# MF921 Topics in Dynamic Asset Pricing

## Week 7

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Chapter 22 American Options (II)

Numerical Methods for American Options

# Recall

Consider the valuation of a finite-horizon American option with payoff $\varphi(S(\tau))$:

$$V(x,t) := \sup_{\tau \geq t} E^0 \left[ e^{-r(\tau - t)} \varphi(S(\tau)) \mid S(t) = x \right],$$

where $S(t) =$ stock price, $\varphi(\cdot) =$ payoff function (e.g. $(K - S)^+$ for a put), $\tau =$ stopping time (the exercise time). So $V(x,t)$ is the value function represent the highest discounted expected payoff you can achieve by optimally choosing when to stop.

First, the optimal reward and moving boundary functions together solve the following free-boundary (to be determined) problem:

$$\begin{cases} \mathcal{A}V - rV = 0, & x > b(t), \quad 0 \leq t \leq T \\ V(b(t),t) = \varphi(b(t)), & \text{(value matching)} \\ V_x(b(t),t) = \varphi'(b(t)), & \text{(smooth pasting)} \\ V(x,T) = \varphi(x), & x > 0 \\ V(0,t) = \varphi(0), & V(\infty,t) = 0 \end{cases}$$

# Recall

Second, the free boundary problem can be written as a partial differential complementarity problem (without the free boundary) as

$$\begin{gathered}\mathcal{A}V - rV \leq 0, \quad x > 0, \quad t \in [0, T) \\ (\mathcal{A}V - rV)\{V(x,t) - \varphi(x)\} = 0, \quad x > 0, \quad t \in [0, T) \\ V(x,t) \geq \varphi(x), \quad x > 0, \quad t \in [0, T)\end{gathered}$$

along with the terminal and boundary conditions

$$\begin{gathered}V(x,T) = \varphi(x), \quad x > 0, \\ \lim_{x \to \infty} V(x,t) = 0, \quad V(0,t) = \varphi(0), \quad t \in [0, T)\end{gathered}$$

This is called the complementarity problem because the two inequalities cannot be strict inequalities simultaneously. The partial differential complementarity problem can be solved numerically by converting it to a matrix linear complementarity problem by using the finite difference method.

# Recall

Third, the partial differential complementarity problem can be rewritten as a variational inequality problem:

$$\min(-\mathcal{A}V + rV, V(x,t) - \varphi(x)) = 0, \quad x > 0, \quad t \in [0,T) \qquad (*)$$

along with the terminal and boundary conditions

$$V(x,T) = \varphi(x), \quad x > 0,$$

$$\lim_{x\to\infty} V(x,t) = 0, \quad V(0,t) = \varphi(0), \quad t \in [0,T).$$

Indeed, there are only two possibility by the partial differential complementarity problem: either $-\mathcal{A}V + rV > 0$ or $-\mathcal{A}V + rV = 0$. In the first case we must have $V(x,t) = \varphi(x)$, and in the second case $V(x,t) \geq \varphi(x)$. Thus, in both cases we have $(*)$. Hence the partial differential complementarity problem implies the variational inequality problem. Conversely, $(*)$ implies that $V(x,t) \geq \varphi(x)$ and $-\mathcal{A}V + rV \geq 0$. Furthermore, if $-\mathcal{A}V + rV > 0$ then $V(x,t) - \varphi(x) = 0$ by $(*)$. Therefore, the variational inequality problem also implies the partial differential complementarity problem.

# Recall

An efficient way to solve the partial differential complementarity problem is to transform it ot a linear complementarity problem via the finite difference method.

Consider the following (matrix) linear complementarity problem: Find a vector $x \in \mathbb{R}^{1\times n}$

$$Ax \geq b,\ x \geq c,\ (x-c)^T(Ax-b) = 0,$$

for given $A \in \mathbb{R}^{n\times n}$, $c, b \in \mathbb{R}^{1\times n}$, where, for two column vectors $x$ and $y$, $x \geq y$ means $x_i \geq y_i$ for each $i$. The linear complementarity problem has a unique solution for all column vectors $b$ and $c$ if and only if $A \in \mathbb{R}^{n\times n}$ is a P-matrix.

Note that if $A$ is symmetric, then $A$ is a P-matrix if and only if $A$ is positive definite. Many matrices that arise in finite-difference and finite-element methods are diagonally dominant.

# Recall

More precisely, the matrix $A$ is diagonally dominant if

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|, \; \forall i,$$

where $a_{ij}$ denotes the entry of $A$ in the $i$th row and $j$th column. If the above inequality is strict, it is called strictly diagonally dominant. A well-known result is that a symmetric strictly diagonally dominant matrix with real positive diagonal entries is positive definite.

The linear complementarity problem can be solved by using many methods, including pivoting methods (e.g., Lemke's algorithm), quadratic programming, successive over relaxation (SOR), projected SOR, etc. There are also several Matlab and Python codes available online.

# Recall

The linear complementarity problem is a particular case of the nonlinear complementarity problem, which is to find a vector $z \in \mathbb{R}^{1\times n}$

$$f(x) \geq b,\ z \geq c,\ (z-c)^T(f(x)-b) = 0,$$

where $f$ is a given function $\mathbb{R}^n \longmapsto \mathbb{R}^n$, $c, b \in \mathbb{R}^{1\times n}$. One way to solve the nonlinear complementarity problem is to use the linear approximation of $f(x)$ to get an iterative algorithm by solving a sequence of the complementarity problem.

The nonlinear complementarity problem is also related to a variational inequality problem: Given a non-empty set $K \in \mathbb{R}^n$, a function $g : \mathbb{R}^n \longmapsto \mathbb{R}^n$, $b \in \mathbb{R}^{1\times n}$, find a $z^* \in K$ such that

$$\min_{y\in K}(y - z^*)^T g(z^*) \geq 0.$$

It can be shown that with

$$K = \mathbb{R}_+^{1\times n},\ z^* = z - c,\ g(x) = f(x) - b,$$

the nonlinear complementarity problem and the variational inequality problem have the same solution.

# Finite Difference Methods

The partial differential complementarity problem for $\psi(x, t)$ without the free boundary,

$$\frac{\partial \psi}{\partial t} + \frac{\sigma^2 x^2}{2} \frac{\partial^2 \psi}{\partial x^2} + rx \frac{\partial \psi}{\partial x} - r\psi \leq 0$$

$$\left( \frac{\partial \psi}{\partial t} + \frac{\sigma^2 x^2}{2} \frac{\partial^2 \psi}{\partial x^2} + rx \frac{\partial \psi}{\partial x} - r\psi \right) \{\psi(t, x) - g(x)\} = 0$$

$$\psi(x, t) \geq g(x)$$

$$\psi(x, T) = g(x)$$

$$\lim_{x \to \infty} \psi(x, t) = 0, \quad \psi(0, t) = g(0),$$

Where $g(x)$ is the payoff function. Although the algorithm also works for the case where $g(x)$ may be $g(x, t)$ depending on $t$, for notational simplicity we shall focus on the case of time-independent $g(x)$. First, we can transform the problem based on the standard heat equation by changing variables.

# Finite Difference Methods

$$S = e^x, \quad \tau = (T-t)\frac{\sigma^2}{2}$$

$$u(x,\tau) = \exp\left\{\frac{1}{2}(c-1)x + \frac{1}{4}(c+1)^2\tau\right\} \cdot \psi(S,t), \quad c = \frac{2r}{\sigma^2},$$

the Black-Scholes partial differential complementarity problem becomes a standard heat partial differential complementarity problem, $x \in (-\infty, \infty)$:

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} \geq 0$$

$$\left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}\right)\{u(\tau,x) - f(x)\} = 0$$

$$u(x,\tau) \geq f(x,\tau), \quad u(x,0) = f(x,0)$$

$$\lim_{x\to\infty} u(x,\tau) = 0$$

where $f(x,\tau) = \exp\left\{\frac{1}{2}(c-1)x + \frac{1}{4}(c+1)^2\tau\right\} g(e^x)$.

This is the heat equation complementarity problem, with constant coefficients. We are interested in finding $u(x,\tau)$, where $x \in (-\infty,\infty)$ and $\tau \in [0, T\sigma^2/2]$.

# Finite Difference Methods

We have the transformed the complicated Black-Scholes complementarity PDE into heat-equation complementarity problem:

$$u_\tau - u_{xx} \geq 0, \quad (u_\tau - u_{xx})(u - f) = 0, \quad u \geq f.$$

We can't solve this analytically, so we approximate the derivatives $(u_\tau, u_{xx})$ with finite differences on a discrete grid. By discretizing time and space on a regular mesh with step sizes $\delta\tau$ and $\delta x$, and truncating $x$ in the region $[-N^-\delta x, N^+\delta x]$ for suitably large integers $N^+$ and $N^-$, we let

$$u_{n,m} = u(n\delta x, m\delta\tau), \quad f_{n,m} = f(n\delta x, m\delta\tau), \quad -N^- \leq n \leq N^+, \quad 0 \leq m \leq M,$$

where $M\delta\tau = T\sigma^2/2$. We use the finite difference approximation

$$\begin{aligned}
\frac{\partial u}{\partial \tau} &= \frac{u_{n,m+1} - u_{n,m}}{\delta\tau} + o(\delta\tau), \\
\frac{\partial^2 u}{\partial x^2} &= \theta\left(\frac{u_{n+1,m+1} - 2u_{n,m+1} + u_{n-1,m+1}}{(\delta x)^2}\right) \\
&\quad + (1-\theta)\left(\frac{u_{n+1,m} - 2u_{n,m} + u_{n-1,m}}{(\delta x)^2}\right) + o((\delta x)^2),
\end{aligned}$$

where $0 < \theta < 1$.

# Finite Difference Methods

Note that this is a general way to do the finite difference method; the cases $\theta = 0, 1/2, 1$ yield the explicit, the Crank-Nicolson, and the implicit schemes, respectively.

The explicit scheme does not involve solving linear equations and is known to be numerically stable and convergent whenever

$$\alpha := \delta\tau/(\delta x)^2 \leq 1/2.$$

The implicit and the Crank-Nicolson schemes are always numerically stable and convergent but more numerically intensive than the explicit method. They require solving a system of linear equations on each time step. Usually, the Crank-Nicolson scheme is the most accurate scheme for small time steps, while the implicit scheme works best for large time steps. In general, if $\theta < 1/2$, then the scheme is stable if $\delta\tau/(\delta x)^2 < (1-\theta)/2$.

# Finite Difference Methods

With these approximations, we have

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} \approx \frac{u_{n,m+1} - u_{n,m}}{\delta \tau} - \theta \left( \frac{u_{n+1,m+1} - 2u_{n,m+1} + u_{n-1,m+1}}{(\delta x)^2} \right) - (1-\theta) \left( \frac{u_{n+1,m} - 2u_{n,m} + u_{n-1,m}}{(\delta x)^2} \right).$$

Thus, the requirement that $\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} \geq 0$ becomes for $-N^- + 2 \leq n \leq N^+ - 2$, $0 \leq m \leq M-1$,

$$u_{n,m+1} - \alpha\theta(u_{n+1,m+1} - 2u_{n,m+1} + u_{n-1,m+1}) \geq b_{n,m},$$

where

$$b_{n,m} := u_{n,m} + \alpha(1-\theta)(u_{n+1,m} - 2u_{n,m} + u_{n-1,m}), \quad \alpha = \delta\tau/(\delta x)^2.$$

Note that if we move forward in time, at the $(m+1)$-th step we know $b_{n,m}$ explicitly, via $u_{n,m}$, $u_{n+1,m}$, and $u_{n-1,m}$.

# Finite Difference Methods

The requirement $u(\tau, x) - f(\tau, x) \geq 0$ becomes, for $-N^- \leq n \leq N^+$, $0 \leq m \leq M$,

$$u_{n,m} \geq f_{n,m}.$$

To simplify the operation, we shall translate the complementarity condition to $(m+1)$-th time point as, for $-N^- + 2 \leq n \leq N^+ - 2$, $0 \leq m \leq M - 1$,

$$\{u_{n,m+1} - \alpha\theta(u_{n+1,m+1} - 2u_{n,m+1} + u_{n-1,m+1}) - b_{n,m}\}\{u_{n,m+1} - f_{n,m+1}\} = 0.$$

The initial condition is easy because $u(0, x) = f(0, x)$, yielding

$$u_{n,0} = f_{n,0}.$$

We have to first pay attention to the boundary conditions at $-N^-$ and $N^+$. Based on the definition of option payoffs, as an approximation we can let

$$u_{-N^-,m} = f(-N^-, m), \quad u_{N^+,m} = f(N^+, m), \quad 0 \leq m \leq M.$$

The intuition is that when stock price $S(t) \to \infty$ or $S(t) \to 0$, corresponding to $x \to \infty$ and $x \to -\infty$, the option price at time $t$ should be very close to the option payoff at time $t$.

# The Boundary Conditions and the Initial Condition

The discretized complementarity PDE for the transformed variable $u(x,\tau)$ only works for interior points $n$ where both neighbors $u_{n+1}$ and $u_{n-1}$ exist. At the boundaries, one neighbor lies outside the computational grid.

For the term $u_{n,m+1}$ when $n = N^+ - 1$, we have

$$
\begin{aligned}
& u_{N^+-1,m+1} - u_{N^+-1,m} - \alpha\theta(u_{N^+,m+1} - 2u_{N^+-1,m+1} + u_{N^+-2,m+1}) \\
& \geq \alpha(1-\theta)(u_{N^+,m} - 2u_{N^+-1,m} + u_{N^+-2,m}),
\end{aligned}
$$

for $0 \leq m \leq M-1$, or

$$
u_{N^+-1,m+1} - \alpha\theta(-2u_{N^+-1,m+1} + u_{N^+-2,m+1}) \geq b_{N^+-1,m} + \alpha\theta u_{N^+,m+1},
$$

where

$$
b_{N^+-1,m} := u_{N^+-1,m} + \alpha(1-\theta)(u_{N^+,m} - 2u_{N^+-1,m} + u_{N^+-2,m}),
$$

$$
u_{-N^-,m+1} = f(-N^-, m+1), \quad 0 \leq m \leq M-1.
$$

# The Boundary Conditions and the Initial Condition

Similarly for the term $u_{n,m+1}$ when $n = -N^- + 1$, we have

$$\begin{aligned} &u_{-N^-+1,m+1} - u_{-N^-+1,m} - \alpha\theta(u_{-N^-+2,m+1} - 2u_{-N^-+1,m+1} + u_{-N^-,m+1}) \\ &\geq \alpha(1-\theta)(u_{-N^-+2,m} - 2u_{-N^-+1,m} + u_{-N^-+1,m}), \end{aligned}$$

for $0 \leq m \leq M - 1$, or

$$u_{-N^-+1,m+1} - \alpha\theta(u_{-N^-+2,m+1} - 2u_{-N^-+1,m+1}) \geq b_{-N^-+1,m} + \alpha\theta u_{-N^-,m+1},$$

where

$$b_{-N^-+1,m} := u_{-N^-+1,m} + \alpha(1-\theta)(u_{-N^-+2,m} - 2u_{-N^-+1,m} + u_{-N^-+1,m}).$$

In summary we can define

$$b_{n,m} := u_{n,m} + \alpha(1-\theta)(u_{n+1,m} - 2u_{n,m} + u_{n-1,m}), \quad -N^- + 1 \leq n \leq N^+ - 1.$$

The linear complementarity condition at the two boundaries can be written down readily.

# Standard-Form of the Linear Complementarity Problem

Introduce the column vectors

$$u^{(m)} := (u_{-N^-+1,m}, \dots, u_{N^+-1,m})^T,$$
$$f^{(m)} := (f_{-N^-+1,m}, \dots, f_{N^+-1,m})^T,$$
$$b^{(m)} := (b_{-N^-+1,m}, \dots, b_{N^+-1,m})^T,$$

So each vector represents one time slice of all the interior spatial nodes. At the boundaries, we define a correction vector to represent how the known boundary values affect the equations for the first and last interior points.

$$\xi^{(m)} := (\alpha\theta u_{-N^-,m+1}, 0, 0, \dots, 0, 0, \alpha\theta u_{N^+,m+1})^T,$$

and a $(N^+ + N^- - 2) \times (N^+ + N^- - 2)$ square, tridiagonal, symmetric matrix

$$C = \begin{pmatrix} 1+2\alpha\theta & -\alpha\theta & 0 & \cdots & 0 \\ -\alpha\theta & 1+2\alpha\theta & -\alpha\theta & \cdots & 0 \\ 0 & -\alpha\theta & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1+2\alpha\theta & -\alpha\theta \\ 0 & 0 & \cdots & -\alpha\theta & 1+2\alpha\theta \end{pmatrix}.$$

# Standard-Form of the Linear Complementarity Problem

Then the inequalities:

$$u_{n,m+1} - \alpha\theta(u_{n+1,m+1} - 2u_{n,m+1} + u_{n-1,m+1}) \geq b_{n,m}$$
$$u_{N^+-1,m+1} - \alpha\theta(-2u_{N^+-1,m+1} + u_{N^+-2,m+1}) \geq b_{N^+-1,m} + \alpha\theta u_{N^+,m+1}$$
$$u_{-N^-+1,m+1} - \alpha\theta(u_{-N^-+2,m+1} - 2u_{-N^-+1,m+1}) \geq b_{-N^-+1,m} + \alpha\theta u_{-N^-,m+1},$$

become to:

$$Cu^{(m+1)} \geq b^{(m)} + \xi^{(m)}.$$

The inequality $u_{n,m} \geq f_{n,m}$ can be written as $u^{(m+1)} \geq f^{(m+1)}$. The linear complementarity condition becomes

$$\left(u^{(m+1)} - f^{(m+1)}\right)^T \left(Cu^{(m+1)} - b^{(m)} - \xi^{(m)}\right) = 0.$$

# Standard-Form of the Linear Complementarity Problem

In summary, the algorithm goes from $m = 0$ to $m = M - 1$, as follows.
**Initialization:** At maturity $(m = 0)$:

$$u^{(0)} = f^{(0)} = \text{payoff}.$$

**Backward time-stepping (for $m = 0, 1, \ldots, M - 1$):**

1. **Compute known quantities:**

   $$b^{(m)} = u^{(m)} + \alpha(1 - \theta)(\text{Laplacian of } u^{(m)}),$$

   and boundary correction $\xi^{(m)}$.
2. **Set up right-hand side:**

   $$q = b^{(m)} + \xi^{(m)}.$$
3. **Solve LCP:**

   $$\begin{cases} Cu^{(m+1)} \geq q, \\ u^{(m+1)} \geq f^{(m+1)}, \\ (u^{(m+1)} - f^{(m+1)})^T (Cu^{(m+1)} - q) = 0. \end{cases}$$

   Use Projected Successive Over-Relaxation (or another solver) to find $u^{(m+1)}$.
4. **Repeat until $m = M - 1$.**

# Standard-Form of the Linear Complementarity Problem

Note that since

$$|1 + 2\alpha\theta| = 1 + 2\alpha\theta > |-\alpha\theta| + |-\alpha\theta|,$$

the symmetric matrix $C$ is a strictly diagonally dominant matrix with real positive diagonal entries. Thus, the symmetric matrix $C$ is positive definite, and the linear complementarity problem given by three inequalities has a unique solution.

Projected Successive Over-Relaxation (PSOR)
We could solve that exactly by inversion, but for large systems (like in PDE grids), we often use iterative methods.
We rearrange each equation for its main variable $u_i$:

$$C_{ii}u_i = q_i - \sum_{j \neq i} C_{ij}u_j.$$

Then we update $u_i$:

$$u_i = \frac{1}{C_{ii}} \left( q_i - \sum_{j \neq i} C_{ij}u_j \right).$$

That's the Gauss-Seidel update rule.

# Standard-Form of the Linear Complementarity Problem

For our tridiagonal $C$, each row has only three nonzero entries:

$$-\alpha\theta u_{i-1} + (1 + 2\alpha\theta)u_i - \alpha\theta u_{i+1} = q_i.$$

Rearrange for $u_i$:

$$u_i = \frac{1}{1 + 2\alpha\theta} \left(q_i + \alpha\theta u_{i-1} + \alpha\theta u_{i+1}\right).$$

We also need $u_i \geq f_i$, we simply project each update:

$$u_i^{(k+1)} = \max\left(f_i, \frac{1}{C_{ii}}\left[q_i - \sum_{j<i} C_{ij} u_j^{(k+1)} - \sum_{j>i} C_{ij} u_j^{(k)}\right]\right).$$

After repeat the algorithm until $m = M - 1$, we hold a vector $u^{(M)}$ :

$$u^{(M)} = [u_{-N^-+1,M}, \ldots, u_{N^+-1,M}]^T.$$

That vector contains the transformed values of the option at time $t = 0$ (the valuation date) for different $x = \ln S$ grid points.

# Standard-Form of the Linear Complementarity Problem

So each component corresponds to:

$$u_{n,M} \approx u(x_n, \tau_{\max}) = e^{\frac{1}{2}(c-1)x_n + \frac{1}{4}(c+1)^2 \tau_{\max}} \psi(S_n, 0),$$

where $S_n = e^{x_n}$.
To return to the original (Black-Scholes) variables, recall the transformation:

$$u(x, \tau) = e^{\frac{1}{2}(c-1)x + \frac{1}{4}(c+1)^2 \tau} \psi(S, t), \quad c = \frac{2r}{\sigma^2}.$$

At $t = 0$, $\tau = T\sigma^2/2$.
So the original option price surface is:

$$\psi(S, 0) = e^{-\frac{1}{2}(c-1)\ln S - \frac{1}{4}(c+1)^2 \tau_{\max}} u(\ln S, \tau_{\max}).$$

That gives the American option price at time 0 for each stock price $S$ on the grid. Usually we're interested in the price for one current stock price $S_0$. If $S_0$ lies between two grid nodes $S_i, S_{i+1}$, we interpolate linearly to find $\psi(S_0, 0)$. We can also track at each time step where $u_{n,m} = f_{n,m}$ (the points where equality binds) to reconstruct the exercise boundary $S^*(t)$.

Chapter 10 Arbitrage

# Motivation to Study Arbitrage

**Can there be arbitrage even without options?**

If your model of stock price dynamics itself creates a riskless profit (without any options), then the model is internally inconsistent. If the relationship between the interest rate $r$ and the possible stock movements (up and down) is inconsistent, arbitrage automatically arises.

A valid model (arbitrage-free) must satisfy:

$$d < R < u,$$

where:

- $u =$ up-factor $= S_u/S_0$,
- $d =$ down-factor $= S_d/S_0$,
- $R = 1 + r$.

Only in that range can there exist a risk-neutral probability $p^* = \frac{R-d}{u-d}$ with $0 < p^* < 1$. Arbitrage-free condition ensures no riskless trading profit can be made by combining borrowing/lending and buying/selling the stock. If your model violates this, then the entire pricing framework collapses.

# Basic Setting of a One-Period Security Market

We construct the simplest possible financial market to study arbitrage formally.

- The model is discrete in time, $t = 0$ and $t = 1$, and finite in states (K scenarios), $\Omega = \{\omega_1, \dots, \omega_K\}$, $P(\omega) > 0$.
- $N + 1$ securities: $B(t)$ is bank account (risk-free), $B(0) = 1$, $B(1) = 1 + r$ and $S_1(t), S_2(t), \dots, S_N(t)$ are risky stocks.
- Trading Strategy: $\varphi = (\varphi_0, \varphi_1, \dots, \varphi_N)$ are portfolio weights and $V(0)$ is initial wealth. The resulting wealth process is given by

$$V(t) := \varphi_0 B(t) + \sum_{n=1}^{N} \varphi_n S_n(t).$$

The self-financing condition is defined as:

$$V(1) - V(0) = \phi_0 \Delta B + \sum_{n=1}^{N} \phi_n \Delta S_n,$$

where $\Delta B := B(1) - B(0) = r, \quad \Delta S_n := S_n(1) - S_n(0).$

# Basic Setting of a One-Period Security Market

Define discounted variables:

$$S_n^*(t) := S_n(t)/B(t), \quad V^*(t) := V(t)/B(t) = \phi_0 + \sum_{n=1}^{N} \phi_n S_n^*(t) \quad t = 0, 1,$$

then we can rewrite the self-financing condition as

$$V^*(1) - V^*(0) = \sum_{n=1}^{N} \phi_n \Delta S_n^*, \quad V^*(t) = V(t)/B(t).$$

Define the capital gain process

$$G^* = \sum_{n=1}^{N} \phi_n \Delta S_n^*, \quad V^*(1) = V(0) + G^*.$$

The simple return of $n$th risky asset is defined as $R_n := (S_n(1) - S_n(0))\ S_n(0)$.

# Dominant Trading Strategies

A trading strategy $\phi$ is said to dominate another strategy $\tilde{\phi}$ if their corresponding wealth $V$ and $\tilde{V}$ satisfy:

$$V(0) = \tilde{V}(0), \quad V(1) > \tilde{V}(1),$$

with probability one (i.e. $V(1,\omega) > \tilde{V}(1,\omega)$ for any $\omega \in \Omega$).

**Lemma 1.** The following statements are equivalent.

- There is a dominant strategy.
- There exists a trading strategy such that $V(0) = 0$ and $V(1) > 0$ with probability one.
- There exists a trading strategy such that $V(0) < 0$ and $V(1) \geq 0$ with probability one.

**Proof**:
(a) $\Rightarrow$ (b): Suppose there are two trading strategies $\hat{\phi}$ and $\tilde{\phi}$ such that $\hat{\phi}$ dominates $\tilde{\phi}$. Then consider another trading strategy $\phi = \hat{\phi} - \tilde{\phi}$. We have, via the definition of wealth processes,

$$V(0) = \hat{V}(0) - \tilde{V}(0) = 0, \quad V(1) = \hat{V}(1) - \tilde{V}(1) > 0,$$

with probability one, from which (b) follows.

# Dominant Trading Strategies

(b) $\Rightarrow$ (c): Now start from (b): $V(0) = 0$, $V(1) > 0$ in all states. Since $V^*(1) = G^* > 0$ for all states, the smallest possible gain $\varepsilon = \min_{\omega \in \Omega} G^*(\omega) > 0$ exists. Construct a new strategy:

$$\tilde{\phi}_n = \phi_n, \quad \tilde{\phi}_0 = \phi_0 - \varepsilon.$$

That reduces the initial wealth by $\varepsilon$:

$$\tilde{V}(0) = -\varepsilon < 0, \quad \tilde{V}^*(1, \omega) = -\varepsilon + G^*(\omega) \geq 0.$$

So you borrow money at the beginning ($V(0) < 0$), but never lose at the end ($V(1) \geq 0$). That's (c).
(c) $\Rightarrow$ (a): Now suppose we have (c): $V(0) < 0$ and $V(1) \geq 0$. Construct a new strategy $\tilde{\phi}$:

$$\tilde{\phi}_n = \phi_n, \quad \tilde{\phi}_0 = -\sum_{n=1}^{N} \phi_n S_n(0) = \phi_0 - V(0).$$

Then the new portfolio has:

$$\tilde{V}(0) = 0, \quad \tilde{V}^*(1) = -V(0) + V^*(1) \geq -V(0) > 0.$$

So $\tilde{\phi}$ strictly dominates the "do nothing" strategy (holding nothing), satisfying (a).

# Law of one price