MF921 Topics in Dynamic Asset Pricing

Week 7

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Chapter 22 American Options (II)

Numerical Methods for American Options

Recall

Consider the valuation of a finite-horizon American option with payoff $\varphi(S(\tau))$:

$$V(x,t) := \sup_{\tau \geq t} E^0 \left[e^{-r(\tau-t)} \varphi(S(\tau)) \mid S(t) = x \right],$$

where $S(t)$ = stock price, $\varphi(\cdot)$ = payoff function (e.g. $(K-S)^+$ for a put), τ = stopping time (the exercise time). So $V(x,t)$ is the value function represent the highest discounted expected payoff you can achieve by optimally choosing when to stop.

First, the optimal reward and moving boundary functions together solve the following free-boundary (to be determined) problem:

$$\begin{cases} \mathcal{A}V - rV = 0, & x > b(t), \quad 0 \leq t \leq T \\ V(b(t),t) = \varphi(b(t)), & \text{(value matching)} \\ V_x(b(t),t) = \varphi'(b(t)), & \text{(smooth pasting)} \\ V(x,T) = \varphi(x), & x > 0 \\ V(0,t) = \varphi(0), & V(\infty,t) = 0 \end{cases}$$

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Second, the free boundary problem can be written as a partial differential complementarity problem (without the free boundary) as

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\begin{gathered}
\mathcal{A}V - rV \leq 0, \quad x > 0, \quad t \in [0, T) \\
(\mathcal{A}V - rV)\{V(x,t) - \varphi(x)\} = 0, \quad x > 0, \quad t \in [0, T) \\
V(x,t) \geq \varphi(x), \quad x > 0, \quad t \in [0, T)
\end{gathered}
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along with the terminal and boundary conditions

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V(x,T) = \varphi(x), \quad x > 0, \\
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This is called the complementarity problem because the two inequalities cannot be strict inequalities simultaneously. The partial differential complementarity problem can be solved numerically by converting it to a matrix linear complementarity problem by using the finite difference method.

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Third, the partial differential complementarity problem can be rewritten as a variational inequality problem:

$$\min(-\mathcal{A}V + rV, V(x,t) - \varphi(x)) = 0, \quad x > 0, \quad t \in [0,T) \tag{*}$$

along with the terminal and boundary conditions

$$V(x,T) = \varphi(x), \quad x > 0,$$

$$\lim_{x\to\infty} V(x,t) = 0, \quad V(0,t) = \varphi(0), \quad t \in [0,T).$$

Indeed, there are only two possibility by the partial differential complementarity problem: either $-\mathcal{A}V + rV > 0$ or $-\mathcal{A}V + rV = 0$. In the first case we must have $V(x,t) = \varphi(x)$, and in the second case $V(x,t) \geq \varphi(x)$. Thus, in both cases we have $(*)$. Hence the partial differential complementarity problem implies the variational inequality problem. Conversely, $(*)$ implies that $V(x,t) \geq \varphi(x)$ and $-\mathcal{A}V + rV \geq 0$. Furthermore, if $-\mathcal{A}V + rV > 0$ then $V(x,t) - \varphi(x) = 0$ by $(*)$.

Therefore, the variational inequality problem also implies the partial differential complementarity problem.

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Recall

An efficient way to solve the partial differential complementarity problem is to transform it ot a linear complementarity problem via the finite difference method.

Consider the following (matrix) linear complementarity problem: Find a vector $x \in \mathbb{R}^{1\times n}$

$$Ax \geq b,\ x \geq c,\ (x-c)^T(Ax-b)=0,$$

for given $A \in \mathbb{R}^{n\times n}$, $c, b \in \mathbb{R}^{1\times n}$, where, for two column vectors x and y, $x \geq y$ means $x_i \geq y_i$ for each i. The linear complementarity problem has a unique solution for all column vectors b and c if and only if $A \in \mathbb{R}^{n\times n}$ is a P-matrix.

Note that if A is symmetric, then A is a P-matrix if and only if A is positive definite. Many matrices that arise in finite-difference and finite-element methods are diagonally dominant.

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More precisely, the matrix A is diagonally dominant if

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|, \ \forall i,$$

where a_{ij} denotes the entry of A in the ith row and jth column. If the above inequality is strict, it is called strictly diagonally dominant. A well-known result is that a symmetric strictly diagonally dominant matrix with real positive diagonal entries is positive definite.

The linear complementarity problem can be solved by using many methods, including pivoting methods (e.g., Lemke's algorithm), quadratic programming, successive over relaxation (SOR), projected SOR, etc. There are also several Matlab and Python codes available online.

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Recall

The linear complementarity problem is a particular case of the nonlinear complementarity problem, which is to find a vector $z \in \mathbb{R}^{1\times n}$

$$f(x) \geq b,\ z \geq c,\ (z-c)^T(f(x)-b) = 0,$$

where f is a given function $\mathbb{R}^n \longmapsto \mathbb{R}^n$, $c, b \in \mathbb{R}^{1\times n}$. One way to solve the nonlinear complementarity problem is to use the linear approximation of $f(x)$ to get an iterative algorithm by solving a sequence of the complementarity problem.

The nonlinear complementarity problem is also related to a variational inequality problem: Given a non-empty set $K \in \mathbb{R}^n$, a function $g : \mathbb{R}^n \longmapsto \mathbb{R}^n$, $b \in \mathbb{R}^{1\times n}$, find a $z^* \in K$ such that

$$\min_{y\in K}(y-z^*)^T g(z^*) \geq 0.$$

It can be shown that with

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Finite Difference Methods

The partial differential complementarity problem for $\psi(x,t)$ without the free boundary,

$$\frac{\partial \psi}{\partial t} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 \psi}{\partial x^2} + rx\frac{\partial \psi}{\partial x} - r\psi \leq 0$$

$$\left(\frac{\partial \psi}{\partial t} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 \psi}{\partial x^2} + rx\frac{\partial \psi}{\partial x} - r\psi\right)\{\psi(t,x) - g(x)\} = 0$$

$$\psi(x,t) \geq g(x)$$

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$$\lim_{x\to\infty} \psi(x,t) = 0, \quad \psi(0,t) = g(0),$$

Where $g(x)$ is the payoff function. Although the algorithm also works for the case where $g(x)$ may be $g(x,t)$ depending on t, for notational simplicity we shall focus on the case of time-independent $g(x)$. First, we can transform the problem based on the standard heat equation by changing variables.

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Finite Difference Methods

$$S = e^x, \quad \tau = (T-t)\frac{\sigma^2}{2}$$

$$u(x,\tau) = \exp\left\{\frac{1}{2}(c-1)x + \frac{1}{4}(c+1)^2\tau\right\} \cdot \psi(S,t), \quad c = \frac{2r}{\sigma^2},$$

the Black-Scholes partial differential complementarity problem becomes a standard heat partial differential complementarity problem, $x \in (-\infty, \infty)$:

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} \geq 0$$

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This is the heat equation complementarity problem, with constant coefficients. We are interested in finding $u(x,\tau)$, where $x \in (-\infty,\infty)$ and $\tau \in [0, T\sigma^2/2]$.

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We have the transformed the complicated Black-Scholes complementarity PDE into heat-equation complementarity problem:

$$u_\tau - u_{xx} \geq 0, \quad (u_\tau - u_{xx})(u - f) = 0, \quad u \geq f.$$

We can't solve this analytically, so we approximate the derivatives (u_τ, u_{xx}) with finite differences on a discrete grid. By discretizing time and space on a regular mesh with step sizes $\delta\tau$ and δx, and truncating x in the region $[-N^-\delta x, N^+\delta x]$ for suitably large integers N^+ and N^-, we let

$$u_{n,m} = u(n\delta x, m\delta\tau), \quad f_{n,m} = f(n\delta x, m\delta\tau), \quad -N^- \leq n \leq N^+, \quad 0 \leq m \leq M,$$

where $M\delta\tau = T\sigma^2/2$.
We use the finite difference approximation

$$\begin{aligned}
\frac{\partial u}{\partial \tau} &= \frac{u_{n,m+1} - u_{n,m}}{\delta\tau} + o(\delta\tau), \\
\frac{\partial^2 u}{\partial x^2} &= \theta\left(\frac{u_{n+1,m+1} - 2u_{n,m+1} + u_{n-1,m+1}}{(\delta x)^2}\right) \\
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where $0 < \theta < 1$.

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Finite Difference Methods

Note that this is a general way to do the finite difference method; the cases $\theta = 0, 1/2, 1$ yield the explicit, the Crank-Nicolson, and the implicit schemes, respectively.

The explicit scheme does not involve solving linear equations and is known to be numerically stable and convergent whenever

$$\alpha := \delta\tau/(\delta x)^2 \leq 1/2.$$

The implicit and the Crank-Nicolson schemes are always numerically stable and convergent but more numerically intensive than the explicit method. They require solving a system of linear equations on each time step. Usually, the Crank-Nicolson scheme is the most accurate scheme for small time steps, while the implicit scheme works best for large time steps. In general, if $\theta < 1/2$, then the scheme is stable if $\delta\tau/(\delta x)^2 < (1-\theta)/2$.

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With these approximations, we have

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} \approx \frac{u_{n,m+1} - u_{n,m}}{\delta\tau} - \theta\left(\frac{u_{n+1,m+1} - 2u_{n,m+1} + u_{n-1,m+1}}{(\delta x)^2}\right) - (1-\theta)\left(\frac{u_{n+1,m} - 2u_{n,m} + u_{n-1,m}}{(\delta x)^2}\right).$$

Thus, the requirement that $\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} \geq 0$ becomes for $-N^- + 2 \leq n \leq N^+ - 2$, $0 \leq m \leq M - 1$,

$$u_{n,m+1} - \alpha\theta(u_{n+1,m+1} - 2u_{n,m+1} + u_{n-1,m+1}) \geq b_{n,m},$$

where

$$b_{n,m} := u_{n,m} + \alpha(1-\theta)(u_{n+1,m} - 2u_{n,m} + u_{n-1,m}), \quad \alpha = \delta\tau/(\delta x)^2.$$

Note that if we move forward in time, at the $(m+1)$-th step we know $b_{n,m}$ explicitly, via $u_{n,m}$, $u_{n+1,m}$, and $u_{n-1,m}$.

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The requirement $u(\tau, x) - f(\tau, x) \geq 0$ becomes, for $-N^- \leq n \leq N^+$, $0 \leq m \leq M$,

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To simplify the operation, we shall translate the complementarity condition to $(m+1)$-th time point as, for $-N^- + 2 \leq n \leq N^+ - 2$, $0 \leq m \leq M - 1$,

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The initial condition is easy because $u(0, x) = f(0, x)$, yielding

$$u_{n,0} = f_{n,0}.$$

We have to first pay attention to the boundary conditions at $-N^-$ and N^+. Based on the definition of option payoffs, as an approximation we can let

$$u_{-N^-,m} = f(-N^-, m), \quad u_{N^+,m} = f(N^+, m), \quad 0 \leq m \leq M.$$

The intuition is that when stock price $S(t) \to \infty$ or $S(t) \to 0$, corresponding to $x \to \infty$ and $x \to -\infty$, the option price at time t should be very close to the option payoff at time t.

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To simplify the operation, we shall translate the complementarity condition to $(m+1)$-th time point as, for $-N^- + 2 \leq n \leq N^+ - 2$, $0 \leq m \leq M - 1$,

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The initial condition is easy because $u(0, x) = f(0, x)$, yielding

$$u_{n,0} = f_{n,0}.$$

We have to first pay attention to the boundary conditions at $-N^-$ and N^+. Based on the definition of option payoffs, as an approximation we can let

$$u_{-N^-,m} = f(-N^-, m), \quad u_{N^+,m} = f(N^+, m), \quad 0 \leq m \leq M.$$

The intuition is that when stock price $S(t) \to \infty$ or $S(t) \to 0$, corresponding to $x \to \infty$ and $x \to -\infty$, the option price at time t should be very close to the option payoff at time t.

The Boundary Conditions and the Initial Condition

The discretized complementarity PDE for the transformed variable $u(x,\tau)$ only works for interior points n where both neighbors u_{n+1} and u_{n-1} exist. At the boundaries, one neighbor lies outside the computational grid.

For the term $u_{n,m+1}$ when $n = N^+ - 1$, we have

$$\begin{aligned} & u_{N^+-1,m+1} - u_{N^+-1,m} - \alpha\theta(u_{N^+,m+1} - 2u_{N^+-1,m+1} + u_{N^+-2,m+1}) \\ & \geq \alpha(1-\theta)(u_{N^+,m} - 2u_{N^+-1,m} + u_{N^+-2,m}), \end{aligned}$$

for $0 \leq m \leq M-1$, or

$$u_{N^+-1,m+1} - \alpha\theta(-2u_{N^+-1,m+1} + u_{N^+-2,m+1}) \geq b_{N^+-1,m} + \alpha\theta u_{N^+,m+1},$$

where

$$b_{N^+-1,m} := u_{N^+-1,m} + \alpha(1-\theta)(u_{N^+,m} - 2u_{N^+-1,m} + u_{N^+-2,m}),$$

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The Boundary Conditions and the Initial Condition

Similarly for the term $u_{n,m+1}$ when $n = -N^- + 1$, we have

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$$b_{-N^-+1,m} := u_{-N^-+1,m} + \alpha(1-\theta)(u_{-N^-+2,m} - 2u_{-N^-+1,m} + u_{-N^-+1,m}).$$

In summary we can define

$$b_{n,m} := u_{n,m} + \alpha(1-\theta)(u_{n+1,m} - 2u_{n,m} + u_{n-1,m}), \quad -N^- + 1 \leq n \leq N^+ - 1.$$

The linear complementarity condition at the two boundaries can be written down readily.

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The linear complementarity condition at the two boundaries can be written down readily.

Standard-Form of the Linear Complementarity Problem

Introduce the column vectors

$$u^{(m)} := (u_{-N^-+1,m}, \ldots, u_{N^+-1,m})^T,$$
$$f^{(m)} := (f_{-N^-+1,m}, \ldots, f_{N^+-1,m})^T,$$
$$b^{(m)} := (b_{-N^-+1,m}, \ldots, b_{N^+-1,m})^T,$$

So each vector represents one time slice of all the interior spatial nodes. At the boundaries, we define a correction vector to represent how the known boundary values affect the equations for the first and last interior points.

$$\xi^{(m)} := (\alpha\theta u_{-N^-,m+1}, 0, 0, \ldots, 0, 0, \alpha\theta u_{N^+,m+1})^T,$$

and a $(N^+ + N^- - 2) \times (N^+ + N^- - 2)$ square, tridiagonal, symmetric matrix

$$C = \begin{pmatrix} 1+2\alpha\theta & -\alpha\theta & 0 & \cdots & 0 \\ -\alpha\theta & 1+2\alpha\theta & -\alpha\theta & \cdots & 0 \\ 0 & -\alpha\theta & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1+2\alpha\theta & -\alpha\theta \\ 0 & 0 & \cdots & -\alpha\theta & 1+2\alpha\theta \end{pmatrix}.$$

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Then the inequalities:

$$u_{n,m+1} - \alpha\theta(u_{n+1,m+1} - 2u_{n,m+1} + u_{n-1,m+1}) \geq b_{n,m}$$
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become to:

$$Cu^{(m+1)} \geq b^{(m)} + \xi^{(m)}.$$

The inequality $u_{n,m} \geq f_{n,m}$ can be written as $u^{(m+1)} \geq f^{(m+1)}$. The linear complementarity condition becomes

$$\left(u^{(m+1)} - f^{(m+1)}\right)^T \left(Cu^{(m+1)} - b^{(m)} - \xi^{(m)}\right) = 0.$$

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In summary, the algorithm goes from $m = 0$ to $m = M - 1$, as follows.
Initialization: At maturity $(m = 0)$:

$$u^{(0)} = f^{(0)} = \text{payoff}.$$

Backward time-stepping (for $m = 0, 1, \ldots, M - 1$):

1. **Compute known quantities:**

$$b^{(m)} = u^{(m)} + \alpha(1 - \theta)(\text{Laplacian of } u^{(m)}),$$

   and boundary correction $\xi^{(m)}$.

2. **Set up right-hand side:**

$$q = b^{(m)} + \xi^{(m)}.$$

3. **Solve LCP:**

$$\begin{cases} Cu^{(m+1)} \geq q, \\ u^{(m+1)} \geq f^{(m+1)}, \\ (u^{(m+1)} - f^{(m+1)})^T (Cu^{(m+1)} - q) = 0. \end{cases}$$

   Use Projected Successive Over-Relaxation (or another solver) to find $u^{(m+1)}$.

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Note that since

$$|1 + 2\alpha\theta| = 1 + 2\alpha\theta > |-\alpha\theta| + |-\alpha\theta|,$$

the symmetric matrix C is a strictly diagonally dominant matrix with real positive diagonal entries. Thus, the symmetric matrix C is positive definite, and the linear complementarity problem given by three inequalities has a unique solution.

Projected Successive Over-Relaxation (PSOR)
We could solve that exactly by inversion, but for large systems (like in PDE grids), we often use iterative methods.
We rearrange each equation for its main variable u_i:

$$C_{ii}u_i = q_i - \sum_{j \neq i} C_{ij}u_j.$$

Then we update u_i:

$$u_i = \frac{1}{C_{ii}} \left(q_i - \sum_{j \neq i} C_{ij}u_j \right).$$

That's the Gauss-Seidel update rule.

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Standard-Form of the Linear Complementarity Problem

For our tridiagonal C, each row has only three nonzero entries:

$$-\alpha\theta u_{i-1} + (1 + 2\alpha\theta)u_i - \alpha\theta u_{i+1} = q_i.$$

Rearrange for u_i:

$$u_i = \frac{1}{1 + 2\alpha\theta}\left(q_i + \alpha\theta u_{i-1} + \alpha\theta u_{i+1}\right).$$

We also need $u_i \geq f_i$, we simply project each update:

$$u_i^{(k+1)} = \max\left(f_i, \frac{1}{C_{ii}}\left[q_i - \sum_{j<i} C_{ij}u_j^{(k+1)} - \sum_{j>i} C_{ij}u_j^{(k)}\right]\right).$$

After repeat the algorithm until $m = M - 1$, we hold a vector $u^{(M)}$:

$$u^{(M)} = [u_{-N^-+1,M}, \ldots, u_{N^+-1,M}]^T.$$

That vector contains the transformed values of the option at time $t = 0$ (the valuation date) for different $x = \ln S$ grid points.

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Standard-Form of the Linear Complementarity Problem

So each component corresponds to:

$$u_{n,M} \approx u(x_n, \tau_{\max}) = e^{\frac{1}{2}(c-1)x_n + \frac{1}{4}(c+1)^2 \tau_{\max}} \psi(S_n, 0),$$

where $S_n = e^{x_n}$.
To return to the original (Black-Scholes) variables, recall the transformation:

$$u(x, \tau) = e^{\frac{1}{2}(c-1)x + \frac{1}{4}(c+1)^2 \tau} \psi(S, t), \quad c = \frac{2r}{\sigma^2}.$$

At $t = 0$, $\tau = T\sigma^2/2$.
So the original option price surface is:

$$\psi(S, 0) = e^{-\frac{1}{2}(c-1)\ln S - \frac{1}{4}(c+1)^2 \tau_{\max}} u(\ln S, \tau_{\max}).$$

That gives the American option price at time 0 for each stock price S on the grid. Usually we're interested in the price for one current stock price S_0. If S_0 lies between two grid nodes S_i, S_{i+1}, we interpolate linearly to find $\psi(S_0, 0)$. We can also track at each time step where $u_{n,m} = f_{n,m}$ (the points where equality binds) to reconstruct the exercise boundary $S^*(t)$.

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Chapter 10 Arbitrage

Motivation to Study Arbitrage

Can there be arbitrage even without options?

If your model of stock price dynamics itself creates a riskless profit (without any options), then the model is internally inconsistent. If the relationship between the interest rate r and the possible stock movements (up and down) is inconsistent, arbitrage automatically arises.

A valid model (arbitrage-free) must satisfy:

$$d < R < u,$$

where:

- $u =$ up-factor $= S_u/S_0$,
- $d =$ down-factor $= S_d/S_0$,
- $R = 1 + r$.

Only in that range can there exist a risk-neutral probability $p^* = \frac{R-d}{u-d}$ with $0 < p^* < 1$. Arbitrage-free condition ensures no riskless trading profit can be made by combining borrowing/lending and buying/selling the stock. If your model violates this, then the entire pricing framework collapses.

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Basic Setting of a One-Period Security Market

We construct the simplest possible financial market to study arbitrage formally.

- The model is discrete in time, $t = 0$ and $t = 1$, and finite in states (K scenarios), $\Omega = \{\omega_1, \ldots, \omega_K\}$, $P(\omega) > 0$.
- $N + 1$ securities: $B(t)$ is bank account (risk-free), $B(0) = 1$, $B(1) = 1 + r$ and $S_1(t), S_2(t), \ldots, S_N(t)$ are risky stocks.
- Trading Strategy: $\varphi = (\varphi_0, \varphi_1, \ldots, \varphi_N)$ are portfolio weights and $V(0)$ is initial wealth. The resulting wealth process is given by

$$V(t) := \varphi_0 B(t) + \sum_{n=1}^{N} \varphi_n S_n(t).$$

The self-financing condition is defined as:

$$V(1) - V(0) = \phi_0 \Delta B + \sum_{n=1}^{N} \phi_n \Delta S_n,$$

where $\Delta B := B(1) - B(0) = r, \quad \Delta S_n := S_n(1) - S_n(0).$

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Basic Setting of a One-Period Security Market

Define discounted variables:

$$S_n^*(t) := S_n(t)/B(t), \quad V^*(t) := V(t)/B(t) = \phi_0 + \sum_{n=1}^{N} \phi_n S_n^*(t) \quad t = 0, 1,$$

then we can rewrite the self-financing condition as

$$V^*(1) - V^*(0) = \sum_{n=1}^{N} \phi_n \Delta S_n^*, \quad V^*(t) = V(t)/B(t).$$

Define the capital gain process

$$G^* = \sum_{n=1}^{N} \phi_n \Delta S_n^*, \quad V^*(1) = V(0) + G^*.$$

The simple return of nth risky asset is defined as $R_n := (S_n(1) - S_n(0))\ S_n(0)$.

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Dominant Trading Strategies

A trading strategy ϕ is said to dominate another strategy $\tilde{\phi}$ if their corresponding wealth V and $\tilde{V}$ satisfy:

$$V(0) = \tilde{V}(0), \quad V(1) > \tilde{V}(1),$$

with probability one (i.e. $V(1,\omega) > \tilde{V}(1,\omega)$ for any $\omega \in \Omega$).

Lemma 1. The following statements are equivalent.

- There is a dominant strategy.
- There exists a trading strategy such that $V(0) = 0$ and $V(1) > 0$ with probability one.
- There exists a trading strategy such that $V(0) < 0$ and $V(1) \geq 0$ with probability one.

Proof:
(a) $\Rightarrow$ (b): Suppose there are two trading strategies $\hat{\phi}$ and $\tilde{\phi}$ such that $\hat{\phi}$ dominates $\tilde{\phi}$. Then consider another trading strategy $\phi = \hat{\phi} - \tilde{\phi}$. We have, via the definition of wealth processes,

$$V(0) = \hat{V}(0) - \tilde{V}(0) = 0, \quad V(1) = \hat{V}(1) - \tilde{V}(1) > 0,$$

with probability one, from which (b) follows.

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Dominant Trading Strategies

(b) $\Rightarrow$ (c): Now start from (b): $V(0) = 0$, $V(1) > 0$ in all states. Since $V^*(1) = G^* > 0$ for all states, the smallest possible gain $\varepsilon = \min_{\omega \in \Omega} G^*(\omega) > 0$ exists. Construct a new strategy:

$$\tilde{\phi}_n = \phi_n, \quad \tilde{\phi}_0 = \phi_0 - \varepsilon.$$

That reduces the initial wealth by ε:

$$\tilde{V}(0) = -\varepsilon < 0, \quad \tilde{V}^*(1, \omega) = -\varepsilon + G^*(\omega) \geq 0.$$

So you borrow money at the beginning ($V(0) < 0$), but never lose at the end ($V(1) \geq 0$). That's (c).

(c) $\Rightarrow$ (a): Now suppose we have (c): $V(0) < 0$ and $V(1) \geq 0$. Construct a new strategy $\tilde{\phi}$:

$$\tilde{\phi}_n = \phi_n, \quad \tilde{\phi}_0 = -\sum_{n=1}^{N} \phi_n S_n(0) = \phi_0 - V(0).$$

Then the new portfolio has:

$$\tilde{V}(0) = 0, \quad \tilde{V}^*(1) = -V(0) + V^*(1) \geq -V(0) > 0.$$

So $\tilde{\phi}$ strictly dominates the "do nothing" strategy (holding nothing), satisfying (a).

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Law of one price

The law says that there should not exist two trading strategies ϕ and ϕ' such that

$$V(0) > V'(0), \quad V(1,\omega) = V'(1,\omega), \quad \forall \omega \in \Omega.$$

Lemma: No dominant strategies implies the law of one price. In other words, the law of one price is a weaker assumption than the assumption of no dominant strategies.

Proof

We shall prove this by contradiction. Suppose the law of one price is violated. Then there are two strategies ϕ and ϕ' such that

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Now consider a new strategy $\tilde{\phi} = \phi' - \phi$. Then

$$\tilde{V}(0) = V'(0) - V(0) < 0, \quad \tilde{V}(1,\omega) = V'(1,\omega) - V(1,\omega) = 0 \geq 0.$$

Therefore, part (c) in the equivalence of dominant strategies is satisfied, and there exists dominant strategies. A contradiction.

Note: Law of one price not imply no dominant strategies.

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Arbitrage

An arbitrage opportunity arises when there exists a trading strategy such that:

$$V(0) = 0, \quad V(1) \geq 0, \quad \mathbb{E}[V(1)] > 0.$$

If there exists a dominant strategy then there must be an arbitrage opportunity.
However, the converse is not true.

Proposition: No arbitrage opportunities $\Rightarrow$ no dominant strategies $\Rightarrow$ law of one price.

Lemma: The following three statements are equivalent.

(a) ϕ is an arbitrage opportunity

(b) The discounted wealth V^* associated with ϕ satisfies $V^*(0) = 0$, $V^*(1) \geq 0$, and $\mathbb{E}(V^*(1)) > 0$;

(c) The associated capital gain process G^* satisfies $G^* \geq 0$ and $\mathbb{E}[G^*] > 0$.

Proof: First of all, (a) and (b) are equivalent simply because $B(1) > 0$.
Next, consider statement (c). Suppose ϕ is an arbitrage opportunity. We have from (b)

$$G^* = V^*(1) - V(0) = V^*(1) \geq 0,$$

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Conversely, suppose $G^* \geq 0$ and $\mathbb{E}[G^*] > 0$ for some strategy ϕ. Introduce a new strategy $\tilde{\phi} = (\tilde{\phi}_0, \phi_1, \ldots, \phi_N)$, where for the saving account $\tilde{\phi}_0 = -\sum_{n=1}^N \phi_n S_n^*(0)$. Then we see

$$\tilde{V}^*(0) = \tilde{\phi}_0 + \sum_{n=1}^N \phi_n S_n^*(0) = 0,$$

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$$\tilde{V}^*(1) = \tilde{\phi}_0 + \sum_{n=1}^N \phi_n S_n^*(1) = \tilde{\phi}_0 + \sum_{n=1}^N \phi_n S_n^*(0) + G^* = G^*.$$

Thus,

$$\tilde{V}^*(0) = 0, \quad \tilde{V}^*(1) \geq 0, \quad \mathbb{E}[\tilde{V}^*(1)] > 0,$$

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Linear Pricing Measure

A linear pricing measure is a non-negative vector $(\pi(\omega_1), \ldots, \pi(\omega_K))$ such that for every trading strategy ϕ, we must have

$$V(0) = \sum_{j=1}^{K} \pi(\omega_j) V^*(1, \omega_j),$$

where recall that $\Omega = (\omega_1, \ldots, \omega_K)$. Note that we only know that $\pi \geq 0$. But some of the $\pi(\omega_i)$ could be zero.

Lemma:

We must have $\sum_{j=1}^{K} \pi(\omega_j) = 1$. Thus, π must be a probability measure, and for every trading strategy ϕ

$$V(0) = E_\pi[V^*(1)].$$

In other words, if there exists a linear pricing measure, then the measure must be a probability measure, and the price of any asset at time 0 can be written as the expectation of the discounted final payoff of the asset, where the expectation is taken with respect to the measure π.

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Linear Pricing Measure

Proof:
Consider a strategy ϕ with $\phi_1 = \phi_2 = \cdots = \phi_N = 0$. In other words, the strategy only invests in the saving account and not in any stocks. Then $V^*(1) \equiv V(0)$.

$$V(0) = \sum_{j=1}^{K} \pi(\omega_j) V^*(1) = \sum_{j=1}^{K} \pi(\omega_j) V(0),$$

from which $\sum_{j=1}^{K} \pi(\omega_j) = 1$, and the conclusion follows.

Theorem 1: There are no dominant strategies if and only if there exists a linear pricing (probability) measure.

Proof later.

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Risk-Neutral Measure

A probability measure Q is called a risk-neutral measure if Q satisfies two requirements:

(i) $Q(\omega_i) > 0$, for all $i = 1, \ldots, K$.

(ii) $E_Q(S_n^*(1)) = S_n(0)$ for all $n = 1, \ldots, N$.

Under a risk-neutral measure, every asset has the same return as that of the saving account. Therefore, under this measure, investor does not give any higher return for higher risk, thus risk-neutral.

Note that in terms of the simple return, under the risk neutral probability Q, we must have

$$E_Q[R_n] = r, \quad R_n = \frac{S_n(1) - S_n(0)}{S_n(0)}.$$

because starting from the discounted-price condition (ii):

$$E_Q[S_n^*(1)] = S_n(0), \quad S_n^*(1) = \frac{S_n(1)}{B(1)}.$$

Since $B(1) = 1 + r$,

$$E_Q\left[\frac{S_n(1)}{1+r}\right] = S_n(0) \Rightarrow E_Q[S_n(1)] = S_n(0)(1+r) \Rightarrow E_Q\left[\frac{S_n(1) - S_n(0)}{S_n(0)}\right] = r.$$

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Risk-Neutral Measure

A risk neutral probability measure Q is not the real probability measure P observed in markets! However, it is a very important mathematical tool to derive pricing formulae.

Theorem 2: Absence of arbitrage opportunity is equivalent to the existence of a risk-neutral measure.

To prove Theorem 2, we shall show first that the existence of a risk neutral measure guarantees no arbitrage opportunity. Suppose there exist a risk-neutral measure Q. Then for any trading strategy ϕ, we must have

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by part (ii) in the definition of risk-neutral measure. Now suppose there is an arbitrage opportunity. Then there is a trading strategy ϕ such that

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$$E_Q[G^*] = \sum_{j=1}^{K} G^*(\omega_j) Q(\omega_j) \geq G^*(\omega_k) Q(\omega_k) > 0,$$

as $Q(\omega_k) > 0$. A contradiction to the fact that $E_Q[G^*] = 0$. The proof is terminated. The harder part is to show that no arbitrage implies the existence of a risk neutral measure. The proof will give later.

Corollary: In the binomial tree model there is no arbitrage for trading stocks if and only if $u > R > d$.

Proof: This is because there is a

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and $p^* \in (0,1)$ if and only if $u > R > d$.

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Linear Programming and Linear Pricing Measures

There are two optimization problems in linear programming:

(1) Primal Problem

$$\max_{x_1,\ldots,x_n} \; c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

subject to:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m$$
$$x_j \ge 0, \quad j = 1, \ldots, n$$

(2) Dual Problem

$$\min_{y_1,\ldots,y_m} \; b_1y_1 + b_2y_2 + \cdots + b_my_m$$

subject to:

$$a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \ge c_1$$
$$a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \ge c_2$$
$$\vdots$$
$$a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \ge c_n$$
$$y_i \ge 0, \quad i = 1, \ldots, m$$

In matrix notation, we have (1) $\max_X c^\top X$, subject to $AX \le b$, $X \ge 0$ and (2) $\min_Y Y^\top b$, subject to $A^\top Y \ge c$, $Y \ge 0$. (1) and (2) are called dual problems to each other. Note that every equality constraint can be translated into two inequality constraints (e.g. $w = c$ means exactly $w \ge c$ and $-w \ge -c$)

Linear Programming and Linear Pricing Measures

There are two optimization problems in linear programming:

(1) Primal Problem

$$\max_{x_1,\dots,x_n} \; c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

subject to:

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1\\
&a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2\\
&\vdots\\
&a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m\\
&x_j \ge 0, \quad j = 1,\dots,n
\end{aligned}$$

(2) Dual Problem

$$\min_{y_1,\dots,y_m} \; b_1y_1 + b_2y_2 + \cdots + b_my_m$$

subject to:

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Linear Programming and Linear Pricing Measures

Two key results in linear programming are:
Result 1 (Feasibility): For any LP problem, one of the following must hold:

(a) The LP has a finite optimal solution.

(b) The optimal solution is either ∞ or $-\infty$.

(c) The feasible region is empty, and thus the LP has no optimal solution.

Result 2 (Strong Duality): If a LP problem has a finite optimal solution, then so does its dual, and the optimal values for both the primary and the dual problem must be equal. The proof idea uses the geometry of convex sets (via the Farkas lemma or separating hyperplane theorem)

Another Result: (Weak Duality): For the primal-dual pair define above, if we take any feasible x and any feasible y, we have the weak duaility $c^\top x \leq b^\top y$. Proof follows from the two inequalities: $c^\top x \leq y^\top Ax \leq y^\top b = b^\top y$.

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Linear Programming and Linear Pricing Measures

Strong Duality Theorem (Linear Programming): If the primal LP

$$\max_x c^\top x \quad \text{s.t. } Ax \le b,\ x \ge 0$$

has a finite optimal solution x^*, then the dual LP

$$\min_y b^\top y \quad \text{s.t. } A^\top y \ge c,\ y \ge 0$$

also has an optimal solution y^*, and $c^\top x^* = b^\top y^*$.

This dual relationship is exactly how arbitrage-free pricing is framed:

- The primal problem corresponds to "finding a trading strategy to make profit" (maximize wealth)
- The dual problem corresponds to "finding consistent state prices or probabilities" (minimize cost)
- no arbitrage $\Leftrightarrow$ both sides match.

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Linear Programming and Linear Pricing Measures

Theorem 1: There are no dominant strategies if and only if there exists a linear pricing (probability) measure.

Proof: Let $S = (1, S_1^*(0), \dots, S_N^*(0))^T$ and let

$$Z = \begin{pmatrix} 1 & \cdots & 1 \\ S_1^*(1,\omega_1) & \cdots & S_1^*(1,\omega_K) \\ \vdots & \ddots & \vdots \\ S_N^*(1,\omega_1) & \cdots & S_N^*(1,\omega_K) \end{pmatrix}$$

be all the possible outcomes of asset prices at time 1. Denote h to be any trading strategy $h = (\phi_0, \phi_1, \dots, \phi_N)$. Then the wealth process is given by

$$V(0) = hS, \quad V^*(1) = hZ.$$

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Conversely, if there is no dominant strategy, then the value of the linear programming problem $\min_h hS$, subject to $hZ \geq 0$, has to be non-negative. But, since one can always choose no trading at all, the value of the linear programming problem has to be zero. Consequently, its dual problem must also have a value zero. This, in conjunction with part (a) of Exercise 4, implies that there exists a non-negative vector π, $Z\pi = S$, which is the linear pricing measure. The proof is terminated.

Theorem 2: Absence of arbitrage opportunity is equivalent to the existence of a risk-neutral measure.

Farkas' Lemma
Let A be an $m \times n$ matrix and b be an m-dimensional column vector. Then either

$$Ax = b, \quad x \geq 0, \quad x = (x_1, \ldots, x_n)^\top$$

has a solution, or

$$yA \leq 0, \quad yb > 0, \quad y = (y_1, \ldots, y_m)$$

has a solution, but not both.

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Linear Programming and Linear Pricing Measures

Now we start to prove that no arbitrage implies the existence of a risk neutral measure. Let A be the $(K+1)\times(2N+K)$ matrix:

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \Delta S_1^*(\omega_1) & -\Delta S_1^*(\omega_1) & \Delta S_2^*(\omega_1) & \cdots & -\Delta S_N^*(\omega_1) \\ \Delta S_1^*(\omega_2) & -\Delta S_1^*(\omega_2) & \Delta S_2^*(\omega_2) & \cdots & -\Delta S_N^*(\omega_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Delta S_1^*(\omega_K) & -\Delta S_1^*(\omega_K) & \Delta S_2^*(\omega_K) & \cdots & -\Delta S_N^*(\omega_K) \end{pmatrix} \circ B,$$

where the notation $\circ$ means pasting two matrices together, and

$$B = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{pmatrix}.$$

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Lemma: Denote $b = (1, 0, \ldots, 0)^T$ be a $(K + 1)$ column vector. Then

$$Ax = b,\ x \geq 0,\ x = (x_1, \ldots, x_{2N+K})$$

has a solution if and only if there exists an arbitrage opportunity.

Proof: The equation $Ax = b$ leads to a system of equations:

(a) The first equation of the system says

$$x_{2N+1} + \cdots + x_{2N+K} = 1,$$

which implies that $x_{2N+k} > 0$ for at least one value of k.

(b) For $i \geq 2$ the ith equation is

$$\Delta S_1^*(\omega_{i-1})(x_1 - x_2) + \cdots + \Delta S_N^*(\omega_{i-1})(x_{2N-1} - x_{2N}) = x_{2N+i-1}.$$

($\Rightarrow$) Suppose $(x_1, x_2, \ldots, x_{2N+K})^T$ is a solution. Let a trading strategy be $\phi_n = x_{2n-1} - x_{2n}$ for $n = 1, 2, \ldots, N$. Then, by the observation (b) above, we have the discounted gain is

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($\Leftarrow$) Conversely, suppose there exists an arbitrage opportunity ϕ, in which case $\mathbb{E}[G^*] > 0$, and

$$G^* = \sum_{n=1}^{N} \phi_n \Delta S_n^* \geq 0.$$

There's at least one state ω_i such that $G^*(\omega_i) > 0$. Therefore, we can find a positive constant λ such that

$$\lambda \sum_{k=1}^{K} G^*(\omega_k) = \lambda \sum_{k=1}^{K} \sum_{n=1}^{N} \phi_n \Delta S_n^*(\omega_k) = 1.$$

This can be done by taking $\lambda = 1/\sum_{k=1}^{K} G^*(\omega_k)$.

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Linear Programming and Linear Pricing Measures

(i) If $\phi_n = 0$, set $x_{2n-1} = x_{2n} = 0$, in which case $x_{2n-1} - x_{2n} = 0 = \lambda\phi_n$.

(ii) If $\phi_n > 0$, set $x_{2n-1} = \lambda\phi_n$ and $x_{2n} = 0$, in which case $x_{2n-1} - x_{2n} = \lambda\phi_n$.

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(iv) For $k = 1, 2, \dots, K$, set

$$x_{2N+k} = \Delta S_1^*(\omega_k)(x_1 - x_2) + \cdots + \Delta S_N^*(\omega_k)(x_{2N-1} - x_{2N}).$$

Then we have all $x_i \geq 0, i = 1, 2, \dots, 2N$, and

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By the above lemma and Farkas's lemma, the absence of arbitrage implies that there is a solution to the equation

$$yA \leq 0, \quad yb > 0, \quad y = (y_0, y_1, \ldots, y_K).$$

Since $b = (1, 0, \ldots, 0)^T$, we have $y_0 = yb > 0$, and $yA \leq 0$ implies three sets of inequalities:

$$\sum_{j=1}^{K} \Delta S_n^*(\omega_j) y_j \leq 0, \quad \sum_{j=1}^{K} -\Delta S_n^*(\omega_j) y_j \leq 0, \quad n = 1, \ldots, N,$$

$$y_0 - y_n \leq 0, \quad n = 1, \ldots, K.$$

Therefore, the first two sets of inequalities above must be equalities, i.e.

$$\sum_{j=1}^{K} \Delta S_n^*(\omega_j) y_j = 0, \quad n = 1, \ldots, N; \quad 0 < y_0 \leq y_n, \quad n = 1, \ldots, K.$$

In summary, under these weights, each asset's discounted price is a martingale. The weights $y_j > 0$ can be normalized to sum to one, giving a probability measure $Q(\omega_j) = y_j / \sum_i y_i$. Thus, Q is a risk-neutral probability measure.

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