MF921 Topics in Dynamic Asset Pricing

Week 5

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Black-Scholes (II): Dominance-Free Interval and Risk Neutral Pricing

A General Brownian Market Model

Given a complete probablity space $(\Omega, \mathcal{F}, \mathbb{P})$. $W(t) = (W_1(t), \ldots, W_d(t))^\top$, independent d-dimensional Brownian motion. The filtraion $\mathcal{F}_t^W = \sigma(W(s) : 0 \leq s \leq t)$ which is complete and right-continuous.

A financial market $\mathcal{M}$ with 1 bond and d stocks under a finite horizon $[0, T]$:

$$dS_0(t) = r(t)S_0(t)dt, \quad S_0(0) = 1$$

$$dS_i(t) = S_i(t)\left(b_i(t)\,dt + \sum_{j=1}^{d} \sigma_{ij}(t)\,dW_j(t)\right), \quad \text{for } i \in 1, 2...., d$$

- $r(t)$: interest rate
- $b(t) = (b_1, \ldots, b_d)$: appreciation rates
- $\sigma(t) = (\sigma_{ij}(t))$: volatility matrix
- $r(t)$, $b(t)$ and $\sigma(t)$ all progressively measurable with respect to $\{\mathcal{F}_t\}$ and bounded uniformly in $(t, \omega) \in [0, T] \times \Omega$.

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Remarks on the Modell

- The model is very general: processes can be dependent, time-varying, and even non-Markovian
- The model does not cover stochastic volatility or jump models. Therefore, dominance arguments do not apply for stochastic volatility or jump models.
- In general, the option price is not pinned to a single number. Instead, there exists an interval $[h_{low}, h_{up}]$. If the option price is within the interval then there will be no dominance, outside the interval there will be dominance opportunity.
- Ideal market assumption: Infinite divisibility of assets, no transaction costs or taxes, no borrowing/short-selling constraints and same interest rate for borrowing and lending

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Introducing auxiliary processes

Relative risk (Sharpe ratio):

$$\theta(t) = \sigma^{-1}(t)\,(b(t) - r(t)\mathbf{1})\,, \quad \mathbf{1} = (1, 1, \ldots\ldots, 1)^T$$

Exponential martingale (RN derivative):

$$Z(t) = \exp\left(-\int_0^t \theta^\top(s)\,dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds\right)$$

Discount factor:

$$\gamma(t) = \exp\left(-\int_0^t r(s)\,ds\right)$$

Brownian motion with drift:

$$W_0(t) = W(t) + \int_0^t \theta(s)\,ds, \quad 0 \le t \le T$$

$\sigma(t)$ invertible, inverses bounded. Ensures bounded $\theta(t)$ and $Z(t)$ is a true martingale. These tools set up the risk-neutral framework for pricing options.

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Investor Setup

Investor type: "small investor" $\rightarrow$ cannot affect market prices.
Trading Strategy defined as:

$$\phi(t) = (\phi_1(t), \ldots, \phi_d(t))$$

Where $\phi_0(t)$ is number of bonds held, $\phi_i(t)$ is number of shares of stock i held. If $\phi_0 < 0$ or $\phi_i < 0$ interpret as short position(loan). All decisions are adapted to $\mathcal{F}_t$.
Cumulative consumption process defined as:

$$C(t) = \int_0^t c(s)ds, \quad c(s) \geq 0$$

$C(t)$ is non-decreasing (consumption can only increase). $C(0) = 0, C(T) < \infty$ almost surely. Models the total amount consumed by the investor up to time t

To ensure consistency, we need to specify what trading strategies are allowed in the market. Two rules impose in the market, self-financing condition and exclusion of doubling strategies.

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Self-Financing Condition

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$$\sum_{i=0}^{d} \phi_i(t) S_i(t) = \sum_{i=0}^{d} \phi_i(0) S_i(0) + \sum_{i=0}^{d} \int_0^t \phi_i(u) dS_i(u) - C(t), \quad 0 \le t \le T$$

Changes in wealth = trading gains – consumption, no outside cash flows.

A portfolio process is defined as $\pi(\cdot) = (\pi_1(\cdot), \dots, \pi_d(\cdot))$, where $\pi_i(t) = \phi_i(t) S_i(t)$ means that total amount of money invested in the ith risky asset. The total wealth $X(t)$ is equal to:

$$X(t) = \sum_{i=0}^{d} \phi_i(t) S_i(t) = \sum_{i=1}^{d} \pi_i(t) + \phi_0(t) S_0(t)$$

Both the wealth process $X(\cdot)$ and the portfolio $\pi(\cdot)$ can clearly take both positive and negative values.

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Self-Financing Condition

For a given initial capital x and a portfolio process $\pi(\cdot)$, the self-financing condition is translated to:

$$X(t) = X(0) + \sum_{i=0}^{d} \int_0^t \phi_i(u) dS_i(u) - C(t)$$

In terms of the differential form, we have:

$$dX(t) = \sum_{i=0}^{d} \phi_i(t) dS_i(t) - dC(t)$$

Express in terms of portfolio π and plug in dynamics of assets:

$$\begin{aligned} dX(t) &= \frac{X(t) - \sum_{i=1}^{d} \pi_i(t)}{S_0(t)} dS_0(t) + \sum_{i=1}^{d} \frac{\pi_i(t)}{S_i(t)} dS_i(t) - dC(t) \\ &= X(t) r(t) dt + \sum_{i=1}^{d} \pi_i(t) \left[(b_i(t) - r(t)) dt + \sum_{j=1}^{d} \sigma_{ij}(t) dW_j(t) \right] - dC(t) \end{aligned}$$

Self-Financing Condition

By vector notation, Brownian motion with drift and Sharpe ratio, we can rewrite as:

$$dX(t) = X(t)r(t)dt + \pi^\top(t)\sigma(t)dW_0(t) - dC(t), \quad X(0) = x$$

Since $d\gamma(t) \cdot dX(t) = 0$, by Itô formula:

$$\begin{aligned} d(\gamma(t)X(t)) &= \gamma(t)dX(t) - r(t)\gamma(t)X(t)dt \\ &= \gamma(t)\pi^\top(t)\sigma(t)dW_0(t) - \gamma(t)dC(t) \end{aligned}$$

Therefore, we have the wealth equation:

$$\gamma(t)X(t) = x - \int_0^t \gamma(s)dC(s) + \int_0^t \gamma(s)\pi^\top(s)\sigma(s)dW_0(s)$$

For a triple (x, π, C), if there exists a unique strong solution $X(\cdot)$, it's called the wealth process. For the stochastic integral to be well-defined:

$$\int_0^T \|\pi(t)\|^2 dt < \infty$$

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Risk-Neutral Probability via Girsanov Theorem

Definition of Risk-Neutral Measure:

$$\mathbb{P}^0(A) := \mathbb{E}[Z(T)1_A], \quad A \in \mathcal{F}_T$$

$Z(T)$ is the exponential martingale. By Girsanov's Theorem, $W_0(t) = W(t) + \int_0^t \theta(s)ds$ is a standard Brownian motion under $\mathbb{P}^0$.

Thus rewirte the wealth process:

$$\begin{aligned} N_0(t) &= \gamma(t)X(t) + \int_0^t \gamma(s)dC(s) \\ &= x + \int_0^t \gamma(s)\pi^\top(s)\sigma(s)dW_0(s) \end{aligned}$$

A continuous $\mathbb{P}^0$-local martingale.

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Stock dynamics under risk-neutral measure:

$$dS_i(t) = S_i(t)\left[b_i(t)dt - \sum_{j=1}^{d}\sigma_{ij}(t)\theta_j(t)dt + \sum_{j=1}^{d}\sigma_{ij}(t)dW_0^{(j)}(t)\right]$$

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$$\begin{aligned} d(\gamma(t)S_i(t)) &= S_i(t)dr(t) + \gamma(t)dS_i(t) \\ &= \gamma(t)S_i(t)\sum_{j=1}^{d}\sigma_{ij}(t)dW_0^{(j)}(t) \end{aligned}$$

Hence, the discounted stock processes $\gamma(\cdot)S_i(\cdot)$ are local martingales. This also confirms our intuition that every asset $S_i(t)$ should have a growth rate $r(t)$ in the risk neutral world.

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Doubling Strategies and Admissibility

Doubling Strategy: Double investment after each loss, leads to arbitrarily large wealth at T. Requires wealth process $X(t)$ unbounded from below. Need to exclude, because creates arbitrage opportunities and violates no-arbitrage principle.

A uniform boundedness condition is needed to prevent the doubling strategy. Wealth must satisfy:

$$X^{x,\pi,C}(t) \geq -\Lambda, \quad 0 \leq t \leq T$$

With $\mathbb{E}^0[\Lambda^p] < \infty$ for some $p > 1$.

In summary, the admissibility on (π, C) essentially requires the portfolio to be self-financing and not to be a doubling strategy.

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Some Properties

(i) Supermartingale Property:

If (π, C) is admissible, $N_0(t)$ is bound below. $N_0(t)$ is a $\mathbb{P}^0$-supermartingale. Consequently:

$$\mathbb{E}^0 \left[\gamma(T)X(T) + \int_0^T \gamma(t)dC(t) \right] \leq x$$

Expected discounted terminal wealth and consumption less than or equal to initial wealth to ensures no arbitrage.

(ii) Scaling Property:

Wealth dynamics are linear in (x, π, C). For any $a \neq 0$:

$$X^{ax,a\pi,aC}(t) = a \cdot X^{x,\pi,C}(t)$$

In particular, $a > 0$ outcome is scaled wealt and $a = -1$ is mirror strategy with wealth is $-X^{x,\pi,C}(t)$. The intuition is that the model is homogeneous of degree 1 in wealth.

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Dominant Opportunities and Dominance-Free Interval

Consider European Contingent Claim (ECC), payoff at maturity is $\psi(T) \geq 0$. For example the Call option payoff $(S_1(T) - K)^+$. To ensure that option price is finite, we assume that $\mathbb{E}[(\psi(T))^{1+\epsilon}] < \infty,\ \forall \epsilon > 0$.

Price at time 0 is $\psi(0)$. Question: what is the fair value of $\psi(0)$? Price too low $\rightarrow$ buyer arbitrage, price too high $\rightarrow$ seller arbitrage. Therefore, the correct price must lie in an interval that rules out dominance.

The main purpose of this section is to find out what $\psi(0)$ should be in the market $\mathcal{M}$ with the ECC, denoted by (M, ψ) for short, with ψ standing for the pair $(\psi(0), \psi(T))$.

A dominance opportunity exists in market $(\mathcal{M}, \psi)$ if we start with some initial wealth $x \geq 0$ (or $x \leq 0$ depending on position). We take an admissible strategy (π, C) and add a position in the ECC (long if $a = -1$, short if $a = +1$).

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A Definition of Dominate Opportunity

Condition:

$$x + a \cdot \psi(0) < 0$$

yet at maturity:

$$\mathbb{P}\{X^{x,\pi,C}(T) + a \cdot \psi(T) \geq 0\} = 1$$

You start with negative initial wealth, but end up with a guaranteed nonnegative payoff. This is a riskless profit strictly better than bond returns.

Moreover, the scaling property will result the unlimited arbitrage. Therefore, dominance opportunities must be excluded in rational, well-behaved markets.

Definition: The admissible price $\psi(0)$ must lie in an dominance-free interval $[a, b]$, such that:

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Lower and Upper Hedging Classes

Upper Hedging Class $\mathcal{U}$:

$$\mathcal{U} := \{x \geq 0 : \exists(\hat{\pi}, \hat{C}) \in \mathcal{A},\ X^{x,\hat{\pi},\hat{C}}(0) = x,\ X^{x,\hat{\pi},\hat{C}}(T) \geq \psi(T)\ a.s.\}$$

Starting with capital x, you can construct an admissible strategy whose terminal wealth is always at least as large as the claim payoff $\psi(T)$ So, the minimum of $\mathcal{U}$ gives the upper bound for the fair price. $\mathcal{U}$ may be empty (think about quadratic payoff).

Lower Hedging Class $\mathcal{L}$:

$$\mathcal{L} := \{x \geq 0 : \exists(\check{\pi}, \check{C}) \in \mathcal{A},\ X^{x,\check{\pi},\check{C}}(0) = -x,\ X^{x,\check{\pi},\check{C}}(T) \geq -\psi(T)\ a.s.\}$$

With initial wealth $-x$ (i.e. receiving x up front), you can construct a strategy whose terminal wealth is always greater than or equal to $-\psi(T)$. So, the maximum of $\mathcal{L}$ gives the lower bound for the fair price.

Observe that both sets are intervals (connected):

- If $x_1 \in \mathcal{L}$ and $0 \leq y_1 \leq x_1$, then $y_1 \in \mathcal{L}$
- If $x_2 \in \mathcal{U}$ and $y_2 \geq x_2$, then $y_2 \in \mathcal{U}$

Thus it would be interesting to look at the endpoints of the intervals.

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Thus it would be interesting to look at the endpoints of the intervals.

Upper and Lower Hedging Prices

Upper bound: $h_{\textbf{up}} := \inf \mathcal{U}$. Because it's the cheapest initial capital needed to guarantee covering the claim $\psi(T)$. $\inf \mathcal{U}$ is the minimal fair one for the seller.

Lower bound: $h_{\text{low}} := \sup \mathcal{L}$. Because it's the maximum initial amount from which a buyer can still hedge against the debt of paying $\psi(T)$. So $\sup \mathcal{L}$ is the highest "safe" price for the buyer.

The intuition suggests that the lower dominate price cannot be bigger than the upper hedging price.

Let $u_0 := \mathbb{E}^0[\gamma(T)\psi(T)]$. Show that $u_0 < \infty$. In fact, we can show a strong result that

$$\mathbb{E}^0[\{\gamma(T)\psi(T)\}^a] < \infty, \quad 1 < a < 1 + \epsilon.$$

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Proof:
Note that $u_0 = \mathbb{E}^0[\gamma(T)\psi(T)] = \mathbb{E}[\gamma(T)\psi(T)Z(T)]$. Let $c < \infty$ be a fixed constant such that $\|\theta(t)\| \leq c,\ \gamma(T) \leq c$. Also let $p = 1+\epsilon,\ 1/p + 1/q = 1.$ We have:

$$\begin{aligned}
&\mathbb{E}\left[e^{-q\int_0^T \theta^\top(s)dW(s) - \frac{1}{2}q\int_0^T \|\theta(s)\|^2 ds}\right] \\
&= \mathbb{E}\left[e^{-q\int_0^T \theta^\top(s)dW(s) - \frac{1}{2}q^2\int_0^T \|\theta(s)\|^2 ds} \cdot e^{\frac{1}{2}(q^2-q)\int_0^T \|\theta(s)\|^2 ds}\right] \\
&\leq \mathbb{E}\left[e^{-q\int_0^T \theta^\top(s)dW(s) - \frac{1}{2}q^2\int_0^T \|\theta(s)\|^2 ds}\right] e^{\frac{1}{2}(q^2-q)c^2T} \text{(Typo: c)} \\
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\begin{aligned}
u_0 &\leq c\mathbb{E}(\psi(T)Z(T)) \\
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&\leq c(\mathbb{E}(\psi(T))^p)^{1/p} \cdot e^{(q-1)c^2T/2} < \infty
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For $\mathbb{E}^0[\{\gamma(T)\psi(T)\}^a]$. Follow the same steps, change the measure and choose Hölder exponents, $p = \frac{1+\varepsilon}{a} > 1, \quad q = \frac{p}{p-1}$. Then:

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\mathbb{E}[\psi^a Z] \leq (\mathbb{E}[\psi^{ap}])^{1/p} (\mathbb{E}[Z^q])^{1/q} = \left(\mathbb{E}[\psi^{1+\varepsilon}]\right)^{1/p} \cdot (\mathbb{E}[Z^q])^{1/q}
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The first factor is finite by assumption; the second is bounded exactly as in previous part. Thus:

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\mathbb{E}^0[\{\gamma(T)\psi(T)\}^a] \leq c^a \left(\mathbb{E}[\psi(T)^{1+\varepsilon}]\right)^{1/p} \exp\left\{\frac{1}{2}(q-1)c^2T\right\} < \infty, \quad \forall 1 < a < 1+\varepsilon
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An Inequality for the Upper and Lower Prices

At any time $t \in [0, T]$, we have $0 \leq h_{\text{low}} \leq u_0 \leq h_{\text{up}}\ \ a.s.$, where $u_0 = \mathbb{E}^0[\gamma(T)\psi(T)]$.

For upper bound: If $\mathcal{U}$ is empty, $h_{\text{up}} = +\infty$ and inequality is trivial. If $\mathcal{U} \neq \emptyset$, then by definition of $\mathcal{U}$, for every $x \in \mathcal{U}$, there exists some admissible portfolio $(\hat{\pi}, \hat{C})$ with:

$$X^{\hat{\pi},\hat{C}}(0) = x, \quad X^{\hat{\pi},\hat{C}}(T) \geq \psi(T)$$

Apply the supermartingale property to the discounted wealth process:

$$x \geq \mathbb{E}^0 \left[\gamma(T) X^{\hat{\pi},\hat{C}}(T) + \int_0^T \gamma(s) d\hat{C}(s) \right]$$

Since $X^{\hat{\pi},\hat{C}}(T) \geq \psi(T)$, we get $x \geq \mathbb{E}^0[\gamma(T)\psi(T)]$. This mean that any initial capital in $\mathcal{U}$ must be at least u_0, $u_0 \leq h_{\text{up}}$.

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For lower bound: Similarly, we can show that $0 \leq h_{\text{low}} \leq u_0$. Indeed, since the set $\mathcal{L}$ contains $x = 0$, it is nonempty. For any $x \geq 0$ in this set, again by the supermartingale property, almost surely.

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By the definition of $\mathcal{L}$. Hence, $x \leq \mathbb{E}^0\left[\gamma(T)\psi(T)\right]$ and $0 \leq h_{\text{low}} \leq u_0$, almost surely.

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Dominance Outside the Interval

Claim: for any ECC price $\psi(0) > h_{\mathsf{up}}$, there exists a dominant opportunity in (M, ψ); similarly for any ECC price $\psi(0) < h_{\mathsf{low}}$.

Suppose that $\psi(0) > h_{\mathsf{up}}$. Then for any $x_1 \in (h_{\mathsf{up}}, \psi(0))$ we know that $x_1 \in \mathcal{U}$, since h_{up} is the left endpoint of a connected interval $\mathcal{U}$. By the definition of $\mathcal{U}$, there exists a $(\hat{\pi}, \hat{C}) \in \mathcal{A}$ such that:

$$X^{\hat{\pi},\hat{C}}(0) - \psi(0) = x_1 - \psi(0) < 0$$

because $x_1 < \psi(0)$, and

$$X^{\hat{\pi},\hat{C}}(T) - \psi(T) \geq \psi(T) - \psi(T) = 0$$

Hence the definition of the dominance is satisfied with $a = -1$.

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Dominance Outside the Interval

Suppose $\psi(0) < h_{\mathsf{low}}$. Then for any $x_1 \in (\psi(0), h_{\mathsf{low}})$ we know that $x_1 \in \mathcal{L}$, since h_{low} is the right endpoint of a connected interval $\mathcal{L}$. By definition of $\sup \mathcal{L}$, there exists an admissible strategy $(\check{\pi}, \check{C})$ such that:

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Consider:

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No dominance within the Interval

Show that for any $\psi(0) \in [h_{\text{low}}, h_{\text{up}}]$ there is no dominant opportunity in $(\mathcal{M}, \psi)$.
Proof:
Suppose there is a dominant opportunity with $\psi(0) \in [h_{\text{low}}, h_{\text{up}}]$.
Case 1: The dominant opportunity satisfies the definition with $a = -1$. In this case, there exist an initial wealth $x \in [0, \infty)$ and a pair $(\pi_1, C_1) \in \mathcal{A}$, such that:

$$x - \psi(0) = X^{\pi_1, C_1}(0) - \psi(0) < 0$$

where $x < \psi(0)$, and

$$X^{\pi_1, C_1}(T) - \psi(T) \geq 0, \text{ a.s.}$$

From the definition of $\mathcal{U}$ we know that $x \in \mathcal{U}$, where $x \geq h_{\text{up}}$, by the definition of h_{up}. Therefore, $h_{\text{up}} \leq x < \psi(0)$; a contradiction, since by assumption $h_{\text{up}} \geq \psi(0)$.

Case 2: The dominant opportunity satisfies the definition with $a = 1$. In this case, there exist an initial wealth $x \in (\infty, 0]$ and a pair $(\pi_2, C_2) \in \mathcal{A}$, such that:

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The interval $[h_{\text{low}}, h_{\text{up}}]$ is the range of possible fair prices for ECC. Inside this interval, no dominance opportunities exist. Therefore, this interval is called the dominance-free interval.

Remark: No arbitrage $\Rightarrow$ No dominance $\Rightarrow$ Law of one price. But the reverse is not always true. In particular, there are examples showing that at the two endpoints h_{low} and h_{up} there may be arbitrage opportunities but no dominant opportunities.

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The Unique Dominance-Free Price in the Ideal Market

In an ideal (complete, unconstrained) market where every contingent claim can be perfectly replicated and unlimited long/short positions are allowed. The interval of possible fair prices, $[h_{\text{low}}, h_{\text{up}}]$, collapses to a single price:

$$h_{\text{low}} = h_{\text{up}} = u_0$$

$$\psi(0) = u_0 = \mathbb{E}^0[\gamma(T)\psi(T)]$$

Furthermore, we can show that, corresponding to the Black-Scholes price u_0, there is a "hedging portfolio" process $\pi(\cdot)$ (hence also a corresponding trading process $\phi(\cdot)$) and a consumption process $C(\cdot) \equiv 0$, such that

$$X^{u_0,\pi,0}(T) = \psi(T) \qquad (*)$$

and with the opposite portfolio $-\pi(\cdot)$ (hence the opposite trading strategy $-\phi(\cdot)$), we have

$$X^{-u_0,-\pi,0}(T) = -\psi(T) \qquad (**)$$

Hence the price for the ECC has to be u_0, if no dominance is allowed in $\mathcal{M}$. This price is called the *dominance-free price*, also known as *Black-Scholes price*.

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Martingale Representation Theorem

To prove $(*)$ and $(**)$, we need the martingale representation theorem.

Suppose W_t is a d-dimensional Brownian motion. Let $\mathcal{F}_t$ be the history (filtration) generated by W_t. If M is a d-dimensional continuous martingale with $M_0 = 0$, then there exists a vector of (progressively measurable) processes:

$$\theta_s = (\theta_s^{(1)}, \dots, \theta_s^{(d)})$$

such that with probability one:

$$M_t = \sum_{j=1}^{d} \int_0^t \theta_s^{(j)} dW_s^{(j)}$$

Any square-integrable martingale can be written as a stochastic integral of Brownian motion. In general, the martingale representation theorem is only an existence results.

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Martingale Representation Theorem

Consider a (Doob) martingale $X_t = \mathbb{E}(W_T^3|\mathcal{F}_t)$, $T \geq t$.

(i) Compute the conditional expectation to show that

$$X_t = 3W_t \cdot (T-t) + W_t^3.$$

(ii) Compute the stochastic integral to show that

$$X_t = \int_0^t 3(T - s + W_s^2) dW_s.$$

Therefore, in the martingale representation theorem $\theta_s = 3(T - s + W_s^2)$.

Proof:
(i)

$$\begin{aligned}
\mathbb{E}(W_T^3|\mathcal{F}_t) &= \mathbb{E}((W_t + (W_T - W_t))^3|\mathcal{F}_t) \\
&= \mathbb{E}(W_t^3|\mathcal{F}_t) + \mathbb{E}(3W_t^2(W_T - W_t)|\mathcal{F}_t) + \mathbb{E}(3W_t(W_T - W_t)^2|\mathcal{F}_t) + \mathbb{E}((W_T - W_t)^3|\mathcal{F}_t) \\
&= W_t^3 + 3W_t(T-t)
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Using that W_t is $\mathcal{F}_t$-measurable and the independent increment $W_T - W_t \perp \mathcal{F}_t$ with odd moments vanish.

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(ii)
Define $f(t,x) := 3x(T-t) + x^3$. Then $X_t = f(t, W_t)$.
Apply Itô's formula to $f(t, W_t)$:

$$\begin{aligned} dX_t &= \left(f_t + \frac{1}{2} f_{xx}\right) dt + f_x dW_t \\ &= (-3W_t + 3W_t)\, dt + \left(3(T-t) + 3W_t^2\right) dW_t \\ &= 3\left(T - t + W_t^2\right) dW_t \end{aligned}$$

Integrate from 0 to t and $X_0 = \mathbb{E}[W_T^3] = 0$, thus:

$$X_t = \int_0^t 3\left(T - s + W_s^2\right) dW_s$$

If you can make the discounted payoff a martingale under some measure, then by the MRT there exists a process θ_s so that the trading strategy replicates the payoff exactly with probability 1. MRT requires continuous martingale. If the market has jumps, MRT generally fails, so you can't in general replicate an option payoff exactly.

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Generalized Martingale Representation Theorem

In this setting, every square-integrable martingale M_t has the form:

$$M_t = M_0 + \int_0^t \theta_s dW_s + \int_0^t \int_{\mathbb{R}} \phi_s(x)\tilde{N}(ds, dx)$$

where:

- θ_s is the Brownian-driven part
- $\tilde{N}(ds, dx) = N(ds, dx) - \nu(dx)ds$ is the compensated Poisson measure
- $\phi_s(x)$ tells you how sensitive M_t is to jumps of size x

In a jump-diffusion market, you generally cannot replicate payoffs perfectly using only continuous trading. You would need extra instruments tied to the jump risk (like insurance contracts, catastrophe bonds, or options directly on jumps).

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Proofs of $(*)$ and $(**)$

Consider the discounted final payoff of the option: $Q(T) := \gamma(T)\psi(T)$. Note that from the definition of u_0, we have $\mathbb{E}^0[Q(T)] = u_0$. Define for $s \in [0, T]$:

$$\zeta(s) := \frac{1}{\gamma(s)}\mathbb{E}^0[Q(T)|\mathcal{F}_s] \geq 0$$

Then almost surely at time 0 and at time T we have:

$$\zeta(0) = u_0, \quad \zeta(T) = \frac{1}{\gamma(T)}\mathbb{E}^0[\gamma(T)\psi(T)|\mathcal{F}_T] = \psi(T)$$

It is tempting to apply the martingale representation theorem to $\mathbb{E}^0[Q(T)|\mathcal{F}_s]$, which is martingale adapted to $\{W_0(t)\}$, to find a replication strategy for the option payoff $\psi(T)$. However, what we need is a replication strategy adapted to $\{W(t)\}$ not $\{W_0(t)\}$.

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Theorem:

There exists a (progressively measurable) process $\eta(\cdot)$, $\int_0^T \|\eta(u)\|^2 du < \infty$, adapted to $\{W(t)\}$ such that, almost surely,

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Note that the standard martingale representation theorem does not apply here, as we want $\eta(\cdot)$ to be adapted to $\mathcal{F}_t$, the P-argumented filtration generated by $\{W(t)\}$ not by $\{W_0(t)\}$. The filtration generated by $\{W_0(t)\}$ is different from $\mathcal{F}_t$ due to the null sets of $\{W_0(t)\}$.

Proof: Let $M^0(t) = \gamma(t)\zeta(t) = E^0[Q(T)|\mathcal{F}_t]$, where $\mathcal{F}_t$ is the filtration generated by the process $\{W(t)\}$. Then $M^0(t)$ is a martingale with respect to $\mathcal{F}_t$ under the new probability P^0, because for $t \geq s$

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$$E^0[Y|\mathcal{F}_s] = \frac{1}{Z(s)} E[Y \cdot Z(t)|\mathcal{F}_s]$$

Since $M^0(t) \in \mathcal{F}_t$, we have by Bayes formula for $t \geq s$

$$E[M(t)|\mathcal{F}_s] = E[Z(t)M^0(t)|\mathcal{F}_s] = Z(s)E^0[M^0(t)|\mathcal{F}_s] = Z(s)M^0(s) = M(s)$$

from which the conclusion follows.
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Now since $M^0(t) = M(t) \cdot 1/Z(t)$, we have:

$$d(M^0(t)) = \frac{1}{Z(t)} dM(t) + M(t) d\left(\frac{1}{Z(t)}\right) + dM(t) d\left(\frac{1}{Z(t)}\right)$$

By Itô's formula:

$$d\left(\frac{1}{Z(t)}\right) = \frac{1}{Z(t)} \left[\theta^*(t) dW(t) + \|\theta(t)\|^2 dt\right]$$

Therefore:

$$d(M^0(t)) = \frac{1}{Z(t)} \left\{\xi^*(t) dW(t) + M(t)\theta^*(t) dW(t) + M(t)\|\theta(t)\|^2 dt + \xi^*(t)\theta(t) dt\right\}$$

Since

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Thus, the martingale representation for $M^0(t)$ holds with respect to $W_0(t)$

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The Unique price

To prove $h_{\text{low}} = h_{\text{up}} = u_0$, we only need to show that $u_0 \in U$ and $u_0 \in L$.

First we shall show that $u_0 \in U$. By theorem, Define a process $\hat{\pi}$ by $\hat{\pi}^*(u) := \frac{1}{\gamma(u)}\eta^*(u)(\sigma(u))^{-1}$. Then $\eta^*(u) = \gamma(u)\hat{\pi}^*(u)\sigma(u)$, and from theorem $\gamma(s)\zeta(s) = u_0 + \int_0^s \gamma(u)\hat{\pi}^*(u)\sigma(u)dW_0(u)$

Recall that the wealth equation is given by:

$$\gamma(t)X(t) = x - \int_0^t \gamma(s)dC(s) + \int_0^t \gamma(s)\pi^*(s)\sigma(s)dW_0(s)$$

Therefore, $\zeta(s)$ satisfies the wealth equation with the initial wealth u_0, the consumption $\hat{C}(s) \equiv 0$, and the portfolio strategy $\hat{\pi}$:

$$X^{u_0,\hat{\pi},\hat{C}}(s) = \zeta(s), \quad \int_0^T ||\hat{\pi}(t)||^2 dt < \infty$$

Since almost surely $\zeta(s) \geq 0$, we have $\left(\hat{\pi}, \hat{C}\right) \in A(u_0)$, hanks to the assumption of the ideal market that there is no constraints on $\hat{\pi}$. Therefore, we get $u_0 \in U$.

The Unique price

To prove $h_{\text{low}} = h_{\text{up}} = u_0$, we only need to show that $u_0 \in U$ and $u_0 \in L$.

First we shall show that $u_0 \in U$. By theorem, Define a process $\hat{\pi}$ by $\hat{\pi}^*(u) := \frac{1}{\gamma(u)}\eta^*(u)(\sigma(u))^{-1}$. Then $\eta^*(u) = \gamma(u)\hat{\pi}^*(u)\sigma(u)$, and from theorem $\gamma(s)\zeta(s) = u_0 + \int_0^s \gamma(u)\hat{\pi}^*(u)\sigma(u)dW_0(u)$

Recall that the wealth equation is given by:

$$\gamma(t)X(t) = x - \int_0^t \gamma(s)dC(s) + \int_0^t \gamma(s)\pi^*(s)\sigma(s)dW_0(s)$$

Therefore, $\zeta(s)$ satisfies the wealth equation with the initial wealth u_0, the consumption $\hat{C}(s) \equiv 0$, and the portfolio strategy $\hat{\pi}$:

$$X^{u_0,\hat{\pi},\hat{C}}(s) = \zeta(s), \quad \int_0^T ||\hat{\pi}(t)||^2 dt < \infty$$

Since almost surely $\zeta(s) \geq 0$, we have $\left(\hat{\pi}, \hat{C}\right) \in A(u_0)$, hanks to the assumption of the ideal market that there is no constraints on $\hat{\pi}$. Therefore, we get $u_0 \in U$.

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To show $u_0 \in L$, notice that with $\tilde{\pi} = -\hat{\pi}$. we have $X^{\tilde{\pi},0}(\cdot) = -X^{\hat{\pi},0}(\cdot)$ To check the admissibility, it only remains to prove that $X^{\tilde{\pi},0}(\cdot)$ is bounded away from below, thanks again to the ideal market assumption that there is no constraints on π.

To achieve this, note that:

$$\begin{aligned}
\inf_{0\leq s\leq T} X^{\tilde{\pi},0}(s) &= -\sup_{0\leq s\leq T}(X^{\hat{\pi},0}(s)) = -\sup_{0\leq s\leq T}(\zeta(s)) \\
&= -\sup_{0\leq s\leq T}\left\{\frac{1}{\gamma(s)}E^0[Q(T)|\mathcal{F}_s]\right\} \\
&\geq -\text{const.}\sup_{0\leq s\leq T}E^0[Q(T)|\mathcal{F}_s]
\end{aligned}$$

where the inequality follows via the boundedness of $\gamma(t)$. Let $M(t) = E^0[Q(T)|\mathcal{F}_t]$. Then $M(t)$ is a martingale with respect to P^0. By the Doob maximal inequality for martingale, we get, for some $p > 1$.

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\begin{aligned}
E^0\left[\left(\sup_{0\le s\le T} E^0[Q(T)|\mathcal{F}_s]\right)^p\right] &= E^0\left[\left(\sup_{0\le s\le T} M(s)\right)^p\right] \\
&\le \left(\frac{p}{p-1}\right)^p E^0[(M(T))^p] \\
&= \left(\frac{p}{p-1}\right)^p E^0[(Q(T))^p] < \infty
\end{aligned}
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where the last inequality has already been proven in Problem 1. Thus, $\inf_{0\le s\le T} X^{\tilde{\pi},0}(s)$ is bounded below by a random variable $-\Lambda$, such that $E^0(\Lambda^p) < \infty$, for some $p > 1$.

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BS Formula in Markets with Constant Coefficients

Market with constant drift b, interest rate r, and volatility σ. One stock $S(t)$ ($d = 1$). To price a European call option with payoff $\psi(T) = (S(T) - K)^+$. The option price is:

$$u_0 = E^0 \left(e^{-rT} (S(T) - K)^+ \right)$$

where E^0 is expectation under the risk-neutral measure P^0. Under P^0:

$$dS(t) = S(t)[rdt + \sigma dW_0(t)]$$

Therefore, $S(T)$ is equal, in distribution, to $S(0) \exp\left\{ \left(r - \frac{\sigma^2}{2} \right) T + Z\sigma\sqrt{T} \right\}$, where Z is a standard normal random variable.

$$u_0 = e^{-rT} \int_{-\infty}^{\infty} \left(S(0) \exp\left\{ \left(r - \frac{\sigma^2}{2} \right) T + z\sigma\sqrt{T} \right\} - K \right)^+ \varphi(z) dz$$

where $\varphi(z)$ is the standard normal density.

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BS Formula in Markets with Constant Coefficients

We can evaluate the integral to show that

$$u_0 = E^0\left(e^{-rT}(S(T)-K)^+\right) = S(0)\cdot\Phi(\mu_+) - Ke^{-rT}\Phi(\mu_-)$$

which is the celebrated Black-Scholes formula for European call option. Here

$$\mu_\pm := \frac{1}{\sigma\sqrt{T}}[\log(S(0)/K) + (r \pm (\sigma^2/2))T]$$

and we set $\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-u^2/2}du$.

Indeed, since $S(0)e^{(r-\sigma^2/2)T+\sigma\sqrt{T}z} \geq K$ is equivalent to $z \geq -\mu_-$, we get

$$\begin{aligned}
u_0 &= e^{-rT}\int_{-\mu_-}^{\infty}\left(S(0)e^{(r-\sigma^2/2)T+\sigma\sqrt{T}z} - K\right)\varphi(z)dz \\
&= S(0)\frac{1}{\sqrt{2\pi}}\int_{-\mu_-}^{\infty} e^{-\sigma^2T/2+\sigma\sqrt{T}z}e^{-z^2/2}dz - Ke^{-rT}\Phi(\mu_-) \\
&= S(0)\frac{1}{\sqrt{2\pi}}\int_{-\mu_-}^{\infty}\exp\left(-\left(z-\sigma\sqrt{T}\right)^2/2\right)dz - Ke^{-rT}\Phi(\mu_-)
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\end{aligned}$$

BS Formula in Markets with Constant Coefficients

Letting $x = z - \sigma\sqrt{T}$ yields

$$u_0 = S(0)\frac{1}{\sqrt{2\pi}}\int_{-\mu_- - \sigma\sqrt{T}}^{\infty} \exp\left(-x^2/2\right) dx - Ke^{-rT}\Phi(\mu_-)$$
$$= S(0)\Phi\left(\mu_- + \sigma\sqrt{T}\right) - Ke^{-rT}\Phi(\mu_-)$$

from which the conclusion follows, because

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