NO.5-2 定积分计算（上）

套路一 定积分的常规计算技巧

例题 1 利用 N-L 公式，直接计算下列定积分

(1) $\int_0^1 \frac{x^2\arcsin x}{\sqrt{1-x^2}}\mathrm{d}x$

(2) $\int_0^3 \arcsin\sqrt{\frac{x}{1+x}}\,\mathrm{d}x$

(3) $\int_1^{16} \arctan\sqrt{\sqrt{x}-1}\,\mathrm{d}x$

(4) $\int_0^{\frac{\pi}{2}} \sin x\cdot\ln\sin x\,\mathrm{d}x$

解： (1) $\int_0^1 \frac{x^2\arcsin x}{\sqrt{1-x^2}}\mathrm{d}x = \int_0^1 x^2\arcsin x\,\mathrm{d}\arcsin x$

令 $\arcsin x = t$

原式 $=\int_0^{\frac{\pi}{2}} t\sin^2 t\,\mathrm{d}t = \int_0^{\frac{\pi}{2}} t\sin^2 t\,\mathrm{d}t = \frac{1}{2}\int_0^{\frac{\pi}{2}} t(1-\cos 2t)\,\mathrm{d}t = \frac{t^2}{4}\Big|_0^{\frac{\pi}{2}} - \frac{1}{2}\int_0^{\frac{\pi}{2}} t\cos 2t\,\mathrm{d}t$

$= \frac{\pi^2}{16} - \frac{1}{2}\left(\frac{t\sin 2t}{2}\Big|_0^{\frac{\pi}{2}} - \frac{1}{2}\int_0^{\frac{\pi}{2}}\sin 2t dt\right) = \frac{\pi^2}{16} - \frac{1}{8}\cos 2t\Big|_0^{\frac{\pi}{2}} = \frac{\pi^2}{16} + \frac{1}{4}$；

(2) 令 $t = \arcsin\sqrt{\frac{x}{1+x}}$， $x = \tan^2 t$

$\int_0^{\frac{\pi}{3}} t\,\mathrm{d}\tan^2 t = t\tan t^2\Big|_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}}\tan^2 t\,\mathrm{d}t = \pi - (\tan t - t)\Big|_0^{\frac{\pi}{3}} = \pi - \left(\sqrt{3} - \frac{\pi}{3}\right)$

$= \frac{4\pi}{3} - \sqrt{3}$；

(3) 令 $t = \arctan\sqrt{\sqrt{x}-1}$

则 $\tan t = \sqrt{\sqrt{x}-1} \Longrightarrow \tan^2 t + 1 = \sqrt{x} \Longrightarrow x = \sec^4 t$

$\int_0^{\frac{\pi}{3}} t\,\mathrm{d}\sec^4 t = t\sec^4 t\Big|_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}}\sec^4 t\,\mathrm{d}t = \frac{16\pi}{3} - \int_0^{\frac{\pi}{3}}\sec^2 t\,\mathrm{d}\tan t$

$= \frac{16\pi}{3} - \int_0^{\frac{\pi}{3}}\tan^2 t + 1 d\tan t = \frac{16\pi}{3} - \left(\frac{\tan^3 t}{3} + \tan t\right)\Big|_0^{\frac{\pi}{3}} = \frac{16\pi}{3} - 2\sqrt{3}$；

(4) $\int \sin x\cdot\ln\sin x\,\mathrm{d}x = -\cos x\cdot\ln\sin x + \int\frac{\cos x^2}{\sin x}\mathrm{d}x = -\cos x\cdot\ln\sin x + \int\frac{1-\sin x^2}{\sin x}\mathrm{d}x$

$= -\cos x\cdot\ln\sin x + \cos x - \ln(\csc x - \cot x)$

$$\int_0^{\frac{\pi}{2}}\sin x\cdot\ln\sin x\,\mathrm{d}x=0-\lim_{x\to 0}-\cos x\cdot\ln\sin x+\cos x+\ln(\csc x-\cot x)$$

$$=-1+\lim_{x\to 0}\cos x\cdot\ln\sin x-\ln(\csc x-\cot x)$$

$$=-1+\lim_{x\to 0}\left(1-\frac{x^2}{2}+o(x^2)\right)\cdot\ln\sin x-\ln(\csc x-\cot x)$$

$$=-1+\lim_{x\to 0}\ln\sin x-\ln(\csc x-\cot x)-\frac{1}{2}\lim_{x\to 0}x^2\cdot\ln\sin x$$

$$=-1+\lim_{x\to 0}\ln\frac{\sin^2 x}{1-\cos x}-\frac{1}{2}\lim_{x\to 0}\frac{\frac{\cos x}{\sin x}}{\frac{1}{x^2}}$$

$$=-1+\ln\left(\lim_{x\to 0}\frac{\sin^2 x}{1-\cos x}\right)-\frac{1}{2}\lim_{x\to 0}\frac{x^2\cos x}{\sin x}=\ln 2-1\text{；}$$

例题 2 计算定积分$\int_0^{2\pi}\frac{1}{1+\cos^2 x}\mathrm{d}x$

解： $\int_0^{2\pi}\frac{1}{1+\cos^2 x}\mathrm{d}x=4\int_0^{\frac{\pi}{2}}\frac{1}{1+\cos^2 x}\mathrm{d}x=4\int_0^{\frac{\pi}{2}}\frac{\sec^2 x}{\sec^2 x+1}\mathrm{d}x$

$$=4\int_0^{\frac{\pi}{2}}\frac{\sec^2 x}{\sec^2 x+1}\mathrm{d}x=4\int_0^{\frac{\pi}{2}}\frac{1}{\tan^2 x+2}\mathrm{d}\tan x=2\sqrt{2}\arctan\left(\frac{\tan x}{\sqrt{2}}\right)\Big|_0^{\frac{\pi}{2}}=\sqrt{2}\pi\text{；}$$

类题 1 计算定积分$\int_0^{\pi}\frac{1}{1+\sin^2 x}\mathrm{d}x$

解： $\int_0^{\pi}\frac{1}{1+\sin^2 x}\mathrm{d}x=2\int_0^{\frac{\pi}{2}}\frac{1}{1+\sin^2 x}\mathrm{d}x\xrightarrow{t=\frac{\pi}{2}-x}2\int_0^{\frac{\pi}{2}}\frac{1}{1+\cos^2 x}\mathrm{d}x=\frac{\sqrt{2}}{2}$

类题 2 以下计算是否正确？为什么？如果错了，请将其更正。

$$\int_{-1}^{1}\left(\arctan\frac{1}{x}\right)'\mathrm{d}x=\arctan\frac{1}{x}\Big|_{-1}^{1}=\arctan 1-\arctan(-1)=\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)=\frac{\pi}{2}$$

解： 错误，积分区间存在无定义的点。

正确做法：$\int_{-1}^{1}\left(\arctan\frac{1}{x}\right)'\mathrm{d}x=\int_{-1}^{0}\left(\arctan\frac{1}{x}\right)'\mathrm{d}x+\int_0^{1}\left(\arctan\frac{1}{x}\right)'\mathrm{d}x$

$$=\lim_{x\to 0^-}\arctan\frac{1}{x}+\frac{\pi}{4}+\frac{\pi}{4}-\lim_{x\to 0^+}\arctan\frac{1}{x}=-\frac{\pi}{2}\text{；}$$

类题 3 设$f(x)=\frac{(x+1)^2(x-1)}{x^3(x-2)}$，计算$I=\int_{-1}^{3}\frac{f'(x)}{1+f^2(x)}\mathrm{d}x$

解： $I=\int_{-1}^{3}\frac{f'(x)}{1+f^2(x)}\mathrm{d}x=\int_{-1}^{0}\frac{f'(x)}{1+f^2(x)}\mathrm{d}x+\int_0^{2}\frac{f'(x)}{1+f^2(x)}\mathrm{d}x+\int_2^{3}\frac{f'(x)}{1+f^2(x)}\mathrm{d}x$

$$=\left(\lim_{x\to 0^-}\arctan f(x)-\arctan f(-1)\right)+\left(\lim_{x\to 2^-}\arctan f(x)-\lim_{x\to 0^+}\arctan f(x)\right)$$

$$+\left(\arctan f(3)-\lim_{x\to 2^+}\arctan f(x)\right)$$

$$=\left(-\frac{\pi}{2}\right)+\left(-\frac{\pi}{2}-\frac{\pi}{2}\right)+\left(\arctan\frac{32}{27}-\frac{\pi}{2}\right)=-2\pi+\arctan\frac{32}{27};$$

例题 3 设 $I_n=\int_0^{2\pi}\sin^n x\,\mathrm{d}x$，$J_n=\int_0^{2\pi}\cos^n x\,\mathrm{d}x$，请背住下列常用结论。

(1) 对于任意正整数 n，均有 $I_n=J_n$

(2) 当 n 是偶数时，$I_n=4\int_0^{\frac{\pi}{2}}\sin^n x\,\mathrm{d}x=4\int_0^{\frac{\pi}{2}}\cos^n x\,\mathrm{d}x$

(3) 当 n 是奇数时，$I_n=0$

例题 4 计算 $\int\frac{\sin 10x}{\sin x}\mathrm{d}x$

解： $\int\frac{\sin 10x}{\sin x}\mathrm{d}x=\int 2\sum_{i=1}^{5}\cos(2k-1)x\,\mathrm{d}x=2\sum_{i=1}^{5}\frac{\sin(2k-1)x}{2k-1}+C$；

例题 5 计算 $I_n=\int_0^{\pi}\frac{\sin(2n+1)x}{\sin x}\mathrm{d}x$

解： $I_n-I_{n-1}=\int_0^{\pi}\frac{\sin(2n+1)x}{\sin x}\mathrm{d}x-\int_0^{\pi}\frac{\sin(2n-1)x}{\sin x}\mathrm{d}x$

$$=\int_0^{\pi}\frac{\sin(2n+1)x-\sin(2n-1)x}{\sin x}\mathrm{d}x=\int_0^{\pi}\frac{2\cos 2nx\cdot\sin x}{\sin x}\mathrm{d}x=\int_0^{\pi}2\cos 2nx\,\mathrm{d}x=\left.\frac{\sin 2nx}{n}\right|_0^{\pi}=0$$

所以 $I_n=I_0=\int_0^{\pi}\frac{\sin x}{\sin x}\mathrm{d}x=\pi$；

类题：请证明：$a_n=\int_0^{\pi}\frac{\sin(2n+1)x}{\sin x}\mathrm{d}x=\pi$

解： 同理可得。

例题 6 若 $a_n=\int_0^{\frac{\pi}{2}}\left(\frac{\sin nx}{\sin x}\right)^2dx$，计算 a_n；

解： $a_{n+1}-a_n=\int_0^{\frac{\pi}{2}}\frac{\sin^2(n+1)x-\sin^2 nx}{\sin^2 x}\mathrm{d}x$

$$=\int_0^{\frac{\pi}{2}}\frac{(\sin(n+1)x+\sin nx)(\sin(n+1)x-\sin nx)}{\sin^2 x}\mathrm{d}x$$

$$=\int_0^{\frac{\pi}{2}}\frac{2\sin\frac{2n+1}{2}x\cos\frac{x}{2}\cdot 2\cos\frac{2n+1}{2}x\sin\frac{x}{2}}{\sin^2 x}\mathrm{d}x$$

$$=\int_0^{\frac{\pi}{2}}\frac{\sin x\sin(2n+1)x}{\sin^2 x}\mathrm{d}x=\int_0^{\frac{\pi}{2}}\frac{\sin(2n+1)x}{\sin x}\mathrm{d}x$$

令 $I_n=\int_0^{\frac{\pi}{2}}\frac{\sin(2n+1)x}{\sin x}\mathrm{d}x$，$I_n-I_{n-1}=\int_0^{\frac{\pi}{2}}\frac{\sin(2n+1)x}{\sin x}\mathrm{d}x-\int_0^{\frac{\pi}{2}}\frac{\sin(2n-1)x}{\sin x}\mathrm{d}x$

$$=\int_0^{\frac{\pi}{2}}\frac{\sin(2n+1)x-\sin(2n-1)x}{\sin x}\mathrm{d}x=\int_0^{\frac{\pi}{2}}\frac{2\cos 2nx\cdot\sin x}{\sin x}\mathrm{d}x=\int_0^{\frac{\pi}{2}}2\cos 2nx\,\mathrm{d}x=\left.\frac{\sin 2nx}{n}\right|_0^{\frac{\pi}{2}}=0$$

所以$I_n = I_0 = \frac{\pi}{2}$，$a_{n+1} - a_n = \frac{\pi}{2}$，因为$a_1 = \int_0^{\frac{\pi}{2}} \left(\frac{\sin x}{\sin x}\right)^2 dx = \frac{\pi}{2}$，所以$a_n = \frac{n\pi}{2}$；

例题 7 计算积分$\int_0^1 (1-x)^{100} x \, dx$

解： $\int_0^1 (1-x)^{100} x \, dx \xrightarrow{t=1-x} \int_0^1 t^{100}(1-t) dt = \int_0^1 t^{100}(1-t) dt = \left(\frac{t^{101}}{101} - \frac{t^{102}}{102}\right)\Bigg|_0^1$

$= \frac{1}{101} - \frac{1}{102} = \frac{1}{101 \times 102} = \frac{1}{10302}$；

例题 8 计算积分$\int_0^2 x(x-1)(x-2) \, dx$

解： $\int_0^2 x(x-1)(x-2) dx \xrightarrow{t=x-1} \int_{-1}^1 (t+1)t(t-1) dt$

令$f(t) = (t+1)t(t-1)$，

$f(-t) = (-t+1)(-t)(-t-1) = -(t+1)t(t-1) = -f(t)$

所以$f(t)$是关于t的奇函数，$\int_{-1}^1 (t+1)t(t-1) dt = 0$；

类题 计算积分$\int_0^{2n} x(x-1)(x-2)\cdots(x-n)\cdots[x-(2n-1)](x-2n) \, dx$

解： $\int_0^{2n} x(x-1)(x-2)\ldots(x-n)\ldots[x-(2n-1)](x-2n) dx$

$\xrightarrow{t=x-n} \int_{-n}^{n} (t+n)(t+(n-1))\ldots t\ldots(t-(n-1))(t-n) dt$

$= \int_{-n}^{n} t(t^2-n^2)(t^2-(n-1)^2)\ldots\ldots(t^2-1) dt = 0$；

例题 9 $\int_0^1 \left|x - \frac{1}{2}\right|^5 x^n (1-x)^n dx$

解： $\xrightarrow{t=x-\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} |t|^5 \left(t+\frac{1}{2}\right)^n \left(\frac{1}{2}-t\right)^n dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} |t|^5 \left(\frac{1}{4}-t^2\right)^n dt$

$= 2\int_0^{\frac{1}{2}} t^5 \left(\frac{1}{4}-t^2\right)^n dt = \int_0^{\frac{1}{2}} t^4 \left(\frac{1}{4}-t^2\right)^n dt^2 \xrightarrow{u=\frac{1}{4}-t^2} \int_0^{\frac{1}{4}} \left(\frac{1}{4}-u\right)^2 u^n du$

$= \int_0^{\frac{1}{4}} \left(\frac{1}{16} - \frac{u}{2} + u^2\right) u^n du = \left(\frac{u^{n+1}}{16(n+1)} - \frac{u^{n+2}}{2(n+2)} + \frac{u^{n+3}}{n+3}\right)\Bigg|_0^{\frac{1}{4}}$

$= \frac{1}{4^{n+3}}\left(\frac{1}{(n+1)} - \frac{2}{(n+2)} + \frac{1}{(n+3)}\right)$；