A Block Decomposition Algorithm for Sparse Optimization

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Outline of This Talk

- Introduction
- Comparisons and Contributions
- The Proposed Algorithm
- Optimality Analysis
- Convergence Analysis
- Experimental Validation

Introduction

Introduction

Optimization Problem:

$$\boxed{\min_{\mathbf{x}} \; f(\mathbf{x}), \; s.t. \; \|\mathbf{x}\|_0 \leq s} \text{ or } \boxed{\min_{\mathbf{x}} \; f(\mathbf{x}) + \lambda \|\mathbf{x}\|_0,}$$

- $f(\mathbf{x})$: smooth, convex, its gradient is L-Lipschitz continuous
- They can be rewritten as the following problem

$$\min_{\mathbf{x}} \; F(\mathbf{x}) \triangleq f(\mathbf{x}) + h(\mathbf{x}), \text{ with } h(\mathbf{x}) \triangleq h_{\text{cons}} \text{ or } h_{\text{regu}}.$$

$$h_{\text{cons}}(\mathbf{x}) \triangleq I_{\Psi}(\mathbf{x}), \; \Psi \triangleq \{\mathbf{x} | \|\mathbf{x}\|_0 \leq s\}, \; I_{\Psi}(\mathbf{x}) = \begin{cases} 0, & \mathbf{x} \in \Psi \\ \infty, & \mathbf{x} \notin \Psi \end{cases}$$

- These two problems are equivalent in the sense that $\lambda \propto \frac{1}{s}$.
- Applications: compressive sensing, sparse coding, subspace clustering

Application: Compressed Sensing

Signal recovery problem under noisy observations:

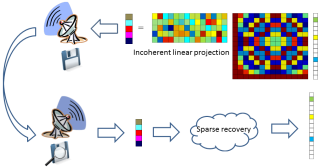
$$\min_{\mathbf{x}\in\mathbb{R}^n} \frac{1}{2}\|\Phi\mathbf{x}-\mathbf{y}\|_2^2, \; s.t. \; \|\mathbf{x}\|_0 \leq s$$

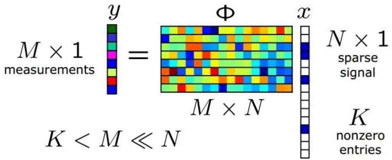
Remarks.

1. Signal recovery problem:

$$\min_{\mathbf{x}\in\mathbb{R}^n} \|\mathbf{x}\|_0, \; s.t. \; \Phi\mathbf{x}=\mathbf{y}$$

2. Many applications: magnetic resonance imaging, single-pixel camera





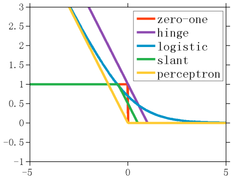
Application: Sparse Logistic Regression

Sparse logistic regression model:

$$\min_{\mathbf{x}\in\mathbb{R}^n} \sum_{i=1}^{m} \log(1+\exp(-\mathbf{y}_i\langle \mathbf{s}_i, \mathbf{x}\rangle)), \ \ s.t. \ \|\mathbf{x}\|_0 \leq s$$

Remarks.

1. Tackle both binary and multiclass classification problems
2. Other alternative loss functions $f(\mathbf{x}) = \sum_{i=1}^{m} l(\mathbf{x}, s_i, y_i)$
   - Zero-one loss: $I(f(s_i) = y_i)$ (I is the indicator function)
   - Slant loss: $\min(1, \min(0, 0.5 - f(s_i)y_i)))$
   - Hinge loss: $\max(0, 1 - f(s_i)y_i)$
   - Perceptron loss: $\max(0, -f(s_i)y_i)$



Comparisons and Contributions

Comparisons of the Methods

1. Relaxed approximation method
   - convex: $f(\mathbf{x}) + \lambda\|\mathbf{x}\|_1$, $f(\mathbf{x}) + \lambda\|\mathbf{x}\|_{\text{tok}-k}$
   - nonconvex: $f(\mathbf{x}) + \lambda\|\mathbf{x}\|_p$, reweighted ℓ_1 norm

   $\Rightarrow$ our method directly controls the sparsity of the solution

2. Greedy pursuit method
   - the solution MUST be initialized to zero
   - $S = \emptyset$, $S = S \cup i_1$, $\min_{\mathbf{x}_S}\ f(\mathbf{x})$, $S = S \cup i_2$, $\min_{\mathbf{x}_S}\ f(\mathbf{x})$, ...

   $\Rightarrow$ our method is a greedy coordinate descent algorithm without forcing the initial solution to zero

Comparisons of the Methods

3. Combinatorial search method: global optimization methods
   - cutting plane methods
   - branch-and-cut methods

   $\Rightarrow$ our method leverages the effectiveness of combinatorial search method methods

4. Gradient projection method
   - $\mathbf{x}^{k+1} = \text{Prox}_{\gamma h}(\mathbf{x}^k - \gamma \nabla f(\mathbf{x}^k))$
   - with $\text{Prox}_{\bar{h}}(\mathbf{a}) = \arg\min_{\mathbf{x}} \ \frac{1}{2}\|\mathbf{x} - \mathbf{a}\|_2^2 + \bar{h}(\mathbf{x})$

   $\Rightarrow$ our method significantly outperforms gradient projection method

Contributions

Contributions of this paper are three-fold:

1. **Algorithmically**: a new decomposition method, use combinatorial search and coordinate descent
2. **Theoretically**:
   - optimality analysis: optimality hierarchy, finds stronger stationary points (e.g. it finds stronger stationary points than Beck and Eldar's method [1])
   - convergence analysis: convergence, convergence rate
3. **Empirically**: state-of-the-art performance (e.g. generally outperforms OMP method)

[1] Sparsity Constrained Nonlinear Optimization: Optimality Conditions and Algorithms, SIAM Journal on Optimization, 2014.

The Proposed Algorithm

The Proposed Algorithm

Input: the size of the working set k, an initial feasible solution $\mathbf{x}^0$.
Set $t = 0$.
while *not converge* **do**

S1 Employ some strategy to find a working set B of size k.
Denote $\bar{B} \triangleq \{1, ..., n\} \setminus B$

S2 Solve the following subproblem globally using combinatorial search:

$$\mathbf{x}^{t+1} \Leftarrow \arg\min_{\mathbf{z}} \; F(\mathbf{z}) + \frac{\theta}{2}\|\mathbf{z} - \mathbf{x}^t\|^2, \; s.t. \; \mathbf{z}_{\bar{B}} = \mathbf{x}^t_{\bar{B}}$$

S3 Increment t by 1

end

Algorithm 1: A Decomposition Algorithm for Sparse Optimization

Comments on the Algorithm

S2 The main procedure:

$$\mathbf{x}^{t+1} \Leftarrow \arg\min_{\mathbf{z}} \ F(\mathbf{z}) + \frac{\theta}{2}\|\mathbf{z} - \mathbf{x}^t\|^2, \ s.t. \ \mathbf{z}_{\bar{B}} = \mathbf{x}^t_{\bar{B}}$$

- **Two New Strategies**: an exhaustive search; a proximal point strategy
- **Solving the Subproblem Globally**: k unknown decision variables, a simple exhaustive search
  An example: $F(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \langle \mathbf{x}, \mathbf{p} \rangle + \lambda\|\mathbf{x}\|_0$

S1 Finding the working set

- **Random strategy**. Select one combination (which contains k coordinates) from the whole working set of size C_n^k uniformly.
- **Greedy strategy**. Pick top-k coordinates that lead to the greatest descent when one variable is changed (the rest variables are fixed) based on the current solution $\mathbf{x}^t$.

Optimality Analysis

Optimality Analysis

Basic Stationary Point

A solution $\breve{\mathbf{x}}$ is called a basic stationary point if the following holds. $h \triangleq h_{\text{regu}}$: $\breve{\mathbf{x}} = \arg\min_{\mathbf{y}\in[-\rho\mathbf{1},\ \rho\mathbf{1}]} \ f(\mathbf{y}),\ s.t.\ \mathbf{y}_Z = \mathbf{0}$; $h \triangleq h_{\text{cons}}$: $\breve{\mathbf{x}} = \arg\min_{\mathbf{y}} f(\mathbf{y}),\ s.t.\ |S| \leq k,\ \mathbf{y}_Z = \mathbf{0}$. Here, $S \triangleq \{i | \breve{\mathbf{x}}_i \neq 0\}$, $Z \triangleq \{j | \breve{\mathbf{x}}_j = 0\}$.

L-Stationary Point

A solution $\grave{\mathbf{x}}$ is an L-stationary point if it holds that: $\grave{\mathbf{x}} = \arg\min_{\mathbf{x}} \ g(\mathbf{x}, \grave{\mathbf{x}}) + h(\mathbf{x})$ with $g(\mathbf{x}, \mathbf{z}) \triangleq f(\mathbf{z}) + \langle \nabla f(\mathbf{z}), \mathbf{x} - \mathbf{z} \rangle + \frac{L}{2} \|\mathbf{x} - \mathbf{z}\|_2^2$.

Block-k Stationary Point

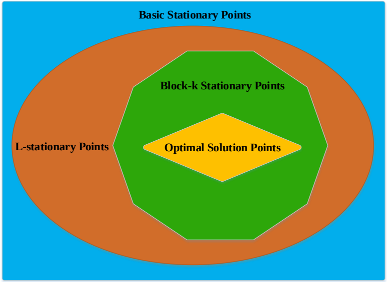
A solution $\bar{\mathbf{x}}$ is a block-k stationary point if it holds that: $\bar{\mathbf{x}} \in \arg\min_{\mathbf{z}\in\mathbb{R}^n} \ \mathcal{P}(\mathbf{z}; \bar{\mathbf{x}}, B) \triangleq \{F(\mathbf{z}),\ s.t.\ \mathbf{z}_{\bar{B}} = \bar{\mathbf{x}}_{\bar{B}}\},\ \forall |B| = k,\ \bar{B} \triangleq \{1, ..., n\} \setminus B$.

Optimality Hierarchy

Relations between the three types of stationary point.

We have the following optimality hierarchy:

$\boxed{\text{Basic Stat. Point}} \overset{(1)}{\Leftarrow} \boxed{L\text{-Stat. Point}} \overset{(2)}{\Leftarrow} \boxed{\text{Block-1 Stat. Point}} \overset{(3)}{\Leftarrow} \boxed{\text{Block-2 Stat. Point}} \Leftarrow \ldots \Leftarrow \boxed{\text{Block-}n\text{ Stat. Point}} \overset{(4)}{\Leftrightarrow} \boxed{\text{Optimal Point}}$.



A Running Example

Optimization Problems:

$$\min_{\mathbf{x}\in\mathbb{R}^n} \ \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{x}^T\mathbf{p}, \ \ s.t. \ \|\mathbf{x}\|_0 \leq 4$$

$$\min_{\mathbf{x}\in\mathbb{R}^n} \ \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{x}^T\mathbf{p} + 0.01\|\mathbf{x}\|_0$$

$n = 6$, $\mathbf{Q} = \mathbf{c}\mathbf{c}^T + \mathbf{I}$, $\mathbf{p} = \mathbf{1}$, $\mathbf{c} = [1\ 2\ 3\ 4\ 5\ 6]^T$.

Number of points satisfying optimality conditions.

	Basic-Stat.	L-Stat.	Block-1 Stat.	Block-2 Stat.	Block-3 Stat.	Block-4 Stat.	Block-5 Stat.	Block-6 Stat.
$h \triangleq h_{\text{cons.}}$	57	14	–	2	1	1	1	1
$h \triangleq h_{\text{regu.}}$	64	56	9	3	1	1	1	1

Convergence Analysis

Global Convergence.

Letting $\mathbf{x}^t$ be the sequence generated by Algorithm 1, we have the following results.

(i) It holds that:

$$F(\mathbf{x}^{t+1}) - F(\mathbf{x}^t) \leq -\frac{\theta}{2}\|\mathbf{x}^{t+1} - \mathbf{x}^t\|^2, \ \lim_{t\to\infty} \mathbb{E}[\|\mathbf{x}^{t+1} - \mathbf{x}^t\|] = 0.$$

(ii) As $t \to \infty$, $\mathbf{x}^t$ converges to the block-k stationary point $\bar{\mathbf{x}}$ in expectation.

Convergence rate when $h \triangleq f_{\text{cons}}$

Optimization problem:

$$\min_{\mathbf{x}} \; f(\mathbf{x}), \; s.t. \; \|\mathbf{x}\|_0 \leq k$$

Convergence Rate for Sparsity Constrained Optimization

Let $f(\cdot)$ be a σ-strongly convex function. We assume that $f(\cdot)$ is Lipschitz continuous such that $\forall t$, $\|\nabla f(\mathbf{x}^t)\|_2^2 \leq \kappa$ for some positive constant κ. Denote $\alpha \triangleq \frac{n\theta}{k\sigma}/(1+\frac{n\theta}{k\sigma})$. We have the following results:

$$\mathbb{E}[F(\mathbf{x}^t) - F(\bar{\mathbf{x}})] \leq (F(\mathbf{x}^0) - F(\bar{\mathbf{x}}))\alpha^t + \frac{\kappa}{2\theta}\frac{\alpha}{1-\alpha},$$

and

$$\mathbb{E}[\frac{\sigma}{4}\|\mathbf{x}^{t+1} - \bar{\mathbf{x}}\|_2^2] \leq \frac{n2\theta}{k}(F(\mathbf{x}^0) - F(\bar{\mathbf{x}}))\alpha^t + \frac{n}{k}\frac{\kappa}{1-\alpha}.$$

Convergence Rate When $h \triangleq f_{\text{regu}}$ (a)

Optimization problem:

$$\min_{\mathbf{x}} \; f(\mathbf{x}) + \lambda \|\mathbf{x}\|_0$$

Convergence Rate for Sparse Regularized Optimization.

(a) It holds that $|\mathbf{x}_i^t| \geq \delta$ for all i with $\mathbf{x}_i^t \neq 0$. Whenever $\mathbf{x}^{t+1} \neq \mathbf{x}^t$, we have $\|\mathbf{x}^{t+1} - \mathbf{x}^t\|_2^2 \geq \frac{k\delta^2}{n}$ and the objective value is decreased at least by D. The solution changes at most $\bar{J}$ times in expectation for finding a block-k stationary point $\bar{\mathbf{x}}$. Here δ, D, and $\bar{J}$ are respectively defined as:

$$\delta \triangleq \min(\rho, \sqrt{2\lambda/(\theta + L)}, \min(|\mathbf{x}^0|)), \;\; D \triangleq \frac{k\theta\delta^2}{2n}, \;\;\; \bar{J} \triangleq \frac{F(\mathbf{x}^0) - F(\bar{\mathbf{x}})}{D}.$$

Convergence Rate When $h \triangleq f_{\text{regu}}$ (b)

Optimization problem:

$$\min_{\mathbf{x}} \; f(\mathbf{x}) + \lambda \|\mathbf{x}\|_0$$

Convergence Rate for Sparse Regularized Optimization.

(b) Assume that $f(\cdot)$ is generally convex, and it holds that $\forall t, \; \|\mathbf{x}^t\|_\infty \leq \rho$. If the support set of $\mathbf{x}^t$ does not changes for all $t = 0, 1, ..., \infty$, Algorithm 1 takes at most V_1 iterations in expectation to converge to a stationary point $\bar{\mathbf{x}}$ satisfying $F(\mathbf{x}^t) - F(\bar{\mathbf{x}}) \leq \epsilon$. Moreover, Algorithm 1 takes at most $V_1 \times \bar{J}$ in expectation to converge to a stationary point $\bar{\mathbf{x}}$ satisfying $F(\mathbf{x}^t) - F(\bar{\mathbf{x}}) \leq \epsilon$. Here, V_1 is defined as:

$$V_1 = \max(\tfrac{4\nu^2}{\theta}, \sqrt{\tfrac{2\nu^2(F(\mathbf{x}^0) - F(\bar{\mathbf{x}}))}{\theta}})/\epsilon, \;\; \text{with } \nu \triangleq \tfrac{2n\rho\sqrt{k}\theta}{k}.$$

Convergence Rate When $h \triangleq f_{\mathrm{regu}}$ (c)

Optimization problem:

$$\min_{\mathbf{x}} \; f(\mathbf{x}) + \lambda \|\mathbf{x}\|_0$$

Convergence Rate Sparse Regularized Optimization.

(c) Assume that $f(\cdot)$ is σ-strongly convex. If the support set of $\mathbf{x}^t$ does not changes for all $t = 0, 1, ..., \infty$, Algorithm 1 takes at most V_2 iterations in expectation to converge to a stationary point $\bar{\mathbf{x}}$ satisfying $F(\mathbf{x}^t) - F(\bar{\mathbf{x}}) \leq \epsilon$. Moreover, Algorithm 1 takes at most $V_2 \times \bar{J}$ in expectation to converge to a stationary point $\bar{\mathbf{x}}$ satisfying $F(\mathbf{x}^t) - F(\bar{\mathbf{x}}) \leq \epsilon$. Here, V_2 is defined as:

$$V_2 = \log_\alpha(\epsilon/(F(\mathbf{x}^0) - F(\bar{\mathbf{x}}))), \text{ with } \alpha \triangleq \tfrac{n\theta}{k\sigma}/(1 + \tfrac{n\theta}{k\sigma}).$$

Improved Convergence Rate When $h \triangleq f_{\text{regu}}$

Optimization problem:

$$\min_{\mathbf{x}} \; f(\mathbf{x}) + \lambda \|\mathbf{x}\|_0, \; s.t. \; \|\mathbf{x}\|_\infty \leq \rho$$

Improved Convergence Rate for Sparse Regularized Optimization.

(a) Assume that $f(\cdot)$ is generally convex, Algorithm 1 takes at most N_1 iterations in expectation to converge to a block-k stationary point $\bar{\mathbf{x}}$ satisfying $F(\mathbf{x}^t) - F(\bar{\mathbf{x}}) \leq \epsilon$, where
$N_1 = (\frac{\bar{J}}{D} + \frac{1}{\epsilon}) \times \max(\frac{4\nu^2}{\theta}, \sqrt{\frac{2\nu^2(F(\mathbf{x}^0) - F(\bar{\mathbf{x}}) - D)}{\theta}})$.

(b) Assume that $f(\cdot)$ is σ-strongly convex, Algorithm 1 takes at most N_2 iterations in expectation to converge to a block-k stationary point $\bar{\mathbf{x}}$ satisfying $F(\mathbf{x}^t) - F(\bar{\mathbf{x}}) \leq \epsilon$, where
$N_2 = \bar{J} \log_\alpha(\frac{D}{(F(\mathbf{x}^0) - F(\bar{\mathbf{x}}))}) + \log_\alpha(\frac{\epsilon}{F(\mathbf{x}^0) - D - F(\bar{\mathbf{x}})})$.

Experimental Validation

Sparsity Constrained Least Squares Problem

Optimization problems:

$$\min_{\mathbf{x}\in\mathbb{R}^n} \ \frac{1}{2}\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2^2, \ s.t. \ \|\mathbf{x}\|_0 \leq s$$

Compared Methods:

- Proximal Gradient Method (PGM)
- Accerlated Proximal Gradient Method (APGM)
- Quadratic Penalty Method (QPM)
- Subspace Pursuit (SSP)
- Regularized Orthogonal Matching Pursuit (ROMP)
- Orthogonal Matching Pursuit (OMP)
- Compressive Sampling Matched Pursuit (CoSaMP)
- Convex ℓ_1 Approximation Method (CVX-ℓ_1)
- Proposed Decomposition Method (DEC-RiGj)

Sparsity Constrained Least Squares Problem

Conclusions:

- **DEC** is more effective than {PGM, APGM}. As k becomes larger, more higher accuracy is achieved.
- **DEC**-**R**0**G**2 converges quickly but it leads to worse solution quality than **DEC**-**R**2**G**0. A combined strategy is preferred.
- **DEC** generally takes less than 30 seconds to converge.

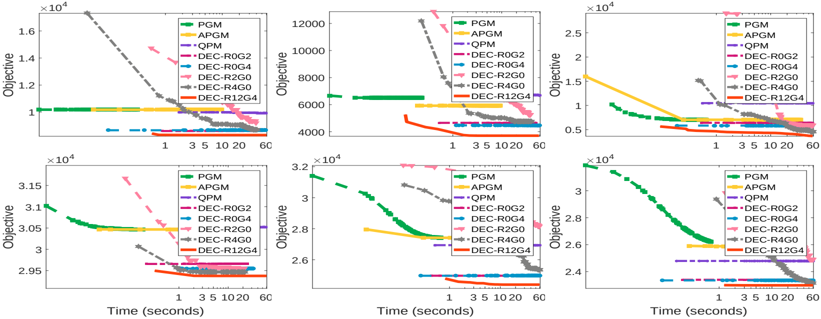


Figure: Convergence curve and computional efficiency for solving sparsity constrained least squares problems on different data sets with different s.

Sparsity Constrained Least Squares Problem

Conclusions:

- IHT methods {PGM, APGM, QPM} lead to bad performance.
- OMP and ROMP sometimes they achieve bad accuracy.
- **DEC** significantly and consistently outperforms the greedy methods on many data sets.

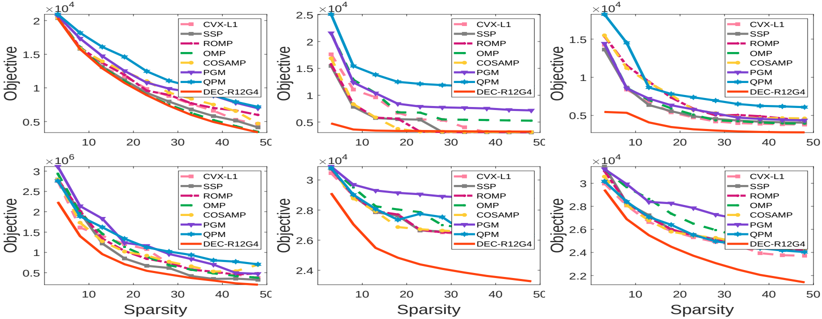


Figure: Experimental results on sparsity constrained least squares problems on different data sets with varying the sparsity of the solution.

Sparse Regularized Least Squares Problem

Optimization problems:

$$\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2 + \lambda\|\mathbf{x}\|_0$$

Compared Methods:

- PGM-ℓ_0: PGM for ℓ_0 Problem
- APGM-ℓ_0: Accerlated PGM for ℓ_0 Problem
- PGM-ℓ_1: PGM for ℓ_1 Problem
- PGM-ℓ_p: PGM for ℓ_p Problem ($p = \frac{1}{2}$)
- Proposed Decomposition Method (DEC-RiGj)

Conclusions:

- PGM-ℓ_p achieves better performance than PGM-ℓ_1.
- **DEC** generally outperforms the other methods in all data sets.

	PGM-ℓ_0	APGM-ℓ_0	PGM-ℓ_1	PGM-ℓ_p	DEC-R10G2
results on random-256-1024					
$\lambda = 10^0$	6.9e+2	2.4e+4	7.8e+2	4.0e+2	4.8e+2
$\lambda = 10^1$	2.3e+3	3.8e+4	3.3e+3	1.9e+3	2.2e+3
$\lambda = 10^2$	2.0e+4	1.3e+5	1.8e+4	1.1e+4	9.4e+3
$\lambda = 10^3$	2.5e+4	1.0e+6	2.4e+4	2.4e+4	2.4e+4
results on random-256-2048					
$\lambda = 10^0$	1.3e+3	2.7e+4	1.4e+3	6.0e+2	5.4e+2
$\lambda = 10^1$	2.9e+3	4.5e+4	4.9e+3	2.2e+3	2.2e+3
$\lambda = 10^2$	2.2e+4	2.3e+5	2.1e+4	1.1e+4	9.5e+3
$\lambda = 10^3$	2.7e+4	2.1e+6	2.6e+4	2.7e+4	2.6e+4
results on e2006-5000-1024					
$\lambda = 10^0$	8.5e+3	3.3e+4	1.1e+4	1.8e+4	7.3e+3
$\lambda = 10^1$	9.4e+3	4.2e+4	3.2e+4	3.2e+4	8.6e+3
$\lambda = 10^2$	3.2e+4	1.3e+5	3.2e+4	3.2e+4	1.3e+4
$\lambda = 10^3$	1.8e+4	1.1e+6	3.2e+4	3.2e+4	1.1e+4
results on e2006-5000-2048					
$\lambda = 10^0$	3.1e+3	3.4e+4	4.4e+3	1.4e+4	2.6e+3
$\lambda = 10^1$	5.2e+3	5.3e+4	1.2e+4	1.2e+4	4.5e+3
$\lambda = 10^2$	3.2e+4	2.4e+5	3.2e+4	3.2e+4	7.0e+3
$\lambda = 10^3$	1.8e+4	2.1e+6	3.2e+4	3.2e+4	1.3e+4

	PGM-ℓ_0	APGM-ℓ_0	PGM-ℓ_1	PGM-ℓ_p	DEC-R10G2
results on random-256-1024-C					
$\lambda = 10^0$	9.6e+2	5.7e+6	1.0e+3	1.0e+3	8.9e+2
$\lambda = 10^1$	8.1e+3	3.5e+6	1.0e+4	8.2e+3	7.3e+3
$\lambda = 10^2$	5.8e+4	6.2e+6	8.9e+4	5.4e+4	5.1e+4
$\lambda = 10^3$	2.5e+5	5.3e+6	3.7e+5	2.2e+5	2.0e+5
results on random-256-2048-C					
$\lambda = 10^0$	1.9e+3	5.7e+6	2.0e+3	1.9e+3	1.2e+3
$\lambda = 10^1$	1.7e+4	7.7e+6	2.0e+4	1.6e+4	9.2e+3
$\lambda = 10^2$	8.4e+4	4.2e+6	1.6e+5	6.4e+4	5.3e+4
$\lambda = 10^3$	2.5e+5	9.6e+6	6.3e+5	2.5e+5	2.4e+5
results on e2006-5000-1024-C					
$\lambda = 10^0$	3.0e+4	3.3e+4	2.8e+4	2.9e+4	2.2e+4
$\lambda = 10^1$	3.2e+4	4.2e+4	3.2e+4	3.2e+4	2.3e+4
$\lambda = 10^2$	3.2e+4	1.3e+5	3.2e+4	3.2e+4	2.9e+4
$\lambda = 10^3$	3.2e+4	1.1e+6	3.2e+4	3.2e+4	3.2e+4
results on e2006-5000-2048-C					
$\lambda = 10^0$	2.9e+4	3.4e+4	2.6e+4	2.7e+4	1.7e+4
$\lambda = 10^1$	3.2e+4	5.3e+4	3.2e+4	3.2e+4	2.1e+4
$\lambda = 10^2$	3.2e+4	2.4e+5	3.2e+4	3.2e+4	2.7e+4
$\lambda = 10^3$	3.2e+4	2.1e+6	3.2e+4	3.2e+4	3.2e+4

Table: Comparisons of objective values of all the methods for solving the sparse regularized least squares problem. The 1^{st}, 2^{nd}, and 3^{rd} best results are colored with red, blue and green, respectively.

Sparse Regularized Least Squares Problem

Conclusions:

- **DEC** takes several times longer to converge. (70 seconds)
- The computational time is acceptable and pays off as **DEC** achieves significantly higher accuracy.
- The main bottleneck: small-sized subproblems ($\mathcal{O}(2^k)$).
- k is a parameter to balance the efficacy and efficiency.

	PGM-ℓ_0	APGM-ℓ_0	PGM-ℓ_1	PGM-ℓ_p	DEC-R10G2
r.-256-1024	12 ± 3	13 ± 3	5 ± 3	15 ± 3	36 ± 3
r.-256-2048	11 ± 3	11 ± 3	9 ± 3	16 ± 3	66 ± 7
e.-5000-1024	12 ± 3	11 ± 3	8 ± 3	14 ± 3	45 ± 3
e.-5000-2048	12 ± 3	10 ± 3	12 ± 3	5 ± 3	65 ± 8

Table: Comparisons of average times (in seconds) of all the methods on different data sets for solving the sparse regularized least squares problem.

Thank You!