# MF921 Topics in Dynamic Asset Pricing

Week 10

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Chapter 14 Viscosity Solutions and HJB Equations

# Definition of Viscosity Solutions

We start with an open domain

$$\Omega \subset \mathbb{R}^d,$$

and a function $u(t,x)$ satisfying a nonlinear second-order PDE

$$F(t,x,u(t,x),D_t u(t,x),D_x u(t,x),D_x^2 u(t,x)) = 0, \quad (t,x) \in [0,T) \times \Omega.$$

Where :

- $D_t u$: time derivative $\partial u/\partial t$
- $D_x u = \nabla_x u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_d}\right)^T$
- $D_x^2 u$: the Hessian matrix, with entries $(D_x^2 u)_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$

with the terminal condition

$$u(T,x) = g(x).$$

This is typical for backward PDEs (as in HJB equations). For infinite-horizon problems, there's no finite terminal time $T$, so this condition disappears.

# Definition of Viscosity Solutions

Before defining viscosity solutions, we require $F$ to behave "nicely" under perturbations, this ensures the notion of viscosity sub/supersolutions makes sense.

- (i) Ellipticity condition: for symmetric matrices $M, \hat{M}$:

$$M \leq \hat{M} \Rightarrow F(t,x,u,q,p,M) \geq F(t,x,u,q,p,\hat{M}), \quad (t,x) \in [0,T) \times \Omega.$$

  So ellipticity ensures $F$ is nonincreasing in the second derivative argument.

- (ii) Parabolicity condition: for the time derivative variable $q$:

$$q \leq \hat{q} \Rightarrow F(t,x,u,q,p,M) \geq F(t,x,u,\hat{q},p,M), \quad (t,x) \in [0,T) \times \Omega.$$

A main motivation for viscosity: many HJB equations (or other nonlinear PDEs) have nonsmooth solutions — the value function $v(t,x)$ is typically not differentiable. So we can't plug $v$ into the PDE in the classical sense (because $Dv$ and $D^2v$ don't exist everywhere).

Viscosity theory solves this by testing the PDE against smooth functions that touch $v$ locally.

# Definition of Viscosity Solutions

Definition. Assume both the ellipticity and parabolicity conditions are satisfied.

- A continuous function $u : \Omega \to \mathbb{R}$ is a viscosity subsolution of the above PDE if for any $C^1 \times C^2$ function $\phi$ that touches $u$ from above and any local maximum point $(t, y) \in [0, T) \times \Omega$ of $u - \phi$ we have

$$F(t, y, u(t, y), D_t\phi(t, y), D_x\phi(t, y), D_x^2\phi(t, y)) \leq 0,$$

  and

$$u(T, x) \leq g(x).$$

- A continuous function $u : \Omega \to \mathbb{R}$ is a viscosity supersolution of the above PDE if for any $C^1 \times C^2$ function $\phi : [0, T) \times \Omega \to \mathbb{R}$ and any local minimum point $(t, y) \in [0, T) \times \Omega$ of $u - \phi$ we have

$$F(t, y, u(t, y), D_t\phi(t, y), D_x\phi(t, y), D_x^2\phi(t, y)) \geq 0,$$

  and

$$u(T, x) \geq g(x).$$

$\phi$ is called a test function. If $u$ is both a viscosity subsolution and a viscosity supersolution, then $u$ is called a viscosity solution (necessarily with $u(T, x) = g(x)$).

# Definition of Viscosity Solutions

Lemma 1

(i) A classical solution is a viscosity solution.

(ii) A $C^1 \times C^2$ viscosity solution is a classical solution.

**Proof**

$(i)$ Suppose $u$ is a classical solution, i.e., $C^1 \times C^2$ and satisfying the PDE. For any test function $\phi$ and any local maximum point $(t, y) \in [0, T) \times \Omega$ of $u - \phi$ we have

$$D_x u(t,y) = D_x \phi(t,y), \qquad D_x^2 u(t,y) \le D_x^2 \phi(t,y),$$

because $\Omega \subset \mathbb{R}^d$ is an open domain, and the first–order inequality holds,

$$D_t u(t,y) \le D_t \phi(t,y),$$

because the maximum point may be at the boundary $t = 0$. Thus,

$$\begin{aligned}
&F\big(t, y, u(t,y), D_t\phi(t,y), D_x\phi(t,y), D_x^2\phi(t,y)\big) \\
&\quad = F\big(t, y, u(t,y), D_t\phi(t,y), D_x u(t,y), D_x^2\phi(t,y)\big) \\
&\quad \le F\big(t, y, u(t,y), D_t\phi(t,y), D_x u(t,y), D_x^2 u(t,y)\big) \\
&\quad \le F\big(t, y, u(t,y), D_t u(t,y), D_x u(t,y), D_x^2 u(t,y)\big) \\
&\quad = 0,
\end{aligned}$$

where the first and second inequalities follow from the ellipticity and parabolicity

# Definition of Viscosity Solutions

(ii) Suppose $u$ is a viscosity solution and is $C^1 \times C^2$. Then we can take $\phi = u$. We have any point $(t, y) \in [0, T) \times \Omega$ is both a local maximum point and local minimum point of $u - \phi$. Thus,

$$\begin{aligned} & F\big(t, y, u(t, y), D_t u(t, y), D_x u(t, y), D_x^2 u(t, y)\big) \\ & \quad = F\big(t, y, u(t, y), D_t \phi(t, y), D_x \phi(t, y), D_x^2 \phi(t, y)\big) \\ & \quad = 0, \end{aligned}$$

where the last equality comes from the definitions of subsolution and supersolution. This shows that $u$ is a classical solution.

**Remark:**

$1)$: For the infinite-horizon problem, the first term $t$ and $D_t$ is dropped from $F$, i.e., we have

$$F(x, u(x), D_x u(x), D_x^2 u(x)) = 0, \quad x \in \Omega,$$

the terminal condition disappears, and we do not need the parabolicity condition. For a finite-horizon deterministic control problem, the term $D_x^2$ is dropped from $F$, i.e., we have

$$F(t,x,u(t,x),D_t u(t,x),D_x u(t,x))=0,\quad (t,x)\in[0,T)\times\Omega,$$

and we do not need the ellipticity condition. For an infinite-horizon deterministic control problem, we have

$$F(x,u(x),D_x u(x))=0,\quad x\in\Omega,$$

for which both ellipticity and parabolicity conditions are not needed.

$2)$: For any viscosity subsolution, we can always choose the new test function $\hat{\phi}$ touches $u$ at one point, which is the local maximum point of $u-\hat{\phi}$, and $\hat{\phi}$ is above the subsolution $u$. Indeed, for any viscosity subsolution and a test function at any local maximum point $(t_0,y_0)\in[0,T)\times\Omega$ of $u-\phi$, we have

$$u(t,x)-\phi(t,x)\le u(t_0,y_0)-\phi(t_0,y_0),\quad \forall(t,x)\in N_{(t_0,y_0)},$$

where $N_{(t_0,y_0)}$ is a sufficiently small neighborhood of $(t_0,y_0)$ within $[0,T)\times\Omega$. We can define a new test function

$$\hat{\phi}(t,x)=\phi(t,x)+u(t_0,y_0)-\phi(t_0,y_0).$$

Then

$$u(t_0,y_0)=\hat{\phi}(t_0,y_0),$$
$$u(t,x)-\hat{\phi}(t,x)=u(t,x)-\phi(t,x)-u(t_0,y_0)+\phi(t_0,y_0)\le 0,\quad \forall(t,x)\in N_{(t_0,y_0)}.$$

# Definition of Viscosity Solutions

Thus, the new test function $\hat{\phi}$ touches $u$ at one point $(t_0, y_0)$, which is the local maximum point of $u - \hat{\phi}$, and $\hat{\phi}$ is above the subsolution $u$. Similarly, for any supersolution $u$, there is a test function $\hat{\phi}$ that touches $u$ at a local minimum point of $u - \hat{\phi}$, and $\hat{\phi}$ is below the supersolution $u$.

$3)$: The fact that $u$ is a viscosity solution to the PDE $F = 0$ does not imply that $u$ is a viscosity solution to the PDE $-F = 0$.

# Connection with the HJB Equation: An Overview

For a finite-horizon stochastic control for diffusion process,

$$v(t,x) = \sup_{u_t \in U} E\left[\int_0^T e^{-\beta s} c(X_s, u_s) ds + e^{-\beta T} g(X_T) \middle| X_t = x\right],$$

$$dX_t = b(X_t, u_t)dt + \sigma(X_t, u_t)dW_t, \quad t \geq 0,$$

where $b : \mathbb{R}^d \times U \to \mathbb{R}^d$ is the drift and $\sigma(X_t, u_t)$ is the volatility matrix. The HJB equation is given by

$$-\frac{\partial v(t,x)}{\partial t} + \beta v(t,x) - \sup_{u \in U}[A_u v(t,x) + c(x,u)] = 0,$$

$$v(T,x) = g(x),$$

where

$$\begin{aligned} A_u v(t,x) &= \sum_{i=1}^d b_i(x,u) \frac{\partial v}{\partial x_i} + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d a_{ij}(x,u) \frac{\partial^2 v}{\partial x_i \partial x_j} \\ &= b^\top(x,u) \cdot Dv + \frac{1}{2} \mathsf{Su}\left(a(x,u) \circ D^2 v\right), \end{aligned}$$

# Connection with the HJB Equation: An Overview

and the matrix $a(x,u) = \sigma(x,u)\sigma^\top(x,u)$ is assumed to be positive semi-definite, $\circ$ denotes the Hadamard product, and $\mathsf{Su}(A)$ denotes the sum of all the elements in the matrix $A$. In this case the finite-horizon HJB equation can be written as

$$F(t,x,r,q,p,M) = -q + \beta r - \sup_{u\in U}\left[b^\top(x,u)\cdot p + \frac{1}{2}\mathsf{Su}\left(a(x,u)\circ M\right) + c(x,u)\right].$$

The parabolicity condition is automatically satisfied thanks to the $-q$ term above. Next, we shall prove that the ellipticity condition is satisfied.

By the Schur product theorem, the Hadamard product of two positive semidefinite matrices is positive semidefinite. Thus, $a(x,u)\circ(\hat{M}-M)$ is positive semidefinite, if $\hat{M}-M$ is positive semidefinite. Therefore,

$$\mathsf{Su}(a(x,u)\circ(\hat{M}-M)) = 1^T\left(a(x,u)\circ(\hat{M}-M)\right)1 \geq 0,$$

where $1^T = (1,...,1)$ is a $d$-dimensional row vector, yielding

$$b^T(x,u)\cdot p + \frac{1}{2}\mathsf{Su}\left(a(x,u)\circ M\right) + c(x,u) \leq b^T(x,u)\cdot p + \frac{1}{2}\mathsf{Su}\left(a(x,u)\circ\hat{M}\right) + c(x,u).$$

# Connection with the HJB Equation: An Overview

Taking sup gives

$$b^T(x,u)\cdot p+\frac{1}{2}\mathsf{Su}\left(a(x,u)\circ M\right)+c(x,u)\le \sup_{u\in U}\left[b^T(x,u)\cdot p+\frac{1}{2}\mathsf{Su}\left(a(x,u)\circ \hat{M}\right)+c(x,u)\right]$$

Repeating the sup leads to

$$\sup_{u\in U}\left[b^T(x,u)\cdot p+\frac{1}{2}\mathsf{Su}\left(a(x,u)\circ M\right)+c(x,u)\right]\le \sup_{u\in U}\left[b^T(x,u)\cdot p+\frac{1}{2}\mathsf{Su}\left(a(x,u)\circ \hat{M}\right)+c(x,u)\right].$$

Hence

$$F(t,x,r,q,p,M)\ge F(t,x,r,q,p,\hat{M}),$$

and the ellipticity condition holds, so that we can define the viscosity solution for the finite-horizon HJB equation.

he viscosity solution is connected to the HJB equation. With suitable regularity conditions, it can be shown that the value function of stochastic control problem is the unique bounded viscosity solution of the finite-horizon HJB equation.

# Infinite-Horizon Deterministic Control and the First-Order HJB equation

Consider the following deterministic control problem

$$v(x) = \sup_{u_t \in U} \int_0^\infty e^{-\beta t} c(X_t, u_t) dt,$$

with state dynamics

$$dX_t = f(X_t, u_t)dt, \quad t \geq 0, \quad X(0) = x.$$

The HJB equation is given by

$$\beta v(x) - \sup_{u \in U} [Dv(x) \cdot f(x,u) + c(x,u)] = 0.$$

Regularity Conditions:

- The set of controls $U$ is compact.
- The state dynamic function $f : \mathbb{R}^d \times U \to \mathbb{R}^d$ is continuous. Furthermore, $f$ is Lipschitz continuous (a strong form of uniform continuous) in the first variable, i.e., there exists a constant $L > 0$ such that

$$\|f(x,u) - f(y,u)\| \leq L\|x - y\|, \quad x, y \in \mathbb{R}^d, \quad u \in U.$$

# Infinite-Horizon Deterministic Control and the First-Order HJB equation

- The cost function $c : \mathbb{R}^d \times U \to \mathbb{R}$ is continuous and uniformly bounded. Furthermore, $c$ is Lipschitz continuous in the first variable, there exists a constant $L > 0$ such that

$$|c(x,u) - c(y,u)| \leq L\|x - y\|, \quad x, y \in \mathbb{R}^d, \quad u \in U.$$

We assume the above regularity conditions hold for all the results below.
Note that the HJB equation is a special case of the nonlinear first order PDE

$$F(x, u(x), Du(x)) = 0, \quad x \in \Omega. \tag{14.6}$$

Since the elliptical and parabolic conditions are automatically satisfied for the first order PDE (14.6), we can define viscosity solutions to (14.6).

Theorem 1. The value function $v$ is the unique uniformly bounded viscosity solution to the HJB equation.
Note that unbounded viscosity solutions may exist, and they are not related to the value function of the control problem.

# Continuity and the Dynamic Programming Principle

First recall the Gronwall inequality: Let $A(t)$, $\Psi(t)$ and $\chi(t)$ be real continuous functions defined on $t \in [a, b]$ with $A(t) \geq 0$. If

$$A(t) \leq \Psi(t) + \int_a^t A(s)\chi(s)ds, \quad t \in [a, b],$$

then

$$A(t) \leq \Psi(t) + \int_a^t \chi(s)\Psi(s) \exp\left[\int_s^t \chi(u)du\right] ds, \quad t \in [a, b].$$

Lemma 2: Suppose $X_t^x$ and $X_t^y$ are the state processes with the same control policy $u_t$ but different starting values $x$ and $y$, i.e., $X_t^x(u_t)$ has the dynamic

$$dX_t = f(X_t, u_t)dt, \quad X_0 = x.$$

Then we have

$$\|X_t^x - X_t^y\| \leq e^{Lt}\|x - y\|, \quad 0 \leq t \leq T.$$

# Continuity and the Dynamic Programming Principle

[Proof]
We have

$$X_t^x = x + \int_0^t f(X_s^x, u_s)ds, \quad X_t^y = y + \int_0^t f(X_s^y, u_s)ds.$$

Thus,

$$X_t^x - X_t^y = x - y + \int_0^t f(X_s^x, u_s) - f(X_s^y, u_s)ds,$$

from which we get

$$\|X_t^x - X_t^y\| \le \|x - y\| + \int_0^t \|f(X_s^x, u_s) - f(X_s^y, u_s)\|ds$$

$$\le \|x - y\| + L\int_0^t \|X_s^x - X_s^y\|ds.$$

# Continuity and the Dynamic Programming Principle

Then the Gronwall inequality gives

$$\begin{aligned}
\|X_t^x - X_t^y\| &\leq \|x-y\| + \int_0^t L\|x-y\|e^{L(t-s)}ds \\
&= \|x-y\| + L\|x-y\|e^{Lt}\left(\frac{1}{L}(1-e^{-Lt})\right) \\
&= \|x-y\| + \|x-y\|\left(e^{Lt}-1\right) \\
&= \|x-y\|e^{Lt},
\end{aligned}$$

from which the conclusion follows.

Proposition 1 : The value function $v$ is uniformly bounded and uniformly continuous.

[Proof]
The uniformly boundedness follows from the fact that $|c(x,u)| \leq C$ and $\int_0^\infty e^{-\beta t} C dt < \infty$.
Fix an arbitrary $\varepsilon > 0$. Consider $x, y \in \mathbb{R}^d$. There exists a control policy $u_t^\varepsilon$ such that

$$v(y) \leq \int_0^\infty e^{-\beta t} c(X_t^y, u_t^\varepsilon) dt + \varepsilon,$$

# Continuity and the Dynamic Programming Principle

$X_t^y$ means that $X_t^y$ starts from $y$ (i.e., $X_0^y = y$), thanks to the definition of sup. Since $u_t^\varepsilon$ is one of the control policies, we also have

$$v(x) \geq \int_0^\infty e^{-\beta t} c(X_t^x, u_t^\varepsilon) dt.$$

Thus,

$$v(y) - v(x) \leq \int_0^\infty e^{-\beta t} [c(X_t^y, u_t^\varepsilon) - c(X_t^x, u_t^\varepsilon)] \, dt + \varepsilon.$$

Since $c$ is uniformly bounded, we can find $T > 0$, independent of $x$ and $y$, such that

$$\int_T^\infty e^{-\beta t} [c(X_t^y, u_t^\varepsilon) - c(X_t^x, u_t^\varepsilon)] \, dt < \varepsilon.$$

Therefore,

$$\begin{aligned} v(y) - v(x) &\leq \int_0^T e^{-\beta t} [c(X_t^y, u_t^\varepsilon) - c(X_t^x, u_t^\varepsilon)] \, dt + 2\varepsilon \\ &\leq L \int_0^T e^{-\beta t} \| X_t^y - X_t^x \| \, dt + 2\varepsilon, \end{aligned}$$

via the Lipschitz condition.

Switching $x$ and $y$ in the above argument leads to

$$v(x) - v(y) \leq L \int_0^T e^{-\beta t} \|X_t^y - X_t^x\| \, dt + 2\varepsilon.$$

In summary,

$$|v(x) - v(y)| \leq L \int_0^T e^{-\beta t} \|X_t^y - X_t^x\| \, dt + 2\varepsilon.$$

Using Lemma 2, we have

$$|v(x) - v(y)| \leq L \|y - x\| \int_0^T e^{(L-\beta)t} \, dt + 2\varepsilon.$$

Thus, there exists a $\delta$ (independent of $x$ and $y$) such that

$$|v(x) - v(y)| \leq 3\varepsilon,$$

as long as $\|y - x\| \leq \delta$. Thus, $v$ is uniformly continuous.

# Continuity and the Dynamic Programming Principle

Proposition 2: (Dynamic Programming Principle) For every $x \in \mathbb{R}^d$ and $t \geq 0$,

$$v(x) = \sup_{u \in \mathcal{U}} \left[ \int_0^t e^{-\beta s} c(X_s, u_s) ds + e^{-\beta t} v(X_t) \right].$$

# The Verification Result

Lemma 3: For every $x \in \mathbb{R}^d$ and every open neighborhood of $x$, denoted by $N_x$, there exists $t_x > 0$ small enough, such that for any control policy $u$, the state dynamics given by

$$dX_t = f(X_t, u_t)dt, \quad X_0 = x,$$

satisfies

$$X_t \in N_x, \quad \forall 0 \le t \le t_x.$$

Intuitively, $f(x, u)$ is bounded and Lipschitz continuous, your system can't move too far away from its starting point $x$ in an infinitesimally short time.

[Proof]
Since $f$ is continuous and $U$ is compact, we have for every $x \in \mathbb{R}^d$,

$$M_x = \sup_{u \in U} \|f(x, u)\| < \infty.$$

Because

$$X_t^x = x + \int_0^t f(X_s^x, u_s)ds = x + \int_0^t f(x, u_s)ds + \int_0^t [f(X_s^x, u_s) - f(x, u_s)]ds,$$

we have, via the Lipschitz continuity,

$$\|X_t^x - x\| \le M_x t + L \int_0^t \|X_s^x - x\|ds.$$

# The Verification Result

The Gronwall inequality yields that

$$\begin{aligned}
\|X_t^x - x\| &\le M_x t + \int_0^t L M_x s \exp[L(t-s)]ds \\
&\le M_x t + L M_x t \int_0^t \exp[L(t-s)]ds \\
&\le M_x t + M_x t(e^{Lt} - 1) \\
&= M_x t e^{Lt},
\end{aligned}$$

from which the conclusion follows.

Proposition 3: The value function $v$ is a viscosity supersolution to the HJB equation

$$\beta v(x) - \sup_{u \in U}[Dv(x) \cdot f(x,u) + c(x,u)] = 0.$$

[Proof]
Consider a $C^1$ test function $\phi$, and let $y$ be local minimum of $v - \phi$. There exists a neighborhood of $y$, say $N_y$, such that

$$v(x) - \phi(x) \ge v(y) - \phi(y), \quad x \in N_y.$$

# The Verification Result

Thus,

$$v(x) - v(y) \geq \phi(x) - \phi(y), \quad x \in N_y.$$

By Lemma 3, for any control policy $u_t \in U$ there exists $t_y$ small enough such that

$$X_0 = y, \quad X_t \in N_y, \quad \forall 0 \leq t \leq t_y.$$

Thus,

$$v(X_t^y) - v(y) \geq \phi(X_t^y) - \phi(y), \quad \forall 0 \leq t \leq t_y.$$

For any control policy $u_t \in U$ and $u_0 = u$, the dynamic programming principle from Proposition 2 yields

$$v(y) \geq \int_0^t e^{-\beta s} c(X_s^y, u_s) ds + e^{-\beta t} v(X_t^y).$$

Therefore, adding the two inequalities leads to, $0 \leq t \leq t_y$,

$$v(X_t^y) \geq \int_0^t e^{-\beta s} c(X_s^y, u_s) ds + e^{-\beta t} v(X_t^y) + \phi(X_t^y) - \phi(y).$$

# The Verification Result

Hence for $0 \leq t \leq t_y$,

$$v(X_t^y)\left(\frac{1-e^{-\beta t}}{t}\right) - \frac{1}{t}\int_0^t e^{-\beta s} c(X_s^y, u_s)ds \geq \frac{\phi(X_t^y) - \phi(y)}{t}. \tag{14.7}$$

Letting $t \to 0$,

- From Proposition 1, we know $v$ is uniformly continuous. As $t \to 0$, the trajectory $X_t^y$ starting at $y$ satisfies $X_t^y \to y$ (because $f$ is continuous). Hence

$$v(X_t^y) \longrightarrow v(y).$$

- $$\frac{1-e^{-\beta t}}{t} \to \beta$$

  This is the Taylor expansion of $e^{-\beta t}$:

$$e^{-\beta t} = 1 - \beta t + o(t) \Longrightarrow \frac{1-e^{-\beta t}}{t} \to \beta.$$

# The Verification Result

- $$\frac{1}{t}\int_0^t e^{-\beta s} c(X_s^y, u_s)\, ds \to c(y, u_0)$$

  For small $t$, the control $u_s$ is nearly constant at its initial value $u_0$, and $X_s^y \approx y + f(y, u_0)s$, so $X_s^y \to y$. By continuity of $c$,

  $$e^{-\beta s} c(X_s^y, u_s) \to c(y, u_0).$$

  Then by the mean value theorem for integrals, the average converges.

- $$\frac{\phi(X_t^y) - \phi(y)}{t} \to D\phi(y) \cdot f(y, u_0)$$

  By first-order Taylor expansion:

  $$\phi(X_t^y) = \phi(y) + D\phi(y) \cdot (X_t^y - y) + o(\|X_t^y - y\|).$$

  But

  $$\frac{X_t^y - y}{t} = \frac{1}{t}\int_0^t f(X_s^y, u_s)\, ds \to f(y, u_0).$$

  So indeed:

  $$\frac{\phi(X_t^y) - \phi(y)}{t} \to D\phi(y) \cdot f(y, u_0).$$

# The Verification Result

Take limits on both sides of (14.7) as $t \to 0$:
Left-hand side:

$$v(X_t^y)\left(\frac{1-e^{-\beta t}}{t}\right) - \frac{1}{t}\int_0^t e^{-\beta s} c(X_s^y, u_s)\, ds \longrightarrow \beta v(y) - c(y, u_0).$$

Right-hand side:

$$\frac{\phi(X_t^y) - \phi(y)}{t} \longrightarrow D\phi(y)\cdot f(y, u_0).$$

Now combine them and rearrange:

$$\beta v(y) \geq D\phi(y)\cdot f(y, u_0) + c(y, u_0).$$

Since $u_0$ is arbitrary, we get

$$\beta v(y) \geq \max_{u\in U}\{D\phi(y)\cdot f(y, u) + c(y, u)\}.$$

In other words

$$\beta v(y) - \sup_{u\in U}\{D\phi(y)\cdot f(y, u) + c(y, u)\} \geq 0,$$

which shows that $v$ is a supersolution of the HJB equation.

# The Verification Result

Proposition 4: The value function $v$ is a viscosity subsolution to the HJB equation

$$\beta v(x) - \sup_{u \in U}[Dv(x) \cdot f(x,u) + c(x,u)] = 0.$$

[Proof]
Consider a $C^1$ test function $\phi$, and let $y$ be local maximum of $v - \phi$. There exists a neighborhood of $y$, say $N_y$, such that

$$v(x) - \phi(x) \leq v(y) - \phi(y), \quad x \in N_y.$$

Thus,

$$v(x) - v(y) \leq \phi(x) - \phi(y), \quad x \in N_y.$$

By Lemma 3, for any control policy $u_t \in U$ there exists $t_y$ small enough such that

$$X_0 = y, \quad X_t \in N_y, \quad \forall 0 \leq t \leq t_y.$$

Thus,

$$v(X_t^y) - v(y) \leq \phi(X_t^y) - \phi(y), \quad \forall 0 \leq t \leq t_y.$$

# The Verification Result

For any small $\varepsilon > 0$, the DPP ensures existence of an $\varepsilon$-optimal control policy $u_s^{\varepsilon,t}$ such that

$$v(y) \leq \int_0^t e^{-\beta s} c(X_s^y, u_s^{\varepsilon,t})\, ds + e^{-\beta t} v(X_t^y) + \varepsilon t.$$

This is the "reverse" inequality of the supersolution case, since $v(y)$ is a supremum and this control nearly attains that supremum.

Therefore, adding the two inequalities leads to, $0 \leq t \leq t_y$,

$$v(X_t^y) \leq \int_0^t e^{-\beta s} c(X_s^y, u_s^{\varepsilon,t})\, ds + e^{-\beta t} v(X_t^y) + \phi(X_t^y) - \phi(y) + \varepsilon t.$$

Rearranging:

$$v(X_t^y)(1 - e^{-\beta t}) - \int_0^t e^{-\beta s} c(X_s^y, u_s^{\varepsilon,t})\, ds \leq \phi(X_t^y) - \phi(y) + \varepsilon t. \tag{14.8}$$

The complication here is that the control policy $u_s^{\varepsilon,t}$ depends on $t$. Thus, we have to be careful when taking the limit $t \to 0$ in (14.8). For $0 \leq t \leq t_y$, we have by the Lipschitz condition of $f$:

# The Verification Result

$$\|f(X_s^y, u_s^{\varepsilon,t}) - f(y, u_s^{\varepsilon,t})\| \leq L\|X_s^y - y\|,$$

and

$$\phi(X_t^y) - \phi(y) = \int_0^t D\phi(X_s^y)\cdot f(X_s^y, u_s^{\varepsilon,t})\,ds = \int_0^t D\phi(y)\cdot f(y, u_s^{\varepsilon,t})\,ds + o(t), \quad (14.9)$$

because

$$\begin{aligned}
&\frac{1}{t}\left|\int_0^t \{D\phi(X_s^y)\cdot f(X_s^y, u_s^{\varepsilon,t}) - D\phi(y)\cdot f(y, u_s^{\varepsilon,t})\}\,ds\right| \\
&= \frac{1}{t}\left|\int_0^t \{D\phi(X_s^y)\cdot (f(X_s^y, u_s^{\varepsilon,t}) - f(y, u_s^{\varepsilon,t})) + (D\phi(X_s^y) - D\phi(y))\cdot f(y, u_s^{\varepsilon,t})\}\,ds\right| \\
&\leq \frac{1}{t}\int_0^t |D\phi(X_s^y)\cdot (f(X_s^y, u_s^{\varepsilon,t}) - f(y, u_s^{\varepsilon,t}))|\,ds + \frac{1}{t}\int_0^t |(D\phi(X_s^y) - D\phi(y))\cdot f(y, u_s^{\varepsilon,t})|\,ds \\
&\leq \frac{1}{t}\int_0^t \|D\phi(X_s^y)\|\cdot\|f(X_s^y, u_s^{\varepsilon,t}) - f(y, u_s^{\varepsilon,t})\|\,ds + \frac{1}{t}\int_0^t |(D\phi(X_s^y) - D\phi(y))\cdot f(y, u_s^{\varepsilon,t})|\,ds \\
&\leq \frac{L}{t}\int_0^t \|D\phi(X_s^y)\|\cdot\|X_s^y - y\|\,ds + M_y\frac{1}{t}\int_0^t \|D\phi(X_s^y) - D\phi(y)\|\,ds
\end{aligned}$$

as $t \to 0$, where $M_y = \sup_{u \in U} \|f(y,u)\|$, because $D\phi(X_s^y) \to D\phi(y)$ and $X_s^y \to y$ as $s \to 0$.

Thus, (14.8) and (14.9) yield

$$v(X_t^y)(1-e^{-\beta t}) - \int_0^t e^{-\beta s} c(X_s^y, u_s^{\varepsilon,t})\, ds - \int_0^t D\phi(y) \cdot f(y, u_s^{\varepsilon,t})\, ds \le o(t) + t\varepsilon. \quad (14.10)$$

Similarly, the Lipschitz condition of $c$ gives

$$\int_0^t e^{-\beta s} c(X_s^y, u_s^{\varepsilon,t})\, ds = \int_0^t c(y, u_s^{\varepsilon,t})\, ds + o(t). \quad (14.11)$$

and the uniform continuity of $v$ leads to

$$v(X_t^y)(1 - e^{-\beta t}) = \beta t v(y) + o(t) = \int_0^t \beta v(y)\, ds + o(t). \quad (14.12)$$

Combining (14.10)-(14.12) yields

$$\int_0^t \left[\beta v(y) - D\phi(y) \cdot f(y, u_s^{\varepsilon,t}) - c(y, u_s^{\varepsilon,t})\right] ds \le t\varepsilon + o(t).$$

# The Verification Result

Thus, the inf of the integrand inside must satisfy

$$\inf_{u \in U} [\beta v(y) - D\phi(y) \cdot f(y,u) - c(y,u)] \leq \varepsilon.$$

In other words,

$$\beta v(y) - \sup_{u \in U} \{D\phi(y) \cdot f(y,u) + c(y,u)\} \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have that $v$ is a subsolution of the HJB equation.

# Uniqueness via the Comparison Principle

Theorem 2: (The Comparison Principle). Let $u_1$ be any viscosity subsolution to the HJB equation and $u_2$ be any viscosity supersolution to the HJB equation. If $u_1$ and $u_2$ are both uniformly bounded, then $u_1 \leq u_2$.

$$\beta v(x) - \sup_{u \in U}[Dv(x) \cdot f(x,u) + c(x,u)] = 0.$$

Theorem 2 is the most difficult part of the proof of Theorem 1. The proof of Theorem 2 relies on a doubling variable technique from Kruzkov (1970), which introduces a new function $\Psi$ of two variables in $u_1(x)$ and $u_2(y)$. The function $\Psi$ allows us to treat $u_2$ (resp. $u_1$) as a constant with respect to the test function of $u_1$ (resp. $u_2$).

We shall prove Theorem 2 by contradiction. Suppose there exists $\delta > 0$ and $z \in \mathbb{R}^d$ such that $u_1(z) - u_2(z) \geq 2\delta$. We will try to derive a contradiction from this assumption using the "doubling variable" trick.

Consider a (doubling variable) function $\Psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ defined as

$$\Psi(x, y; \varepsilon, \alpha, p) = -u_1(x) + u_2(y) + \frac{\|x-y\|^2}{2\varepsilon} + \alpha\left[\left(1 + \|x\|^2\right)^p + \left(1 + \|y\|^2\right)^p\right],$$

where the constants $\varepsilon, \alpha, p > 0$. The key idea of the proof is to show a contradiction by using the fact that, when $\varepsilon \to 0$ and $\alpha \to 0$, the minimum of $\Psi(x, y; \varepsilon, \alpha, p)$ should be achieved when $x = y$ with the value $u_2(x) - u_1(x)$. The term $\left(1 + \|x\|^2\right)^p + \left(1 + \|y\|^2\right)^p$ is added to ensure that $\Psi$ has a minimum point.

Choose $\alpha$ small enough so that $2\alpha \left(1 + \|z\|^2\right)^p \leq \delta$

$$\Psi(z, z; \varepsilon, \alpha, p) = -u_1(z) + u_2(z) + 2\alpha \left(1 + \|z\|^2\right)^p \leq -2\delta + \delta = -\delta.$$

We choose $p > 0$ so small that

$$p\,(2L + M) \leq \beta,$$

where $M = \sup_{u \in U} \|f(0, u)\|$. Note that $\alpha$ and $p$ do not depend on $\varepsilon$.

The definition of $\Psi$ implies that

$$\lim_{\|x\| \to \infty} \Psi(x, y; \varepsilon, \alpha, p) \to \infty, \quad \lim_{\|y\| \to \infty} \Psi(x, y; \varepsilon, \alpha, p) \to \infty,$$

# Uniqueness via the Comparison Principle

Because $u_1$ and $u_2$ are bounded by the assumption in the theorem. Therefore, there must exist a pair $(\bar{x}, \bar{y}) \in K_\varepsilon$ that attain the minimum of the continuous function $\Psi$, where $K_\varepsilon$ is a compact set. In particular,

$$\Psi(\bar{x}, \bar{y}; \varepsilon, \alpha, p) = \inf_{x,y} \Psi(x, y; \varepsilon, \alpha, p) \leq \Psi(z, z; \varepsilon, \alpha, p) \leq -\delta.$$

Lemma 4: For any $\varepsilon > 0$, we have

$$\beta\Psi(\bar{x}, \bar{y}; \varepsilon, \alpha, p) \geq -\frac{L}{\varepsilon}\|\bar{x} - \bar{y}\|^2 - L\|\bar{x} - \bar{y}\| - \varepsilon. \tag{14.15}$$

[Proof]
Introduce the notation

$$H(x, p) = \sup_{u \in U}[f(x, u) \cdot p + c(x, u)].$$

Then the HJB equation can be written as

$$\beta v(x) - H(x, Dv(x)) = 0.$$

(i) Consider a test function

$$\phi(x) = u_2(\bar{y}) + \frac{\|x - \bar{y}\|^2}{2\varepsilon} + \alpha\left[\left(1 + \|x\|^2\right)^p + \left(1 + \|\bar{y}\|^2\right)^p\right].$$

Note that

$$\Psi(x, \bar{y}; \varepsilon, \alpha, p) = -u_1(x) + \phi(x).$$

Thus, $-u_1(x) + \phi(x)$ attains its minimum at $\bar{x}$. Hence, $u_1(x) - \phi(x)$ attains its maximum at $\bar{x}$. By the definition of the subsolution,

$$\beta u_1(\bar{x}) - H(\bar{x}, D\phi(\bar{x})) \leq 0.$$

However,

$$D\phi(x) = \frac{x - \bar{y}}{\varepsilon} + \alpha \left[ 2p \left( 1 + \|x\|^2 \right)^{p-1} x \right].$$

Thus,

$$\beta u_1(\bar{x}) \leq H(\bar{x}, D\phi(\bar{x})) = H\left( \bar{x}, \frac{\bar{x} - \bar{y}}{\varepsilon} + \alpha \left[ 2p \left( 1 + \|\bar{x}\|^2 \right)^{p-1} \bar{x} \right] \right),$$

from which we get, by the definition of sup in $H(x, p)$

$$\beta u_1(\bar{x}) \leq f(\bar{x}, u_\varepsilon) \cdot \left\{ \frac{\bar{x} - \bar{y}}{\varepsilon} + \alpha \left[ 2p \left( 1 + \|\bar{x}\|^2 \right)^{p-1} \bar{x} \right] \right\} + c(\bar{x}, u_\varepsilon) + \varepsilon, \quad \exists u_\varepsilon \in U. \tag{14.16}$$

# Uniqueness via the Comparison Principle

Similarly, consider a test function

$$\tilde{\phi}(y) = u_1(\bar{x}) - \frac{\|\bar{x} - y\|^2}{2\varepsilon} - \alpha\left[\left(1 + \|\bar{x}\|^2\right)^p + \left(1 + \|y\|^2\right)^p\right].$$

Note that

$$\Psi(\bar{x}, y; \varepsilon, \alpha, p) = u_2(y) - \tilde{\phi}(y).$$

Thus, $u_2(y) - \tilde{\phi}(y)$ attains its minimum at $\bar{y}$. By the definition of the supersolution,

$$\beta u_2(\bar{y}) - H(\bar{y}, D\tilde{\phi}(\bar{y})) \geq 0.$$

However,

$$D\tilde{\phi}(y) = \frac{\bar{x} - y}{\varepsilon} + \alpha\left[2p\left(1 + \|y\|^2\right)^{p-1} y\right].$$

Thus,

$$\beta u_2(\bar{y}) \geq H\left(\bar{y}, \frac{\bar{x} - \bar{y}}{\varepsilon} + \alpha\left[2p\left(1 + \|\bar{y}\|^2\right)^{p-1} \bar{y}\right]\right),$$

from which we get

$$\beta u_2(\bar{y}) \geq f(\bar{y}, u) \cdot \left\{\frac{\bar{x} - \bar{y}}{\varepsilon} + \alpha\left[2p\left(1 + \|\bar{y}\|^2\right)^{p-1} \bar{y}\right]\right\} + c(\bar{y}, u), \quad \forall u \in U. \quad (14.17)$$

# Uniqueness via the Comparison Principle

Combine (14.16) and (14.17) gives

$$
\begin{aligned}
&\beta\left(u_1(\bar{x}) - u_2(\bar{y})\right) \\
\leq & f(\bar{x}, u_\varepsilon) \cdot \left\{ \frac{\bar{x} - \bar{y}}{\varepsilon} + \alpha \left[ 2p \left(1 + \|\bar{x}\|^2\right)^{p-1} \bar{x} \right] \right\} + c(\bar{x}, u_\varepsilon) \\
& - f(\bar{y}, u_\varepsilon) \cdot \left\{ \frac{\bar{x} - \bar{y}}{\varepsilon} + \alpha \left[ 2p \left(1 + \|\bar{y}\|^2\right)^{p-1} \bar{y} \right] \right\} - c(\bar{y}, u_\varepsilon) + \varepsilon \\
= & \left(f(\bar{x}, u_\varepsilon) - f(\bar{y}, u_\varepsilon)\right) \cdot \frac{\bar{x} - \bar{y}}{\varepsilon} + c(\bar{x}, u_\varepsilon) - c(\bar{y}, u_\varepsilon) + \varepsilon \\
& + \alpha 2p \left(1 + \|\bar{x}\|^2\right)^{p-1} f(\bar{x}, u_\varepsilon) \cdot \bar{x} - \alpha 2p \left(1 + \|\bar{y}\|^2\right)^{p-1} f(\bar{y}, u_\varepsilon) \cdot \bar{y}.
\end{aligned}
$$

Note that for the first two terms of the above equation,

$$
\begin{aligned}
\left| \left(f(\bar{x}, u_\varepsilon) - f(\bar{y}, u_\varepsilon)\right) \cdot \frac{\bar{x} - \bar{y}}{\varepsilon} \right| &\leq \frac{1}{\varepsilon} \|f(\bar{x}, u_\varepsilon) - f(\bar{y}, u_\varepsilon)\| \cdot \|\bar{x} - \bar{y}\| \\
&\leq \frac{L}{\varepsilon} \|\bar{x} - \bar{y}\|^2, \\
|c(\bar{x}, u) - c(\bar{y}, u)| &\leq L \|\bar{x} - \bar{y}\|.
\end{aligned}
$$