MF921 Topics in Dynamic Asset Pricing

Week 3

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Part I

Option Pricing Under a Double Exponential Jump Diffusion Model

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This paper aims to show that a double exponential jump diffusion model can lead to an analytic approximation for finite-horizon American options and analytical solutions for popular path-dependent options (such as lookback, barrier, and perpetual American options). We will focus on lookback and barrier options here.

Part II

Pricing Path-Dependent Options with Jump Risk via Laplace Transforms

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Show the analytical solutions for two-dimensional Laplace transforms of barrier option prices, as well as an approximation based on Laplace transforms for the prices of finite-time horizon American options, under a double exponential jump diffusion model.

Background

Recall The Double Exponential Jump Diffusion Model:

$$\frac{dS(t)}{S(t^-)} = \mu dt + \sigma dW(t) + d\left(\sum_{i=1}^{N(t)}(V_i - 1)\right)$$

- $W(t)$: Brownian motion under the real-world measure.
- $N(t)$: Poisson process with rate λ.
- V_i: multiplicative jump sizes, i.i.d. random variables.
- $Y = \log(V)$, the jump sizes follow double exponential law:

$$f_Y(y) = p\eta_1 e^{-\eta_1 y}\mathbf{1}_{y\geq 0} + q\eta_2 e^{\eta_2 y}\mathbf{1}_{y<0}$$

with parameters:

- $p, q \geq 0, p + q = 1$: probabilities of upward/downward jumps.
- $\eta_1 > 1$: rate for upward jumps.
- $\eta_2 > 0$: rate for downward jumps.

Background Con.

For option pricing, we switch to a risk-neutral measure P^*, so that the discounted price process is a martingale:

$$E^{P^*}[e^{-rt}S(t)] = S(0)$$

Under P^*, the dynamics adjust:

$$\frac{dS(t)}{S(t^-)} = (r - \lambda^*(t)\zeta^*)dt + \sigma dW^*(t) + d\left(\sum_{i=1}^{N^*(t)} (V_i^* - 1)\right)$$

where:

- $W^*(t)$: Brownian motion under P^*,
- $N^*(t)$: Poisson process with intensity λ^*,
- $V^* = e^{Y^*}$: jump multiplier with new parameters $(p^*, q^*, \eta_1^*, \eta_2^*)$,
- $\zeta^* = E^{P^*}[V^*] - 1 = \frac{p^*\eta_1^*}{\eta_1^*-1} + \frac{q^*\eta_2^*}{\eta_2^*+1} - 1$ is mean percentage jump size.

The log-price process:

$$X(t) = \log\left(\frac{S(t)}{S(0)}\right) = \left(r - \frac{1}{2}\sigma^2 - \lambda^*\zeta^*\right)t + \sigma W^*(t) + \sum_{i=1}^{N^*(t)} Y_i^*, \quad X(0) = 0$$

Intuition of the Pricing Formula

- Without jumps, the model reduces to geometric Brownian motion. Pricing American, barrier, and lookback options is straightforward. First passage times are tractable, and closed-form formulas are well known (as what we show in last week).
- With jumps, however, analytical pricing becomes difficult because the process can cross barriers by jumping over them (Overshoot Problem).

Intuition of the Pricing Formula Con

Define the first passage time:

$$\tau_b := \inf\{t \geq 0 : X(t) \geq b\}, \quad b > 0$$

In a jump diffusion, when the process crosses b, it may overshoot: $X(\tau_b) - b > 0$.

This overshoot creates complications to compute the distribution of the first passage times analytically:

- Need the distribution of overshoot $X(\tau_b) - b$.
- Need the joint dependence between overshoot and τ_b.
- Need correlation between overshoot and the terminal state $X(T)$

Note: Double exponential distribution assumption has a memoryless property, this property simplifies the overshoot distribution and allows tractable Laplace transforms of first passage times.

Some Useful Formulas

The double exponential jump diffusion process is a special case of Lévy processes with two-sided jumps, whose characteristic exponent admits the (unique) representation:

$$\phi(\theta) = E[e^{i\theta X_1}] = \exp\left\{ i\gamma\theta - \frac{1}{2}A\theta^2 + \int_{-\infty}^{\infty} (e^{i\theta y} - 1 - i\theta y I_{\{|y|\leq 1\}})\Pi(dy) \right\}$$

where the generating triplet (γ, A, Π) is given by:

- $A = \sigma^2$
- $\Pi(dy) = \lambda \cdot f_Y(y)dy = \lambda p \eta_1 e^{-\eta_1 y} I_{\{y\geq 0\}} dy + \lambda q \eta_2 e^{\eta_2 y} I_{\{y<0\}} dy$
- $\gamma = \mu + \lambda E[V I_{\{|V|\leq 1\}}] = \mu + \lambda p \left(\frac{1-e^{-\eta_1}}{\eta_1} - e^{-\eta_1} \right) - \lambda q \left(\frac{1-e^{-\eta_2}}{\eta_2} - e^{-\eta_2} \right)$

Some Useful Formulas Con

Moment Generating Function of the log-price process, $X(t)$:

$$\mathbb{E}^* \left[e^{\theta X(t)} \right] = \exp\{G(\theta)t\}$$

Where the function $G(\cdot)$ is defined as:

$$G(x) = x\left(r - \frac{1}{2}\sigma^2 - \lambda\zeta\right) + \frac{1}{2}x^2\sigma^2 + \lambda\left(\frac{p\eta_1}{\eta_1 - x} + \frac{q\eta_2}{\eta_2 + x} - 1\right)$$

Note: Lemma 3.1 in Kou and Wang (2003) shows that the equation $G(x) = \alpha, \forall\alpha > 0$, has exactly four roots: $\beta_{1,\alpha}$, $\beta_{2,\alpha}$, $-\beta_{3,\alpha}$, and $-\beta_{4,\alpha}$, where:

$$0 < \beta_{1,\alpha} < \eta_1 < \beta_{2,\alpha} < \infty$$
$$0 < \beta_{3,\alpha} < \eta_2 < \beta_{4,\alpha} < \infty$$

These roots determine the structure of Laplace transforms for first passage times.

Some Useful Formulas Con

Infinitesimal Generator of the log-price process, $X(t)$:

$$(\mathcal{L}V)(x) = \frac{1}{2}\sigma^2 V''(x) + \left(r - \frac{1}{2}\sigma^2 - \lambda\zeta\right) V'(x) + \lambda \int_{-\infty}^{\infty} \left(V(x+y) - V(x)\right) f_Y(y)\, dy$$

The generator describes how expectations of functions of $X(t)$ evolve in time:

$$\frac{d}{dt}\mathbb{E}[V(X_t)] = \mathbb{E}[(\mathcal{L}V)(X_t)]$$

They provide the mathematical foundation to derive option pricing formulas.

Lookback Options

Consider a lookback put option with an initial "prefixed maximum" $M \geq S(0)$:

$$
\begin{aligned}
LP(T) &= \mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}\left(\max\{M, \max_{0\leq t\leq T} S(t)\} - S(T)\right)\right] \\
&= \mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}\max\{M, \max_{0\leq t\leq T} S(t)\}\right] - S(0)
\end{aligned}
$$

You need the joint distribution of $\max S(t)$ and $S(T)$, which is complicated for jump processes. Laplace transforms convert a complicated path integral over time into a function of roots of $G(x)$ which we can solve algebraically.

Lookback Options Con.

Theorem:

Using the notations $\beta_{1,\alpha+r}$ and $\beta_{2,\alpha+r}$ as in early silde, the Laplace transform of the lookback put is given by:

$$\hat{L}(T) = \int_0^\infty e^{-\alpha T} \mathrm{LP}(T) dT = \frac{S(0)A_\alpha}{C_\alpha} \left(\frac{S(0)}{M} \right)^{\beta_{1,\alpha+r}-1} + \frac{S(0)B_\alpha}{C_\alpha} \left(\frac{S(0)}{M} \right)^{\beta_{2,\alpha+r}-1} + \frac{M}{\alpha+r} - \frac{S(0)}{\alpha}$$

For all $\alpha > 0$; here:

$$A_\alpha = \frac{(\eta_1 - \beta_{1,\alpha+r})\beta_{2,\alpha+r}}{\beta_{1,\alpha+r} - 1}$$
$$B_\alpha = \frac{(\beta_{2,\alpha+r} - \eta_1)\beta_{1,\alpha+r}}{\beta_{2,\alpha+r} - 1}$$
$$C_\alpha = (\alpha + r)\eta_1(\beta_{2,\alpha+r} - \beta_{1,\alpha+r})$$

Laplace inversion:

$$LP(T) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\alpha T}\hat{L}(\alpha)d\alpha$$

This is the Bromwich inversion integral, intractable in closed form. So we approximate it numerically.
The most widely used numerical methods for Laplace transform inversion is Gaver-Stehfest (GS) algorithm. Suppose you know the Laplace transform of a function $f(t)$:

$$F(\alpha) = \int_0^\infty e^{-\alpha t} f(t)dt$$

The GS method approximates $f(t)$ by evaluating $F(\alpha)$ at carefully chosen points along the real line.

Lookback Options Con.

The formula is:

$$f(t) \approx \frac{\ln(2)}{t} \sum_{k=1}^{n} w_k F\left(\frac{k\ln(2)}{t}\right)$$

where:

- n is an even integer (typically converges nicely even for n between 5 and 10).
- w_k are weights (depending only on n and k):

$$w_k = (-1)^{\frac{n}{2}+k} \sum_{j=\lceil k/2 \rceil}^{\min(k,n/2)} \frac{j^{n/2}(2j)!}{(\frac{n}{2}-j)!j!(j-1)!(k-j)!(2j-k)!}$$

Intuition: The algorithm generates a sequence $f_n(x)$ such that $f_n(x) \to f(x), n \to \infty$

Lookback Options Con.

Another method:

We shall invert the transform in the complex domain by using the Euler inversion algorithm (EUL) developed by Abate and Whitt (1995), rather than in the real domain by the Gaver–Stehfest algorithm (GS).

The main reason for this is that the EUL inversion (which is carried out in the complex-domain) does not require the high numerical precision of the GS: A precision of 12 digits will suffice for the EUL, compared with the 80 digits accuracy required by the GS. The EUL algorithm is made possible partly due to an explicit formula for the roots of $G(x)$.

The statistical results show that the difference between the EUL and GS results are small. So the the EUL implementation is preferable, since it's simple to implement, and it converges fast without requiring high numerical precision as in the GS.

Barrier Options

Consider the up-and-in call (UIC) option with the barrier level H ($H > S(0)$):

$$UIC = E^{\mathbb{P}^*}[e^{-rT}(S(T) - K)^+ I\{\max_{0 \leq t \leq T} S(t) \geq H\}]$$

For any given probability P, define:

$$\Psi(\mu, \sigma, \lambda, p, \eta_1, \eta_2; a, b, T) := P[Z(T) \geq a, \max_{0 \leq t \leq T} Z(t) \geq b]$$

where under $P, Z(t)$ is a double exponential jump diffusion process with drift μ, volatility σ, and jump rate λ, i.e., $Z(t) = \mu t + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i$, and Y has a double exponential distribution with density $f_Y(y) \sim p \cdot \eta_1 e^{-\eta_1 y} 1_{\{y \geq 0\}} + q \cdot \eta_2 e^{y \eta_2} 1_{\{y < 0\}}$.

Barrier Options Con.

Theorem:

The price of the UIC option is obtained as:

$$\begin{aligned}\text{UIC} =& S(0)\Psi\left(r+\frac{1}{2}\sigma^2-\lambda\zeta,\sigma,\tilde{\lambda},\tilde{p},\tilde{\eta}_1,\tilde{\eta}_2;\ \log\left(\frac{K}{S(0)}\right),\log\left(\frac{H}{S(0)}\right),T\right)\\ &-Ke^{-rT}\cdot\Psi\left(r-\frac{1}{2}\sigma^2-\lambda\zeta,\sigma,\lambda,p,\eta_1,\eta_2;\ \log\left(\frac{K}{S(0)}\right),\log\left(\frac{H}{S(0)}\right),T\right)\end{aligned}$$

where $\tilde{p}=(p/(1+\zeta))\cdot(\eta_1/(\eta_1-1)),\tilde{\eta}_1=\eta_1-1,\tilde{\eta}_2=\eta_2+1,\tilde{\lambda}=\lambda(\zeta+1)$, with $\zeta=E^{P^*}[V]-1=\frac{p\eta_1}{\eta_1-1}+\frac{q\eta_2}{\eta_2+1}-1$. The Laplace transforms of Ψ is computed explicitly in Kou and Wang (2003).

Barrier Options

Rewrite the pricing formula of up-and-in call option(UIC) in early silde:

$$UIC(k,T) = E^{\mathbb{P}^*}\left[e^{-rT}\left(S(T)-e^{-k}\right)^+ I[\tau_b < T]\right]$$

where $H > S(0)$ is the barrier level, $k = -\log(K)$ the transformed strike and $b = \log(H/S(0))$. In previous paper, we obtain:

$$UIC(k,T) = S(0)\tilde{\Psi}_{UI}(k,T) - Ke^{-rT}\Psi_{UI}(k,T)$$

where:

$$\Psi_{UI}(k,T) = P^*(S(T) \geq e^{-k}, \tau_b < T), \quad \widetilde{\Psi}_{UI}(k,T) = \widetilde{P}(S(T) \geq e^{-k}, \tau_b < T)$$

Remark: Here we will relies on a two-dimensional Laplace transform for botth the option price and the probabilities. The formulae after doing two-dimensional transforms become much simpler than the one-dimensional formulae in Kou and Wang (2003), which involve many special functions.

Barrier Options Con.

Theorem: For ξ and α such that $0 < \xi < \eta_1 - 1$ and $\alpha > \max(G(\xi+1) - r, 0)$ (such a choice of ξ and α is possible for all small enough ξ as $G(1) - r = -\delta < 0$). The Laplace transform with respect to k and T of $UIC(k,T)$ is given by

$$\begin{aligned}\tilde{f}_{UIC}(\xi,\alpha) &= \int_0^\infty \int_{-\infty}^\infty e^{-\xi k - \alpha T} UIC(k,T) dk dT \\ &= \frac{H^{\xi+1}}{\xi(\xi+1)} \frac{1}{r+\alpha - G(\xi+1)} \left(A(r+\alpha) \frac{\eta_1}{\eta_1 - (\xi+1)} + B(r+\alpha) \right)\end{aligned}$$

where

$$\begin{aligned}A(h) &:= E^*\left[e^{-h\tau_b} \mathbf{1}_{\{X(\tau_b) > b\}}\right] = \frac{(\eta_1 - \beta_{1,h})(\beta_{2,h} - \eta_1)}{\eta_1(\beta_{2,h} - \beta_{1,h})} \left[e^{-b\beta_{1,h}} - e^{-b\beta_{2,h}}\right] \\ B(h) &:= E^*\left[e^{-h\tau_b} \mathbf{1}_{\{X(\tau_b = b)\}}\right] = \frac{\eta_1 - \beta_{1,h}}{\beta_{2,h} - \beta_{1,h}} e^{-b\beta_{1,h}} + \frac{\beta_{2,h} - \eta_1}{\beta_{2,h} - \beta_{1,h}} e^{-b\beta_{2,h}}\end{aligned}$$

with $b = \log(H/S(0))$.

Barrier Options Con.

If $0 < \xi < \eta_1$ and $\alpha > \max(G(\xi), 0)$ (again this choice of ξ and α is possible for all ξ small enough as $G(0) = 0$), then the Laplace transform with respect to k and T of $\Psi_{UI}(k, T)$ is:

$$\begin{aligned}
\tilde{f}_{\Psi_{UI}}(\xi, \alpha) &= \int_0^{\infty} \left(\int_{-\infty}^{\infty} e^{-\xi k - \alpha T} \Psi_{UI}(k, T) dk \right) dT \\
&= \frac{H^{\xi}}{\xi} \frac{1}{\alpha - G(\xi)} \left(A(\alpha) \frac{\eta_1}{\eta_1 - \xi} + B(\alpha) \right)
\end{aligned}$$

The Laplace transforms with respect to k and T of $\tilde{\Psi}_{UI}(k, T)$ is given similarly with $\tilde{G}$ replacing G and the functions $\tilde{A}$ and $\tilde{B}$ defined similarly. To perform the inversion, They use the two-sided Euler method(EUL) as in Petrella (2004). Compare with GS method, EUL use standard double precision, get stable answers, and convergence is much faster.

Barrier Options Con.

Statistical results:

The author price up-and-in calls using the two-dimensional transform herein and compare the results with the one-dimensional transform in Kou and Wang (2003) (KW from now on and in the table) and Monte Carlo simulation (MC).

For the two-dimensional transform, the price obtained in both inverting $\tilde{f}_{UIC}(\xi, \alpha)$ and inverting $\tilde{f}_{\Psi_{UI}}(\xi, \alpha)$.

The two-dimensional Laplace transform approach is more efficient, simpler, and numerically stable compared to Kou-Wang's GS inversion method, while achieving accuracy comparable to or better than Monte Carlo at a fraction of the computational cost.