

# Physics Notes

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## 1 Measurement

### 1.1 Instrument Uncertainty

All instruments have uncertainties:

1. Analogue Instruments: Half the smallest measurement unit
2. Digital Instruments: The smallest significant figure
3. Human reaction time:  $\pm 0.10$ s

### 1.2 Significant Figures

1. Adding or subtracting: Follow term with least *decimal place*
2. Multiplying or Dividing: Follow term with least *significant figure*

### 1.3 Propagation of error

For any  $f(a, \dots)$  the general formula for  $\Delta f$  is:

$$\Delta f = \sqrt{\left(\frac{\partial f}{\partial a} \Delta a\right)^2 + \dots}$$

Some specific examples:

1.  $f = a \pm b$   

$$\Delta f = \sqrt{(\Delta a)^2 + (\Delta b)^2}$$
2.  $f = ab$  or  $f = \frac{a}{b}$   

$$\frac{\Delta f}{f} = \sqrt{\left(\frac{\Delta a}{a}\right)^2 + \left(\frac{\Delta b}{b}\right)^2}$$

## 2 Mechanics

### 2.1 Statics

When all objects are motionless (or have constant velocity),

$$\sum \mathbf{F}_{net} = 0$$

$$\sum \boldsymbol{\tau}_{net} = 0$$

Four basic forces to consider:

**Tension** Pulling force felt by a rope, string, etc. Every piece of rope feels a pulling force in both directions.

**Friction** Parallel to surface of contact, can be static or kinetic.

**Normal** Perpendicular to surface of contact, prevents object from falling through surface.

**Gravity** Force acting between two objects with mass. Always acts downwards for objects on surface of earth.

### 2.2 Kinematics

$$\mathbf{v} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{x}}{\Delta t} = \frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2 \mathbf{x}}{dt^2} = \dot{\mathbf{v}} = \ddot{\mathbf{x}}$$

### 2.2.1 Polar Coordinates

Differentiation of unit vectors:

$$\dot{\hat{r}} = \dot{\theta} \hat{\theta}$$

$$\dot{\hat{\theta}} = -\dot{\theta} \hat{r}$$

Velocity and acceleration in polar form:

$$\mathbf{r} = r \hat{r}$$

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r} \hat{r} + r \dot{\theta} \hat{\theta}$$

$$\mathbf{a} = \dot{\mathbf{v}} = (\ddot{r} - \dot{\theta}^2 r) \hat{r} + (r \ddot{\theta} + 2\dot{r} \dot{\theta}) \hat{\theta}$$

### 2.3 Dynamics

$$\mathbf{F} = m \ddot{\mathbf{x}}$$

$$\mathbf{F}_{action} = -\mathbf{F}_{reaction}$$

Free body diagram techniques:

1.  $\Sigma \mathbf{F}_{net} = 0$  for massless pulleys
2. Conservation of string

Solving differential equations in 1-dimension:

1.  $F = f(t)$

$$m \int_{v_0}^{v(t)} dv' = \int_{t_0}^t f(t') dt'$$

$$m \int_{x_0}^{x(t)} dx' = \int_{t_0}^t v(t') dt'$$

2.  $F = f(x)$

$$a = \frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = v \frac{dv}{dx}$$

$$m \int_{v_0}^{v(x)} v' dv' = \int_{x_0}^x f(x') dx'$$

3.  $F = f(v)$

$$m \int_{v_0}^{v(t)} \frac{dv'}{f(v')} = \int_{t_0}^t dt'$$

#### 2.3.1 Friction

Kinetic and static friction:

$$\mathbf{f}_k = \mu_k \mathbf{N}$$

$$\mathbf{f}_s \leq \mu_s \mathbf{N}$$

Static friction does no work.

#### 2.3.2 Constraining Forces

For any rigid body, there are 6 degrees of freedom ( $DF$ ). There can be constraining forces ( $C$ ) acting on the body.

- Statics:  $C + DF = 6$
- Dynamics  $C + DF \geq 6$

There are 3 assumptions made for a body moving without any constraint:

1.  $\mathbf{f}_{ij} \parallel \mathbf{r}_{ij}$
2.  $\mathbf{r}_{ij}$  is constant for any 2 points in a rigid body
3.  $\mathbf{f}_{12} + \mathbf{f}_{21} = 0$

### 2.3.3 Fictitious Forces

For any vector  $\mathbf{A}$  in a moving frame, we calculate its time derivative in a frame rotating at  $\omega$  respect to the stationary frame:

$$\left. \frac{d\mathbf{A}}{dt} \right|_{\text{stat}} = \left. \frac{d\mathbf{A}}{dt} \right|_{\text{mov}} + \boldsymbol{\omega} \times \mathbf{A}$$

Let  $\mathbf{r}$  be the position vector of the object in an accelerated frame and  $\mathbf{R}$  be the vector to the origin of the accelerated frame, then the possible forces that acts on  $\mathbf{r}$  in the moving frame are:

$$\frac{d^2 \mathbf{r}}{dt^2} = \frac{\mathbf{F}}{m} - \frac{d^2 \mathbf{R}}{dt^2} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2\boldsymbol{\omega} \times \mathbf{v} - \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}$$

1. Translational force:  $-m \frac{d^2 \mathbf{R}}{dt^2}$
2. Centrifugal force:  $-m \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$
3. Coriolis force:  $-2m \boldsymbol{\omega} \times \mathbf{v}$
4. Azimuthal force:  $-m \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}$

### 2.4 Conservation Laws

**Energy**  $W_{NC} = 0$

**Momentum**  $\Sigma \mathbf{F}_{net} = 0$

**Angular Momentum**  $\Sigma \boldsymbol{\tau}_{net} = 0$

### 2.5 Energy

For a force in one dimension:

$$m \mathbf{\dot{r}} \frac{d\mathbf{\dot{r}}}{dr} = \mathbf{F}(\mathbf{r})$$

$$\frac{1}{2} m |\mathbf{\dot{r}}|^2 = E + \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}'$$

We can then define *potential energy*:

$$U(\mathbf{r}) = - \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}'$$

Work-Energy theorem:

$$W_{AB} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}'$$

$$W_{\text{total}} = \Delta KE$$

Conservative forces are forces that only depend on *position*. For conservative forces:

$$\oint \mathbf{F} \cdot d\mathbf{r} = 0$$

$$\nabla \times \mathbf{F} = 0$$

$$\mathbf{F} = -\nabla U$$

$$W_C = -\Delta U$$

For non-conservative forces:

$$W_{NC} = \Delta(K + U) = \Delta E$$

Where  $E$  is defined as the mechanical energy of the system.

### 2.5.1 Virial Theorem

If we have a collection of particles at positions  $\mathbf{r}_i$ , and each of them experiences a force  $\mathbf{F}_i$ , their average kinetic energy is given by:

$$\langle T \rangle = -\frac{1}{2} \left\langle \sum \mathbf{F}_i \cdot \mathbf{r}_i \right\rangle$$

For one particle:

$$\langle T \rangle = -\frac{1}{2} \left\langle \frac{dU}{dr} \cdot \mathbf{r} \right\rangle$$

### 2.5.2 Power

Power is the rate of work done per unit time:

$$P = \frac{dW}{dt}$$

Mechanical power:

$$P = \frac{d}{dt} \oint \mathbf{F} \cdot d\mathbf{r} = \frac{d}{dt} \oint \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \mathbf{F} \cdot \mathbf{v}$$

### 2.6 Momentum

Momentum is defined as:

$$\mathbf{p} = m\mathbf{v}$$

When there is no net force on the system,

$$\sum \mathbf{F}_{net} = 0 \Rightarrow \frac{d\mathbf{p}}{dt} = 0 \\ \Rightarrow \mathbf{p} \text{ is conserved}$$

Impulse is defined as:

$$\mathcal{I} = \int_{t_1}^{t_2} \mathbf{F}(t) dt = \int_{t_1}^{t_2} \frac{d\mathbf{p}}{dt} dt \\ \mathcal{I} = \mathbf{p}(t_2) - \mathbf{p}(t_1) = \Delta \mathbf{p}$$

For perfectly elastic collisions of two objects in 1-D, relative velocity is constant.

$$\mathbf{v}_1 - \mathbf{v}_2 = -(\mathbf{v}'_1 - \mathbf{v}'_2)$$

For other collisions in 1-D, we have the coefficient of restitution  $e$ :

$$e = -\frac{\mathbf{v}'_2 - \mathbf{v}'_1}{\mathbf{v}_2 - \mathbf{v}_1} \quad 0 \leq e \leq 1$$

### 2.7 Lagrangian Mechanics

The Lagrangian method is based on the *principle of stationary action*.

$$\mathcal{L}(\dot{x}, x, t) = T - V \\ \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$

### 2.7.1 Multiple Coordinates

If we have a Lagrangian in  $n$  coordinates  $\mathcal{L}(t, q_1, \dot{q}_1, \dots, q_n, \dot{q}_n)$ , we simply get  $n$  Euler-Lagrange equations:

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) = \frac{\partial \mathcal{L}}{\partial q_i}$$

### 2.7.2 Forces of Constraint

If we have an equation of constraint  $f(\mathbf{x}) = 0$ , we can use Lagrange multipliers to get the equations of motion, and along with the constraint equations solve for  $\lambda$ :

$$\frac{\partial \mathcal{L}}{\partial x_i} + \lambda \frac{\partial f}{\partial x_i} = \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}_i} \right)$$

The the forces of constraint are:

$$F_i^c = \lambda \frac{\partial f}{\partial x_i}$$

### 2.7.3 Conservation of Energy

If we take the total time derivative of the Lagrangian, we get:

$$\frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial t} + \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \dot{q}_i \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right)$$

If the Lagrangian is explicitly independent of time, we have the following conserved quantity, which is the energy of the system:

$$\frac{d}{dt} \left[ \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L} \right] = 0$$

### 2.7.4 Noether's Theorem

A "symmetry" is a change of coordinates that does not result in a first order change in the Lagrangian. For each symmetry, there is a conserved quantity. If the Lagrangian is invariant in first order under the change of coordinates:

$$q_i \rightarrow q_i + \epsilon K_i(q)$$

The following quantity is conserved:

$$\frac{d}{dt} \left[ K_i(q) \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right]$$

### 2.8 Hamiltonian Mechanics

The Hamiltonian  $\mathcal{H}(\mathbf{p}, \mathbf{q}, t)$  for multiple coordinates is defined as:

$$\mathcal{H} = p_i \dot{q}_i - \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) \\ p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \quad 4$$

The following equations of motion can then be obtained:

$$\frac{\partial \mathcal{H}}{\partial q_i} = -\dot{p}_i \\ \frac{\partial \mathcal{H}}{\partial p_i} = \dot{q}_i \\ \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}$$

### 2.8.1 Liouville's Theorem

The Hamiltonian formulation gives two first order ordinary differential equations, which can always be uniquely solved when given initial conditions  $(\mathbf{p}_0, \mathbf{q}_0)$ . Thus no two phase space orbits with different initial conditions cross, and consequently any volume in phase space is constant under time evolution.

### 2.8.2 Poisson Brackets

The Poisson bracket binary operation is defined as:

$$\{f(p, q, t), g(p, q, t)\} = \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial q}$$

If  $p$  and  $q$  are the solutions to Hamiltonian's equations:

$$\dot{q} = \{q, \mathcal{H}\}$$

$$\dot{p} = \{p, \mathcal{H}\}$$

$$\frac{d}{dt} f(p, q, t) = \{f, \mathcal{H}\} + \frac{\partial f}{\partial t}$$

### 2.9 Central Forces

For any two objects subject to a central force,

$$F(r) = \mu \ddot{r} - \mu r \dot{\theta}^2 \\ L = \mu r^2 \dot{\theta}$$

Where  $\mu = (m_1 m_2) / (m_1 + m_2)$  is their reduced mass. Because angular momentum  $L$  is constant, we can look at central forces systems in 1-dimension.

$$V_{\text{eff}}(r) = \frac{L^2}{2\mu r^2} + V(r)$$

$$E = V_{\text{eff}} + \frac{1}{2} \mu \dot{r}^2$$

If we let  $q(\theta) = \frac{1}{r}$ , we get the following equation in polar coordinates:

$$q''(\theta) + q(\theta) + \frac{\mu}{L^2 q^2} F(r) = 0$$

### 2.9.1 Gravity

For any two point masses of  $m_1$  and  $m_2$  in empty space, the gravitational force between them is:

$$\mathbf{F} = \frac{G m_1 m_2}{|\mathbf{r}|^2} \hat{\mathbf{r}}$$

Where  $\mathbf{r}$  is the position vector of one mass respect to the other, and  $G$  is the gravitational constant.

$$F = mg$$

For a mass  $m$  at the Earth's surface, where  $g = 9.81 \text{ m/s}^2$  pointing downwards.

## 2.10 Uniform Circular Motion

For a point mass moving in uniform circular motion, we define:

$$\omega = \frac{v}{r}$$

The centripetal acceleration  $a$  and the force required to keep the object in its circular path:

$$a = \frac{v^2}{r} = \omega^2 r$$

$$F = \frac{mv^2}{r} = m\omega^2 r$$

## 2.11 Rotational Dynamics (Constant $\hat{L}$ )

### 2.11.1 Angular Momentum

The angular momentum of a point mass is defined as:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

For a flat object lying on a 2-D plane rotating with angular speed  $\omega$ :

$$\mathbf{L} = \int \mathbf{r} \times \mathbf{p} = \int r^2 \omega \hat{\mathbf{z}} dm$$

If we define the *moment of inertia* about the  $z$ -axis to be  $I_z = \int (x^2 + y^2) dm$ , we have:

$$L_z = I_z \omega$$

$$T = \int \frac{1}{2} m v^2 = \int \frac{r^2 \omega^2}{2} dm = \frac{1}{2} I_z \omega^2$$

For the  $z$ -component of  $\mathbf{L}$  and kinetic energy  $T$ .

### 2.11.2 General Motion

For an object with a moving center of mass, and rotating at  $\omega$  about it,

$$\mathbf{L} = \mathbf{r}_{\text{CM}} \times \mathbf{p}_{\text{CM}} + I_{\text{CM}} \omega \hat{\mathbf{z}}$$

$$T = \frac{1}{2} m v_{\text{CM}}^2 + \frac{1}{2} I_{\text{CM}} \omega^2$$

### 2.11.3 Torque

Torque is defined as:

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

Using an origin satisfying any of the following conditions to calculate  $\mathbf{L}$ ,

1. The origin is the center of mass

2. The origin is not accelerating
3.  $(\mathbf{R} - \mathbf{r}_0)$  is parallel to  $\mathbf{r}_0$ , the position of the origin in a fixed coordinate system

$$\frac{d\mathbf{L}}{dt} = \sum \boldsymbol{\tau}_{\text{ext}}$$

When there is no external torque, we have the conservation of angular momentum.

$$\boldsymbol{\tau}_{\text{ext}} = I \alpha$$

Where  $\alpha = \frac{d\omega}{dt}$  is the angular acceleration.

### 2.11.4 Angular Impulse

Angular impulse is defined as:

$$\mathcal{I}_\theta = \int_{t_1}^{t_2} \boldsymbol{\tau}(t) dt = \Delta \mathbf{L}$$

If we have a force  $\mathbf{F}(t)$  applied at a constant distance  $R$  from the origin,

$$\boldsymbol{\tau}(t) = \mathbf{R} \times \mathbf{F}(t)$$

$$\mathcal{I}_\theta = \mathbf{R} \times \mathcal{I}$$

$$\Delta \mathbf{L} = \mathbf{R} \times (\Delta \mathbf{p})$$

### 2.11.5 Parallel-axis Theorem

Let an object of mass  $M$  rotate about its center of mass with the same frequency  $\omega$  as the center of mass rotates about the origin (with radius  $R$ ):

$$L_z = (MR^2 + I_{\text{CM}}) \omega$$

Thus if the moment of inertia of an object is  $I_0$  about a particular axis, its moment of inertia about a parallel axis separated by  $R$  is:

$$I = MR^2 + I_0$$

### 2.11.6 Perpendicular-axis Theorem

For flat 2-D objects in the  $x$ - $y$  plane, and orthogonal axes  $x$ ,  $y$  and  $z$ :

$$I_z = I_x + I_y$$

### 2.11.7 Moments of Inertia

Center of mass for an object of mass  $M$ :

$$\mathbf{R}_{\text{CM}} = \frac{\int \mathbf{r} dm}{M}$$

Common moments of inertia (taken about center of mass unless stated):

1. Point mass at  $r$  from axis:  $mr^2$
2. Rod of length  $L$  about center:  $\frac{1}{12} mL^2$
3. Rod of length  $L$  about one end:  $\frac{1}{3} mL^2$

4. Solid disk of radius  $r$  perpendicular to axis:  $\frac{1}{2} mr^2$
5. Hollow sphere with radius  $r$ :  $\frac{2}{3} mr^2$
6. Solid sphere with radius  $r$ :  $\frac{2}{5} mr^2$

## 2.12 General Rotational Motion

For any body moving in space, its motion can be written as a sum of its translational motion and a rotation about an axis at a particular time.

### 2.12.1 Angular Velocity

The angular velocity vector  $\boldsymbol{\omega}$  points along the axis of rotation, with a magnitude equal to its angular speed. Its direction is determined by convention of the right hand rule. For an object rotating at  $\boldsymbol{\omega}$ , the time derivative of any vector  $\mathbf{r}$  fixed in the body frame is:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \times \mathbf{r}$$

Angular velocities add like vectors. Let  $S_1$ ,  $S_2$  and  $S_3$  be coordinate systems. If  $S_1$  rotates with  $\boldsymbol{\omega}_{1,2}$  with respect to  $S_2$ , and  $S_2$  rotates with  $\boldsymbol{\omega}_{2,3}$  with respect to  $S_3$ , then  $S_1$  rotates instantaneously with respect to  $S_3$  at:

$$\boldsymbol{\omega}_{1,3} = \boldsymbol{\omega}_{1,2} + \boldsymbol{\omega}_{2,3}$$

### 2.12.2 Angular Momentum

$$\mathbf{L} = \int \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) dm = \mathbf{I} \boldsymbol{\omega}$$

$\mathbf{I}$  is the moment of inertia tensor:

$$\begin{pmatrix} \int (y^2 + z^2) & -\int xy & -\int zx \\ -\int xy & \int (z^2 + x^2) & -\int yz \\ -\int zx & -\int yz & \int (x^2 + y^2) \end{pmatrix}$$

The kinetic energy of the object is given by:

$$T = \int \frac{1}{2} |\boldsymbol{\omega} \times \mathbf{r}|^2 dm = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{I} \boldsymbol{\omega} = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{L}$$

To find the angular momentum for an object of mass  $M$  in general motion, let the position of its center of mass be  $\mathbf{R}$ , its velocity be  $\mathbf{V}$ . Then:

$$\mathbf{L} = M(\mathbf{R} \times \mathbf{V}) + \mathbf{L}_{\text{CM}}$$

The kinetic energy of the object is:

$$T = \frac{1}{2} M V^2 + \frac{1}{2} \boldsymbol{\omega}' \mathbf{L}_{\text{CM}}$$

Where  $\omega'$  and  $\mathbf{L}_{\text{CM}}$  are measured about the center of mass along axes parallel to the fixed-frame axes.

### 2.12.3 Principle Axes

A principle axis is an axis of rotation  $\hat{\omega}$  such that  $\mathbf{I}\hat{\omega} = I\hat{\omega}$ . An object can rotate about a principle axis at constant angular velocity with no external torque. An orthonormal set of principle axis exists for every object.

### 2.12.4 Euler's Equations

When an object is instantaneously rotating about an axis  $\omega$ , we can relate the rate of change of angular momentum in the frame of the principle axes and the lab frame by:

$$\frac{d\mathbf{L}}{dt}_{\text{lab}} = \frac{d\mathbf{L}}{dt}_{\text{body}} + \omega \times \mathbf{L}$$

This gives us Euler's equations, where  $\omega_i$  and  $\tau_i$  are components of  $\omega$  and torque projected onto the principle axes respectively:

$$\tau_1 = I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3$$

$$\tau_2 = I_1\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1$$

$$\tau_3 = I_1\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2$$

## 3 Special Relativity

### 3.1 Postulates

1. The speed of light has the same value in all inertial frames
2. Physical laws remain the same in all inertial frames

### 3.2 Kinematics

#### 3.2.1 Lorentz Transform

$$x = \gamma(x' + \beta ct')$$

$$y = y'$$

$$z = z'$$

$$ct = \gamma(\beta x' + ct')$$

Where  $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$  and  $\beta = \frac{v}{c}$ .

#### 3.2.2 Fundamental Effects

##### Length contraction

$$l' = \frac{l}{\gamma}$$

Where  $l$  is the proper length.

##### Time dilation

$$t' = \gamma t$$

Where  $t$  is the proper time.

### Loss of simultaneity

$$\Delta t = \frac{Lv}{c^2}$$

Two events separated by  $L$  and  $\Delta t$  in the rest frame will appear simultaneous to an observer moving at  $v$ .

### Longitudinal velocity addition

$$v'_x = \frac{u + v}{1 + uv/c^2}$$

Where  $u$  is the velocity of an object in the frame traveling at  $v$  respect to the lab frame, and  $v'_x$  is the  $x$ -velocity of the object viewed by the lab frame.

### Transverse velocity addition

$$v'_y = \frac{u_y}{\gamma_v(1 + u_x v/c^2)}$$

Where  $u_y$  and  $u_x$  are velocity components of an object in the frame traveling at  $v$  respect to the lab frame, and  $v'_y$  is the  $y$ -velocity of the object viewed by the lab frame.

### Longitudinal Doppler effect

$$f' = f \sqrt{\frac{1 + \beta}{1 - \beta}}$$

Where  $f'$  is the frequency observed of a moving source emitting at frequency  $f$  in its rest frame.

### 3.2.3 Minkowski Diagrams

Space-time diagrams with  $x$  and  $ct$  axes. Some properties are:

1. Light travels at  $45^\circ$  to horizontal.
2.  $x'$  and  $ct'$  axes of another moving frame are  $\theta$  to the  $x$  and  $ct$  axes respectively, with

$$\tan(\theta) = \beta$$

3. Units on axes of the moving and stationary frames are related by:

$$\frac{x'}{x} = \frac{ct'}{ct} = \sqrt{\frac{1 + \beta^2}{1 - \beta^2}}$$

## 3.3 Dynamics

### 3.3.1 Momentum

$$\mathbf{p} = \gamma_v m \mathbf{v} = \frac{m \mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

### 3.3.2 Energy

$$E^2 = p^2 c^2 + m^2 c^4$$

For massive particles:

$$E = \gamma m c^2 = \frac{m c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

For massless particles (such as photons):

$$E = pc = \frac{hc}{\lambda}$$

## 3.4 4-vectors

A 4-vector  $\vec{A} = (A_0, A_1, A_2, A_3)$  is a quantity that transforms as follows:

$$A'_0 = \gamma(A_0 + \beta A_1)$$

$$A'_1 = \gamma(A_1 + \beta A_0)$$

$$A'_2 = A_2$$

$$A'_3 = A_3$$

The dot product of two 4-vectors is invariant under Lorentz transformations:

$$\begin{aligned} \vec{A} \cdot \vec{B} &= A_1 B_1 + A_2 B_2 + A_3 B_3 - A_0 B_0 \\ &= \vec{A}' \cdot \vec{B}' \end{aligned}$$

### 3.4.1 Different 4-vectors

**4-position** ( $cdt, dx, dy, dz$ ) 4-vectors originate from the invariant interval  $ds$ .

$$\begin{aligned} d\vec{s}^2 &= (cdt, dx, dy, dz)^2 \\ &= dx^2 + dy^2 + dz^2 - c^2 dt^2 \end{aligned}$$

**4-velocity**  $\gamma_v(c, \mathbf{v})$  To obtain other 4-vectors, we can multiply invariant quantities to the 4-position vector, such as proper time:

$$\begin{aligned} d\tau &= \frac{dt}{\gamma} \\ \vec{v} &= \frac{ds}{d\tau} \\ &= \gamma_v \left( c, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right) \end{aligned}$$

**4-momentum**  $\left( \frac{E}{c}, \mathbf{p} \right)$  As mass is invariant,

$$\begin{aligned} \vec{p} &= m \vec{v} \\ &= (\gamma_v m \mathbf{v}, \gamma_v m c) \\ &= \left( \frac{E}{c}, \mathbf{p} \right) \end{aligned}$$

For photons in  $x$ -direction, the 4-momentum vector is:

$$\vec{p} = \left( \frac{h}{\lambda}, \frac{h}{\lambda}, 0, 0 \right)$$

**4-wave**  $(\frac{\omega}{c}, \mathbf{k})$  For electromagnetic waves,

$$k = \frac{2\pi}{\lambda} = \frac{\omega}{c}$$

$$\mathbf{p} = \frac{h}{\lambda} = \hbar \mathbf{k}$$

$$E = hf = \hbar\omega$$

$$\vec{p} = \hbar \left( \frac{\omega}{c}, \mathbf{k} \right)$$

$$\vec{k} = \frac{\vec{p}}{\hbar}$$

**4-acceleration**  $\gamma_v^4 \left( va_x, a_x, \frac{a_y}{\gamma_v^2}, \frac{a_z}{\gamma_v^2} \right)$

$$\vec{a} = \frac{d\vec{v}}{d\tau}$$

$$= \gamma_v^4 \left( va_x, a_x, \frac{a_y}{\gamma_v^2}, \frac{a_z}{\gamma_v^2} \right)$$

**4-force**  $\gamma_v \left( \frac{1}{c} \frac{dE}{dt}, \mathbf{f} \right)$

$$\vec{F} = \frac{d\vec{p}}{d\tau}$$

$$= \gamma_v \left( \frac{dE}{dt} \frac{1}{c}, \mathbf{f} \right)$$

## 4 Electricity and Magnetism

### 4.1 Electrostatics

**Coulomb's law** The force between a point charge  $q$  and test charge  $Q$ :

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{Qq}{r^2} \hat{\mathbf{r}}$$

Where  $\mathbf{r} = \mathbf{r} - \mathbf{r}'$  is the displacement vector from  $Q$  at  $\mathbf{r}$  and  $q$  at  $\mathbf{r}'$ .

**Superposition principle** The interaction between any two charges is unaffected by any other charges

#### 4.1.1 Electric Field

The electric field of a point charge is defined as:

$$\mathbf{E} = \frac{\mathbf{F}}{Q} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}$$

For a continuous volume charge distribution  $\rho(\mathbf{r}')$ , we can use the superposition principle to get:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}')}{r^2} \hat{\mathbf{r}} d\tau'$$

Taking the divergence of  $\mathbf{E}$ , we get Gauss' law:

$$\nabla \cdot \mathbf{E} = \frac{\rho(\mathbf{r})}{\epsilon_0}$$

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

Taking the curl of  $\mathbf{E}$ :

$$\nabla \times \mathbf{E} = 0$$

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0$$

For any surface charge in an electric field  $\mathbf{E}$ , the field felt by an area element on the surface is:

$$\mathbf{E}_{\text{felt}} = \frac{1}{2} (\mathbf{E}_{\text{above}} + \mathbf{E}_{\text{below}})$$

#### 4.1.2 Electric Potential

As the line integral of the electrostatic field is path independent, we can define the potential at a point  $\mathbf{r}$ :

$$V(\mathbf{r}) = - \int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}$$

Where  $\mathcal{O}$  is a standard reference point, usually set to infinity. The potential of a point charge can then be found, and with the superposition principle we can find the potential of any charge distribution:

$$V = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}')}{r} d\tau'$$

Taking the gradient of the potential:

$$\mathbf{E} = -\nabla V$$

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

#### 4.1.3 Work and Energy

The work needed to bring a charge  $Q$  from infinity to a point  $\mathbf{a}$  is:

$$W = \int_{\infty}^{\mathbf{a}} \mathbf{F} \cdot d\mathbf{l}$$

$$= -Q \int_{\infty}^{\mathbf{a}} \mathbf{E} \cdot d\mathbf{l}$$

$$= QV(\mathbf{a})$$

The energy in a continuous charge distribution is:

$$W = \frac{1}{2} \int \rho V d\tau$$

$$= \frac{\epsilon}{2} \int E^2 d\tau$$

Where the integral is taken over all space.

#### 4.1.4 Conductors

A perfect conductor has an unlimited supply of free charges.

1.  $\mathbf{E} = 0$  and  $\rho = 0$  inside a conductor
2. Any conductor is an equipotential
3. Just outside a conductor,  $\mathbf{E}$  is perpendicular to the surface.

If we charge up two conductors with  $+Q$  and  $-Q$ , the potential between them is proportional to the charge  $Q$

(because the electric field is proportional to  $Q$ ), and we define the constant of proportionality capacitance:

$$C = \frac{Q}{V}$$

The work done by charging a capacitor is:

$$W = \int_0^Q \frac{q}{C} dq = \frac{Q^2}{2C}$$

$$= \frac{1}{2} CV^2$$

#### 4.1.5 Image Charges

In certain special cases, a charge placed next to a grounded conductor has equivalents.

1. A point charge and a conducting sheet: An opposite charge in the mirror image position.
2. A point charge and a conducting sphere, or an infinite line charge and conducting cylinder: Opposite image charge and charge forms the Apollonius sphere/cylinder.

#### 4.1.6 Uniqueness Theorems

**First uniqueness theorem** The solution to Laplace's equation ( $\nabla^2 V = 0$ ) in some volume  $\mathcal{V}$  is uniquely determined if  $V$  is specified on the boundary surface  $\mathcal{S}$ .

**Second uniqueness theorem** In a volume  $\mathcal{V}$  surrounded by conductors and containing a specified charge density  $\rho$ , the electric field is uniquely determined if the total charge on each conductor is given.

## 4.2 Magnetostatics

### 4.2.1 Lorentz Force Law

The force felt by:

1. A point charge  $q$  moving at velocity  $\mathbf{v}$  through a magnetic field  $\mathbf{B}$ :

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$$

2. A line current  $I$ :

$$\mathbf{F} = I \int (d\mathbf{l} \times \mathbf{B})$$

3. A general volume current  $\mathbf{J}$  per unit area perpendicular to flow:

$$\mathbf{F} = \int (\mathbf{J} \times \mathbf{B}) d\tau$$

### 4.2.2 Biot-Savart Law

The magnetic field created by a steady line current:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l} \times \hat{\mathbf{r}}}{r^2}$$

### 4.2.3 Magnetic Fields

Unlike in electrostatics, conductors do not screen magnetic fields. The magnetic field is divergence-free:

$$\nabla \cdot \mathbf{B} = 0$$

$$\oint \mathbf{B} \cdot d\mathbf{a} = 0$$

Taking the curl of the magnetic field gives Ampere's Law:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$

The energy stored in an magnetic field is:

$$U = \frac{1}{2\mu_0} \int B^2 d\tau$$

### 4.2.4 Magnetic Vector Potential

For any magnetic field  $\mathbf{B}$ , we define the vector potential  $\mathbf{A}$  such that:

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Taking the curl of the magnetic field and applying Ampere's law, we get:

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}}{r} dV$$

## 4.3 Electrodynamics

### 4.3.1 Electric Currents

For  $N_e$  particles of charge  $e$  moving at an average velocity  $\langle \mathbf{v} \rangle$ , the current density is:

$$\mathbf{J} = -N_e e \langle \mathbf{v} \rangle$$

If  $\rho$  is the charge density, the current  $I$  is given by:

$$I = -\frac{\partial}{\partial t} \int_V \rho d\tau$$

When the total charge is conserved, we have the continuity equation:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho(t)}{\partial t}$$

### 4.3.2 Electromotive Force

For an electric field applied in a material:

$$\mathbf{J} = \sigma \mathbf{E}$$

Where  $\sigma$  is the conductivity constant depending on the material. This leads to Ohm's law:

$$V = IR$$

$$R = \frac{l}{\sigma A}$$

The power delivered:

$$P = VI = I^2 R$$

The electromotive force (emf)  $\mathcal{E}$  is the line integral of the force per unit charge driving the current:

$$\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l}$$

$\mathcal{E} = V$  for an ideal source.

### 4.3.3 Faraday's Law

Faraday's law states that a changing magnetic flux  $\Phi$  induces an electric field:

$$\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt}$$

$$\nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt}$$

### 4.3.4 Inductance

If we have two current loops 1 and 2, the flux  $\Phi_2$  through loop 2 is proportional to the current through loop 1:

$$\Phi_2 = M_{21} I_1$$

Where  $M_{21} = M_{12}$  is the mutual inductance between these two loops. We can also define an self inductance  $L$ , for a single loop:

$$\Phi = LI$$

$$\mathcal{E} = -L \frac{dI}{dt}$$

When a steady current  $I$  is flowing through an inductor with inductance  $L$ , the energy stored in the inductor is:

$$U = \frac{1}{2} LI^2$$

### 4.3.5 Displacement Current

In order for the continuity equation to hold under changing magnetic fields, we must consider another displacement current when using Ampere's law:

$$\mathbf{J}_D = \epsilon_0 \frac{d\mathbf{E}}{dt}$$

$$\nabla \times \mathbf{B} = \mu_0 (\mathbf{J} + \mathbf{J}_D)$$

## 4.4 Electric Circuits

## 5 Oscillations and Waves

Many questions involve solving linear differential equations. For such equations, linear combinations of solutions will also be a solution.

### 5.1 Oscillations

#### 5.1.1 Simple Harmonic Motion

We have a spring force,  $F = -kx$ .

$$\ddot{x} + \omega^2 x = 0, \text{ where } \omega = \sqrt{\frac{k}{m}}$$

$$x(t) = A \cos(\omega t + \phi)$$

#### 5.1.2 Damped Oscillators

In addition to the spring force, we now have a drag force  $F_f = -bv$ , and the total force  $F = -kx - b\dot{x}$ .

$$\ddot{x} + 2\gamma \dot{x} + \omega^2 x = 0$$

Where  $2\gamma = b/m$  and  $\omega^2 = k/m$ . Let  $\Omega = \sqrt{\gamma^2 - \omega^2}$ .

$$x(t) = e^{-\gamma t} (Ae^{\Omega t} + Be^{-\Omega t})$$

**Underdamping** ( $\Omega^2 < 0$ )

$$x(t) = e^{-\gamma t} (Ae^{i\tilde{\omega}t} + Be^{-i\tilde{\omega}t}) = e^{-\gamma t} C \cos(\tilde{\omega}t + \phi)$$

Where  $\tilde{\omega} = \sqrt{\omega^2 - \gamma^2}$ . The system will oscillate with its amplitude decreasing over time. The frequency of oscillations will be smaller than in the undamped case.

**Overdamping** ( $\Omega^2 > 0$ )

$$x(t) = Ae^{-(\gamma-\Omega)t} + Be^{-(\gamma+\Omega)t}$$

The system will not oscillate, and the motion will go to zero for large  $t$ .

**Critical damping** ( $\Omega^2 = 0$ ) We have  $\gamma = \omega$ , and:

$$\ddot{x} + 2\gamma \dot{x} + \gamma^2 x = 0$$

In this special case,  $x = te^{-\gamma t}$  is also a solution:

$$x(t) = e^{-\gamma t} (A + Bt)$$

Systems with critical damping go to zero the quickest.

#### 5.1.3 Driven Oscillators

We have to solve differential equations of this form:

$$\ddot{x} + 2\gamma \dot{x} + ax = \sum_{n=1}^N C_n e^{i\omega_n t}$$



We first find particular solutions for each  $n$ , by guessing solutions of the form  $x_{p_n}(t) = Ae^{i\omega_n t}$ :

$$-A\omega_n^2 + 2iA\gamma\omega_n + Aa = C_n$$

$$x_{p_n}(t) = \frac{C_n}{-\omega_n^2 + 2i\gamma\omega_n + a} e^{i\omega_n t}$$

Using the superposition principle, the final solution is a linear combination of the general solution and the particular solutions, with the combination constants determined by initial conditions.

### 5.1.4 Coupled Oscillators

Normal modes are states of a system where all parts are moving with the same frequency. General strategy to find normal modes:

1. Write down the  $n$  equations of motions corresponding to the  $n$  degrees of freedom the system has.
2. Substitute  $x_i = A_i e^{i\omega t}$  into the differential equations to get a system of linear equations in  $A_i$ , with  $i = 1, 2, \dots, n$
3. Non-trivial solutions exist if and only if the determinant of the matrix is zero. Solve for  $\omega$ , and subsequently find  $A_i$

The motion of the system can then be decomposed into linear combinations of its normal modes.

### 5.1.5 Small Oscillations

For an object at a local minimum of a potential well, we can expand  $V(x)$  about the equilibrium point:

$$V(x) = V(x_0) + V'(x_0)(x - x_0) + \frac{1}{2!} V''(x_0)(x - x_0)^2 + \dots$$

As  $V(x_0)$  is an additive constant, and  $V'(x_0) = 0$  by definition of equilibrium,

$$V(x) \approx \frac{1}{2} V''(x_0)(x - x_0)^2$$

$$F = -\frac{dV}{dx} = -V''(x_0)(x - x_0)$$

$$\omega = \sqrt{\frac{V''(x_0)}{m}}$$

## 5.2 Wave Equation

A wave is a disturbance of a continuous medium that propagates with a fixed shape at constant velocity. In one dimension:

$$u(z, t) = u(z - vt, 0) = f(z - vt)$$

All such functions  $f$  are the solutions to the wave equation:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2}$$

Where  $v$  is the speed of propagation.

### 5.2.1 String with Fixed Ends

If the equation is subject to the following initial and boundary conditions:

$$u_x(0, t) = u_x(L, t) = 0$$

$$u(x, 0) = f(x)$$

$$u_t(x, 0) = g(x)$$

The solution for these conditions is:

$$u(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi}{L} x \cdot$$

$$\left( a_n \sin \frac{n\pi\alpha}{L} t + b_n \cos \frac{n\pi\alpha}{L} t \right)$$

$$a_n = \frac{2}{n\pi\alpha} \int_0^L g(x) \sin \frac{n\pi x}{L} dx$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

### 5.2.2 D'Alembert's Solution

For an infinite string, it can be proved that any solution to the wave equation can be written as a superposition of two waves of velocity  $v$ , one travelling to the left, the other travelling to the right. For the initial conditions:

$$u(x, 0) = f(x)$$

$$u_t(x, 0) = g(x)$$

The solution of the wave equation is:

$$u(x, t) = \frac{1}{2} \left[ f(x + vt) + f(x - vt) + \frac{1}{v} \int_{x-vt}^{x+vt} g(x') dx' \right]$$

### 5.2.3 Electromagnetic Waves

Maxwell's equations in vacuum:

$$\nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt}$$

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{d\mathbf{E}}{dt}$$

$$\nabla \cdot \mathbf{B} = 0$$

Plugging the equations in and simplifying, we get:

$$\nabla^2 \mathbf{B} = \frac{1}{\mu_0 \epsilon_0} \frac{d\mathbf{B}}{dt}$$

$$\nabla^2 \mathbf{E} = \frac{1}{\mu_0 \epsilon_0} \frac{d\mathbf{E}}{dt}$$

For Maxwell's equations to hold, the  $\mathbf{E}$  and  $\mathbf{B}$  fields and their direction of propagation are mutually perpendicular. Also, the amplitudes  $E_0$  and  $B_0$  are related by:  $B_0 = \frac{1}{c} E_0$ .

### 5.2.4 Poynting Vector

The Poynting vector  $\mathbf{S}$  is defined as:

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$$

This vector points in the direction of propagation of the wave, and  $\mathbf{S} \cdot d\mathbf{a}$  is the energy per unit time passing through  $d\mathbf{a}$ .

## 6 Optics

### 6.1 Geometric Optics

Results from Fermat's principle of least time:

$$\theta_{\text{incidence}} = \theta_{\text{reflection}}$$

$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$

Sign convention:

- Light rays travel from left to right
- $f$  is positive if surface makes rays more convergent
- Distances are measured from the surface (left is negative)
- $s_o$  is negative for real objects
- $s_i$  is positive for real images
- $y$  above optical axis is positive

$$\frac{1}{s_o} + \frac{1}{f} = \frac{1}{s_i}$$

$$M = \frac{y_i}{y_o} = -\frac{s_i}{s_o}$$

For thin lenses and mirrors:

$$\frac{1}{f} = \frac{1}{R}$$

For composite thin lenses:

$$\frac{1}{f} = (n-1) \left( \frac{1}{R_1} + \frac{1}{R_2} \right)$$

Lens formed by interface of two materials with different  $n$ :

$$\frac{n_2 - n_1}{R} = \frac{n_2}{s_i} + \frac{n_1}{s_o}$$

### 6.2 Polarization

For polarized light:

$$E = E_0 \cos \theta$$

$$I = I_0 \cos^2 \theta$$

For unpolarized light:

$$\langle I \rangle = I_0 \langle \cos^2 \theta \rangle = \frac{I_0}{2}$$

Brewster angle at which all reflected light at an interface is polarized:

$$\tan \theta_i = \frac{n_t}{n_i}$$

### 6.3 Physical Optics

Interference is the superposition of wave amplitudes when waves overlap.

#### 6.3.1 Double Slit:

Occurs when slits are of negligible width, distance between slits comparable to wavelength, such that diffraction effects are insignificant. For bright fringes:

$$d\sin\theta = m\lambda$$
$$y_m = R \frac{m\lambda}{d} \quad m \in \mathbb{Z}$$

For incident medium's refractive index  $n_i$ , reflection medium's refractive index  $n_r$ , if  $n_i < n_r$ , the reflected wave undergoes a  $\frac{\pi}{2}$  phase shift.

#### 6.3.2 Single Slit:

Occurs when size of slit is comparable to wavelength. Location of dark fringes when wavelets at distance  $\frac{a}{2}$  destructively interfere:

$$\sin\theta = \frac{m\lambda}{d}$$
$$y_m = x \frac{m\lambda}{a} \quad m \in \mathbb{Z}$$

#### 6.3.3 Intensity in Diffraction Patterns

For double slit interference:

$$I = I_{\max} \cos^2 \left( \frac{\pi d \sin\theta}{\lambda} \right)$$

For single slit diffraction:

$$I = I_{\max} \left[ \frac{\sin(\pi a \sin\theta / \lambda)}{\pi a \sin\theta / \lambda} \right]^2$$

Double slit including effects of diffraction:

$$I = I_{\max} \cos^2 \left( \frac{\pi d \sin\theta}{\lambda} \right) \cdot \left[ \frac{\sin(\pi a \sin\theta / \lambda)}{\pi a \sin\theta / \lambda} \right]^2$$

## 7 Thermodynamics

If two objects are in thermal equilibrium with a third system, then they are in equilibrium with each other.

### 7.1 Thermal Expansion

For linear expansion, the change in length is:

$$\Delta L = \alpha L_0 \Delta T$$

Where  $\alpha$  is the coefficient of linear expansion. For area expansion, use approximately  $2\alpha$ . For volume expansion, use approximately  $3\alpha$ .

## 7.2 Kinetic Theory of Gases

### 7.2.1 Ideal Gas Law

An ideal gas' molecules are treated as non-interacting point particles. For an ideal gas of  $N$  particles at pressure  $P$ , volume  $V$  and temperature  $T$ :

$$PV = NK_B T$$

For a non-ideal gas, the Van der Waals correction to the ideal gas law is:

$$\left( P + a \left( \frac{n}{V} \right)^2 \right) (V - bn) = nRT$$

Where  $a$  and  $b$  are constants.

### 7.2.2 Internal Energy

Different gases at the same temperature have the same average kinetic energy. Thus we define temperature of a substance to be its average kinetic energy. For a monatomic ideal gas:

$$\frac{1}{2} m \langle v^2 \rangle = \frac{3}{2} kT$$

For a gas molecule with  $r$  atoms, its total kinetic energy, center of mass kinetic energy and internal vibrational/rotational energy are given by:

$$E_{\text{Total}} = \frac{3r}{2} kT$$
$$E_{\text{COM}} = \frac{3}{2} kT$$
$$E_{\text{Internal}} = \frac{3(r-1)}{2} kT$$

The equipartition theorem states that each degree of freedom a molecule has contributes an extra  $\frac{1}{2} kT$  of kinetic energy.

### 7.2.3 Maxwell Distribution

For an ideal gas, the distribution of its velocities is:

$$f(v) = 4\pi v^2 \left( \frac{m}{2\pi kT} \right)^{\frac{3}{2}} e^{-\frac{mv^2}{2kT}}$$

From this distribution, we can get the average speed of a particle:

$$\langle v \rangle = \sqrt{\frac{8kT}{\pi m}}$$

The most probable velocity is the maximum point of the distribution:

$$v_{\text{mp}} = \sqrt{\frac{2kT}{m}}$$

For any two particles, their average relative speed is:

$$\langle v_{\text{rel}} \rangle = \sqrt{2} \langle v \rangle = \sqrt{\frac{16kT}{\pi m}}$$

From this, we can get the mean free path of a particle, the average distance a particle travels before hitting another particle:

$$l_m = \frac{1}{4\pi\sqrt{2}r^2n}$$

Where  $n$  is the number density of the particle and  $r$  is its radius.

### 7.2.4 Diffusion

For a substance undergoing diffusion due to a concentration gradient  $\frac{dc}{dx}$ , the diffusive flux  $J$  is:

$$J = DA \frac{dc}{dx}$$

## 7.3 Heat Transfer

For heat transfer through a material with length  $l$ , area  $A$  and thermal conductivity  $K$  between two heat reservoirs  $T_1 > T_2$ :

$$\frac{dQ}{dt} = \frac{KA(T_1 - T_2)}{l}$$

For a blackbody at temperature  $T$  radiating heat away:

$$\frac{dQ}{dt} = \sigma AT^4$$

The heat transferred by changing the temperature of a solid of mass  $m$  with heat capacity  $c$  is:

$$\Delta Q = mc\Delta T$$

## 7.4 Thermodynamic Processes

In all the process described below, the heat  $Q$  that goes into the gas is positive, and the work done on the gas  $W$  is positive. The first law of thermodynamics states that the change of internal energy  $U$  is:

$$U = Q + W$$

$$U(\gamma - 1) = NkT$$

Where  $\gamma = C_p/C_v$  is the ideal gas constant and  $C_v = C_p - k$ .

### 7.4.1 Isochoric

In this constant volume process:

$$W = 0$$

$$Q = NC_v \Delta T$$

$$U = Q$$

### 7.4.2 Isobaric

In a constant pressure volume expansion from  $V_1$  to  $V_2$ :

$$W = P(V_1 - V_2)$$

$$Q = NC_p \Delta T$$

$$U = NC_v \Delta T$$

### 7.4.3 Isothermal

For an isothermal expansion from  $V_1$  to  $V_2$ :

$$W = NkT \ln \left( \frac{V_1}{V_2} \right)$$

$$Q = -W$$

$$U = 0$$

### 7.4.4 Adiabatic

For an adiabatic process,

$$W = - \int P dV$$

$$Q = 0$$

$$U = W$$

Integrating the work done, we get the following relation:

$$PV^\gamma = \text{constant}$$

## 7.5 Heat Engines

The efficiency of a heat engine that takes in  $Q_H$  and gives out  $Q_L$  while doing work  $W$ , its efficiency is given by:

$$\eta = \frac{|W|}{|Q_H|}$$

$$= 1 - \frac{|Q_L|}{|Q_H|}$$

The efficiency of a heat pump that uses  $W$  to pump  $Q_L$  from the col reservoir is:

$$\eta = \frac{|Q_L|}{|W|}$$

All reversible engines operating between the same two temperatures have the same efficiency as a Carnot engine, as you can fit many infinitesimally small Carnot cycles into any reversible cycle:

$$\eta_{\text{carnot}} = 1 - \frac{T_L}{T_H}$$

## 7.6 Second Law of Thermodynamics

- A process whose only net result is to take heat from a reservoir and convert it to heat is impossible.

- No heat engine can working between two temperatures  $T_1$  and  $T_2$  can have a higher efficiency than a reversible engine.

## 7.7 Entropy

### 7.7.1 Macroscopic Definition

Entropy is the measure of disorder. If heat is added reversibly into a system at temperature  $T$ , the increase in entropy in the system is:

$$dS = \frac{dQ}{T}$$

Entropy is a state function that doesn't depend on the path travelled. The total entropy change in the system and surroundings for a reversible process is zero. For an irreversible process, the total entropy change is always positive. At  $T=0$ ,  $S=0$ . This is the third law of thermodynamics.

### 7.7.2 Microscopic Definition

Boltzmann defined entropy of a system by counting the number of indistinguishable microstates  $w$  inside:

$$S = k \ln w$$

## 8 Quantum Mechanics

### 8.1 Blackbody Radiation

Ideal blackbodies have a continuous emission spectrum, with the energy density  $\rho(\lambda, T)$  as a function of wavelength and temperature given by Planck's distribution:

$$\rho(\lambda, T) = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{hc/\lambda KT} - 1}$$

Integrating over this distribution, we find the total emission power as a function of temperature:

$$R(T) = \sigma T^4$$

Where  $\sigma$  is the Stefan-Boltzmann constant. Finding the wavelength with the peak emission at a given temperature, we arrive at Wien's law:

$$\lambda_{\text{max}} T = b$$

Where  $b$  is a constant.

### 8.2 Schrödinger's Equation

$\Psi(x, t)$  is a complex wave function of time and position, the one-dimensional Schrödinger's equation is given by:

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V\Psi$$

If we denote the complex conjugate of the wave function to be  $\Psi^*$ , the conjugate of Schrödinger's equation is:

$$-i\hbar \frac{\partial \Psi^*}{\partial t} = \frac{\hbar^2}{2m} \nabla^2 \Psi^* - V\Psi^*$$

At time  $t$ , the probability of finding a particle from  $x=a$  to  $x=b$  is:

$$\int_a^b |\Psi(r, t)|^2 dr = \int_a^b \Psi \Psi^* dr$$

### 8.2.1 Normalization

All wave functions must be normalized, so that the probability of finding the particle over all space is 1:

$$\int_{-\infty}^{\infty} |\Psi(r, t)|^2 dr = 1$$

Once a function is normalized, it remains normalized as time evolves:

$$\frac{d}{dt} \int_{-\infty}^{\infty} \Psi \Psi^* dr = 0$$

### 8.2.2 Expectation Values

An expectation value of an observed quantity is the average of the measurement performed on many "copies" of the system at the same time.

$$\langle x \rangle = \int_{-\infty}^{\infty} x |\Psi(x, t)|^2 dx$$

$$= \int_{-\infty}^{\infty} \Psi^* x \Psi dx$$

$$\langle p_x \rangle = m \frac{d\langle x \rangle}{dt}$$

$$= \int_{-\infty}^{\infty} \Psi^* \left( -i\hbar \frac{\partial}{\partial x} \right) \Psi dx$$

In general, the expectation value of any quantity is:

$$\langle Q(r, p) \rangle = \int \Psi^* Q(r, -i\hbar \nabla) \Psi dr$$

### 8.3 Time Independent Solution

If  $V$  is independent of time, we solve Schrödinger's equation by separating variables. Let:

$$\Psi(x, t) = \psi(x) \phi(t)$$

Then the equation can be written as:

$$i\hbar \psi \frac{\partial \phi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \phi + V \phi \psi$$

$$\left( \frac{i\hbar \partial \phi}{\phi \partial t} \right) + \left( \frac{\hbar^2}{2m \psi} \nabla^2 - V(x) \right) = 0$$

As the two terms in the equation are independent of each other and they

sum to zero, they must be constant. If we let:

$$E = \frac{i\hbar}{\phi} \frac{\partial \phi}{\partial t}$$

$$\phi(t) = e^{-iEt/\hbar}$$

The time independent solution is given by:

$$-\frac{\hbar^2}{2m} \nabla^2 + V(x) \psi = E \psi$$

If we define the Hamiltonian operator  $\hat{H} = -\frac{\hbar^2}{2m} \nabla^2 + V$ ,

$$\hat{H} \psi = E \psi$$

The separated solutions can then be combined:

$$\Psi(\mathbf{r}, t) = \sum_E C_E(t_0) e^{-iE(t-t_0)/\hbar} \psi_E(\mathbf{r})$$

### 8.3.1 Integral Form

If we integrate the time-independent Schrödinger's equation about  $\pm\epsilon$  for small  $\epsilon$ , the  $E\psi$  term disappears and we get:

$$\left[ \frac{d\psi}{dx} \right]_{-\epsilon}^{+\epsilon} = \frac{2m}{\hbar^2} \int_{-\epsilon}^{+\epsilon} V(x) \psi(x) dx$$

From this we see that the derivative of  $\psi$  is continuous if  $V(x)$  is finite.

## 8.4 Momentum Space Wavefunction

The position space and momentum space representations of a wavefunction can be interchanged with a Fourier transform:

$$\psi(x, t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{iP_x x/\hbar} \Phi(P_x, t) dP_x$$

$$\phi(P_x, t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-iP_x x/\hbar} \Psi(x, t) dx$$

## 8.5 1-D Examples

### 8.5.1 Free Particle

For the free particle  $V(x)=0$ . We have the general solution:

$$\psi(x) = A e^{ikx} + B e^{-ikx}$$

After normalization, we have the following solution:

$$\Psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i(kx - \frac{\hbar k^2}{2m} t)} dk$$

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x, 0) e^{-ikx} dx$$

### 8.5.2 Infinite Well

For an infinite well,

$$V(x) = \begin{cases} 0 & |x| < a \\ \infty & |x| > a \end{cases}$$

The general solution is the same as that of the free particle, but since  $\psi(a) = \psi(-a) = 0$ , we have

$$\psi_n(x) = \begin{cases} \frac{1}{\sqrt{a}} \cos\left(\frac{n\pi x}{2a}\right) & n=1, 3, \dots \\ \frac{1}{\sqrt{a}} \sin\left(\frac{n\pi x}{2a}\right) & n=2, 4, \dots \end{cases}$$

The energy  $E_n$  is proportional to  $n^2$ :

$$E_n = \frac{\hbar^2 \pi^2 n^2}{8ma^2}$$

### 8.5.3 Finite Well

For a finite well,

$$V(x) = \begin{cases} -V_0 & |x| < a \\ 0 & |x| > a \end{cases}$$

For  $-V_0 \leq E < 0$ , we have the following equations:

$$\frac{d^2 \psi(x)}{dx^2} + \alpha^2 \psi(x) = 0,$$

$$\alpha = \sqrt{\frac{2m}{\hbar^2} (V_0 - |E|)}$$

$$\frac{d^2 \psi(x)}{dx^2} - \beta^2 \psi(x) = 0,$$

$$\beta = \sqrt{\frac{2m|E|}{\hbar^2}}$$

For  $x > 0$ , the even solutions are:

$$\psi(x) = \begin{cases} A \cos(\alpha x) & x \in [0, a] \\ C e^{-\beta x} & x > a \end{cases}$$

The odd solutions are:

$$\psi(x) = \begin{cases} B \sin(\alpha x) & x \in [0, a] \\ C e^{-\beta x} & x > a \end{cases}$$

After solving for boundary conditions, we get:

$$\alpha \tan(\alpha a) = \beta$$

$$\alpha \cot(\alpha a) = -\beta$$

### 8.5.4 Harmonic Oscillator

For a harmonic oscillator,

$$\hat{H} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{m}{2} \omega^2 x^2$$

Substituting  $\xi = \sqrt{\frac{m\omega}{\hbar}} x$ , we get solutions:

$$\psi_n(x) = \left( \frac{m\omega}{\pi\hbar} \right)^{\frac{1}{4}} H_n(\xi) e^{-\xi^2/2} \frac{1}{\sqrt{2^n n!}}$$

$$E_n = \left( n + \frac{1}{2} \right) \hbar \omega, n=0, 1, \dots$$

Where  $H_n$  are the Hermite polynomials:

$$\begin{aligned} H_0 &= 1 & H_1 &= 2\xi \\ H_2 &= 4\xi^2 - 2 & H_3 &= 8\xi^3 - 12\xi \\ H_4 &= 16\xi^4 - 48\xi^2 + 12 \end{aligned}$$

## 8.6 Linear Algebra Formalism

### 8.6.1 Postulate 1

To an ensemble of physical systems, one can assign a wave/state function which contains all the information that can be known of that ensemble  $\psi$ . This function is in general complex. One can multiply it with an arbitrary complex number without changing its physical meaning. For  $N$  particles  $\psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N)$ ,  $\psi^* \psi$  gives the probability density that #1 is at  $\mathbf{r}_1$ , #2 is at  $\mathbf{r}_2$ , etc. We represent the state vector as  $|\psi\rangle$  and the adjoint vector as  $\langle\psi|$ :

$$\langle\psi_1|\psi_2\rangle = \int \psi_1^* \psi_2 d\mathbf{r} = \langle\psi_2|\psi_1\rangle^*$$

$$\langle\psi_1|c\psi_2\rangle = c\langle\psi_1|\psi_2\rangle$$

$$\langle c\psi_1|\psi_2\rangle = c^* \langle\psi_1|\psi_2\rangle$$

$$\langle\psi_3|\psi_1 + \psi_2\rangle = \langle\psi_3|\psi_1\rangle + \langle\psi_3|\psi_2\rangle$$

$$\langle\psi|\psi\rangle = 1$$

$\psi_1$  and  $\psi_2$  are orthogonal if and only if  $\langle\psi_1|\psi_2\rangle = 0$ .

### 8.6.2 Postulate 2

The principle of superposition: if  $\psi_1$  and  $\psi_2$  are solutions to the Schrödinger's equation, so is  $\psi = c_1 \psi_1 + c_2 \psi_2$ .

### 8.6.3 Postulate 3

Every dynamical variable is associated with a linear operator.  $\mathcal{A} = \mathcal{A}(\mathbf{r}_1, \dots, \mathbf{r}_n, \mathbf{p}_1, \dots, \mathbf{p}_n, t) = \hat{\mathcal{A}}(\mathbf{r}_1, \dots, \mathbf{r}_n, i\hbar \nabla_1, \dots, i\hbar \nabla_n, t)$ .

### 8.6.4 Postulate 4

The only result of a precise measurement is one of the eigenvalues of the associated linear operator  $\hat{\mathcal{A}}$ :

$$\hat{\mathcal{A}} \psi_n = a_n \psi_n$$

To have real eigenvalues,  $\hat{\mathcal{A}}$  must be Hermitian:

$$\langle\psi_n|\hat{\mathcal{A}}\psi_n\rangle = \langle\hat{\mathcal{A}}\psi_n|\psi_n\rangle = a_n \langle\psi_n|\psi_n\rangle$$

### 8.6.5 Postulate 5

If a system of measurements is made of the dynamical variable  $A$  on an ensemble of identical systems each described by the same  $|\psi\rangle$ , the average value is given by:

$$\langle \hat{A} \rangle = \frac{\langle \psi | \hat{A} \psi \rangle}{\langle \psi | \psi \rangle}$$