

MF921 Topics in Dynamic Asset Pricing

Week 7

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Chapter 22 American Options (II)

Numerical Methods for American Options

Consider the valuation of a finite-horizon American option with payoff $\varphi(S(\tau))$:

$$V(x, t) := \sup_{\tau \geq t} E^0 \left[e^{-r(\tau-t)} \varphi(S(\tau)) \mid S(t) = x \right],$$

where $S(t)$ = stock price, $\varphi(\cdot)$ = payoff function (e.g. $(K - S)^+$ for a put), τ = stopping time (the exercise time). So $V(x, t)$ is the value function represent the highest discounted expected payoff you can achieve by optimally choosing when to stop.

First, the optimal reward and moving boundary functions together solve the following free-boundary (to be determined) problem:

$$\begin{cases} \mathcal{A}V - rV = 0, & x > b(t), \quad 0 \leq t \leq T \\ V(b(t), t) = \varphi(b(t)), & \text{(value matching)} \\ V_x(b(t), t) = \varphi'(b(t)), & \text{(smooth pasting)} \\ V(x, T) = \varphi(x), & x > 0 \\ V(0, t) = \varphi(0), & V(\infty, t) = 0 \end{cases}$$

Recall

Second, the free boundary problem can be written as a partial differential complementarity problem (without the free boundary) as

$$\begin{aligned} \mathcal{A}V - rV &\leq 0, \quad x > 0, \quad t \in [0, T) \\ (\mathcal{A}V - rV)\{V(x, t) - \varphi(x)\} &= 0, \quad x > 0, \quad t \in [0, T) \\ V(x, t) &\geq \varphi(x), \quad x > 0, \quad t \in [0, T) \end{aligned}$$

along with the terminal and boundary conditions

$$\begin{aligned} V(x, T) &= \varphi(x), \quad x > 0, \\ \lim_{x \rightarrow \infty} V(x, t) &= 0, \quad V(0, t) = \varphi(0), \quad t \in [0, T) \end{aligned}$$

This is called the complementarity problem because the two inequalities cannot be strict inequalities simultaneously. The partial differential complementarity problem can be solved numerically by converting it to a matrix linear complementarity problem by using the finite difference method.

Recall

Third, the partial differential complementarity problem can be rewritten as a variational inequality problem:

$$\min(-\mathcal{A}V + rV, V(x, t) - \varphi(x)) = 0, \quad x > 0, \quad t \in [0, T) \quad (*)$$

along with the terminal and boundary conditions

$$V(x, T) = \varphi(x), \quad x > 0,$$

$$\lim_{x \rightarrow \infty} V(x, t) = 0, \quad V(0, t) = \varphi(0), \quad t \in [0, T).$$

Indeed, there are only two possibility by the partial differential complementarity problem: either $-\mathcal{A}V + rV > 0$ or $-\mathcal{A}V + rV = 0$. In the first case we must have $V(x, t) = \varphi(x)$, and in the second case $V(x, t) \geq \varphi(x)$. Thus, in both cases we have (*). Hence the partial differential complementarity problem implies the variational inequality problem. Conversely, (*) implies that $V(x, t) \geq \varphi(x)$ and $-\mathcal{A}V + rV \geq 0$. Furthermore, if $-\mathcal{A}V + rV > 0$ then $V(x, t) - \varphi(x) = 0$ by (*). Therefore, the variational inequality problem also implies the partial differential complementarity problem.

An efficient way to solve the partial differential complementarity problem is to transform it to a linear complementarity problem via the finite difference method.

Consider the following (matrix) linear complementarity problem: Find a vector $x \in \mathbb{R}^{1 \times n}$

$$Ax \geq b, x \geq c, (x - c)^T (Ax - b) = 0,$$

for given $A \in \mathbb{R}^{n \times n}$, $c, b \in \mathbb{R}^{1 \times n}$, where, for two column vectors x and y , $x \geq y$ means $x_i \geq y_i$ for each i . The linear complementarity problem has a unique solution for all column vectors b and c if and only if $A \in \mathbb{R}^{n \times n}$ is a P-matrix.

Note that if A is symmetric, then A is a P-matrix if and only if A is positive definite. Many matrices that arise in finite-difference and finite-element methods are diagonally dominant.

More precisely, the matrix A is diagonally dominant if

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|, \forall i,$$

where a_{ij} denotes the entry of A in the i th row and j th column. If the above inequality is strict, it is called strictly diagonally dominant. A well-known result is that a symmetric strictly diagonally dominant matrix with real positive diagonal entries is positive definite.

The linear complementarity problem can be solved by using many methods, including pivoting methods (e.g., Lemke's algorithm), quadratic programming, successive over relaxation (SOR), projected SOR, etc. There are also several Matlab and Python codes available online.

The linear complementarity problem is a particular case of the nonlinear complementarity problem, which is to find a vector $z \in \mathbb{R}^{1 \times n}$

$$f(x) \geq b, z \geq c, (z - c)^T (f(x) - b) = 0,$$

where f is a given function $\mathbb{R}^n \mapsto \mathbb{R}^n$, $c, b \in \mathbb{R}^{1 \times n}$. One way to solve the nonlinear complementarity problem is to use the linear approximation of $f(x)$ to get an iterative algorithm by solving a sequence of the complementarity problem.

The nonlinear complementarity problem is also related to a variational inequality problem: Given a non-empty set $K \in \mathbb{R}^n$, a function $g : \mathbb{R}^n \mapsto \mathbb{R}^n$, $b \in \mathbb{R}^{1 \times n}$, find a $z^* \in K$ such that

$$\min_{y \in K} (y - z^*)^T g(z^*) \geq 0.$$

It can be shown that with

$$K = \mathbb{R}_+^{1 \times n}, z^* = z - c, g(x) = f(x) - b,$$

the nonlinear complementarity problem and the variational inequality problem have the same solution.

Finite Difference Methods

The partial differential complementarity problem for $\psi(x, t)$ without the free boundary,

$$\frac{\partial \psi}{\partial t} + \frac{\sigma^2 x^2}{2} \frac{\partial^2 \psi}{\partial x^2} + rx \frac{\partial \psi}{\partial x} - r\psi \leq 0$$
$$\left(\frac{\partial \psi}{\partial t} + \frac{\sigma^2 x^2}{2} \frac{\partial^2 \psi}{\partial x^2} + rx \frac{\partial \psi}{\partial x} - r\psi \right) \{\psi(t, x) - g(x)\} = 0$$

$$\psi(x, t) \geq g(x)$$

$$\psi(x, T) = g(x)$$

$$\lim_{x \rightarrow \infty} \psi(x, t) = 0, \quad \psi(0, t) = g(0),$$

Where $g(x)$ is the payoff function. Although the algorithm also works for the case where $g(x)$ may be $g(x, t)$ depending on t , for notational simplicity we shall focus on the case of time-independent $g(x)$. First, we can transform the problem based on the standard heat equation by changing variables.

Finite Difference Methods

$$S = e^x, \quad \tau = (T - t) \frac{\sigma^2}{2}$$
$$u(x, \tau) = \exp \left\{ \frac{1}{2}(c - 1)x + \frac{1}{4}(c + 1)^2 \tau \right\} \cdot \psi(S, t), \quad c = \frac{2r}{\sigma^2},$$

the Black-Scholes partial differential complementarity problem becomes a standard heat partial differential complementarity problem, $x \in (-\infty, \infty)$:

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} \geq 0$$
$$\left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} \right) \{u(\tau, x) - f(x)\} = 0$$
$$u(x, \tau) \geq f(x, \tau), \quad u(x, 0) = f(x, 0)$$
$$\lim_{x \rightarrow \infty} u(x, \tau) = 0$$

where $f(x, \tau) = \exp \left\{ \frac{1}{2}(c - 1)x + \frac{1}{4}(c + 1)^2 \tau \right\} g(e^x)$.

This is the heat equation complementarity problem, with constant coefficients. We are interested in finding $u(x, \tau)$, where $x \in (-\infty, \infty)$ and $\tau \in [0, T\sigma^2/2]$.

Finite Difference Methods

We have transformed the complicated Black-Scholes complementarity PDE into heat-equation complementarity problem:

$$u_\tau - u_{xx} \geq 0, \quad (u_\tau - u_{xx})(u - f) = 0, \quad u \geq f.$$

We can't solve this analytically, so we approximate the derivatives (u_τ, u_{xx}) with finite differences on a discrete grid. By discretizing time and space on a regular mesh with step sizes $\delta\tau$ and δx , and truncating x in the region $[-N^-\delta x, N^+\delta x]$ for suitably large integers N^+ and N^- , we let

$$u_{n,m} = u(n\delta x, m\delta\tau), \quad f_{n,m} = f(n\delta x, m\delta\tau), \quad -N^- \leq n \leq N^+, \quad 0 \leq m \leq M,$$

where $M\delta\tau = T\sigma^2/2$. We use the finite difference approximation

$$\begin{aligned} \frac{\partial u}{\partial \tau} &= \frac{u_{n,m+1} - u_{n,m}}{\delta\tau} + o(\delta\tau), \\ \frac{\partial^2 u}{\partial x^2} &= \theta \left(\frac{u_{n+1,m+1} - 2u_{n,m+1} + u_{n-1,m+1}}{(\delta x)^2} \right) \\ &\quad + (1 - \theta) \left(\frac{u_{n+1,m} - 2u_{n,m} + u_{n-1,m}}{(\delta x)^2} \right) + o((\delta x)^2), \end{aligned}$$

where $0 < \theta < 1$. Note that this is a general way to do the finite difference method; the cases $\theta = 0, 1/2, 1$ yield the explicit, the Crank-Nicolson, and the implicit schemes, respectively.

The explicit scheme does not involve solving linear equations and is known to be numerically stable and convergent whenever

$$\alpha := \delta\tau/(\delta x)^2 \leq 1/2.$$