Multivariate Causal Models

Congyuan Duan

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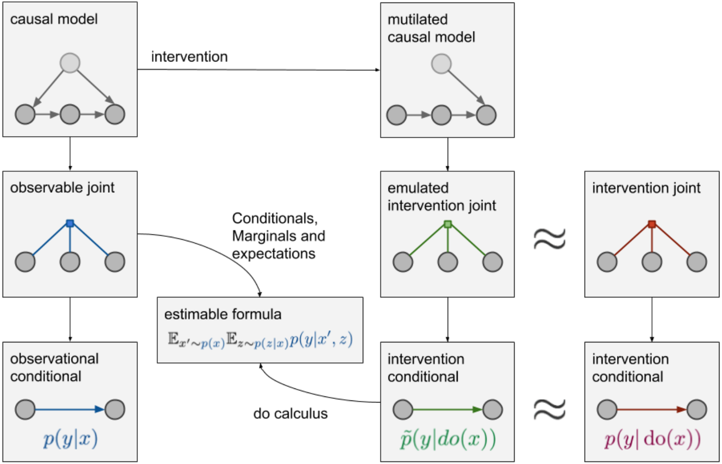
Calculating intervention distributions

Problem: the intervention distribution is unknown, and we do not have data from it.



Independence assumption

If we intervene on a variable, then the other mechanisms remain invariant. Given an SCM $\mathfrak{C}$, and intervening on X_k but not on X_j, we have

$$p^{\tilde{\mathfrak{C}}}(x_j | x_{pa(j)}) = p^{\mathfrak{C}}(x_j | x_{pa(j)})$$

This allows us to compute statements about intervention distributions even though we have never seen data from it.

Truncated factorization

Consider an SCM $\mathfrak{C}$ with structural assignments

$$X_j := f_j(X_{pa(j)}, N_j), \qquad j = 1, \ldots, d,$$

and density $p^{\mathfrak{C}}$. Because of the Markov property, we have[13]

$$p^{\mathfrak{C}}(x_1, \ldots, x_d) = \prod_{j=1}^{d} p^{\mathfrak{C}}(x_j \,|\, x_{pa(j)}).$$

Now consider the SCM $\tilde{\mathfrak{C}}$ that evolves from $\mathfrak{C}$ after $do\left(X_k := \tilde{N}_k\right)$, where $\tilde{N}_k$ allows for the density $\tilde{p}$. Again, it follows from the Markov assumption that

$$\begin{aligned} p^{\mathfrak{C}; do\left(X_k := \tilde{N}_k\right)}(x_1, \ldots, x_d) &= \prod_{j \neq k} p^{\mathfrak{C}; do\left(X_k := \tilde{N}_k\right)}(x_j \,|\, x_{pa(j)}) \cdot p^{\mathfrak{C}; do\left(X_k := \tilde{N}_k\right)}(x_k) \\ &= \prod_{j \neq k} p^{\mathfrak{C}}(x_j \,|\, x_{pa(j)}) \tilde{p}(x_k). \end{aligned} \tag{6.8}$$

In the last step, we make use of the powerful invariance (6.7). Equation (6.8) allows us to compute an interventional statement (left-hand side) from observational quantities (right-hand side). As a special case, we obtain

$$p^{\mathfrak{C}; do(X_k := a)}(x_1, \ldots, x_d) = \begin{cases} \prod_{j \neq k} p^{\mathfrak{C}}(x_j \,|\, x_{pa(j)}) & \text{if } x_k = a \\ 0 & \text{otherwise.} \end{cases} \tag{6.9}$$

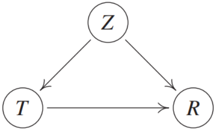
Special case

Conditioning and intervening are different operations, but they become identical for variables that do not have any parents.

$$\begin{aligned} p^{\mathfrak{C}}(x_2, \ldots, x_d \,|\, x_1 = a) &= \frac{p(x_1 = a) \prod_{j=2}^{d} p^{\mathfrak{C}}(x_j \,|\, x_{pa(j)})}{p(x_1 = a)} \\ &= p^{\mathfrak{C}; do(X_1 := a)}(x_2, \ldots, x_d). \end{aligned}$$

Example: Kidney stones, continued

	Overall	Patients with small stones	Patients with large stones
Treatment *a*: Open surgery	78% (273/350)	**93%** (81/87)	**73%** (192/263)
Treatment *b*: Percutaneous nephrolithotomy	**83%** (289/350)	87% (234/270)	69% (55/80)



we should base our choice of treatment on a comparison between

$$\mathbb{E}^{\mathfrak{C}_A} R = P^{\mathfrak{C}_A}(R=1) = P^{\mathfrak{C};do(T:=A)}(R=1)$$

and

$$\mathbb{E}^{\mathfrak{C}_B} R = P^{\mathfrak{C}_B}(R=1) = P^{\mathfrak{C};do(T:=B)}(R=1).$$

Example: Kidney stones, continued

$$
\begin{aligned}
P^{\mathfrak{C}_A}(R=1) &= \sum_{z=0}^{1} P^{\mathfrak{C}_A}(R=1, T=A, Z=z) \\
&= \sum_{z=0}^{1} P^{\mathfrak{C}_A}(R=1 \mid T=A, Z=z)\, P^{\mathfrak{C}_A}(T=A, Z=z) \\
&= \sum_{z=0}^{1} P^{\mathfrak{C}_A}(R=1 \mid T=A, Z=z)\, P^{\mathfrak{C}_A}(Z=z) \\
&\overset{(6.7)}{=} \sum_{z=0}^{1} P^{\mathfrak{C}}(R=1 \mid T=A, Z=z)\, P^{\mathfrak{C}}(Z=z).
\end{aligned}
$$

The last equation is called ”adjusting” for the variable Z.

Example: Kidney stones, continued

	Overall	Patients with small stones	Patients with large stones
Treatment *a*: Open surgery	78% (273/350)	**93%** (81/87)	**73%** (192/263)
Treatment *b*: Percutaneous nephrolithotomy	**83%** (289/350)	87% (234/270)	69% (55/80)

With the data in the table

$$P^{\mathfrak{C}_A}(R=1) \approx 0.93 \cdot \frac{357}{700} + 0.73 \cdot \frac{343}{700} = 0.832.$$

$$P^{\mathfrak{C}_B}(R=1) \approx 0.87 \cdot \frac{357}{700} + 0.69 \cdot \frac{343}{700} \approx 0.782,$$

This example also shows the difference between conditioning and intervention.

$$p^{\mathfrak{C};do(T:=t)}(r) = \sum_z p^{\mathfrak{C}}(r|z,t) p^{\mathfrak{C}}(z) \neq \sum_z p^{\mathfrak{C}}(r|z,t) p^{\mathfrak{C}}(z|t) = p^{\mathfrak{C}}(r|t).$$

$$P^{\mathfrak{C}}(R=1 \mid T=A) - P^{\mathfrak{C}}(R=1 \mid T=B) = 0.78 - 0.83,$$

Valid adjustment set and confounding

Definition 6.38 (Valid adjustment set) *Consider an SCM $\mathfrak{C}$ over nodes $\mathbf{V}$ and let $Y \notin \mathbf{PA}_X$ (otherwise we have $p^{\mathfrak{C};do(X:=x)}(y) = p^{\mathfrak{C}}(y)$). We call a set $\mathbf{Z} \subseteq \mathbf{V} \setminus \{X, Y\}$ a valid adjustment set for the ordered pair (X, Y) if*

$$p^{\mathfrak{C};do(X:=x)}(y) = \sum_{\mathbf{z}} p^{\mathfrak{C}}(y \mid x, \mathbf{z})\, p^{\mathfrak{C}}(\mathbf{z}). \tag{6.13}$$

Here, the sum (could also be an integral) is over the range of $\mathbf{Z}$, that is, over all values $\mathbf{z}$ that $\mathbf{Z}$ can take.

When the empty set is not a valid adjustment set, we say that the causal effect is confounded.

Definition 6.39 (Confounding) *Consider an SCM $\mathfrak{C}$ over nodes $\mathbf{V}$ with a directed path from X to Y, $X, Y \in \mathbf{V}$. The causal effect from X to Y is called confounded if*

$$p^{\mathfrak{C};do(X:=x)}(y) \neq p^{\mathfrak{C}}(y \mid x). \tag{6.14}$$

Otherwise, the causal effect is called "unconfounded."

Valid adjustment set and confounding

In the last example, for any set Z,

$$\begin{aligned} p^{\mathfrak{C};do(X:=x)}(y) &= \sum_z p^{\mathfrak{C};do(X:=x)}(y,z) \\ &= \sum_z p^{\mathfrak{C};do(X:=x)}(y|x,z)\, p^{\mathfrak{C};do(X:=x)}(z) \end{aligned}$$

If we have

$$\begin{aligned} p^{\mathfrak{C};do(X:=x)}(y|x,z) &= p^{\mathfrak{C}}(y|x,z) \\ p^{\mathfrak{C};do(X:=x)}(z) &= p^{\mathfrak{C}}(z) \end{aligned}$$

it follows that Z is a valid adjustment set. We thus need to address the question of which conditionals remain invariant under the intervention $do(X := x)$.

Characterization of invariant conditionals

We construct a new SCM $\mathfrak{C}^*$ that equals $\mathfrak{C}$ but has one more variable I that indicates whether the intervention took place or not. I is a parent of X_k and does not have any other neighbors.

$$
\begin{aligned}
I &:= N_I \\
X_j &:= f_j(\mathbf{PA}_j, N_j) \quad \text{for } j \neq k \\
X_k &:= \begin{cases} f_k(\mathbf{PA}_k, N_k) & \text{if } I = 0 \\ x_k & \text{otherwise} \end{cases}
\end{aligned}
$$

$I = 0$ corresponds to the observational setting and $I = 1$ to the interventional setting. We have

$$
\begin{aligned}
p^{\mathfrak{C}^*}(x_1, \cdots, x_d | I = 0) &= p^{\mathfrak{C}^*; do(I:=0)}(x_1, \cdots, x_d) \\
&= p^{\mathfrak{C}^*}(x_1, \cdots, x_d)
\end{aligned}
$$

and

$$
p^{\mathfrak{C}^*}(x_1, \cdots, x_d | I = 1) = p^{\mathfrak{C}^*; do(X_k:=x_k)}(x_1, \cdots, x_d)
$$

Characterization of invariant conditionals

Using the Markov condition for $\mathfrak{C}^*$, it thus follows for variables A and a set of variables B that

$$\begin{aligned} A \perp\!\!\!\perp_{\mathcal{G}^*} I \,|\, \mathbf{B} \quad &\Longrightarrow \quad p^{\mathfrak{C}^*}(a \,|\, \mathbf{b}, I = 0) = p^{\mathfrak{C}^*}(a \,|\, \mathbf{b}, I = 1) \\ &\Longrightarrow \quad p^{\mathfrak{C}}(a \,|\, \mathbf{b}) = p^{\mathfrak{C}; do(X_k := x_k)}(a \,|\, \mathbf{b}). \end{aligned}$$

So we need to find Z such that

$$Y \perp\!\!\!\perp_{\mathcal{G}^*} I \,|\, X, \mathbf{Z} \qquad \text{and} \qquad \mathbf{Z} \perp\!\!\!\perp_{\mathcal{G}^*} I.$$

Valid adjustment sets

Proposition 6.41 (Valid adjustment sets) *Consider an SCM over variables* $\mathbf{X}$ *with* $X, Y \in \mathbf{X}$ *and* $Y \notin \mathbf{PA}_X$. *Then, the following three statements are true.*

(i) ***"parent adjustment"***:

$$\mathbf{Z} := \mathbf{PA}_X$$

is a valid adjustment set for (X, Y).

(ii) ***"backdoor criterion"***: *Any* $\mathbf{Z} \subseteq \mathbf{X} \setminus \{X, Y\}$ *with*

- $\mathbf{Z}$ *contains no descendant of* X *AND*
- $\mathbf{Z}$ *blocks all paths from* X *to* Y *entering* X *through the backdoor* $(X \leftarrow \ldots,$ *see Figure 6.5*$)$

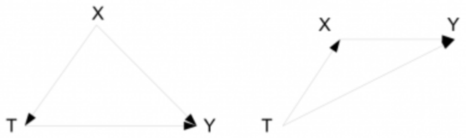
is a valid adjustment set for (X, Y).

(iii) ***"toward necessity"***: *Any* $\mathbf{Z} \subseteq \mathbf{X} \setminus \{X, Y\}$ *with*

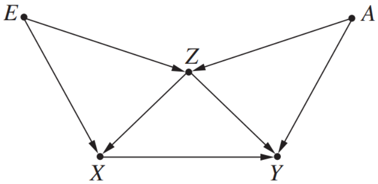
- $\mathbf{Z}$ *contains no descendant of any node on a directed path from* X *to* Y *(except for descendants of* X *that are not on a directed path from* X *to* Y*) AND*
- $\mathbf{Z}$ *blocks all non-directed paths from* X *to* Y

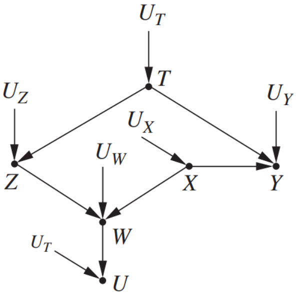
is a valid adjustment set for (X, Y).

Backdoor criterion



(a) (b)





Valid adjustment sets

Suppose the valid adjustment set Z is obtained via the backdoor criterion. We can then add any node Z_0 to Z that satisfies $Z_0 \perp\!\!\!\perp Y | X, Z$ because then

$$\begin{aligned}\sum_{\mathbf{z}, z_0} p(y \mid x, \mathbf{z}, z_0) p(\mathbf{z}, z_0) &= \sum_{\mathbf{z}} p(y \mid x, \mathbf{z}) \sum_{z_0} p(\mathbf{z}, z_0) \\ &= \sum_{\mathbf{z}} p(y \mid x, \mathbf{z}) p(\mathbf{z}).\end{aligned}$$

In fact, Proposition (iii) characterizes all valid adjustment sets.

Example

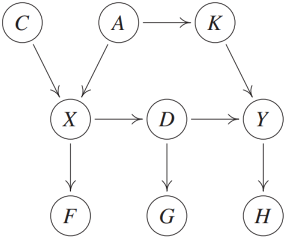


Figure 6.5: Only the path $X \leftarrow A \rightarrow K \rightarrow Y$ is a "backdoor path" from X to Y. The set $\mathbf{Z} = \{K\}$ satisfies the backdoor criterion (see Proposition 6.41 (ii)); but $\mathbf{Z} = \{F, C, K\}$ is also a valid adjustment set for (X, Y); see Proposition 6.41 (iii).

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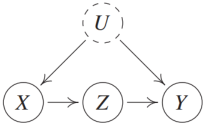
Front-door adjustment

Example 6.46 (Front-door adjustment) Let $\mathfrak{C}$ be an SCM with corresponding graph



If we do not observe U, we cannot apply the backdoor criterion. In fact, there is no valid adjustment set. But still, provided that $p^{\mathfrak{C}}(x,z) > 0$, the *do*-calculus provides us with

$$p^{\mathfrak{C};do(X:=x)}(y) = \sum_z p^{\mathfrak{C}}(z\,|\,x) \sum_{\tilde{x}} p^{\mathfrak{C}}(y\,|\,\tilde{x},z)\; p^{\mathfrak{C}}(\tilde{x}). \tag{6.23}$$

The fact that observing Z in addition to X and Y here reveals causal information nicely shows that causal relations can also be explored by observing the "channel" (here Z) that carries the "signal" from X to Y. □

Front-door adjustment

Definition 3.4.1 (Front-Door) *A set of variables Z is said to satisfy the front-door criterion relative to an ordered pair of variables (X, Y) if*

1. *Z intercepts all directed paths from X to Y.*
2. *There is no unblocked path from X to Z.*
3. *All backdoor paths from Z to Y are blocked by X.*

Theorem 3.4.1 (Front-Door Adjustment) *If Z satisfies the front-door criterion relative to (X, Y) and if $P(x, z) > 0$, then the causal effect of X on Y is identifiable and is given by the formula*

$$P(y|do(x)) = \sum_z P(z|x) \sum_{x'} P(y|x', z)P(x') \tag{3.16}$$

Three rules

Given a graph $\mathcal{G}$ and disjoint subsets X, Y, Z and W, we have

1. "Insertion/deletion of observations":

$$p^{\mathfrak{C};do(\mathbf{X}:=\mathbf{x})}(\mathbf{y}\,|\,\mathbf{z},\mathbf{w}) = p^{\mathfrak{C};do(\mathbf{X}:=\mathbf{x})}(\mathbf{y}\,|\,\mathbf{w})$$

   if $\mathbf{Y}$ and $\mathbf{Z}$ are d-separated by $\mathbf{X},\mathbf{W}$ in a graph where incoming edges in $\mathbf{X}$ have been removed.

2. "Action/observation exchange":

$$p^{\mathfrak{C};do(\mathbf{X}:=\mathbf{x},\mathbf{Z}=\mathbf{z})}(\mathbf{y}\,|\,\mathbf{w}) = p^{\mathfrak{C};do(\mathbf{X}:=\mathbf{x})}(\mathbf{y}\,|\,\mathbf{z},\mathbf{w})$$

   if $\mathbf{Y}$ and $\mathbf{Z}$ are d-separated by $\mathbf{X},\mathbf{W}$ in a graph where incoming edges in $\mathbf{X}$ and outgoing edges from $\mathbf{Z}$ have been removed.

3. "Insertion/deletion of actions":

$$p^{\mathfrak{C};do(\mathbf{X}:=\mathbf{x},\mathbf{Z}=\mathbf{z})}(\mathbf{y}\,|\,\mathbf{w}) = p^{\mathfrak{C};do(\mathbf{X}:=\mathbf{x})}(\mathbf{y}\,|\,\mathbf{w})$$

   if $\mathbf{Y}$ and $\mathbf{Z}$ are d-separated by $\mathbf{X},\mathbf{W}$ in a graph where incoming edges in $\mathbf{X}$ and $\mathbf{Z}(\mathbf{W})$ have been removed. Here, $\mathbf{Z}(\mathbf{W})$ is the subset of nodes in $\mathbf{Z}$ that are not ancestors of any node in $\mathbf{W}$ in a graph that is obtained from $\mathcal{G}$ after removing all edges into $\mathbf{X}$.

Do-calculus

We call an intervention distribution $p^{\mathfrak{C};do(X:=x)}(y)$ **identifiable** if it can be computed from the observational distribution and the graph structure. If there is a valid adjustment set for $(X; Y)$, $p^{\mathfrak{C};do(X:=x)}(y)$ is certainly identifiable.

Theorem 6.45 (Do-calculus) *The following statements hold.*

(i) The rules are complete; that is, all identifiable intervention distributions can be computed by an iterative application of these three rules [Huang and Valtorta, 2006, Shpitser and Pearl, 2006].

(ii) In fact, there is an algorithm, proposed by Tian [2002] that is guaranteed [Huang and Valtorta, 2006, Shpitser and Pearl, 2006] to find all identifiable intervention distributions.

(iii) There is a necessary and sufficient graphical criterion for identifiability of intervention distributions [Shpitser and Pearl, 2006, Corollary 3], based on so-called hedges [see also Huang and Valtorta, 2006].

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Example: the eye doctor

For each unit u we can observe either $B_u(t=1)$ or $B_u(t=0)$ and never both of them at the same time. They are called **potential outcomes**.

Unit u	Treatment T	Pot. Outcome $B_u(t=0)$	Pot. Outcome $B_u(t=1)$	Unit-Level Causal Effect $B_u(t=1)-B_u(t=0)$
1	1	1	0	-1
2	0	1	0	-1
3	1	1	0	-1
⋮				
43	1	1	0	-1
44	0	0	1	1
45	0	1	0	-1
⋮				
119	1	1	0	-1
120	1	0	1	1
121	0	1	0	-1
⋮				
200	0	1	0	-1

Example: the eye doctor

The unit-level causal effect is

$$B_u(t=1) - B_u(t=0)$$

The average causal effect is

$$CE = \frac{1}{n}\sum_{u=1}^{n} B_u(t=1) - B_u(t=0)$$

Assume that in a completely randomized experiment, units $u \in U_0$ received treatment $T = 0$ and units $u \in U_1$ treatment $T = 1$, then

$$\widehat{CE} = \frac{1}{\# U_1}\sum_{u \in U_1} B_u(t=1) - \frac{1}{\# U_0}\sum_{u \in U_0} B_u(t=0)$$

is an unbiased estimator. However, this solution is not reasonable due to the existence of **confounders**.

Potential outcome framework

The core problem is: how to estimate the average causal effect $ACE = E(Y_i(1) - Y_i(0))$.
If $Z \perp\!\!\!\perp (Y(0), Y(1))$, it can be estimated as

$$\begin{aligned} ACE(Z \to Y) &= E(Y_i(1)) - E(Y_i(0)) \\ &= E(Y_i(1)|Z_i = 1) - E(Y_i(0)|Z_i = 0) \\ &= E(Y_i|Z_i = 1) - E(Y_i|Z_i = 0), \end{aligned}$$

However, $Z \perp\!\!\!\perp (Y(0), Y(1))$ is violated due to the confounders.

Assumptions

- **Stable Unit Treatment Value Assumption(SUTVA)**: The potential outcomes for any unit do not vary with the treatment assigned to other units, and, for each unit, there are no different forms or versions of each treatment level, which lead to different potential outcomes.
- **Ignorability**: Given the background variable X, treatment assignment Z is independent of the potential outcomes.

Under these assumptions, the average causal effect can be written as

$$
\begin{aligned}
ACE &= E(Y(1)) - E(Y(0)) \\
&= E[E(Y(1) \mid X)] - E[E(Y(0) \mid X)] \\
&= E[E(Y(1) \mid X, Z=1)] - E[E(Y(0) \mid X, Z=0)] \\
&= E[E(Y \mid X, Z=1)] - E[E(Y \mid X, Z=0)].
\end{aligned}
$$

Relation between Potential Outcomes and SCMs

In SCMs, we can represent potential outcomes using the language of counterfactuals.

$$\underbrace{B_u(t=\tilde{t})}_{\text{potential outcome}} \quad = \quad \underbrace{B \text{ in the SCM } \mathfrak{C}|\mathbf{N}=\mathbf{n}_u; do\,(T:=\tilde{t})}_{\text{counterfactual SCM}},$$

The two representations are equivalent. Any theorem that holds for counterfactual SCMs holds in the world of potential outcomes and vice versa.

Reference

Pearl J, Glymour M, Jewell N P. Causal inference in statistics: A primer[M]. John Wiley & Sons, 2016.
Yao L, Chu Z, Li S, et al. A Survey on Causal Inference[J]. ACM Transactions on Knowledge Discovery from Data (TKDD), 2021, 15(5): 1-46.
https://www.math.pku.edu.cn/teachers/yaoy/math112230/