu)}{\alpha+\beta} + \frac{2\beta^2(t\sigma+\mu)^2}{(\alpha+\beta)^2} + O(\frac{\beta^3}{(\alpha+\beta)^3})) \\
&+ 2\beta(1-\frac{2\alpha(t\sigma+\mu)}{\alpha+\beta} + \frac{2\alpha^2(t\sigma+\mu)^2}{(\alpha+\beta)^2} + O(\frac{\alpha^3}{(\alpha+\beta)^3})) \\
&= 2\alpha + 2\beta + \frac{4\alpha\beta(t\sigma+\mu)^2}{\alpha+\beta}
\end{aligned}
$$

So $P(t)$ becomes

$$P(t) = \frac{1}{\sqrt{2\pi}}(1+\frac{2\alpha\beta(t\sigma+\mu)^2}{(\alpha+\beta)^2})^{-(\alpha+\beta)}dt$$

Use logarithms

$$
\begin{aligned}
-(\alpha+\beta)\log\left(1+\frac{2\alpha\beta(t\sigma+\mu)^2}{(\alpha+\beta)^2}\right) &= -(\alpha+\beta)\left(\frac{2\alpha\beta(t\sigma+\mu)^2}{(\alpha+\beta)^2} - \frac{1}{2}\left(\frac{2\alpha\beta(t\sigma+\mu)^2}{(\alpha+\beta)^2}\right)^2 + \right. \\
&\left. \sum_{r=3}^{\infty}\frac{(-1)^{r+1}}{r}\left(\frac{2\alpha\beta(t\sigma+\mu)^2}{(\alpha+\beta)^2}\right)^r\right) \\
&= -\frac{t^2\sigma^2+2t\sigma\mu+\mu^2}{2\sigma^2} + \frac{2\alpha^2\beta^2(t\sigma+\mu)^4}{(\alpha+\beta)^3} + \sum_{r=3}^{\infty}(-1)^r\frac{[2\alpha\beta(t\sigma+\mu)^2]^r}{r(\alpha+\beta)^{2r-1}} \\
&= -\frac{t^2}{2} - \frac{t\mu}{\sigma} - \frac{\mu^2}{2\sigma^2} + \frac{2\alpha^2\beta^2}{(\alpha+\beta)^2}\frac{(t\sigma+\mu)^4}{\alpha+\beta} + \sum_{r=3}^{\infty}(-1)^r\frac{[2\alpha\beta(t\sigma+\mu)^2]^r}{r(\alpha+\beta)^{2r-1}}
\end{aligned}
$$

Let $U = \mu/\sigma$, obviously,

$$\mu = \frac{1}{2}\log\left[(1-\frac{1}{2\alpha})/(1-\frac{1}{2\beta})\right] = \frac{1}{2}\left[\log(1-\frac{1}{2\alpha}) - \log(1-\frac{1}{2\beta})\right] < \frac{1}{2}(\frac{1}{2\beta}-\frac{1}{2\alpha}) < \frac{1}{4}(\frac{1}{\beta}+\frac{1}{\alpha}) = \sigma^2$$

so $U < \sigma$. Since $\lim\sigma = 0$,$\lim\mu = 0$, we have $\lim U = 0$ and $\lim U^2 = 0$. Consider $\frac{\alpha^2+\beta^2(t\sigma+\mu)^4}{(\alpha+\beta)^3} = \frac{(t\sigma+\mu)^4}{16(\alpha+\beta)\sigma^4} = \frac{(t+U)^4}{16(\alpha+\beta)}$, and $\lim\frac{(t+U)^4}{16(\alpha+\beta)} = 0$. Similarly, $\lim\sum_{r=3}^{\infty}(-1)^r\frac{[2\alpha\beta(t\sigma+\mu)^2]^r}{r(\alpha+\beta)^{2r-1}} = 0$.
Hence,

$$P(t) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{t^2}{2})dt$$

This completes the proof.
Since $\log(\pi/(1-\pi)) = 2\delta - \log(\beta/\alpha)$, this is a linear transformation, so its mean and variance are

$$
\begin{aligned}
&2\times\frac{1}{2}\log\left[(1-\frac{1}{2\alpha})/(1-\frac{1}{2\beta})\right] - \log(\beta/\alpha) = \log\left[(\alpha-\frac{1}{2})/(\beta-\frac{1}{2})\right] \\
&4\times\frac{1}{4}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right) = \left(\frac{1}{\alpha}+\frac{1}{\beta}\right)
\end{aligned}
$$

6.4 d

From question c, we can use normal approximation for π and ρ, so the mean and variance of $\log(\pi/(1-\pi))$ and $\log(\rho/(1-\rho))$ are

$$\mu_\pi = \log\left[(\alpha_0 + x - \frac{1}{2})/(\beta_0 + m - x - \frac{1}{2})\right]$$
$$\mu_\rho = \log\left[(\gamma_0 + y - \frac{1}{2})/(\delta_0 + n - y - \frac{1}{2})\right]$$
$$\sigma_\pi^2 = \left(\frac{1}{\alpha_0 + x} + \frac{1}{\beta_0 + m - x}\right)$$
$$\sigma_\rho^2 = \left(\frac{1}{\gamma_0 + y} + \frac{1}{\delta_0 + n - y}\right)$$

Since they are independent, the mean and variance of $\log(\pi/(1-\pi)) - \log(\rho/(1-\rho))$ is

$$\mu_1 = \log\left[\frac{(\alpha_0 + x - \frac{1}{2})(\delta_0 + n - y - \frac{1}{2})}{(\beta_0 + m - x - \frac{1}{2})(\gamma_0 + y - \frac{1}{2})}\right]$$
$$\sigma_1^2 = \frac{1}{\alpha_0 + x} + \frac{1}{\beta_0 + m - x} + \frac{1}{\gamma_0 + y} + \frac{1}{\delta_0 + n - y}$$

7 Reference

[1]Clyde, M., Rundel, M. C., Rundel, C., Banks, D., Chai, C., & Huang, L. (2020). An Introduction to Bayesian Thinking-A Companion to the Statistics with R Course. GitHub repository: GitHub.

[2]Williamson, P. P. (2009). Bayesian Alternatives to the F Test for Two Population Variances. Communications in Statistics-Theory and Methods, 38(14), 2288-2301.

[3]Aroian, L. A. (1941). A study of RA Fisher's z distribution and the related F distribution. The Annals of Mathematical Statistics, 12(4), 429-448.

[4]Wikipedia contributors. (2021, September 15). Multivariate t-distribution. In Wikipedia, The Free Encyclopedia. Retrieved 08:38, October 8, 2021, from https://en.wikipedia.org/w/index.php?title=Multivariate_t-distribution&oldid=1044413090