MF921 Topics in Dynamic Asset Pricing

Stochastic Analysis & Stochastic Calculus in Quantitative Finance

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Week 2

Option Pricing via the Change of Numeraire Argument

Change of Numeraire: Motivation and Key Idea

In option pricing, we usually price under the risk-neutral measure using the money market account $B(t) = e^{rt}$ as the numeraire. But sometimes payoffs become simpler if we change the unit of measurement (the numeraire). Instead of measuring in "dollars," measure in "shares of stock".

The key idea is :

- Pick any strictly positive traded asset $N(t)$ as the numeraire.
- Then define a new probability measure $\tilde{\mathbb{P}}$ such that $\frac{S(t)}{N(t)}$ is a martingale under $\tilde{\mathbb{P}}$. No-arbitrage is preserved.

We first look at the details how this work (Radon Nikodym derivative & Girsanov Theorem) and then apply the scheme to price different type of options.

Change of Numeraire

Given $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}^*)$ with d-dim Brownian W:

- Money market account (baseline numeraire): $dB_t = B_t r_t dt$
- Traded asset $S(t)$: $dS_t = S_t(r_t dt + \sigma_t \, dW_t)$, $\quad \frac{S_t}{B_t}$ is a martingale.
- Derivative pricing rule: for payoff X_T at maturity T, $V_0 = \mathbb{E}^{\mathbb{P}^*}\left[\frac{X_T}{B_T}\right]$

Our goal is to pick another strictly positive traded asset $N(t)$ and define a new measure $\tilde{\mathbb{P}}$ such that $\frac{S(t)}{N(t)}$ is a martingale for every traded asset $S(t)$.

Change of Numeraire Con.

Oberve $\dfrac{S(t)}{B(t)}$ is a martingale under $\mathbb{P}^*$. We want $\dfrac{S(t)}{N(t)}$ to be a martingale under $\tilde{\mathbb{P}}$.
Define $\tilde{\mathbb{P}}$ via the Radon−Nikodym derivative with respect to $\mathbb{P}^*$:

$$\left.\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^*}\right|_{\mathcal{F}_T} = Z_T := \frac{N(T)/B(T)}{N(0)/B(0)}$$

By construction, $\frac{N(T)}{B(T)}$ is a martingale under $\mathbb{P}^*$, $Z_T > 0$ and $\mathbb{E}^{\mathbb{P}^*}[Z_T] = 1$ and take any payoff X_T:

$$V(0) = N(0)\,\mathbb{E}^{\tilde{\mathbb{P}}}\left[\frac{X_T}{N(T)}\right] = N(0)\mathbb{E}^{\mathbb{P}^*}\left[\frac{X_T}{N(T)}\,Z_T\right] = \mathbb{E}^{\mathbb{P}^*}\left[\frac{X_T}{B(T)}\right]$$

So the choice of Radon−Nikodym derivative guarantees the prices are consistent under both measures and no arbitrage is preserved.

Change of Numeraire Con.

What is the $dS(t)$ looks like under meausre $\tilde{\mathbb{P}}$?

Note: Under Q, we have $\begin{cases} dS(t) = r(t)S(t)\,dt + \sigma(t)S(t)\,dW(t) \\ dN(t) = r(t)N(t)\,dt + \gamma(t)N(t)\,dW(t) \end{cases}$

Denote $\widehat{N}_t = \frac{N_t}{B_t}$, apply Itô we get $\frac{d\widehat{N}_t}{\widehat{N}_t} = \gamma_t\,dW_t$, $\widehat{N}_t = \widehat{N}_0 e^{\left(\int_0^t \gamma_s \cdot dW_s - \frac{1}{2}\int_0^t \|\gamma_s\|^2\,ds\right)}$.

Oberve that $Z_t = \frac{\widehat{N}_t}{\widehat{N}_0} = e^{\left(\int_0^t \gamma_s \cdot dW_s - \frac{1}{2}\int_0^t \|\gamma_s\|^2\,ds\right)}$

Girsanov's theorem says: if we define a new measure $\tilde{\mathbb{P}}$ via this Z_t, then the process

$$\tilde{W}(t) \;=\; W(t) - \int_0^t \gamma_s dt$$

is a Brownian motion under $\tilde{\mathbb{P}}$. Substitute into $dS(t)$ to get the $\tilde{\mathbb{P}}$ dynamics:

$$dS(t) = S(t)\left[(r(t) + \sigma(t)\cdot\gamma(t))\,dt + \sigma(t)\cdot d\tilde{W}(t)\right]$$

$$S(t) = S_0 \exp\left(\int_0^t \left(r(s) + \sigma(s)\cdot\gamma(s) - \tfrac{1}{2}\|\sigma(s)\|^2\right)ds + \int_0^t \sigma(s)\cdot d\tilde{W}(s)\right)$$

Black-Scholes Formula

Given r, σ are constant, we have $S(T) = S(0)\exp\left\{(r - \frac{1}{2}\sigma^2)T + \sigma W(T)\right\}$.

The no-arbitrage price for the call option:

$$
\begin{aligned}
\psi_c(0) &= \mathbb{E}^{\mathbb{P}^*}(e^{-rT}(S(T)-K)^+) \\
&= \mathbb{E}^{\mathbb{P}^*}(e^{-rT}(S(T)-K)I(S(T)\geq K)) \\
&= \mathbb{E}^{\mathbb{P}^*}(e^{-rT}S(T)I(S(T)\geq K)) - Ke^{-rT}\mathbb{P}^*(S(T)\geq K) \\
&= I - Ke^{-rT}\cdot II
\end{aligned}
$$

For II:

$$
\begin{aligned}
II = \mathbb{P}^*(S(T)\geq K) = 1 - \Phi\left(\frac{\log(K/S(0)) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\
= \Phi\left(\frac{\log(S(0)/K) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)
\end{aligned}
$$

Note: Φ is the CDF of the standard normal distribution.

For I, we apply the change of numeraire and use stock itself as numeraire. Then based on the eraly definition we have $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^*}\Big|_{\mathcal{F}_T} = Z_T := e^{-rT}\frac{S(T)}{S(0)}$ and $\gamma_t = \sigma$. Therefore, under $\tilde{\mathbb{P}}$ we have the following dynamics of $S(t)$:

$$\frac{dS_t}{S_t} = rdt + \sigma^2 dt + \sigma d\tilde{W}_t,\ S(t) = S(0)\exp\left\{(r+\sigma^2/2)t + \sigma\tilde{W}_t\right\}$$

Then we can rewrite I:

$$\begin{aligned} I = S(0)\mathbb{E}^{\mathbb{P}^*}\left(e^{-rT}\frac{S(T)}{S(0)}I(S(T)\geq K)\right) &= S(0)\mathbb{E}^{\tilde{\mathbb{P}}}(I(S(T)\geq K)) \\ &= S(0)\tilde{\mathbb{P}}(S(T)\geq K) \\ &= S(0)\Phi\left(\frac{\log(S(0)/K) + (r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right). \end{aligned}$$

Putting together, we have the price of the call option is given by:

$$I - Ke^{-rT}\cdot II = S(0)\Phi(d_+) - Ke^{-rT}\Phi(d_-)$$

where $d_\pm = \frac{\log(S(0)/K)+(r\pm\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$.

One Dimensional Barrier Options

Barrier options are path-dependent derivatives whose payoff is activated (knock-in) or extinguished (knock-out) if the underlying asset crosses a pre-specified barrier. They extend vanilla calls/puts by adding a barrier condition.
We first study continuously monitored barriers and derive Merton's closed-form pricing formulas (1973) for single-barrier options.
Given $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}^*)$ with 1-dim Brownian W. The Market setting following:

$$dB(t) = B(t)rdt \ , \ dS(t) = rS(t)dt + \sigma S(t)dW(t)$$

A continuously monitored barrier option has payoff $=$ vanilla option payoff $\times$ indicator of the barrier condition. For example:

- Up-and-out call:
$$V_0 = \mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}(S(T)-K)^+ I\left\{\max_{0\leq t\leq T} S(t) \leq H\right\}\right], \quad H > S(0)$$
- Down-and-in put:
$$V_0 = \mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}(K-S(T))^+ I\left\{\min_{0\leq t\leq T} S(t) \leq H\right\}\right], \quad H < S(0)$$

Study the case of the down-and-in call option (DAIC) with strike K, barrier $H < S(0)$:

$$\text{DAIC} = e^{-rT}\mathbb{E}^{\mathbb{P}^*}\left[(S(T)-K)^+ I\left\{\min_{0\leq t\leq T} S(t) \leq H\right\}\right]$$

One Dimensional Barrier Options Con.

For notation simplicity, denote a drifted Brownian motion:

$$W_{\mu,\sigma}(t) = \mu t + \sigma W(t), \quad M_t = \max_{0\le s\le t} W_{\mu,\sigma}(s).$$

Some useful results from the reflection principle for a Brownian motion with a drift:
(i) When $x \le y$, $y > 0, \sigma > 0$:

- $P(W_{\mu,\sigma}(t) \le x, M_t \ge y) = e^{2\mu y/\sigma^2}\Phi\left(\frac{x-2y-\mu t}{\sigma\sqrt{t}}\right)$
- $P(W_{\mu,\sigma}(t) \le x, M_t \le y) = \Phi\left(\frac{x-\mu t}{\sigma\sqrt{t}}\right) - e^{2\mu y/\sigma^2}\Phi\left(\frac{x-2y-\mu t}{\sigma\sqrt{t}}\right)$

(ii) When $x \ge y > 0$, $\sigma > 0$:

- $P(W_{\mu,\sigma}(t) \le x, M_t \le y) = P(M_t \le y) = \Phi\left(\frac{y-\mu t}{\sigma\sqrt{t}}\right) - e^{2\mu y/\sigma^2}\Phi\left(\frac{-y-\mu t}{\sigma\sqrt{t}}\right)$
- $P(W_{\mu,\sigma}(t) \le x, M_t \ge y) = P(W_{\mu,\sigma}(t) \le x) - P(W_{\mu,\sigma}(t) \le x, M_t \le y) =$ $\Phi\left(\frac{x-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{y-\mu t}{\sigma\sqrt{t}}\right) + e^{2\mu y/\sigma^2}\Phi\left(\frac{-y-\mu t}{\sigma\sqrt{t}}\right)$

(iii) When $x \ge y$, $y < 0$, $\sigma > 0$:

- $P\left(W_{\mu,\sigma}(t) \ge x, \min_{0\le s\le t} W_{\mu,\sigma}(s) \le y\right) = e^{2\mu y/\sigma^2}\Phi\left(\frac{-x+2y+\mu t}{\sigma\sqrt{t}}\right)$

One Dimensional Barrier Options Con.

Back to the valuation of DAIC:

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}(S(T)-K)^+ I\left(\min_{0\le t\le T} S(t)\le H\right)\right]\\
&=\mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}(S(T)-K) I\left(S(T)\ge K, \min_{0\le t\le T} S(t)\le H\right)\right]\\
&=\mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}S(T) I\left(S(T)\ge K, \min_{0\le t\le T} S(t)\le H\right)\right]\\
&\quad -Ke^{-rT}P^*\left(S(T)\ge K, \min_{0\le t\le T} S(t)\le H\right)\\
&=I-Ke^{-rT}\cdot II
\end{aligned}
$$

For II:

$$
\begin{aligned}
II &= P^*\left(S(T)\ge K, \min_{0\le t\le T} S(t)\le H\right)\\
&= P\left\{W_{r-\frac{\sigma^2}{2},\sigma}(T)\ge \log(K/S(0)), \min_{0\le t\le T} W_{r-\frac{\sigma^2}{2},\sigma}(t)\le \log(H/S(0))\right\}\\
&= \exp\left\{\frac{2(r-\sigma^2/2)}{\sigma^2}\log(H/S(0))\right\}\cdot\Phi\left(\frac{2\log(H/S(0))-\log(K/S(0))+(r-\sigma^2/2)T}{\sigma\sqrt{T}}\right)
\end{aligned}
$$

One Dimensional Barrier Options Con.

For I, by changing of numeraire we can get:

$$
\begin{aligned}
I &= S(0)\mathbb{E}^{\mathbb{P}^*}\left(e^{-rT}\frac{S(T)}{S(0)}\cdot I\left\{S(T)\geq K,\min_{0\leq t\leq T}S(t)\leq H\right\}\right)\\
&= S(0)\tilde{P}\left(S(T)\geq K,\min_{0\leq t\leq T}S(t)\leq H\right)\\
&= S(0)P\left\{W_{r+\frac{\sigma^2}{2},\sigma}(T)\geq \log(K/S(0)),\min_{0\leq t\leq T}W_{r+\frac{\sigma^2}{2},\sigma}(t)\leq \log(H/S(0))\right\}\\
&= S(0)\cdot(H/S(0))^{\frac{2r}{\sigma^2}+1}\Phi\left(\frac{\log(\{H^2/S(0)\}/K)+(r+\sigma^2/2)T}{\sigma\sqrt{T}}\right)\\
&= (H/S(0))^{\frac{2r}{\sigma^2}-1}(H^2/S(0))\Phi\left(\frac{\log(\{H^2/S(0)\}/K)+(r+\sigma^2/2)T}{\sigma\sqrt{T}}\right)
\end{aligned}
$$

Putting the two terms together, we get $I - Ke^{-rT}\cdot II = (H/S(0))^{\frac{2r}{\sigma^2}-1}\text{BSC}(H^2/S(0))$.
Where BSC(x) is the Black-Scholes formula for a call option with the initial stock price being x:

$$
\text{BSC}(x) = x\Phi(d_+) - Ke^{-rT}\Phi(d_-) \text{ with } d_\pm = \frac{\log(x/K)+(r\pm\sigma^2/2)T}{\sigma\sqrt{T}}
$$

Exchange Options

Given $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}^*)$ with 2-dim independent Brownian, $W_1(t)$ and $W_2(t)$. We have two traded assets $S_1(t)$ and $S_2(t)$ with the following dynamics:

$$\frac{dS_1(t)}{S_1(t)} = rdt + \sigma_1 dW_1(t)$$
$$\frac{dS_2(t)}{S_2(t)} = rdt + \sigma_2 \left\{\rho dW_1(t) + \sqrt{1-\rho^2} dW_2(t)\right\}$$

The exchange option gives the holder the right, but not the obligation, to exchange asset S_2 for asset S_1 at maturity T. The price of this option as following:

$$\begin{aligned}
u(0) &= \mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}(S_1(T) - S_2(T))^+\right] \\
&= S_2(0)\mathbb{E}^{\mathbb{P}^*}\left[\frac{e^{-rT}S_2(T)}{S_2(0)}\left(\frac{S_1(T)}{S_2(T)} - 1\right)^+\right] \\
&= S_2(0)\mathbb{E}^{\tilde{\mathbb{P}}}\left[\left(\frac{S_1(T)}{S_2(T)} - 1\right)^+\right] \\
&= S_2(0)\mathbb{E}^{\tilde{\mathbb{P}}}\left[(F(T) - 1)^+\right]
\end{aligned}$$

Apply Itô, we have the Radon−Nikodym derivative for numeraire:

$$\left.\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^*}\right|_{\mathcal{F}_T} = Z_T^2 := \frac{e^{-rT}S_2(T)}{S_2(0)} = \exp\left[\sigma_2\left\{\rho W_1(T) + \sqrt{1-\rho^2}W_2(T)\right\} - \frac{T}{2}\sigma_2^2\right]$$

$$\left.\frac{d\hat{\mathbb{P}}}{d\mathbb{P}^*}\right|_{\mathcal{F}_T} = Z_T^1 := \frac{e^{-rT}S_1(T)}{S_1(0)} = \frac{e^{-rT}S_1(T)}{S_1(0)} = \exp\left(\sigma_1 W_1(T) - \frac{1}{2}\sigma_1^2 T\right)$$

By Girsanov theorem, under new measure $\tilde{\mathbb{P}}$:

$$\tilde{W}_1(t) = W_1(t) - \rho\sigma_2 t, \;\; \tilde{W}_2(t) = W_2(t) - \sigma_2\sqrt{1-\rho^2}t$$

Apply Itô, we can get $d\ln S_1$, $d\ln S_2$:

$$\begin{aligned} d\ln F(t) &= d\ln S_1(t) - d\ln S_2(t) \\ &= \left[-\tfrac{1}{2}\sigma_1^2 - \tfrac{1}{2}\sigma_2^2 + \rho\sigma_1\sigma_2\right]dt + (\sigma_1 - \rho\sigma_2)d\tilde{W}_1 - \sigma_2\sqrt{1-\rho^2}d\tilde{W}ć_2. \end{aligned}$$

Apply Itô to $g(x) = e^x$ with $x = \ln F(t)$:

$$\frac{dF_t}{F_t} = d(\ln F_t) + \frac{1}{2}d < \ln F >_t = (\sigma_1 - \rho\sigma_2)d\tilde{W}_{1t} - \sigma_2\sqrt{1-\rho^2}d\tilde{W}_{2t}$$

Exchange Options Con.

Denote $\sigma = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$, $\tilde{W}(t) := \frac{1}{\sigma}\left\{(\sigma_1 - \rho\sigma_2)\tilde{W}_1(t) - \sigma_2\sqrt{1-\rho^2}\tilde{W}_2(t)\right\}$
Observe that $\tilde{W}$ is a standard Brownian motion under $\tilde{\mathbb{P}}$. We have $\frac{dF(t)}{F(t)} = \sigma d\tilde{W}(t)$, observe that $F_T = F_0 \exp\left(-\frac{1}{2}\sigma^2 T + \sigma\sqrt{T}Z\right), Z \sim N(0,1)$ under $\tilde{\mathbb{P}}$. Similarly, we have $F_T = F_0 \exp\left(\frac{1}{2}\sigma^2 T + \sigma\sqrt{T}Z\right), Z \sim N(0,1)$ under $\hat{\mathbb{P}}$.

Then we can rewrite $u(0)$:

$$
\begin{aligned}
u(0) &= S_2(0)\mathbb{E}^{\tilde{\mathbb{P}}}\left[(F(T)-1)^+\right] \\
&= S_2(0)\mathbb{E}^{\tilde{\mathbb{P}}}\left[(F(T)-1)I(F(T)>1)\right] \\
&= S_2(0)\left[\mathbb{E}^{\tilde{\mathbb{P}}}[F_T I\{F_T>1\}] - \tilde{\mathbb{P}}(F_T>1)\right] \\
&= S_2(0)\left[\mathbb{E}^{\mathbb{P}^*}[\frac{e^{-rT}S_2(T)}{S_2(0)}\frac{S_1(T)}{S_2(T)}I\{F_T>1\}] - \tilde{\mathbb{P}}(F_T>1)\right] \\
&= S_2(0)\left[\frac{1}{S_2(0)}\mathbb{E}^{\mathbb{P}^*}[\frac{e^{-rT}S_1(T)}{S_1(0)}S_1(0)I\{F_T>1\}] - \tilde{\mathbb{P}}(F_T>1)\right]
\end{aligned}
$$

Exchange Options Con.

$$
\begin{aligned}
=&S_2(0)\left[\frac{S_1(0)}{S_2(0)}\mathbb{E}^{\hat{\mathbb{P}}}[I\{F_T>1\}]-\tilde{\mathbb{P}}(F_T>1)\right]\\
=&S_1(0)\hat{\mathbb{P}}[I\{F_T>1\}]-S_2(0)\tilde{\mathbb{P}}(F_T>1)\\
=&S_1(0)\Phi(d_+)-S_2(0)\Phi(d_-)
\end{aligned}
$$

Where:

$$
d_\pm=\frac{\log(F(0))\pm\frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}=\frac{\log(S_1(0)/S_2(0))\pm\frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}.
$$

- (i) If the second asset is cash, or $S_2(t)=Ke^{-r(T-t)}$, then the formula degenerates to the Black-Scholes formula.
- (ii) The hedging strategy is given by long $\Phi(d_+)$ shares of the first asset and short $\Phi(d_-)$ shares of the second asset.

Two-Dimensional Barrier Options

Suppose we have two Wiener processes, $X(t)$ and $Y(t)$, governed by the following dynamics

$$dX(t) = \mu_1 dt + \sigma_1 dW_1(t), \quad X(0) = 0, \quad \sigma_1 > 0,$$
$$dY(t) = \mu_2 dt + \sigma_2 \left\{ \rho dW_1(t) + \sqrt{1-\rho^2} dW_2(t) \right\}, \quad Y(0) = 0, \quad \sigma_2 > 0,$$

where the W_1 and W_2 are two independent standard Brownian motions.
For $b > 0$, consider the first passage time of the process $Y(t)$:

$$\tau_b^Y = \inf\{t \geq 0 : Y(t) = b > 0\}.$$

We shall prove that the joint distribution between $X(T)$ and the first passage time of $Y(t)$ is given by:

$$P(X(T) < a, \tau_b^Y > T) = P\left(X(T) < a, \max_{0 \leq t \leq T} Y(t) < b\right)$$

$$= \Phi_2\left(\frac{a - \mu_1 T}{\sigma_1\sqrt{T}}, \frac{b - \mu_2 T}{\sigma_2\sqrt{T}}; \rho\right) - e^{2\mu_2 b/\sigma_2^2} \Phi_2\left(\frac{a - \mu_1 T - 2\rho b \sigma_1/\sigma_2}{\sigma_1\sqrt{T}}, \frac{-b - \mu_2 T}{\sigma_2\sqrt{T}}; \rho\right)$$

where $b > 0$ and $\Phi_2(x, y; \rho)$ denotes the bivariate normal distribution given by

$$\Phi_2(x, y; \rho) = P(Z_1 \leq x, Z_2 \leq y),$$

with Z_1 and Z_2 being two standard normal random variables with correlation ρ.

Two-Dimensional Barrier Options Con.

Remark:

- Above equation holds for both $a \geq b$ and $a \leq b$, as long as $b > 0$. That's more general than the 1D reflection principle formulas which needed to be split into separate cases depending on $a \leq b$ or $a \geq b$.
- when $\rho = 1$, $\mu_1 = \mu_2 = \mu$, $\sigma_1 = \sigma_2 = \sigma$, the two dimensional case reduces to the one-dimensional case, as it becomes:

$$
\begin{aligned}
&P\left(X(T) < a, \max_{0 \leq t \leq T} X(t) < b\right) \\
&= \Phi_2\left(\frac{a - \mu T}{\sigma\sqrt{T}}, \frac{b - \mu T}{\sigma\sqrt{T}}; 1\right) - e^{2\mu b/\sigma^2}\Phi_2\left(\frac{a - \mu T - 2b}{\sigma\sqrt{T}}, \frac{-b - \mu T}{\sigma\sqrt{T}}; 1\right) \\
&= P\left\{Z \leq \frac{a - \mu T}{\sigma\sqrt{T}}, Z \leq \frac{b - \mu T}{\sigma\sqrt{T}}\right\} - e^{2\mu b/\sigma^2} P\left\{Z \leq \frac{a - \mu T - 2b}{\sigma\sqrt{T}}, Z \leq \frac{-b - \mu T}{\sigma\sqrt{T}}\right\} \\
&= P\left\{Z \leq \min\left\{\frac{a - \mu T}{\sigma\sqrt{T}}, \frac{b - \mu T}{\sigma\sqrt{T}}\right\}\right\} - e^{2\mu b/\sigma^2} P\left\{Z \leq \min\left\{\frac{a - \mu T - 2b}{\sigma\sqrt{T}}, \frac{-b - \mu T}{\sigma\sqrt{T}}\right\}\right\}
\end{aligned}
$$

Which incorporates two cases in one dimensional case.

Next we proof the formula of the joint distribution between $X(T)$ and the first passage time of $Y(t)$:
[Proof]
Consider the case of $\sigma_1 = \sigma_2 = 1$. Define a new process $V(t)$ to decouple X and Y:

$$V(t) := X(t) - \rho Y(t)$$

First check independence between V and Y:

$$\begin{aligned} dV(t)dY(t) &= (dX(t) - \rho dY(t))dY(t) \\ &= \left((1-\rho^2)dW_1 - \rho\sqrt{1-\rho^2}dW_2\right) \cdot \left(\rho dW_1 + \sqrt{1-\rho^2}dW_2\right) \\ &= (1-\rho^2)\rho(dW_1)^2 - \rho(1-\rho^2)(dW_2)^2 \\ &= (1-\rho^2)\rho dt - (1-\rho^2)\rho dt = 0 \end{aligned}$$

Since $V(T) = X(T) - \rho Y(T)$, it is Gaussian. Its mean is:

$$\mathbb{E}[V(T)] = \mu_1 T - \rho\mu_2 T$$

Its variance is:

$$\begin{aligned}\mathsf{Var}(V(T)) &= \mathsf{Var}(X(T)) + \rho^2\mathsf{Var}(Y(T)) - 2\rho\mathsf{Cov}(X(T), Y(T))\\ &= T + \rho^2 T - 2\rho^2 T = (1-\rho^2)T\end{aligned}$$

Thus:

$$V(T) \sim N((\mu_1 - \rho\mu_2)T, (1-\rho^2)T).$$

Incidentally, the same logic applying to two standard normal random variables with correlation ρ also leads to a representation for the bivariate normal distribution:

$$\Phi_2(\alpha, \beta; \rho) = \int_{z_2=-\infty}^{\beta}\int_{z_1=-\infty}^{\alpha} \frac{1}{\sqrt{1-\rho^2}}\varphi\left(\frac{z_1 - \rho z_2}{\sqrt{1-\rho^2}}\right)\varphi(z_2)\, dz_1\, dz_2.$$

Where $\varphi(\cdot)$ is the standard normal density function, $\varphi(z) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)$.

Two-Dimensional Barrier Options Con.

Now, in terms of $V(T)$, we can rewrite $P(X(T) < a, \tau_b^Y > T)$ as:

$$P(X(T) < a, \tau_b^Y > T)$$
$$= \int_{x=-\infty}^{a} \int_{y=-\infty}^{b} P(X(T) \in dx, Y(T) \in dy, \tau_b^Y > T)$$

Note:the transformation is linear with determinant 1 and the independence of V and Y

$$= \int_{x=-\infty}^{a} \int_{y=-\infty}^{b} P(V(T) \in dx - \rho dy) P(Y(T) \in dy, \tau_b^Y > T)$$

There are two terms inside the integrand. For the first term since $V(T)$ has a normal distribution with mean $\mu_1 T - \rho \mu_2 T$ and variance $(1-\rho^2)T$, we have:

$$P(V(T) \in dx - \rho dy) = \frac{1}{\sqrt{(1-\rho^2)T}} \varphi\left(\frac{x - \rho y - \mu_1 T + \rho \mu_2 T}{\sqrt{(1-\rho^2)T}}\right),$$

For the second term, recall the eraly result in one-dim, When $x \leq y$, $y > 0, \sigma > 0$:

$$P(W_{\mu,\sigma}(t) \leq x, M_t \leq y) = \Phi\left(\frac{x - \mu t}{\sigma\sqrt{t}}\right) - e^{2\mu y/\sigma^2}\Phi\left(\frac{x - 2y - \mu t}{\sigma\sqrt{t}}\right)$$

We have for all $y < b, b > 0$:

$$P(Y(T) \leq y, \tau_b^Y > T) = \Phi\left(\frac{y - \mu_2 T}{\sqrt{T}}\right) - e^{2\mu_2 b}\Phi\left(\frac{y - 2b - \mu_2 T}{\sqrt{T}}\right).$$

Differentiating the above equation yields:

$$P(Y(T) \in dy, \tau_b^Y > T) = \frac{1}{\sqrt{T}}\varphi\left(\frac{y - \mu_2 T}{\sqrt{T}}\right) - \frac{1}{\sqrt{T}}e^{2\mu_2 b}\varphi\left(\frac{y - 2b - \mu_2 T}{\sqrt{T}}\right).$$

Plugging the above two terms into:

$$\int_{x=-\infty}^{a}\int_{y=-\infty}^{b} P(V(T) \in dx - \rho dy)P(Y(T) \in dy, \tau_b^Y > T)$$

Two-Dimensional Barrier Options Con.

$$P(X(T) < a, \tau_b^Y > T) = I - II$$

where:

$$I = \int_{x=-\infty}^{a} \int_{y=-\infty}^{b} \frac{1}{\sqrt{(1-\rho^2)T}} \varphi\left(\frac{x - \rho y - \mu_1 T + \rho \mu_2 T}{\sqrt{(1-\rho^2)T}}\right) \frac{1}{\sqrt{T}} \varphi\left(\frac{y - \mu_2 T}{\sqrt{T}}\right) dy dx,$$

$$II = e^{2\mu_2 b} \int_{x=-\infty}^{a} \int_{y=-\infty}^{b} \frac{1}{\sqrt{(1-\rho^2)T}} \varphi\left(\frac{x - \rho y - \mu_1 T + \rho \mu_2 T}{\sqrt{(1-\rho^2)T}}\right) \frac{1}{\sqrt{T}} \varphi\left(\frac{y - 2b - \mu_2 T}{\sqrt{T}}\right) dy dx,$$

and $\varphi(x) = e^{-x^2/2}/\sqrt{2\pi}$ is the standard normal density function.

With $\tilde{x} = \frac{x-\mu_1 T}{\sqrt{T}}, \quad \tilde{y} = \frac{y-\mu_2 T}{\sqrt{T}}$, Then:

$$dx = \sqrt{T} d\tilde{x}, \quad dy = \sqrt{T} d\tilde{y}$$

$$x \le a \iff \tilde{x} \le \frac{a - \mu_1 T}{\sqrt{T}}, \quad y \le b \iff \tilde{y} \le \frac{b - \mu_2 T}{\sqrt{T}}$$

we have

$$I = \int_{x=-\infty}^{a} \int_{y=-\infty}^{b} \frac{1}{\sqrt{(1-\rho^2)T}} \varphi\left(\frac{\tilde{x}-\rho\tilde{y}}{\sqrt{(1-\rho^2)}}\right) \frac{1}{\sqrt{T}} \varphi(\tilde{y}) dy dx$$

$$= \int_{-\infty}^{\frac{a-\mu_1 T}{\sqrt{T}}} \int_{-\infty}^{\frac{b-\mu_2 T}{\sqrt{T}}} \frac{1}{\sqrt{(1-\rho^2)}} \varphi\left(\frac{\tilde{x}-\rho\tilde{y}}{\sqrt{(1-\rho^2)}}\right) \varphi(\tilde{y}) d\tilde{y} d\tilde{x}$$

By the conditional-Gaussian factorization, this integrand is exactly the joint pdf of a standard bivariate normal (Z_1, Z_2) with correlation ρ:

$$f_{Z_1,Z_2}(\tilde{x},\tilde{y}) = \frac{1}{\sqrt{1-\rho^2}} \varphi\left(\frac{\tilde{x}-\rho\tilde{y}}{\sqrt{1-\rho^2}}\right) \varphi(\tilde{y})$$

Hence the double integral is, by definition,

$$I = \Phi_2\left(\frac{a-\mu_1 T}{\sqrt{T}}, \frac{b-\mu_2 T}{\sqrt{T}}; \rho\right)$$

Two-Dimensional Barrier Options Con.

Similarly, with

$$\hat{x} = \frac{x - \mu_1 T - 2\rho b}{\sqrt{T}}, \quad \hat{y} = \frac{y - 2b - \mu_2 T}{\sqrt{T}}$$

simplifying the term II yields

$$\begin{aligned}
II &= \int_{x=-\infty}^{a} \int_{y=-\infty}^{b} \frac{1}{\sqrt{(1-\rho^2)T}} \varphi\left(\frac{\hat{x} - \rho\hat{y}}{\sqrt{(1-\rho^2)}}\right) \frac{1}{\sqrt{T}} e^{2\mu_2 b} \varphi(\hat{y}) dy dx \\
&= \int_{-\infty}^{\frac{a-\mu_1 T - 2\rho b}{\sqrt{T}}} \int_{-\infty}^{\frac{-b-\mu_2 T}{\sqrt{T}}} \frac{1}{\sqrt{(1-\rho^2)}} \varphi\left(\frac{\hat{x} - \rho\hat{y}}{\sqrt{(1-\rho^2)}}\right) e^{2\mu_2 b} \varphi(\hat{y}) dy dx \\
&= e^{2\mu_2 b} \Phi_2\left(\frac{a - \mu_1 T - 2\rho b}{\sqrt{T}}, \frac{-b - \mu_2 T}{\sqrt{T}}; \rho\right),
\end{aligned}$$

from which the result follows. The general case can be reduced to this particular case by letting:

$$\tilde{X}(t) = X(t)/\sigma_1, \quad \tilde{Y}(t) = Y(t)/\sigma_2,$$
$$\tilde{b} = b/\sigma_2, \quad \tilde{a} = a/\sigma_1, \quad \tilde{\mu}_1 = \mu_1/\sigma_1, \quad \tilde{\mu}_2 = \mu_2/\sigma_2.$$

Two-Dimensional Barrier Options Con.

Given the joint distribution between $X(T)$ and the first passage time of $Y(t)$ by :

$$P(X(T) < a, \tau_b^Y > T) = P\left(X(T) < a, \max_{0\le t\le T} Y(t) < b\right)$$

$$= \Phi_2\left(\frac{a-\mu_1 T}{\sigma_1\sqrt{T}}, \frac{b-\mu_2 T}{\sigma_2\sqrt{T}}; \rho\right) - e^{\frac{2\mu_2 b}{\sigma_2^2}}\Phi_2\left(\frac{a-\mu_1 T - 2\rho b\sigma_1/\sigma_2}{\sigma_1\sqrt{T}}, \frac{-b-\mu_2 T}{\sigma_2\sqrt{T}}; \rho\right)$$

Remark:
(i) Using the facts that $P(X(T) > a, \tau_b^Y > T) = P(-X(T) < -a, \tau_b^Y > T)$, that the correlation between $-X(t)$ and $Y(t)$ is $-\rho$,we can show that for $b > 0$(the following equation will use for next example to price of an up-and-out option):

$$P(X(T) > a, \tau_b^Y > T) = P\left(-X(T) < -a, \max_{0\le t\le T} Y(t) < b\right)$$

$$= \Phi_2\left(-\frac{a-\mu_1 T}{\sigma_1\sqrt{T}}, \frac{b-\mu_2 T}{\sigma_2\sqrt{T}}; -\rho\right) - e^{\frac{2\mu_2 b}{\sigma_2^2}}\Phi_2\left(-\frac{a-\mu_1 T - 2\rho b\sigma_1/\sigma_2}{\sigma_1\sqrt{T}}, \frac{-b-\mu_2 T}{\sigma_2\sqrt{T}}; -\rho\right)$$

(ii) Using the fact that $P(X(T) < a, \tau_{-b}^Y > T) = P(X(T) < a, \tau_b^{-Y} > T)$, that the correlation between $X(t)$ and $-Y(t)$ is $-\rho$, we can show that for $b > 0$:

$$P(X(T) < a, \tau_{-b}^Y > T) = P\left(X(T) < a, \min_{0\le t\le T} Y(t) > -b\right) = P\left(X(T) < a, \max_{0\le t\le T} (-Y(t)) < b\right)$$

$$= \Phi_2\left(\frac{a-\mu_1 T}{\sigma_1\sqrt{T}}, \frac{b+\mu_2 T}{\sigma_2\sqrt{T}}; -\rho\right) - e^{\frac{2\mu_2 b}{\sigma_2^2}}\Phi_2\left(\frac{a-\mu_1 T + 2\rho b\sigma_1/\sigma_2}{\sigma_1\sqrt{T}}, \frac{-b+\mu_2 T}{\sigma_2\sqrt{T}}; -\rho\right)$$

Two-Dimensional Barrier Options Con.

Let's calculate the price of an up-and-out call option, we have the following set up:

$$U_0 = e^{-rT}\mathbb{E}^{\mathbb{P}^*}\left[(S_1(T)-K)^+ I\left\{\max_{0\le t\le T} S_2(t)\le H\right\}\right], \quad S_i(t)=S_i(0)e^{X_i(t)}, X_1=X, X_2=Y$$

Under the risk-neutral measure $\mathbb{P}^*$,

$$dX(t)=\mu_1 dt+\sigma_1 dW_1(t), \quad \mu_1 = r-\frac{1}{2}\sigma_1^2,$$

$$dY(t)=\mu_2 dt+\sigma_2\left\{\rho dW_1(t)+\sqrt{1-\rho^2}dW_2(t)\right\}, \quad \mu_2 = r-\frac{1}{2}\sigma_2^2,$$

with W_1, W_2 independent.
Write the barrier level in log space

$$b := \log\frac{H}{S_2(0)}, \quad \text{and } a := \log\frac{K}{S_1(0)}$$

Then we have:

$$\{\max S_2(t)\le H\} = \left\{\max S_2(0)e^{Y(t)}\le H\right\} = \left\{\max_{0\le t\le T} Y(t)\le b\right\}$$

$$\{S_1(T)>K\} = \{S_1(0)e^{X(t)}>K\} = \{X(T)>a\}$$

Two-Dimensional Barrier Options Con.

$$
\begin{aligned}
U_0 &= e^{-rT}\mathbb{E}^{\mathbb{P}^*}\left[(S_1(T)-K)^+ I\left\{\max_{0\le t\le T} S_2(t)\le H\right\}\right] \\
&= e^{-rT}\mathbb{E}^{\mathbb{P}^*}\left[(S_1(T)-K) I\left\{S_1(T)>K \max_{0\le t\le T} S_2(t)\le H\right\}\right] \\
&= e^{-rT}\mathbb{E}^{\mathbb{P}^*}\left[S_1(T) I\left\{X(T)>a,\max_{0\le t\le T} Y(t)\le b\right\}\right] - Ke^{-rT}\mathbb{P}^*(X(T)>a,\max_{0\le t\le T} Y(t)\le b) \\
&= I - Ke^{-rT}\cdot II
\end{aligned}
$$

Apply Itô, we have the Radon−Nikodym derivative for numeraire:

$$
\left.\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^*}\right|_{\mathcal{F}_T} = Z_T := \frac{e^{-rT}S_1(T)}{S_1(0)} = \exp\left[\sigma_1 W_1(T) - \frac{1}{2}\sigma_1^2 T\right]
$$

By Girsanov theorem, under new measure $\tilde{\mathbb{P}}$:

$$
\tilde{W}_1(t) = W_1(t) - \sigma_1 t,\ \ \tilde{W}_2(t) = W_2(t)
$$

And the drifts of X and Y under $\tilde{\mathbb{P}}$ become:

$$
\mu_1^{(1)} = \mu_1 + \sigma_1^2 = r + \frac{1}{2}\sigma_1^2, \quad \mu_2^{(1)} = \mu_2 + \rho\sigma_1\sigma_2 = r - \frac{1}{2}\sigma_2^2 + \rho\sigma_1\sigma_2
$$

For I, using change of numeraire and the formula in remark (i) :

$$
\begin{aligned}
I =&\mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}S_1(T)I\left\{X(T)>a,\ \max_{0\le t\le T}Y(t)\le b\right\}\right]\\
=&\mathbb{E}^{\mathbb{P}^*}\left[\frac{e^{-rT}S_1(T)}{S_1(0)}S_1(0)I\left\{X(T)>a,\ \max_{0\le t\le T}Y(t)\le b\right\}\right]\\
=&S_1(0)\tilde{\mathbb{P}}\left\{X(T)>a,\ \max_{0\le t\le T}Y(t)\le b\right\}
\end{aligned}
$$

$$
= S_1(0)\left[\Phi_2\left(-\frac{a-\mu_1^{(1)}T}{\sigma_1\sqrt{T}},\frac{b-\mu_2^{(2)}T}{\sigma_2\sqrt{T}};-\rho\right)-e^{\frac{2\mu_2^{(2)}b}{\sigma_2^2}}\Phi_2\left(-\frac{a-\mu_1^{(1)}T-2\rho b\sigma_1/\sigma_2}{\sigma_1\sqrt{T}},\frac{-b-\mu_2^{(2)}T}{\sigma_2\sqrt{T}};-\rho\right)\right]
$$

For II, apply the formula in remakr(i) directly:

$$
\begin{aligned}
II =&\mathbb{P}^*(X(T)>a,\ \max_{0\le t\le T}Y(t)\le b)\\
=&\Phi_2\left(-\frac{a-\mu_1T}{\sigma_1\sqrt{T}},\frac{b-\mu_2T}{\sigma_2\sqrt{T}};-\rho\right)-e^{\frac{2\mu_2b}{\sigma_2^2}}\Phi_2\left(-\frac{a-\mu_1T-2\rho b\sigma_1/\sigma_2}{\sigma_1\sqrt{T}},\frac{-b-\mu_2T}{\sigma_2\sqrt{T}};-\rho\right)
\end{aligned}
$$

Part II: Chapter 17

Introduction to Stochastic Calculus for Jump Processes

Counting Processes

A counting process $N(t)$: tracks the number of events up to time t. We have the following Key properties of any counting process:

- $N(t) \geq 0$
- Takes integer values
- Non-decreasing ($N(s) \leq N(t)$ if $s < t$)
- Increment $N(t) - N(s)$ counts events in $(s, t]$

To make analysis tractable, the following assumptions are typically imposed:

- Independent increments: the number of events occurring in disjoint time intervals is statistically independent
- Stationary increments: distribution of increments depends only on interval length, not location.

These two properties are also underpin the definition of Brownian motion.

Two Equivalent Definitions of Poisson Processes

First definition(I) of a Poisson process:
Counting process $\{N(t), t \geq 0\}$ with rate $\lambda > 0$ such that :

1. $N(0) = 0$
2. The process exhibits independent and stationary increments
3. For each $t \geq 0$, the random variable $N(t)$ follows a Poisson distribution:

$$\mathbb{P}[N(t) = n] = e^{-\lambda t}\frac{(\lambda t)^n}{n!} \quad \text{for } n = 0, 1, 2, \ldots, \quad \mathbb{E}[N(t)] = \lambda t$$

Second definition(II) of a Poisson process:
Counting process $\{N(t), t \geq 0\}$ with rate $\lambda > 0$ such that :

1. $N(0) = 0$
2. The process exhibits independent and stationary increments
3. $\mathbb{P}[N(h) = 1] = \lambda h + o(h)$
4. $\mathbb{P}[N(h) \geq 2] = o(h)$

Two Equivalent Definitions of Poisson Processes Con.

We show the equivalence of the two definitions:
Proof:
(i) $I \Rightarrow II$
For small h, use Taylor expansion of the exponential:

$$\mathbb{P}(N(h)=1)=e^{-\lambda h}(\lambda h)=(1-\lambda h+\frac{1}{2}(\lambda h)^2+o(h^2))(\lambda h)=\lambda h+o(h)$$

Similarly:

$$\mathbb{P}(N(h)\geq 2)=1-\mathbb{P}(0)-\mathbb{P}(1)=1-\left(1-\lambda h+\frac{1}{2}(\lambda h)^2+o(h^2)\right)-(\lambda h+o(h))=o(h)$$

(ii) $II \Rightarrow I$(Some Intuition)
Partition $(0,t]$ into m small subintervals of length $h=t/m$ and define increments $X_i=N(ih)-N((i-1)h)$:

$$N(t)=\sum_{i=1}^{m} X_i$$

Let $m\to\infty$, by the small-interval conditions:

$$\mathbb{P}\{X_i=1\}=\lambda h+o(h),\quad \mathbb{P}\{X_i=0\}=1-\lambda h+o(h),\quad \mathbb{P}\{X_i\geq 2\}=o(h)$$

Probability of ≥ 2 events in a subinterval is negligible $(o(h))$. So each X_i behaves like a Bernoulli(λh). due to the independent increments property, the X_i are mutually independent. Since the sum of independent and identically distributed Bernoulli random variables follows a binomial distribution, it follows that $N(t)$ is approximately binomial with parameters m and $p = \lambda h$.

As $m \to \infty$ and $p \to 0$, the classical Poisson approximation to the binomial distribution implies that $N(t)$ converges in distribution to a Poisson random variable with rate

$$mp = m\lambda h = m\lambda \frac{t}{m} = \lambda t,$$

which precisely corresponds to the distribution given in equation of the requirement three in the first definition. Hence, this approximation provides intuition for the equivalence of the two definitions.

Remark: Powerful tool for modeling infrequent extreme events. In financial contexts, poisson processes can capture market shocks and discontinuities missed by continuous-path models. Important for pricing derivatives sensitive to jump risk.