

Chapter 1: Classical Cryptography

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1 Summary

This chapter mainly introduced 7 simple cryptosystems and cryptanalysis of these cryptosystems, as shown in the following table.

| \mathcal{L} | \mathcal{P} | \mathcal{K} | \mathcal{C} |
|---------------------|---|---------------|--|
| Language Alphabet | Plaintext Space | Key Space | Cyphertext Space |
| Cryptosystem | K | | E_K |
| Shift Cipher | $\mathcal{K} = \mathcal{C} = \mathcal{K}$ | | $E_K(x) = (x + K) \bmod \mathcal{L} $ |
| Substitution Cipher | $\mathcal{P} = \mathcal{C}, \mathcal{K} = Sym(\mathcal{L})$ | | $E_\pi(x) = \pi(x)$ |
| Affine Cipher | $\mathcal{P} = \mathcal{C}, \mathcal{K} \subset \mathcal{P} \times \mathcal{P}$ | | $E_K(x) = (ax + b) \bmod \mathcal{L} $ |
| Vigenere Cipher | $\mathcal{K} = \mathcal{C} = \mathcal{K}$ | | $E_K(x_1, x_2, \dots, x_m) = (x_1 + k_1, x_2 + k_2, \dots, x_m + k_m) \bmod \mathcal{L} $ |
| Hill Cipher | $\mathcal{P} = \mathcal{C}, \mathcal{K} = GL(n, \mathcal{L})$ | | $E_K(x) = x \cdot K$ |
| Permutation Cipher | $\mathcal{P} = \mathcal{C}, \mathcal{K} = Sym(m)$ | | $E_\pi(x_1, x_2, \dots, x_m) = (x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(m)})$ |
| Stream Cipher | $\mathcal{P} = \mathcal{C} = \mathcal{K} = \mathcal{L}, z_{i+m} = \sum_{j=0}^{m-1} c_j z_j$ | | $E_K(x_j) = x_j + z_j \bmod \mathcal{L} $ |

| Cryptosystem | Cryptanalysis |
|---------------------|--|
| Shift Cipher | frequency of occurrence of letters |
| Substitution Cipher | frequency of occurrence of letters |
| Affine Cipher | frequency of occurrence of letters |
| Vigenere Cipher | Kasiski test & computation of indices of coincidence |
| Hill Cipher | matrix inverse(known plaintext) |
| Permutation Cipher | matrix inverse(known plaintext) |
| Stream Cipher | matrix inverse(known plaintext) |

2 Excercise

- 1.5 $Q \rightarrow a$, Plaintext: lookupintheairitsabirditsaplaneitssuperman
- 1.6 $30 = 2 \cdot 3 \cdot 5, \phi(30) = (2-1)(3-1)(5-1) = 8, |\mathcal{K}| = 8 \cdot 30 = 240$
 $100 = 2^2 \cdot 5^2, \phi(100) = (4-2)(25-5) = 40, |\mathcal{K}| = 40 \cdot 100 = 4000$
 $1225 = 5^2 \cdot 7^2, \phi(1225) = (25-5)(49-7) = 840, |\mathcal{K}| = 840 \cdot 1225 = 1029000$
- 1.8 m=28: 1,3,5,9,11,13,15,17,19,23,25,27
m=33: 1,2,4,5,7,8,10,13,14,16,17,19,20,23,25,26,28,29,31,32
m=35: 1,2,3,4,6,8,9,11,12,13,16,17,18,19,22,23,24,26,27,29,31,32,33,34
- 1.10 (a) : $5^{-1} = 6, d_K(y) = 6y + 1910$
(b) : $d_K(e_K(x)) = 6e_K(x) + 19 = 6(5x + 21) + 19 = 30x + 145 \equiv x \pmod{29}$
- 1.11 (a) If an encryption function e_K is identical to the decryption function, then the key is called an invulutory key. In this question, it requires $a^{-1} \equiv a, -a^{-1} \cdot b \equiv b \pmod{n}$
i.e. $a^{-1} \equiv a, (a+1) \cdot b \equiv 0 \pmod{n}$
(b) (1,0),(4,0),(4,3),(4,6),(4,9),(4,12),(11,0),(11,5),(11,10),(14,0),(14,1),(14,2),(14,3),(14,4),(14,5),(14,6),(14,7),(14,8),(14,9),(14,10),(14,11),(14,12),(14,13),(14,14)
(c) Applying (a), $a^2 \equiv 1 \Leftrightarrow pq|(a+1)(a-1) \bmod pq$

$\Leftrightarrow a \equiv \pm 1 \text{ or } p|(a-1) \& q|(a+1) \text{ or } q|(a-1) \& p|(a+1)$

If $a \equiv 1$, applying (a), we get $2b \equiv 0 \pmod{pq} \therefore p \text{ and } q \text{ are primes}, \therefore b \equiv 0 \pmod{pq}$, resulting in 1 solutions

If $a \equiv -1$, $(a+1)b \equiv 0$ always holds, resulting in pq solutions

If $p|(a-1) \& q|(a+1)$, then applying (a), we get $p|b \Rightarrow b = kp, k = 0, 1, \dots, q-1$, resulting in q solutions

If $q|(a-1) \& p|(a+1)$, then applying (a), we get $q|b \Rightarrow b = kq, k = 0, 1, \dots, p-1$, resulting in p solutions

There are $p+q+pq+1$ involuntary keys in total in total.

- 1.12 (a) First column is a 2-element vector. It only need to be non-zero vector, resulting in $p^2 - 1$ choices.
 Second column is a 2-element vector independent of the first column, resulting in $p^2 - p$ choices
 Combined, we get $(p^2 - 1)(p^2 - p)$ invertible matrices.

$$(b) \# \text{ of invertible } m \times m \text{ matrices over } Z_p = \prod_{k=0}^{m-1} (p^m - p^k)$$

- 1.13 If $n = pq$ (p and q are primes, $p \neq q$), then

(1) if $p \neq q$, then applying Chinese remainder theorem, we get

$$GL(2, \mathbb{Z}_{pq}) = GL(2, \mathbb{Z}_p) \times GL(2, \mathbb{Z}_q) \Rightarrow |GL(2, \mathbb{Z}_{pq})| = |GL(2, \mathbb{Z}_p)| \cdot |GL(2, \mathbb{Z}_q)|$$

By Exercise 1.12, $GL(2, \mathbb{Z}_{pq}) = (p^2 - 1)(p^2 - p)(q^2 - 1)(q^2 - q)$

(2) If $p = q$, $\therefore \mathbb{Z}_{p^2} \cong \mathbb{Z}/p^2$, \therefore we can identify invertible $m \times m$ matrices over \mathbb{Z}_n with $Aut_{group}((\mathbb{Z}/n)^m)$
 $|Aut(H_p)| = (p^2 - 1)(p^2 - p)p^4$, see [this page](#)

$$\Rightarrow |GL(2, \mathbb{Z}_{p^2})| = (p^2 - 1)(p^2 - p)p^4$$

n=6: $6 = 2 \cdot 3$, 288 matrices

n=9: $9 = 3^2$, 3888 matrices

n=26: $26 = 2 \cdot 13$, 157248 matrices

- 1.14 (a) calculating determinant both sides $\Rightarrow \det(A) = \det(A)^{-1} \Rightarrow \det(A)^2 \equiv 1 \pmod{26}$
 $\Rightarrow \det(A) \equiv \pm 1 \pmod{26}$

(b) Applying (a) and Corollary 1.4, we get

If $\det(A) \equiv 1$, then $a_{22} = a_{11} = a, a_{12} = -a_{21} = b \Rightarrow a^2 + b^2 \equiv 1 \pmod{26}$

$\Rightarrow 22$ solutions for $(a, b) : (0, 1), (1, 0), (0, 25), (25, 0), (2, 7), (7, 2), (2, 19), (19, 2), (6, 11), (11, 6), (6, 15), (15, 6), (7, 24), (24, 7), (11, 20), (20, 11), (12, 13), (13, 12), (13, 14), (14, 13), (15, 20), (20, 15)$

If $\det(A) \equiv -1$, then $a_{11} = -a_{22} = a, a_{12} = a_{21} = b \Rightarrow a^2 + b^2 \equiv 25 \pmod{26}$

$\Rightarrow 24$ solutions for $(a, b) : (0, 5), (5, 0), (0, 21), (21, 0), (3, 4), (4, 3), (3, 22), (22, 3), (4, 23), (23, 4), (8, 13), (13, 8), (9, 10), (10, 9), (9, 16), (16, 9), (10, 17), (17, 10), (13, 18), (18, 13), (16, 17), (17, 16), (22, 23), (23, 22)$

So there are 46 involuntary keys.

- 1.15 (a) $\det(A) \equiv 17, 17^{-1} \equiv 23, A^{-1} \equiv \begin{pmatrix} 11 & 25 \\ 11 & 20 \end{pmatrix}$

$$(b) \det(A) \equiv 5, 5^{-1} \equiv 21, A^{-1} \equiv \begin{pmatrix} 25 & 11 & 22 \\ 10 & 13 & 4 \\ 17 & 24 & 1 \end{pmatrix}$$

- 1.16 (a) $x \ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8$

$$\pi^{-1}(x) \ 2 \ 4 \ 6 \ 1 \ 8 \ 3 \ 5 \ 7$$

(b) gentle mendonotread each others mail

- 1.17 (a) represent permutation π by matrix A, $\Rightarrow A = A^T$, then π is involuntary, $\Leftrightarrow A = A^{-1} = A^T$

\Leftrightarrow if $\pi(i) = (j)$, then $\pi(j) = (i)$

(b) Choosing $2j$ elements to form j pairs that are changed by π , and other elements are not changed.

If $m=2k$: there are $1 + \sum_{j=1}^k \binom{m}{2j} (2j-1)!!$ symmetric matrices

If $m=2k+1$: there are $1 + \sum_{j=1}^k \binom{m}{2j} (2j-1)!!$ symmetric matrices

So, $m=2$: 2 keys, $m=3$: 4 keys, $m=4$: 10 keys, $m=5$: 41 keys, $m=6$: 76 keys

- 1.18 0000: period 1; other keys: period 5

- 1.19 0000: period 1; other keys: period 6

- 1.20 K is fixed, so σ_i only depends on σ_{i-1}, z_i only depends on σ_i , since Σ is finite set. According to pegionhole principle, there are at least two elements of the same value in $\{\sigma_0, \dots, \sigma_{|\Sigma|}\}$, $\Rightarrow \exists i \neq j, s.t. \sigma_i = \sigma_j$
 \Rightarrow sequence σ_i has period at most $|\Sigma|$, keystream z_i has period at most $|\Sigma|$

- 1.22 (a) If $\exists q_i \leq q_j$ and $i \leq j$, then let $q'_i = q_j, q'_j = q_i \Rightarrow p_i q_i + p_j q_j - p_i q'_i - p_j q'_j = p_i q_i + p_j q_j - p_i q_j - p_j q_i = (p_i - p_j)(q_i - q_j) < 0$, $\therefore \sum_{i=1}^n p_i q'_i$ is maximized when $q'_1 \geq \dots \geq q'_n$
(b) Applying (a), Equation(1.1) is maximized when $\frac{f_i+g}{n'} = p_i \Leftrightarrow g = k_i$
- 1.23 By testing we get m=3, $e_K((x_1, x_2, x_3)) = (x_1, x_2, x_3) \cdot \begin{pmatrix} 3 & 21 & 20 \\ 4 & 15 & 23 \\ 6 & 14 & 5 \end{pmatrix}$
- 1.24 In number form, we write 0 3 8 18 15 11 0 24 4 3 4 16 20 0 19 8 14 13 \rightarrow 3 18 17 12 18 8 14 15 11 23 11 9 1 25 20 11 11 12, subtracting the last 3 number from the first 15 number(3 in a group), we get 18 15 21 10 1 24 18 10 17 21 16 3 12 12 6 \rightarrow 18 7 5 1 7 22 3 4 25 12 0 23 16 14 8
Taking the middle 9 elements we get $\begin{pmatrix} 1 & 7 & 22 \\ 3 & 4 & 25 \\ 12 & 0 & 23 \end{pmatrix} = \begin{pmatrix} 10 & 1 & 24 \\ 18 & 10 & 17 \\ 21 & 16 & 3 \end{pmatrix} \cdot L$, where right mat is invertible
 $\Rightarrow L = \begin{pmatrix} 3 & 6 & 4 \\ 5 & 15 & 18 \\ 17 & 8 & 5 \end{pmatrix}$
For b part,take the first 3 plaintext $\Rightarrow (0 \ 3 \ 18) \cdot L + b \equiv (3 \ 18 \ 17) \Rightarrow b = (20 \ 11 \ 3)$
- 1.25 Frequent ciphertext diagram: TX:4, LM:3
Frequent plaintext diagram: th,he,in,er,an,re,ed,on,es,st,en,at,to,nt,ha,nd,ou,ea,ng,as,or,ti,is,et,it,ar,te,se, hi,of. Trough programming, we get in \rightarrow TX, th \rightarrow LM
plaintext = 'the king was in his counting house counting out his money the queen was in the parlour eating bread and honey z'
1.26 (a) ommitted
(b) 42 letters divided into 7 groups(6 letters a group), then for each group, apply this method m=2,n=6
plaintext: marymaryquitecontraryhowdoesyourgardengrow
- 1.27 (a)(b) ommitted
(c) Applying (b) and evaluate the i-th element of v_h , we get $z_{h+i-1} = \sum_{j=0}^{h-2} \alpha_j z_{j+1+i-1}, i = 1, \dots, m$
 $\Rightarrow z_{h+i-1} = \sum_{j=0}^{h-2} \alpha_j z_{j+i}, i = 1, \dots, m.$
If $i > m$, assume $z_{h+k-1} = \sum_{j=0}^{h-2} \alpha_j z_{j+k}$ when $k < i$
 $\Rightarrow z_{h+i-1} = \sum_{j=0}^{m-1} c_j z_{h+i+j-m-1} = \sum_{j=0}^{m-1} c_j \sum_{l=0}^{h-2} \alpha_l z_{l+i+j-m} = \sum_{l=0}^{h-2} \alpha_l \sum_{j=0}^{m-1} c_j z_{l+i+j-m} = \sum_{l=0}^{h-2} \alpha_l z_{l+i} \text{ mod } 2$
(d) If $h \leq m$, consider initial vector $(0, \dots, 0, 1)$, applying (c), we get $z_m = \sum_{l=0}^{h-2} \alpha_l z_{l+m+1-h} = 0 \text{ mod } 2$
Contradiction to $z_m = 1 \text{ mod } 2$! $\therefore h = m + 1$, the matrix is invertible.
1.28 Testing K from A to Z, we get K=19, plaintext: thereisnotimelikethepresent
1.29 (a) m letters a row, and for $i=2,3,\dots$, replace the i-th row by: $\text{mod}(i^{\text{th}} \text{ row} - i + 1, 26)$, then use computation of indices of coincidence just like in vigenere cipher to determine key length and keyword.
(b) 246 characters, (Reminder: key length doesn't have to divide 246)
keylength:5 Using the same method as p.35(textbook), we can get key is PRIME
plaintext: the most famous cryptologist in history owes his fame less to what he did than to what he said and to the sensational way in which he said it and this was most perfectly in character for Herbert Yardley who perhaps the most engaginig articulate and technicolored personality in the business
- 1.30 K=k, plaintext: the first deposit consisted of one thousand and fourteen pounds of gold