MF921 Topics in Dynamic Asset Pricing

Week 6

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Chapter 22 American Options (II)

Brownian Market Setup

Given a complete probablity space $(\Omega, \mathcal{F}, \mathbb{P})$. $W(t) = (W_1(t), \ldots, W_d(t))^\top$, independent d-dimensional Brownian motion. The filtraion $\mathcal{F}_t^W = \sigma(W(s) : 0 \leq s \leq t)$ which is complete and right-continuous.

A financial market $\mathcal{M}$ with 1 bond and d stocks under a finite horizon $[0, T]$:

$$dS_0(t) = r(t) S_0(t) dt, \quad S_0(0) = 1$$

$$dS_i(t) = S_i(t) \left(b_i(t)\, dt + \sum_{j=1}^{d} \sigma_{ij}(t)\, dW_j(t) \right), \quad \text{for } i \in 1, 2...., d$$

- $r(t)$: interest rate
- $b(t) = (b_1, \ldots, b_d)$: appreciation rates
- $\sigma(t) = (\sigma_{ij}(t))$: volatility matrix
- $r(t)$, $b(t)$ and $\sigma(t)$ all progressively measurable with respect to $\{\mathcal{F}_t\}$ and bounded uniformly in $(t, \omega) \in [0, T] \times \Omega$.

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Introducing auxiliary processes

Relative risk (Sharpe ratio):

$$\theta(t) = \sigma^{-1}(t)\,(b(t) - r(t)\mathbf{1})\,, \quad \mathbf{1} = (1, 1, \ldots\ldots, 1)^T$$

Exponential martingale (RN derivative):

$$Z(t) = \exp\left(-\int_0^t \theta^\top(s)\,dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds\right)$$

Discount factor:

$$\gamma(t) = \exp\left(-\int_0^t r(s)\,ds\right)$$

Brownian motion with drift:

$$W_0(t) = W(t) + \int_0^t \theta(s)\,ds, \quad 0 \le t \le T$$

$\sigma(t)$ invertible, inverses bounded. Ensures bounded $\theta(t)$ and $Z(t)$ is a true martingale.

These tools set up the risk-neutral framework for pricing options.

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Self-Financing Condition

A portfolio process is defined as $\pi(\cdot) = (\pi_1(\cdot), \dots, \pi_d(\cdot))$, where $\pi_i(t) = \phi_i(t)S_i(t)$ means that total amount of money invested in the ith risky asset. The self-financing condition leads to the wealth equation:

$$\gamma(t)X(t) = x - \int_0^t \gamma(s)dC(s) + \int_0^t \gamma(s)\pi^\top(s)\sigma(s)dW_0(s)$$

where $C(t)$ is the cumulative consumption process, for the stochastic integral to be well-defined:

$$\int_0^T \|\pi(t)\|^2 dt < \infty$$

Risk-Neutral Probability via Girsanov Theorem

Definition of Risk-Neutral Measure:

$$\mathbb{P}^0(A) := \mathbb{E}[Z(T)1_A], \quad A \in \mathcal{F}_T$$

$Z(T)$ is the exponential martingale. By Girsanov's Theorem, $W_0(t) = W(t) + \int_0^t \theta(s)ds$ is a standard Brownian motion under $\mathbb{P}^0$.

Thus rewirte the wealth process:

$$N_0(t) = \gamma(t)X(t) + \int_0^t \gamma(s)dC(s)$$
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A continuous $\mathbb{P}^0$-local martingale.

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Risk-Neutral Probability via Girsanov Theorem

Stock dynamics under risk-neutral measure:

$$dS_i(t) = S_i(t)\left[b_i(t)dt - \sum_{j=1}^{d}\sigma_{ij}(t)\theta_j(t)dt + \sum_{j=1}^{d}\sigma_{ij}(t)dW_0^{(j)}(t)\right]$$

Since $b(t) - \sigma(t)\theta(t) = r(t)\mathbf{1}$:

$$dS_i(t) = S_i(t)\left[r(t)dt + \sum_{j=1}^{d}\sigma_{ij}(t)dW_0^{(j)}(t)\right], \quad i = 1, \ldots, d$$

Since $d\gamma(t) \cdot dS_i(t) = 0$, apply Itô:

$$\begin{aligned} d(\gamma(t)S_i(t)) &= S_i(t)dr(t) + \gamma(t)dS_i(t) \\ &= \gamma(t)S_i(t)\sum_{j=1}^{d}\sigma_{ij}(t)dW_0^{(j)}(t) \end{aligned}$$

Hence, the discounted stock processes $\gamma(\cdot)S_i(\cdot)$ are local martingales. This also confirms our intuition that every asset $S_i(t)$ should have a growth rate $r(t)$ in the risk neutral world.

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Doubling Strategies and Admissibility

Doubling Strategy: Double investment after each loss, leads to arbitrarily large wealth at T. Requires wealth process $X(t)$ unbounded from below. Need to exclude, because creates arbitrage opportunities and violates no-arbitrage principle.

A uniform boundedness condition is needed to prevent the doubling strategy. Wealth must satisfy:

$$X^{x,\pi,C}(t) \geq -\Lambda, \quad 0 \leq t \leq T$$

With $\mathbb{E}^0[\Lambda^p] < \infty$ for some $p > 1$.

Supermartingale Property:

If (π, C) is admissible, $N_0(t)$ is bound below. $N_0(t)$ is a $\mathbb{P}^0$-supermartingale. Consequently:

$$\mathbb{E}^0\left[\gamma(T)X(T) + \int_0^T \gamma(t)dC(t)\right] \leq x$$

Expected discounted terminal wealth and consumption less than or equal to initial wealth to ensures no arbitrage.

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Dominance-Free Interval for American Options

At $t = 0$, two agents agree: **Seller** promises to pay the buyer $\psi(\tau, \omega) \geq 0$ at a stopping time $\tau \in \mathcal{S}$ (chosen by the buyer). **Buyer** pays upfront amount $x \geq 0$ to seller.

$\psi(t, \omega)$: F-adapted, continuous process representing possible payoff. Integrability condition:

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} \left(\gamma_0(t)\psi(t)\right)^{1+\varepsilon}\right] < \infty, \quad \varepsilon > 0$$

ensures finite expected discounted payoff. Such a process $\psi(\cdot)$ defines an American Contingent Claim (ACC).

Question: *What should the buyer pay at $t = 0$ for this option?* i.e. find the arbitrage-free price allowing both sides to hedge. We will look at the situation of each agent separately

Seller receives x at $t = 0$ and seeks a self-financing portfolio $(\hat{\pi}, \hat{C})$ such that he can always meet the buyer's demand:

$$X^{x,\hat{\pi},\hat{C}}(\tau) \geq \psi(\tau), \quad \forall \tau \in \mathcal{S}, \text{ a.s.}$$

Dominance-Free Interval for American Options

The smallest initial capital x that makes this possible is

$$h_{up} := \inf \left\{ x \geq 0 : \exists (\hat{\pi}, \hat{C}) \in \mathcal{A}_0(x) \text{ s.t. } X^{x,\hat{\pi},\hat{C}}(\tau) \geq \psi(\tau), \, \forall \tau \in \mathcal{S} \right\}$$

Buyer pays x at $t = 0 \Rightarrow$ and searches for a stopping time $\hat{\tau}$ and portfolio $(\hat{\pi}, \hat{C})$ such that the payment that he receives allows him to recover the debt he incurred at $t = 0$ by purchasing the ACC:

$$X^{-x,\hat{\pi},\hat{C}}(\hat{\tau}) + \psi(\hat{\tau}) \geq 0, \quad \text{a.s.}$$

The largest x that allows this is

$$h_{low} := \sup \left\{ x \geq 0 : \exists (\hat{\pi}, \hat{C}) \in \mathcal{A}_0(-x) \text{ s.t. } X^{-x,\hat{\pi},\hat{C}}(\hat{\tau}) + \psi(\hat{\tau}) \geq 0 \right\}$$

Note: Seller needs to hedge against any stopping time $\tau \in \mathcal{S}$, whereas the buyer need only hedge for some stopping time $\tilde{\tau} \in \mathcal{S}$. We need to justify defined "upper" and "lower" hedging price.

Dominance-Free Interval for American Options

Consider the decreasing function

$$u(t) =: \sup_{\tau \in S_{t,T}} E^0[\gamma_0(\tau)\psi(\tau)], \quad 0 \le t \le T$$

We must have

$$0 \le \psi(0) \le h_{\text{low}} \le u(0) \le h_{\text{up}} \le \infty$$

If h_{up} is empty set, then $h_{\text{up}} = \infty$ and $h_{\text{up}} \ge u(0)$ holds trivially; if not, let x be an arbitrary element of this set. Under the risk-neutral measure P^0, from the supermartingale property of discounted wealth and seller hedging condition.

$$x \ge E^0 \left[\gamma_0(\tau) X^{x,\tilde{\pi},\tilde{C}}(\tau) + \int_{(0,\tau]} \gamma_0(t) d\tilde{C}(t) \right] \ge E^0[\gamma_0(\tau)\psi(\tau)]; \quad \forall \tau \in S$$

Taking the supremum over all stopping times gives $x \ge \sup_{\tau \in \mathcal{S}} E^0[\gamma_0(\tau)\psi(\tau)] = u(0)$. Thus, any admissible initial capital x that allows perfect hedging must satisfy $x \ge u(0)$. Therefore $h_{\text{up}} \ge u(0)$.

Dominance-Free Interval for American Options

On the other hand, the number $\psi(0)$ clearly belongs to the set of h_{low} (take $x = \psi(0) \geq 0$, $\tilde{\pi} = 0$, $\tilde{\pi}(\cdot) \equiv 0$, $\tilde{C}(\cdot) \equiv 0$). Applying again the discounted wealth

supermartingale property and buyer hedging condition, we get:

$$-x \geq E^0 \left[\gamma_0(\tilde{\tau}) X^{-x,\tilde{\pi},\tilde{C}}(\tilde{\tau}) + \int_{(0,\tilde{\tau}]} \gamma_0(t) d\tilde{C}(t) \right] \geq -E^0[\gamma_0(\tilde{\tau})\psi(\tilde{\tau})] \geq -u(0)$$

$x \leq u(0)$, therefore $h_{low} \leq u(0)$.
The integrability condition, and the boundedness of the process $\theta(\cdot)$ implies that

$$\mathsf{E}^0 \left[\sup_{0 \leq t \leq T} (\gamma_0(t)\psi(t)) \right] = \mathsf{E} \left[Z_0(T) \cdot \sup_{0 \leq t \leq T} (\gamma_0(t)\psi(t)) \right]$$
$$\leq (\mathsf{E}(Z_0(T))^q)^{1/q} \cdot \left(\mathsf{E} \sup_{0 \leq t \leq T} (\gamma_0(t)\psi(t))^p \right)^{1/p} < \infty$$

with $p = 1 + \epsilon > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. In particular, $u(0) < \infty$. It can be shown that $[h_{\mathrm{low}}, h_{\mathrm{up}}]$ forms a dominance-free interval.

Theorem In an ideal market

$$h_{\mathsf{up}} = h_{\mathsf{low}} = u(0) =: \sup_{\tau \in S} E^0[\gamma_0(\tau)\psi(\tau)] < \infty$$

Furthermore, there exists a pair $(\hat{\pi}, \hat{C}) \in A_0(u(0))$ such that, with

$$\hat{X}_0(t) := \frac{1}{\gamma_0(t)} \text{ess} \sup_{\tau \in S_{t,T}} E^0[\gamma_0(\tau)\psi(\tau)|F(t)], \quad 0 \leq t \leq T$$

$$\check{\sigma} =: \inf\{t \in [0,T)/\hat{X}_0(t) = \psi(t)\} \wedge T$$

and $\hat{\pi}(\cdot) \equiv -\hat{\pi}(\cdot)$, we have almost surely:

$$X^{u(0),\hat{\pi},\hat{C}}(t) = \hat{X}_0(t) \geq \psi(t), \quad \forall\, 0 \leq t \leq T$$
$$X^{u(0),\hat{\pi},\hat{C}}(t) = -X^{-u(0),\hat{\pi},0}(t) > \psi(t), \quad \forall\, 0 \leq t < tilde\tau$$
$$\hat{C}(\tilde{\tau}) = 0, \quad X^{u(0),\hat{\pi},\hat{C}}(\tilde{\tau}) = -X^{-u(0),\hat{\pi},0}(\tilde{\tau}) = \psi(\tilde{\tau})$$