MF921 Topics in Dynamic Asset Pricing

Week 4

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Background

Recall The Double Exponential Jump Diffusion Model:

$$\frac{dS(t)}{S(t^-)} = \mu dt + \sigma dW(t) + d\left(\sum_{i=1}^{N(t)}(V_i - 1)\right)$$

- $W(t)$: Brownian motion under the real-world measure.
- $N(t)$: Poisson process with rate λ.
- V_i: multiplicative jump sizes, i.i.d. random variables.
- $Y = \log(V)$, the jump sizes follow double exponential law:

$$f_Y(y) = p\eta_1 e^{-\eta_1 y}\mathbf{1}_{y\geq 0} + q\eta_2 e^{\eta_2 y}\mathbf{1}_{y<0}$$

with parameters:

- $p, q \geq 0, p + q = 1$: probabilities of upward/downward jumps.
- $\eta_1 > 1$: rate for upward jumps.
- $\eta_2 > 0$: rate for downward jumps.

Background Con.

For option pricing, we switch to a risk-neutral measure P^*, so that the discounted price process is a martingale:

$$E^{P^*}[e^{-rt}S(t)] = S(0)$$

Under P^*, the dynamics adjust:

$$\frac{dS(t)}{S(t^-)} = (r - \lambda^*(t)\zeta^*)dt + \sigma dW^*(t) + d\left(\sum_{i=1}^{N^*(t)} (V_i^* - 1)\right)$$

where:

- $W^*(t)$: Brownian motion under P^*,
- $N^*(t)$: Poisson process with intensity λ^*,
- $V^* = e^{Y^*}$: jump multiplier with new parameters $(p^*, q^*, \eta_1^*, \eta_2^*)$,
- $\zeta^* = E^{P^*}[V^*] - 1 = \frac{p^*\eta_1^*}{\eta_1^*-1} + \frac{q^*\eta_2^*}{\eta_2^*+1} - 1$ is mean percentage jump size.

The log-price process:

$$X(t) = \log\left(\frac{S(t)}{S(0)}\right) = \left(r - \frac{1}{2}\sigma^2 - \lambda^*\zeta^*\right)t + \sigma W^*(t) + \sum_{i=1}^{N^*(t)} Y_i^*, \quad X(0) = 0$$

Some Useful Formulas Con

Moment Generating Function of the log-price process, $X(t)$:

$$\mathbb{E}^{\mathbb{P}^*}\left[e^{\theta X(t)}\right] = \exp\{G(\theta)t\}$$

Where the function $G(\cdot)$ is defined as:

$$G(x) = x\left(r - \frac{1}{2}\sigma^2 - \lambda\zeta\right) + \frac{1}{2}x^2\sigma^2 + \lambda\left(\frac{p\eta_1}{\eta_1 - x} + \frac{q\eta_2}{\eta_2 + x} - 1\right)$$

Note: Lemma 3.1 in Kou and Wang (2003) shows that the equation $G(x) = \alpha, \forall\alpha > 0$, has exactly four roots: $\beta_{1,\alpha}$, $\beta_{2,\alpha}$, $-\beta_{3,\alpha}$, and $-\beta_{4,\alpha}$, where:

$$0 < \beta_{1,\alpha} < \eta_1 < \beta_{2,\alpha} < \infty$$
$$0 < \beta_{3,\alpha} < \eta_2 < \beta_{4,\alpha} < \infty$$

These roots determine the structure of Laplace transforms for first passage times.

Some Useful Formulas Con

Infinitesimal Generator of the log-price process, $X(t)$:

$$(\mathcal{L}V)(x) = \frac{1}{2}\sigma^2 V''(x) + \left(r - \frac{1}{2}\sigma^2 - \lambda\zeta\right) V'(x) + \lambda \int_{-\infty}^{\infty} \left(V(x+y) - V(x)\right) f_Y(y)\, dy$$

The generator describes how expectations of functions of $X(t)$ evolve in time:

$$\frac{d}{dt}\mathbb{E}[V(X_t)] = \mathbb{E}[(\mathcal{L}V)(X_t)]$$

They provide the mathematical foundation to derive option pricing formulas.

Part I

Option Pricing Under a Double Exponential Jump Diffusion Model

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Proof of two Theorems. The Laplace transform of lookback option and barrier option.

Lookback Options

Consider a lookback put option with an initial "prefixed maximum" $M \geq S(0)$:

$$
\begin{aligned}
LP(T) &= \mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}\left(\max\{M, \max_{0\leq t\leq T} S(t)\} - S(T)\right)\right] \\
&= \mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}\max\{M, \max_{0\leq t\leq T} S(t)\}\right] - S(0)
\end{aligned}
$$

You need the joint distribution of $\max S(t)$ and $S(T)$, which is complicated for jump processes. Laplace transforms convert a complicated path integral over time into a function of roots of $G(x)$ which we can solve algebraically.

Lookback Options Con.

Theorem:

Using the notations $\beta_{1,\alpha+r}$ and $\beta_{2,\alpha+r}$ as in early silde, the Laplace transform of the lookback put is given by:

$$\hat{L}(T) = \int_0^\infty e^{-\alpha T}\mathrm{LP}(T)dT = \frac{S(0)A_\alpha}{C_\alpha}\left(\frac{S(0)}{M}\right)^{\beta_{1,\alpha+r}-1} + \frac{S(0)B_\alpha}{C_\alpha}\left(\frac{S(0)}{M}\right)^{\beta_{2,\alpha+r}-1} + \frac{M}{\alpha+r} - \frac{S(0)}{\alpha}$$

For all $\alpha > 0$; here:

$$A_\alpha = \frac{(\eta_1 - \beta_{1,\alpha+r})\beta_{2,\alpha+r}}{\beta_{1,\alpha+r} - 1}$$
$$B_\alpha = \frac{(\beta_{2,\alpha+r} - \eta_1)\beta_{1,\alpha+r}}{\beta_{2,\alpha+r} - 1}$$
$$C_\alpha = (\alpha + r)\eta_1(\beta_{2,\alpha+r} - \beta_{1,\alpha+r})$$

Proof

Lemma : $\lim_{y\to\infty} e^{y}\mathbb{P}^*[M_X(T)\geq y]=0,\ \forall T\geq 0.\ M_X(T):=\max_{0\leq t\leq T} X(t)$

[Proof]

Given $\theta\in(-\eta_2,\ \eta_1)$, $\mathbb{E}^{\mathbb{P}^*}[e^{\theta X(t)}]<\infty$, by stationary independent increments we can get that the process $\{e^{\theta X(t)-G(\theta)t}; t\geq 0\}$ is a martingale.

Observe that $G(x)$ is continuous and $G(1)=r>0$, thus we can fix an $\theta\in(1,\ \eta_1)$ such that $G(\theta)>0$. Let $\tau_y=\inf\{t\geq 0: X(t)\geq y\}$. By Optional Sampling Theorem:

$$1=\mathbb{E}^{\mathbb{P}^*}[M_{\tau_y\wedge T}]\mathbb{E}^{\mathbb{P}^*}\left[e^{\theta X(\tau_y\wedge T)}e^{-G(\theta)(\tau_y\wedge T)}\right]\geq e^{\theta y}e^{-G(\theta)T}\mathbb{P}^*(\tau_y\leq T)$$

Thus $e^{\theta y}\mathbb{P}^*(\tau_y\leq T)\leq e^{G(\theta)T}$. Since $\theta>1$, then we have:

$$e^{y}\mathbb{P}^*(M_X(T)\geq y)=e^{(1-\theta)y}\left[e^{\theta y}\mathbb{P}^*(\tau_y\leq T)\right]\leq e^{(1-\theta)y}e^{G(\theta)T}\xrightarrow{y\to\infty}0$$

This will use to justify the boundary term vanishing in the integration-by-parts step later.

Proof Con

Given $s = S(0)$ and M are constants, $\max\limits_{0\le t\le T} S(t) = se^{M_X(T)}$, the lookback put as:

$$LP(T) = \mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}\max\{M, se^{M_X(T)}\}\right] - s$$

Letting $z = \log(M/s) \ge 0$, define:

$$\begin{aligned} L(s, M; T) :&= \mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}\max\{M, se^{M_X(T)}\}\right] \\ &= s\mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}\max\{e^z, e^{M_X(T)}\}\right] \\ &= s\mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}\left(e^{M_X(T)} - e^z\right)\mathbf{1}_{\{M_X(T)\ge z\}}\right] + se^z e^{-rT} \end{aligned}$$

Proof Con

Integration by parts:

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}e^{M_X(T)}\mathbf{1}_{\{M_X(T)\geq z\}}\right] &= e^{-rT}\int_z^\infty e^y d(1-\mathbb{P}^*[M_X(T)\geq y]) \\
&= -e^{-rT}\int_z^\infty e^y d\mathbb{P}^*[M_X(T)\geq y] \\
&= -e^{-rT}\left\{\left(-e^y\mathbb{P}^*[M_X(T)\geq y]\right)\Big|_0^\infty - \int_z^\infty \mathbb{P}^*[M_X(T)\geq y]e^y dy\right\} \\
&= -e^{-rT}\left\{-e^z\mathbb{P}^*[M_X(T)\geq z] - \int_z^\infty \mathbb{P}^*[M_X(T)\geq y]e^y dy\right\} \\
&= \mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}e^z\mathbf{1}_{\{M_X(T)\geq z\}}\right] + e^{-rT}\int_z^\infty e^y\mathbb{P}^*[M_X(T)\geq y]dy;
\end{aligned}$$

Plug back into $L(s,M;T)$:

$$\begin{aligned}
L(s,M;T) &= s\mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}\left(e^{M_X(T)}-e^z\right)\mathbf{1}_{\{M_X(T)\geq z\}}\right] + se^ze^{-rT} \\
&= se^{-rT}\int_z^\infty e^y\mathbb{P}^*[M_X(T)\geq y]dy + Me^{-rT}
\end{aligned}$$

Proof Con

Then take Laplace in maturity and use Fubini Theorem:

$$\begin{aligned}\int_0^\infty e^{-\alpha T} L(s, M; T) dT &= s \int_0^\infty e^{-\alpha T} e^{-rT} \int_z^\infty e^y \mathbb{P}^*[M_X(T) \geq y] dy dT + \frac{M}{\alpha + r} \\ &= s \int_z^\infty e^y \int_0^\infty e^{-(\alpha+r)T} \mathbb{P}^*[M_X(T) \geq y] dT dy + \frac{M}{\alpha + r}\end{aligned}$$

Follows from Kou and Wang (2003) that:

$$\int_0^\infty e^{-(\alpha+r)T} \mathbb{P}^*[M_X(T) \geq y] dT = A_1 e^{-y\beta_{1,\alpha+r}} + B_1 e^{-y\beta_{2,\alpha+r}}$$

$$A_1 = \frac{1}{\alpha + r} \frac{\eta_1 - \beta_{1,\alpha+r}}{\eta_1} \cdot \frac{\beta_{2,\alpha+r}}{\beta_{2,\alpha+r} - \beta_{1,\alpha+r}}, \quad B_1 = \frac{1}{\alpha + r} \frac{\beta_{2,\alpha+r} - \eta_1}{\eta_1} \cdot \frac{\beta_{1,\alpha+r}}{\beta_{2,\alpha+r} - \beta_{1,\alpha+r}}.$$

Proof Con

Note that $\beta_{2,\alpha+r} > \eta_1 > 1$, $\beta_{1,\alpha+r} > \beta_{1,r} = 1$. Therefore:

$$\begin{aligned}
\int_0^\infty e^{-\alpha T} L(s,M;T)dT &= s\int_z^\infty e^y A_1 e^{-y\beta_{1,\alpha+r}}dy + s\int_z^\infty e^y B_1 e^{-y\beta_{2,\alpha+r}}dy + \frac{M}{\alpha+r} \\
&= sA_1\frac{e^{-z(\beta_{1,\alpha+r}-1)}}{\beta_{1,\alpha+r}-1} + sB_1\frac{e^{-z(\beta_{2,\alpha+r}-1)}}{\beta_{2,\alpha+r}-1} + \frac{M}{\alpha+r} \\
&= s\frac{A_\alpha}{C_\alpha}e^{-z(\beta_{1,\alpha+r}-1)} + s\frac{B_\alpha}{C_\alpha}e^{-z(\beta_{2,\alpha+r}-1)} + \frac{M}{\alpha+r}.
\end{aligned}$$

This yields the Laplace transform we obtained in the Theorem:

$$\begin{aligned}
\int_0^\infty e^{-\alpha T} L(s,M;T)dT &= s\int_z^\infty e^y A_1 e^{-y\beta_{1,\alpha+r}}dy + s\int_z^\infty e^y B_1 e^{-y\beta_{2,\alpha+r}}dy + \frac{M}{\alpha+r} \\
&= sA_1\frac{e^{-z(\beta_{1,\alpha+r}-1)}}{\beta_{1,\alpha+r}-1} + sB_1\frac{e^{-z(\beta_{2,\alpha+r}-1)}}{\beta_{2,\alpha+r}-1} + \frac{M}{\alpha+r} \\
&= s\frac{A_\alpha}{C_\alpha}e^{-z(\beta_{1,\alpha+r}-1)} + s\frac{B_\alpha}{C_\alpha}e^{-z(\beta_{2,\alpha+r}-1)} + \frac{M}{\alpha+r}
\end{aligned}$$

This yields the Laplace transform we obtained in the Theorem.

Barrier Options

Consider the up-and-in call (UIC) option with the barrier level H ($H > S(0)$):

$$UIC = E^{\mathbb{P}^*}[e^{-rT}(S(T) - K)^+ I\{\max_{0\leq t\leq T} S(t) \geq H\}]$$

For any given probability P, define:

$$\Psi(\mu, \sigma, \lambda, p, \eta_1, \eta_2; a, b, T) := P[Z(T) \geq a, \max_{0\leq t\leq T} Z(t) \geq b]$$

where under $P, Z(t)$ is a double exponential jump diffusion process with drift μ, volatility σ, and jump rate λ, i.e., $Z(t) = \mu t + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i$, and Y has a double exponential distribution with density $f_Y(y) \sim p \cdot \eta_1 e^{-\eta_1 y} 1_{\{y\geq 0\}} + q \cdot \eta_2 e^{y\eta_2} 1_{\{y<0\}}$.

Theorem:

The price of the UIC option is obtained as:

$$
\begin{aligned}
\text{UIC} =& S(0)\Psi\left(r+\frac{1}{2}\sigma^2-\lambda\zeta,\sigma,\tilde{\lambda},\tilde{p},\tilde{\eta}_1,\tilde{\eta}_2;\ \log\left(\frac{K}{S(0)}\right),\log\left(\frac{H}{S(0)}\right),T\right)\\
&-Ke^{-rT}\cdot\Psi\left(r-\frac{1}{2}\sigma^2-\lambda\zeta,\sigma,\lambda,p,\eta_1,\eta_2;\ \log\left(\frac{K}{S(0)}\right),\log\left(\frac{H}{S(0)}\right),T\right)
\end{aligned}
$$

where $\tilde{p}=(p/(1+\zeta))\cdot(\eta_1/(\eta_1-1)),\tilde{\eta}_1=\eta_1-1,\tilde{\eta}_2=\eta_2+1,\tilde{\lambda}=\lambda(\zeta+1)$, with $\zeta=E^{P^*}[V]-1=\frac{p\eta_1}{\eta_1-1}+\frac{q\eta_2}{\eta_2+1}-1$. The Laplace transforms of Ψ is computed explicitly in Kou and Wang (2003).

Proof

Theorem 3.1 in Kou and Wang (2003):
For any $\alpha \in (0,\infty)$, let $\beta_{1,\alpha}$ and $\beta_{2,\alpha}$ be the only two positive roots of the equation

$$\alpha = G(\beta),$$

where $0 < \beta_{1,\alpha} < \eta_1 < \beta_{2,\alpha} < \infty$. Then we have the following results concerning the Laplace transforms of τ_b and X_{τ_b}:

$$\mathbb{E}[e^{-\alpha\tau_b}] = \frac{\eta_1 - \beta_{1,\alpha}}{\eta_1}\frac{\beta_{2,\alpha}}{\beta_{2,\alpha}-\beta_{1,\alpha}}e^{-b\beta_{1,\alpha}} + \frac{\beta_{2,\alpha}-\eta_1}{\eta_1}\frac{\beta_{1,\alpha}}{\beta_{2,\alpha}-\beta_{1,\alpha}}e^{-b\beta_{2,\alpha}}$$

$$\mathbb{E}[e^{-\alpha\tau_b}\mathbf{1}_{\{X_{\tau_b}-b>y\}}] = e^{-\eta_1 y}\frac{(\eta_1-\beta_{1,\alpha})(\beta_{2,\alpha}-\eta_1)}{\eta_1(\beta_{2,\alpha}-\beta_{1,\alpha})}[e^{-b\beta_{1,\alpha}} - e^{-b\beta_{2,\alpha}}] \text{ for all } y \geq 0$$

$$\mathbb{E}[e^{-\alpha\tau_b}\mathbf{1}_{\{X_{\tau_b}=b\}}] = \frac{\eta_1 - \beta_{1,\alpha}}{\beta_{2,\alpha}-\beta_{1,\alpha}}e^{-b\beta_{1,\alpha}} + \frac{\beta_{2,\alpha}-\eta_1}{\beta_{2,\alpha}-\beta_{1,\alpha}}e^{-b\beta_{2,\alpha}}$$

The definition of τ_b are same and definition of X equilivent to Z here.

Proof Con.

Based on the Theorem 3.1, we can get explicit formula of the Laplace transform for $\Psi(\mu, \sigma, \lambda, p, \eta_1, \eta_2; a, b, T)$ (write as $\Psi(a, b, T)$ for simplicity). Define the Laplace transform in T:

$$\Phi_\alpha(a,b) := \int_0^\infty e^{-\alpha T}\Psi(a,b,T)dT = \int_0^\infty e^{-\alpha T}\mathbb{E}[1_{\{\tau_b \le T\}}1_{\{X(T)\ge a\}}]dT$$

We first use Fubini Theorem and strong Markov property can rewrite the formula, then split by overshoot and use tail-integration eventuality we will end up with the following:

$$\Phi_\alpha(a,b) = A_1 e^{-\beta_{1,\alpha}(a-b)}\left(\mathbb{E}\left[e^{-\alpha\tau_b}\mathbf{1}_{\{X_{\tau_b}=b\}}\right] + \beta_{1,\alpha}\int_0^{a-b} e^{\beta_{1,\alpha}y}\mathbb{E}\left[e^{-\alpha\tau_b}\mathbf{1}_{\{X_{\tau_b}-b>y\}}\right]dy\right)$$
$$+ B_1 e^{-\beta_{2,\alpha}(a-b)}\left(\mathbb{E}\left[e^{-\alpha\tau_b}\mathbf{1}_{\{X_{\tau_b}=b\}}\right] + \beta_{2,\alpha}\int_0^{a-b} e^{\beta_{2,\alpha}y}\mathbb{E}\left[e^{-\alpha\tau_b}\mathbf{1}_{\{X_{\tau_b}-b>y\}}\right]dy\right)$$

where:

$$A_1 = \frac{1}{\alpha}\frac{\eta_1-\beta_{1,\alpha}}{\eta_1}\cdot\frac{\beta_{2,\alpha}}{\beta_{2,\alpha}-\beta_{1,\alpha}}, \quad B_1 = \frac{1}{\alpha}\frac{\beta_{2,\alpha}-\eta_1}{\eta_1}\cdot\frac{\beta_{1,\alpha}}{\beta_{2,\alpha}-\beta_{1,\alpha}}$$

Proof Con.

Back to the proof of the Theorem for Barrier option. First rewrite UIC as:

$$
\begin{aligned}
UIC &= E^{\mathbb{P}^*}[e^{-rT}(S(T)-K)^+ I\{\max_{0\le t\le T} S(t)\ge H\}] \\
&= \mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}S(T)\mathbf{1}_{\{S(T)\ge K, \max_{0\le t\le T} S(t)\ge H\}}\right] \\
&- Ke^{-rT}\mathbb{P}^*\left[S(T)\ge K, \max_{0\le t\le T} S(t)\ge H\right] \\
&= I - Ke^{-rT}\cdot II
\end{aligned}
$$

Since the log–price $X(t) = \log(S(t)/S(0))$:

$$
S(T)\ge K \iff X(T)\ge \log\frac{K}{S(0)}, \quad \max_{0\le t\le T} S(t)\ge H \iff \max_{0\le t\le T} X(t)\ge \log\frac{H}{S(0)}.
$$

Based on the definition of $\Psi(\mu,\sigma,\lambda,p,\eta_1,\eta_2;a,b,T)$:

$$
II = \Psi\left(r-\frac{1}{2}\sigma^2-\lambda\zeta, \sigma, \lambda, p, \eta_1, \eta_2; \log\left(\frac{K}{S(0)}\right), \log\left(\frac{H}{S(0)}\right), T\right)
$$

Proof Con.

For the first term, we can use a change of numeraire argument. More precisely, introduce a new probability $\tilde{\mathbb{P}}$ defined as:

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^*}\bigg|_{t=T} = e^{-rT}\frac{S(T)}{S(0)} = e^{-rT}e^{X(T)} = \exp\left\{\left(-\frac{1}{2}\sigma^2 - \lambda\zeta\right)T + \sigma W(T) + \sum_{i=1}^{N(T)} Y_i\right\}$$

Note that this is a well-defined probability as $\mathbb{E}^{\mathbb{P}^*}[e^{-rt}(S(t)/S(0))] = 1$. Then we reparametrize everything we need under $\tilde{\mathbb{P}}$. The Brownian part by Girsanov is $\tilde{W}(t) = W(t) - \sigma t$. The diffusion drift shifts by $+\sigma^2$, $\tilde{\mu} = r + \frac{1}{2}\sigma^2 - \lambda\zeta$. The modified jump rate and the jump distribution:

$$\text{New rate : } \tilde{\lambda} = \lambda\mathbb{E}^{\mathbb{P}^*}[e^Y] = \lambda(1+\zeta), \quad \text{where } \zeta = \mathbb{E}^{\mathbb{P}^*}[e^Y] - 1 = \frac{p\eta_1}{\eta_1 - 1} + \frac{q\eta_2}{\eta_2 + 1} - 1$$

$$\text{New jump density: } \tilde{f}_Y(y) = \frac{e^y}{\mathbb{E}^{\mathbb{P}^*}[e^Y]} f_Y(y) = \tilde{p}\tilde{\eta}_1 e^{-\tilde{\eta}_1 y}\mathbf{1}_{y\geq 0} + \tilde{q}\tilde{\eta}_2 e^{\tilde{\eta}_2 y}\mathbf{1}_{y<0}$$

$$\text{With: } \tilde{\eta}_1 = \eta_1 - 1,\ \tilde{\eta}_2 = \eta_2 + 1,\ \tilde{p} = p\left(\frac{p\eta_1}{\eta_1 - 1} + \frac{q\eta_2}{\eta_2 + 1}\right)^{-1}\frac{\eta_1}{\eta_1 - 1},\ \tilde{q} = q\left(\frac{p\eta_1}{\eta_1 - 1} + \frac{q\eta_2}{\eta_2 + 1}\right)^{-1}\frac{\eta_2}{\eta_2 + 1}$$

For I:

$$\begin{aligned}
I &= S(0)\mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}\frac{S(T)}{S(0)}\cdot\mathbf{1}_{\{S(T)\geq K, \max_{0\leq t\leq T} S(t)\geq H\}}\right] \\
&= S(0)\tilde{\mathbb{P}}\left[S(T)\geq K, \min_{0\leq t\leq T} S(t)\leq H\right] \\
&= S(0)\Psi\left(r + \frac{1}{2}\sigma^2 - \lambda\zeta, \sigma, \tilde{\lambda}, \tilde{p}, \tilde{\eta}_1, \tilde{\eta}_2; \log\left(\frac{K}{S(0)}\right), \log\left(\frac{H}{S(0)}\right), T\right)
\end{aligned}$$

Part II

Pricing Path-Dependent Options with Jump Risk via Laplace Transforms

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Derive the Laplace transforms for casepricing of European call and put options. Derive the two dimensional Laplace transform for barrier option.

European call and put options

The price of a European call and put with maturity T and strike K, is given:

$$C_T(k) = e^{-rT}\mathbb{E}^{\mathbb{P}^*}[(S(T)-K)^+] = e^{-rT}\mathbb{E}^{\mathbb{P}^*}\left[(S(0)e^{X(T)} - e^{-k})^+\right]$$
$$P_T(k') = e^{-rT}\mathbb{E}^{\mathbb{P}^*}[(K-S(T))^+] = e^{-rT}\mathbb{E}^{\mathbb{P}^*}\left[(e^{k'} - S(0)e^{X(T)})^+\right]$$

By the change of numeraire argument w.r.t $S(t)$:

$$C_T(k) = S(0)\tilde{\Psi}_C(k) - e^{-k}e^{-rT}\Psi_C(k)$$
$$P_T(k') = e^{k'}e^{-rT}\Psi_P(k') - S(0)\tilde{\Psi}_P(k')$$

where:

$$\Psi_C(k) = \mathbb{P}^*(S(T) \geq e^{-k}),\ \tilde{\Psi}_C(k) = \tilde{\mathbb{P}}(S(T) \geq e^{-k})$$
$$\Psi_P(k') = \mathbb{P}^*(S(T) < e^{k'}),\ \tilde{\Psi}_P(k') = \tilde{\mathbb{P}}(S(T) < e^{k'})$$

European call and put options Con.

Lemma. The Laplace transform with respect to k of $C_T(k)$ and with respect to k' for the put option $P_T(k')$ are given by:

$$\tilde{f}_C(\xi) := \int_{-\infty}^{\infty} e^{-\xi k} C_T(k) dk = e^{-rT} \frac{S(0)^{\xi+1}}{\xi(\xi+1)} \exp(G(\xi+1)T), \quad \xi > 0$$

$$\tilde{f}_P(\xi) := \int_{-\infty}^{\infty} e^{-\xi k'} P_T(k') dk' = e^{-rT} \frac{S(0)^{-\xi-1}}{\xi(\xi-1)} \exp(G(-(\xi-1))T), \quad \xi > 1$$

The Laplace transforms with respect to k of $\Psi_C(k)$ and k' of $\Psi_P(k')$ are:

$$\tilde{f}_{\Psi_C}(\xi) := \int_{-\infty}^{\infty} e^{-\xi k} \Psi_C(k) dk = \frac{S(0)^{\xi}}{\xi} \exp(G(\xi)T), \quad \xi > 0$$

$$\tilde{f}_{\Psi_P}(\xi) := \int_{-\infty}^{\infty} e^{-\xi k'} \Psi_P(k') dk' = e^{-rT} \frac{S(0)^{-\xi}}{\xi} \exp(G(-\xi)T), \quad \xi > 0$$

European call and put options Con.

Proof:
The Laplace transform for the call option is:

$$\hat{f}_C(\xi) = e^{-rT} \int_{-\infty}^{\infty} e^{-\xi k} \mathbb{E}^{\mathbb{P}^*} \left[(S(0)e^{X(T)} - e^{-k})^+ \right] dk$$

Applying the Fubini theorem yields for every $\xi > 0$:

$$\begin{aligned}
\hat{f}_C(\xi) &= e^{-rT} \mathbb{E}^{\mathbb{P}^*} \left[\int_{-\infty}^{\infty} e^{-\xi k} \left(S(0)e^{X(T)} - e^{-k} \right)^+ dk \right] \\
&= e^{-rT} \mathbb{E}^{\mathbb{P}^*} \left[\int_{-X(T)-\log S(0)}^{\infty} e^{-\xi k} \left(S(0)e^{X(T)} - e^{-k} \right) dk \right] \\
&= e^{-rT} \mathbb{E}^{\mathbb{P}^*} \left[S(0)e^{X(T)} e^{\xi(X(T)+\log S(0))} \frac{1}{\xi} - e^{(\xi+1)(X(T)+\log S(0))} \frac{1}{\xi+1} \right] \\
&= e^{-rT} \frac{S(0)^{\xi+1}}{\xi(\xi+1)} \mathbb{E}^{\mathbb{P}^*} \left[e^{(\xi+1)X(T)} \right] \overset{\text{MGF}}{=} e^{-rT} \frac{S(0)^{\xi+1}}{\xi(\xi+1)} e^{G(\xi+1)T}
\end{aligned}$$

Similary, we can get the Laplace transform for the put option.

European call and put options Con.

The Laplace transforms with respect to k of $\Psi_C(k)$:

$$\begin{aligned}
\hat{f}_{\Psi_C}(\xi) &= \int_{-\infty}^{\infty} e^{-\xi k} \mathbb{E}^{\mathbb{P}^*}\left[\mathbf{1}_{\{S(T)\geq e^{-k}\}}\right] dk \\
&= \int_{-\infty}^{\infty} e^{-\xi k} \mathbb{E}^{\mathbb{P}^*}\left[\mathbf{1}_{\{k\geq -\log S(T)\}}\right] dk \\
&= \mathbb{E}^{\mathbb{P}^*}\left[\int_{-\log S(T)}^{\infty} e^{-\xi k} dk\right] \\
&= \frac{1}{\xi}\mathbb{E}^{\mathbb{P}^*}\left[S(T)^{\xi}\right] = \frac{S(0)^{\xi}}{\xi}\mathbb{E}^{\mathbb{P}^*}\left[e^{\xi X(T)}\right] \overset{\text{MGF}}{=} \frac{S(0)^{\xi}}{\xi} e^{G(\xi)T}
\end{aligned}$$

Similary, we can get the Laplace transform for $\Psi_P(k')$.

Barrier Options

Rewrite the pricing formula of up-and-in call option(UIC) in early silde:

$$UIC(k,T) = E^{\mathbb{P}^*}\left[e^{-rT}\left(S(T) - e^{-k}\right)^+ I[\tau_b < T]\right]$$

where $H > S(0)$ is the barrier level, $k = -\log(K)$ the transformed strike and $b = \log(H/S(0))$. In previous paper, we obtain:

$$UIC(k,T) = S(0)\tilde{\Psi}_{UI}(k,T) - Ke^{-rT}\Psi_{UI}(k,T)$$

where:

$$\Psi_{UI}(k,T) = P^*(S(T) \geq e^{-k}, \tau_b < T), \quad \widetilde{\Psi}_{UI}(k,T) = \widetilde{P}(S(T) \geq e^{-k}, \tau_b < T)$$

Barrier Options Con.

Theorem: For ξ and α such that $0 < \xi < \eta_1 - 1$ and $\alpha > \max(G(\xi+1) - r, 0)$ (such a choice of ξ and α is possible for all small enough ξ as $G(1) - r = -\delta < 0$). The Laplace transform with respect to k and T of $UIC(k,T)$ is given by

$$\begin{aligned}\tilde{f}_{UIC}(\xi,\alpha) &= \int_0^\infty \int_{-\infty}^\infty e^{-\xi k - \alpha T} UIC(k,T) dk dT \\ &= \frac{H^{\xi+1}}{\xi(\xi+1)} \frac{1}{r+\alpha-G(\xi+1)} \left(A(r+\alpha) \frac{\eta_1}{\eta_1 - (\xi+1)} + B(r+\alpha) \right)\end{aligned}$$

where

$$A(h) := E^{\mathbb{P}^*}\left[e^{-h\tau_b} \mathbf{1}_{\{X(\tau_b) > b\}} \right] = \frac{(\eta_1 - \beta_{1,h})(\beta_{2,h} - \eta_1)}{\eta_1(\beta_{2,h} - \beta_{1,h})} \left[e^{-b\beta_{1,h}} - e^{-b\beta_{2,h}} \right]$$

$$B(h) := E^{\mathbb{P}^*}\left[e^{-h\tau_b} \mathbf{1}_{\{X(\tau_b = b)\}} \right] = \frac{\eta_1 - \beta_{1,h}}{\beta_{2,h} - \beta_{1,h}} e^{-b\beta_{1,h}} + \frac{\beta_{2,h} - \eta_1}{\beta_{2,h} - \beta_{1,h}} e^{-b\beta_{2,h}}$$

with $b = \log(H/S(0))$.

Barrier Options Con.

If $0 < \xi < \eta_1$ and $\alpha > \max(G(\xi), 0)$ (again this choice of ξ and α is possible for all ξ small enough as $G(0) = 0$), then the Laplace transform with respect to k and T of $\Psi_{UI}(k, T)$ is:

$$\begin{aligned}\tilde{f}_{\Psi_{UI}}(\xi, \alpha) &= \int_0^{\infty} \left(\int_{-\infty}^{\infty} e^{-\xi k - \alpha T} \Psi_{UI}(k, T) dk \right) dT \\ &= \frac{H^{\xi}}{\xi} \frac{1}{\alpha - G(\xi)} \left(A(\alpha) \frac{\eta_1}{\eta_1 - \xi} + B(\alpha) \right)\end{aligned}$$

The Laplace transforms with respect to k and T of $\tilde{\Psi}_{UI}(k, T)$ is given similarly with $\tilde{G}$ replacing G and the functions $\tilde{A}$ and $\tilde{B}$ defined similarly.

Barrier Options Con.

Proof: Follow the pricing formula of UIC and the Fubini theorem:

$$
\begin{aligned}
\tilde{f}_{UIC}(\xi,\alpha) &= \int_0^\infty \int_{-\infty}^\infty e^{-\xi k-(r+\alpha)T}\mathbb{E}^{\mathbb{P}^*}\left[\left(S(T)-e^{-k}\right)^+ \mathbf{1}_{\{\tau_b<T\}}\right] dkdT \\
&= \mathbb{E}^{\mathbb{P}^*}\left[\int_0^\infty e^{-(r+\alpha)T}\mathbf{1}_{\{\tau_b<T\}}\left(\int_{-\log S(T)}^\infty e^{-\xi k}\left(S(T)-e^{-k}\right)dk\right)dT\right] \\
&= \frac{1}{\xi(\xi+1)}\mathbb{E}^{\mathbb{P}^*}\left[\int_0^\infty e^{-(r+\alpha)T}\mathbf{1}_{\{\tau_b<T\}}S(T)^{\xi+1}dT\right] \\
(T=\tau_b+t \text{ with } t>0) &= \frac{1}{\xi(\xi+1)}\mathbb{E}^{\mathbb{P}^*}\left[e^{-(r+\alpha)\tau_b}\int_0^\infty e^{-(r+\alpha)t}S(t+\tau_b)^{\xi+1}dt\right] \\
&= \frac{1}{\xi(\xi+1)}\mathbb{E}^{\mathbb{P}^*}\left[e^{-(r+\alpha)\tau_b}\mathbb{E}^{\mathbb{P}^*}\left[\int_0^\infty e^{-(r+\alpha)t}S(t+\tau_b)^{\xi+1}dt\middle|\mathcal{F}_{\tau_b}\right]\right] \\
(i) &= \frac{1}{\xi(\xi+1)}\mathbb{E}^{\mathbb{P}^*}\left[e^{-(r+\alpha)\tau_b}S(\tau_b)^{\xi+1}\int_0^\infty e^{-(r+\alpha)t}\mathbb{E}^{\mathbb{P}^*}\left[e^{(\xi+1)X(t)}\right]dt\right]
\end{aligned}
$$

Where i is based on $S(\tau_b+t)^{\xi+1}=(S(\tau_b))^{\xi+1}\cdot e^{(\xi+1)(X(\tau_b+t)-X(\tau_b))}$ and strong Markov property that $\mathbb{E}^{\mathbb{P}^*}\left[e^{(\xi+1)(X(\tau_b+t)-X(\tau_b))}\middle|\mathcal{F}_{\tau_b}\right]=\mathbb{E}^{\mathbb{P}^*}\left[e^{(\xi+1)X(t)}\right]$

Barrier Options Con.

Con.

$$
\begin{aligned}
(\text{MGF}) &= \frac{1}{\xi(\xi+1)}\mathbb{E}^{\mathbb{P}^*}\left[e^{-(r+\alpha)\tau_b}S(\tau_b)^{\xi+1}\int_0^\infty e^{-(r+\alpha)t}e^{G(\xi+1)t}dt\right] \\
&= \frac{1}{\xi(\xi+1)}\frac{1}{r+\alpha-G(\xi+1)}\mathbb{E}^{\mathbb{P}^*}\left[e^{-(r+\alpha)\tau_b}S(\tau_b)^{\xi+1}\right] \\
&= \frac{1}{\xi(\xi+1)}\frac{1}{r+\alpha-G(\xi+1)}\left\{\mathbb{E}^{\mathbb{P}^*}\left[e^{-(r+\alpha)\tau_b}H^{\xi+1}\mathbf{1}_{\{X(\tau_b)>b\}}\right]\mathbb{E}^{\mathbb{P}^*}\left[e^{(\xi+1)\chi^+}\right]\right. \\
&\left. + \mathbb{E}^{\mathbb{P}^*}\left[e^{-(r+\alpha)\tau_b}H^{\xi+1}\mathbf{1}_{\{X(\tau_b)=b\}}\right]\right\} \\
&= \frac{H^{\xi+1}}{\xi(\xi+1)}\frac{1}{r+\alpha-G(\xi+1)}\left\{A(r+\alpha)\frac{\eta_1}{\eta_1-(\xi+1)}+B(r+\alpha)\right\}
\end{aligned}
$$

Where $\chi^+ \sim \text{Exp}(\eta_1)$, and $A(h) := \mathbb{E}^{\mathbb{P}^*}[e^{-h\tau_b}\mathbf{1}_{\{X(\tau_b)>b\}}]$, $B(h) := \mathbb{E}^{\mathbb{P}^*}[e^{-h\tau_b}\mathbf{1}_{\{X(\tau_b)=b\}}]$. $A(h)$ and $B(h)$ explicitly (via first-passage Laplace transforms) with $\beta_{1,h}, \beta_{2,h}$ being the two positive roots of $G(\beta) = h$:

$$
A(h) = \frac{(\eta_1-\beta_{1,h})(\beta_{2,h}-\eta_1)}{\eta_1(\beta_{2,h}-\beta_{1,h})}\left(e^{-b\beta_{1,h}} - e^{-b\beta_{2,h}}\right)
$$

$$
B(h) = \frac{\eta_1-\beta_{1,h}}{\beta_{2,h}-\beta_{1,h}}e^{-b\beta_{1,h}} + \frac{\beta_{2,h}-\eta_1}{\beta_{2,h}-\beta_{1,h}}e^{-b\beta_{2,h}}
$$

Barrier Options Con.

For the Laplace transform of the probability Ψ_{UI}, apply the same trick we have:

$$
\begin{aligned}
\hat{f}_{\Psi_{UI}}(\xi,\alpha) &= \int_0^\infty \left[\int_{-\infty}^{\infty} e^{-\xi k-\alpha T}\cdot \mathbb{E}^{\mathbb{P}^*}\left\{\mathbf{1}_{\{k>-\log(S(T)),\tau_b<T\}}\right\} dk\right] dT \\
&= \mathbb{E}^{\mathbb{P}^*}\left\{\int_{\tau_b}^{\infty}\left[\int_{-\log S(T)}^{\infty} e^{-\xi k-\alpha T} dk\right] dT\right\} \\
&= \frac{1}{\xi}\mathbb{E}^{\mathbb{P}^*}\left\{\int_{\tau_b}^{\infty} S(T)^{\xi} e^{-\alpha T} dT\right\} \\
&= \frac{1}{\xi}\mathbb{E}^{\mathbb{P}^*}\left\{e^{-\alpha\tau_b}\int_0^\infty \{S(t+\tau_b)\}^{\xi} e^{-\alpha t} dt\right\} \\
&= \frac{1}{\xi}\mathbb{E}^{\mathbb{P}^*}\left\{e^{-\alpha\tau_b}\mathbb{E}^{\mathbb{P}^*}\left[\int_0^\infty S(\tau_b+t)^{\xi} e^{-\alpha t} dt\middle|\mathcal{F}_{\tau_b}\right]\right\} \\
&= \frac{1}{\xi}\frac{1}{\alpha-G(\xi)}\mathbb{E}^{\mathbb{P}^*}\left\{e^{-\alpha\tau_b}[S(\tau_b)]^{\xi}\right\} \\
&= \frac{1}{\xi}\frac{1}{\alpha-G(\xi)}\left\{\mathbb{E}^{\mathbb{P}^*}\left[e^{-\alpha\tau_b}H^{\xi}\mathbf{1}_{[X(\tau_b)>b]}\right]\mathbb{E}^{\mathbb{P}^*}\left[e^{\xi\chi^+}\right]+\mathbb{E}^{\mathbb{P}^*}\left[e^{-\alpha\tau_b}H^{\xi}\mathbf{1}_{[X(\tau_b)=b]}\right]\right\} \\
&= \frac{H^{\xi}}{\xi}\frac{1}{\alpha-G(\xi)}\left\{A(\alpha)\frac{\eta_1}{\eta_1-\xi}+B(\alpha)\right\}
\end{aligned}
$$

A Two-Sided Laplace inversion Algorithm
With Computable Error Bounds And Its
Applications In Financial Engineering

Ning Cai
Steven Kou
ZongJian Liu

Derive the Laplace transforms for casepricing of European call and put options. Derive the two dimensional Laplace transform for barrier option.

Two-sided Laplace Transform

For a function $f(t)$ defined on the whole real line $(-\infty, \infty)$, its two-sided Laplace transform is:

$$L_f(s) = \int_{-\infty}^{\infty} e^{-st} f(t)\, dt$$

Where $s = \sigma + i\omega$ is a complex variable. The region of absolute convergence (ROAC) is the set of s such that:

$$\int_{-\infty}^{\infty} e^{-\sigma t} |f(t)|\, dt < \infty$$

Define:

$$\sigma_l = \inf\left\{\sigma \in \mathbb{R} : \int_{-\infty}^{\infty} e^{-\sigma t} |f(t)|\, dt < \infty\right\} \quad \sigma_u = \sup\left\{\sigma \in \mathbb{R} : \int_{-\infty}^{\infty} e^{-\sigma t} |f(t)|\, dt < \infty\right\}$$

Then the ROAC is the strip (σ_l, σ_u) in the complex plane (independent of the imaginary part ω).

Ambiguity of inversion

It is worth pointing out that the same two-sided Laplace transforms with different ROACs may correspond to different original functions; for example, consider three functions:

$$f_1(t) = \begin{cases} e^{2t} - e^{-3t}, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad f_2(t) = \begin{cases} -e^{-3t}, & t \geq 0 \\ -e^{2t}, & t < 0 \end{cases} \quad f_3(t) = \begin{cases} 0, & t \geq 0 \\ e^{-3t} - e^{2t}, & t < 0 \end{cases}$$

Laplace transforms are all $L(s) = 5/(s^2 + s - 6)$ but with different ROACs: $(2, +\infty)$, $(-3, 2)$, and $(-\infty, -3)$, respectively. Consequently, when inverting a two-sided Laplace transform $L_f(s)$, we should first specify a particular ROAC. Moreover, the examples imply that the ROAC of a two-sided Laplace transform may not include the imaginary axis $\{s : \text{Re}(s) = 0\}$.

One-sided Laplace transforms and Fourier transforms

If $f(t) = 0$ for any $t < 0$ then its two-sided Laplace transform is reduced to:

$$L_f(s) = \int_0^{+\infty} e^{-st} f(t)dt \quad \text{for } \mathrm{Re}(s) \in \text{ROAC}$$

and we call it the one-sided Laplace transform. The Fourier transform $\mathcal{F}_f(\omega) : \mathbb{R} \mapsto \mathbb{C}$ of the function $f(t)$ is defined by

$$\mathcal{F}_f(\omega) := \int_{-\infty}^{\infty} e^{-i\omega t} f(t)dt \equiv L_f(i\omega) \quad \text{for any } \omega \in \mathbb{R}.$$

Because the ROAC of a two-sided Laplace transform may not include the imaginary axis $\{s : \mathrm{Re}(s) = 0\}$, the Fourier transform of a function may not exist. For example, the Fourier transforms of $f_1(t)$ and $f_3(t)$ in the example do not exist, whereas their two-sided Laplace transforms are well defined. Therefore, the Fourier transform is a special case of the two-sided Laplace transform.

The Two-sided Laplace inversion formula

We want to invert the two-sided Laplace transform:

$$L_f(s) = \int_{-\infty}^{\infty} e^{-st} f(t) dt$$

to recover $f(t)$. The challenge is the inversion involves infinite integrals. We must approximate them. That's where discretization and truncation errors come in. Our two-sided Laplace inversion formula involves parameters C and N for the purpose of controlling the discretization and truncation errors, respectively.

Assumption: The function $e^{-\sigma t} f(t)$ is of bounded variation on $\mathbb{R}$ for any $\sigma \in$ ROAC. This ensures the inversion formula is well-defined and stable. Recall a real-valued function $g : [a, b] \to \mathbb{R}$ is said to be of bounded variation if the total variation:

$$V_a^b(g) := \sup \left\{ \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})| : a = x_0 < x_1 < \cdots < x_n = b \right\}$$

is finite.

The Two-sided Laplace inversion formula

Theorem: Consider a function $f(t)$ normalized such that $2f(t) = f(t+0) + f(t-0)$ for any real t, where $f(t \pm 0) := \lim_{\varepsilon \downarrow 0} f(t \pm \varepsilon)$. Then, under assumption, for any t and $\sigma \in$ ROAC,

$$f(t) = f_A(t, \sigma, C, N) + e_T(t, \sigma, C, N) - e_D(t, \sigma, C)$$

where the output of the inversion algorithm is:

$$f_A(t, \sigma, C, N) := \frac{e^{\sigma t} L_f(\sigma)}{2(|t| + C)} + \frac{e^{\sigma t}}{|t| + C} \sum_{k=1}^{N} \left[(-1)^k \mathrm{Re} \left(\exp\left\{ -\frac{\mathrm{sgn}(t) C k \pi i}{t + \mathrm{sgn}(t) C} \right\} \times L_f\left(\sigma + \frac{k\pi i}{t + \mathrm{sgn}(t) C} \right) \right) \right]$$

$C \geq 0$ is a constant such that $|t| + C \neq 0$, $N > 0$ is a positive integer, and $\mathrm{sgn}(x)$ equals 1 if $x \geq 0$ and equals -1 otherwise. The terms $e_T(t, \sigma, C, N)$ and $e_D(t, \sigma, C)$ represent the truncation error and the discretization error, respectively:

$$e_T(t, \sigma, C, N) := \frac{e^{\sigma t}}{|t| + C} \sum_{k=N+1}^{+\infty} \left[(-1)^k \mathrm{Re} \left(\exp\left\{ -\frac{\mathrm{sgn}(t) C k \pi i}{t + \mathrm{sgn}(t) C} \right\} \times L_f\left(\sigma + \frac{k\pi i}{t + \mathrm{sgn}(t) C} \right) \right) \right]$$

and

$$e_D(t, \sigma, C) := \sum_{k=-\infty \ k \neq 0}^{+\infty} e^{-2\sigma k (t + \mathrm{sgn}(t) C)} f(2k(t + \mathrm{sgn}(t) C) + t)$$

The Two-sided Laplace inversion formula

Proof: It suffices to show that, for any $t \neq 0$ and $\sigma \in$ ROAC:

$$f(t) = \frac{e^{\sigma t} L_f(\sigma)}{2|t|} + \frac{e^{\sigma t}}{|t|} \sum_{k=1}^{\infty} (-1)^k \text{Re}\left(L_f\left(\sigma + \frac{k\pi i}{t}\right)\right) - \sum_{k=-\infty\ k\neq 0}^{+\infty} e^{-2\sigma k t} f((2k+1)t)$$

Indeed, to evaluate $f(t)$, we can alternatively apply above equation to the function $g_+(y) := f(y - C)$ at the point $t + C$ if $t \geq 0$ or to the function $g_-(y) := f(y + C)$ at the point $t - C$ if $t < 0$. This give theorem immediately. The reason why shift wich C is the denominator $|t| + C$ in the theorem stabilizes the formula (avoids singularity at t=0).
Now we prove above equation. Because $\sigma \in$ ROAC, substituting $\sigma + i\omega$ for s in the Bromwich contour integral:

$$f(t) = \frac{1}{2\pi i} \lim_{T\to+\infty} \int_{\sigma - iT}^{\sigma + iT} e^{ts} L_f(s) ds$$

By Euler's formula we obtain:

$$\begin{aligned} f(t) &= \frac{e^{\sigma t}}{2\pi} \lim_{T\to+\infty} \int_{T}^{-T} [\cos(\omega t) + i\sin(\omega t)] L_f(\sigma + i\omega) d\omega \\ &= \frac{e^{\sigma t}}{2\pi} \int_0^{+\infty} \{[\cos(\omega t) + i\sin(\omega t)] L_f(\sigma + i\omega) + [\cos(\omega t) - i\sin(\omega t)] L_f(\sigma - i\omega)\} d\omega \\ &= \frac{e^{\sigma t}}{\pi} \int_0^{+\infty} [\cos(\omega t) \text{Re}(L_f(\sigma + i\omega)) + \sin(\omega t) \text{Im}(L_f(\sigma - i\omega))] d\omega \end{aligned}$$

The Two-sided Laplace inversion formula

Con.
where the last equality holds because Re$(L_f(\sigma+i\omega))=$ Re$(L_f(\sigma-i\omega))$ and Im$(L_f(\sigma+i\omega))=-$Im$(L_f(\sigma-i\omega))$. By the trapezoidal rule, for $h>0$, we define $\tilde{f}(t)$ as

$$\tilde{f}(t):=\frac{he^{\sigma t}}{2\pi}L_f(\sigma)+\frac{he^{\sigma t}}{\pi}\sum_{k=1}^{+\infty}\cos(kht)\,\mathrm{Re}(L_f(\sigma+ikh))+\frac{he^{\sigma t}}{\pi}\sum_{k=1}^{+\infty}\sin(kht)\,\mathrm{Im}(L_f(\sigma-ikh))$$

Because $t\neq 0$, letting $h=\pi/|t|$ yields:

$$\tilde{f}(t)=\frac{e^{\sigma t}L_f(\sigma)}{2|t|}+\frac{e^{\sigma t}}{|t|}\sum_{k=1}^{\infty}(-1)^k\,\mathrm{Re}\left(L_f\left(\sigma+\frac{k\pi i}{t}\right)\right)$$

The trapezoidal rule: The trapezoidal rule approximates an integral $\int_0^\infty g(\omega)d\omega$ by sampling g at equally spaced points, $\frac{h}{2}g(0)+h\sum_{k=1}^{\infty}g(kh)$.
To analyze the discretization error $\tilde{f}(t)-f(t)$ generated by the trapezoidal rule, we define $g(x):=e^{-\sigma x}f(x)$ for any fixed $\sigma\in\mathrm{ROAC}$. Under the condition of Theorem, $g(x)$ is absolutely integrable over $\mathbb{R}$, is of bounded variation on $\mathbb{R}$, and satisfies $2g(x)=g(x+0)+g(x-0)$ for any x. Then, for any fixed $t\neq 0$, $g^*(x):=g(t+x/h)=g(t+|t|x/\pi)$ also has these three properties.

The Two-sided Laplace inversion formula

Con.
Applying the Poisson summation formula to $g^*(x)$:

$$\sum_{k=-\infty}^{+\infty} g^*(2\pi k) = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} \int_{-\infty}^{+\infty} g^*(z) e^{-ikz} dz$$

Poisson summation formula: For a "nice" function $g(x)$ (say, integrable and smooth enough), the Poisson summation formula says:

$$\sum_{k=-\infty}^{\infty} g(k) = \sum_{m=-\infty}^{\infty} \hat{g}(2\pi m)$$

Where $\hat{g}(\xi)$ is the Fourier transform of $g(x)$, $\hat{g}(\xi) = \int_{-\infty}^{\infty} g(x) e^{-i\xi x} dx$. The left-hand side is a sum of the function's samples at the integers. The right-hand side is a sum of its Fourier transform sampled at multiples of 2π.

The Two-sided Laplace inversion formula

Con.
The left-hand side:

$$
\begin{aligned}
g^*(2\pi k) &= \sum_{k=-\infty}^{+\infty} g(t+2|t|k) \\
(t<0,\ \text{reindex } k \to -k) &= \sum_{k=-\infty}^{+\infty} g((2k+1)t) \\
\Big(\text{Recall } g(x) := e^{-\sigma x} f(x)\Big) &= \sum_{k=-\infty}^{+\infty} e^{-\sigma(2k+1)t} f((2k+1)t) \\
&= e^{-\sigma t}\left[f(t) + \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} e^{-2\sigma k t} f((2k+1)t) \right]
\end{aligned}
$$

The Two-sided Laplace inversion formula

Con.
The right-hand side:

$$\frac{1}{2\pi}\sum_{k=-\infty}^{+\infty}\int_{-\infty}^{+\infty} g^*(z)e^{-ikz}dz = \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty}\int_{-\infty}^{+\infty} g\left(t+\frac{|t|z}{\pi}\right)e^{-ikz}dz$$

$$\left(y = t+\frac{|t|}{\pi}z\right) = \frac{1}{2|t|}\sum_{k=-\infty}^{+\infty}\left[\left(\int_{-\infty}^{+\infty} g(y)e^{-ik\pi y/|t|}dy\right)e^{ik\pi t/|t|}\right]$$

$$\left(L_f(s) = \int_{-\infty}^{\infty} e^{-st}f(t)dt\right) = \frac{1}{2|t|}\sum_{k=-\infty}^{+\infty} L_f\left(\sigma+\frac{ik\pi}{|t|}\right)e^{ik\pi t/|t|}$$

$$= \frac{1}{2|t|}\sum_{k=-\infty}^{+\infty}(-1)^k L_f\left(\sigma+\frac{ik\pi}{t}\right)$$

$$= \frac{1}{2|t|}L_f(\sigma) + \frac{1}{|t|}\sum_{k=1}^{+\infty}(-1)^k \text{Re}\left(L_f\left(\sigma+\frac{ik\pi}{t}\right)\right)$$

The last step, separate the $k=0$ term and use conjugate symmetry, for real f, $L_f(\sigma - i\alpha) = \overline{L_f(\sigma+i\alpha)}$. Link the formula for LHS and RHS we proof the formula of $f(x)$. The proof is completed by comparing $f(t)$ with $f\tilde{(t)}$.

The Two-sided Laplace inversion formula

Lemma:
Consider a function $f(x) \in C^1$. If there exist a constant c and a monotone function $\bar{f}(x)$ such that $f(x) = e^{cx}\bar{f}(x)$, then then for any σ in the ROAC:

$$g_\sigma(x) := e^{-\sigma x} f(x) = e^{(c-\sigma)x}\bar{f}(x)$$

is of bounded variation on $\mathbb{R}$.

Proof:
It suffices to show that, for any $\sigma \in$ ROAC, $e^{-\sigma x} f(x)$ satisfies (i) $e^{-\sigma x} f(x) \in C^1$; (ii) $\int_{-\infty}^{+\infty} e^{-\sigma x}|f(x)|dx < +\infty$; and (iii) $\int_{-\infty}^{+\infty} |(e^{-\sigma x} f(x))'|dx < +\infty$. Conditions (i) and (ii) hold obviously. As regards (iii), since $f(x) = e^{cx}\bar{f}(x)$ with $\bar{f}(x)$ a monotone function, without loss of generality, we assume that $\bar{f}(x)$ is nondecreasing, i.e. $\bar{f}'(x) \geq 0$. Then:

$$\begin{aligned}
\int_{-\infty}^{+\infty} |(e^{-\sigma x} f(x))'|dx &= \int_{-\infty}^{+\infty} |(c-\sigma)e^{(c-\sigma)x}\bar{f}(x) + e^{(c-\sigma)x}\bar{f}'(x)|dx \\
&\leq |c-\sigma| \int_{-\infty}^{+\infty} |e^{(c-\sigma)x}\bar{f}(x)|dx + \int_{-\infty}^{+\infty} e^{(c-\sigma)x}\bar{f}'(x)dx \\
&= |c-\sigma| \int_{-\infty}^{+\infty} |e^{-\sigma x} f(x)|dx - (c-\sigma) \int_{-\infty}^{+\infty} e^{-\sigma x} f(x)dx
\end{aligned}$$

The second term in last equality by using integrate by part and the boundary term vanishes.

Discretization errors

In practice, one chooses a closed interval $[\sigma_1^*, \sigma_u^*] \subset$ ROAC to do the numerical inversion. Without loss of generality, we assume that $\sigma_1^* \sigma_u^* \neq 0$, and that σ_1^* and σ_u^* are both finite. The following theorem shows that introducing the discretization parameter C can make the discretization error decay exponentially for $\sigma \in (\sigma_1^*, \sigma_u^*)$.

Theorem: If there exists a nonnegative function $\delta(\cdot)$ such that, for any $\sigma \in [\sigma_1^*, \sigma_u^*]$, we have:

$$e^{-\sigma y}|f(y)| \leq \delta(\sigma) < +\infty \quad \text{for any } y$$

then, for any fixed $t \in \mathbb{R}$, $\sigma \in (\sigma_1^*, \sigma_u^*)$, and $C > 0$, we have the error bound :

$$|e_D(t, \sigma, C)| \leq \frac{\rho(\sigma, t)}{e^{\theta(\sigma)C} - 1}$$

where $\theta(\sigma) := 2\min\{\sigma_u^* - \sigma, \sigma - \sigma_1^*\} > 0$ and

$$\rho(\sigma, t) := \begin{cases} \delta(\sigma_u^*)e^{(2\sigma - \sigma_u^*)t} + \delta(\sigma_1^*)e^{(3\sigma_1^* - 2\sigma)t} & \text{if } t \geq 0 \\ \delta(\sigma_1^*)e^{(2\sigma - \sigma_1^*)t} + \delta(\sigma_u^*)e^{(3\sigma_u^* - 2\sigma)t} & \text{if } t < 0 \end{cases}$$

Discretization errors

Proof:
From $e^{-\sigma y}|f(y)| \leq \delta(\sigma) < +\infty$ for any y, we obtain:

$$|f(y)| \leq \begin{cases} \delta(\sigma_1^*)e^{\sigma_1^* y} & \text{for any } y \geq 0 \\ \delta(\sigma_u^*)e^{\sigma_u^* y} & \text{for any } y \leq 0 \end{cases} \tag{1}$$

We proceed to discuss two cases, $t \geq 0$ and $t < 0$. If $t \geq 0$ then, for any $C \geq 0$, we have:

$$2k(t + \text{sgn}(t)C) + t = 2k(t + C) + t \begin{cases} \geq 0 & \text{if } k > 0 \text{ and } t \geq 0 \\ \leq 0 & \text{if } k < 0 \text{ and } t \geq 0 \end{cases}$$

Then by (1), for any fixed $t \geq 0$, $\sigma \in (\sigma_1^*, \sigma_u^*)$, and $C > 0$:

$$\begin{aligned} |e_D(t, \sigma, C)| &:= \left| \sum_{k=-\infty\ k\neq 0}^{+\infty} e^{-2\sigma k(t+\text{sgn}(t)C)} f(2k(t + \text{sgn}(t)C) + t) \right| \\ &= \left| \sum_{k=-\infty\ k\neq 0}^{+\infty} e^{-2k(t+C)\sigma} f(2k(t + C) + t) \right| \\ &\leq \sum_{k=-\infty\ k\neq 0}^{+\infty} e^{-2k(t+C)\sigma} |f(2k(t + C) + t)| \end{aligned}$$

Discretization errors

Con.

$$
\begin{aligned}
(by(1)) &\leq \delta(\sigma_u^*) \sum_{k=-\infty}^{-1} e^{-2k(t+C)\sigma} e^{\sigma_u^*[2k(t+C)+t]} + \delta(\sigma_1^*) \sum_{k=1}^{\infty} e^{-2k(t+C)\sigma} e^{\sigma_1^*[2k(t+C)+t]} \\
(i) &= \delta(\sigma_u^*) e^{(2\sigma-\sigma_u^*)t} \frac{e^{-2C(\sigma_u^*-\sigma)}}{1-e^{-2(t+C)(\sigma_u^*-\sigma)}} + \delta(\sigma_1^*) e^{(3\sigma_1^*-2\sigma)t} \frac{e^{-2C(\sigma-\sigma_1^*)}}{1-e^{-2(t+C)(\sigma-\sigma_1^*)}} \\
&\leq \delta(\sigma_u^*) e^{(2\sigma-\sigma_u^*)t} \frac{e^{-2C(\sigma_u^*-\sigma)}}{1-e^{-2C(\sigma_u^*-\sigma)}} + \delta(\sigma_1^*) e^{(3\sigma_1^*-2\sigma)t} \frac{e^{-2C(\sigma-\sigma_1^*)}}{1-e^{-2C(\sigma-\sigma_1^*)}} \\
&= \delta(\sigma_u^*) e^{(2\sigma-\sigma_u^*)t} \frac{1}{e^{2C(\sigma_u^*-\sigma)}-1} + \delta(\sigma_1^*) e^{(3\sigma_1^*-2\sigma)t} \frac{1}{e^{2C(\sigma-\sigma_1^*)}-1} \\
&\leq \frac{\rho(\sigma,t)}{e^{\theta(\sigma)C}-1}
\end{aligned}
$$

For (i), the second term is geometric series has ratio $r_2 = e^{-2(t+C)(\sigma-\sigma_1^*)} \in (0,1)$. The first term is reindex $m=-k$, then it is geometric series has ratio $r_1 = e^{-2(t+C)(\sigma_u^*-\sigma)} \in (0,1)$. For $t \geq 0$, this is exactly with the upper $\rho(\sigma,t)$ branch.
If $t<0$, we have, for any $C \geq 0$:

$$
2k(t+\text{sgn}(t)C)+t = 2k(t-C)+t \begin{cases} \geq 0 & \text{if } k<0 \text{ and } t<0, \\ \leq 0 & \text{if } k>0 \text{ and } t<0. \end{cases}
$$

Discretization errors

By (1), the discretization error for any $t < 0$, $\sigma \in (\sigma_1^*, \sigma_u^*)$, and $C > 0$ can be bounded as follows:

$$
\begin{aligned}
|e_D(t,\sigma,C)| &= \left| \sum_{k=-\infty\ k\neq 0}^{+\infty} e^{-2k(t-C)\sigma} f(2k(t-C)+t) \right| \\
&\leq \sum_{k=-\infty\ k\neq 0}^{+\infty} e^{-2k(t-C)\sigma} |f(2k(t-C)+t)| \\
&\leq \delta(\sigma_u^*) \sum_{k=1}^{\infty} e^{-2k(t-C)\sigma} e^{\sigma_u^*[2k(t-C)+t]} + \delta(\sigma_1^*) \sum_{k=-\infty}^{-1} e^{-2k(t-C)\sigma} e^{\sigma_1^*[2k(t-C)+t]} \\
&= \delta(\sigma_u^*) e^{(3\sigma_u^*-2\sigma)t} \frac{e^{-2C(\sigma_u^*-\sigma)}}{1-e^{-2(C-t)(\sigma_u^*-\sigma)}} + \delta(\sigma_1^*) e^{(2\sigma-\sigma_1^*)t} \frac{e^{-2C(\sigma-\sigma_1^*)}}{1-e^{-2(C-t)(\sigma-\sigma_1^*)}} \\
&\leq \delta(\sigma_u^*) e^{(3\sigma_u^*-2\sigma)t} \frac{e^{-2C(\sigma_u^*-\sigma)}}{1-e^{-2C(\sigma_u^*-\sigma)}} + \delta(\sigma_1^*) e^{(2\sigma-\sigma_1^*)t} \frac{e^{-2C(\sigma-\sigma_1^*)}}{1-e^{-2C(\sigma-\sigma_1^*)}} \\
&= \delta(\sigma_u^*) e^{(3\sigma_u^*-2\sigma)t} \frac{1}{e^{2C(\sigma_u^*-\sigma)}-1} + \delta(\sigma_1^*) e^{(2\sigma-\sigma_1^*)t} \frac{1}{e^{2C(\sigma-\sigma_1^*)}-1} \\
&\leq \frac{\rho(\sigma,t)}{e^{\theta(\sigma)C}-1}
\end{aligned}
$$

Discretization errors

The error bound is computable since the function $\delta(\sigma)$ can be specified explicitly in many applications. This allows practitioners to pick a sufficiently large parameter C to control discretization error.

However, while the bound can be very small, it is hard to know how close it is to the "true" discretization error, because the exact error formula depends on the original function $f(\cdot)$, which is often unknown.

The tightness of the bound varies across cases, depending heavily on $\delta(\sigma)$ and the specific function considered.

Despite this uncertainty, numerical experiments in the paper show that by choosing a large enough C, the discretization error can be reduced to a level that guarantees the desired accuracy. Even though the bound may not always be tight, it is practically useful because it provides a reliable way to ensure small errors in computation.