ECE4960 – Project Assignment 5
A Generic 1D/2D Parabolic PDE Solver for Heat Equation with Finite Difference

1. Internal Document

In this project, a generic parabolic PDE solver is implemented, which could take in any arbitrary initial values of any length, with the finite difference discretization rules of forward, backward, and trapezoidal Euler. This PDE solver is used to simulate and solve the naïve heat conduction equations:

General form: $\frac{\partial T}{\partial t} = \frac{K}{c_\rho \rho} \nabla^2 T$

1D form: $\frac{\partial T}{\partial t} = \frac{K}{c_\rho \rho} \frac{\partial^2 T}{\partial x^2}$

2D form: $\frac{\partial T}{\partial t} = \frac{K}{c_\rho \rho} (\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2})$

Dirichlet boundary conditions are adapted, and $\frac{K}{c_\rho \rho}$ is a constant whose value is around 0.1 - 0.3. In this project, $\frac{K}{c_\rho \rho}$ is set to be 0.15.

- **Program Design Breakdown:**

  The program is broken down into: ***PDESolver.java***, ***DiscretizationRule.java***, ***Main.java***, ***TestHelper.java***, and other inherited helper scripts including *FilIO.java*, *SparseMatrix.java*, *Vector.java*, *Jacobi.java*.

  a) ***PDESolver.java:*** Java class, a generic 1D/2D parabolic PDE solver that takes in any arbitrary length of input, constant D, grid size h, delta_t, start and end time. It also interfaces with ***DiscretizationRule.java***, which provides different finite difference discretization rules for solving parabolic PDEs.

  b) ***DiscretizationRule.java:*** Java interface, provides handlers for different finite difference discretization rules including forward, backward, trapezoid Euler in both 1D and 2D.

  c) ***TestHelper.java:*** Implements white box testing for helper functions used in this project.

  d) ***Main.java***: The main entrance for this project, invokes tests for helper functions, and validates heat equations' simulation results.

  e) ***FilIO.java:*** Imported from previous-written java project, auto-generates output files and test reports.

  f) ***Jacobi.java*:** Imported from previous-written java project, a generic Jacobi-iterative matrix solver, helps solve PDE equations.

  g) ***Vector.java, SparseMatrix.java***: Imported from previous-written java

project, the data structures for implementing PDE solvers. Instead of FullMatrix, SparseMatrix is used because it would be more space efficient when the input vector space is tremendously large.

- **Program Implementation:**

a) **Forward Euler rule:** Its implementation is different from the backward the trapezoidal Euler. The **1D** forward Euler PDE could be expressed as below:

$$n_{i,j+1} = n_{i,j}\left(1 - \frac{2D*\Delta t}{h^2}\right) + \left(n_{i+1,j} + n_{i-1,j}\right)\frac{D*\Delta t}{h^2}$$

The left hand side equation represents time = j+1, and the right hand side represents time = j. Therefore, to calculate the temperatures for next time-step, we only need to do $\boldsymbol{Temperature\ (t = j+1) = Matrix\ A * Temperature(t = j)}$, where A's diagonal elements are $(1 - \frac{2D*\Delta t}{h^2})$, and $n_{i,j}$'s neighbors having the value of $\frac{D*\Delta t}{h^2}$ in A.

For **2D PDE**: The diagonal values are $(1 - \frac{4D*\Delta t}{h^2})$, and $n_{i,j}$'s neighbors are $\frac{D*\Delta t}{h^2}$.

b) **Backward and Trapezoidal Euler rule:** require solving matrixes, instead of solving by matrix multiplication. The PDE equations of backward and trapezoidal Euler in **1D** could be respectively written as:

$$n_{i,j+1}\left(\frac{1}{\Delta t} + \frac{2D}{h^2}\right) - \left(n_{i+1,j+1} + n_{i-1,j+1}\right)\frac{D}{h^2} = n_{i,j}\left(-\frac{1}{\Delta t}\right)$$

$$n_{i,j+1}\left(\frac{1}{\Delta t} + \frac{D}{h^2}\right) - \left(n_{i+1,j+1} + n_{i-1,j+1}\right)\frac{D}{2h^2} = n_{i,j+1}\left(\frac{1}{\Delta t} - \frac{D}{h^2}\right) + \left(n_{i+1,j+1} + n_{i-1,j+1}\right)\frac{D}{2h^2}$$

To calculate the temperatures for next time-step, we only need to do $\boldsymbol{Matrix\ A * Temperature\ (t = j+1) = Matrix\ B * Temperature(t = j)}$, where A's diagonal elements are $(\frac{1}{\Delta t} + \frac{2D}{h^2})$, and $n_{i,j}$'s neighbors having the value of $\frac{D}{h^2}$ in A for Backward Euler rule, and $\left(\frac{1}{\Delta t} + \frac{D}{h^2}\right), \frac{D}{2h^2}$ for Trapezoidal Euler rule. B's diagonal value is $-\frac{1}{\Delta t}$ for backward Euler, $\left(\frac{1}{\Delta t} - \frac{D}{h^2}\right)$ for trapezoidal rule's diagonal value, and $\frac{D}{2h^2}$ for

trapezoid rule's neighbors.

For **2D PDE**: A's diagonal values are $(\frac{1}{\Delta t}+\frac{4D}{h^2})$ and $\left(\frac{1}{\Delta t}+\frac{2D}{h^2}\right)$.

- **Program Validation and Testing strategy:**

The helper functions are tested with white box testing, and their test results and testing strategies are available in "TestHelperReport.txt".
The PDE solver is validated with Wilkinson's Principle, where a problem is approached with different methods, and by checking whether these methods have same or similar output, we could tell whether the program is performing correctly.

a) For 1D heat conduction equation:

The 1D heat conduction could be imagined as the heat is conducting through a thin pole. The initial condition is (0, 0, 0, 0, 0, 0, 10, 10, 10, 0, 0, 0, 0, 0, 0), meaning the center of the pole is hot at the beginning. The time and space plots of three discretization rules are as below:

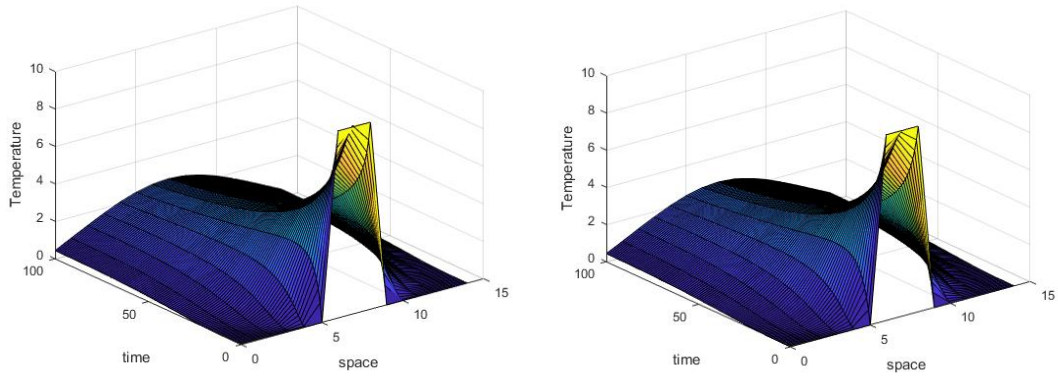


Fig. [1] Left: Backward Euler; Right: Forward Euler

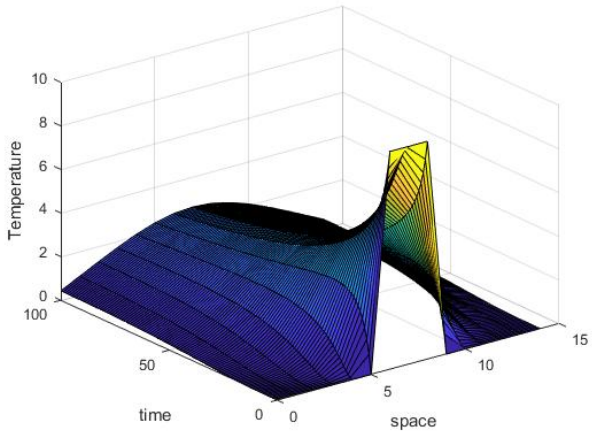


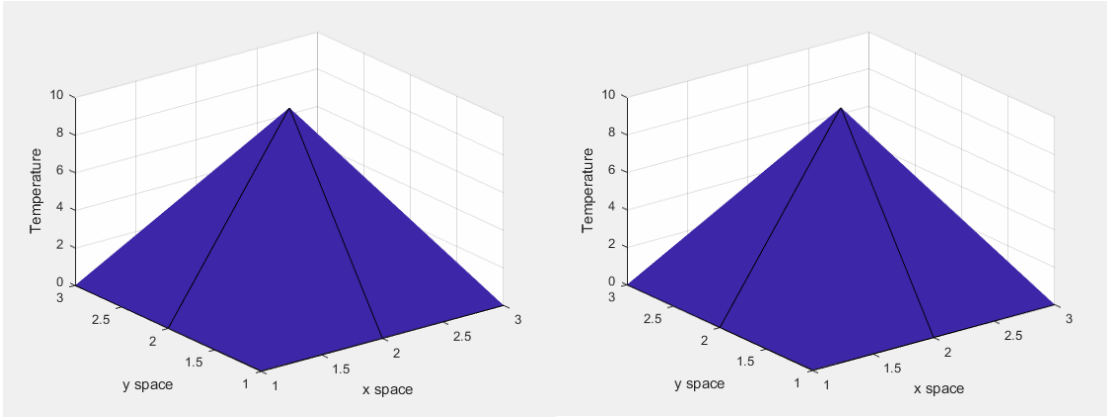
Fig. [2] Trapezoid Euler

It could be seen that the temperature is attenuating with time, and the heat is

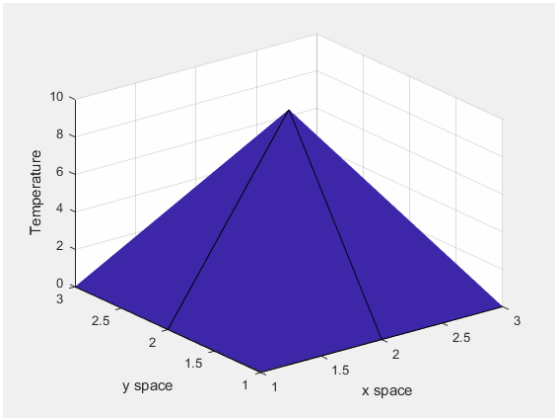
conducted from the center pole to the ends of the pole, which obeys the physical law and common sense.
Also, three rule's outputs are very similar to each other, therefore, the 1D PDE solver is validated.
It's also worth mentioning, if you look at the actual output log files, the trapezoidal output has smoother changes, then the backward Euler, and the forward Euler's output has the least smooth changes.

b) For 2D heat conduction equation:

The 2D heat conduction could be imagined as heat is conducting within a plane. The initial condition is (0, 0, 0, 0, 10, 0, 0, 0, 0), meaning the center of the plane is hot at the beginning. The time and space plots are made into animations, which should be "animating" by looking at the .gif in my git repo:



Animation. [3] Left: Backward Euler; Right: Forward Euler
The animation should be animating in this project's git repo, in directory **Outputs**



Animation. [4] Trapezoid Euler
The animation should be animating in this project's git repo, in directory **Outputs**

It could be seen through the animations that the temperature is attenuating with time, and the heat is conducted from the center to the edges of the plane, which obeys the physical law.
Also, three rule's outputs are very similar to each other, therefore, the 2D PDE

solver is also validated.
It's also worth mentioning, if you look at the actual output log files, the trapezoidal output has smoother changes, then the backward Euler, and the forward Euler's output has least smooth changes.

2. External Document

This program is designed to solve any arbitrary parabolic PDEs in 1D or 2D with finite difference discretization rules including forward, backward and trapezoidal Euler. The API is:

```
PDESolver.solve(new backward1D(), "1D", initial1D, h1, D1, dt1,
startTime1, endTime1);
```

To invoke the solver, users only need to indicate the discretization rule, the space dimension, the initial conditions, the space and time grid sizes of h and delta_t, the constant D, and the time range.
Then the program would automatically output the solutions into user's directory as .txt format files.