

NO.4-1 不定积分解题方法（上）

套路一 有理函数的积分

（一）有理函数积分的通用方法

例题 1 $\int \frac{x+3}{x^2+2x+4} dx$

解: $I = \int \frac{\frac{1}{2}(2x+2)+2}{x^2+2x+4} dx = \frac{1}{2} \ln|x^2+2x+4| + 2 \int \frac{1}{(x+1)^2+(\sqrt{3})^2} dx$
 $= \frac{1}{2} \ln|x^2+2x+4| + \frac{2}{\sqrt{3}} \arctan \frac{x+1}{\sqrt{3}} + C$

例题 2 $\int \frac{x^2}{(a^2+x^2)^2} dx$

解: $I = -\frac{1}{2} \int x d(\frac{1}{x^2+a^2}) = -\frac{x}{2(x^2+a^2)} + \frac{1}{2} \int \frac{1}{x^2+a^2} dx = -\frac{x}{2(x^2+a^2)} + \frac{1}{2a} \arctan \frac{x}{a} + C$

类题 $\int \frac{1}{(a^2+x^2)^2} dx$

解: 使用三角换元, 令 $x = a \tan t$, 则 $dx = a \sec^2 t dt$

$$\begin{aligned} I &= \int \frac{1}{a^4 \sec^4 t} \cdot a \sec^2 t dt = \frac{1}{a^3} \int \cos^2 t dt = \frac{1}{2a^3} \int (\cos 2t + 1) dt = \frac{1}{4a^3} \sin 2t + \frac{t}{2a^3} + C \\ &= \frac{x}{2a^2(x^2+a^2)} + \frac{1}{2a^3} \arctan \frac{x}{a} + C \end{aligned}$$

注: 换元后一定要回代, 回代!!!

例题 3 $\int \frac{3x+6}{(x-1)^2(x^2+x+1)} dx$ (此为 2019 年的一道 10 分大题, 居然只有一个考点, 出题人真是极其无聊)

解: $\frac{3x+6}{(x-1)^2(x^2+x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+x+1}$
 $= \frac{A(x-1)(x^2+x+1) + B(x^2+x+1) + (Cx+D)(x-1)^2}{(x-1)^2(x^2+x+1)}$

可得
$$\begin{cases} A+C=0 \\ 3B=9(x=1 \text{ 代入}) \\ B-2C+D=0 \\ -A+B+D=6(x=0 \text{ 代入}) \end{cases}$$
 解得 $A=-2, B=3, C=2, D=1$

$$I = -2 \ln|x-1| - \frac{1}{3(x-1)} + \ln|x^2+x+1| + C$$

例题 4 $\int \frac{1}{1+x^3} dx$

解: $\frac{1}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1-x}$

可得 $1 = A(x^2 + 1 - x) + (Bx + C)(x + 1)$

$$\begin{cases} A + B = 0 \\ B + C - A = 0 \text{ 解得 } A = \frac{1}{3}, B = -\frac{1}{3}, C = \frac{2}{3} \\ A + C = 1 \end{cases}$$

$$\begin{aligned} I &= \int \frac{1}{(x+1)(x^2+1-x)} dx = \int \left(\frac{1}{3} \cdot \frac{1}{x+1} + \frac{-\frac{1}{3}x + \frac{2}{3}}{x^2+1-x} \right) dx \\ &= \frac{1}{3} \ln|x+1| - \frac{1}{6} \ln|x^2+1-x| + \frac{1}{2} \int \frac{1}{x^2+1-x} dx \\ &= \frac{1}{3} \ln|x+1| - \frac{1}{6} \ln|x^2+1-x| + \frac{1}{2} \int \frac{1}{(x-\frac{1}{2})^2 + \frac{3}{4}} dx \\ &= \frac{1}{3} \ln|x+1| - \frac{1}{6} \ln|x^2+1-x| + \frac{1}{\sqrt{3}} \arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C \end{aligned}$$

例题 5 若不定积分 $\int \frac{x^2+ax+2}{(x+1)(x^2+1)} dx$ 的结果中不含反正切函数，求 a

$$\text{解: } \frac{x^2+ax+2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+D}{x^2+1}$$

$$\text{原积分 } I = A \ln|x| + \frac{B}{2} \ln|x^2+1| + D \arctan x + C$$

可得 $A(x^2+1) + (Bx+D)(x+1) = x^2+ax+2$

$$\begin{cases} A + B = 1 \\ B + D = a \text{ 解得 } A = \frac{3-a}{2}, B = \frac{a-1}{2}, D = \frac{a+1}{2} \\ A + D = 2 \end{cases}$$

若使得原积分不含反正切函数，则 $D = 0$ 即 $a = -1$

(二) 有理函数积分的特殊解法

例题 6 $\int \frac{1}{1-x^4} dx$

$$\text{解: } I = \frac{1}{2} \int \frac{1+x^2+1-x^2}{(1+x^2)(1-x^2)} dx = \frac{1}{2} \int \left(\frac{1}{2} \cdot \frac{1-x+1+x}{1-x^2} + \frac{1}{1+x^2} \right) dx = \frac{1}{4} \ln \left| \frac{x+1}{x-1} \right| + \frac{1}{2} \arctan x + C$$

类题 1 $\int \frac{1}{x^8(1+x^2)} dx$

$$\begin{aligned} \text{解: } I &= - \int \frac{x^8-1-x^8}{x^8(1+x^2)} dx = - \int \frac{(x^4+1)(x^2+1)(x^2-1)}{x^8(1+x^2)} dx + \arctan x + C \\ &= - \int \frac{x^6+x^2-x^4-1}{x^8} dx + \arctan x + C = \frac{1}{x} + \frac{1}{5x^5} - \frac{1}{3x^7} - \frac{1}{7x^9} + \arctan x + C \end{aligned}$$

注: 本题使用倒代换也可，具体操作如下：

$$\text{令 } x = \frac{1}{t}, \text{ 则 } dt = -\frac{1}{x^2} dx$$

$$\begin{aligned}
I &= \int \frac{1}{\frac{1}{t^8} \left(1 + \frac{1}{t^2}\right)} \left(-\frac{1}{t^2}\right) dt = - \int \frac{t^8}{t^2 + 1} dt = - \int [(t^4 + 1)(t^2 - 1) + \frac{1}{t^2 + 1}] dt \\
&= -\frac{1}{7}t^7 - \frac{1}{3}t^3 + \frac{1}{5}t^5 + t - \arctant + C = -\frac{1}{7x^7} - \frac{1}{3x^3} + \frac{1}{5x^5} + \frac{1}{x} - \arctan \frac{1}{x} + C
\end{aligned}$$

我们发现这两种方法求出的原函数竟然不一样!!! 这里我们知道由于方法可能会不定积分的答案不唯一, 答案会相差一个常数, 所以只要求导回去得到原来的被积函数, 答案都是对的。不过这里我们介绍一个恒等式你就明白为什么。

$$\arctan x + \arctan \frac{1}{x} = \begin{cases} \frac{\pi}{2}, & x > 0 \\ -\frac{\pi}{2}, & x < 0 \end{cases}$$

类题 2 $\int \frac{1+x^4}{1+x^6} dx$ (一道神仙题, 最后一步侮辱智商)

$$\text{解: } I = \int \frac{1+x^4-x^2+x^2}{1+x^6} dx = \arctan x + \frac{1}{3} \int \frac{1}{1+(x^3)^2} d(x^3) = \arctan x + \frac{1}{3} \arctan x^3 + C$$

类题 3 $\int \frac{1}{x(x^3+27)} dx$

$$\text{解: } I = \frac{1}{27} \int \frac{x^3+27-x^3}{x(x^3+27)} dx = \frac{1}{27} \ln|x| - \frac{1}{81} \ln|x^3+27| + C$$

例题 7 $\int \frac{1+x^2}{1+x^4} dx$

$$\text{解: } I = \int \frac{1+\frac{1}{x^2}}{\frac{1}{x^2}+x^2} dx = \int \frac{1}{(x-\frac{1}{x})^2+2} d\left(x-\frac{1}{x}\right) = \frac{1}{\sqrt{2}} \arctan \frac{x-\frac{1}{x}}{\sqrt{2}} + C$$

注: 数是连续的, 而其原函数却含有间断点, 引入了无定义点, 此处需要补充定义。

$$\lim_{x \rightarrow 0^+} \frac{1}{\sqrt{2}} \arctan \frac{x-\frac{1}{x}}{\sqrt{2}} = -\frac{\pi}{2\sqrt{2}}$$

$$\lim_{x \rightarrow 0^-} \frac{1}{\sqrt{2}} \arctan \frac{x-\frac{1}{x}}{\sqrt{2}} = \frac{\pi}{2\sqrt{2}}$$

$$\text{故 } F(x) = \begin{cases} \frac{1}{\sqrt{2}} \arctan \frac{x-\frac{1}{x}}{\sqrt{2}} + \frac{\pi}{2\sqrt{2}} + C, & x > 0 \\ C, & x = 0 \\ \frac{1}{\sqrt{2}} \arctan \frac{x-\frac{1}{x}}{\sqrt{2}} - \frac{\pi}{2\sqrt{2}} + C, & x < 0 \end{cases}$$

才是其原函数

类题 1 $\int \frac{1-x^2}{1+x^4} dx$

$$\begin{aligned}
\text{解: } I &= - \int \frac{1 - \frac{1}{x^2}}{\frac{1}{x^2} + x^2} dx = - \int \frac{1}{\left(x + \frac{1}{x}\right)^2 - 2} d\left(x + \frac{1}{x}\right) \\
&= - \frac{1}{2\sqrt{2}} \int \frac{\left(x + \frac{1}{x} + \sqrt{2}\right) - \left(x + \frac{1}{x} - \sqrt{2}\right)}{\left(x + \frac{1}{x}\right)^2 - 2} d\left(x + \frac{1}{x}\right) = - \frac{1}{2\sqrt{2}} \ln \left| \frac{x + \frac{1}{x} - \sqrt{2}}{x + \frac{1}{x} + \sqrt{2}} \right| + C
\end{aligned}$$

类题 2 $\int \frac{1}{1+x^6} dx$

$$\begin{aligned}
\text{解: } I &= \int \frac{1+x^2-x^2}{1+x^6} dx = \int \frac{1}{1+x^4-x^2} dx - \frac{1}{3} \arctan x^3 = \frac{1}{2} \int \frac{1+x^2+1-x^2}{1+x^4-x^2} dx - \frac{1}{3} \arctan x^3 \\
&= \frac{1}{2} \int \frac{1+\frac{1}{x^2}}{x^2 + \frac{1}{x^2} - 1} dx + \frac{1}{2} \int \frac{\frac{1}{x^2}-1}{x^2 + \frac{1}{x^2} - 1} dx - \frac{1}{3} \arctan x^3 \\
&= \frac{1}{2} \int \frac{d(x - \frac{1}{x})}{(x - \frac{1}{x})^2 + 1} - \frac{1}{2} \int \frac{d(x + \frac{1}{x})}{(x + \frac{1}{x})^2 - 3} - \frac{1}{3} \arctan x^3 \\
&= \frac{1}{2} \arctan(x - \frac{1}{x}) - \frac{1}{4\sqrt{3}} \ln \left| \frac{x + \frac{1}{x} - \sqrt{3}}{x + \frac{1}{x} + \sqrt{3}} \right| - \frac{1}{3} \arctan x^3 + C
\end{aligned}$$

套路二 三角有理函数的积分

(一) 三角有理函数积分的通用方法

例题 1 $\int \frac{1}{3+5\cos x} dx$

解: 使用万能公式, 令 $u = \tan \frac{x}{2}$

$$\begin{aligned}
I &= \int \frac{1}{3 + 5 \frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2} du = \int \frac{1}{4-u^2} du = \frac{1}{4} \int \frac{2+u+2-u}{4-u^2} du = \frac{1}{4} \left(\frac{1}{2-u} + \frac{1}{2+u} \right) du \\
&= \frac{1}{4} \ln \left| \frac{u+2}{u-2} \right| + C = \frac{1}{4} \ln \left| \frac{\tan \frac{x}{2} + 2}{\tan \frac{x}{2} - 2} \right| + C
\end{aligned}$$

例题 2 $\int \frac{1}{1+\sin x + \cos x} dx$

解: 使用万能公式, 令 $u = \tan \frac{x}{2}$

$$I = \int \frac{1}{1 + \frac{2u}{1+u^2} + \frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2} du = \int \frac{1}{u+1} du = \ln|u+1| + C = \ln|\tan \frac{x}{2} + 1| + C$$

(二) 三角有理函数积分的特殊解法

例题 3 $\int \frac{1}{1+\cos x} dx$

解: $I = \int \frac{1}{2\cos^2 \frac{x}{2}} dx = \tan \frac{x}{2} + C$

类题 1 $\int \frac{\sin x}{1+\sin x} dx$

解: $I = \int \frac{\frac{2u}{1+u^2}}{1+\frac{2u}{1+u^2}} \cdot \frac{2}{1+u^2} du = \int \frac{4u}{(1+u^2)(u+1)^2} du = 2 \int \frac{(u+1)^2 - (1+u^2)}{(1+u^2)(u+1)^2} du$
 $= 2 \int \frac{1}{1+u^2} du - 2 \int \frac{1}{(u-1)^2} du = 2 \arctan u + \frac{2}{u-1} + C = 2 \arctan(\tan \frac{x}{2}) + \frac{2}{\tan \frac{x}{2} - 1} + C$

类题 2 $\int \frac{1}{\sin x + \cos x} dx$

解: $I = \int \frac{1}{\sqrt{2} \sin(x + \frac{\pi}{4})} dx = \frac{1}{\sqrt{2}} \ln |\csc(x + \frac{\pi}{4}) - \cot(x + \frac{\pi}{4})| + C$

类题 3 $\int \frac{\cos x}{\sin x + \cos x} dx$

解: $I = \int \frac{\cos(x + \frac{\pi}{4} - \frac{\pi}{4})}{\sqrt{2} \sin(x + \frac{\pi}{4})} dx = \frac{1}{\sqrt{2}} \int \frac{\frac{\sqrt{2}}{2} \cos(x + \frac{\pi}{4}) + \frac{\sqrt{2}}{2} \sin(x + \frac{\pi}{4})}{\sin(x + \frac{\pi}{4})} dx = \frac{1}{2} \ln |\sin(x + \frac{\pi}{4})| + \frac{1}{2} x + C$

例题 4 $\int \frac{1}{\sin^2 x \cdot \cos x} dx$

解: $I = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos x} dx = \ln |\sec x + \tan x| - \frac{1}{\sin x} + C$

例题 5 $\int \frac{\cos^3 x - 2 \cos x}{1 + \sin^2 x + \sin^4 x} dx$

解: $I = \int \frac{\cos^2 x - 2}{1 + \sin^2 x + \sin^4 x} d(\sin x) = - \int \frac{1 + \sin^2 x}{1 + \sin^2 x + \sin^4 x} d(\sin x) = - \int \frac{1 + \frac{1}{u^2}}{u^2 + 1 + \frac{1}{u^2}} du$

$= - \int \frac{1}{(u - \frac{1}{u})^2 + 3} d(u - \frac{1}{u}) = - \frac{1}{\sqrt{3}} \arctan(\frac{u - \frac{1}{u}}{\sqrt{3}}) + C = - \frac{1}{\sqrt{3}} \arctan\left(\frac{\sin x - \frac{1}{\sin x}}{\sqrt{3}}\right) + C$

例题 6 $\int \sec^3 x dx$

解: $I = \int \sec x dtan x = \sec x \tan x - \int \sec x \tan^2 x dx = \sec x \tan x - \int \sec x (\sec^2 x - 1) dx$
 $= \sec x \tan x + \int \sec x dx - I$

$$\text{故 } I = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C$$

$$\text{类题 1 } \int \sqrt{1+x^2} dx$$

$$\begin{aligned}\text{解: } I &= x\sqrt{1+x^2} - \int \frac{x^2}{\sqrt{1+x^2}} dx = x\sqrt{1+x^2} - \int \frac{x^2+1-1}{\sqrt{1+x^2}} dx = x\sqrt{1+x^2} - \int (\sqrt{1+x^2} - \frac{1}{\sqrt{1+x^2}}) dx \\ &= x\sqrt{1+x^2} - I + \ln|x + \sqrt{1+x^2}|\end{aligned}$$

$$\text{故 } I = \frac{1}{2}(x\sqrt{1+x^2} + \ln|x + \sqrt{1+x^2}|) + C$$

$$\text{类题 2 请思考如何计算积分 } I_n = \int \sec^n x dx \quad (n \geq 3)$$

$$\begin{aligned}\text{解: } I_n &= \int \sec^{n-2} x d(\tan x) = \tan x \cdot \sec^{n-2} x - (n-2) \int \tan^2 x \cdot \sec^{n-2} x dx \\ &= \tan x \cdot \sec^{n-2} x - (n-2) \int (\sec^2 x - 1) \sec^{n-2} x dx = \tan x \cdot \sec^{n-2} x - (n-2)(I_n - I_{n-2})\end{aligned}$$

$$\text{故 } I_n = \frac{1}{n-1} \tan x \cdot \sec^{n-2} x + \frac{n-2}{n-1} I_{n-2}$$

$$\text{其中 } I_1 = \ln|\sec x + \tan x| + C, \quad I_2 = \tan x + C$$

$$\text{类题 3 请推导出积分 } I_n = \int \tan^n x dx \quad (n \geq 2) \text{ 的递推公式}$$

$$\begin{aligned}\text{解: } I_n &= \int \tan^{n-2} x (\tan^2 x + 1) dx - I_{n-2} = \int \tan^{n-2} x \cdot \sec^2 x dx - I_{n-2} = \int \tan^{n-2} x d(\tan x) - I_{n-2} \\ &= \frac{1}{n-1} \tan^{n-1} x - I_{n-2}\end{aligned}$$

$$\text{其中 } I_0 = x + C, \quad I_1 = -\ln|\cos x| + C$$

$$\text{类题 4 请推导出积分 } I_n = \int \sin^n x dx \quad (n \geq 2) \text{ 的递推公式}$$

$$\begin{aligned}\text{解: } I_n - I_{n+2} &= \int \sin^n x (1 - \sin^2 x) dx = \int \sin^n x \cdot \cos^2 x dx = \frac{1}{n+1} \int \cos x d(\sin^{n+1} x) \\ &= \frac{1}{n+1} \sin^{n+1} x \cos x + \frac{1}{n+1} I_{n+2}\end{aligned}$$

$$\text{故 } I_{n+2} = \frac{n+1}{n+2} I_n - \frac{1}{n+2} \sin^{n+1} x \cos x \text{ 即 } I_n = \frac{n-1}{n} I_{n-2} - \frac{1}{n} \sin^{n-1} x \cos x$$

$$\text{其中 } I_1 = -\cos x + C, \quad I_0 = x + C$$

类题 5 根据类题 4 的结论, 可推导出定积分中大名鼎鼎的“点火公式”——

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \sin^2 x dx = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \dots \cdot \frac{2}{3}, & n \text{ 为奇数} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \dots \cdot \frac{1}{2} \cdot \frac{\pi}{2}, & n \text{ 为偶数} \end{cases}$$

例题 7 $\int \frac{1}{\sin x \cdot \cos^2 x} dx$

解: $I = \int \csc x d(\tan x) = \csc x \tan x + \int \tan x \cdot \csc x \cot x dx = \csc x \tan x + \ln|\csc x - \cot x| + C$

例题 8 $\int \frac{5 + 4 \cos x}{(2 + \cos x)^2 \cdot \sin x} dx$

解: $I = \int \frac{(5 + 4 \cos x) \sin x}{(2 + \cos x)^2 \sin^2 x} dx = - \int \frac{5 + 4 \cos x}{(2 + \cos x)^2 (1 - \cos^2 x)} d(\cos x)$
 $= - \int \frac{(2 + \cos x)^2 + (1 - \cos^2 x)}{(2 + \cos x)^2 (1 - \cos^2 x)} d(\cos x) = - \int \frac{1}{1 - \cos^2 x} d(\cos x) - \int \frac{1}{(2 + \cos x)^2} d(\cos x + 2)$
 $= \frac{1}{2} \ln \left| \frac{\cos x - 1}{\cos x + 1} \right| + \frac{1}{2 + \cos x} + C$

例题 9 $\int \frac{1}{1 + \cos^2 x} dx$

解: $I = \int \frac{1}{\sin^2 x + 2 \cos^2 x} dx = \int \frac{1}{\tan^2 x + 2} d(\tan x) = \frac{1}{\sqrt{2}} \arctan \frac{\tan x}{\sqrt{2}} + C$

例题 10 $\int \frac{1}{(3 \sin x + 2 \cos x)^2} dx$

解: $I = \int \frac{\sec^2 x}{(3 \tan x + 2)^2} dx = \frac{1}{3} \int \frac{1}{(3 \tan x + 2)^2} d(3 \tan x + 2) = - \frac{1}{3(3 \tan x + 2)} + C$

例题 11 $\int \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x} dx$

解: 当 $a \neq 0, b = 0$ 时 $I = \frac{1}{a^2} \int \frac{1}{\sin^2 x} dx = - \frac{1}{a^2} \cot x + C$

当 $a = 0, b \neq 0$ 时 $I = \frac{1}{b^2} \int \frac{1}{\cos^2 x} dx = \frac{1}{b^2} \tan x + C$

当 $a \neq 0, b \neq 0$ 时 $I = \int \frac{1}{a^2 \tan^2 x + b^2} d(\tan x) = \frac{1}{ab} \arctan \left(\frac{a}{b} \tan x \right) + C$

例题 12 $\int \frac{1}{\sin^4 x \cdot \cos^2 x} dx$

解: $I = \int \frac{\sin^2 x + \cos^2 x}{\sin^4 x \cos^2 x} dx = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx + \int \frac{1}{\sin^4 x} dx = \int \left(\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} \right) dx + \int \frac{1}{\sin^4 x} dx$
 $= -\cot x + \tan x - \int \csc^2 x d(\cot x) = -\cot x + \tan x - \int (\cot^2 x + 1) d(\cot x)$
 $= -2 \cot x + \tan x - \frac{1}{3} \cot^3 x + C$

例题 13 $\int \frac{1 + \sin x + \cos x}{1 + \sin^2 x} dx$

解: $I = \int \frac{1}{2 \sin^2 x + \cos^2 x} dx + \int \frac{\sin x}{1 + \sin^2 x} dx + \arctan(\sin x)$
 $= \frac{1}{\sqrt{2}} \arctan(\sqrt{2} \tan x) + \int \frac{\sec x \tan x}{\tan^2 x + \sec^2 x} dx + \arctan(\sin x)$

$$\begin{aligned}
 &= \frac{1}{\sqrt{2}} \arctan(\sqrt{2} \tan x) + \int \frac{1}{2 \sec^2 x - 1} d(\sec x) + \arctan(\sin x) \\
 &= \frac{1}{\sqrt{2}} \arctan(\sqrt{2} \tan x) + \frac{1}{2\sqrt{2}} \ln \left| \frac{\sqrt{2} \sec x - 1}{\sqrt{2} \sec x + 1} \right| + \arctan(\sin x) + C
 \end{aligned}$$

例题 14 $\int \frac{\cos x}{\sin x + \cos x} dx$

$$\text{解: } I = \int \frac{\frac{1}{2}(\sin x + \cos x) + \frac{1}{2}(\cos x - \sin x)}{\sin x + \cos x} dx = \frac{x}{2} + \frac{1}{2} \ln |\sin x + \cos x| + C$$

例题 15 $\int \frac{1}{\sin x \cdot \sin 2x} dx$

$$\begin{aligned}
 \text{解: } I &= \frac{1}{2} \int \frac{1}{\sin^2 x \cos x} dx = -\frac{1}{2} \int \sec x d(\cot x) = -\frac{1}{2} \sec x \cot x + \frac{1}{2} \int \cot x \cdot \sec x \tan x dx \\
 &= -\frac{1}{2} \sec x \cot x + \frac{1}{2} \ln |\sec x + \tan x| + C
 \end{aligned}$$

例题 16 $\int \frac{\cos 2x - \sin 2x}{\sin x + \cos x} dx$

$$\begin{aligned}
 \text{解: } I &= \int \frac{\cos^2 x - \sin^2 x - 2 \sin x \cos x}{\sin x + \cos x} dx = \int (\cos x - \sin x + \frac{1 - (\sin x + \cos x)^2}{\sin x + \cos x}) dx \\
 &= 2 \cos x + \int \frac{1}{\sin x + \cos x} dx = 2 \cos x + \frac{1}{\sqrt{2}} \int \frac{1}{\sin(x + \frac{\pi}{4})} dx \\
 &= 2 \cos x + \frac{1}{\sqrt{2}} \ln |\csc(x + \frac{\pi}{4}) - \cot(x + \frac{\pi}{4})| + C
 \end{aligned}$$

例题 17 $\int \sin 2x \cdot \sin 3x dx$

$$\text{解: } I = \frac{1}{2} \int (\cos x - \cos 5x) dx = \frac{1}{2} \sin x - \frac{1}{10} \sin 5x + C$$

例题 18 $\int \frac{1}{1 + \cos x} dx$

$$\text{解: } I = \int \frac{1}{2 \cos^2 \frac{x}{2}} dx = \tan \frac{x}{2} + C$$

类题 1 $\int \frac{1}{\sin x \cdot \sin 2x} dx$

$$\text{解: } I = \int \frac{\sin^2 x + \cos^2 x}{2 \sin^2 x \cos x} dx = \frac{1}{2} \int \frac{1}{\cos x} dx + \frac{1}{2} \int \frac{d \sin x}{\sin^2 x} = \frac{1}{2} \ln |\sec x + \tan x| - \frac{1}{2 \sin x} + C$$

类题 2 $\int \frac{1}{1 + \sin x + \cos x} dx$

解: 使用万能公式

$$I = \int \frac{1}{1 + \frac{2u}{1+u^2} + \frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2} du = \int \frac{1}{u+1} du = \ln |u+1| + C = \ln |\tan \frac{x}{2} + 1| + C$$

例题 19 $\int \frac{\sin x \cdot \cos x}{\sin x + \cos x} dx$

解: $I = \frac{1}{2} \int \frac{(\sin x + \cos x)^2 - 1}{\sin x + \cos x} dx = \frac{1}{2} \int (\sin x + \cos x) dx - \frac{1}{2} \int \frac{1}{\sin x + \cos x} dx$
 $= \frac{1}{2} \int (\sin x + \cos x) dx - \frac{1}{2\sqrt{2}} \int \frac{1}{\sin(x + \frac{\pi}{4})} dx$
 $= \frac{1}{2}(\sin x - \cos x) - \frac{1}{2\sqrt{2}} \ln|\csc(x + \frac{\pi}{4}) - \cot(x + \frac{\pi}{4})| + C$

注: $(\sin x + \cos x)^2 = 1 + 2 \sin x \cos x = 1 + \sin 2x$

$(\sin x - \cos x)^2 = 1 - 2 \sin x \cos x = 1 - \sin 2x$

例题 20 $\int \frac{\sin x \cdot \cos x}{\sin^4 x + \cos^4 x} dx$

解: $I = \int \frac{\tan x \sec^2 x}{\tan^4 x + 1} dx = \frac{1}{2} \int \frac{1}{\tan^4 x + 1} d(\tan^2 x) = \frac{1}{2} \arctan(\tan^2 x) + C$

例题 21 $\int \frac{1}{\sin^6 x + \cos^6 x} dx$

解: $I = \int \frac{\sec^6 x}{\tan^6 x + 1} dx = \int \frac{\sec^4 x}{\tan^6 x + 1} d(\tan x) = \int \frac{(\tan^2 x + 1)^2}{\tan^6 x + 1} d(\tan x)$
 $= \int \frac{\tan^4 x + 1 + 2 \tan^2 x}{\tan^6 x + 1} d(\tan x) = \int \frac{\tan^4 x + 1 - \tan^2 x + 3 \tan^2 x}{\tan^6 x + 1} d(\tan x)$
 $= \int \frac{1}{\tan^2 x + 1} d(\tan x) + \int \frac{1}{(\tan^3 x)^2 + 1} d(\tan^3 x) = x + \arctan(\tan^3 x) + C$

注: 熟记公式 $\sec^2 x = \tan^2 x + 1$ 和立方和立方差公式

例题 22 $\int \frac{1}{\sin^3 x + \cos^3 x} dx$

解: $I = \int \frac{1}{(\sin x + \cos x)(\sin^2 x + \cos^2 x - \sin x \cos x)} dx = \int \frac{1}{(\sin x + \cos x)(1 - \sin x \cos x)} dx$
 $= \frac{1}{3} \int \frac{(\sin x + \cos x)^2 + 2(1 - \sin x \cos x)}{(\sin x + \cos x)(1 - \sin x \cos x)} dx = \frac{1}{3} \int \frac{\sin x + \cos x}{1 - \sin x \cos x} dx + \frac{2}{3} \int \frac{1}{\sin x + \cos x} dx$
 $= \frac{2}{3} \int \frac{1}{1 + (\sin x - \cos x)^2} d(\sin x - \cos x) + \frac{\sqrt{2}}{3} \int \frac{1}{\sin(x + \frac{\pi}{4})} dx$
 $= \frac{2}{3} \arctan(\sin x - \cos x) + \frac{\sqrt{2}}{3} \ln|\csc(x + \frac{\pi}{4}) - \cot(x + \frac{\pi}{4})| + C$

例题 23 $\int \frac{1}{\sin(x+a)\sin(x+b)} dx$ (其中 $\sin(a-b) \neq 0$)

解: $I = \frac{\int \frac{\sin[x+a] - (x+b)]}{\sin(x+a)\sin(x+b)} dx}{\sin(a-b)} = \frac{\int \frac{\sin(x+a)\cos(x+b) - \cos(x+a)\sin(x+b)}{\sin(x+a)\sin(x+b)} dx}{\sin(a-b)}$
 $= \frac{1}{\sin(a-b)} \ln \left| \frac{\sin(x+b)}{\sin(x+a)} \right| + C$

NO.4-2 不定积分解题方法（下）

套路三 换元法的基本套路

（1）整体换元

例题 1 $\int \sqrt{\frac{x}{x+1}} dx$

解：令 $\sqrt{\frac{x}{x+1}} = t$

$$\begin{aligned} \text{原式} &= \int t d\left(\frac{1}{1-t^2}\right) = \frac{t}{1-t^2} + \int \frac{1}{t^2-1} dt = \frac{t}{1-t^2} + \frac{1}{2} \ln \left| \frac{t-1}{t+1} \right| + C \\ &= (x+1) \sqrt{\frac{x}{1+x}} + \frac{1}{2} \ln \left| \frac{\sqrt{x}-\sqrt{x+1}}{\sqrt{x}+\sqrt{x+1}} \right| + C \end{aligned}$$

类题 1 $\int \frac{1}{x} \sqrt{\frac{x+1}{x}} dx$

解：令 $\sqrt{\frac{x+1}{x}} = t$

$$\begin{aligned} \text{原式} &= \int (t^2-1) t d\left(\frac{1}{t^2-1}\right) = t - \int \frac{1}{t^2-1} (3t^2-3) + 2 dt \\ &= t - \left[3t + 2 \int \frac{1}{t^2-1} dt \right] \\ &= -2t - \ln \left| \frac{t-1}{t+1} \right| + C \\ &= -2\sqrt{\frac{x+1}{x}} - \ln \left| \frac{\sqrt{x+1}-\sqrt{x}}{\sqrt{x}+\sqrt{x+1}} \right| + C \end{aligned}$$

类题 2 请计算 $\int \sqrt{\frac{1-x}{1+x}} dx$ 和 $\int \left(\sqrt{\frac{1-x}{1+x}} + \sqrt{\frac{1+x}{1-x}} \right) dx$ 两个积分

解：(1) 原式 $= \int \frac{1-x}{\sqrt{1-x^2}} dx = \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx = \arcsin x + \sqrt{1-x^2} + C$

(2) 原式 $= \int \left(\frac{1-x}{\sqrt{1-x^2}} + \frac{1+x}{\sqrt{1-x^2}} \right) dx = \int \frac{2}{\sqrt{1-x^2}} dx = 2 \arcsin x + C$

类题 3 $\int \frac{x e^x}{\sqrt{e^x-2}} dx$

解：令 $\sqrt{e^x-2} = t$

$$\begin{aligned} \text{原式} &= \int \frac{\ln(2+t^2)}{t} (2+t^2) \frac{2t}{(2+t^2)} dt = 2 \int \ln(2+t^2) dt = 2t \ln(2+t^2) - 2 \int t \cdot \frac{2t}{(2+t^2)} dt \\ &= 2t \ln(2+t^2) - 4t + 2\sqrt{2} \arctan \frac{t}{\sqrt{2}} + C = 2x \sqrt{e^x-2} - 4\sqrt{e^x-2} + 2\sqrt{2} \arctan \frac{\sqrt{e^x-2}}{\sqrt{2}} + C \end{aligned}$$

类题 4 $\int \frac{1}{\sqrt[3]{(x+1)^2(x-1)^4}} dx$

解：令 $\sqrt[3]{\frac{x+1}{x-1}} = t$

$$\text{原式} = - \int \frac{t(t^3-1)^2 6t^2}{(t^3-1)^2 4t^2} dt = - \int \frac{3}{2} dt = - \frac{3}{2} t + C = - \frac{3}{2} \sqrt[3]{\frac{x+1}{x-1}} + C$$

例题 2 $\int \frac{1}{(1+\sqrt[3]{x})\sqrt{x}} dx$

解：令 $x = t^6$

$$\text{原式} = \int \frac{6t^5}{(1+t^2)t^3} dt = 6 \int \frac{t^2+1-1}{1+t^2} dt = 6t - 6 \arctan t + C = 6x^{\frac{1}{6}} - 6 \arctan x^{\frac{1}{6}} + C$$

类题 $\int \frac{1}{1+e^{\frac{x}{2}}+e^{\frac{x}{3}}+e^{\frac{x}{6}}} dx$

解：令 $t = e^{\frac{x}{6}}$

$$\begin{aligned} \text{原式} &= \int \frac{1}{1+t^3+t^2+t} \frac{6}{t} dt = 6 \int \frac{1}{t(1+t)(1+t^2)} dt = \int \left(\frac{6}{t} - \frac{3}{1+t} - \frac{3t+3}{1+t^2} \right) dt \\ &= 6 \ln t - 3 \ln(1+t) - \frac{3}{2} \ln(1+t^2) - 3 \arctan t + C = x - 3 \ln \left(1 + e^{\frac{x}{6}} \right) - \frac{3}{2} \ln \left(1 + e^{\frac{x}{3}} \right) - 3 \arctan e^{\frac{x}{6}} + C \end{aligned}$$

(2) 三角换元

例题 3 $\int \frac{1}{x^4} \sqrt{4-x^2} dx$

解：令 $x = 2 \sin t$

$$\begin{aligned} \text{原式} &= \int \frac{2 \cos t}{16 \sin t^4} \cdot 2 \cos t dt = \frac{1}{4} \int \frac{\cos t^2}{\sin t^4} dt = \frac{1}{4} \int \frac{\sec t^2}{\tan t^4} dt = \frac{1}{4} \int \frac{1}{\tan t^4} dt \tan t = - \frac{1}{12} \tan t^{-3} + C \\ &= - \frac{1}{12} \tan \arcsin \frac{x}{2}^{-3} + C \end{aligned}$$

例题 4 $\int \frac{1}{\sqrt{(x^2+1)^3}} dx$

解：令 $x = \tan t$

$$\text{原式} = \int \frac{1}{\sec t^3} \sec t^2 dt = \int \cos t dt = \sin t + C = \frac{x}{\sqrt{1+x^2}} + C$$

例题 5 $\int \frac{1}{\sqrt{x(4-x)}} dx$

解：令 $x-2 = 2 \sin t$

$$\text{原式} = \int \frac{1}{\sqrt{4-(x-2)^2}} dx = \int 1 dt = t + C = \arcsin \frac{x-2}{2} + C$$

例题 6 $\int x \sqrt{2x-x^2} dx$

解：令 $x - 1 = \sin t$

$$\begin{aligned} \text{原式} &= \int (\sin t + 1) \cos t^2 dt = - \int \cos t^2 d\cos t + \int \cos t^2 dt = -\frac{1}{3} \cos t^3 + \frac{1}{2}t + \frac{1}{2} \sin t \cos t + C \\ &= -\frac{1}{3} \sqrt{1-(x-1)^2}^3 + \frac{1}{2} \arcsin(x-1) + \frac{1}{2} \sqrt{1-(x-1)^2} (x-1) + C \end{aligned}$$

例题 7 $\int \frac{1}{x^2 \sqrt{x^2 - 1}} dx$

解：令 $x = \sec t$

$$\text{原式} = \int \frac{\sec t \tan t}{\sec t^2 \tan t} dt = \int \frac{1}{\sec t} dt = \int \cos t dt = \sin t + C = \sqrt{1 - \frac{1}{x^2}} + C$$

类题 1 $\int \frac{1}{x \sqrt{2x^2 + 2x + 1}} dx$

解：令 $x = \frac{1}{t}$

$$\begin{aligned} \text{原式} &= - \int \frac{1}{\sqrt{t^2 + 2t + 2}} dt = - \int \frac{1}{\sqrt{1 + (t+1)^2}} d(t+1) \\ &= - \ln |t+1 + \sqrt{1+(t+1)^2}| + C = - \ln \left| \frac{1}{x} + 1 + \sqrt{1 + \left(\frac{1}{x} + 1\right)^2} \right| + C \end{aligned}$$

类题 2 $\int \frac{1}{x^2 \sqrt{2x^2 + 2x + 1}} dx$

解：令 $x = \frac{1}{t}$

$$\begin{aligned} \text{原式} &= - \int \frac{t+1-1}{\sqrt{t^2 + 2t + 2}} dt = \int \frac{1}{\sqrt{1 + (t+1)^2}} d(t+1) - \int \frac{t+1}{\sqrt{t^2 + 2t + 2}} dt \\ &= - \ln |t+1 + \sqrt{1+(t+1)^2}| - \sqrt{t^2 + 2t + 2} + C \\ &= - \ln \left| \frac{1}{x} + 1 + \sqrt{1 + \left(\frac{1}{x} + 1\right)^2} \right| - \frac{\sqrt{2x^2 + 2x + 1}}{x} + C \end{aligned}$$

套路四 分部积分法的基本套路

例题 1 $\int x \cdot \arctan x dx$

解：原式 $= \frac{1}{2} \int \arctan x d(x^2 + 1) = \frac{1}{2} \arctan x \cdot (x^2 + 1) - \frac{x}{2} + C$

类题 $\int x \cdot \ln(1+x^2) \cdot \arctan x dx$

解：原式 $= \frac{1}{2} \int \arctan x \ln(x^2 + 1) d(x^2 + 1)$

$$= \frac{1}{2} \arctan x \cdot (x^2 + 1) \cdot \ln(x^2 + 1) - \frac{1}{2} \int (x^2 + 1) \left[\frac{2x}{x^2 + 1} \arctan x + \ln(x^2 + 1) \frac{1}{x^2 + 1} \right] dx$$

$$\begin{aligned}
&= \frac{1}{2} \arctan x \cdot (x^2 + 1) \cdot \ln(x^2 + 1) - \frac{1}{2} \int \arctan x \, d(x^2 + 1) - \frac{1}{2} \int \ln(x^2 + 1) \, dx \\
&= \frac{1}{2} \arctan x \cdot (x^2 + 1) \cdot \ln(x^2 + 1) - \frac{1}{2} \arctan x (x^2 + 1) + \frac{1}{2} x - \frac{1}{2} x \ln(x^2 + 1) + \frac{1}{2} \int \frac{2x^2}{1+x^2} \, dx \\
&= \frac{1}{2} \arctan x \cdot (x^2 + 1) \cdot \ln(x^2 + 1) - \frac{1}{2} \arctan x (x^2 + 1) + \frac{1}{2} x - \frac{1}{2} x \ln(x^2 + 1) + x - \arctan x + C
\end{aligned}$$

例题 2 $\int e^x \sin x \, dx$

$$\begin{aligned}
\text{解: } \text{原式} &= \int \sin x \, de^x = \sin x e^x - \int e^x \cos x \, dx = \sin x e^x - \int \cos x \, de^x \\
&= \sin x e^x - e^x \cos x - \int e^x \sin x \, dx = \frac{1}{2} e^x \sin x - \frac{1}{2} e^x \cos x + C
\end{aligned}$$

类题 $\int x \cdot e^x \cdot \sin x \, dx$

$$\begin{aligned}
\text{解: } \text{原式} &= \frac{1}{2} \int x \, de^x (\sin x - \cos x) = \frac{x}{2} e^x (\sin x - \cos x) - \frac{1}{2} \int e^x (\sin x - \cos x) \, dx \\
&= \frac{x}{2} e^x (\sin x - \cos x) + \frac{1}{2} e^x \cos x + C
\end{aligned}$$

例题 3 $\int \ln^2(x + \sqrt{1+x^2}) \, dx$

$$\begin{aligned}
\text{解: } \text{原式} &= x \ln(x + \sqrt{1+x^2})^2 - \int \frac{2x \ln((x + \sqrt{1+x^2}))}{\sqrt{1+x^2}} \, dx \\
&= x \ln(x + \sqrt{1+x^2})^2 - \int 2 \ln(x + \sqrt{1+x^2}) \, d\sqrt{1+x^2} \\
&= x \ln(x + \sqrt{1+x^2})^2 - 2\sqrt{1+x^2} \ln(x + \sqrt{1+x^2}) + 2 \int \frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} \, dx \\
&= x \ln(x + \sqrt{1+x^2})^2 - 2\sqrt{1+x^2} \ln(x + \sqrt{1+x^2}) + 2x + C
\end{aligned}$$

套路五 换元法+分部积分

例题 1 $\int e^{2x} \arctan \sqrt{e^x - 1} \, dx$

解: 令 $\sqrt{e^x - 1} = t$

$$\begin{aligned}
\text{原式} &= \int \arctan t \cdot \frac{2t}{1+t^2} (1+t^2)^2 \, dt = \frac{1}{2} \int \arctan t \, d(1+t^2)^2 = \frac{(1+t^2)^2}{2} \arctan t - \frac{1}{2} \int 1+t^2 \, dt \\
&= \frac{(1+t^2)^2}{2} \arctan t - \frac{t^3}{6} - \frac{t}{2} + C = \frac{e^{2x}}{2} \arctan \sqrt{e^x - 1} - \frac{(e^x - 1)^{\frac{3}{2}}}{6} - \frac{\sqrt{e^x - 1}}{2} + C
\end{aligned}$$

类题 1 请计算 $\int \frac{x \cdot e^{\arctan x}}{(1+x^2)^{\frac{3}{2}}} \, dx$ 和 $\int \frac{e^{\arctan x}}{(1+x^2)^{\frac{3}{2}}} \, dx$

解: (1) 令 $x = \tan t$

$$\text{原式} = \int \frac{\tan t \cdot e^t}{\sec t^3} \sec t^2 dt = \int \sin t \cdot e^t dt = \frac{1}{2} e^t \sin t - \frac{1}{2} e^t \cos t + C = \frac{1}{2} e^{\arctan x} \frac{x-1}{\sqrt{1+x^2}} + C$$

(2) 令 $x = \tan t$

$$\text{原式} = \int \frac{e^t}{\sec t^3} \sec t^2 dt = \int \cos t \cdot e^t dt = \frac{1}{2} e^t \sin t + \frac{1}{2} e^t \cos t + C = \frac{1}{2} e^{\arctan x} \frac{x+1}{\sqrt{1+x^2}} + C$$

$$\text{类题 2 } \int \frac{\ln x}{(1+x^2)^{\frac{3}{2}}} dx$$

解：令 $x = \tan t$

$$\begin{aligned} \text{原式} &= \int \frac{\ln \tan t}{\sec t} dt = \int \cos t \ln \tan t dt = \int \ln \tan t d \sin t = \sin t \ln \tan t - \int \frac{1}{\cos t} dt \\ &= \sin t \ln \tan t - \ln |\sec t + \tan t| + C = \frac{x}{\sqrt{x^2+1}} \ln x - \ln |\sqrt{x^2+1} + x| + C \end{aligned}$$

$$\text{类题 3 请计算} \int \frac{\arctan \sqrt{x-1}}{x \sqrt{x-1}} dx \text{ 和} \int \frac{\sqrt{x-1} \arctan \sqrt{x-1}}{x} dx$$

解：(1) 令 $\sqrt{x-1} = t$

$$\begin{aligned} \text{原式} &= 2 \int \frac{t^2 \arctan t}{1+t^2} dt = 2 \int \arctan t dt - 2 \int \frac{\arctan t}{1+t^2} dt = 2 \ln(1+t^2) - (\arctan t)^2 + C \\ &= 2\sqrt{x-1} \arctan \sqrt{x-1} - \ln x - (\arctan \sqrt{x-1})^2 + C \end{aligned}$$

(2) 令 $\sqrt{x-1} = t$

$$\begin{aligned} \text{原式} &= \int \frac{t^2 \arctan t}{1+t^2} dt = 2 \int \arctan t dt - 2 \int \frac{\arctan t}{1+t^2} dt = 2t \arctan t - \ln(1+t^2) + (\arctan t)^2 + C \\ &= 2t \arctan \sqrt{x-1} - \ln x + (\arctan \sqrt{x-1})^2 + C \end{aligned}$$

$$\text{例题 2 } \int \ln \left(1 + \sqrt{\frac{1+x}{x}} \right) dx$$

解：令 $\sqrt{\frac{1+x}{x}} = t$

$$\begin{aligned} \text{原式} &= \int \ln(1+t) d \frac{1}{t^2-1} = \frac{\ln(1+t)}{t^2-1} - \int \frac{1}{t+1} \frac{1}{t^2-1} dt \\ &= \frac{\ln(1+t)}{t^2-1} + \frac{1}{4} \ln \frac{t+1}{t-1} - \frac{1}{2t+2} + C \\ &= x \ln \left(1 + \sqrt{\frac{1+x}{x}} \right) + \frac{1}{2} \ln (\sqrt{1+x} + \sqrt{x}) + \frac{1}{2} x - \frac{1}{2} \sqrt{x+x^2} + C \end{aligned}$$

$$\text{类题 1 } \int \arctan(1+\sqrt{x}) dx$$

解：令 $\sqrt{x} = t$

$$\begin{aligned} \text{原式} &= \int \arctan(1+t) dt^2 = t^2 \arctan(1+t) - \int \frac{t^2+2t+2-2t-2}{t^2+2t+2} dt \\ &= t^2 \arctan(1+t) - t + \ln(t^2+2t+2) + C \end{aligned}$$

类题 2 $\int \sqrt{1+x^2} dx$

解：原式 $= x\sqrt{1+x^2} - \int \frac{x^2}{\sqrt{1+x^2}} dx = x\sqrt{1+x^2} - \int \sqrt{1+x^2} dx + \int \frac{1}{\sqrt{1+x^2}} dx$
 $= x\sqrt{1+x^2} - \text{原式} + \ln|x+\sqrt{1+x^2}| + C = \frac{1}{2}(x\sqrt{1+x^2} + \ln|x+\sqrt{1+x^2}|) + C$

套路六 利用分部积分，对分母进行降阶

例题 1 $\int \frac{x e^x}{(1+x)^2} dx$

解：原式 $= - \int x e^x d\left(\frac{1}{1+x}\right) = - \frac{x e^x}{1+x} + \int \frac{1+x}{1+x} e^x dx = \frac{e^x}{1+x} + C$

类题 1 $\int \frac{x^2 e^x}{(x+2)^2} dx$

解：原式 $= - \int x^2 e^x d\left(\frac{1}{2+x}\right) = - \frac{x^2 e^x}{x+2} + \int \frac{e^x(2x+x^2)}{x+2} dx = - \frac{x^2 e^x}{x+2} + (x-1)e^x + C$

类题 2 $\int \frac{x e^x}{(1+e^x)^2} dx$

解：原式 $= \int \frac{x}{(1+e^x)^2} d(e^x+1) = - \int x d\left(\frac{1}{e^x+1}\right) = - \frac{x}{e^x+1} + \int \frac{1}{e^x+1} dx = - \frac{x}{e^x+1} + \ln \frac{e^x}{e^x+1} + C$

例题 2 $\int \frac{x^2}{(x \sin x + \cos x)^2} dx$

解：原式 $= \int \frac{x}{\cos x} \frac{x \cos x}{(x \sin x + \cos x)^2} dx = \int \frac{x}{\cos x} \frac{1}{(x \sin x + \cos x)^2} d(x \sin x + \cos x)$
 $= - \int \frac{x}{\cos x} d\left(\frac{1}{x \sin x + \cos x}\right) = - \frac{x}{(x \sin x + \cos x) \cos x} + \tan x + C$

例题 3 $\int_0^{+\infty} \frac{e^{-x^2}}{\left(x^2 + \frac{1}{2}\right)^2} dx$

解：令 $x = \frac{1}{t}$

原式 $= 4 \int_0^{+\infty} \frac{e^{-\frac{1}{t^2}} t^4}{(t^2 + 2)^2} \frac{1}{t^2} dt = 2 \int_0^{+\infty} t e^{-\frac{1}{t^2}} d\frac{t^2 + 2}{(t^2 + 2)^2} = 2 \int_0^{+\infty} \frac{e^{-\frac{1}{t^2}}}{t^2} dt = \sqrt{\pi}$

套路七 利用分部积分，实现“积分抵消”

例题 1 $\int \frac{x e^x}{(1+x)^2} dx$

解：原式 $= - \int xe^x d\frac{1}{1+x} = - \frac{xe^x}{1+x} + \int \frac{1+x}{1+x} e^x dx = \frac{e^x}{1+x} + C$

类题 1 $\int \frac{(1+\sin x)e^x}{1+\cos x} dx$

解：原式 $= \int (\frac{1}{1+\cos x} + \frac{\sin x}{1+\cos x}) e^x dx = \frac{\sin x}{1+\cos x} e^x + C$

类题 2 $\int e^x \left(\frac{1-x}{1+x^2} \right)^2 dx$

解：原式 $= \int e^x \frac{x^2 - 2x + 1}{(x^2 + 1)^2} dx = \int e^x \left[\frac{1}{x^2 + 1} - \frac{2x}{(x^2 + 1)^2} \right] dx = \frac{e^x}{x^2 + 1} + C$

类题 3 $\int \frac{x^2 e^x}{(x+2)^2} dx$

解：原式 $= - \int x^2 e^x d\frac{1}{2+x} = - \frac{x^2 e^x}{x+2} + \int \frac{e^x (2x + x^2)}{x+2} dx = - \frac{x^2 e^x}{x+2} + (x-1)e^x + C$

例题 2 $\int \frac{e^{-\sin x} \cdot \sin 2x}{\sin^4(\frac{\pi}{4} - \frac{x}{2})} dx$

解：令 $\sin x = t$

原式 $= 8 \int \frac{\sin x \cos x e^{-\sin x}}{(1-\sin x)^2} dx = 8 \int \frac{te^{-t}}{(t-1)^2} dt = -8 \int te^{-t} d\frac{1}{t-1} = -8 \frac{te^{-t}}{t-1} + 8e^{-t} + C$

$$= -8 \frac{\sin x e^{-\sin x}}{\sin x - 1} + 8e^{-\sin x} + C$$

例题 3 $\int e^{-\frac{x}{2}} \frac{\cos x - \sin x}{\sqrt{\sin x}} dx$

解：原式 $= \int e^{-\frac{x}{2}} \frac{\cos x}{\sqrt{\sin x}} dx - \int e^{-\frac{x}{2}} \sqrt{\sin x} dx = 2 \int e^{-\frac{x}{2}} d\sqrt{\sin x} - \int e^{-\frac{x}{2}} \sqrt{\sin x} dx = 2e^{-\frac{x}{2}} \sqrt{\sin x} + C$

类题 1 $\int e^{\sin x} \frac{x \cos^3 x - \sin x}{\cos^2 x} dx$

解：原式 $= \int x de^{\sin x} - \int e^{\sin x} d\frac{1}{\cos x} = xe^{\sin x} - \int e^{\sin x} dx - \frac{e^{\sin x}}{\cos x} + \int \frac{\cos x}{\cos x} e^{\sin x} dx = xe^{\sin x} - \frac{e^{\sin x}}{\cos x} + C$

类题 2 $\int \left(1+x-\frac{1}{x}\right) e^{x+\frac{1}{x}} dx$

解：原式 $= \int e^{x+\frac{1}{x}} dx + \int \left(x-\frac{1}{x}\right) e^{x+\frac{1}{x}} dx = xe^{x+\frac{1}{x}} - \int xe^{x+\frac{1}{x}} \left(1-\frac{1}{x^2}\right) dx + \int \left(x-\frac{1}{x}\right) e^{x+\frac{1}{x}} dx$
 $= xe^{x+\frac{1}{x}} + C$

例题 4 $\int \left(\ln \ln x + \frac{1}{\ln x}\right) dx$

解：原式 $= x \ln \ln x - \int \frac{1}{\ln x} dx + \int \frac{1}{\ln x} dx = x \ln \ln x + C$

例题 5 已知 $f''(x)$ 连续, $f'(x) \neq 0$, 求 $\int \left[\frac{f(x)}{f'(x)} - \frac{f^2(x)f''(x)}{(f'(x))^3} \right] dx$

$$\text{解: 原式} = \int \frac{-f''f^2 + ff'^2}{f'^3} dx = \int \frac{f}{f'} d \frac{f}{f'} = \frac{1}{2} \left(\frac{f}{f'} \right)^2 + C$$

例题 6 $\int \frac{1 - \ln x}{(x - \ln x)^2} dx$

$$\text{解: 原式} = \int \frac{x - \ln x + 1 - x}{(x - \ln x)^2} dx = \int \frac{1}{x - \ln x} dx + \int \frac{1 - x}{(x - \ln x)^2} dx = \frac{x}{x - \ln x} + C$$

套路八 对复杂因子求导, 期待出现奇迹

例题 1 $\int \frac{\ln x}{\sqrt{1 + [x(\ln x - 1)]^2}} dx$

$$\text{解: 原式} = \int \frac{1}{\sqrt{1 + [x(\ln x - 1)]^2}} dx (\ln x - 1) = \ln \left(x(\ln x - 1) + \sqrt{1 + [x(\ln x - 1)]^2} \right) + C$$

例题 2 $\int \frac{x + 1}{x(1 + xe^x)} dx$

$$\text{解: 原式} = \int \frac{1}{e^x x(1 + xe^x)} dx e^x = \ln \left| \frac{xe^x}{1 + xe^x} \right| + C$$

类题 $\int \frac{1 + x \cos x}{x(1 + xe^{\sin x})} dx$

$$\text{解: 原式} = \int \frac{e^{\sin x}(x \cos x + 1)}{xe^{\sin x}(1 + xe^{\sin x})} dx = \ln \left| \frac{xe^{\sin x}}{1 + xe^{\sin x}} \right| + C$$

例题 3 $\int \frac{1 - \ln x}{(x - \ln x)^2} dx$

$$\text{解: 原式} = \int \frac{x - \ln x + 1 - x}{(x - \ln x)^2} dx = \int \frac{1}{x - \ln x} dx + \int \frac{1 - x}{(x - \ln x)^2} dx = \frac{x}{x - \ln x} + C$$

类题 1 $\int \frac{e^x(x - 1)}{(x - e^x)^2} dx$

$$\text{解: 原式} = \int \frac{e^x(x - 1)}{x^2 \left(1 - \frac{e^x}{x}\right)^2} dx = \frac{1}{1 - \frac{e^x}{x}} + C = \frac{x}{x - e^x} + C$$

类题 2 $\int \frac{x + \sin x \cdot \cos x}{(\cos x - x \cdot \sin x)^2} dx$

$$\text{解: 原式} = \int \frac{x + \sin x \cos x}{(1 - x \tan x)^2 \cos x^2} dx = \int \frac{1}{(x \tan x - 1)^2} dx \tan x = \frac{1}{1 - x \tan x} + C$$