Physics Olympiad Notes

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1 Measurement and Uncertainty

1.1 Uncertainty in Instruments

All instruments have uncertainty:

1. Analogue Instruments: Uncertainty is half the the smallest measurement unit
2. Digital Instruments: Uncertainty is the smallest significant figure
3. Human reaction time: ± 0.10s

1.2 Significant Figures

1. Adding or subtracting: Follow term with least *decimal place*
2. Multiplying or Dividing: Follow term with least *significant figure*

1.3 Propagation of error

For any $f(a, \cdots)$ the general formula for Δf is:

$$\Delta f = \sqrt{\left(\frac{\partial f}{\partial a}\Delta a\right)^2 + \cdots}$$

Some specific examples:

1. $f = a \pm b$

$$\Delta f = \sqrt{(\Delta a)^2 + (\Delta b)^2}$$

2. $f = ab$ or $f = \frac{a}{b}$

$$\frac{\Delta f}{f} = \sqrt{\left(\frac{\Delta a}{a}\right)^2 + \left(\frac{\Delta b}{b}\right)^2}$$

2 Mechanics

2.1 Statics

When all objects are motionless (or have constant velocity),

$$\sum \boldsymbol{F}_{net} = 0$$
$$\sum \boldsymbol{\tau}_{net} = 0$$

Four basic forces to consider:

Tension Pulling force felt by a rope, string, etc. Every piece of rope feels a pulling force in both directions.

Friction Parallel to surface of contact, can be static or kinetic.

Normal Perpendicular to surface of contact, prevents object from falling through surface.

Gravity Force acting between two objects with mass. Always acts downwards for objects on surface of earth.

2.2 Kinematics

$$\boldsymbol{v} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{x}}{\Delta t} = \frac{d\boldsymbol{x}}{dt} = \dot{\boldsymbol{x}}$$
$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \frac{d^2\boldsymbol{x}}{dt^2} = \dot{\boldsymbol{x}} = \ddot{\boldsymbol{x}}$$

Polar Coordinates

Differentiation of unit vectors:

$$\dot{\hat{\boldsymbol{r}}} = \dot{\theta}\hat{\boldsymbol{\theta}}$$
$$\dot{\hat{\boldsymbol{\theta}}} = -\dot{\theta}\hat{\boldsymbol{r}}$$

Velocity and acceleration in polar form:

$$\boldsymbol{r} = r\hat{\boldsymbol{r}}$$
$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$
$$\boldsymbol{a} = \dot{\boldsymbol{v}} = (\ddot{r} - \dot{\theta}^2 r)\hat{\boldsymbol{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}$$

2.3 Dynamics

$$\boldsymbol{F} = m\ddot{\boldsymbol{x}}$$
$$\boldsymbol{F}_{action} = -\boldsymbol{F}_{reaction}$$

Free body diagram techniques:

1. $\Sigma \boldsymbol{F}_{net} = 0$ for massless pulleys
2. Conservation of string

Solving differential equations in 1-dimension:

1. $F = f(t)$

$$m\int_{v_0}^{v(t)} dv' = \int_{t_0}^{t} f(t')dt'$$
$$m\int_{x_0}^{x(t)} dx' = \int_{t_0}^{t} v(t')dt'$$

2. $F = f(x)$

$$a = \frac{dv}{dt} = \frac{dv}{dt}\frac{dx}{dx} = v\frac{dv}{dx}$$
$$m\int_{v_0}^{v(x)} v'dv' = \int_{x_0}^{x} f(x')dx'$$

3. $F = f(v)$

$$m\int_{v_0}^{v(t)} \frac{dv'}{f(v')} = \int_{t_0}^{t} dt'$$

Friction

Kinetic and static friction:

$$\boldsymbol{f_k} = \mu_k \boldsymbol{N}$$
$$\boldsymbol{f_s} \le \mu_s \boldsymbol{N}$$

Static friction does no work.

Constraining Forces

For any rigid body, there are 6 degrees of freedom (DF). There can be constraining forces (C) acting on the body.

- Statics: $C + DF = 6$
- Dynamics $C + DF \ge 6$

There are 3 assumptions made for a body moving without any constraint:

1. $\boldsymbol{f}_{ij} \parallel \boldsymbol{r}_{ij}$
2. $\boldsymbol{r}_{ij}$ is constant for any 2 points in a rigid body
3. $\boldsymbol{f}_{12} + \boldsymbol{f}_{21} = 0$

Fictitious Forces

Force felt by an object in a non-inertial frame. Let $\boldsymbol{r}$ to be the position vector of the object in the accelerated frame and $\boldsymbol{R}$ be the position vector of the accelerated frame, then the possible forces that acts on $\boldsymbol{r}$ are:

$$\frac{\delta^2 \boldsymbol{r}}{\delta t^2} = \frac{\boldsymbol{F}}{m} - \frac{d^2\boldsymbol{R}}{dt^2} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) - 2\boldsymbol{\omega} \times \boldsymbol{v} - \frac{d\boldsymbol{\omega}}{dt}\boldsymbol{r}$$

1. Translational force: $-m\frac{d^2\boldsymbol{R}}{dt^2}$
2. Centrifugal force: $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})$
3. Coriolis force: $-2m\boldsymbol{\omega} \times \boldsymbol{v}$
4. Azimuthal force: $-m\frac{d\boldsymbol{\omega}}{dt}\boldsymbol{r}$

2.4 Conservation Laws

Energy $W_{NC} = 0$

Momentum $\Sigma \boldsymbol{F}_{net} = 0$

Angular Momentum $\Sigma \boldsymbol{\tau}_{net} = 0$

2.5 Energy

For a force in one dimension:

$$mv\frac{dv}{dx} = F(x)$$
$$\frac{1}{2}mv^2 = E + \int_{x_0}^{x} F(x')dx'$$

We can then define *potential energy*:

$$U(x) = -\int_{x_0}^{x} F(x')dx'$$

Work-Energy theorem:

$$W_{AB} = \int_{x_1}^{x_2} F(x')dx'$$
$$W_{\text{total}} = \Delta KE$$

Conservative forces are forces that only depend on *position*. For conservative forces:

$$\oint \boldsymbol{F} \cdot d\boldsymbol{l} = 0$$
$$\boldsymbol{\nabla} \times \boldsymbol{F} = 0$$
$$\boldsymbol{F} = -\boldsymbol{\nabla} U$$
$$W_C = -\Delta U$$

For non-conservative forces:

$$W_{NC} = \Delta(K + U) = \Delta E$$

Where E is defined as the mechanical energy of the system.

Energy Analysis

The Lagrangian method is based on the *principle of stationary action.*

$$\mathcal{L}(\dot{x}, x, t) = T - V$$
$$\frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{x}}) - \frac{\partial \mathcal{L}}{\partial x} = 0$$

The Hamiltonian $\mathcal{H}$ can be used for the conservation of energy:

$$\mathcal{H}(\dot{x}, x, t) = T + V$$
$$\dot{\mathcal{H}} = 0$$

Where T is the kinetic energy, and V is the potential energy of the system.

Power

Power is the rate of work done per unit time:

$$P = \frac{dW}{dt}$$

Mechanical power:

$$P = \frac{d}{dt}\oint \boldsymbol{F} \cdot d\boldsymbol{x} = \frac{d}{dt}\oint \boldsymbol{F} \cdot \frac{d\boldsymbol{x}}{dt}dt = \boldsymbol{F} \cdot \boldsymbol{v}$$

2.6 Momentum

Momentum is defined as:

$$\boldsymbol{p} = m\boldsymbol{v}$$

When there is no net force on the system,

$$\sum \boldsymbol{F}_{net} = 0 \Rightarrow \frac{d\boldsymbol{p}}{dt} = 0 \Rightarrow \boldsymbol{p} \text{ is conserved}$$

Impulse is defined as:

$$I = \int_{t_1}^{t_2} \boldsymbol{F}(t)dt = \int_{t_1}^{t_2} \frac{d\boldsymbol{p}}{dt}dt$$
$$I = \boldsymbol{p}(t_2) - \boldsymbol{p}(t_1) = \Delta \boldsymbol{p}$$

For perfectly elastic collisions of two objects in 1-D, relative velocity is constant.

$$\boldsymbol{v}_1 - \boldsymbol{v}_2 = -(\boldsymbol{v}_1' - \boldsymbol{v}_2')$$

For other collisions in 1-D, we have the coefficient of restitution e:

$$e = -\frac{\boldsymbol{v}_2' - \boldsymbol{v}_1'}{\boldsymbol{v}_2 - \boldsymbol{v}_1} \qquad 0 \le e \le 1$$

2.7 Oscillations

Most questions involve solving linear differential equations. For such equations, linear combinations of solutions will also be a solution.

Simple Harmonic Motion

We have a spring force, $F = -kx$.

$$\ddot{x} + \omega^2 x = 0, \text{ where } \omega = \sqrt{\frac{k}{m}}$$
$$x(t) = A\cos(\omega t + \phi)$$

Damped Harmonic Motion

In addition to the spring force, we now have a drag force $F_f = -bv$, and the total force $F = -kx - b\dot{x}$.

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0$$

Where $2\gamma = b/m$ and $\omega^2 = k/m$. Let $\Omega = \sqrt{\gamma^2 - \omega^2}$.

$$x(t) = e^{-\gamma t}(Ae^{\Omega t} + Be^{-\Omega t})$$

Underdamping $(\Omega^2 < 0)$

$$x(t) = e^{-\gamma t}(Ae^{i\tilde{\omega}t} + Be^{-i\tilde{\omega}t}) = e^{-\gamma t}C\cos(\tilde{\omega}t + \phi)$$

Where $\tilde{\omega} = \sqrt{\omega^2 - \gamma^2}$. The system will oscillate with its amplitude decreasing over time. The frequency of oscillations will be smaller than in the undamped case.

Overdamping $(\Omega^2 > 0)$

$$x(t) = Ae^{-(\gamma - \Omega)t} + Be^{-(\gamma + \Omega)t}$$

The system will not oscillate, and the motion will go to zero for large t.

Critical damping $(\Omega^2 = 0)$

We have $\gamma = \omega$, and:

$$\ddot{x} + 2\gamma\dot{x} + \gamma^2 x = 0$$

In this special case, $x = te^{-\gamma t}$ is also a solution:

$$x(t) = e^{-\gamma t}(A + Bt)$$

Systems with critical damping go to zero the quickest.

Driven Harmonic Motion

We have to solve differential equations of this form:

$$\ddot{x} + 2\gamma\dot{x} + ax = \sum_{n=1}^{N} C_n e^{i\omega_n t}$$

We first find particular solutions for each n, by guessing solutions of the form $x_{p_n}(t) = Ae^{i\omega_n t}$:

$$-A{\omega_n}^2 + 2iA\gamma\omega_n + Aa = C_n$$
$$x_{p_n}(t) = \frac{C_n}{-{\omega_n}^2 + 2i\gamma\omega_n + a} e^{i\omega_n t}$$

Using the superposition principle, the final solution is a linear combination of the general solution and the particular solutions, with the combination constants determined by initial conditions.

Coupled Oscillators

Normal modes are states of a system where all parts are moving with the same frequency. General strategy to find normal modes:

1. Write down the n equations of motions corresponding to the n degrees of freedom the system has.
2. Substitute $x_i = A_i e^{i\omega t}$ into the differential equations to get a system of linear equations in A_i, with $i = 1, 2, \cdots, n$
3. Non-trivial solutions exist if and only if the determinant of the matrix is zero. Solve for ω, and subsequently find A_i

The motion of the system can then be decomposed into linear combinations of its normal modes.

Small Oscillations

For an object at a local minimum of a potential well, we can expand $V(x)$ about the equilibrium point:

$$V(x) = V(x_0) + V'(x_0)(x - x_0) + \frac{1}{2!}V''(x_0)(x - x_0)^2 + \cdots$$

As $V(x_0)$ is an additive constant, and $V'(x_0) = 0$ by definition of equilibrium,

$$V(x) \approx \frac{1}{2}V''(x_0)(x - x_0)^2$$
$$F = -\frac{dV}{dx} = -V''(x_0)(x - x_0)$$
$$\omega = \sqrt{\frac{V''(x_0)}{m}}$$

2.8 Central Forces

For any particle subjected to a central force,

$$F(r) = mr\dot{\theta}^2 - m\ddot{r}$$
$$L = mr^2\dot{\theta}$$

Because angular momentum L is constant, we can look at central forces systems in 1-dimension.

$$V_{\text{eff}}(r) = \frac{L^2}{2mr^2} + V(r)$$
$$E = V_{\text{eff}} + \frac{1}{2}m\dot{r}^2$$

Gravity

For any two point masses of m_1 and m_2 in empty space, the gravitational force between them is:

$$\boldsymbol{F} = \frac{Gm_1m_2}{|\boldsymbol{r}|^2}\hat{\boldsymbol{r}}$$

Where $\boldsymbol{r}$ is the position vector of one mass respect to the other, and G is the gravitational constant.

$$\boldsymbol{F} = mg$$

For a mass m at the Earth's surface, where $g = 9.81m/s^2$ pointing downwards.

3 Rotational Dynamics

4 Special Relativity

4.1 Postulates

1. The speed of light has the same value in all inertial frames
2. Physical laws remain the same in all inertial frames

4.2 Kinematics

Lorentz Transformations

$$x = \gamma(x' + \beta ct')$$
$$y = y'$$
$$z = z'$$
$$ct = \gamma(\beta x' + ct')$$

Where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ and $\beta = \frac{v}{c}$.

Fundamental Effects

Length contraction

$$l' = \frac{l}{\gamma}$$

Where l is the proper length.

Time dilation

$$t' = \gamma t$$

Where t is the proper time.

Loss of simultaneity

$$\Delta t = \frac{Lv}{c^2}$$

Two events separated by L and Δt in the rest frame will appear simultaneous to an observer moving at v.

Longitudinal velocity addition

$$v'_x = \frac{u + v}{1 + uv/c^2}$$

Where u is the velocity of an object in the frame traveling at v respect to the lab frame, and v'_x is the x-velocity of the object viewed by the lab frame.

Transverse velocity addition

$$v'_y = \frac{u_y}{\gamma_v(1 + u_x v/c^2)}$$

Where u_y and u_x are velocity components of an object in the frame traveling at v respect to the lab frame, and v'_y is the y-velocity of the object viewed by the lab frame.

Longitudinal Doppler effect

$$f' = f\sqrt{\frac{1 + \beta}{1 - \beta}}$$

Where f' is the frequency observed of a moving source emitting at frequency f in its rest frame.

Minkowski Diagrams

Space-time diagrams with x and ct axes. Some properties are:

1. Light travels at 45° to horizontal.
2. x' and ct' axes of another moving frame are θ to the x and ct axes respectively, with

$$\tan(\theta) = \beta$$

3. Units on axes of the moving and stationary frames are related by:

$$\frac{x'}{x} = \frac{ct'}{ct} = \sqrt{\frac{1+\beta^2}{1-\beta^2}}$$

4.3 Dynamics

Momentum

$$\boldsymbol{p} = \gamma_v m \boldsymbol{v} = \frac{m\boldsymbol{v}}{\sqrt{1-\frac{v^2}{c^2}}}$$

Energy

$$E^2 = p^2c^2 + m^2c^4$$

For massive particles:

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

For massless particles(such as photons):

$$E = pc = \frac{hc}{\lambda}$$

4.4 4-vectors

A 4-vector $\vec{A} = (A_1, A_2, A_3, A_4)$ is a quantity that transforms as follows:

$$\begin{aligned} A_1' &= \gamma(A_1 + i\beta A_4) \\ A_2' &= A_2 \\ A_3' &= A_3 \\ A_4' &= \gamma(A_4 - i\beta A_1) \end{aligned}$$

The dot product of two 4-vectors is invariant under Lorentz transformations:

$$\vec{A} \cdot \vec{B} = \vec{A}' \cdot \vec{B}'$$

Different types of 4-vectors

4-position $(dx, dy, dz, icdt)$

4-vectors originate from the invariant interval ds.

$$\begin{aligned} \vec{ds}^{\,2} &= (dx, dy, dz, icdt)^2 \\ &= dx^2 + dy^2 + dz^2 - c^2dt^2 \end{aligned}$$

4-velocity $\gamma_v(\boldsymbol{v}, ic)$

To obtain other 4-vectors, we can multiply invariant quantities to the 4-position vector, such as proper time:

$$\begin{aligned} d\tau &= \frac{dt}{\gamma} \\ \vec{v} &= \frac{ds}{d\tau} \\ &= \gamma_v\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}, ic\right) \\ &= \gamma_v(\boldsymbol{v}, ic) \end{aligned}$$

4-momentum $\left(\boldsymbol{p}, i\frac{E}{c}\right)$

As mass is invariant,

$$\begin{aligned} \vec{p} &= m\vec{v} \\ &= (\gamma_v m\boldsymbol{v}, i\gamma_v mc) \\ &= \left(\boldsymbol{p}, i\frac{E}{c}\right) \end{aligned}$$

For photons in x-direction, the 4-momentum vector is:

$$\vec{p} = \left(\frac{h}{\lambda}, 0, 0, i\frac{h}{\lambda}\right)$$

4-wave $\left(\boldsymbol{k}, i\frac{\omega}{c}\right)$

For electromagnetic waves,

$$\begin{aligned} k &= \frac{2\pi}{\lambda} = \frac{\omega}{c} \\ \boldsymbol{p} &= \frac{h}{\lambda} = \hbar\boldsymbol{k} \\ E &= hf = \hbar\omega \\ \vec{p} &= \hbar\left(\boldsymbol{k}, i\frac{\omega}{c}\right) \\ \vec{k} &= \frac{\vec{p}}{\hbar} \end{aligned}$$

4-force $\gamma_v\left(\boldsymbol{f}, \frac{i}{c}\frac{dE}{dt}\right)$

$$\begin{aligned} \vec{F} &= \frac{d\vec{p}}{d\tau} \\ &= \gamma_v\left(\boldsymbol{f}, \frac{d}{dt}\left(i\frac{E}{c}\right)\right) \end{aligned}$$

5 Optics

5.1 Geometric Optics

Results from Fermat's principle of least time:

$$\begin{aligned} \theta_{incidence} &= \theta_{reflection} \\ n_1\sin\theta_1 &= n_2\sin\theta_2 \end{aligned}$$

Sign convention:

- Light rays travel from left to right
- f is positive if surface makes rays more convergent
- Distances are measured from the surface (left is negative)
- s_o is negative for real objects
- s_i is positive for real images
- y above optical axis is positive

$$\begin{aligned} \frac{1}{f} &= \frac{1}{s_i} + \frac{1}{s_o} \\ M &= \frac{y_i}{y_o} = -\frac{s_i}{s_o} \end{aligned}$$

For thin lenses and mirrors:

$$\frac{1}{f} = \frac{2}{R}$$

For composite thin lenses:

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

Lens formed by interface of two materials with different n:

$$\frac{n_2 - n_1}{R} = \frac{n_2}{s_i} + \frac{n_1}{s_o}$$

5.2 Polarization

For polarized light:

$$\begin{aligned} E &= E_0\cos\theta \\ I &= I_0\cos^2\theta \end{aligned}$$

For unpolarized light:

$$\langle I\rangle = I_0\langle\cos^2\theta\rangle = \frac{I_0}{2}$$

Brewster angle at which all reflected light at an interface is polarized:

$$\tan\theta_i = \frac{n_t}{n_i}$$

5.3 Physical Optics

Interference is the superposition of wave amplitudes when waves overlap.

Double Slit Interference:

Occurs when slits are of negligible width, distance between slits comparable to wavelength, such that diffraction effects are insignificant. For bright fringes:

$$d \sin\theta = m\lambda$$
$$y_m = R\frac{m\lambda}{d} \qquad m \in \mathbb{Z}$$

For incident medium's refractive index n_i, reflection medium's refractive index n_r, if $n_i < n_r$, the reflected wave undergoes a $\frac{\pi}{2}$ phase shift.

Single Slit Diffraction:

Occurs when size of slit is comparable to wavelength. Location of dark fringes when wavelets at distance $\frac{a}{2}$ destructively interfere:

$$\sin\theta = \frac{m\lambda}{d}$$
$$y_m = x\frac{m\lambda}{a} \qquad m \in \mathbb{Z}$$

Intensity in Diffraction Patterns

For double slit interference:

$$I = I_{\max} \cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right)$$

For single slit diffraction:

$$I = I_{\max}\left[\frac{\sin(\pi a \sin\theta/\lambda)}{\pi a \sin\theta/\lambda}\right]^2$$

Double slit including effects of diffraction:

$$I = I_{\max} \cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right) \cdot \left[\frac{\sin(\pi a \sin\theta/\lambda)}{\pi a \sin\theta/\lambda}\right]^2$$