MF921 Topics in Dynamic Asset Pricing

Week 4

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Background

Recall The Double Exponential Jump Diffusion Model:

$$\frac{dS(t)}{S(t^-)} = \mu dt + \sigma dW(t) + d\left(\sum_{i=1}^{N(t)}(V_i - 1)\right)$$

- $W(t)$: Brownian motion under the real-world measure.
- $N(t)$: Poisson process with rate λ.
- V_i: multiplicative jump sizes, i.i.d. random variables.
- $Y = \log(V)$, the jump sizes follow double exponential law:

$$f_Y(y) = p\eta_1 e^{-\eta_1 y}\mathbf{1}_{y\geq 0} + q\eta_2 e^{\eta_2 y}\mathbf{1}_{y<0}$$

with parameters:

- $p, q \geq 0, p + q = 1$: probabilities of upward/downward jumps.
- $\eta_1 > 1$: rate for upward jumps.
- $\eta_2 > 0$: rate for downward jumps.

For option pricing, we switch to a risk-neutral measure P^*, so that the discounted price process is a martingale:

$$E^{P^*}[e^{-rt}S(t)] = S(0)$$

Under P^*, the dynamics adjust:

$$\frac{dS(t)}{S(t^-)} = (r - \lambda^*(t)\zeta^*)dt + \sigma dW^*(t) + d\left(\sum_{i=1}^{N^*(t)}(V_i^* - 1)\right)$$

where:

- $W^*(t)$: Brownian motion under P^*,
- $N^*(t)$: Poisson process with intensity λ^*,
- $V^* = e^{Y^*}$: jump multiplier with new parameters $(p^*, q^*, \eta_1^*, \eta_2^*)$,
- $\zeta^* = E^{P^*}[V^*] - 1 = \frac{p^*\eta_1^*}{\eta_1^*-1} + \frac{q^*\eta_2^*}{\eta_2^*+1} - 1$ is mean percentage jump size.

The log-price process:

$$X(t) = \log\left(\frac{S(t)}{S(0)}\right) = \left(r - \frac{1}{2}\sigma^2 - \lambda^*\zeta^*\right)t + \sigma W^*(t) + \sum_{i=1}^{N^*(t)} Y_i^*, \quad X(0) = 0$$

Moment Generating Function of the log-price process, $X(t)$:

$$\mathbb{E}^* \left[e^{\theta X(t)} \right] = \exp\{G(\theta)t\}$$

Where the function $G(\cdot)$ is defined as:

$$G(x) = x\left(r - \frac{1}{2}\sigma^2 - \lambda\zeta\right) + \frac{1}{2}x^2\sigma^2 + \lambda\left(\frac{p\eta_1}{\eta_1 - x} + \frac{q\eta_2}{\eta_2 + x} - 1\right)$$

Note: Lemma 3.1 in Kou and Wang (2003) shows that the equation $G(x) = \alpha, \forall \alpha > 0$, has exactly four roots: $\beta_{1,\alpha}$, $\beta_{2,\alpha}$, $-\beta_{3,\alpha}$, and $-\beta_{4,\alpha}$, where:

$$0 < \beta_{1,\alpha} < \eta_1 < \beta_{2,\alpha} < \infty$$
$$0 < \beta_{3,\alpha} < \eta_2 < \beta_{4,\alpha} < \infty$$

These roots determine the structure of Laplace transforms for first passage times.

Some Useful Formulas Con

Infinitesimal Generator of the log-price process, $X(t)$:

$$(\mathcal{L}V)(x) = \frac{1}{2}\sigma^2 V''(x) + \left(r - \frac{1}{2}\sigma^2 - \lambda\zeta\right) V'(x) + \lambda \int_{-\infty}^{\infty} \left(V(x+y) - V(x)\right) f_Y(y)\, dy$$

The generator describes how expectations of functions of $X(t)$ evolve in time:

$$\frac{d}{dt}\mathbb{E}[V(X_t)] = \mathbb{E}[(\mathcal{L}V)(X_t)]$$

They provide the mathematical foundation to derive option pricing formulas.

Lookback Options

Consider a lookback put option with an initial "prefixed maximum" $M \geq S(0)$:

$$
\begin{aligned}
LP(T) &= \mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}\left(\max\{M, \max_{0\leq t\leq T} S(t)\} - S(T)\right)\right] \\
&= \mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}\max\{M, \max_{0\leq t\leq T} S(t)\}\right] - S(0)
\end{aligned}
$$

You need the joint distribution of $\max S(t)$ and $S(T)$, which is complicated for jump processes. Laplace transforms convert a complicated path integral over time into a function of roots of $G(x)$ which we can solve algebraically.

Theorem:

Using the notations $\beta_{1,\alpha+r}$ and $\beta_{2,\alpha+r}$ as in early silde, the Laplace transform of the lookback put is given by:

$$\hat{L}(T) = \int_0^\infty e^{-\alpha T} \mathrm{LP}(T) dT = \frac{S(0)A_\alpha}{C_\alpha}\left(\frac{S(0)}{M}\right)^{\beta_{1,\alpha+r}-1} + \frac{S(0)B_\alpha}{C_\alpha}\left(\frac{S(0)}{M}\right)^{\beta_{2,\alpha+r}-1} + \frac{M}{\alpha+r} - \frac{S(0)}{\alpha}$$

For all $\alpha > 0$; here:

$$A_\alpha = \frac{(\eta_1 - \beta_{1,\alpha+r})\beta_{2,\alpha+r}}{\beta_{1,\alpha+r} - 1}$$
$$B_\alpha = \frac{(\beta_{2,\alpha+r} - \eta_1)\beta_{1,\alpha+r}}{\beta_{2,\alpha+r} - 1}$$
$$C_\alpha = (\alpha + r)\eta_1(\beta_{2,\alpha+r} - \beta_{1,\alpha+r})$$

Proof

Lemma : $\lim_{y\to\infty} e^y \mathbb{P}^*[M_X(T) \geq y] = 0$, $\forall T \geq 0$. $M_X(T) := \max_{0\leq t\leq T} X(t)$

[Proof]

Given $\theta \in (-\eta_2, \eta_1)$, $\mathbb{E}^*[e^{\theta X(t)}] < \infty$, by stationary independent increments we can get that the process $\{e^{\theta X(t) - G(\theta)t}; t \geq 0\}$ is a martingale.

Observe that $G(x)$ is continuous and $G(1) = r > 0$, thus we can fix an $\theta \in (1, \eta_1)$ such that $G(\theta) > 0$. Let $\tau_y = \inf\{t \geq 0 : X(t) \geq y\}$. By Optional Sampling Theorem:

$$1 = \mathbb{E}^*[M_{\tau_y \wedge T}]\mathbb{E}^*\left[e^{\theta X(\tau_y \wedge T)} e^{-G(\theta)(\tau_y \wedge T)}\right] \geq e^{\theta y} e^{-G(\theta)T} \mathbb{P}^*(\tau_y \leq T)$$

Thus $e^{\theta y}\mathbb{P}^*(\tau_y \leq T) \leq e^{G(\theta)T}$. Since $\theta > 1$, then we have:

$$e^y \mathbb{P}^*(M_X(T) \geq y) = e^{(1-\theta)y}\left[e^{\theta y}\mathbb{P}^*(\tau_y \leq T)\right] \leq e^{(1-\theta)y} e^{G(\theta)T} \xrightarrow{y\to\infty} 0$$

This will use to justify the boundary term vanishing in the integration-by-parts step later.

Proof Con

Given $s = S(0)$ and M are constants, $\max\limits_{0 \le t \le T} S(t) = se^{M_X(T)}$, the lookback put as:

$$LP(T) = \mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}\max\{M, se^{M_X(T)}\}\right] - s$$

Letting $z = \log(M/s) \ge 0$, define:

$$\begin{aligned}
L(s, M; T) :&= \mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}\max\{M, se^{M_X(T)}\}\right] \\
&= s\mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}\max\{e^z, e^{M_X(T)}\}\right] \\
&= s\mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}\left(e^{M_X(T)} - e^z\right)\mathbf{1}_{\{M_X(T) \ge z\}}\right] + se^z e^{-rT}
\end{aligned}$$

Proof Con

Integration by parts:

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}e^{M_X(T)}\mathbf{1}_{\{M_X(T)\geq z\}}\right] &= e^{-rT}\int_z^\infty e^y\, d(1-\mathbb{P}^*[M_X(T)\geq y]) \\
&= -e^{-rT}\int_z^\infty e^y\, d\mathbb{P}^*[M_X(T)\geq y] \\
&= -e^{-rT}\left\{\left(-e^y\mathbb{P}^*[M_X(T)\geq y]\right)\Big|_0^\infty - \int_z^\infty \mathbb{P}^*[M_X(T)\geq y]e^y\, dy\right\} \\
&= -e^{-rT}\left\{-e^z\mathbb{P}^*[M_X(T)\geq z] - \int_z^\infty \mathbb{P}^*[M_X(T)\geq y]e^y\, dy\right\} \\
&= \mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}e^z\mathbf{1}_{\{M_X(T)\geq z\}}\right] + e^{-rT}\int_z^\infty e^y\mathbb{P}^*[M_X(T)\geq y]dy;
\end{aligned}
$$

Plug back into $L(s,M;T)$:

$$
\begin{aligned}
L(s,M;T) &= s\mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}\left(e^{M_X(T)}-e^z\right)\mathbf{1}_{\{M_X(T)\geq z\}}\right] + se^z e^{-rT} \\
&= se^{-rT}\int_z^\infty e^y\mathbb{P}^*[M_X(T)\geq y]dy + Me^{-rT}
\end{aligned}
$$

Proof Con

Then take Laplace in maturity and use Fubini Theorem:

$$\begin{aligned}\int_0^\infty e^{-\alpha T} L(s, M; T) dT &= s \int_0^\infty e^{-\alpha T} e^{-rT} \int_z^\infty e^y \mathbb{P}^*[M_X(T) \geq y] dy dT + \frac{M}{\alpha + r} \\ &= s \int_z^\infty e^y \int_0^\infty e^{-(\alpha+r)T} \mathbb{P}^*[M_X(T) \geq y] dT dy + \frac{M}{\alpha + r}\end{aligned}$$

Follows from Kou and Wang (2003) that:

$$\int_0^\infty e^{-(\alpha+r)T} \mathbb{P}^*[M_X(T) \geq y] dT = A_1 e^{-y\beta_{1,\alpha+r}} + B_1 e^{-y\beta_{2,\alpha+r}}$$

$$A_1 = \frac{1}{\alpha + r} \frac{\eta_1 - \beta_{1,\alpha+r}}{\eta_1} \cdot \frac{\beta_{2,\alpha+r}}{\beta_{2,\alpha+r} - \beta_{1,\alpha+r}}, \quad B_1 = \frac{1}{\alpha + r} \frac{\beta_{2,\alpha+r} - \eta_1}{\eta_1} \cdot \frac{\beta_{1,\alpha+r}}{\beta_{2,\alpha+r} - \beta_{1,\alpha+r}}.$$

Proof Con

Note that $\beta_{2,\alpha+r} > \eta_1 > 1$, $\beta_{1,\alpha+r} > \beta_{1,r} = 1$. Therefore:Note that $\beta_{2,\alpha+r} > \eta_1 > 1$, $\beta_{1,\alpha+r} > \beta_{1,r} = 1$. Therefore,

$$\begin{aligned}\int_0^\infty e^{-\alpha T} L(s,M;T)dT &= s\int_z^\infty e^y A_1 e^{-y\beta_{1,\alpha+r}}dy + s\int_z^\infty e^y B_1 e^{-y\beta_{2,\alpha+r}}dy + \frac{M}{\alpha+r}\\ &= sA_1\frac{e^{-z(\beta_{1,\alpha+r}-1)}}{\beta_{1,\alpha+r}-1} + sB_1\frac{e^{-z(\beta_{2,\alpha+r}-1)}}{\beta_{2,\alpha+r}-1} + \frac{M}{\alpha+r}\\ &= s\frac{A_\alpha}{C_\alpha}e^{-z(\beta_{1,\alpha+r}-1)} + s\frac{B_\alpha}{C_\alpha}e^{-z(\beta_{2,\alpha+r}-1)} + \frac{M}{\alpha+r}.\end{aligned}$$

This yields the Laplace transform we obtained in Theorem 1.

$$\begin{aligned}\int_0^\infty e^{-\alpha T} L(s,M;T)dT &= s\int_z^\infty e^y A_1 e^{-y\beta_{1,\alpha+r}}dy + s\int_z^\infty e^y B_1 e^{-y\beta_{2,\alpha+r}}dy + \frac{M}{\alpha+r}\\ &= sA_1\frac{e^{-z(\beta_{1,\alpha+r}-1)}}{\beta_{1,\alpha+r}-1} + sB_1\frac{e^{-z(\beta_{2,\alpha+r}-1)}}{\beta_{2,\alpha+r}-1} + \frac{M}{\alpha+r}\\ &= s\frac{A_\alpha}{C_\alpha}e^{-z(\beta_{1,\alpha+r}-1)} + s\frac{B_\alpha}{C_\alpha}e^{-z(\beta_{2,\alpha+r}-1)} + \frac{M}{\alpha+r}\end{aligned}$$

This yields the Laplace transform we obtained in the Theorem.