MF921 Topics in Dynamic Asset Pricing

Week 10

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Chapter 14 Viscosity Solutions and HJB Equations

The Verification Result

Lemma 3: For every $x \in \mathbb{R}^d$ and every open neighborhood of x, denoted by N_x, there exists $t_x > 0$ small enough, such that for any control policy u, the state dynamics given by

$$dX_t = f(X_t, u_t)dt, \quad X_0 = x,$$

satisfies

$$X_t \in N_x, \quad \forall 0 \le t \le t_x.$$

Intuitively, $f(x, u)$ is bounded and Lipschitz continuous, your system can't move too far away from its starting point x in an infinitesimally short time.

[Proof]

Since f is continuous and U is compact, we have for every $x \in \mathbb{R}^d$,

$$M_x = \sup_{u \in U} \|f(x, u)\| < \infty.$$

Because

$$X_t^x = x + \int_0^t f(X_s^x, u_s)ds = x + \int_0^t f(x, u_s)ds + \int_0^t [f(X_s^x, u_s) - f(x, u_s)]ds,$$

we have, via the Lipschitz continuity,

$$\|X_t^x - x\| \le M_x t + L \int_0^t \|X_s^x - x\| ds.$$

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The Verification Result

The Gronwall inequality yields that

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\begin{aligned}
\|X_t^x - x\| &\leq M_x t + \int_0^t L M_x s \exp[L(t-s)] ds \\
&\leq M_x t + L M_x t \int_0^t \exp[L(t-s)] ds \\
&\leq M_x t + M_x t (e^{Lt} - 1) \\
&= M_x t e^{Lt},
\end{aligned}
$$

from which the conclusion follows.

Proposition 3: The value function v is a viscosity supersolution to the HJB equation

$$\beta v(x) - \sup_{u \in U} [Dv(x) \cdot f(x,u) + c(x,u)] = 0.$$

[Proof]
Consider a C^1 test function ϕ, and let y be local minimum of $v - \phi$. There exists a neighborhood of y, say N_y, such that

$$v(x) - \phi(x) \geq v(y) - \phi(y), \quad x \in N_y.$$

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$$v(x) - v(y) \geq \phi(x) - \phi(y), \quad x \in N_y.$$

By Lemma 3, for any control policy $u_t \in U$ there exists t_y small enough such that

$$X_0 = y, \quad X_t \in N_y, \quad \forall 0 \leq t \leq t_y.$$

Thus,

$$v(X_t^y) - v(y) \geq \phi(X_t^y) - \phi(y), \quad \forall 0 \leq t \leq t_y.$$

For any control policy $u_t \in U$ and $u_0 = u$, the dynamic programming principle from Proposition 2 yields

$$v(y) \geq \int_0^t e^{-\beta s} c(X_s^y, u_s) ds + e^{-\beta t} v(X_t^y).$$

Therefore, adding the two inequalities leads to, $0 \leq t \leq t_y$,

$$v(X_t^y) \geq \int_0^t e^{-\beta s} c(X_s^y, u_s) ds + e^{-\beta t} v(X_t^y) + \phi(X_t^y) - \phi(y).$$

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Hence for $0 \leq t \leq t_y$,

$$v(X_t^y)\left(\frac{1-e^{-\beta t}}{t}\right) - \frac{1}{t}\int_0^t e^{-\beta s} c(X_s^y, u_s)ds \geq \frac{\phi(X_t^y)-\phi(y)}{t}. \tag{14.7}$$

Letting $t \to 0$,

- From Proposition 1, we know v is uniformly continuous. As $t \to 0$, the trajectory X_t^y starting at y satisfies $X_t^y \to y$ (because f is continuous). Hence

$$v(X_t^y) \longrightarrow v(y).$$

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$$\frac{1-e^{-\beta t}}{t} \to \beta$$

  This is the Taylor expansion of $e^{-\beta t}$:

$$e^{-\beta t} = 1 - \beta t + o(t) \Longrightarrow \frac{1-e^{-\beta t}}{t} \to \beta.$$

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The Verification Result

- $$\frac{1}{t}\int_0^t e^{-\beta s} c(X_s^y, u_s)\, ds \to c(y, u_0)$$
  For small t, the control u_s is nearly constant at its initial value u_0, and $X_s^y \approx y + f(y, u_0)s$, so $X_s^y \to y$. By continuity of c,
  $$e^{-\beta s} c(X_s^y, u_s) \to c(y, u_0).$$
  Then by the mean value theorem for integrals, the average converges.

- $$\frac{\phi(X_t^y) - \phi(y)}{t} \to D\phi(y) \cdot f(y, u_0)$$
  By first-order Taylor expansion:
  $$\phi(X_t^y) = \phi(y) + D\phi(y) \cdot (X_t^y - y) + o(\|X_t^y - y\|).$$
  But
  $$\frac{X_t^y - y}{t} = \frac{1}{t}\int_0^t f(X_s^y, u_s)\, ds \to f(y, u_0).$$
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Take limits on both sides of (14.7) as $t \to 0$:
Left-hand side:

$$v(X_t^y)\left(\frac{1-e^{-\beta t}}{t}\right) - \frac{1}{t}\int_0^t e^{-\beta s} c(X_s^y, u_s)\, ds \longrightarrow \beta v(y) - c(y, u_0).$$

Right-hand side:

$$\frac{\phi(X_t^y) - \phi(y)}{t} \longrightarrow D\phi(y) \cdot f(y, u_0).$$

Now combine them and rearrange:

$$\beta v(y) \geq D\phi(y) \cdot f(y, u_0) + c(y, u_0).$$

Since u_0 is arbitrary, we get

$$\beta v(y) \geq \max_{u \in U}\{D\phi(y) \cdot f(y, u) + c(y, u)\}.$$

In other words

$$\beta v(y) - \sup_{u \in U}\{D\phi(y) \cdot f(y, u) + c(y, u)\} \geq 0,$$

which shows that v is a supersolution of the HJB equation.

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Proposition 4: The value function v is a viscosity subsolution to the HJB equation

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[Proof]
Consider a C^1 test function ϕ, and let y be local maximum of $v - \phi$. There exists a neighborhood of y, say N_y, such that

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For any small $\varepsilon > 0$, the DPP ensures existence of an ε-optimal control policy $u_s^{\varepsilon,t}$ such that

$$v(y) \leq \int_0^t e^{-\beta s} c(X_s^y, u_s^{\varepsilon,t})\, ds + e^{-\beta t} v(X_t^y) + \varepsilon t.$$

This is the "reverse" inequality of the supersolution case, since $v(y)$ is a supremum and this control nearly attains that supremum.

Therefore, adding the two inequalities leads to, $0 \leq t \leq t_y$,

$$v(X_t^y) \leq \int_0^t e^{-\beta s} c(X_s^y, u_s^{\varepsilon,t})\, ds + e^{-\beta t} v(X_t^y) + \phi(X_t^y) - \phi(y) + \varepsilon t.$$

Rearranging:

$$v(X_t^y)(1 - e^{-\beta t}) - \int_0^t e^{-\beta s} c(X_s^y, u_s^{\varepsilon,t})\, ds \leq \phi(X_t^y) - \phi(y) + \varepsilon t. \tag{14.8}$$

The complication here is that the control policy $u_s^{\varepsilon,t}$ depends on t. Thus, we have to be careful when taking the limit $t \to 0$ in (14.8). For $0 \leq t \leq t_y$, we have by the Lipschitz condition of f:

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The Verification Result

$$\|f(X_s^y, u_s^{\varepsilon,t}) - f(y, u_s^{\varepsilon,t})\| \leq L\|X_s^y - y\|,$$

and

$$\phi(X_t^y) - \phi(y) = \int_0^t D\phi(X_s^y) \cdot f(X_s^y, u_s^{\varepsilon,t})\, ds = \int_0^t D\phi(y) \cdot f(y, u_s^{\varepsilon,t})\, ds + o(t), \quad (14.9)$$

because

$$\begin{aligned}
&\frac{1}{t}\left|\int_0^t \{D\phi(X_s^y) \cdot f(X_s^y, u_s^{\varepsilon,t}) - D\phi(y) \cdot f(y, u_s^{\varepsilon,t})\}\, ds\right| \\
&= \frac{1}{t}\left|\int_0^t \{D\phi(X_s^y) \cdot (f(X_s^y, u_s^{\varepsilon,t}) - f(y, u_s^{\varepsilon,t})) + (D\phi(X_s^y) - D\phi(y)) \cdot f(y, u_s^{\varepsilon,t})\}\, ds\right| \\
&\leq \frac{1}{t}\int_0^t |D\phi(X_s^y) \cdot (f(X_s^y, u_s^{\varepsilon,t}) - f(y, u_s^{\varepsilon,t}))|\, ds + \frac{1}{t}\int_0^t |(D\phi(X_s^y) - D\phi(y)) \cdot f(y, u_s^{\varepsilon,t})|\, ds \\
&\leq \frac{1}{t}\int_0^t \|D\phi(X_s^y)\| \cdot \|f(X_s^y, u_s^{\varepsilon,t}) - f(y, u_s^{\varepsilon,t})\|\, ds + \frac{1}{t}\int_0^t |(D\phi(X_s^y) - D\phi(y)) \cdot f(y, u_s^{\varepsilon,t})|\, ds \\
&\leq \frac{L}{t}\int_0^t \|D\phi(X_s^y)\| \cdot \|X_s^y - y\|\, ds + M_y \frac{1}{t}\int_0^t \|D\phi(X_s^y) - D\phi(y)\|\, ds
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$$\phi(X_t^y) - \phi(y) = \int_0^t D\phi(X_s^y) \cdot f(X_s^y, u_s^{\varepsilon,t})\, ds = \int_0^t D\phi(y) \cdot f(y, u_s^{\varepsilon,t})\, ds + o(t), \quad (14.9)$$

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&\le \frac{1}{t}\int_0^t |D\phi(X_s^y) \cdot (f(X_s^y, u_s^{\varepsilon,t}) - f(y, u_s^{\varepsilon,t}))|\, ds + \frac{1}{t}\int_0^t |(D\phi(X_s^y) - D\phi(y)) \cdot f(y, u_s^{\varepsilon,t})|\, ds \\
&\le \frac{1}{t}\int_0^t \|D\phi(X_s^y)\| \cdot \|f(X_s^y, u_s^{\varepsilon,t}) - f(y, u_s^{\varepsilon,t})\|\, ds + \frac{1}{t}\int_0^t |(D\phi(X_s^y) - D\phi(y)) \cdot f(y, u_s^{\varepsilon,t})|\, ds \\
&\le \frac{L}{t}\int_0^t \|D\phi(X_s^y)\| \cdot \|X_s^y - y\|\, ds + M_y \frac{1}{t}\int_0^t \|D\phi(X_s^y) - D\phi(y)\|\, ds
\end{aligned}$$

The Verification Result

as $t \to 0$, where $M_y = \sup_{u \in U} \|f(y,u)\|$, because $D\phi(X_s^y) \to D\phi(y)$ and $X_s^y \to y$ as $s \to 0$.

Thus, (14.8) and (14.9) yield

$$v(X_t^y)(1 - e^{-\beta t}) - \int_0^t e^{-\beta s} c(X_s^y, u_s^{\varepsilon,t})\, ds - \int_0^t D\phi(y) \cdot f(y, u_s^{\varepsilon,t})\, ds \le o(t) + t\varepsilon. \quad (14.10)$$

Similarly, the Lipschitz condition of c gives

$$\int_0^t e^{-\beta s} c(X_s^y, u_s^{\varepsilon,t})\, ds = \int_0^t c(y, u_s^{\varepsilon,t})\, ds + o(t). \quad (14.11)$$

and the uniform continuity of v leads to

$$v(X_t^y)(1 - e^{-\beta t}) = \beta t v(y) + o(t) = \int_0^t \beta v(y)\, ds + o(t). \quad (14.12)$$

Combining (14.10)-(14.12) yields

$$\int_0^t \left[\beta v(y) - D\phi(y) \cdot f(y, u_s^{\varepsilon,t}) - c(y, u_s^{\varepsilon,t})\right] ds \le t\varepsilon + o(t).$$

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The Verification Result

Thus, the inf of the integrand inside must satisfy

$$\inf_{u \in U} [\beta v(y) - D\phi(y) \cdot f(y,u) - c(y,u)] \leq \varepsilon.$$

In other words,

$$\beta v(y) - \sup_{u \in U} \{D\phi(y) \cdot f(y,u) + c(y,u)\} \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have that v is a subsolution of the HJB equation.

Uniqueness via the Comparison Principle

Theorem 2: (The Comparison Principle). Let u_1 be any viscosity subsolution to the HJB equation and u_2 be any viscosity supersolution to the HJB equation. If u_1 and u_2 are both uniformly bounded, then $u_1 \leq u_2$.

$$\beta v(x) - \sup_{u \in U}[Dv(x) \cdot f(x,u) + c(x,u)] = 0.$$

Theorem 2 is the most difficult part of the proof of Theorem 1. The proof of Theorem 2 relies on a doubling variable technique from Kruzkov (1970), which introduces a new function Ψ of two variables in $u_1(x)$ and $u_2(y)$. The function Ψ allows us to treat u_2 (resp. u_1) as a constant with respect to the test function of u_1 (resp. u_2).

We shall prove Theorem 2 by contradiction. Suppose there exists $\delta > 0$ and $z \in \mathbb{R}^d$ such that $u_1(z) - u_2(z) \geq 2\delta$. We will try to derive a contradiction from this assumption using the "doubling variable" trick.

Consider a (doubling variable) function $\Psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ defined as

$$\Psi(x,y;\varepsilon,\alpha,p) = -u_1(x) + u_2(y) + \frac{\|x-y\|^2}{2\varepsilon} + \alpha\left[\left(1+\|x\|^2\right)^p + \left(1+\|y\|^2\right)^p\right],$$

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where the constants $\varepsilon, \alpha, p > 0$. The key idea of the proof is to show a contradiction by using the fact that, when $\varepsilon \to 0$ and $\alpha \to 0$, the minimum of $\Psi(x, y; \varepsilon, \alpha, p)$ should be achieved when $x = y$ with the value $u_2(x) - u_1(x)$. The term $\left(1 + \|x\|^2\right)^p + \left(1 + \|y\|^2\right)^p$ is added to ensure that Ψ has a minimum point.

Choose α small enough so that $2\alpha \left(1 + \|z\|^2\right)^p \leq \delta$

$$\Psi(z, z; \varepsilon, \alpha, p) = -u_1(z) + u_2(z) + 2\alpha \left(1 + \|z\|^2\right)^p \leq -2\delta + \delta = -\delta.$$

We choose $p > 0$ so small that

$$p\,(2L + M) \leq \beta, \tag{14.13}$$

where $M = \sup_{u \in U} \|f(0, u)\|$. Note that α and p do not depend on ε.

The definition of Ψ implies that

$$\lim_{\|x\| \to \infty} \Psi(x, y; \varepsilon, \alpha, p) \to \infty, \qquad \lim_{\|y\| \to \infty} \Psi(x, y; \varepsilon, \alpha, p) \to \infty,$$

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Uniqueness via the Comparison Principle

Because u_1 and u_2 are bounded by the assumption in the theorem. Therefore, there must exist a pair $(\bar{x}, \bar{y}) \in K_\varepsilon$ that attain the minimum of the continuous function Ψ, where K_ε is a compact set. In particular,

$$\Psi(\bar{x}, \bar{y}; \varepsilon, \alpha, p) = \inf_{x,y} \Psi(x, y; \varepsilon, \alpha, p) \leq \Psi(z, z; \varepsilon, \alpha, p) \leq -\delta. \tag{14.14}$$

Lemma 4: For any $\varepsilon > 0$, we have

$$\beta\Psi(\bar{x}, \bar{y}; \varepsilon, \alpha, p) \geq -\frac{L}{\varepsilon}\|\bar{x} - \bar{y}\|^2 - L\|\bar{x} - \bar{y}\| - \varepsilon. \tag{14.15}$$

[Proof]
Introduce the notation

$$H(x, p) = \sup_{u \in U}[f(x, u) \cdot p + c(x, u)].$$

Then the HJB equation can be written as

$$\beta v(x) - H(x, Dv(x)) = 0.$$

(i) Consider a test function

$$\phi(x) = u_2(\bar{y}) + \frac{\|x - \bar{y}\|^2}{2\varepsilon} + \alpha\left[\left(1 + \|x\|^2\right)^p + \left(1 + \|\bar{y}\|^2\right)^p\right].$$

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Thus, $-u_1(x) + \phi(x)$ attains its minimum at $\bar{x}$. Hence, $u_1(x) - \phi(x)$ attains its maximum at $\bar{x}$. By the definition of the subsolution,

$$\beta u_1(\bar{x}) - H(\bar{x}, D\phi(\bar{x})) \leq 0.$$

However,

$$D\phi(x) = \frac{x - \bar{y}}{\varepsilon} + \alpha \left[2p \left(1 + \|x\|^2\right)^{p-1} x \right].$$

Thus,

$$\beta u_1(\bar{x}) \leq H(\bar{x}, D\phi(\bar{x})) = H\left(\bar{x}, \frac{\bar{x} - \bar{y}}{\varepsilon} + \alpha \left[2p \left(1 + \|\bar{x}\|^2\right)^{p-1} \bar{x} \right]\right),$$

from which we get, by the definition of sup in $H(x, p)$

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Similarly, consider a test function

$$\tilde{\phi}(y) = u_1(\bar{x}) - \frac{\|\bar{x} - y\|^2}{2\varepsilon} - \alpha \left[\left(1 + \|\bar{x}\|^2\right)^p + \left(1 + \|y\|^2\right)^p \right].$$

Note that

$$\Psi(\bar{x}, y; \varepsilon, \alpha, p) = u_2(y) - \tilde{\phi}(y).$$

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Uniqueness via the Comparison Principle

Combine (14.16) and (14.17) gives

$$
\begin{aligned}
&\beta\left(u_1(\bar{x})-u_2(\bar{y})\right) \\
\leq& f(\bar{x},u_\varepsilon)\cdot\left\{\frac{\bar{x}-\bar{y}}{\varepsilon}+\alpha\left[2p\left(1+\|\bar{x}\|^2\right)^{p-1}\bar{x}\right]\right\}+c(\bar{x},u_\varepsilon) \\
&-f(\bar{y},u_\varepsilon)\cdot\left\{\frac{\bar{x}-\bar{y}}{\varepsilon}+\alpha\left[2p\left(1+\|\bar{y}\|^2\right)^{p-1}\bar{y}\right]\right\}-c(\bar{y},u_\varepsilon)+\varepsilon \\
=&\left(f(\bar{x},u_\varepsilon)-f(\bar{y},u_\varepsilon)\right)\cdot\frac{\bar{x}-\bar{y}}{\varepsilon}+c(\bar{x},u_\varepsilon)-c(\bar{y},u_\varepsilon)+\varepsilon \\
&+\alpha 2p\left(1+\|\bar{x}\|^2\right)^{p-1}f(\bar{x},u_\varepsilon)\cdot\bar{x}-\alpha 2p\left(1+\|\bar{y}\|^2\right)^{p-1}f(\bar{y},u_\varepsilon)\cdot\bar{y}.
\end{aligned}
$$

Note that for the first two terms of the above equation,

$$
\left|\left(f(\bar{x},u_\varepsilon)-f(\bar{y},u_\varepsilon)\right)\cdot\frac{\bar{x}-\bar{y}}{\varepsilon}\right|\leq\frac{1}{\varepsilon}\|f(\bar{x},u_\varepsilon)-f(\bar{y},u_\varepsilon)\|\cdot\|\bar{x}-\bar{y}\|\leq\frac{L}{\varepsilon}\|\bar{x}-\bar{y}\|^2,
$$

$$
|c(\bar{x},u)-c(\bar{y},u)|\leq L\|\bar{x}-\bar{y}\|.
$$

In addition, we have by Lipschitz

$$
\|f(x,u_\varepsilon)\|\leq\|f(x,u_\varepsilon)-f(0,u_\varepsilon)\|+M\leq L\|x\|+M.
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&-f(\bar{y},u_\varepsilon)\cdot\left\{\frac{\bar{x}-\bar{y}}{\varepsilon}+\alpha\left[2p\left(1+\|\bar{y}\|^2\right)^{p-1}\bar{y}\right]\right\}-c(\bar{y},u_\varepsilon)+\varepsilon\\
=&\left(f(\bar{x},u_\varepsilon)-f(\bar{y},u_\varepsilon)\right)\cdot\frac{\bar{x}-\bar{y}}{\varepsilon}+c(\bar{x},u_\varepsilon)-c(\bar{y},u_\varepsilon)+\varepsilon\\
&+\alpha 2p\left(1+\|\bar{x}\|^2\right)^{p-1}f(\bar{x},u_\varepsilon)\cdot\bar{x}-\alpha 2p\left(1+\|\bar{y}\|^2\right)^{p-1}f(\bar{y},u_\varepsilon)\cdot\bar{y}.
\end{aligned}
$$

Note that for the first two terms of the above equation,

$$
\left|\left(f(\bar{x},u_\varepsilon)-f(\bar{y},u_\varepsilon)\right)\cdot\frac{\bar{x}-\bar{y}}{\varepsilon}\right|\leq\frac{1}{\varepsilon}\|f(\bar{x},u_\varepsilon)-f(\bar{y},u_\varepsilon)\|\cdot\|\bar{x}-\bar{y}\|\leq\frac{L}{\varepsilon}\|\bar{x}-\bar{y}\|^2,
$$

$$
|c(\bar{x},u)-c(\bar{y},u)|\leq L\|\bar{x}-\bar{y}\|.
$$

In addition, we have by Lipschitz

$$
\|f(x,u_\varepsilon)\|\leq\|f(x,u_\varepsilon)-f(0,u_\varepsilon)\|+M\leq L\|x\|+M.
$$

Uniqueness via the Comparison Principle

Combine (14.16) and (14.17) gives

$$\begin{aligned}
&\beta\left(u_1(\bar{x}) - u_2(\bar{y})\right) \\
\leq & f(\bar{x}, u_\varepsilon) \cdot \left\{ \frac{\bar{x}-\bar{y}}{\varepsilon} + \alpha \left[2p \left(1 + \|\bar{x}\|^2\right)^{p-1} \bar{x} \right] \right\} + c(\bar{x}, u_\varepsilon) \\
& - f(\bar{y}, u_\varepsilon) \cdot \left\{ \frac{\bar{x}-\bar{y}}{\varepsilon} + \alpha \left[2p \left(1 + \|\bar{y}\|^2\right)^{p-1} \bar{y} \right] \right\} - c(\bar{y}, u_\varepsilon) + \varepsilon \\
= & \left(f(\bar{x}, u_\varepsilon) - f(\bar{y}, u_\varepsilon)\right) \cdot \frac{\bar{x}-\bar{y}}{\varepsilon} + c(\bar{x}, u_\varepsilon) - c(\bar{y}, u_\varepsilon) + \varepsilon \\
& + \alpha 2p \left(1 + \|\bar{x}\|^2\right)^{p-1} f(\bar{x}, u_\varepsilon) \cdot \bar{x} - \alpha 2p \left(1 + \|\bar{y}\|^2\right)^{p-1} f(\bar{y}, u_\varepsilon) \cdot \bar{y}.
\end{aligned}$$

Note that for the first two terms of the above equation,

$$\left| \left(f(\bar{x}, u_\varepsilon) - f(\bar{y}, u_\varepsilon)\right) \cdot \frac{\bar{x}-\bar{y}}{\varepsilon} \right| \leq \frac{1}{\varepsilon} \|f(\bar{x}, u_\varepsilon) - f(\bar{y}, u_\varepsilon)\| \cdot \|\bar{x}-\bar{y}\| \leq \frac{L}{\varepsilon} \|\bar{x}-\bar{y}\|^2,$$

$$|c(\bar{x}, u) - c(\bar{y}, u)| \leq L \|\bar{x}-\bar{y}\|.$$

In addition, we have by Lipschitz

$$\|f(x, u_\varepsilon)\| \leq \|f(x, u_\varepsilon) - f(0, u_\varepsilon)\| + M \leq L\|x\| + M.$$

Uniqueness via the Comparison Principle

Thus, the third term has an upper bound

$$|f(x,u)\cdot x| \le \|f(x,u)\|\|x\| \le (L\|x\| + M)\|x\| = L\|x\|^2 + M\|x\|.$$

Then

$$L\|x\|^2 + M\|x\| = \frac{L\|x\|^2 + M\|x\|}{1+\|x\|^2}(1+\|x\|^2) \le \left(L + \frac{M}{2}\right)(1+\|x\|^2),$$

Hence

$$\left(1+\|\bar{x}\|^2\right)^{p-1} f(\bar{x}, u_\varepsilon)\cdot \bar{x} \le \left(L + \frac{M}{2}\right)\left(1+\|\bar{x}\|^2\right)^p,$$

$$\left(1+\|\bar{y}\|^2\right)^{p-1} f(\bar{y}, u_\varepsilon)\cdot \bar{y} \le \left(L + \frac{M}{2}\right)\left(1+\|\bar{y}\|^2\right)^p.$$

Putting all pieces together we have

$$\begin{aligned}
&\beta(u_1(\bar{x}) - u_2(\bar{y}))\\
&\le \frac{L}{\varepsilon}\|\bar{x}-\bar{y}\|^2 + L\|\bar{x}-\bar{y}\| + \alpha p(2L+M)\left\{\left(1+\|\bar{x}\|^2\right)^p + \left(1+\|\bar{y}\|^2\right)^p\right\} + \varepsilon\\
&\le \frac{L}{\varepsilon}\|\bar{x}-\bar{y}\|^2 + L\|\bar{x}-\bar{y}\| + \alpha\beta\left\{\left(1+\|\bar{x}\|^2\right)^p + \left(1+\|\bar{y}\|^2\right)^p\right\} + \varepsilon,
\end{aligned}$$

because $p(2L+M) \le \beta$ via (14.13).

Uniqueness via the Comparison Principle

Thus, the third term has an upper bound

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Hence

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&\le \frac{L}{\varepsilon}\|\bar{x}-\bar{y}\|^2 + L\|\bar{x}-\bar{y}\| + \alpha\beta\left\{\left(1+\|\bar{x}\|^2\right)^p + \left(1+\|\bar{y}\|^2\right)^p\right\} + \varepsilon,
\end{aligned}$$

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Uniqueness via the Comparison Principle

In summary, we have

$$\begin{aligned}
&\beta\Psi(\bar{x},\bar{y};\varepsilon,\alpha,p) \\
&= -\beta u_1(\bar{x}) + \beta u_2(\bar{y}) + \frac{\beta\|\bar{x}-\bar{y}\|^2}{2\varepsilon} + \alpha\beta\left[\left(1+\|\bar{x}\|^2\right)^p + \left(1+\|\bar{y}\|^2\right)^p\right] \\
&\geq -\frac{L}{\varepsilon}\|\bar{x}-\bar{y}\|^2 - L\|\bar{x}-\bar{y}\| + \frac{\beta\|\bar{x}-\bar{y}\|^2}{2\varepsilon} - \varepsilon \\
&\geq -\frac{L}{\varepsilon}\|\bar{x}-\bar{y}\|^2 - L\|\bar{x}-\bar{y}\| - \varepsilon,
\end{aligned}$$

from which the lemma is proved.

[Proof of Theorem 2]
Recall the setup, we assumed by contradiction that there exists a point $z \in \mathbb{R}^d$ and some $\delta > 0$ such that

$$u_1(z) - u_2(z) \geq 2\delta.$$

The goal is to show this cannot happen.

Uniqueness via the Comparison Principle

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&\geq -\frac{L}{\varepsilon}\|\bar{x}-\bar{y}\|^2 - L\|\bar{x}-\bar{y}\| + \frac{\beta\|\bar{x}-\bar{y}\|^2}{2\varepsilon} - \varepsilon \\
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The goal is to show this cannot happen.

Uniqueness via the Comparison Principle

To use the doubling-variable technique, we defined the auxiliary function

$$\Psi(x, y; \varepsilon, \alpha, p) = -u_1(x) + u_2(y) + \frac{\|x - y\|^2}{2\varepsilon} + \alpha \left[(1 + \|x\|^2)^p + (1 + \|y\|^2)^p \right],$$

and chose $(\bar{x}, \bar{y})$ such that Ψ attains its minimum:

$$\Psi(\bar{x}, \bar{y}; \varepsilon, \alpha, p) = \inf_{x,y} \Psi(x, y; \varepsilon, \alpha, p).$$

(14.14) and (14.15) imply that

$$-\beta\delta \geq -\frac{L}{\varepsilon}\|\bar{x} - \bar{y}\|^2 - L\|\bar{x} - \bar{y}\| - \varepsilon. \tag{14.18}$$

We shall prove that

$$\frac{\|\bar{x} - \bar{y}\|^2}{\varepsilon} \to 0, \quad \varepsilon \to 0. \tag{14.19}$$

If this holds, then the right-hand side of (14.18) tends to 0, and since $\delta > 0$ and $\beta > 0$, we get $-\beta\delta \geq 0$, an impossibility — hence $u_1 \leq u_2$.

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Uniqueness via the Comparison Principle

Since $(\bar{x}, \bar{y})$ is the argmin of $\Psi(\bar{x}, \bar{y}; \varepsilon, \alpha, p)$, we have

$$2\Psi(\bar{x}, \bar{y}; \varepsilon, \alpha, p) \leq \Psi(\bar{x}, \bar{x}; \varepsilon, \alpha, p) + \Psi(\bar{y}, \bar{y}; \varepsilon, \alpha, p),$$

Plug in the definition Ψ on both side

$$-2u_1(\bar{x}) + 2u_2(\bar{y}) + \frac{\|\bar{x} - \bar{y}\|^2}{\varepsilon} + 2\alpha\left[\left(1 + \|\bar{x}\|^2\right)^p + \left(1 + \|\bar{y}\|^2\right)^p\right]$$

$$\leq -u_1(\bar{x}) + u_2(\bar{x}) - u_1(\bar{y}) + u_2(\bar{y}) + 2\alpha\left[\left(1 + \|\bar{x}\|^2\right)^p + \left(1 + \|\bar{y}\|^2\right)^p\right],$$

Cancel the identical α-terms and rearrange

$$\frac{\|\bar{x} - \bar{y}\|^2}{\varepsilon} \leq u_1(\bar{x}) - u_1(\bar{y}) + u_2(\bar{x}) - u_2(\bar{y}). \tag{14.20}$$

First the boundedness of u_1 and u_2 implies that

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First the boundedness of u_1 and u_2 implies that

$$\|\bar{x}-\bar{y}\|^2 \to 0, \quad \varepsilon \to 0,$$

Uniqueness via the Comparison Principle

because the right side of (14.20) is bounded. Next, since $(\bar{x}, \bar{y})$ is in a compact set K_ε and $\|\bar{x} - \bar{y}\|^2 \to 0$ (as is just proved), the continuity of u_1 and u_2 in the definition of subsolution and supersolution implies that

$$u_1(\bar{x}) - u_1(\bar{y}) + u_2(\bar{x}) - u_2(\bar{y}) \to 0,$$

as $\varepsilon \to 0$. Thus, we get (14.19) via (14.20). In summary, letting $\varepsilon \to 0$ we get $\beta\delta \leq 0$ from (14.18), a contradiction.

[Proof of Theorem 1]
Theorem 1 says that the value function v of the control problem

$$\beta v(x) - \sup_{u \in U}\{Dv(x) \cdot f(x,u) + c(x,u)\} = 0 \tag{14.5}$$

is the unique bounded viscosity solution of the HJB equation.
We already showed in previous propositions that v is a viscosity supersolution, and a viscosity subsolution. So existence is done.

Now we need uniqueness.

Uniqueness via the Comparison Principle

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Now we need uniqueness.

Uniqueness via the Comparison Principle

Assume for contradiction that there's another bounded viscosity solution v_1. Both v and v_1 satisfy (14.5). We will use the Comparison Principle (Theorem 2):

If v_1 is a subsolution and v is a supersolution, and both are bounded, then $v_1 \leq v$. by Theorem 2 we get

$$v_1 \leq v.$$

But v_1 also satisfies the same PDE, so it is both a subsolution and a supersolution. Hence v can be seen as the subsolution and v_1 the supersolution, giving

$$v \leq v_1.$$

Thus,

$$v = v_1.$$

That is, the value function v is unique among all bounded viscosity solutions.

Remark: Instead of requiring continuity in the definition of the sub- solution and supersolution, the subsolution can be relaxed to be upper semi- continuous and the supersoluton to be lower semi-continuous. Theorem 1 still holds with this relaxation.

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