Portfolio Optimization Constrained by Performance Attribution

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https://www.mdpi.com/1911-8074/14/5/201

- This study investigates performance attribution measures as a basis for constraining portfolio optimization.
- We employ optimizations that minimize conditional value-at-risk and investigate two performance attributes, asset allocation (AA) and the selection effect (SE), as constraints on asset weights.
- The test portfolio consists of stocks from the Dow Jones Industrial Average index.
- The results suggest that achieving SE performance thresholds requires larger turnover values than that required for achieving comparable AA thresholds.

The Approach

Consider a managed portfolio p comprised of N assets, consisting of M asset classes with n_i assets in class $i, i = 1, \ldots, M$, such that $\sum_{i=1}^{M} n_i = N$. Let b denote a benchmark portfolio composed of Q assets comprising with same M asset classes. Assume all weights are non-negative and a fully invested portfolio is summation of weights equivalent to one. The quantities AA and SE for asset class i are defined as follows (Biglova and Rachev, 2007)

$$AA_i = \left(w_i^{(p)} - w_i^{(b)}\right)\left(R_i^{(b)} - R^{(b)}\right),$$

$$SE_i = w_i^{(b)}\left(R_i^{(p)} - R_i^{(b)}\right),$$

where $R_i^{(p)} = \sum_{j=1}^{n_i} \frac{w_{ij}^{(p)}}{w_i^{(p)}} E(r_{ij})$, $R_i^{(b)} = \sum_{j=1}^{q_i} \frac{w_{ij}^{(b)}}{w_i^{(b)}} E(r_{ij})$, $R^{(b)} = \sum_{i=1}^{M} \sum_{j=1}^{q_i} w_{ij}^{(b)} E(r_{ij})$, $R^{(b)} = \sum_{i=1}^{M} \sum_{j=1}^{n} w_{ij}^{(p)} E(r_{ij}) = \sum_{j=1}^{M} w_i^{(p)} R_i^{(p)}$ and $E(\cdot)$ denotes expected value.

Portfolio Optimization

We employ CVaR optimization (Rockafellar, 2000) in the study, $\min_{\omega} \mathrm{CVaR}_{\alpha}(\omega) = \min_{\omega,\gamma} \left\{\gamma - \frac{1}{\alpha T}\sum_{t=1}^{T}(\gamma - \omega' \boldsymbol{r}(t))^{+}\right\}$, where $\boldsymbol{r}(t)$ is a finite sample of asset returns. And we also consider soft constraints,

$$\min_{\omega} \mathrm{CVaR}_{\alpha}(\omega) = \min_{\omega,\gamma} \left\{\gamma - \frac{1}{\alpha T}\sum_{t=1}^{T}\left(\gamma - \omega' \boldsymbol{r}(t)\right)^{+} + \beta((c_1(\omega))^{+})^2\right\},$$

where β can be set by the user, and $c_1(\omega) \le 0$ is the optimization constraints.

Optimization Constraints

Consider four portfolio optimization problems:

P_{α}^{0}: **(a)** $w_{ij}^{(p)} \ge 0, \sum_{i,j} w_{ij}^{(p)} = 1$; **(b)** $\mathrm{TO} \le C_{\mathrm{TO}}$

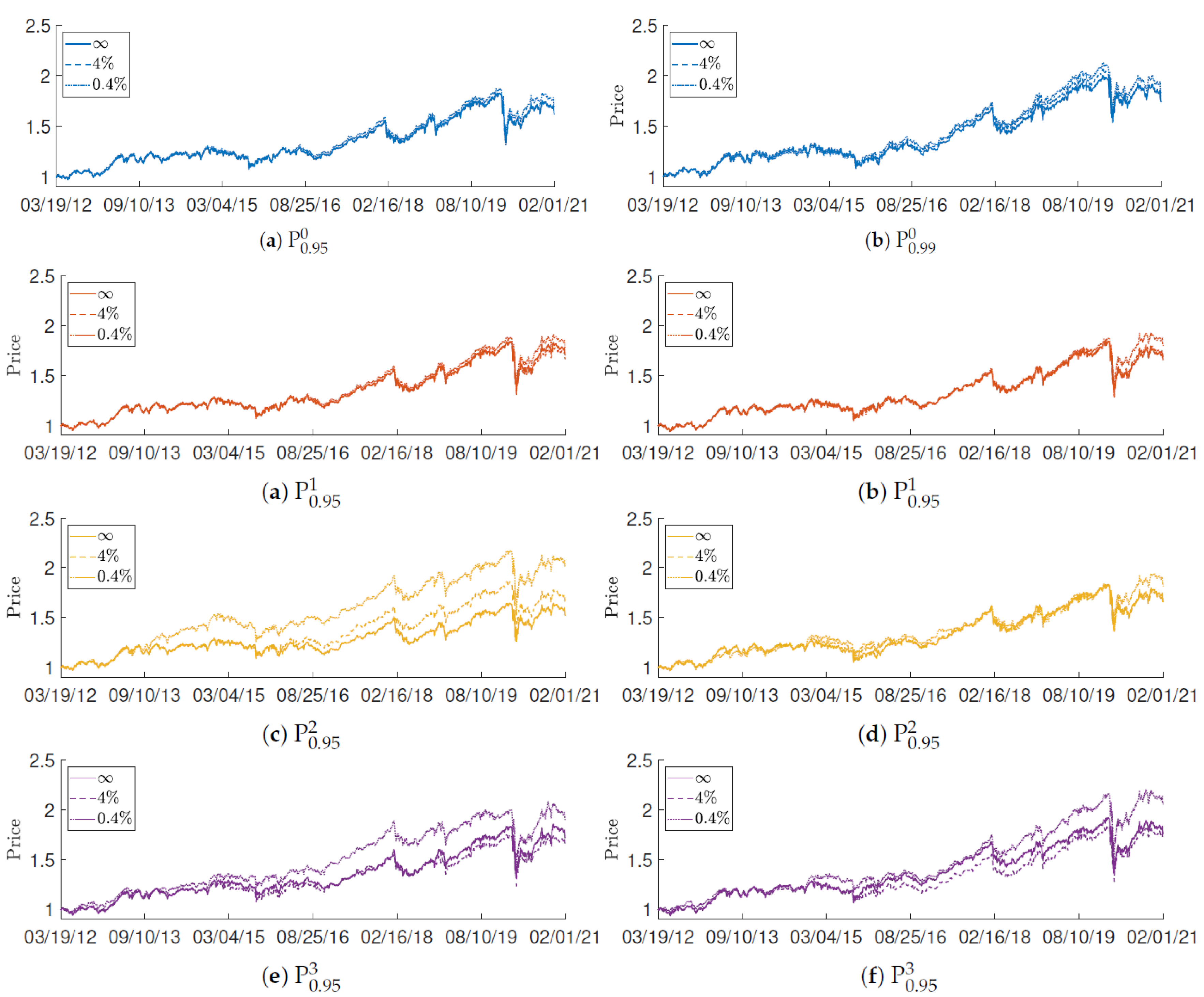
P_{α}^{1}: **(a)** $w_{ij}^{(p)} \ge 0, \sum_{i,j} w_{ij}^{(p)} = 1$; **(b)** $\mathrm{TO} \le C_{\mathrm{TO}}$; and **(c)** $a_1 \le \mathrm{AA} \le b_1$

P_{α}^{2}: **(a)** $w_{ij}^{(p)} \ge 0, \sum_{i,j} w_{ij}^{(p)} = 1$; **(b)** $\mathrm{TO} \le C_{\mathrm{TO}}$ and **(d)** $a_2 \le \mathrm{SE} \le b_2$

P_{α}^{3}: **(a)** $w_{ij}^{(p)} \ge 0, \sum_{i,j} w_{ij}^{(p)} = 1$; **(b)** $\mathrm{TO} \le C_{\mathrm{TO}}$; **(c)** $a_1 \le \mathrm{AA} \le b_1$; and **(d)** $a_2 \le \mathrm{SE} \le b_2$

And TO is the turnover constraint,

$$\mathrm{TO} = \frac{1}{2}\sum_{i=1}^{M}\sum_{j=1}^{n_i} |w_{ij}^{(p)}(t) - w_{ij}^{(p)}(t-1)| \le C_{\mathrm{TO}}$$



(a) $P^0_{0.95}$ (b) $P^0_{0.99}$

(a) $P^1_{0.95}$ (b) $P^1_{0.95}$

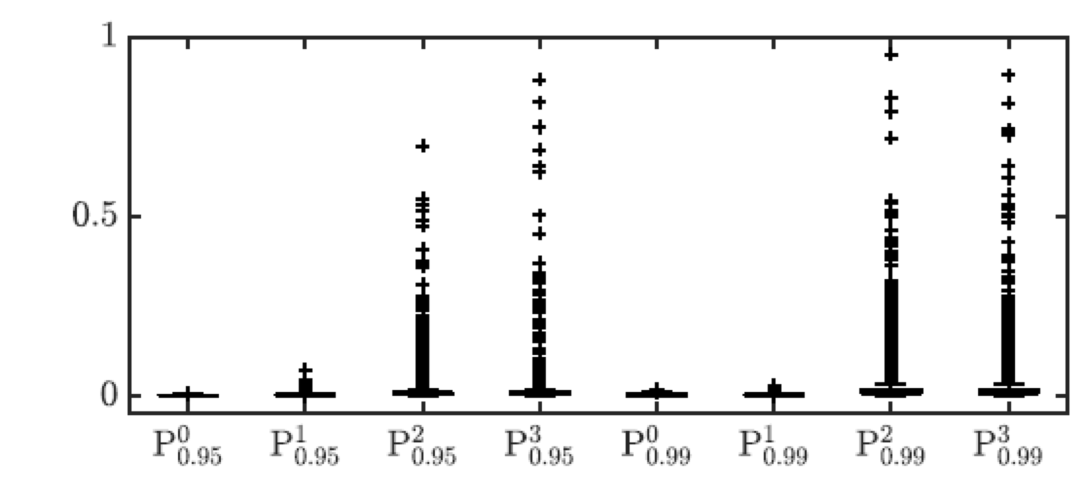
(c) $P^2_{0.95}$ (d) $P^2_{0.95}$

(e) $P^3_{0.95}$ (f) $P^3_{0.95}$

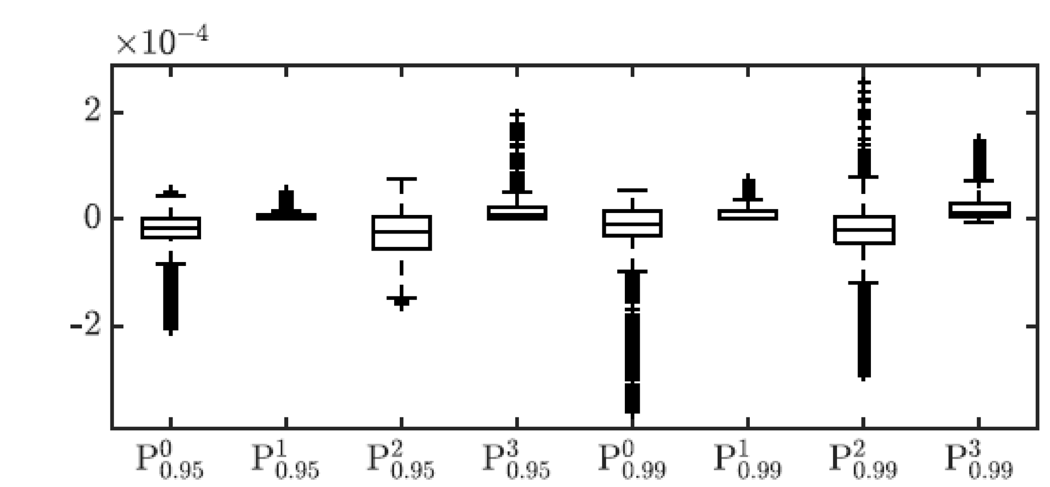
Portfolio	All 'Hard'	TO	TO+AA	TO+SE	TO+SE+AA	no Solution for t
			$\mathrm{TO} < \infty$			
$P^0_{0.95}$	100.00%					0.00%
$P^1_{0.95}$	100.00%		0.00%			0.00%
$P^2_{0.95}$	44.11%			55.84%		0.04%
$P^3_{0.95}$	54.90%			42.99%	1.93%	0.18%
			$\mathrm{TO} \le 4\%$			
$P^0_{0.95}$	99.87%	0.13%				0.00%
$P^1_{0.95}$	98.61%	1.39%	0.00%			0.00%
$P^2_{0.95}$	41.24%	12.58%		46.13%		0.04%
$P^3_{0.95}$	45.54%	19.03%		34.39%	1.03%	0.00%
			$\mathrm{TO} \le 0.4\%$			
$P^0_{0.95}$	78.15%	21.85%				0.00%
$P^1_{0.95}$	51.68%	48.32%	0.00%			0.00%
$P^2_{0.95}$	6.85%	87.51%		5.64%		0.00%
$P^3_{0.95}$	6.76%	90.06%		3.00%	0.18%	0.00%

Biglova, Almira, and Svetlozar T. Rachev. 2007. Portfolio performance attribution. Investment Management and Financial Innovations 4: 7–22.

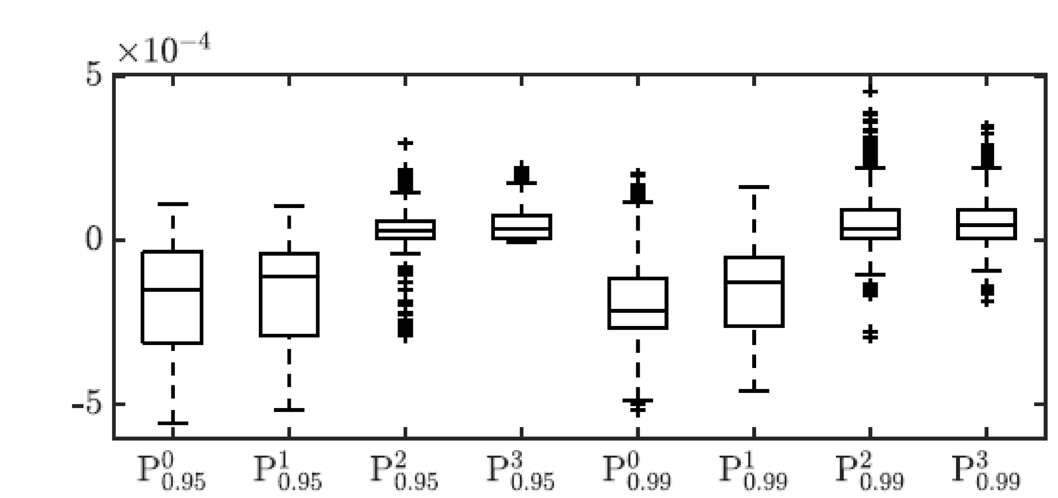
Rockafellar, R. Tyrrell, and Stanislav Uryasev. 2000. Optimization of conditional value-at-risk. Journal of Risk 2: 21–41. [CrossRef]



(a) TO



(b) AA



(c) SE