

0624 习题课

1. 重点知识梳理

(1) 一致收敛的判别法 (Weierstrass, Dirichlet, Abel) (讲义 Lec. 2nd)

Dirichlet 要注意一致趋于 0 的理解 (函数级数和参变量)

例: p. 227 4. (4) $\sum_{n=1}^{\infty} x^n e^{-nx} n^{-x}$ 一致有界

有同: $\sum_{n=1}^{\infty} x^n e^{-nx} = \frac{x^2 e^{-x}}{e^x - 1} = x e^{-x}$ Dirichlet X

$\frac{1}{n^x}$ 对 $x \in (0, +\infty)$ 单调趋于 0
 $\frac{1}{n^x} \rightarrow 1$ as $x \rightarrow 0$ $\forall n \exists x > t. \frac{1}{n^x} > \epsilon$
 不收敛于 0

数项级数 Dirichlet \Rightarrow Abel \checkmark

函数项级数 Dirichlet \nRightarrow Abel

Link: 函数项级数 & 参变量积分

$$\begin{matrix} x & \longleftrightarrow & u \\ n & \longleftrightarrow & x \end{matrix}$$

(2) 一致收敛与连续, 积分换序, 微分换序

函数项级数: 内闭一致收敛 + 连续 \Rightarrow 连续

($\sum f_n(x, n)$)

一点收敛 + 导函数连续 + 导函数内闭一致收敛 \Rightarrow 微分可换序
 (一致收敛 \nRightarrow 导函数一致收敛. 例 $S_n(x) = \frac{\sin nx}{\sqrt{n}}$)

一致收敛 + 闭区间连续 \Rightarrow 积分可换序

无穷区间上极限与积分交换条件:

$f(x, n)$ 在 $x \in [a, +\infty)$ 上收敛于 g

(a) $\forall A > a$ $f(x, n)$ 在 $[a, A]$ 上一致收敛.

(b) $\int_a^{+\infty} f_n(x) dx$ 对 n 一致收敛

则 $\int_a^{+\infty} \lim_{n \rightarrow \infty} f(x, n) dx = \lim_{n \rightarrow \infty} \int_a^{+\infty} f(x, n) dx$

$$\left| \int_a^{+\infty} g(x) dx - \int_a^{+\infty} f(x, n) dx \right| \leq \int_a^A |g(x) - f(x, n)| dx + \left| \int_A^{+\infty} g(x) dx \right| + \left| \int_A^{+\infty} f(x, n) dx \right|$$



含参变量义积分: $\int_a^b f(x, u) dx$

f 连续 \Rightarrow 连续

$f, \frac{\partial f}{\partial u}$ 连续 \Rightarrow 微分可换序

f 连续 + $u \in [a, b]$ 有限闭区间 \Rightarrow 积分可换序

例: $I(r) = \int_0^\pi \ln(1 - 2r \cos x + r^2) dx$

$|r| < 1 \quad I'(r) = \int_0^\pi \frac{-2 \cos x + 2r}{1 - 2r \cos x + r^2} dx$

$I'(0) = \int_0^\pi -2 \cos x dx = 0$

$r \neq 0 \quad I'(r) = \frac{1}{r} \int_0^\pi 1 - \frac{1-r^2}{1 - 2r \cos x + r^2} dx$

$= \frac{1}{r} \left(\pi - (1-r^2) \int_0^\pi \frac{dx}{1 - 2r \cos x + r^2} \right)$

$t = \tan \frac{x}{2} \quad = \frac{1}{r} \left(\pi - \frac{1+r}{1-r} \int_0^{+\infty} \frac{2 dt}{1 + \left(\frac{1+r}{1-r} t\right)^2} \right)$

有得到 $\frac{1+r}{1-r} > 0$ 即 $|r| < 1$ 时
 $= \frac{1}{r} \left(\pi - 2 \arctan \frac{1+r}{1-r} t \Big|_0^{+\infty} \right) = 0$

$\therefore I(r) = 0 \quad (|r| < 1)$

$|r| > 1$ 时 $\rho = \frac{1}{r} \quad |\rho| < 1$

$I(r) = I\left(\frac{1}{r}\right) = \int_0^\pi \ln(1 - 2\rho \cos x + \rho^2) dx = \int_0^\pi \ln \rho^2 dx$

$= -2\pi \ln |\rho| = 2\pi \ln r$

$|r| = 1$ 时, 可直接计算 $I = \int_0^\pi \ln(1 - 2 \cos x) dx = \int_0^\pi \ln(2 + 2 \cos t) dt$

$2I = \int_0^\pi \ln(4 \sin^2 x) dx = 2\pi \ln 2 + 2 \int_0^\pi \ln \sin x dx$

$= 2\pi \ln 2 - 2\pi \ln 2$

$= 0$

也可由连续直接得到结果

$I(\pm 1) = 0$



例: 计算 $\int_0^{+\infty} \frac{x^p}{1+x} dx \quad (0 < p < 1)$

$$\int_0^{+\infty} \frac{x^p}{1+x} = \int_0^1 \frac{x^p}{1+x} dx + \int_1^{+\infty} \frac{x^p}{1+x} dx$$

$\downarrow \quad \quad \quad \downarrow$
 $I_1 \quad \quad \quad I_2$

$$I_1 = \int_0^1 \sum_{n=0}^{\infty} (-1)^n x^{p+n-1} dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+p}$$

$$\left(\sum_{n=2}^{\infty} (-1)^n x^{p+n-1} = \frac{x^{p+1}(1-(-1)^n)}{1+x} \right)$$

$$\leq 2 \frac{x^{p+1}}{1+x} \leq 2x^{p+1}$$

$$I_2 = \int_0^1 \frac{t^{-p}}{1+t} = \int_0^1 \frac{t^{1-p-1}}{1+t} dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1-p} = \sum_{n=1}^{\infty} \frac{(-1)^n}{p-n}$$

$$\therefore \int_0^{+\infty} \frac{x^p}{1+x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{p+n} + \sum_{n=1}^{\infty} \frac{(-1)^n}{p-n}$$

$$f(x) = \cos px$$

Fourier 级数

$$\cos px = \frac{2p \sin px}{\pi} \left(\frac{1}{2p} + \sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{p^2 - n^2} \right)$$

取 $x = p$ 得公式

$$\frac{\pi}{\sin p\pi} = \frac{1}{p} + \sum_{n=1}^{\infty} (-1)^n \frac{2p}{p^2 - n^2}$$

$$\int_0^{+\infty} \frac{x^p}{1+x} dx = \frac{\pi}{\sin p\pi} \quad (0 < p < 1)$$



· 含参变量的常义积分.

设 $f(x, u) \in C([a, b] \times [\alpha, \beta])$

$$\text{令 } g(u) = \int_a^b f(x, u) dx.$$

① $u_0 \in [\alpha, \beta]$:

$$\lim_{u \rightarrow u_0} \int_a^b f(x, u) dx = \int_a^b \lim_{u \rightarrow u_0} f(x, u) dx$$

· 原因: 一致连续.

② 若 $\frac{\partial f}{\partial u} \in C([a, b] \times [\alpha, \beta])$:

$$\frac{d}{du} \int_a^b f(x, u) dx = \int_a^b \frac{\partial}{\partial u} f(x, u) dx$$

· 原因:
$$\int_a^b \frac{f(x, u+h) - f(x, u)}{h} dx.$$

$$= \int_a^b \frac{\partial}{\partial u} f(x, u+\theta h) dx. \xrightarrow{\theta \rightarrow 0} \int_a^b \frac{\partial}{\partial u} f(x, u) dx$$

$$\int_{\alpha}^{\beta} \int_a^b f(x, u) dx du = \int_a^b \int_{\alpha}^{\beta} f(x, u) du dx.$$

· 原因: 由二重积分理论.



含参变量的反常积分.

设 $f(x, u) \in C([a, +\infty) \times [\alpha, \beta])$.

且 $\forall u \in [\alpha, \beta], \int_a^{+\infty} f(x, u) dx$ 收敛

① $u_0 \in [\alpha, \beta]$, 且 $\int_a^{+\infty} f(x, u) dx$ 一致收敛.

$$\lim_{u \rightarrow u_0} \int_a^{+\infty} f(x, u) dx = \int_a^{+\infty} f(x, u_0) dx.$$

原因: $|\int_a^{+\infty} f(x, u) - f(x, u_0) dx|$

$$\leq |\int_a^b f(x, u) - f(x, u_0) dx|$$

$$+ \underbrace{|\int_b^{+\infty} f(x, u) dx|}_{< \frac{\epsilon}{2}} + \underbrace{|\int_b^{+\infty} f(x, u_0) dx|}_{< \frac{\epsilon}{2}}.$$

$$\forall \epsilon > 0, \lim_{u \rightarrow u_0} |\int_a^{+\infty} f(x, u) - f(x, u_0) dx| < \epsilon. \quad \#$$

② $\frac{\partial f}{\partial u} \in C([a, +\infty) \times [\alpha, \beta])$ 且

$$\int_a^{+\infty} \frac{\partial f}{\partial u}(x, u) dx \text{ 一致收敛.}$$

$$\frac{d}{du} \int_a^{+\infty} f(x, u) dx = \int_a^{+\infty} \frac{\partial}{\partial u} f(x, u) dx.$$

$$\text{原因: } \int_a^{+\infty} \frac{f(x, u+h) - f(x, u)}{h} dx.$$

$$= \int_a^{+\infty} \frac{\partial}{\partial u} f(x, u+\theta h) dx$$

$$\stackrel{\Delta \theta \rightarrow 0}{=} \int_a^{+\infty} \frac{\partial f}{\partial u}(x, u) dx.$$



(2) 若 $\int_a^{+\infty} f(x, u) dx$ 一致收敛, 则

$$\int_{\alpha}^{\beta} \int_a^{+\infty} f(x, u) dx du = \int_a^{+\infty} \int_{\alpha}^{\beta} f(x, u) du dx$$

原因: $\forall A > a, \int_{\alpha}^{\beta} \int_a^A f(x, u) dx du$

$$= \int_a^A \int_{\alpha}^{\beta} f(x, u) du dx.$$

> 研究 $A \rightarrow +\infty$ 时, 左式的极限.

$$\left| \int_{\alpha}^{\beta} \int_a^A f(x, u) dx du - \int_{\alpha}^{\beta} \int_a^{+\infty} f(x, u) dx du \right|$$

$$= \left| \int_{\alpha}^{\beta} \left(\int_a^A - \int_a^{+\infty} \right) f(x, u) dx du \right|$$

$$\leq \int_{\alpha}^{\beta} \left| \int_a^{+\infty} f(x, u) dx \right| du.$$

$$\leq \varepsilon \cdot (\beta - \alpha) \rightarrow 0 \quad \#.$$



反常积分一致收敛的判别法.

① 定义: $\lim_{A \rightarrow +\infty} \int_a^{+\infty} f(x, u) dx = 0$, 对 u -致

i.e. $\lim_{A \rightarrow +\infty} \sup_{u \in \textcircled{D}} \left| \int_a^{+\infty} f(x, u) dx \right| = 0$.

这里, \textcircled{D} 是参数 u 的取值范围.

② Cauchy 判别法: $\lim_{A, A' \rightarrow +\infty} \left| \int_A^{A'} f(x, u) dx \right| = 0$, 对 u -致.

* 常用于判别非一致收敛:

取合适的 u, A, A' , 并且 A, A' "充分大".

st $\left| \int_A^{A'} f(x, u) dx \right| \geq \varepsilon_0 > 0$.

③ Weierstrass 判别法.

若 $|f(x, u)| \leq F(x)$, 且 $\int_a^{+\infty} F(x) dx$ 收敛.

则 $\int_a^{+\infty} f(x, u) dx$ 绝对且一致收敛.

④ Dirichlet & Abel 判别法.

$\int_a^{+\infty} f(x, u) g(x, u) dx \xrightarrow{\textcircled{D} \quad \textcircled{A}} \int_a^{+\infty} f(x, u) dx$
 $\int_a^{+\infty} f(x, u) dx$: \textcircled{D} 一致有界 (对 u), \textcircled{A} 一致收敛 (对 u).

$g(x, u) \begin{cases} \text{对 } x \text{ 单调} \\ \text{对 } u \text{ 一致} \downarrow 0 \end{cases} \rightarrow \begin{cases} \text{对 } x \text{ 单调} \\ \text{对 } u \text{ 一致有界} \end{cases}$

记忆

(*) $\int_A^{A'} f(x, u) g(x, u) dx = g(A, u) \int_A^{\xi} f(x, u) dx + g(A', u) \int_{\xi}^{A'} f(x, u) dx$.



另反常积分一致收敛 4个例子

例) $\int_0^{+\infty} \sqrt{u} e^{-ux^2} dx \quad 0 \leq u \leq +\infty$ 不一致收敛

pf: $F(A, u) = \int_A^{+\infty} \sqrt{u} e^{-ux^2} dx = \int_{\sqrt{u}A}^{+\infty} e^{-t^2} dt$

$$\Rightarrow \sup_{u \in [0, +\infty)} F(A, u) \geq \int_0^{+\infty} e^{-t^2} dt > 0$$

$$\Rightarrow \lim_{A \rightarrow +\infty} \sup_{u \in [0, +\infty)} F(A, u) \neq 0$$

从而非一致收敛

注意: $\forall u \in (0, +\infty) \quad \int_0^{+\infty} \sqrt{u} e^{-ux^2} dx = \int_0^{+\infty} e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$

但 $u=0$ 时 $\int_0^{+\infty} \sqrt{u} e^{-ux^2} dx = 0$

说明即便积分关于参数差恒为常值, 也不能保证其一致收敛

另记: 令 $f(u) = \int_0^{+\infty} \sqrt{u} e^{-ux^2} dx$

若 $f(u)$ 在 $[0, +\infty)$ 上一致收敛

则 $\lim_{u \rightarrow 0^+} f(u) = f(0) = 0$ 矛盾!

例) $\int_0^{+\infty} \frac{x \sin ux}{a^2 + x^2} dx \quad (a > 0)$ 在 $u \in (0, +\infty)$ 上不一致收敛

证 1: $\frac{\sin ux}{x} = \frac{x \sin ux}{x^2 + a^2} \frac{a^2 + x^2}{x^2}$

若反常积分一致收敛, 则由 Abel 判别法

在 $u \in (0, +\infty)$ 上一致收敛

证 2. 考虑 $u = \frac{1}{n}$

$$\int_{\frac{n}{4}}^{\frac{3n}{4}} \frac{x \sin ux}{a^2 + x^2} dx$$

$$\int_0^{+\infty} \frac{\sin ux}{x} dx$$

$$\xrightarrow{u = \frac{1}{n}} \geq \frac{\sqrt{2}}{2} \int_{\frac{n}{4}}^{\frac{3n}{4}} \frac{x \sin ux}{a^2 + x^2} dx \xrightarrow{u = \frac{1}{n}} \geq \frac{\sqrt{2}}{2} \ln 9 \neq 0$$



$$例1) \int_0^{+\infty} \frac{x \sin \alpha x}{\alpha(1+x^2)} dx$$

$$① \alpha \in [\eta, +\infty), \eta > 0$$

$$\int_0^{+\infty} \frac{\sin \alpha x}{\alpha} dx \quad \frac{x}{1+x^2}$$

Dirichlet \Rightarrow 一致收敛

$$② \alpha \in (0, \delta)$$

$$A = \frac{n\pi}{4} \quad A' = \frac{3n\pi}{4} \quad \alpha = \frac{1}{n}$$

$$\int_A^{A'} \dots = \int_{\frac{n\pi}{4}}^{\frac{3n\pi}{4}} \frac{x \sin \frac{x}{n}}{\frac{1}{n}(1+x^2)} dx \geq \frac{\sqrt{3}}{2} \int_{\frac{n\pi}{4}}^{\frac{3n\pi}{4}} \frac{x^2}{1+x^2} dx$$

$$\xrightarrow{n \rightarrow \infty} +\infty \Rightarrow \text{不一致收敛}$$

$$例1) \int_1^{+\infty} e^{-(x-\frac{1}{\alpha})^2/\alpha^2} dx, (\alpha \in (0, 1])$$

$$\text{解: } \int_A^{+\infty} e^{-(x-\frac{1}{\alpha})^2/\alpha^2} dx = \alpha \int_{\frac{A-\frac{1}{\alpha}}{\alpha}}^{+\infty} e^{-t^2} dt$$

$$\text{分段 } \alpha \in (0, \varepsilon) \quad \text{则 } \leq \alpha \int_{-\infty}^{+\infty} e^{-t^2} dt \leq \varepsilon \sqrt{\pi}$$

$$\alpha \in (\varepsilon, 1] \quad \text{则 } \leq 1 \cdot \int_{\frac{A-\frac{1}{\varepsilon}}{\varepsilon}}^{+\infty} e^{-t^2} dt \xrightarrow{A \rightarrow +\infty} 0$$

\therefore 一致收敛



最后关于反常积分计算讲两个例子 (Gamma 函数 etc.)

例) $\int_a^b \frac{dx}{\sqrt{(x-a)(b-x)}}$ 显然收敛

计算 $\because \frac{x-a}{b-a} \quad \frac{b-x}{b-a}$ 区数都为 1

$$\therefore \text{设 } \frac{x-a}{b-a} = \sin^2 \theta$$

$$x = a \cos^2 \theta + b \sin^2 \theta$$

$$dx = 2(b-a) \sin \theta \cos \theta d\theta$$

$$\text{积分} = \int_0^{\frac{\pi}{2}} 2 d\theta = \pi$$

例) $\int_0^1 \frac{dt}{\sqrt{1-t^4}}$ 与 $\int_0^1 \frac{dt}{\sqrt{1+t^4}}$ 比值

$$x = t^4 \quad \frac{1}{4} x^{-\frac{3}{4}}$$

$$I_1 = \int_0^1 \frac{1}{\sqrt{1-x}} dx = \frac{1}{4} B\left(\frac{1}{4}, \frac{3}{2}\right) = \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{\Gamma(\frac{1}{4}) \Gamma(\frac{3}{2})}{\Gamma(\frac{3}{4})} = \frac{1}{8} \Gamma\left(\frac{1}{4}\right)^2 \cdot \frac{\sqrt{\pi}}{\pi / \sin \frac{\pi}{4}} = \frac{\sqrt{2}}{2} \Gamma\left(\frac{1}{4}\right)^2$$

$$I_2 = \int_1^{+\infty} \frac{1}{\sqrt{1+x^4}} dx \quad \therefore I_2 = \frac{1}{2} \int_0^{+\infty} \frac{1}{\sqrt{1+x^4}} dx = \frac{1}{2} \int_0^1 u^{\frac{1}{2}} \frac{1}{4} \left(\frac{1-u}{u}\right)^{-\frac{3}{4}} \left(\frac{1}{u^2}\right) du$$

$$\uparrow$$

$$x = \frac{1}{t}$$

$$\mu = \frac{1}{1+x^4}$$

$$x = \left(\frac{1}{u} - 1\right)^{\frac{1}{4}}$$

$$= \frac{1}{8} \int_0^1 u^{-\frac{3}{4}} (1-u)^{-\frac{3}{4}} du$$

$$= \frac{1}{8} B\left(\frac{1}{4}, \frac{1}{4}\right)$$

$$= \frac{1}{8} \frac{\Gamma\left(\frac{1}{4}\right)^2}{\sqrt{\pi}}$$

$$\therefore I_1 / I_2 = \frac{\sqrt{2}}{2}$$

另习题 $\int_0^{\frac{\pi}{2}} \tan^x x dx$ 略

