MF921 Topics in Dynamic Asset Pricing

Week 4

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Background

Recall The Double Exponential Jump Diffusion Model:

$$\frac{dS(t)}{S(t^-)} = \mu dt + \sigma dW(t) + d\left(\sum_{i=1}^{N(t)}(V_i - 1)\right)$$

- $W(t)$: Brownian motion under the real-world measure.
- $N(t)$: Poisson process with rate λ.
- V_i: multiplicative jump sizes, i.i.d. random variables.
- $Y = \log(V)$, the jump sizes follow double exponential law:

$$f_Y(y) = p\eta_1 e^{-\eta_1 y}\mathbf{1}_{y\geq 0} + q\eta_2 e^{\eta_2 y}\mathbf{1}_{y<0}$$

with parameters:

- $p, q \geq 0, p + q = 1$: probabilities of upward/downward jumps.
- $\eta_1 > 1$: rate for upward jumps.
- $\eta_2 > 0$: rate for downward jumps.

Background Con.

For option pricing, we switch to a risk-neutral measure P^*, so that the discounted price process is a martingale:

$$E^{P^*}[e^{-rt}S(t)] = S(0)$$

Under P^*, the dynamics adjust:

$$\frac{dS(t)}{S(t^-)} = (r - \lambda^*(t)\zeta^*)dt + \sigma dW^*(t) + d\left(\sum_{i=1}^{N^*(t)}(V_i^* - 1)\right)$$

where:

- $W^*(t)$: Brownian motion under P^*,
- $N^*(t)$: Poisson process with intensity λ^*,
- $V^* = e^{Y^*}$: jump multiplier with new parameters $(p^*, q^*, \eta_1^*, \eta_2^*)$,
- $\zeta^* = E^{P^*}[V^*] - 1 = \frac{p^*\eta_1^*}{\eta_1^*-1} + \frac{q^*\eta_2^*}{\eta_2^*+1} - 1$ is mean percentage jump size.

The log-price process:

$$X(t) = \log\left(\frac{S(t)}{S(0)}\right) = \left(r - \frac{1}{2}\sigma^2 - \lambda^*\zeta^*\right)t + \sigma W^*(t) + \sum_{i=1}^{N^*(t)} Y_i^*, \quad X(0) = 0$$

Moment Generating Function of the log-price process, $X(t)$:

$$\mathbb{E}^{\mathbb{P}^*}\left[e^{\theta X(t)}\right] = \exp\{G(\theta)t\}$$

Where the function $G(\cdot)$ is defined as:

$$G(x) = x\left(r - \frac{1}{2}\sigma^2 - \lambda\zeta\right) + \frac{1}{2}x^2\sigma^2 + \lambda\left(\frac{p\eta_1}{\eta_1 - x} + \frac{q\eta_2}{\eta_2 + x} - 1\right)$$

Note: Lemma 3.1 in Kou and Wang (2003) shows that the equation $G(x) = \alpha, \forall\alpha > 0$, has exactly four roots: $\beta_{1,\alpha}$, $\beta_{2,\alpha}$, $-\beta_{3,\alpha}$, and $-\beta_{4,\alpha}$, where:

$$0 < \beta_{1,\alpha} < \eta_1 < \beta_{2,\alpha} < \infty$$
$$0 < \beta_{3,\alpha} < \eta_2 < \beta_{4,\alpha} < \infty$$

These roots determine the structure of Laplace transforms for first passage times.

Infinitesimal Generator of the log-price process, $X(t)$:

$$(\mathcal{L}V)(x) = \frac{1}{2}\sigma^2 V''(x) + \left(r - \frac{1}{2}\sigma^2 - \lambda\zeta\right) V'(x) + \lambda \int_{-\infty}^{\infty} \left(V(x+y) - V(x)\right) f_Y(y)\, dy$$

The generator describes how expectations of functions of $X(t)$ evolve in time:

$$\frac{d}{dt}\mathbb{E}[V(X_t)] = \mathbb{E}[(\mathcal{L}V)(X_t)]$$

They provide the mathematical foundation to derive option pricing formulas.

Option Pricing Under a Double Exponential Jump Diffusion Model

S.G. Kou
Hui Wang

Proof of two Theorems. The Laplace transform of lookback option and barrier option.

Lookback Options

Consider a lookback put option with an initial "prefixed maximum" $M \geq S(0)$:

$$
\begin{aligned}
LP(T) &= \mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}\left(\max\{M, \max_{0\le t\le T} S(t)\} - S(T)\right)\right] \\
&= \mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}\max\{M, \max_{0\le t\le T} S(t)\}\right] - S(0)
\end{aligned}
$$

You need the joint distribution of $\max S(t)$ and $S(T)$, which is complicated for jump processes. Laplace transforms convert a complicated path integral over time into a function of roots of $G(x)$ which we can solve algebraically.

Theorem:

Using the notations $\beta_{1,\alpha+r}$ and $\beta_{2,\alpha+r}$ as in early silde, the Laplace transform of the lookback put is given by:

$$\hat{L}(T) = \int_0^\infty e^{-\alpha T} \mathrm{LP}(T) dT = \frac{S(0)A_\alpha}{C_\alpha} \left(\frac{S(0)}{M}\right)^{\beta_{1,\alpha+r}-1} + \frac{S(0)B_\alpha}{C_\alpha} \left(\frac{S(0)}{M}\right)^{\beta_{2,\alpha+r}-1} + \frac{M}{\alpha+r} - \frac{S(0)}{\alpha}$$

For all $\alpha > 0$; here:

$$A_\alpha = \frac{(\eta_1 - \beta_{1,\alpha+r})\beta_{2,\alpha+r}}{\beta_{1,\alpha+r} - 1}$$

$$B_\alpha = \frac{(\beta_{2,\alpha+r} - \eta_1)\beta_{1,\alpha+r}}{\beta_{2,\alpha+r} - 1}$$

$$C_\alpha = (\alpha + r)\eta_1(\beta_{2,\alpha+r} - \beta_{1,\alpha+r})$$

Proof

Lemma : $\lim_{y\to\infty} e^y \mathbb{P}^*[M_X(T) \geq y] = 0$, $\forall T \geq 0$. $M_X(T) := \max_{0\leq t\leq T} X(t)$

[Proof]

Given $\theta \in (-\eta_2, \eta_1)$, $\mathbb{E}^{\mathbb{P}^*}[e^{\theta X(t)}] < \infty$, by stationary independent increments we can get that the process $\{e^{\theta X(t) - G(\theta)t}; t \geq 0\}$ is a martingale.

Observe that $G(x)$ is continuous and $G(1) = r > 0$, thus we can fix an $\theta \in (1, \eta_1)$ such that $G(\theta) > 0$. Let $\tau_y = \inf\{t \geq 0 : X(t) \geq y\}$. By Optional Sampling Theorem:

$$1 = \mathbb{E}^{\mathbb{P}^*}[M_{\tau_y \wedge T}] \mathbb{E}^{\mathbb{P}^*}\left[e^{\theta X(\tau_y \wedge T)} e^{-G(\theta)(\tau_y \wedge T)}\right] \geq e^{\theta y} e^{-G(\theta)T} \mathbb{P}^*(\tau_y \leq T)$$

Thus $e^{\theta y}\mathbb{P}^*(\tau_y \leq T) \leq e^{G(\theta)T}$. Since $\theta > 1$, then we have:

$$e^y \mathbb{P}^*(M_X(T) \geq y) = e^{(1-\theta)y}\left[e^{\theta y}\mathbb{P}^*(\tau_y \leq T)\right] \leq e^{(1-\theta)y} e^{G(\theta)T} \xrightarrow{y\to\infty} 0$$

This will use to justify the boundary term vanishing in the integration-by-parts step later.

Proof Con

Given $s = S(0)$ and M are constants, $\max_{0 \le t \le T} S(t) = se^{M_X(T)}$, the lookback put as:

$$LP(T) = \mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}\max\{M, se^{M_X(T)}\}\right] - s$$

Letting $z = \log(M/s) \ge 0$, define:

$$\begin{aligned}
L(s, M; T) :&= \mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}\max\{M, se^{M_X(T)}\}\right] \\
&= s\mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}\max\{e^z, e^{M_X(T)}\}\right] \\
&= s\mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}\left(e^{M_X(T)} - e^z\right)\mathbf{1}_{\{M_X(T)\ge z\}}\right] + se^z e^{-rT}
\end{aligned}$$

Proof Con

Integration by parts:

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}e^{M_X(T)}\mathbf{1}_{\{M_X(T)\geq z\}}\right] &= e^{-rT}\int_z^\infty e^y d(1-\mathbb{P}^*[M_X(T)\geq y]) \\
&= -e^{-rT}\int_z^\infty e^y d\mathbb{P}^*[M_X(T)\geq y] \\
&= -e^{-rT}\left\{\left(-e^y\mathbb{P}^*[M_X(T)\geq y]\right)\Big|_0^\infty - \int_z^\infty \mathbb{P}^*[M_X(T)\geq y]e^y dy\right\} \\
&= -e^{-rT}\left\{-e^z\mathbb{P}^*[M_X(T)\geq z] - \int_z^\infty \mathbb{P}^*[M_X(T)\geq y]e^y dy\right\} \\
&= \mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}e^z\mathbf{1}_{\{M_X(T)\geq z\}}\right] + e^{-rT}\int_z^\infty e^y\mathbb{P}^*[M_X(T)\geq y]dy;
\end{aligned}
$$

Plug back into $L(s,M;T)$:

$$
\begin{aligned}
L(s,M;T) &= s\mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}\left(e^{M_X(T)}-e^z\right)\mathbf{1}_{\{M_X(T)\geq z\}}\right] + se^z e^{-rT} \\
&= se^{-rT}\int_z^\infty e^y\mathbb{P}^*[M_X(T)\geq y]dy + Me^{-rT}
\end{aligned}
$$

Proof Con

Then take Laplace in maturity and use Fubini Theorem:

$$\begin{aligned}\int_0^\infty e^{-\alpha T} L(s, M; T)dT &= s\int_0^\infty e^{-\alpha T} e^{-rT}\int_z^\infty e^y \mathbb{P}^*[M_X(T) \geq y]dydT + \frac{M}{\alpha + r} \\ &= s\int_z^\infty e^y \int_0^\infty e^{-(\alpha+r)T}\mathbb{P}^*[M_X(T) \geq y]dTdy + \frac{M}{\alpha + r}\end{aligned}$$

Follows from Kou and Wang (2003) that:

$$\int_0^\infty e^{-(\alpha+r)T}\mathbb{P}^*[M_X(T) \geq y]dT = A_1 e^{-y\beta_{1,\alpha+r}} + B_1 e^{-y\beta_{2,\alpha+r}}$$

$$A_1 = \frac{1}{\alpha + r}\frac{\eta_1 - \beta_{1,\alpha+r}}{\eta_1}\cdot\frac{\beta_{2,\alpha+r}}{\beta_{2,\alpha+r} - \beta_{1,\alpha+r}}, \quad B_1 = \frac{1}{\alpha + r}\frac{\beta_{2,\alpha+r} - \eta_1}{\eta_1}\cdot\frac{\beta_{1,\alpha+r}}{\beta_{2,\alpha+r} - \beta_{1,\alpha+r}}.$$

Proof Con

Note that $\beta_{2,\alpha+r} > \eta_1 > 1$, $\beta_{1,\alpha+r} > \beta_{1,r} = 1$. Therefore:

$$\begin{aligned}\int_0^\infty e^{-\alpha T} L(s, M; T) dT &= s \int_z^\infty e^y A_1 e^{-y\beta_{1,\alpha+r}} dy + s \int_z^\infty e^y B_1 e^{-y\beta_{2,\alpha+r}} dy + \frac{M}{\alpha + r} \\ &= sA_1 \frac{e^{-z(\beta_{1,\alpha+r}-1)}}{\beta_{1,\alpha+r} - 1} + sB_1 \frac{e^{-z(\beta_{2,\alpha+r}-1)}}{\beta_{2,\alpha+r} - 1} + \frac{M}{\alpha + r} \\ &= s\frac{A_\alpha}{C_\alpha} e^{-z(\beta_{1,\alpha+r}-1)} + s\frac{B_\alpha}{C_\alpha} e^{-z(\beta_{2,\alpha+r}-1)} + \frac{M}{\alpha + r}.\end{aligned}$$

This yields the Laplace transform we obtained in the Theorem:

$$\begin{aligned}\int_0^\infty e^{-\alpha T} L(s, M; T) dT &= s \int_z^\infty e^y A_1 e^{-y\beta_{1,\alpha+r}} dy + s \int_z^\infty e^y B_1 e^{-y\beta_{2,\alpha+r}} dy + \frac{M}{\alpha + r} \\ &= sA_1 \frac{e^{-z(\beta_{1,\alpha+r}-1)}}{\beta_{1,\alpha+r} - 1} + sB_1 \frac{e^{-z(\beta_{2,\alpha+r}-1)}}{\beta_{2,\alpha+r} - 1} + \frac{M}{\alpha + r} \\ &= s\frac{A_\alpha}{C_\alpha} e^{-z(\beta_{1,\alpha+r}-1)} + s\frac{B_\alpha}{C_\alpha} e^{-z(\beta_{2,\alpha+r}-1)} + \frac{M}{\alpha + r}\end{aligned}$$

This yields the Laplace transform we obtained in the Theorem.

Barrier Options

Consider the up-and-in call (UIC) option with the barrier level H ($H > S(0)$):

$$UIC = E^{\mathbb{P}^*}[e^{-rT}(S(T) - K)^+ I\{\max_{0 \le t \le T} S(t) \ge H\}]$$

For any given probability P, define:

$$\Psi(\mu, \sigma, \lambda, p, \eta_1, \eta_2; a, b, T) := P[Z(T) \ge a, \max_{0 \le t \le T} Z(t) \ge b]$$

where under $P, Z(t)$ is a double exponential jump diffusion process with drift μ, volatility σ, and jump rate λ, i.e., $Z(t) = \mu t + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i$, and Y has a double exponential distribution with density $f_Y(y) \sim p \cdot \eta_1 e^{-\eta_1 y} 1_{\{y \ge 0\}} + q \cdot \eta_2 e^{y \eta_2} 1_{\{y < 0\}}$.

Barrier Options Con.

Theorem:

The price of the UIC option is obtained as:

$$\begin{aligned}\text{UIC} =& S(0)\Psi\left(r+\frac{1}{2}\sigma^2-\lambda\zeta,\sigma,\tilde{\lambda},\tilde{p},\tilde{\eta}_1,\tilde{\eta}_2;\ \log\left(\frac{K}{S(0)}\right),\log\left(\frac{H}{S(0)}\right),T\right)\\ &-Ke^{-rT}\cdot\Psi\left(r-\frac{1}{2}\sigma^2-\lambda\zeta,\sigma,\lambda,p,\eta_1,\eta_2;\ \log\left(\frac{K}{S(0)}\right),\log\left(\frac{H}{S(0)}\right),T\right)\end{aligned}$$

where $\tilde{p}=(p/(1+\zeta))\cdot(\eta_1/(\eta_1-1)),\tilde{\eta}_1=\eta_1-1,\tilde{\eta}_2=\eta_2+1,\tilde{\lambda}=\lambda(\zeta+1)$, with $\zeta=E^{P^*}[V]-1=\frac{p\eta_1}{\eta_1-1}+\frac{q\eta_2}{\eta_2+1}-1$. The Laplace transforms of Ψ is computed explicitly in Kou and Wang (2003).

Proof

Theorem 3.1 in Kou and Wang (2003):
For any $\alpha \in (0, \infty)$, let $\beta_{1,\alpha}$ and $\beta_{2,\alpha}$ be the only two positive roots of the equation

$$\alpha = G(\beta),$$

where $0 < \beta_{1,\alpha} < \eta_1 < \beta_{2,\alpha} < \infty$. Then we have the following results concerning the Laplace transforms of τ_b and X_{τ_b}:

$$\mathbb{E}[e^{-\alpha\tau_b}] = \frac{\eta_1 - \beta_{1,\alpha}}{\eta_1}\frac{\beta_{2,\alpha}}{\beta_{2,\alpha} - \beta_{1,\alpha}}e^{-b\beta_{1,\alpha}} + \frac{\beta_{2,\alpha} - \eta_1}{\eta_1}\frac{\beta_{1,\alpha}}{\beta_{2,\alpha} - \beta_{1,\alpha}}e^{-b\beta_{2,\alpha}}$$

$$\mathbb{E}[e^{-\alpha\tau_b}\mathbf{1}_{\{X_{\tau_b}-b>y\}}] = e^{-\eta_1 y}\frac{(\eta_1 - \beta_{1,\alpha})(\beta_{2,\alpha} - \eta_1)}{\eta_1(\beta_{2,\alpha} - \beta_{1,\alpha})}[e^{-b\beta_{1,\alpha}} - e^{-b\beta_{2,\alpha}}] \text{ for all } y \geq 0$$

$$\mathbb{E}[e^{-\alpha\tau_b}\mathbf{1}_{\{X_{\tau_b}=b\}}] = \frac{\eta_1 - \beta_{1,\alpha}}{\beta_{2,\alpha} - \beta_{1,\alpha}}e^{-b\beta_{1,\alpha}} + \frac{\beta_{2,\alpha} - \eta_1}{\beta_{2,\alpha} - \beta_{1,\alpha}}e^{-b\beta_{2,\alpha}}$$

The definition of τ_b are same and definition of X equilivent to Z here.

Proof Con.

Based on the Theorem 3.1, we can get explicit formula of the Laplace transform for $\Psi(\mu, \sigma, \lambda, p, \eta_1, \eta_2; a, b, T)$ (write as $\Psi(a, b, T)$ for simplicity). Define the Laplace transform in T:

$$\Phi_\alpha(a, b) := \int_0^\infty e^{-\alpha T} \Psi(a, b, T) dT = \int_0^\infty e^{-\alpha T} \mathbb{E}[1_{\{\tau_b \leq T\}} 1_{\{X(T) \geq a\}}] dT$$

We first use Fubini Theorem and strong Markov property can rewrite the formula, then split by overshoot and use tail-integration eventuality we will end up with the following:

$$\begin{aligned}\Phi_\alpha(a, b) = A_1 e^{-\beta_{1,\alpha}(a-b)} &\left(\mathbb{E}\left[e^{-\alpha\tau_b} \mathbf{1}_{\{X_{\tau_b} = b\}} \right] + \beta_{1,\alpha} \int_0^{a-b} e^{\beta_{1,\alpha} y} \mathbb{E}\left[e^{-\alpha\tau_b} \mathbf{1}_{\{X_{\tau_b} - b > y\}} \right] dy \right) \\ + B_1 e^{-\beta_{2,\alpha}(a-b)} &\left(\mathbb{E}\left[e^{-\alpha\tau_b} \mathbf{1}_{\{X_{\tau_b} = b\}} \right] + \beta_{2,\alpha} \int_0^{a-b} e^{\beta_{2,\alpha} y} \mathbb{E}\left[e^{-\alpha\tau_b} \mathbf{1}_{\{X_{\tau_b} - b > y\}} \right] dy \right)\end{aligned}$$

where:

$$A_1 = \frac{1}{\alpha} \frac{\eta_1 - \beta_{1,\alpha}}{\eta_1} \cdot \frac{\beta_{2,\alpha}}{\beta_{2,\alpha} - \beta_{1,\alpha}}, \quad B_1 = \frac{1}{\alpha} \frac{\beta_{2,\alpha} - \eta_1}{\eta_1} \cdot \frac{\beta_{1,\alpha}}{\beta_{2,\alpha} - \beta_{1,\alpha}}$$

Proof Con.

Back to the proof of the Theorem for Barrier option. First rewrite UIC as:

$$
\begin{aligned}
UIC &= E^{\mathbb{P}^*}[e^{-rT}(S(T)-K)^+ I\{\max_{0\le t\le T} S(t)\ge H\}] \\
&= \mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}S(T)\mathbf{1}_{\{S(T)\ge K, \max_{0\le t\le T} S(t)\ge H\}}\right] \\
&- Ke^{-rT}\mathbb{P}^*\left[S(T)\ge K, \max_{0\le t\le T} S(t)\ge H\right] \\
&= I - Ke^{-rT}\cdot II
\end{aligned}
$$

Since the log–price $X(t)=\log(S(t)/S(0))$:

$$
S(T)\ge K \iff X(T)\ge \log\frac{K}{S(0)}, \quad \max_{0\le t\le T} S(t)\ge H \iff \max_{0\le t\le T} X(t)\ge \log\frac{H}{S(0)}.
$$

Based on the definition of $\Psi(\mu,\sigma,\lambda,p,\eta_1,\eta_2;a,b,T)$:

$$
II = \Psi\left(r-\frac{1}{2}\sigma^2-\lambda\zeta,\sigma,\lambda,p,\eta_1,\eta_2;\log\left(\frac{K}{S(0)}\right),\log\left(\frac{H}{S(0)}\right),T\right)
$$

Proof Con.

For the first term, we can use a change of numeraire argument. More precisely, introduce a new probability $\tilde{\mathbb{P}}$ defined as:

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^*}\bigg|_{t=T} = e^{-rT}\frac{S(T)}{S(0)} = e^{-rT}e^{X(T)} = \exp\left\{\left(-\frac{1}{2}\sigma^2 - \lambda\zeta\right)T + \sigma W(T) + \sum_{i=1}^{N(T)} Y_i\right\}$$

Note that this is a well-defined probability as $\mathbb{E}^{\mathbb{P}^*}[e^{-rt}(S(t)/S(0))] = 1$. Then we reparametrize everything we need under $\tilde{\mathbb{P}}$. The Brownian part by Girsanov is $\tilde{W}(t) = W(t) - \sigma t$. The diffusion drift shifts by $+\sigma^2$, $\tilde{\mu} = r + \frac{1}{2}\sigma^2 - \lambda\zeta$. The modified jump rate and the jump distribution:

$$\text{New rate}: \ \tilde{\lambda} = \lambda\mathbb{E}^{\mathbb{P}^*}[e^Y] = \lambda(1+\zeta), \quad \text{where } \zeta = \mathbb{E}^{\mathbb{P}^*}[e^Y] - 1 = \frac{p\eta_1}{\eta_1 - 1} + \frac{q\eta_2}{\eta_2 + 1} - 1$$

$$\text{New jump density: } \tilde{f}_Y(y) = \frac{e^y}{\mathbb{E}^{\mathbb{P}^*}[e^Y]} f_Y(y) = \tilde{p}\tilde{\eta}_1 e^{-\tilde{\eta}_1 y}\mathbf{1}_{y\geq 0} + \tilde{q}\tilde{\eta}_2 e^{\tilde{\eta}_2 y}\mathbf{1}_{y<0}$$

With: $\tilde{\eta}_1 = \eta_1 - 1,\ \tilde{\eta}_2 = \eta_2 + 1,\ \tilde{p} = p\left(\frac{p\eta_1}{\eta_1 - 1} + \frac{q\eta_2}{\eta_2 + 1}\right)^{-1}\frac{\eta_1}{\eta_1 - 1},\ \tilde{q} = q\left(\frac{p\eta_1}{\eta_1 - 1} + \frac{q\eta_2}{\eta_2 + 1}\right)^{-1}\frac{\eta_2}{\eta_2 + 1}$

For I:

$$\begin{aligned} I &= S(0)\mathbb{E}^{\mathbb{P}^*}\left[e^{-rT}\frac{S(T)}{S(0)}\cdot \mathbf{1}_{\{S(T)\geq K, \max_{0\leq t\leq T} S(t)\geq H\}}\right] \\ &= S(0)\tilde{\mathbb{P}}\left[S(T)\geq K, \min_{0\leq t\leq T} S(t) \leq H\right] \\ &= S(0)\Psi\left(r + \frac{1}{2}\sigma^2 - \lambda\zeta, \sigma, \tilde{\lambda}, \tilde{p}, \tilde{\eta}_1, \tilde{\eta}_2; \log\left(\frac{K}{S(0)}\right), \log\left(\frac{H}{S(0)}\right), T\right) \end{aligned}$$

Part II

Pricing Path-Dependent Options with Jump Risk via Laplace Transforms

Steven Kou
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Derive the Laplace transforms for casepricing of European call and put options. Derive the two dimensional Laplace transform for barrier option.

European call and put options

The price of a European call and put with maturity T and strike K, is given:

$$C_T(k) = e^{-rT}\mathbb{E}^{\mathbb{P}^*}[(S(T)-K)^+] = e^{-rT}\mathbb{E}^{\mathbb{P}^*}\left[(S(0)e^{X(T)} - e^{-k})^+\right]$$
$$P_T(k') = e^{-rT}\mathbb{E}^{\mathbb{P}^*}[(K-S(T))^+] = e^{-rT}\mathbb{E}^{\mathbb{P}^*}\left[(e^{k'} - S(0)e^{X(T)})^+\right]$$

By the change of numeraire argument w.r.t $S(t)$:

$$C_T(k) = S(0)\tilde{\Psi}_C(k) - e^{-k}e^{-rT}\Psi_C(k)$$
$$P_T(k') = e^{k'}e^{-rT}\Psi_P(k') - S(0)\tilde{\Psi}_P(k')$$

where:

$$\Psi_C(k) = \mathbb{P}^*(S(T) \geq e^{-k}),\ \tilde{\Psi}_C(k) = \tilde{\mathbb{P}}(S(T) \geq e^{-k})$$
$$\Psi_P(k') = \mathbb{P}^*(S(T) < e^{k'}),\ \tilde{\Psi}_P(k') = \tilde{\mathbb{P}}(S(T) < e^{k'})$$

European call and put options Con.

Lemma. The Laplace transform with respect to k of $C_T(k)$ and with respect to k' for the put option $P_T(k')$ are given by:

$$\tilde{f}_C(\xi) := \int_{-\infty}^{\infty} e^{-\xi k} C_T(k) dk = e^{-rT} \frac{S(0)^{\xi+1}}{\xi(\xi+1)} \exp(G(\xi+1)T), \quad \xi > 0$$

$$\tilde{f}_P(\xi) := \int_{-\infty}^{\infty} e^{-\xi k'} P_T(k') dk' = e^{-rT} \frac{S(0)^{-\xi-1}}{\xi(\xi-1)} \exp(G(-(\xi-1))T), \quad \xi > 1$$

The Laplace transforms with respect to k of $\Psi_C(k)$ and k' of $\Psi_P(k')$ are:

$$\tilde{f}_{\Psi_C}(\xi) := \int_{-\infty}^{\infty} e^{-\xi k} \Psi_C(k) dk = \frac{S(0)^{\xi}}{\xi} \exp(G(\xi)T), \quad \xi > 0$$

$$\tilde{f}_{\Psi_P}(\xi) := \int_{-\infty}^{\infty} e^{-\xi k'} \Psi_P(k') dk' = e^{-rT} \frac{S(0)^{-\xi}}{\xi} \exp(G(-\xi)T), \quad \xi > 0$$

European call and put options Con.

Proof:
The Laplace transform for the call option is:

$$\hat{f}_C(\xi) = e^{-rT} \int_{-\infty}^{\infty} e^{-\xi k} \mathbb{E}^{\mathbb{P}^*} \left[(S(0)e^{X(T)} - e^{-k})^+ \right] dk$$

Applying the Fubini theorem yields for every $\xi > 0$:

$$\begin{aligned}
\hat{f}_C(\xi) &= e^{-rT} \mathbb{E}^{\mathbb{P}^*} \left[\int_{-\infty}^{\infty} e^{-\xi k} \left(S(0)e^{X(T)} - e^{-k} \right)^+ dk \right] \\
&= e^{-rT} \mathbb{E}^{\mathbb{P}^*} \left[\int_{-X(T)-\log S(0)}^{\infty} e^{-\xi k} \left(S(0)e^{X(T)} - e^{-k} \right) dk \right] \\
&= e^{-rT} \mathbb{E}^{\mathbb{P}^*} \left[S(0)e^{X(T)} e^{\xi(X(T)+\log S(0))} \frac{1}{\xi} - e^{(\xi+1)(X(T)+\log S(0))} \frac{1}{\xi+1} \right] \\
&= e^{-rT} \frac{S(0)^{\xi+1}}{\xi(\xi+1)} \mathbb{E}^{\mathbb{P}^*} \left[e^{(\xi+1)X(T)} \right] \overset{\text{MGF}}{=} e^{-rT} \frac{S(0)^{\xi+1}}{\xi(\xi+1)} e^{G(\xi+1)T}
\end{aligned}$$

Similary, we can get the Laplace transform for the put option.

European call and put options Con.

The Laplace transforms with respect to k of $\Psi_C(k)$:

$$\begin{aligned}
\hat{f}_{\Psi_C}(\xi) &= \int_{-\infty}^{\infty} e^{-\xi k} \mathbb{E}^{\mathbb{P}^*}\left[\mathbf{1}_{\{S(T)\geq e^{-k}\}}\right] dk \\
&= \int_{-\infty}^{\infty} e^{-\xi k} \mathbb{E}^{\mathbb{P}^*}\left[\mathbf{1}_{\{k\geq -\log S(T)\}}\right] dk \\
&= \mathbb{E}^{\mathbb{P}^*}\left[\int_{-\log S(T)}^{\infty} e^{-\xi k} dk\right] \\
&= \frac{1}{\xi}\mathbb{E}^{\mathbb{P}^*}\left[S(T)^{\xi}\right] = \frac{S(0)^{\xi}}{\xi}\mathbb{E}^{\mathbb{P}^*}\left[e^{\xi X(T)}\right] \overset{\text{MGF}}{=} \frac{S(0)^{\xi}}{\xi} e^{G(\xi)T}
\end{aligned}$$

Similary, we can get the Laplace transform for $\Psi_P(k')$.

Barrier Options

Rewrite the pricing formula of up-and-in call option(UIC) in early silde:

$$UIC(k,T) = E^{\mathbb{P}^*}\left[e^{-rT}\left(S(T) - e^{-k}\right)^+ I[\tau_b < T]\right]$$

where $H > S(0)$ is the barrier level, $k = -\log(K)$ the transformed strike and $b = \log(H/S(0))$. In previous paper, we obtain:

$$UIC(k,T) = S(0)\tilde{\Psi}_{UI}(k,T) - Ke^{-rT}\Psi_{UI}(k,T)$$

where:

$$\Psi_{UI}(k,T) = P^*(S(T) \geq e^{-k}, \tau_b < T), \quad \widetilde{\Psi}_{UI}(k,T) = \widetilde{P}(S(T) \geq e^{-k}, \tau_b < T)$$

Barrier Options Con.

Theorem: For ξ and α such that $0 < \xi < \eta_1 - 1$ and $\alpha > \max(G(\xi+1) - r, 0)$ (such a choice of ξ and α is possible for all small enough ξ as $G(1) - r = -\delta < 0$). The Laplace transform with respect to k and T of $UIC(k,T)$ is given by

$$\begin{aligned}\tilde{f}_{UIC}(\xi,\alpha) &= \int_0^\infty \int_{-\infty}^\infty e^{-\xi k - \alpha T} UIC(k,T) dk dT \\ &= \frac{H^{\xi+1}}{\xi(\xi+1)} \frac{1}{r + \alpha - G(\xi+1)} \left(A(r+\alpha) \frac{\eta_1}{\eta_1 - (\xi+1)} + B(r+\alpha) \right)\end{aligned}$$

where

$$\begin{aligned}A(h) &:= E^{\mathbb{P}^*}\left[e^{-h\tau_b} \mathbf{1}_{\{X(\tau_b)>b\}}\right] = \frac{(\eta_1 - \beta_{1,h})(\beta_{2,h} - \eta_1)}{\eta_1(\beta_{2,h} - \beta_{1,h})}\left[e^{-b\beta_{1,h}} - e^{-b\beta_{2,h}}\right] \\ B(h) &:= E^{\mathbb{P}^*}\left[e^{-h\tau_b} \mathbf{1}_{\{X(\tau_b=b)\}}\right] = \frac{\eta_1 - \beta_{1,h}}{\beta_{2,h} - \beta_{1,h}} e^{-b\beta_{1,h}} + \frac{\beta_{2,h} - \eta_1}{\beta_{2,h} - \beta_{1,h}} e^{-b\beta_{2,h}}\end{aligned}$$

with $b = \log(H/S(0))$.

Barrier Options Con.

If $0 < \xi < \eta_1$ and $\alpha > \max(G(\xi), 0)$ (again this choice of ξ and α is possible for all ξ small enough as $G(0) = 0$), then the Laplace transform with respect to k and T of $\Psi_{UI}(k, T)$ is:

$$
\begin{aligned}
\tilde{f}_{\Psi_{UI}}(\xi, \alpha) &= \int_0^{\infty} \left(\int_{-\infty}^{\infty} e^{-\xi k - \alpha T} \Psi_{UI}(k, T) dk \right) dT \\
&= \frac{H^{\xi}}{\xi} \frac{1}{\alpha - G(\xi)} \left(A(\alpha) \frac{\eta_1}{\eta_1 - \xi} + B(\alpha) \right)
\end{aligned}
$$

The Laplace transforms with respect to k and T of $\tilde{\Psi}_{UI}(k, T)$ is given similarly with $\tilde{G}$ replacing G and the functions $\tilde{A}$ and $\tilde{B}$ defined similarly.

Barrier Options Con.

Proof: Follow the pricing formula of UIC and the Fubini theorem:

$$\tilde{f}_{UIC}(\xi,\alpha) = \int_0^\infty \int_{-\infty}^\infty e^{-\xi k-(r+\alpha)T}\mathbb{E}^{\mathbb{P}^*}\left[\left(S(T)-e^{-k}\right)^+ \mathbf{1}_{\{\tau_b<T\}}\right] dkdT$$

$$= \mathbb{E}^{\mathbb{P}^*}\left[\int_0^\infty e^{-(r+\alpha)T}\mathbf{1}_{\{\tau_b<T\}}\left(\int_{-\log S(T)}^\infty e^{-\xi k}\left(S(T)-e^{-k}\right)dk\right)dT\right]$$

$$= \frac{1}{\xi(\xi+1)}\mathbb{E}^{\mathbb{P}^*}\left[\int_0^\infty e^{-(r+\alpha)T}\mathbf{1}_{\{\tau_b<T\}}S(T)^{\xi+1}dT\right]$$

$$(T=\tau_b+t \text{ with } t>0) = \frac{1}{\xi(\xi+1)}\mathbb{E}^{\mathbb{P}^*}\left[e^{-(r+\alpha)\tau_b}\int_0^\infty e^{-(r+\alpha)t}S(t+\tau_b)^{\xi+1}dt\right]$$

$$= \frac{1}{\xi(\xi+1)}\mathbb{E}^{\mathbb{P}^*}\left[e^{-(r+\alpha)\tau_b}\mathbb{E}^{\mathbb{P}^*}\left[\int_0^\infty e^{-(r+\alpha)t}S(t+\tau_b)^{\xi+1}dt\middle|\mathcal{F}_{\tau_b}\right]\right]$$

$$(i) = \frac{1}{\xi(\xi+1)}\mathbb{E}^{\mathbb{P}^*}\left[e^{-(r+\alpha)\tau_b}S(\tau_b)^{\xi+1}\int_0^\infty e^{-(r+\alpha)t}\mathbb{E}^{\mathbb{P}^*}\left[e^{(\xi+1)X(t)}\right]dt\right]$$

Where i is based on $S(\tau_b+t)^{\xi+1} = (S(\tau_b))^{\xi+1}\cdot e^{(\xi+1)(X(\tau_b+t)-X(\tau_b))}$ and strong Markov property that $\mathbb{E}^{\mathbb{P}^*}\left[e^{(\xi+1)(X(\tau_b+t)-X(\tau_b))}\middle|\mathcal{F}_{\tau_b}\right] = \mathbb{E}^{\mathbb{P}^*}\left[e^{(\xi+1)X(t)}\right]$

Con.

$$
\begin{aligned}
(\text{MGF}) &= \frac{1}{\xi(\xi+1)}\mathbb{E}^{\mathbb{P}^*}\left[e^{-(r+\alpha)\tau_b}S(\tau_b)^{\xi+1}\int_0^\infty e^{-(r+\alpha)t}e^{G(\xi+1)t}dt\right] \\
&= \frac{1}{\xi(\xi+1)}\frac{1}{r+\alpha-G(\xi+1)}\mathbb{E}^{\mathbb{P}^*}\left[e^{-(r+\alpha)\tau_b}S(\tau_b)^{\xi+1}\right] \\
&= \frac{1}{\xi(\xi+1)}\frac{1}{r+\alpha-G(\xi+1)}\left\{\mathbb{E}^{\mathbb{P}^*}\left[e^{-(r+\alpha)\tau_b}H^{\xi+1}\mathbf{1}_{\{X(\tau_b)>b\}}\right]\mathbb{E}^{\mathbb{P}^*}\left[e^{(\xi+1)\chi^+}\right]\right. \\
&\left.+\ \mathbb{E}^{\mathbb{P}^*}\left[e^{-(r+\alpha)\tau_b}H^{\xi+1}\mathbf{1}_{\{X(\tau_b)=b\}}\right]\right\} \\
&= \frac{H^{\xi+1}}{\xi(\xi+1)}\frac{1}{r+\alpha-G(\xi+1)}\left\{A(r+\alpha)\frac{\eta_1}{\eta_1-(\xi+1)}+B(r+\alpha)\right\}
\end{aligned}
$$

Where $\chi^+ \sim \text{Exp}(\eta_1)$, and $A(h) := \mathbb{E}^{\mathbb{P}^*}[e^{-h\tau_b}\mathbf{1}_{\{X(\tau_b)>b\}}]$, $B(h) := \mathbb{E}^{\mathbb{P}^*}[e^{-h\tau_b}\mathbf{1}_{\{X(\tau_b)=b\}}]$. Kou-Wang compute $A(h)$ and $B(h)$ explicitly (via first-passage Laplace transforms) with $\beta_{1,h}, \beta_{2,h}$ being the two positive roots of $G(\beta) = h$:

$$
A(h) = \frac{(\eta_1-\beta_{1,h})(\beta_{2,h}-\eta_1)}{\eta_1(\beta_{2,h}-\beta_{1,h})}\left(e^{-b\beta_{1,h}}-e^{-b\beta_{2,h}}\right)
$$

$$
B(h) = \frac{\eta_1-\beta_{1,h}}{\beta_{2,h}-\beta_{1,h}}e^{-b\beta_{1,h}}+\frac{\beta_{2,h}-\eta_1}{\beta_{2,h}-\beta_{1,h}}e^{-b\beta_{2,h}}
$$

Barrier Options Con.

For the Laplace transform of the probability Ψ_{UI}, apply the same trick we have:

$$
\begin{aligned}
\hat{f}_{\Psi_{UI}}(\xi,\alpha) &= \int_0^\infty \left[\int_{-\infty}^\infty e^{-\xi k-\alpha T}\cdot \mathbb{E}^{\mathbb{P}^*}\left\{\mathbf{1}_{\{k>-\log(S(T)),\tau_b<T\}}\right\}dk\right]dT \\
&= \mathbb{E}^{\mathbb{P}^*}\left\{\int_{\tau_b}^\infty\left[\int_{-\log S(T)}^\infty e^{-\xi k-\alpha T}dk\right]dT\right\} \\
&= \frac{1}{\xi}\mathbb{E}^{\mathbb{P}^*}\left\{\int_{\tau_b}^\infty S(T)^\xi e^{-\alpha T}dT\right\} \\
&= \frac{1}{\xi}\mathbb{E}^{\mathbb{P}^*}\left\{e^{-\alpha\tau_b}\int_0^\infty \{S(t+\tau_b)\}^\xi e^{-\alpha t}dt\right\} \\
&= \frac{1}{\xi}\mathbb{E}^{\mathbb{P}^*}\left\{e^{-\alpha\tau_b}\mathbb{E}^{\mathbb{P}^*}\left[\int_0^\infty S(\tau_b+t)^\xi e^{-\alpha t}dt\middle|\mathcal{F}_{\tau_b}\right]\right\} \\
&= \frac{1}{\xi}\frac{1}{\alpha-G(\xi)}\mathbb{E}^{\mathbb{P}^*}\left\{e^{-\alpha\tau_b}[S(\tau_b)]^\xi\right\} \\
&= \frac{1}{\xi}\frac{1}{\alpha-G(\xi)}\left\{\mathbb{E}^{\mathbb{P}^*}\left[e^{-\alpha\tau_b}H^\xi\mathbf{1}_{[X(\tau_b)>b]}\right]\mathbb{E}^{\mathbb{P}^*}\left[e^{\xi\chi^+}\right]+\mathbb{E}^{\mathbb{P}^*}\left[e^{-\alpha\tau_b}H^\xi\mathbf{1}_{[X(\tau_b)=b]}\right]\right\} \\
&= \frac{H^\xi}{\xi}\frac{1}{\alpha-G(\xi)}\left\{A(\alpha)\frac{\eta_1}{\eta_1-\xi}+B(\alpha)\right\}
\end{aligned}
$$