MF921 Topics in Dynamic Asset Pricing

Stochastic Analysis & Stochastic Calculus in Quantitative Finance

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Week 2

Change of Numeraire: Motivation and Key Idea

In option pricing, we usually price under the risk-neutral measure using the money market account $B(t) = e^{rt}$ as the numeraire. But sometimes payoffs become simpler if we change the unit of measurement (the numeraire). Instead of measuring in dollars, measure in shares of stock.

The key idea is :

- Pick any strictly positive traded asset $N(t)$ as the numeraire.
- Then define a new probability measure $\tilde{\mathbb{P}}$ such that $\frac{S(t)}{N(t)}$ is a martingale under $\tilde{\mathbb{P}}$. No-arbitrage is preserved.

We first look at the details how this work (Radon Nikodym derivative & Girsanov Theorem) and then apply the scheme to price different type of options.

Change of Numeraire

Given $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}^*)$ with d-dim Brownian W:

- Money market account (baseline numeraire): $dB_t = B_t r_t dt$
- Traded asset $S(t)$: $dS_t = S_t(r_t dt + \sigma_t\, dW_t)$, $\quad \frac{S_t}{B_t}$ is a martingale.
- Derivative pricing rule: for payoff X_T at maturity T, $V_0 = \mathbb{E}^{\mathbb{P}^*}\left[\frac{X_T}{B_T}\right]$

Our goal is to pick another strictly positive traded asset $N(t)$ and define a new measure $\tilde{\mathbb{P}}$ such that $\frac{S(t)}{N(t)}$ is a martingale for every traded asset $S(t)$.

Change of Numeraire Con.

Oberve $\dfrac{S(t)}{B(t)}$ is a martingale under $\mathbb{P}^*$. We want $\dfrac{S(t)}{N(t)}$ to be a martingale under $\tilde{\mathbb{P}}$. Define $\tilde{\mathbb{P}}$ via the Radon$-$Nikodym derivative with respect to $\mathbb{P}^*$:

$$\left.\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^*}\right|_{\mathcal{F}_T} = Z_T := \quad \frac{N(T)/B(T)}{N(0)/B(0)}$$

By construction, $\frac{N(T)}{B(T)}$ is a martingale under $\mathbb{P}^*$, $Z_T > 0$ and $\mathbb{E}^{\mathbb{P}^*}[Z_T] = 1$ and take any payoff X_T:

$$V(0) \;=\; N(0)\,\mathbb{E}^{\tilde{\mathbb{P}}}\left[\frac{X_T}{N(T)}\right] = N(0)\mathbb{E}^{\mathbb{P}^*}\left[\frac{X_T}{N(T)}\,Z_T\right] = \mathbb{E}^{\mathbb{P}^*}\left[\frac{X_T}{B(T)}\right]$$

So the choice of Radon$-$Nikodym derivative guarantees the prices are consistent under both measures and no arbitrage is preserved.

Change of Numeraire Con.

What is the $dS(t)$ looks like under meausre $\tilde{\mathbb{P}}$?

Note: Under Q, we have $\begin{cases} dS_t = r_t S_t\, dt + \sigma_t S_t\, dW_t \\ dN_t = r_t N_t\, dt + \gamma_t N_t\, dW_t \end{cases}$

Denote $\widehat{N}_t = \frac{N_t}{B_t}$, apply Itô we get $\frac{d\widehat{N}_t}{\widehat{N}_t} = \gamma_t\, dW_t$, $\widehat{N}_t = \widehat{N}_0 e^{\left(\int_0^t \gamma_s \cdot dW_s - \frac{1}{2}\int_0^t \|\gamma_s\|^2\, ds\right)}$.

Oberve that $Z_t = \frac{\widehat{N}_t}{\widehat{N}_0} = e^{\left(\int_0^t \gamma_s \cdot dW_s - \frac{1}{2}\int_0^t \|\gamma_s\|^2\, ds\right)}$

Girsanov's theorem says: if we define a new measure $\tilde{\mathbb{P}}$ via this Z_t, then the process

$$W_t^N \;=\; W_t - \int_0^t \gamma_s dt$$

is a Brownian motion under $\tilde{\mathbb{P}}$. Substitute into dS_t to get the $\tilde{\mathbb{P}}$ dynamics:

$$dS_t = S_t\Big[(r_t + \sigma_t \cdot \gamma_t)\, dt + \sigma_t \cdot dW_t^N\Big]$$

$$S_t = S_0 \exp\left(\int_0^t \left(r_s + \sigma_s \cdot \gamma_s - \tfrac{1}{2}\|\sigma_s\|^2\right) ds + \int_0^t \sigma_s \cdot dW_s^N\right)$$

Black-Scholes Formula

Given r, σ are constant, we have $S(T) = S(0)\exp\left\{(r - \frac{1}{2}\sigma^2)T + \sigma W(T)\right\}$.
The no-arbitrage price for the call option:

$$\begin{aligned}
\psi_c(0) &= \mathbb{E}^{\mathbb{P}^*}(e^{-rT}(S(T) - K)^+) \\
&= \mathbb{E}^{\mathbb{P}^*}(e^{-rT}(S(T) - K)I(S(T) \geq K)) \\
&= \mathbb{E}^{\mathbb{P}^*}(e^{-rT}S(T)I(S(T) \geq K)) - Ke^{-rT}\mathbb{P}^*(S(T) \geq K) \\
&= I - Ke^{-rT} \cdot II
\end{aligned}$$

For II:

$$\begin{aligned}
II = \mathbb{P}^*(S(T) \geq K) = 1 - \Phi\left(\frac{\log(K/S(0)) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\
= \Phi\left(\frac{\log(S(0)/K) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)
\end{aligned}$$

Note: Φ is the CDF of the standard normal distribution.

Black-Scholes Formula Con.

For I, we apply the change of numeraire and use stock itself as numeraire. Then based on the eraly definition we have $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^*}\Big|_{\mathcal{F}_T} = Z_T := e^{-rT}\frac{S(T)}{S(0)}$ and $\gamma_t = \sigma$. Therefore, under $\tilde{\mathbb{P}}$ we have the following dynamics of $S(t)$:

$$\frac{dS_t}{S_t} = rdt + \sigma^2 dt + \sigma d\tilde{W}_t,\ S(t) = S(0)\exp\left\{(r+\sigma^2/2)t + \sigma\tilde{W}_t\right\}$$

Then we can rewrite I:

$$\begin{aligned} I = S(0)\mathbb{E}^{\mathbb{P}^*}\left(e^{-rT}\frac{S(T)}{S(0)}I(S(T)\geq K)\right) &= S(0)\mathbb{E}^{\tilde{\mathbb{P}}}(I(S(T)\geq K)) \\ &= S(0)\tilde{\mathbb{P}}(S(T)\geq K) \\ &= S(0)\Phi\left(\frac{\log(S(0)/K)+(r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right). \end{aligned}$$

Putting together, we have the price of the call option is given by:

$$I - Ke^{-rT}\cdot II = S(0)\Phi(d_+) - Ke^{-rT}\Phi(d_-)$$

where $d_\pm = \frac{\log(S(0)/K)+(r\pm\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$.

One Dimensional Barrier Options

Barrier options are path-dependent derivatives whose payoff is activated (knock-in) or extinguished (knock-out) if the underlying asset crosses a pre-specified barrier. They extend vanilla calls/puts by adding a barrier condition.
We first study continuously monitored barriers and derive Mertons closed-form pricing formulas (1973) for single-barrier options.

Given $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}^*)$ with 1-dim Brownian W. The Market setting following:

$$dB_t = B_t r dt \ , \ dS_t = rS_t dt + \sigma S_t dW_t$$

A continuously monitored barrier option has payoff $=$ vanilla option payoff $\times$ indicator of the barrier condition. For example:

- Up-and-out call: $V_0 = \mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}(S_T - K)^+ I\left\{\max_{0\leq t\leq T} S_t \leq H\right\}\right], \quad H > S_0$
- Down-and-in put: $V_0 = \mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}(K - S_T)^+ I\left\{\min_{0\leq t\leq T} S_t \leq H\right\}\right], \quad H < S_0$

Study the case of the down-and-in call option (DAIC) with strike K, barrier $H < S_0$:

$$\text{DAIC} = e^{-rT}\mathbb{E}^{\mathbb{P}^*}\left[(S_T - K)^+ I\left\{\min_{0\leq t\leq T} S_t \leq H\right\}\right]$$

One Dimensional Barrier Options Con.

For notation simplicity, denote a drifted Brownian motion:

$$W_{\mu,\sigma}(t) = \mu t + \sigma W(t), \quad M_t = \max_{0 \le s \le t} W_{\mu,\sigma}(s).$$

Some useful results from the reflection principle for a Brownian motion with a drift:
(i) When $x \le y$, $y > 0, \sigma > 0$:

- $P(W_{\mu,\sigma}(t) \le x, M_t \ge y) = e^{2\mu y/\sigma^2} \Phi\left(\frac{x-2y-\mu t}{\sigma\sqrt{t}}\right)$
- $P(W_{\mu,\sigma}(t) \le x, M_t \le y) = \Phi\left(\frac{x-\mu t}{\sigma\sqrt{t}}\right) - e^{2\mu y/\sigma^2} \Phi\left(\frac{x-2y-\mu t}{\sigma\sqrt{t}}\right)$

(ii) When $x \ge y > 0$, $\sigma > 0$:

- $P(W_{\mu,\sigma}(t) \le x, M_t \le y) = P(M_t \le y) = \Phi\left(\frac{y-\mu t}{\sigma\sqrt{t}}\right) - e^{2\mu y/\sigma^2} \Phi\left(\frac{-y-\mu t}{\sigma\sqrt{t}}\right)$
- $P(W_{\mu,\sigma}(t) \le x, M_t \ge y) = P(W_{\mu,\sigma}(t) \le x) - P(W_{\mu,\sigma}(t) \le x, M_t \le y) = \Phi\left(\frac{x-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{y-\mu t}{\sigma\sqrt{t}}\right) + e^{2\mu y/\sigma^2} \Phi\left(\frac{-y-\mu t}{\sigma\sqrt{t}}\right)$

(iii) When $x \ge y$, $y < 0$, $\sigma > 0$:

- $P\left(W_{\mu,\sigma}(t) \ge x, \min_{0 \le s \le t} W_{\mu,\sigma}(s) \le y\right) = e^{2\mu y/\sigma^2} \Phi\left(\frac{-x+2y+\mu t}{\sigma\sqrt{t}}\right)$

One Dimensional Barrier Options Con.

Back to the valuation of DAIC:

$$X + T$$