Introduction to Algorithms

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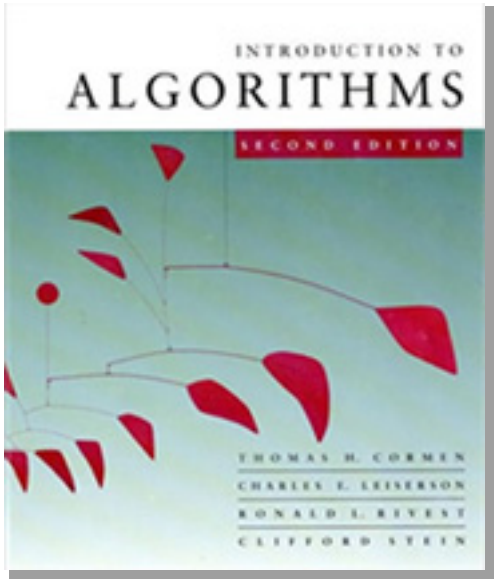

LECTURE 2

Asymptotic Notation

- *O*-, Ω-, and Θ-notation

Recurrences

- Substitution method
- Iterating the recurrence
- Recursion tree
- Master method

Prof. Erik Demaine

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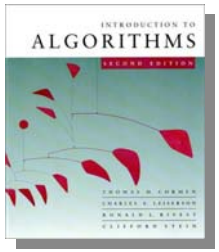

Asymptotic notation

O-notation (upper bounds):

We write $f(n) = O(g(n))$ if there exist constants $c > 0$, $n_0 > 0$ such that $0 \le f(n) \le cg(n)$ for all $n \ge n_0$.

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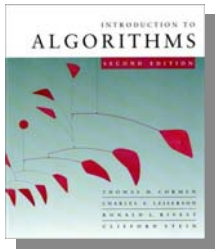

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EXAMPLE: $2n^2 = O(n^3)$ $\quad (c = 1, n_0 = 2)$

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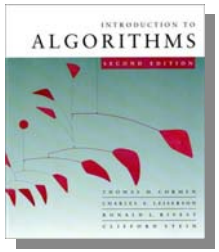

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functions, not values

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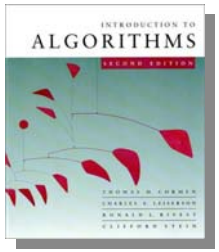

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functions, not values

funny, "one-way" equality

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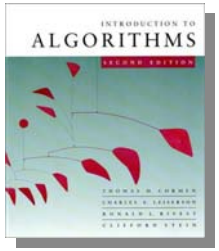

Set definition of O-notation

$O(g(n)) = \{ f(n) :$ there exist constants $c > 0$, $n_0 > 0$ such that $0 \le f(n) \le cg(n)$ for all $n \ge n_0 \}$

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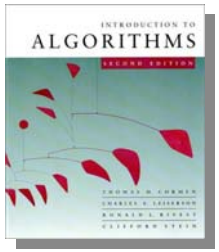

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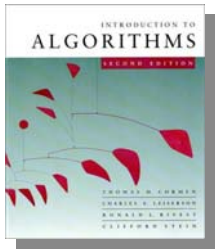

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EXAMPLE: $2n^2 \in O(n^3)$

(*Logicians:* $\lambda n.2n^2 \in O(\lambda n.n^3)$, but it's convenient to be sloppy, as long as we understand what's *really* going on.)

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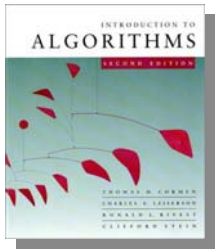

Macro substitution

Convention: A set in a formula represents an anonymous function in the set.

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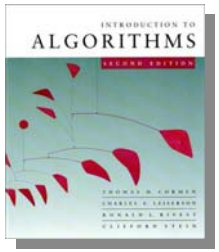

Macro substitution

Convention: A set in a formula represents an anonymous function in the set.

EXAMPLE: $f(n) = n^3 + O(n^2)$

means

$f(n) = n^3 + h(n)$

for some $h(n) \in O(n^2)$.

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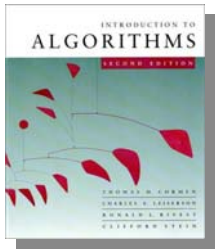

Macro substitution

Convention: A set in a formula represents an anonymous function in the set.

EXAMPLE: $n^2 + O(n) = O(n^2)$

means

for any $f(n) \in O(n)$:

$n^2 + f(n) = h(n)$

for some $h(n) \in O(n^2)$.

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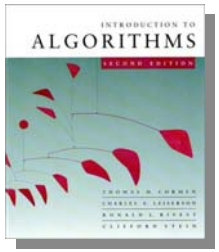

Ω-notation (lower bounds)

O-notation is an *upper-bound* notation. It makes no sense to say $f(n)$ is at least $O(n^2)$.

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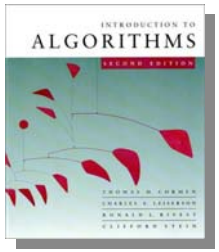

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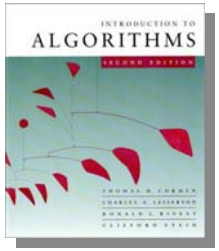

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$\Omega(g(n)) = \{ f(n) :$ there exist constants $c > 0, n_0 > 0$ such that $0 \le cg(n) \le f(n)$ for all $n \ge n_0 \}$

EXAMPLE: $\sqrt{n} = \Omega(\lg n)$ $(c = 1, n_0 = 16)$

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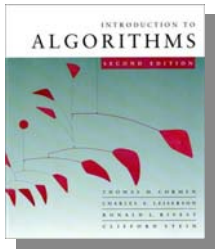

Θ-notation (tight bounds)

$$\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$$

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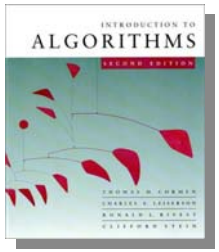

Θ-notation (tight bounds)

$$\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$$

EXAMPLE: $\frac{1}{2}n^2 - 2n = \Theta(n^2)$

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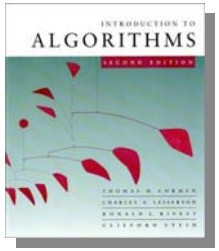

o-notation and ω-notation

O-notation and Ω-notation are like ≤ and ≥.
o-notation and ω-notation are like < and >.

$o(g(n)) = \{ f(n) :$ for any constant $c > 0$, there is a constant $n_0 > 0$ such that $0 \le f(n) < cg(n)$ for all $n \ge n_0 \}$

EXAMPLE: $2n^2 = o(n^3)$ $(n_0 = 2/c)$

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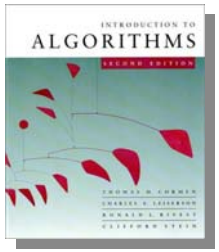

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$\omega(g(n)) = \{ f(n) :$ for any constant $c > 0$,
there is a constant $n_0 > 0$
such that $0 \le cg(n) < f(n)$
for all $n \ge n_0 \}$

EXAMPLE: $\sqrt{n} = \omega(\lg n)$ $\quad (n_0 = 1+1/c)$

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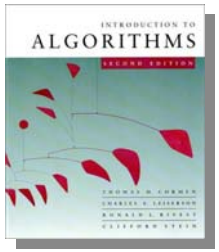

Solving recurrences

- The analysis of merge sort from ***Lecture 1*** required us to solve a recurrence.
- Recurrences are like solving integrals, differential equations, etc.
  - Learn a few tricks.
- ***Lecture 3***: Applications of recurrences to divide-and-conquer algorithms.

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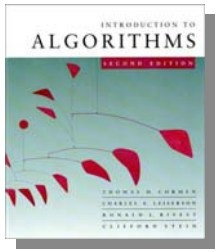

Substitution method

The most general method:

1. ***Guess*** the form of the solution.
2. ***Verify*** by induction.
3. ***Solve*** for constants.

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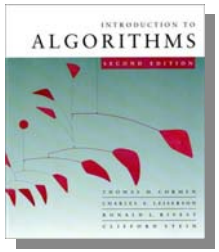

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1. ***Guess*** the form of the solution.
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EXAMPLE: $T(n) = 4T(n/2) + n$

- [Assume that $T(1) = \Theta(1)$.]
- Guess $O(n^3)$. (Prove O and Ω separately.)
- Assume that $T(k) \le ck^3$ for $k < n$.
- Prove $T(n) \le cn^3$ by induction.

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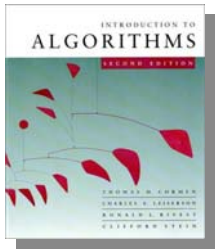

Example of substitution

$$
\begin{aligned}
T(n) &= 4T(n/2) + n \\
&\le 4c(n/2)^3 + n \\
&= (c/2)n^3 + n \\
&= cn^3 - ((c/2)n^3 - n) \quad \leftarrow \textit{desired} - \textit{residual} \\
&\le cn^3 \quad \leftarrow \textit{desired}
\end{aligned}
$$

whenever $(c/2)n^3 - n \ge 0$, for example, if $c \ge 2$ and $n \ge 1$.

residual

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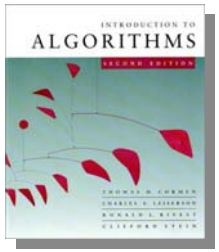

Example (continued)

- We must also handle the initial conditions, that is, ground the induction with base cases.
- ***Base:*** $T(n) = \Theta(1)$ for all $n < n_0$, where n_0 is a suitable constant.
- For $1 \le n < n_0$, we have "$\Theta(1)$" $\le cn^3$, if we pick c big enough.

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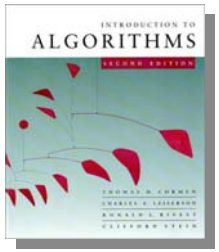

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This bound is not tight!

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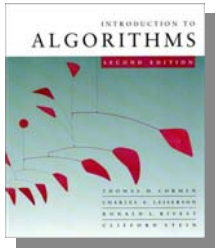

A tighter upper bound?

We shall prove that $T(n) = O(n^2)$.

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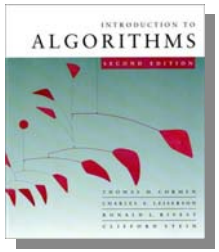

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We shall prove that $T(n) = O(n^2)$.

Assume that $T(k) \le ck^2$ for $k < n$:

$$\begin{aligned} T(n) &= 4T(n/2) + n \\ &\le 4c(n/2)^2 + n \\ &= cn^2 + n \\ &= O(n^2) \end{aligned}$$

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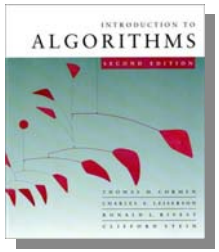

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Wrong! We must prove the I.H.

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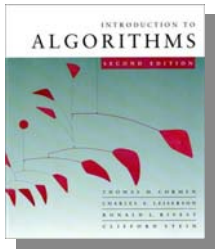

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&= cn^2 + n \\
&= O(n^2) \quad \textbf{\textit{Wrong!}}\ \text{We must prove the I.H.} \\
&= cn^2 - (-n) \quad [\ \text{desired} - \text{residual}\ ] \\
&\le cn^2 \quad \text{for } \textbf{\textit{no}} \text{ choice of } c > 0. \ \text{Lose!}
\end{aligned}
$$

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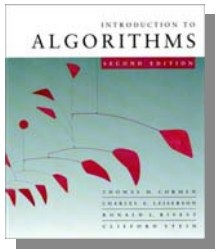

A tighter upper bound!

IDEA: Strengthen the inductive hypothesis.

- ***Subtract*** a low-order term.

Inductive hypothesis: $T(k) \le c_1 k^2 - c_2 k$ for $k < n$.

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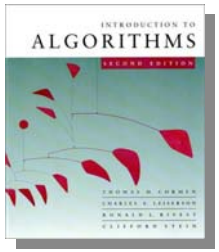

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$$\begin{aligned} T(n) &= 4T(n/2) + n \\ &= 4(c_1(n/2)^2 - c_2(n/2)) + n \\ &= c_1 n^2 - 2c_2 n + n \\ &= c_1 n^2 - c_2 n - (c_2 n - n) \\ &\le c_1 n^2 - c_2 n \quad \text{if } c_2 \ge 1. \end{aligned}$$

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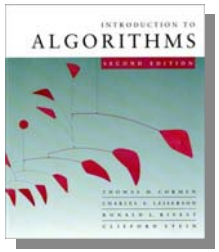

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Pick c_1 big enough to handle the initial conditions.

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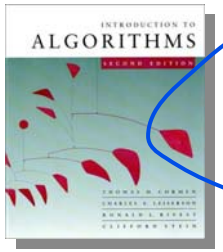

Recursion-tree method

- A recursion tree models the costs (time) of a recursive execution of an algorithm.
- The recursion-tree method can be unreliable, just like any method that uses ellipses (…).
- The recursion-tree method promotes intuition, however.
- The recursion tree method is good for generating guesses for the substitution method.

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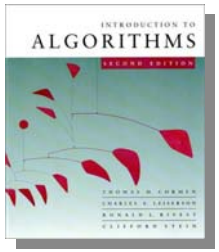

Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

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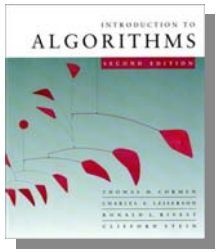

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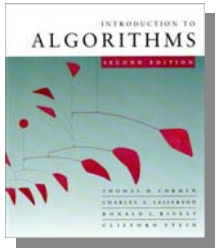

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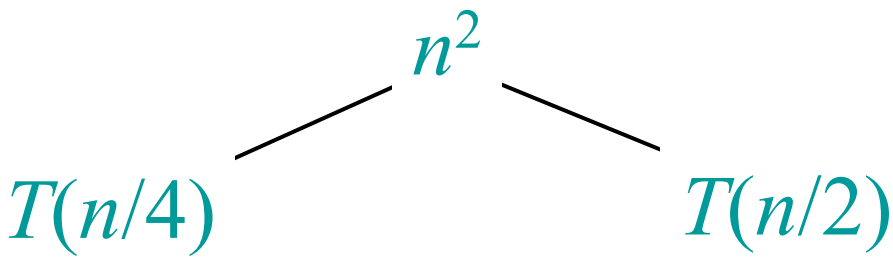

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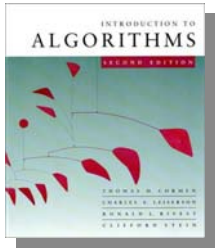

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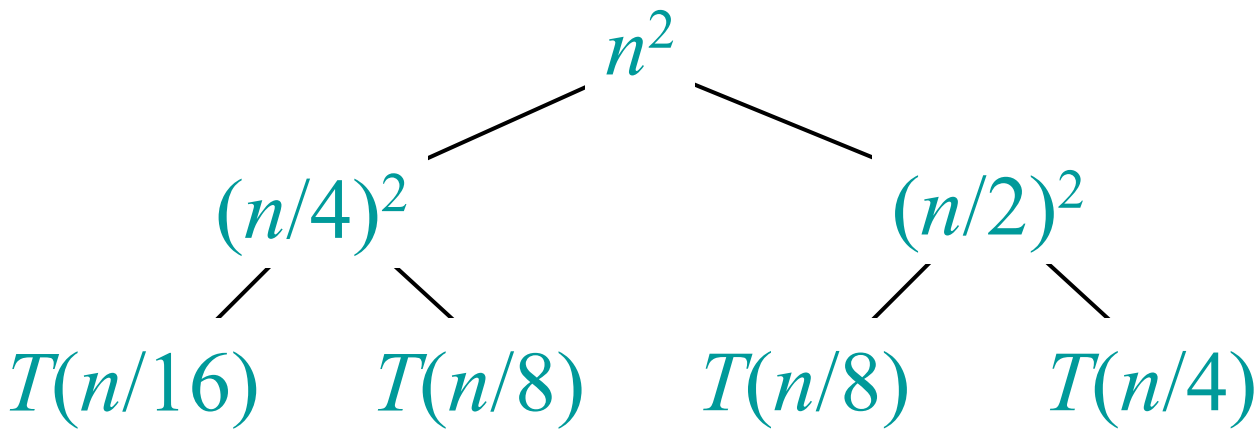

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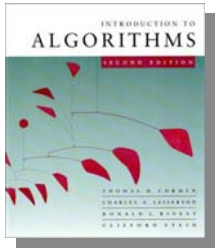

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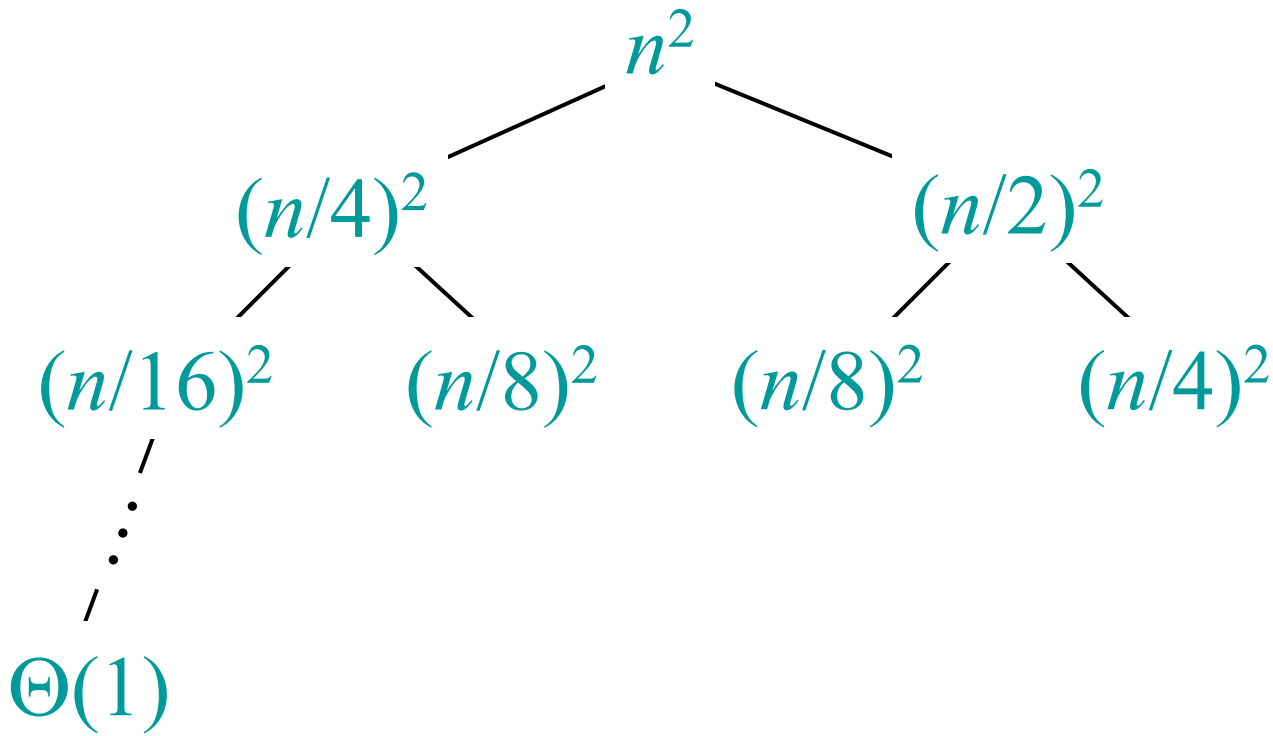

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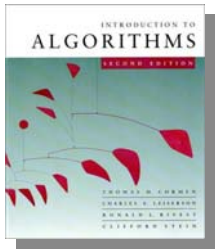

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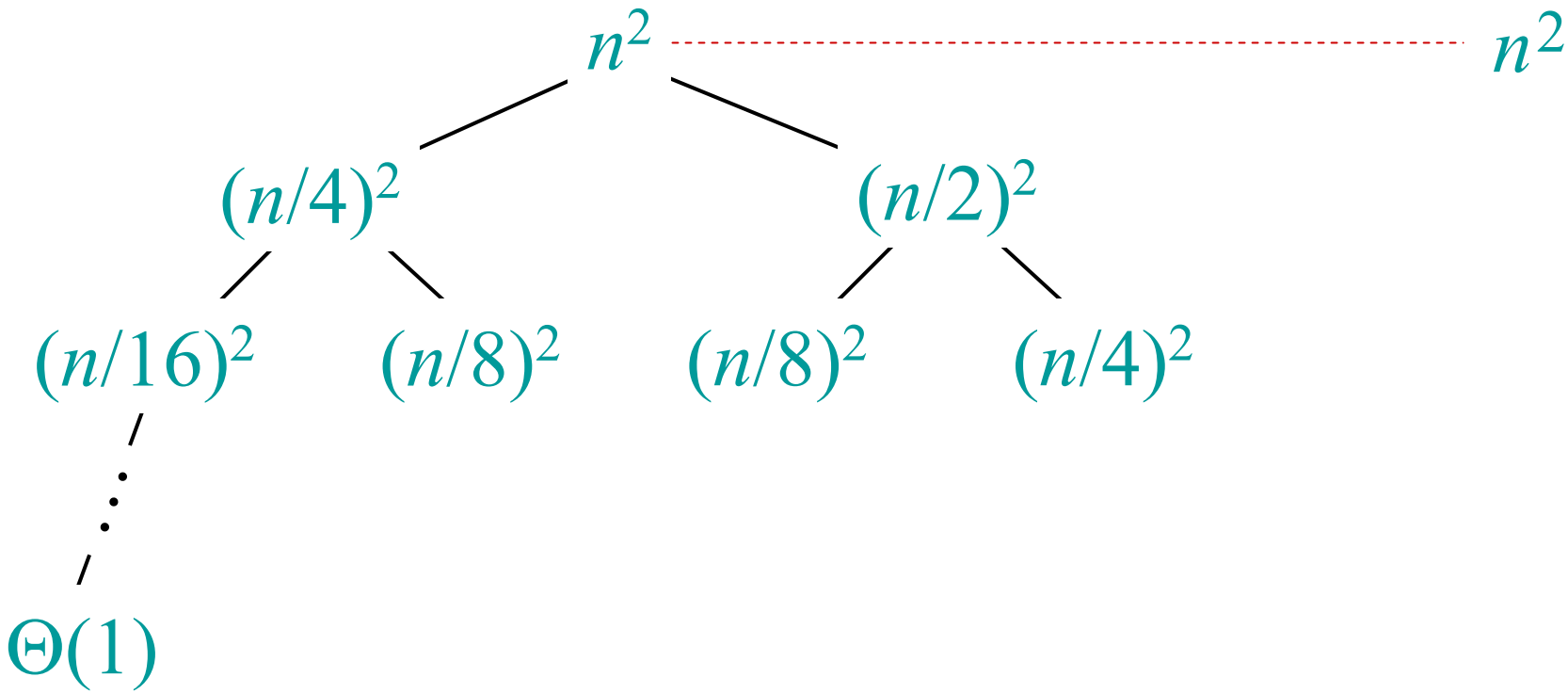

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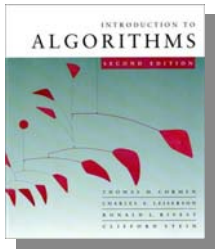

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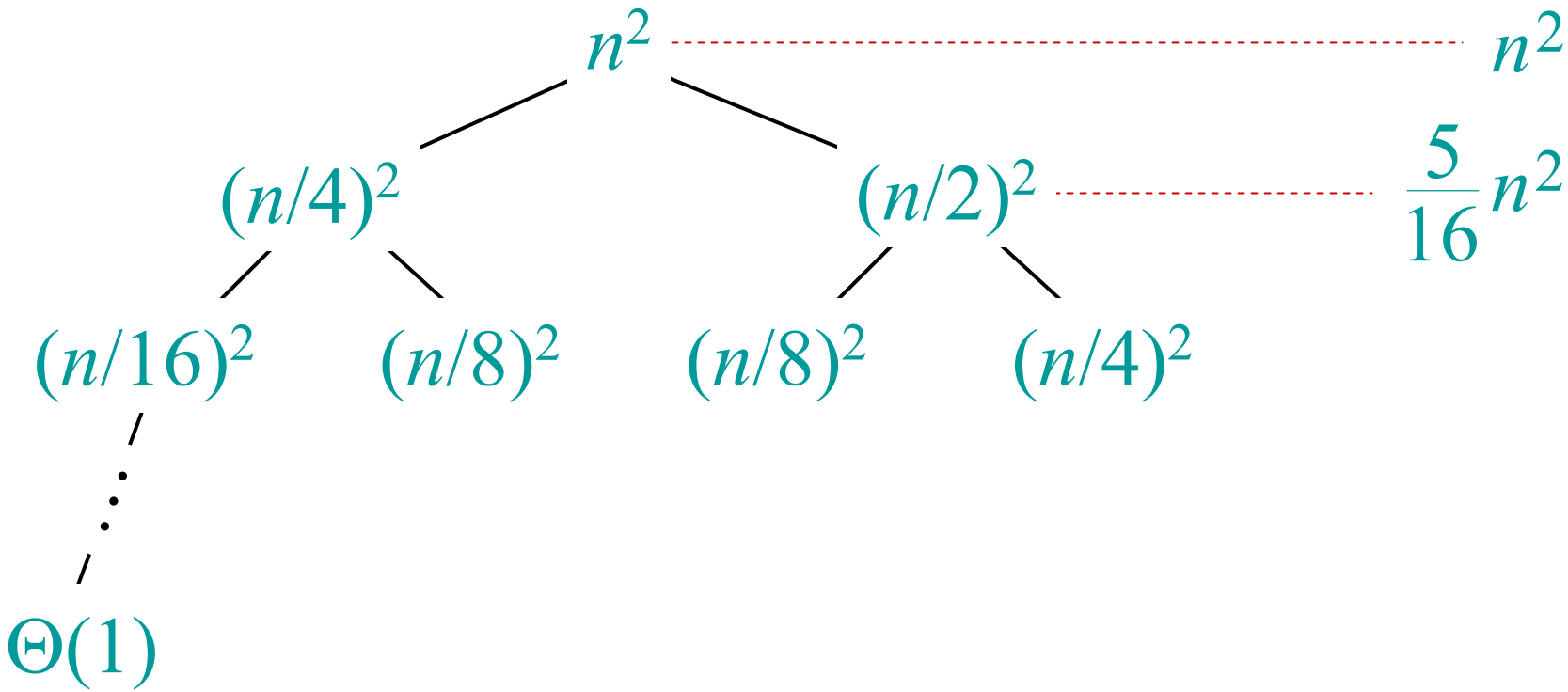

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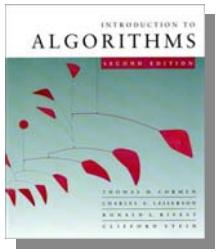

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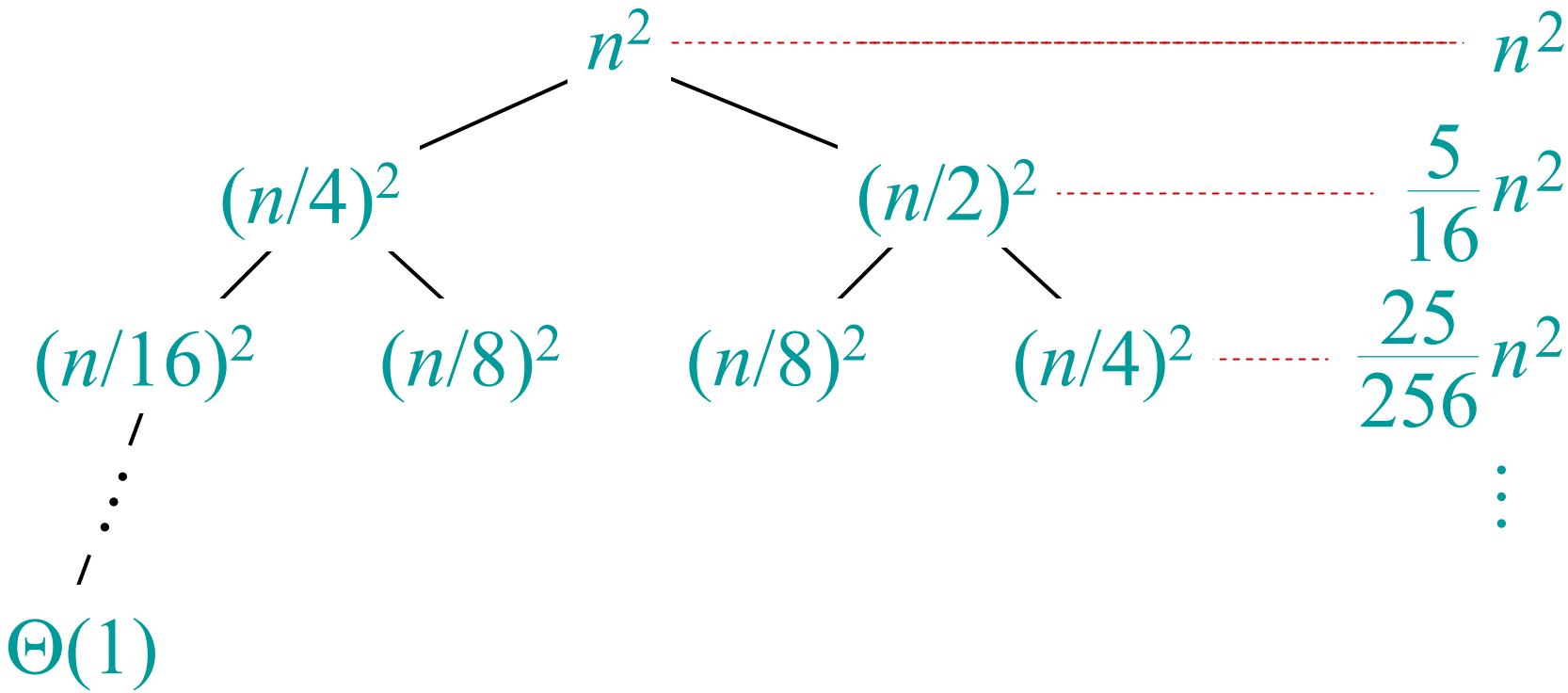

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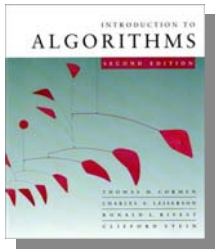

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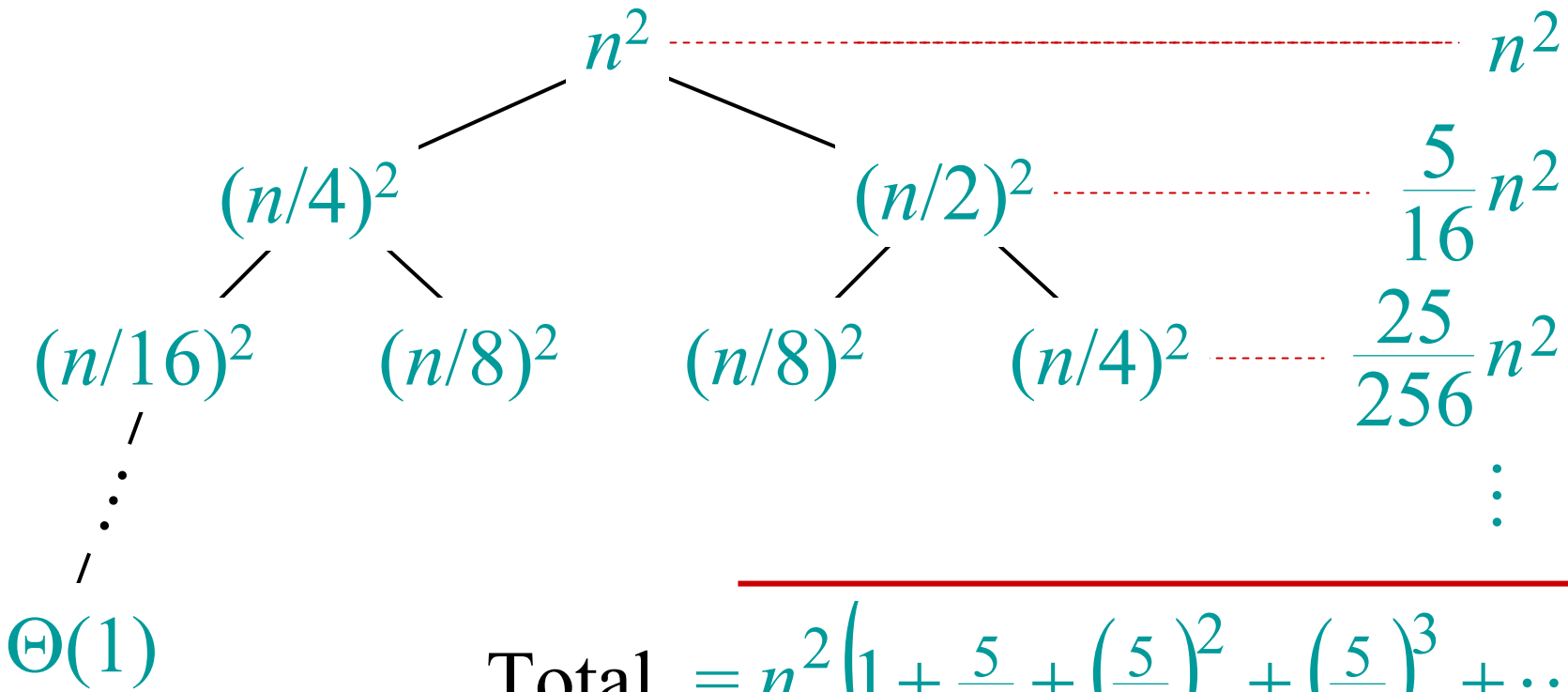

$$\text{Total} = n^2\left(1 + \frac{5}{16} + \left(\frac{5}{16}\right)^2 + \left(\frac{5}{16}\right)^3 + \cdots\right)$$

$$= \Theta(n^2) \quad \textit{geometric series}$$

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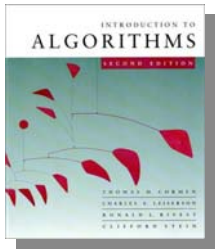

The master method

The master method applies to recurrences of the form

$$T(n) = a\,T(n/b) + f(n)\,,$$

where $a \geq 1$, $b > 1$, and f is asymptotically positive.

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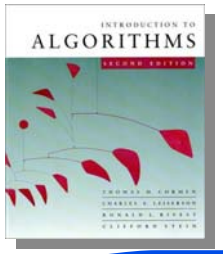

Three common cases

Compare $f(n)$ with $n^{\log_b a}$:

1. $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially slower than $n^{\log_b a}$ (by an n^{ε} factor).

   Solution: $T(n) = \Theta(n^{\log_b a})$.

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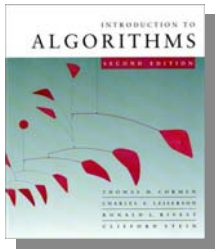

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   Solution: $T(n) = \Theta(n^{\log_b a})$.

2. $f(n) = \Theta(n^{\log_b a} \lg^k n)$ for some constant $k \geq 0$.
   - $f(n)$ and $n^{\log_b a}$ grow at similar rates.

   Solution: $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$.

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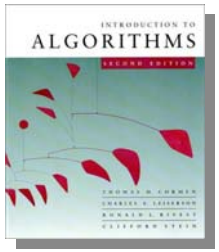

Three common cases (cont.)

Compare $f(n)$ with $n^{\log_b a}$:

3. $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially faster than $n^{\log_b a}$ (by an n^{ε} factor),

   and $f(n)$ satisfies the ***regularity condition*** that $a f(n/b) \le c f(n)$ for some constant $c < 1$.

Solution: $T(n) = \Theta(f(n))$.

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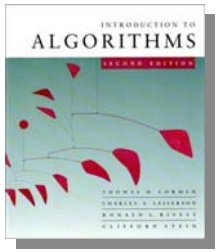

Examples

Ex. $T(n) = 4T(n/2) + n$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n.$

CASE 1: $f(n) = O(n^{2-\varepsilon})$ for $\varepsilon = 1$.

$\therefore\ T(n) = \Theta(n^2).$

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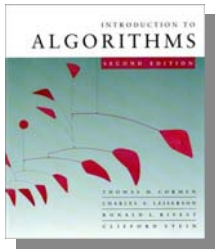

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$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n^2.$

CASE 2: $f(n) = \Theta(n^2 \lg^0 n)$, that is, $k = 0$.

$\therefore\ T(n) = \Theta(n^2 \lg n).$

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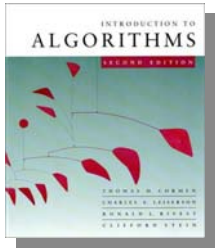

Examples

Ex. $T(n) = 4T(n/2) + n^3$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n^3.$

CASE 3: $f(n) = \Omega(n^{2+\varepsilon})$ for $\varepsilon = 1$

and $4(n/2)^3 \le cn^3$ (reg. cond.) for $c = 1/2$.

$\therefore\ T(n) = \Theta(n^3).$

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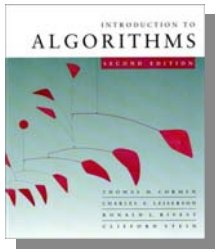

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Ex. $T(n) = 4T(n/2) + n^2/\lg n$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n^2/\lg n.$

Master method does not apply. In particular, for every constant $\varepsilon > 0$, we have $n^\varepsilon = \omega(\lg n)$.

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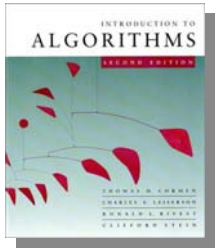
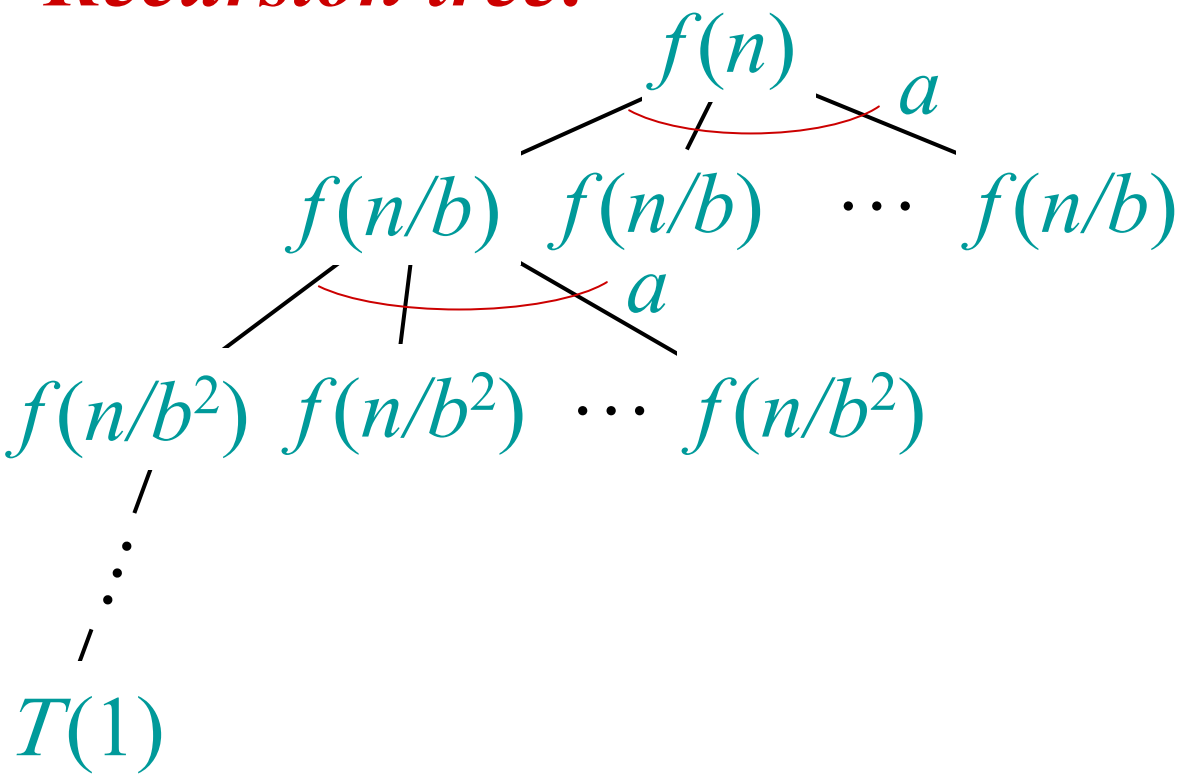


Idea of master theorem

Recursion tree:



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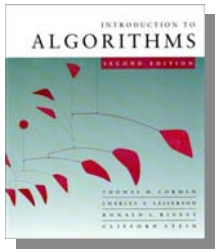

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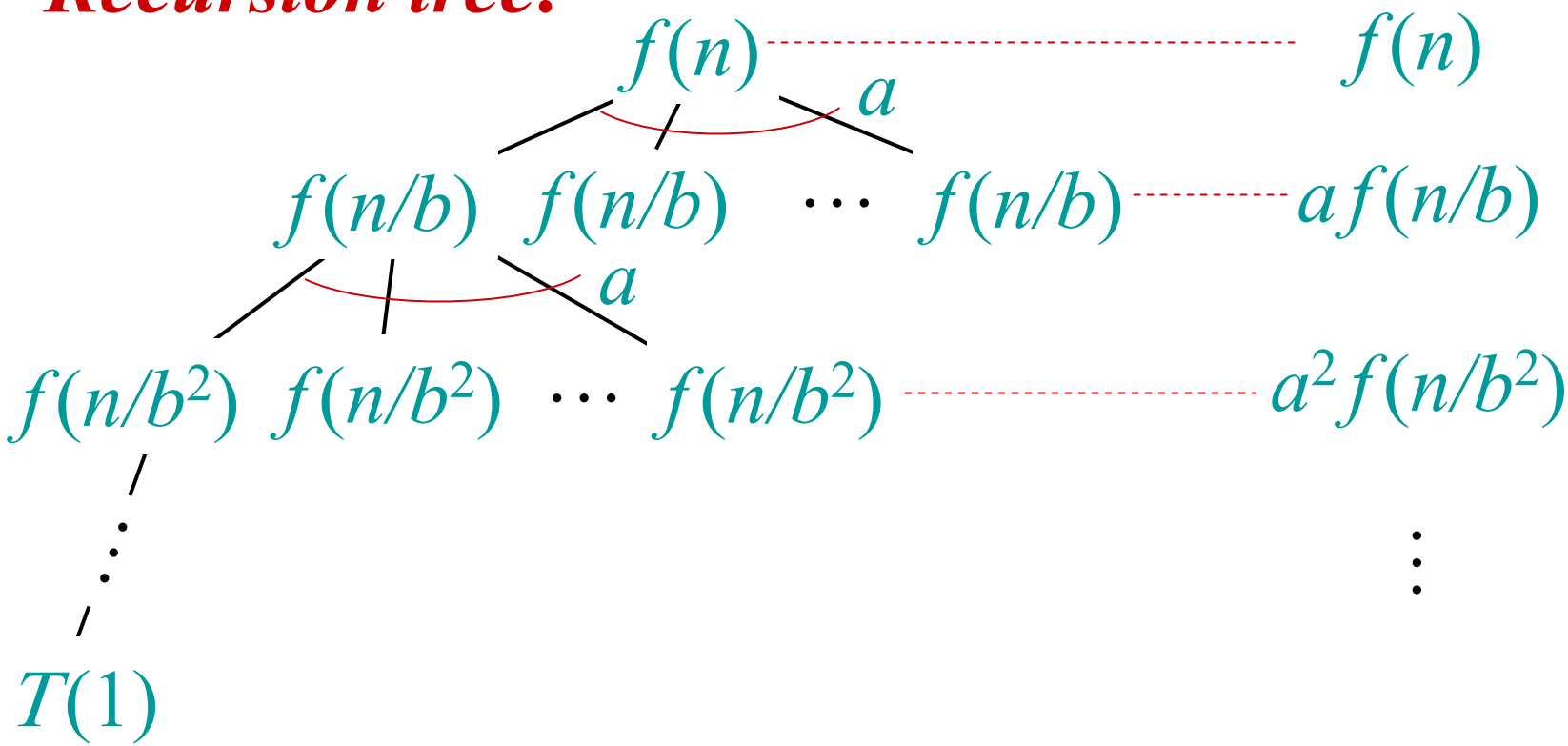

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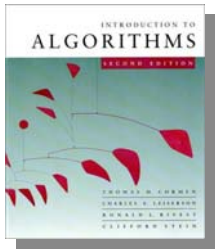

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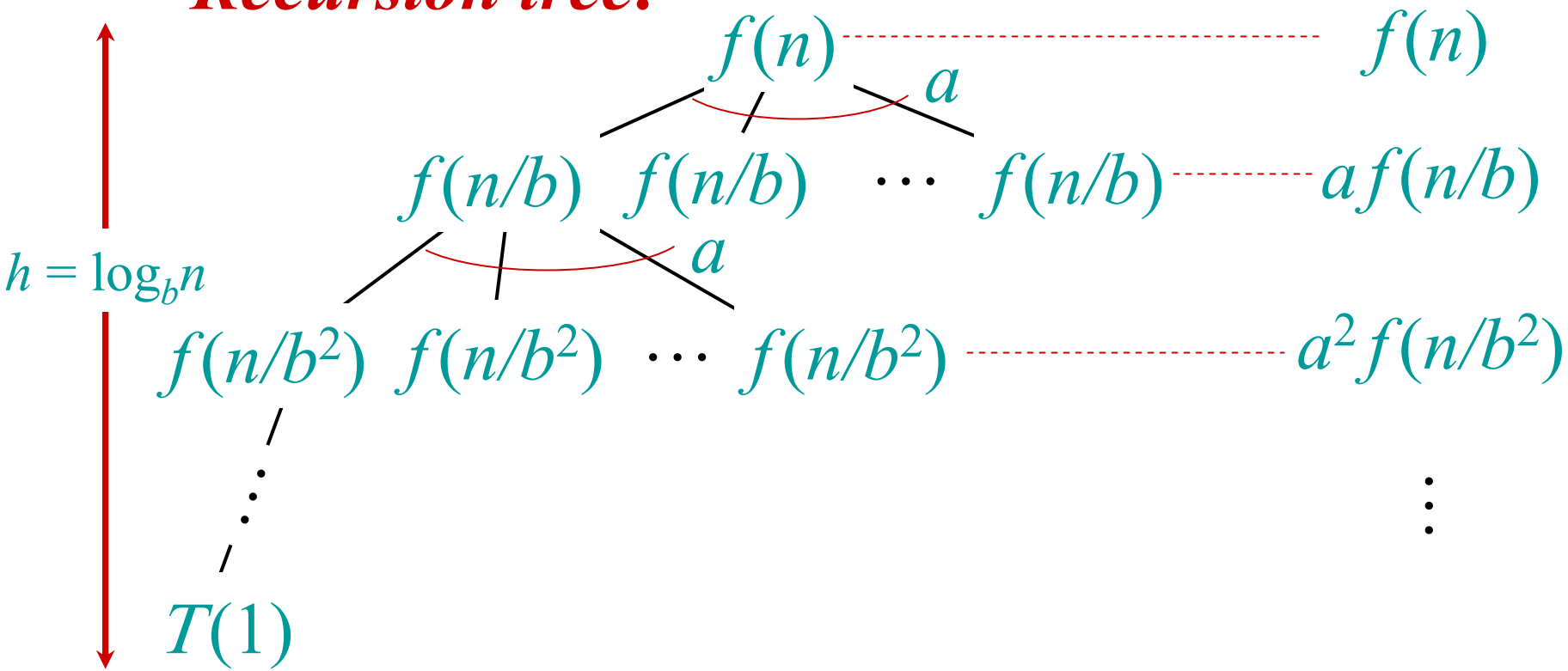

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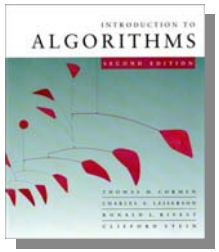

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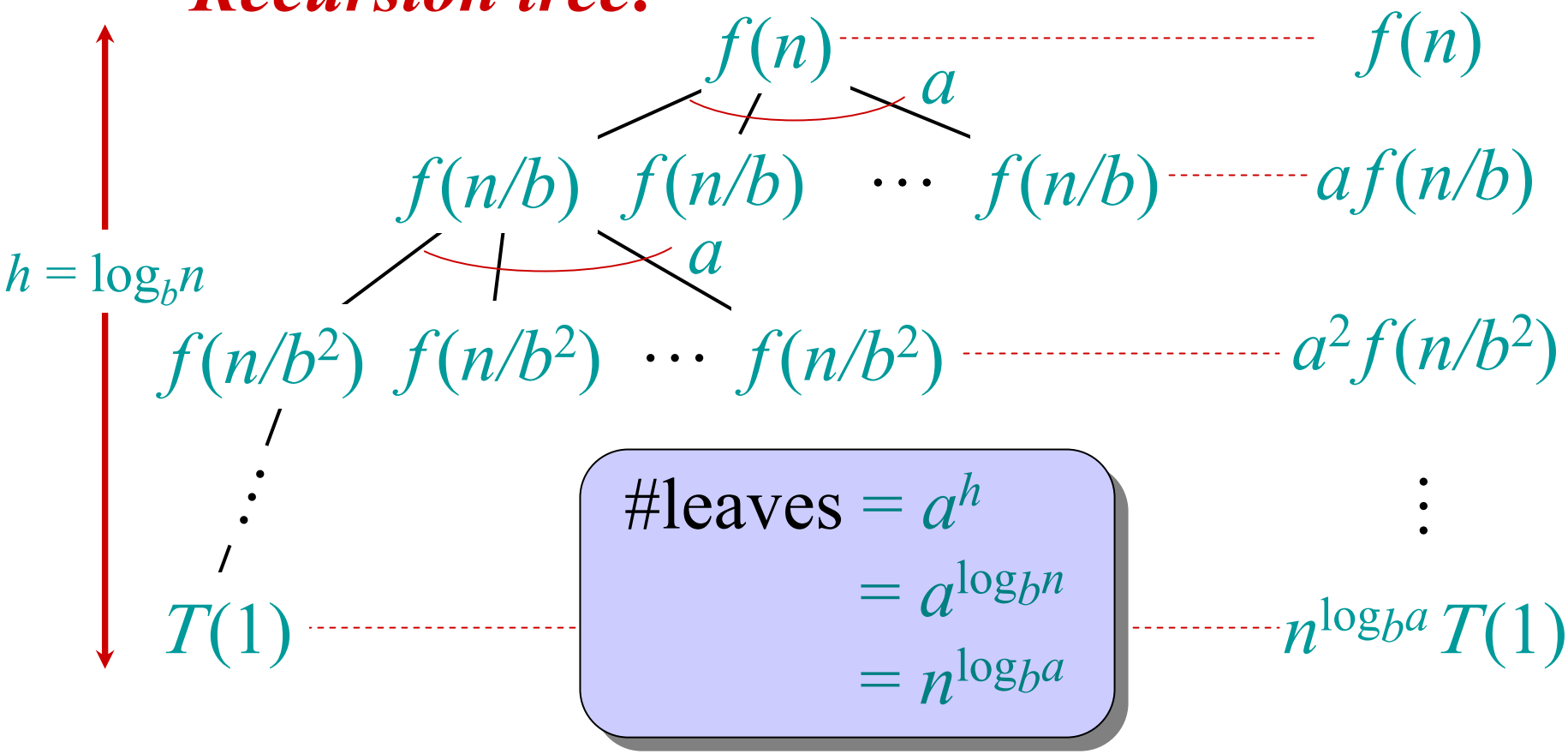

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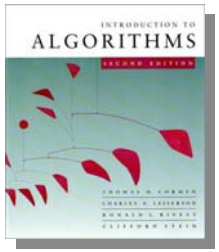

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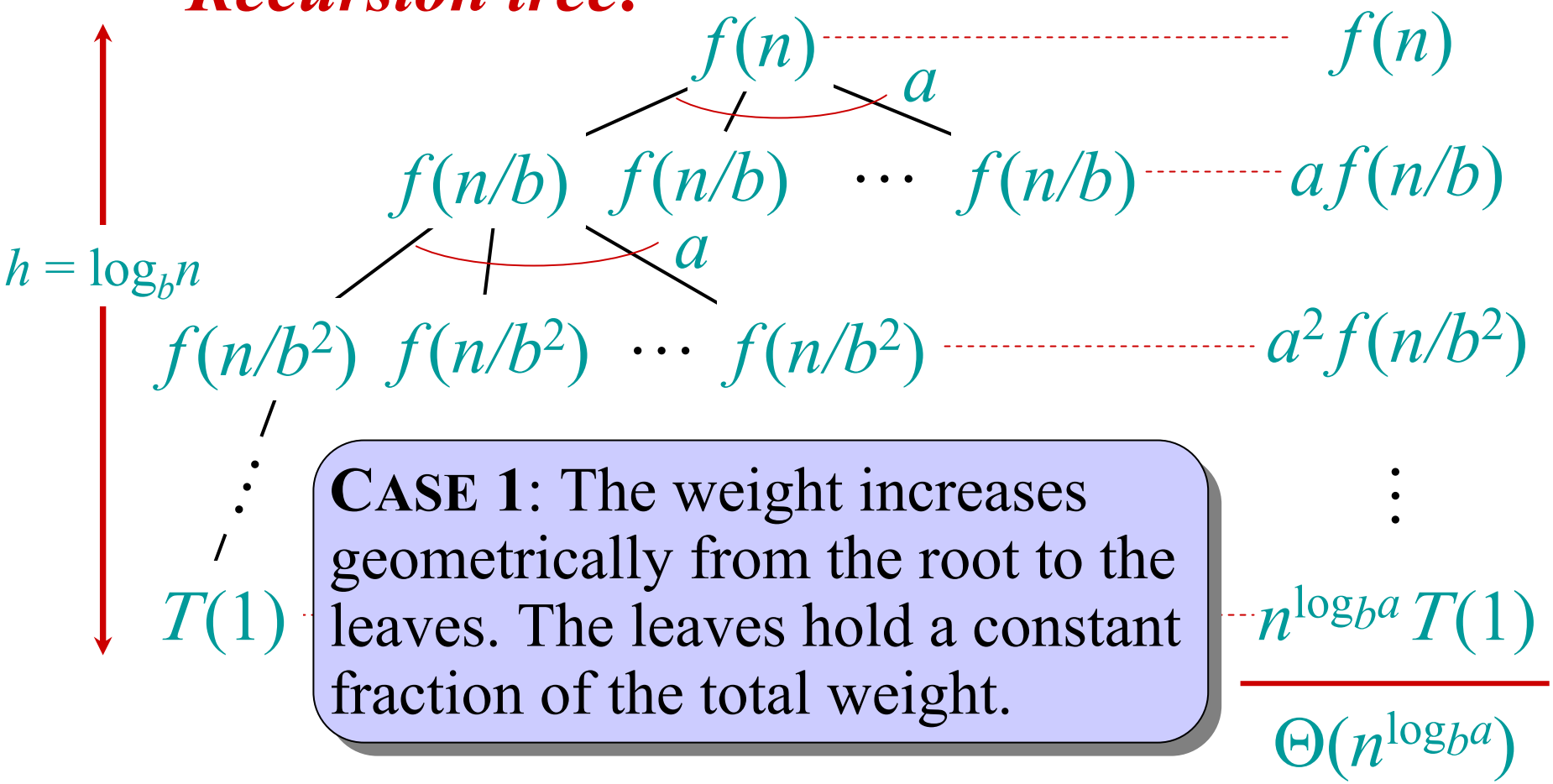

CASE 1: The weight increases geometrically from the root to the leaves. The leaves hold a constant fraction of the total weight.

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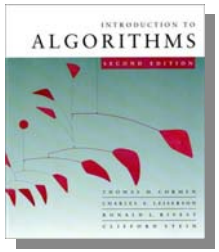

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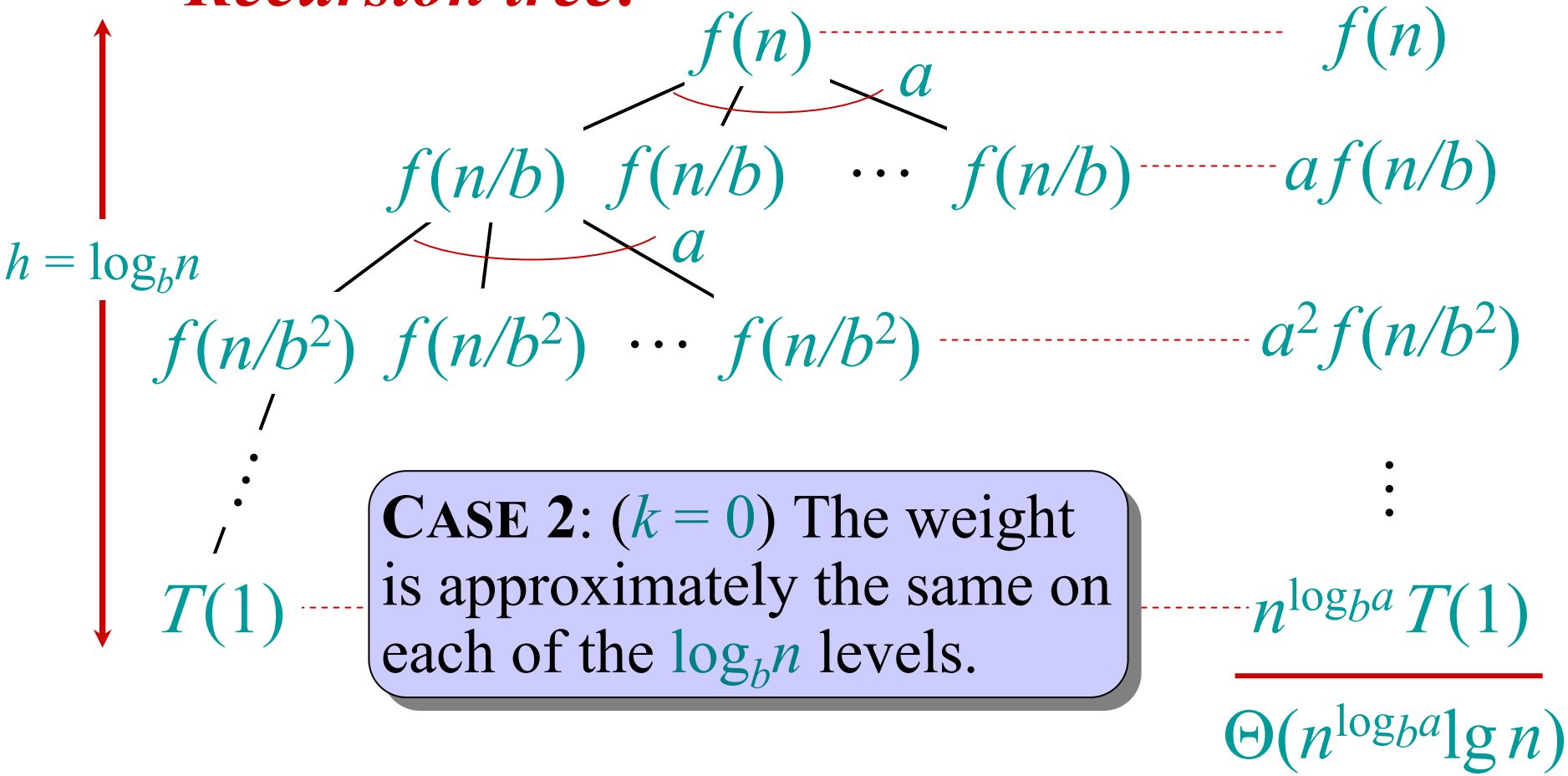

CASE 2: ($k = 0$) The weight is approximately the same on each of the $\log_b n$ levels.

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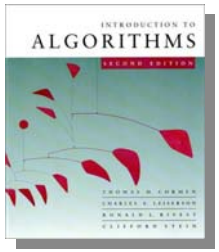

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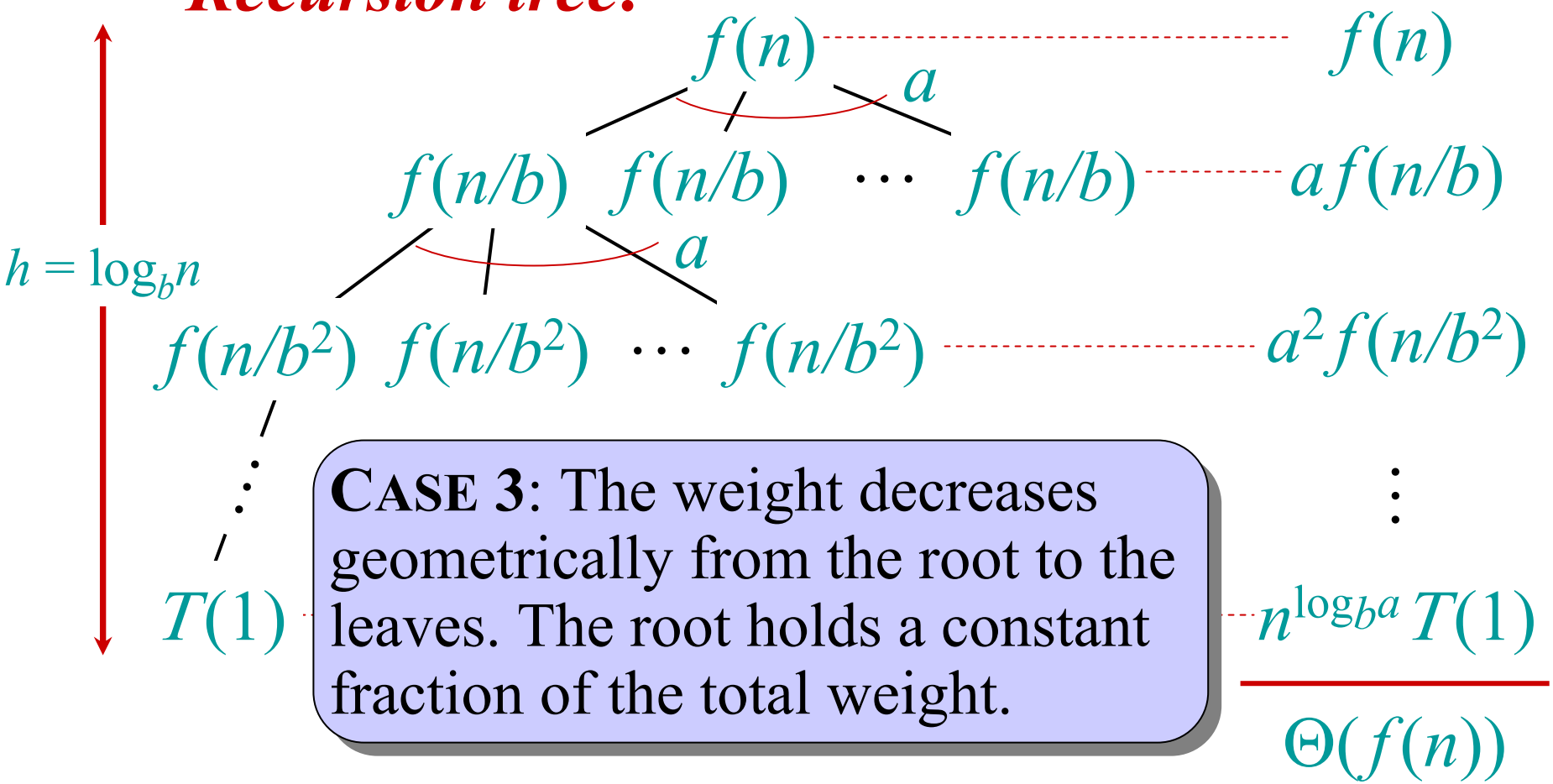

CASE 3: The weight decreases geometrically from the root to the leaves. The root holds a constant fraction of the total weight.

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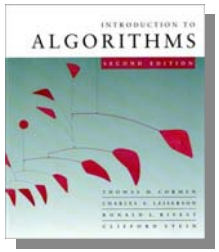
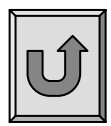


Appendix: geometric series

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x} \quad \text{for } x \neq 1$$

$$1 + x + x^2 + \cdots = \frac{1}{1 - x} \quad \text{for } |x| < 1$$

Return to last slide viewed.



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