MF921 Topics in Dynamic Asset Pricing

Week 3

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Part I

Option Pricing Under a Double Exponential Jump Diffusion Model

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This paper aims to show that a double exponential jump diffusion model can lead to an analytic approximation for finite-horizon American options and analytical solutions for popular path-dependent options (such as lookback, barrier, and perpetual American options). We will focus on lookback and barrier options here.

Background

Recall The Double Exponential Jump Diffusion Model:

$$\frac{dS(t)}{S(t^-)} = \mu dt + \sigma dW(t) + d\left(\sum_{i=1}^{N(t)}(V_i - 1)\right)$$

- $W(t)$: Brownian motion under the real-world measure.
- $N(t)$: Poisson process with rate λ.
- V_i: multiplicative jump sizes, i.i.d. random variables.
- $Y = \log(V)$, the jump sizes follow double exponential law:

$$f_Y(y) = p\eta_1 e^{-\eta_1 y}\mathbf{1}_{y\geq 0} + q\eta_2 e^{\eta_2 y}\mathbf{1}_{y<0}$$

with parameters:

- $p, q \geq 0, p + q = 1$: probabilities of upward/downward jumps.
- $\eta_1 > 1$: rate for upward jumps.
- $\eta_2 > 0$: rate for downward jumps.

Background Con.

For option pricing, we switch to a risk-neutral measure P^*, so that the discounted price process is a martingale:

$$E^{P^*}[e^{-rt}S(t)] = S(0)$$

Under P^*, the dynamics adjust:

$$\frac{dS(t)}{S(t^-)} = (r - \lambda^*(t)\zeta^*)dt + \sigma dW^*(t) + d\left(\sum_{i=1}^{N^*(t)} (V_i^* - 1)\right)$$

where:

- $W^*(t)$: Brownian motion under P^*,
- $N^*(t)$: Poisson process with intensity λ^*,
- $V^* = e^{Y^*}$: jump multiplier with new parameters $(p^*, q^*, \eta_1^*, \eta_2^*)$,
- $\zeta^* = E^{P^*}[V^*] - 1 = \frac{p^*\eta_1^*}{\eta_1^* - 1} + \frac{q^*\eta_2^*}{\eta_2^* + 1} - 1$ is mean percentage jump size.

The log-price process:

$$X(t) = \log\left(\frac{S(t)}{S(0)}\right) = \left(r - \frac{1}{2}\sigma^2 - \lambda^*\zeta^*\right)t + \sigma W^*(t) + \sum_{i=1}^{N^*(t)} Y_i^*, \quad X(0) = 0$$

Intuition of the Pricing Formula

- Without jumps, the model reduces to geometric Brownian motion. Pricing American, barrier, and lookback options is straightforward. First passage times are tractable, and closed-form formulas are well known (as what we show in last week).
- With jumps, however, analytical pricing becomes difficult because the process can cross barriers by jumping over them (Overshoot Problem).

Intuition of the Pricing Formula Con

Define the first passage time:

$$\tau_b := \inf\{t \geq 0 : X(t) \geq b\}, \quad b > 0$$

In a jump diffusion, when the process crosses b, it may overshoot: $X(\tau_b) - b > 0$.

This overshoot creates complications to compute the distribution of the first passage times analytically:

- Need the distribution of overshoot $X(\tau_b) - b$.
- Need the joint dependence between overshoot and τ_b.
- Need correlation between overshoot and the terminal state $X(T)$

Note: Double exponential distribution assumption has a memoryless property, this property simplifies the overshoot distribution and allows tractable Laplace transforms of first passage times.

Some Useful Formulas

The double exponential jump diffusion process is a special case of Lévy processes with two-sided jumps, whose characteristic exponent admits the (unique) representation:

$$\phi(\theta) = E[e^{i\theta X_1}] = \exp\left\{ i\gamma\theta - \frac{1}{2}A\theta^2 + \int_{-\infty}^{\infty} (e^{i\theta y} - 1 - i\theta y I_{\{|y|\leq 1\}})\Pi(dy) \right\}$$

where the generating triplet (γ, A, Π) is given by:

- $A = \sigma^2$
- $\Pi(dy) = \lambda \cdot f_Y(y)dy = \lambda p \eta_1 e^{-\eta_1 y} I_{\{y \geq 0\}} dy + \lambda q \eta_2 e^{\eta_2 y} I_{\{y<0\}} dy$
- $\gamma = \mu + \lambda E[V I_{\{|V|\leq 1\}}] = \mu + \lambda p \left(\frac{1-e^{-\eta_1}}{\eta_1} - e^{-\eta_1} \right) - \lambda q \left(\frac{1-e^{-\eta_2}}{\eta_2} - e^{-\eta_2} \right)$

Some Useful Formulas Con

Moment Generating Function of the log-price process, $X(t)$:

$$\mathbb{E}^* \left[e^{\theta X(t)} \right] = \exp\{G(\theta)t\}$$

Where the function $G(\cdot)$ is defined as:

$$G(x) = x\left(r - \frac{1}{2}\sigma^2 - \lambda\zeta\right) + \frac{1}{2}x^2\sigma^2 + \lambda\left(\frac{p\eta_1}{\eta_1 - x} + \frac{q\eta_2}{\eta_2 + x} - 1\right)$$

Note: Lemma 3.1 in Kou and Wang (2003) shows that the equation $G(x) = \alpha, \forall\alpha > 0$, has exactly four roots: $\beta_{1,\alpha}$, $\beta_{2,\alpha}$, $-\beta_{3,\alpha}$, and $-\beta_{4,\alpha}$, where:

$$0 < \beta_{1,\alpha} < \eta_1 < \beta_{2,\alpha} < \infty$$
$$0 < \beta_{3,\alpha} < \eta_2 < \beta_{4,\alpha} < \infty$$

These roots determine the structure of Laplace transforms for first passage times.

Some Useful Formulas Con

Infinitesimal Generator of the log-price process, $X(t)$:

$$(\mathcal{L}V)(x) = \frac{1}{2}\sigma^2 V''(x) + \left(r - \frac{1}{2}\sigma^2 - \lambda\zeta\right) V'(x) + \lambda \int_{-\infty}^{\infty} \left(V(x+y) - V(x)\right) f_Y(y)\, dy$$

The generator describes how expectations of functions of $X(t)$ evolve in time:

$$\frac{d}{dt}\mathbb{E}[V(X_t)] = \mathbb{E}[(\mathcal{L}V)(X_t)]$$

They provide the mathematical foundation to derive option pricing formulas.

Lookback Options

Consider a lookback put option with an initial "prefixed maximum" $M \geq S(0)$:

$$\begin{aligned} LP(T) &= \mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}\left(\max\{M, \max_{0\leq t\leq T} S(t)\} - S(T)\right)\right] \\ &= \mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}\max\{M, \max_{0\leq t\leq T} S(t)\}\right] - S(0) \end{aligned}$$

You need the joint distribution of $\max S(t)$ and $S(T)$, which is complicated for jump processes. Laplace transforms convert a complicated path integral over time into a function of roots of $G(x)$ which we can solve algebraically.

Lookback Options Con.

Theorem:

Using the notations $\beta_{1,\alpha+r}$ and $\beta_{2,\alpha+r}$ as in early silde, the Laplace transform of the lookback put is given by:

$$\hat{L}(T) = \int_0^\infty e^{-\alpha T} \mathrm{LP}(T) dT = \frac{S(0)A_\alpha}{C_\alpha} \left(\frac{S(0)}{M} \right)^{\beta_{1,\alpha+r}-1} + \frac{S(0)B_\alpha}{C_\alpha} \left(\frac{S(0)}{M} \right)^{\beta_{2,\alpha+r}-1} + \frac{M}{\alpha+r} - \frac{S(0)}{\alpha}$$

For all $\alpha > 0$; here:

$$A_\alpha = \frac{(\eta_1 - \beta_{1,\alpha+r})\beta_{2,\alpha+r}}{\beta_{1,\alpha+r} - 1}$$

$$B_\alpha = \frac{(\beta_{2,\alpha+r} - \eta_1)\beta_{1,\alpha+r}}{\beta_{2,\alpha+r} - 1}$$

$$C_\alpha = (\alpha + r)\eta_1(\beta_{2,\alpha+r} - \beta_{1,\alpha+r})$$

Lookback Options Con.

Laplace inversion:

$$LP(T) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\alpha T}\hat{L}(\alpha)d\alpha$$

This is the Bromwich inversion integral, intractable in closed form. So we approximate it numerically.
The most widely used numerical methods for Laplace transform inversion is Gaver-Stehfest (GS) algorithm. Suppose you know the Laplace transform of a function $f(t)$:

$$F(\alpha) = \int_0^\infty e^{-\alpha t} f(t)dt$$

The GS method approximates $f(t)$ by evaluating $F(\alpha)$ at carefully chosen points along the real line.

Lookback Options Con.

The formula is:

$$f(t) \approx \frac{\ln(2)}{t} \sum_{k=1}^{n} w_k F\left(\frac{k\ln(2)}{t}\right)$$

where:

- n is an even integer (typically converges nicely even for n between 5 and 10).
- w_k are weights (depending only on n and k):

$$w_k = (-1)^{\frac{n}{2}+k} \sum_{j=\lceil k/2 \rceil}^{\min(k,n/2)} \frac{j^{n/2}(2j)!}{(\frac{n}{2}-j)!j!(j-1)!(k-j)!(2j-k)!}$$

Intuition: The algorithm generates a sequence $f_n(x)$ such that $f_n(x) \to f(x), n \to \infty$

Barrier Options

Consider the up-and-in call (UIC) option with the barrier level H ($H > S(0)$):

$$UIC = E^{\mathbb{P}^*}[e^{-rT}(S(T) - K)^+ I\{\max_{0\le t\le T} S(t) \ge H\}]$$

For any given probability P, define:

$$\Psi(\mu, \sigma, \lambda, p, \eta_1, \eta_2; a, b, T) := P[Z(T) \ge a, \max_{0\le t\le T} Z(t) \ge b]$$

where under $P, Z(t)$ is a double exponential jump diffusion process with drift μ, volatility σ, and jump rate λ, i.e., $Z(t) = \mu t + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i$, and Y has a double exponential distribution with density $f_Y(y) \sim p \cdot \eta_1 e^{-\eta_1 y} 1_{\{y\ge 0\}} + q \cdot \eta_2 e^{y\eta_2} 1_{\{y<0\}}$.

Theorem:

The price of the UIC option is obtained as:

$$\begin{aligned}\text{UIC} =&S(0)\Psi\left(r+\frac{1}{2}\sigma^2-\lambda\zeta,\sigma,\tilde{\lambda},\tilde{p},\tilde{\eta}_1,\tilde{\eta}_2;\ \log\left(\frac{K}{S(0)}\right),\log\left(\frac{H}{S(0)}\right),T\right)\\&-Ke^{-rT}\cdot\Psi\left(r-\frac{1}{2}\sigma^2-\lambda\zeta,\sigma,\lambda,p,\eta_1,\eta_2;\ \log\left(\frac{K}{S(0)}\right),\log\left(\frac{H}{S(0)}\right),T\right)\end{aligned}$$

where $\tilde{p}=(p/(1+\zeta))\cdot(\eta_1/(\eta_1-1)),\tilde{\eta}_1=\eta_1-1,\tilde{\eta}_2=\eta_2+1,\tilde{\lambda}=\lambda(\zeta+1)$, with $\zeta=E^{P^*}[V]-1=\frac{p\eta_1}{\eta_1-1}+\frac{q\eta_2}{\eta_2+1}-1$. The Laplace transforms of Ψ is computed explicitly in Kou and Wang (2003).

Numerical Results

Setting: For the lookback put option the predetermined maximum is $M = 110$.
For the UIC option the barrier and the strike price are given by $H = 120$ and $K = 100$.
For others $T = 1, r = 5\%, \sigma = 0.2, p = 0.3, \frac{1}{\eta_1} = 0.02, \frac{1}{\eta_2} = 0.04, \lambda = 3, S(0) = 100$.

Table 3 The Prices of Lookback Put and UIC Options

		Lookback put		Up-and-in call	
n		$\lambda = 0.01$	$\lambda = 3$	$\lambda = 0.01$	$\lambda = 3$
1		17.20214	18.58516	10.32186	10.98416
2		16.55458	17.83041	9.81060	10.62657
3		16.14481	17.35415	9.49290	10.31446
4		15.95166	17.13035	9.34838	10.13851
5		15.87823	17.04556	9.29672	10.07120
6		15.85473	17.01851	9.28173	10.05461
7		15.84823	17.01105	9.27813	10.05280
8		15.84664	17.00923	9.27740	10.05309
9		15.84629	17.00884	9.27726	10.05315
10		15.84622	17.00877	9.27724	10.05307
Total CPU time		0.541 sec	0.711 sec	2.849 min	2.815 min
Brownian motion case		15.84226	N.A.	9.27451	N.A.
		Monte Carlo simulation			
200 points	point est.	15.39	16.29	9.14	9.82
CPU time: 8 min	95% C.I.	(15.22, 15.56)	(16.06, 16.52)	(8.90, 9.38)	(9.56, 10.08)
2,000 points	point est.	15.65	16.78	9.24	10.05
CPU time: 37 min	95% C.I.	(15.47, 15.83)	(16.59, 16.97)	(9.00, 9.48)	(9.79, 10.31)

Note. The Monte Carlo results are based on 16,000 simulation runs.

Part II

Pricing Path-Dependent Options with Jump Risk via Laplace Transforms

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Show the analytical solutions for two-dimensional Laplace transforms of barrier option prices, as well as an approximation based on Laplace transforms for the prices of finite-time horizon American options, under a double exponential jump diffusion model.