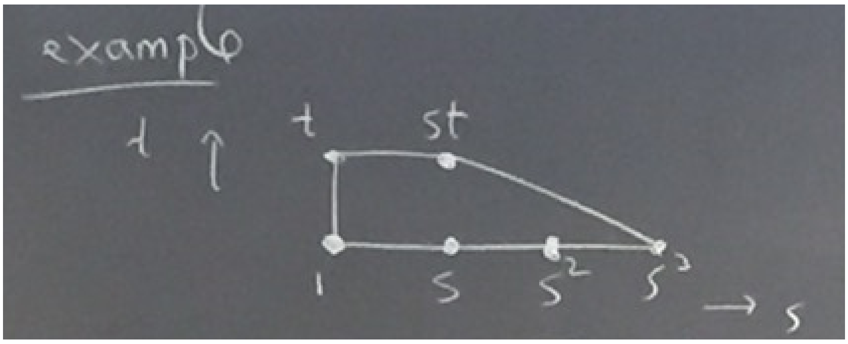
MATH7670 LECTURE NOTES

1. Lecture 01

What is a toric variety(over $\mathbb{C}$)? simplest: $\mathbb{P}^n$, also $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_r}$(via Segre embdedding).

Example 1 (Hirzebruch surface)**.** A polygon



$$\varphi : \mathbb{A}^2_{s,t} \to \mathbb{P}^5$$
$$(s,t) \mapsto (1, s, s^2, s^3, t, st)$$

Consider

$$\overline{\varphi(\mathbb{A}^2)} = X \subseteq \mathbb{P}^5.$$

X is an example of a toric variety. The edges correspond to curves $C_1, \ldots, C_4$ on X, $C_1 \cap C_2 = \text{pt}$, $C_2 \cap C_3 = \text{pt}$, etc. Pic$X$ is generated by $[C_1], [C_2]$

Toric varieties allow explicit constructions of many concepts in A.G.

- singularities, smoothness, resolution of singularities
- divisors, geometry of divisors Pic$X$, ClX.
- cohomology of divisors
- vector bundles, projective bundles, sheaf of differentials
- Serre duality, Riemann-Roch

Special features:

- defined using: cones, fans, polyhedron
- action of the torus
$$T = (\mathbb{C}^*)^n.$$
- homogenerous coordinates on a toric variety.

Caution. Our toric varieties are always normal.

First few weeks:

- cones, F-M,
- affine toric varieties: define/ideals/points/singularities, maps, T-action.
- projective, more general, toric varieties

Date: January 24, 2019.

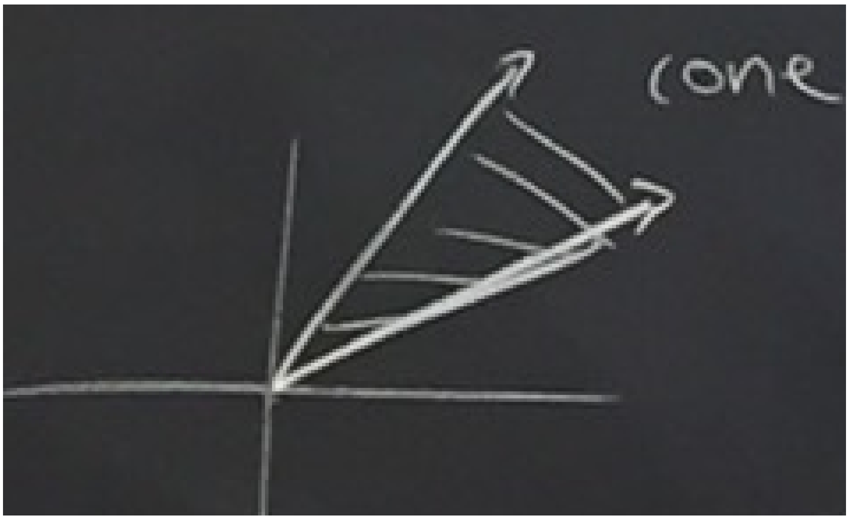
1.1. **Cones.** Let V be a finite dimensional $\mathbb{R}$-vector space.

Definition. Let $S \subseteq V$ be a subset, nonempty.

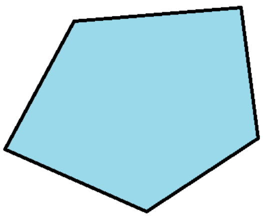
(1) S is a **(convex) cone** if $\forall \mathbf{x}, \mathbf{y} \in S$, $\forall \alpha, \beta \in \mathbb{R}$, $\alpha, \beta \geq 0$, then $\alpha\mathbf{x} + \beta\mathbf{y} \in S$.
(2) S is a **convex set** if $\forall \mathbf{x}, \mathbf{y} \in S$, $\forall \alpha, \beta \in \mathbb{R}$, $\alpha, \beta \geq 0$, $\alpha + \beta = 1$, then $\alpha\mathbf{x} + \beta\mathbf{y} \in S$.

Example 2. Some examples when $V = \mathbb{R}^2$:

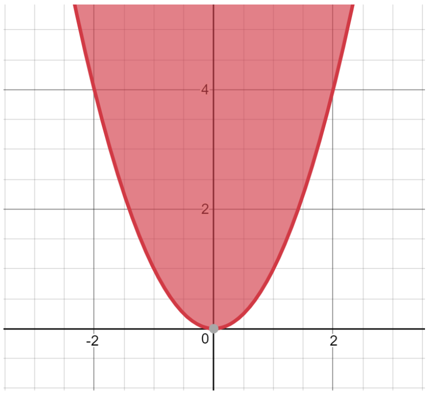
(1) A convex cone:



(2) Convex sets:



(A) A polygon



(B) $y \geq x^2$

Example 3. Key examples:

(1) If $A \in \mathbb{R}^{m\times n}$, write $A = [A_1, A_2 \dots, A_n]$, define

$$\begin{aligned} \mathrm{vcone}(A) &:= \{x_1A_1 + \cdots + x_nA_n \in \mathbb{R}^n, x_1, \dots, x_n \geq 0\} \\ &= \{A\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq 0, \mathbf{x} \in \mathbb{R}^n\} \end{aligned}$$

A cone of this form is called finitely generated or V-cone.

(2) obtain cone via intersections of half-spaces. If

$$B = \begin{bmatrix} B_1 \\ \vdots \\ B_d \end{bmatrix}, B_i \in (\mathbb{R}^n)^*$$

Define

$$\begin{aligned}\text{hcone}(B) &:= \{\mathbf{y} \in (\mathbb{R}^n)^* : \langle B_1, \mathbf{y}\rangle = B_1\mathbf{y} \le 0, \dots, \langle B_n, \mathbf{y}\rangle = B_n\mathbf{y} \le 0\} \\ &= \{\mathbf{y} \in (\mathbb{R}^m)^* : B\mathbf{y} \le 0\}\end{aligned}$$

(3) if $A \in \mathbb{R}^{m\times n}$, define

$$\text{conv}(A) := \{x_1A_1 + \cdots + x_nA_n : x_i \ge 0, \sum x_i = 1\}$$

A set of this form is a polytope.

(4) if $A \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$

$$P(A, \mathbf{b}) := \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \le \mathbf{b}\}$$

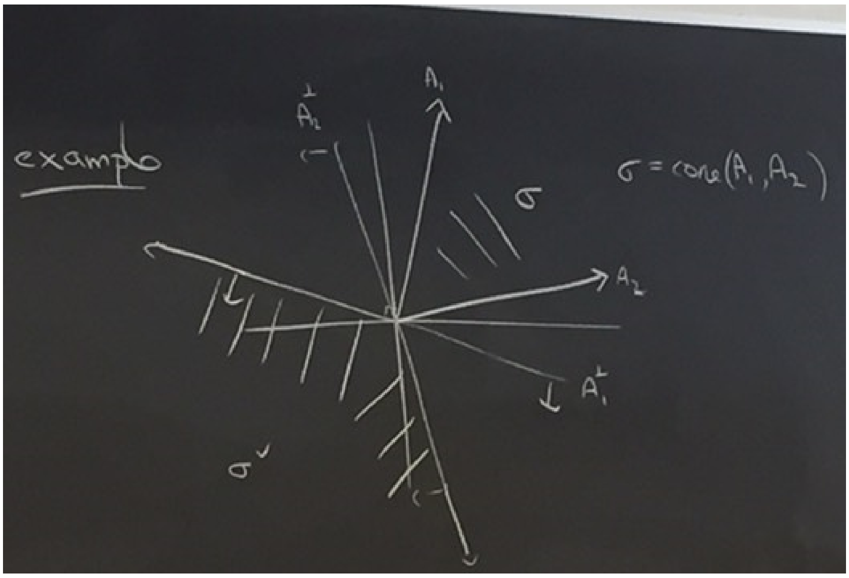
A set of this form is a polyhedron.

Definition (Key construction: dual cone)**.** If $\sigma \in V$ is a cone, define its **dual cone**

$$\sigma^\vee = \{\mathbf{y} \in V^* : \langle \mathbf{y}, \mathbf{x}\rangle \le 0, \forall \mathbf{x} \in \sigma\}.$$

(Check: $\sigma^\vee$ is also a cone.)

Example 4. $V = \mathbb{R}^2$. Let $\sigma = \text{cone}(A_1, A_2)$. Then $\sigma^\vee = \text{cone}(A_1^\perp, A_2^\perp)$.



Remark. if $\sigma = \text{vcone}(A)$, $A \in \mathbb{R}^{m\times n}$, then $\sigma^\vee = \text{hcone}(A^T)$. This is because

$$\begin{aligned}\sigma^\vee &= \{\mathbf{y} \in (\mathbb{R}^m)^* : \langle \mathbf{y}, \mathbf{x}\rangle \le 0, \forall \mathbf{x} \in \sigma\} \\ &= \{\mathbf{y} : \langle \mathbf{y}, A_1\rangle \le 0, \dots, \langle \mathbf{y}, A_n\rangle \le 0\}.\end{aligned}$$

Some desires:

(1) show that σ is a hcone $\iff$ σ is a vcone
(2) find $\sigma^\vee$, i.e. find B s.t. $\sigma^\vee = \text{vcone}(B)$
(3) $\sigma^{\vee\vee} = \sigma$?

1.2. **Fourier-Motzkin elimination.** Let $\text{vcone}(A) = \{A\mathbf{x} : \mathbf{x} \geq 0\}$, consider

$$C = \{\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix} : \mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in \mathbb{R}^m, \mathbf{y} = A\mathbf{x}, \mathbf{x} \geq 0\}$$
$$= \{\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix} : \begin{bmatrix} A & -I \\ -A & I \\ -I & 0\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix} \leq 0\}$$

is an hcone $\subseteq \mathbb{R}^{n+m}$. Consider $\pi_{\mathbf{y}}$

$$\pi_{\mathbf{y}} : \mathbb{R}^{n+m} \to \mathbb{R}^m,$$
$$\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix} \mapsto \mathbf{y}.$$

Then $\pi_{\mathbf{y}}(C) = \sigma$.
Generalize: let $P = P(A, \mathbf{b}) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}\} \subseteq \mathbb{R}^n$. Fix $1 \leq k \leq n$ column index ($A \in \mathbb{R}^{m\times n}$), define

$$\pi_k : \mathbb{R}^n \to \mathbb{R}^n$$
$$\begin{bmatrix}x_1\\ \vdots \\ x_n\end{bmatrix} \mapsto \begin{bmatrix}x_1\\ \vdots \\ 0 \\ \vdots \\ x_n\end{bmatrix}.$$

The image of π_k is $\{x_k = 0\} = \mathbb{R}^{n-1} \subseteq \mathbb{R}^n$. Then $\pi_k(P) \subseteq \mathbb{R}^{n-1} \subseteq \mathbb{R}^n$. Is $\pi_k(P)$ a polyhedral?

Theorem 1.1 (Fourier-Motzkin elimination)**.** *Fix $A \in \mathbb{R}^{m\times n}$, $1 \leq k \leq n$, then there exists $C \in \mathbb{R}^{r\times m}$, some r, s.t.*

(1) $C \geq 0$
(2) CA has column k equal to 0
(3) if $P = P(A, \mathbf{b})$, then $\pi_k(P) = P(A', \mathbf{b}')$ where $A' = CA$, $\mathbf{b}' = C\mathbf{b}$.

Example 5.

$$A = \begin{bmatrix}1 & 1 & -1\\ 2 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & -2 & 0\end{bmatrix}, \sigma = \{\mathbf{x} : A\mathbf{x} \leq 0\} \subseteq \mathbb{R}^3$$

and

$$\pi_3 : \mathbb{R}^3 \to \mathbb{R}^3,$$
$$\begin{bmatrix}x_1\\ x_2 \\ x_3\end{bmatrix} \mapsto \begin{bmatrix}x_1\\ x_2 \\ 0\end{bmatrix}$$

$$\begin{cases} x_1 + x_2 - x_3 \leq 0 \\ 2x_1 - x_2 - x_3 \leq 0 \\ x_1 + x_3 \leq 0 \\ x_1 - 2x_2 \leq 0 \end{cases} \tag{1}$$

parition $\{1,2,3,4\}$:

$$Z := \{4\}$$
$$N := \{1,2\}$$
$$P := \{3\}$$

$$\begin{cases} x_1 - 2x_2 \leq 0 \\ 2x_1 + x_2 \leq 0 \\ 3x_1 - x_2 \leq 0 \end{cases} \tag{2}$$

if $\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$ satisfies (2), find $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ satisfies (1):

$$\begin{aligned} x_3 &\geq x_1 + x_2 \\ x_3 &\geq 2x_1 - x_2 \\ x_3 &\leq -x_1 \end{aligned}$$

Let

$$\begin{aligned} &L = \max(x_1 + x_2, 2x_1 - x_2), \\ &U = \min(-x_1), \end{aligned}$$

then any x_3 s.t. $L \leq x_3 \leq U$ satisfies $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \sigma$.

2. Lecture 02

Recall: F-M elimination (Theorem1.1). (HW: prove it.)
We had defined $\sigma^\vee$: $\sigma \subseteq V = \mathbb{R}^m$

$$\sigma^\vee = \{\mathbf{y} \in V^* : \langle \mathbf{y}, \mathbf{x}\rangle \leq 0, \forall \mathbf{x} \in \sigma\}.$$

Goals:

- hcones = vcones
- $\sigma^{\vee\vee} = \sigma$

2.1. "Farkas jungle".

Corollary 1. *Let $A \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, then either*

(1) $\exists \mathbf{x} \in \mathbb{R}^n$ s.t. $A\mathbf{x} \leq \mathbf{b}$, or
(2) $\exists \mathbf{y} \in \mathbb{R}^m$ s.t. $\mathbf{y}^T A = A^T\mathbf{y} = 0$, $\mathbf{y} \geq 0$ but $\mathbf{y}^T\mathbf{b} = \langle \mathbf{y}, \mathbf{b}\rangle < 0$,

but not both.

Proof. not both: If we have $A\mathbf{x} \leq \mathbf{b}$, $\mathbf{y}^T A = 0$, $\mathbf{y} \geq 0$, $\mathbf{y}^T\mathbf{b} < 0$, then

$$0 = \mathbf{y}^T A\mathbf{x} \leq \mathbf{y}^T\mathbf{b} < 0.$$

Contradiction!
Suppose $A\mathbf{x} \leq \mathbf{b}$ has no solution. Produce $\mathbf{y}$: Apply F-M to columns $n, n-1, \ldots, 1$, obtain $C \in \mathbb{R}^{N\times m}$, $C \geq 0$ and $CA = 0$. Then we get a system $0 = CAX \leq C\mathbf{b} = \mathbf{b}'$ having no solutions. So some $b_i' = 0$. Let $\mathbf{y}^T = i$-th row of C. So

$$\begin{aligned} \mathbf{y} &\geq 0, \\ \mathbf{y}^T A &= 0, \\ \mathbf{y}^T\mathbf{b} &< 0. \end{aligned}$$

□

Corollary 2. *Let $A \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, then either*

(1) $\exists \mathbf{x} \in \mathbb{R}^n$ s.t. $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq 0$ or,
(2) $\exists \mathbf{y} \in \mathbb{R}^m$ s.t. $\mathbf{y}^T A \leq 0$, $\mathbf{y}^T\mathbf{b} > 0$,

but not both.

Proof. Not both: $A\mathbf{x} = \mathbf{b}$,

$$\mathbf{y}^T A\mathbf{x} = \mathbf{y}^T\mathbf{b} > 0$$

Contradiction!
Assume (1) fails, i.e.

$$\{\mathbf{x} \in \mathbb{R}^n : \begin{bmatrix} A \\ -A \\ -I \end{bmatrix} \mathbf{x} \leq \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \\ 0 \end{bmatrix}\} = \emptyset$$

by Corollary 1, $\exists \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \end{bmatrix} \geq 0$ s.t.

$$\begin{bmatrix} \mathbf{y}_1^T & \mathbf{y}_2^T & \mathbf{y}_3^T \end{bmatrix} \begin{bmatrix} A \\ -A \\ -I \end{bmatrix} = 0,$$

$$\begin{bmatrix} \mathbf{y}_1^T & \mathbf{y}_2^T & \mathbf{y}_3^T \end{bmatrix} \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \\ 0 \end{bmatrix} < 0,$$

so

$$\begin{aligned}(\mathbf{y}_1^T - \mathbf{y}_2^T)A = \mathbf{y}_3^T \geq 0,\\ (\mathbf{y}_1^T - \mathbf{y}_2^T)\mathbf{b} < 0.\end{aligned}$$

Let $\mathbf{y} = \mathbf{y}_1^T - \mathbf{y}_2^T$. □

Corollary 3 (Farkas lemma)**.** *Let $\sigma = vcone(A)$, $A \in \mathbb{R}^{m\times n}$, let $\mathbf{b} \in \mathbb{R}^m$, then either*

(1) $\mathbf{b} \in \sigma$ or,
(2) $\exists \mathbf{y} \in \sigma^\vee$ s.t. $\mathbf{y}^T\mathbf{b} > 0$.

Corollary 4. *Given $A \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, let $\sigma = vcone(A) \subset \mathbb{R}^m$. If $\mathbf{b} \notin \sigma$, then $\exists \mathbf{y} \in \sigma^\vee \subseteq (\mathbb{R}^n)^*$ s.t. $\mathbf{y}^T\sigma \leq 0$, $\mathbf{y}^T\mathbf{b} > 0$. i.e.*

$$\sigma \subseteq H_{\mathbf{y}}^{\leq 0} := \{\mathbf{x} \in \mathbb{R}^m : \mathbf{y}^T\mathbf{x} \leq 0\}$$

and

$$\mathbf{b} \in H_{\mathbf{y}}^{>0} := \{\mathbf{x} \in \mathbb{R}^m : \mathbf{y}^T\mathbf{x} > 0\}.$$

Notation: $\mathbf{y}^\perp := H_{\mathbf{y}}^{=0} := \{\mathbf{x} \in \mathbb{R}^m : \mathbf{y}^T\mathbf{x} = 0\}$.

Corollary 5. *let $\sigma = vcone(A) \subseteq \mathbb{R}^m$, $A \in \mathbb{R}^{m\times n}$, then $\sigma = \sigma^{\vee\vee}$.*

Proof. (1) $\sigma \subseteq \sigma^{\vee\vee}$: this is essentially formal.

$$\begin{aligned}\mathbf{y} \in \sigma^\vee &\iff \mathbf{y}^T A_1 \leq 0, \dots, \mathbf{y}^T A_n \leq 0\\ &\iff \mathbf{y} \in \text{hcone}(A).\end{aligned}$$

need to show $A_i \in \text{hcone}(A^T)^\vee = \sigma^{\vee\vee}$, if $\mathbf{y} \in \sigma^\vee$, then $\mathbf{y}^T A_i \leq 0 \implies A_i \in \sigma^{\vee\vee}$.

(2) $\sigma \supseteq \sigma^{\vee\vee}$ is the main part:
If $\mathbf{b} \in \sigma^{\vee\vee}$, but $\mathbf{b} \notin \sigma$, Farkas $\implies \exists \mathbf{y} \in \sigma^\vee$ s.t. $\mathbf{y}^T\mathbf{b} > 0$ contradiction since this must be $< 0(\mathbf{b} \in \sigma^{\vee\vee})$.

□

Corollary 6. *Let $\sigma = hcone(A^T) \subseteq \mathbb{R}^m$, $A \in \mathbb{R}^{m\times n}$ be a polyhedral cone. Then*

$$\sigma^\vee = vcone(A).$$

Proof. Let $\tau = \text{vcone}(A)$, then $\sigma = \tau^\vee$, then $\sigma^\vee = \tau^{\vee\vee} = \tau$. □

Is the proof of Corollary 6 circular reasoning? [No. Because from last class we know that $\tau = \text{vcone}(A) \implies \tau^\vee = \text{hcone}(A^T)$.]

Theorem 2.1 (Weyl-Minkowski)**.** *let $\sigma \subseteq \mathbb{R}^m$ be a cone, then σ is finitely generated $\iff$ σ is polyhedral.*

Proof. σ is f.g. $\implies$ σ is polyhedral: this is just F-M.
σ is polyhedral $\implies$ σ is f.g.: σ is polyhedral $\implies$ $\sigma^\vee$ f.g. $\implies$ $\sigma^\vee$ is polyhedral $\implies$ $\sigma^{\vee\vee}$ is f.g. $\implies$ σ f.g. □

Definition. let $\sigma = \text{vcone}(A) = \text{vcone}(A_1, \dots, A_n)$ is an **irredundant representation** of σ if $\forall i \in 1, \dots, n$,

$$\text{vcone}\{A_j : j \neq i\} \neq \sigma$$

Similarly, if $\sigma = \text{hcone}(B) = \text{hcone}(B_1, \dots, B_d)^T$, say this is irredundant if $\forall i \in 1, \dots, d$,

$$\text{vcone}\{B_j : j \neq i\} \neq \sigma$$

What does F-M gives us?

$$A_{m\times n} \xrightarrow[\begin{bmatrix} A \\ -A \\ -I \end{bmatrix}]{\text{FM}} B_{m\times r} \to C_{m\times s}$$

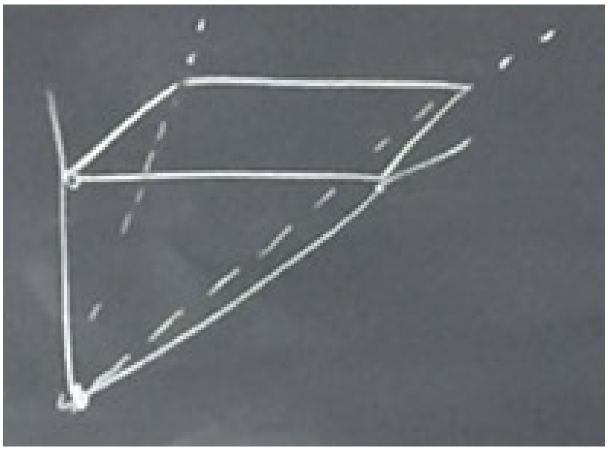
s.t. $\sigma = \text{vcone}(A) = \text{hcone}(B^T) = \text{vcone}(C)$. We can arrange that B, C give irredundant representation of $\sigma^\vee = \text{vcone}(B)$ and $\sigma^{\vee\vee} = \text{vcone}(C)$.

Example 6 (M2 code e.g. in Handout #1)**.** vcone(A) is a cone$\subseteq \mathbb{R}^4$ generated by $2\mathbf{e}_i+\mathbf{e}_j$ for $i \neq j, 1 \leq i, j \leq 4$. $B = FM(A)$ is a 4×8 matrix, whose columns generate $\sigma^\vee$.

2.2. **faces of σ, faces of $\sigma^\vee$.**

Definition. A **face** of a polyhedral cone $\sigma \subseteq \mathbb{R}^m$ is a subset $\tau \subset \sigma$ of the form $\tau = \sigma \cap u^\perp$ for some $u \in \sigma^\vee$.

Example 7. A 3d cone:



Remark.
- σ is a face of σ
- the smallest face is

$$\sigma \cap (-\sigma)(= \text{lineality space})$$

- a face $\tau \leq \sigma$($\leq$ means a face of) is also a polyhedral cone. if $\sigma = \text{vcone}(A_1, \dots, A_n)$ and $u \in \sigma^\vee$, then $\tau = \text{vcone}(\{A_i : u \cdot A_i = 0\})$.[#o6 in hand out#1:columns are gens of σ, rows are gens of $\sigma^\vee$, can figure out all faces($u_i^T v_j = 0$ in entry (i, j))]