MF921 Topics in Dynamic Asset Pricing

Week 10

Yuanhui Zhao

Boston University

Chapter 14 Viscosity Solutions and HJB Equations

Definition of Viscosity Solutions

We start with an open domain

$$\Omega \subset \mathbb{R}^d,$$

and a function $u(t,x)$ satisfying a nonlinear second-order PDE

$$F(t,x,u(t,x),D_t u(t,x),D_x u(t,x),D_x^2 u(t,x)) = 0, \quad (t,x) \in [0,T) \times \Omega.$$

Where :

- $D_t u$: time derivative $\partial u/\partial t$
- $D_x u = \nabla_x u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_d}\right)^T$
- $D_x^2 u$: the Hessian matrix, with entries $(D_x^2 u)_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$

with the terminal condition

$$u(T,x) = g(x).$$

This is typical for backward PDEs (as in HJB equations). For infinite-horizon problems, there's no finite terminal time T, so this condition disappears.

Definition of Viscosity Solutions

Before defining viscosity solutions, we require F to behave "nicely" under perturbations, this ensures the notion of viscosity sub/supersolutions makes sense.

- (i) Ellipticity condition: for symmetric matrices $M, \hat{M}$:

$$M \leq \hat{M} \Rightarrow F(t,x,u,q,p,M) \geq F(t,x,u,q,p,\hat{M}), \quad (t,x) \in [0,T) \times \Omega.$$

  So ellipticity ensures F is nonincreasing in the second derivative argument.

- (ii) Parabolicity condition: for the time derivative variable q:

$$q \leq \hat{q} \Rightarrow F(t,x,u,q,p,M) \geq F(t,x,u,\hat{q},p,M), \quad (t,x) \in [0,T) \times \Omega.$$

A main motivation for viscosity: many HJB equations (or other nonlinear PDEs) have nonsmooth solutions — the value function $v(t,x)$ is typically not differentiable. So we can't plug v into the PDE in the classical sense (because Dv and D^2v don't exist everywhere).

Viscosity theory solves this by testing the PDE against smooth functions that touch v locally.

Definition of Viscosity Solutions

Definition. Assume both the ellipticity and parabolicity conditions are satisfied.

- A continuous function $u : \Omega \to \mathbb{R}$ is a viscosity subsolution of the above PDE if for any $C^1 \times C^2$ function ϕ that touches u from above and any local maximum point $(t, y) \in [0, T) \times \Omega$ of $u - \phi$ we have

$$F(t, y, u(t, y), D_t\phi(t, y), D_x\phi(t, y), D_x^2\phi(t, y)) \leq 0,$$

  and

$$u(T, x) \leq g(x).$$

- A continuous function $u : \Omega \to \mathbb{R}$ is a viscosity supersolution of the above PDE if for any $C^1 \times C^2$ function $\phi : [0, T) \times \Omega \to \mathbb{R}$ and any local minimum point $(t, y) \in [0, T) \times \Omega$ of $u - \phi$ we have

$$F(t, y, u(t, y), D_t\phi(t, y), D_x\phi(t, y), D_x^2\phi(t, y)) \geq 0,$$

  and

$$u(T, x) \geq g(x).$$

  ϕ is called a test function. If u is both a viscosity subsolution and a viscosity supersolution, then u is called a viscosity solution (necessarily with $u(T, x) = g(x)$).

Lemma 1

(i) A classical solution is a viscosity solution.

(ii) A $C^1 \times C^2$ viscosity solution is a classical solution.

Proof

(i) Suppose u is a classical solution, i.e., $C^1 \times C^2$ and satisfying the PDE. For any test function ϕ and any local maximum point $(t, y) \in [0, T) \times \Omega$ of $u - \phi$ we have

$$D_x u(t, y) = D_x \phi(t, y), \qquad D_x^2 u(t, y) \le D_x^2 \phi(t, y),$$

because $\Omega \subset \mathbb{R}^d$ is an open domain, and the first–order inequality holds,

$$D_t u(t, y) \le D_t \phi(t, y),$$

because the maximum point may be at the boundary $t = 0$. Thus,

$$\begin{aligned} & F\big(t, y, u(t, y), D_t\phi(t, y), D_x\phi(t, y), D_x^2\phi(t, y)\big) \\ & = F\big(t, y, u(t, y), D_t\phi(t, y), D_x u(t, y), D_x^2\phi(t, y)\big) \\ & \le F\big(t, y, u(t, y), D_t\phi(t, y), D_x u(t, y), D_x^2 u(t, y)\big) \\ & \le F\big(t, y, u(t, y), D_t u(t, y), D_x u(t, y), D_x^2 u(t, y)\big) \\ & = 0, \end{aligned}$$

where the first and second inequalities follow from the ellipticity and parabolicity

Definition of Viscosity Solutions

(ii) Suppose u is a viscosity solution and is $C^1 \times C^2$. Then we can take $\phi = u$. We have any point $(t, y) \in [0, T) \times \Omega$ is both a local maximum point and local minimum point of $u - \phi$. Thus,

$$\begin{aligned} &F\big(t, y, u(t,y), D_t u(t,y), D_x u(t,y), D_x^2 u(t,y)\big) \\ &\quad = F\big(t, y, u(t,y), D_t \phi(t,y), D_x \phi(t,y), D_x^2 \phi(t,y)\big) \\ &\quad = 0, \end{aligned}$$

where the last equality comes from the definitions of subsolution and supersolution. This shows that u is a classical solution.

Remark:
$1)$: For the infinite-horizon problem, the first term t and D_t is dropped from F, i.e., we have

$$F(x, u(x), D_x u(x), D_x^2 u(x)) = 0, \quad x \in \Omega,$$

the terminal condition disappears, and we do not need the parabolicity condition. For a finite-horizon deterministic control problem, the term D_x^2 is dropped from F, i.e., we have

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$$F(t, x, u(t,x), D_t u(t,x), D_x u(t,x)) = 0, \quad (t,x) \in [0,T) \times \Omega,$$

and we do not need the ellipticity condition. For an infinite-horizon deterministic control problem, we have

$$F(x, u(x), D_x u(x)) = 0, \quad x \in \Omega,$$

for which both ellipticity and parabolicity conditions are not needed.

$2)$: For any viscosity subsolution, we can always choose the new test function $\hat{\phi}$ touches u at one point, which is the local maximum point of $u - \hat{\phi}$, and $\hat{\phi}$ is above the subsolution u. Indeed, for any viscosity subsolution and a test function at any local maximum point $(t_0, y_0) \in [0,T) \times \Omega$ of $u - \phi$, we have

$$u(t,x) - \phi(t,x) \le u(t_0, y_0) - \phi(t_0, y_0), \quad \forall (t,x) \in N_{(t_0, y_0)},$$

where $N_{(t_0,y_0)}$ is a sufficiently small neighborhood of (t_0, y_0) within $[0,T) \times \Omega$. We can define a new test function

$$\hat{\phi}(t,x) = \phi(t,x) + u(t_0,y_0) - \phi(t_0,y_0).$$

Then

$$u(t_0,y_0) = \hat{\phi}(t_0,y_0),$$

$$u(t,x) - \hat{\phi}(t,x) = u(t,x) - \phi(t,x) - u(t_0,y_0) + \phi(t_0,y_0) \le 0, \quad \forall (t,x) \in N_{(t_0,y_0)}.$$

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Thus, the new test function $\hat{\phi}$ touches u at one point (t_0, y_0), which is the local maximum point of $u - \hat{\phi}$, and $\hat{\phi}$ is above the subsolution u. Similarly, for any supersolution u, there is a test function $\hat{\phi}$ that touches u at a local minimum point of $u - \hat{\phi}$, and $\hat{\phi}$ is below the supersolution u.

$3)$: The fact that u is a viscosity solution to the PDE $F = 0$ does not imply that u is a viscosity solution to the PDE $-F = 0$.