MF921 Topics in Dynamic Asset Pricing

Week 5

Yuanhui Zhao

Boston University

Black-Scholes (II): Dominance-Free Interval and Risk Neutral Pricing

A General Brownian Market Model

Given a complete probablity space $(\Omega, \mathcal{F}, \mathbb{P})$. $W(t) = (W_1(t), \ldots, W_d(t))^\top$, independent d-dimensional Brownian motion. The filtraion $\mathcal{F}_t^W = \sigma(W(s) : 0 \leq s \leq t)$ which is complete and right-continuous.

A financial market $\mathcal{M}$ with 1 bond and d stocks under a finite horizon $[0, T]$:

$$dS_0(t) = r(t)S_0(t)dt, \quad S_0(0) = 1$$

$$dS_i(t) = S_i(t)\left(b_i(t)\,dt + \sum_{j=1}^{d} \sigma_{ij}(t)\,dW_j(t)\right), \quad \text{for } i \in 1, 2...., d$$

- $r(t)$: interest rate
- $b(t) = (b_1, \ldots, b_d)$: appreciation rates
- $\sigma(t) = (\sigma_{ij}(t))$: volatility matrix
- $r(t)$, $b(t)$ and $\sigma(t)$ all progressively measurable with respect to $\{\mathcal{F}_t\}$ and bounded uniformly in $(t, \omega) \in [0, T] \times \Omega$.

Remarks on the Modell

- The model is very general: processes can be dependent, time-varying, and even non-Markovian
- The model does not cover stochastic volatility or jump models. Therefore, dominance arguments do not apply for stochastic volatility or jump models.
- In general, the option price is not pinned to a single number. Instead, there exists an interval $[h_{low}, h_{up}]$. If the option price is within the interval then there will be no dominance, outside the interval there will be dominance opportunity.
- Ideal market assumption: Infinite divisibility of assets, no transaction costs or taxes, no borrowing/short-selling constraints and same interest rate for borrowing and lending

Introducing auxiliary processes

Relative risk (Sharpe ratio):

$$\theta(t) = \sigma^{-1}(t)\,(b(t) - r(t)\mathbf{1})\,, \quad \mathbf{1} = (1, 1, \ldots\ldots, 1)^T$$

Exponential martingale (RN derivative):

$$Z(t) = \exp\left(-\int_0^t \theta^\top(s)\,dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds\right)$$

Discount factor:

$$\gamma(t) = \exp\left(-\int_0^t r(s)\,ds\right)$$

Brownian motion with drift:

$$W_0(t) = W(t) + \int_0^t \theta(s)\,ds, \quad 0 \le t \le T$$

$\sigma(t)$ invertible, inverses bounded. Ensures bounded $\theta(t)$ and $Z(t)$ is a true martingale. These tools set up the risk-neutral framework for pricing options.

Investor Setup

Investor type: "small investor" $\rightarrow$ cannot affect market prices.
Trading Strategy defined as:

$$\phi(t) = (\phi_1(t), \ldots, \phi_d(t))$$

Where $\phi_0(t)$ is number of bonds held, $\phi_i(t)$ is number of shares of stock i held. If $\phi_0 < 0$ or $\phi_i < 0$ interpret as short position(loan). All decisions are adapted to $\mathcal{F}_t$.
Cumulative consumption process defined as:

$$C(t) = \int_0^t c(s)ds, \quad c(s) \geq 0$$

$C(t)$ is non-decreasing (consumption can only increase). $C(0) = 0, C(T) < \infty$ almost surely. Models the total amount consumed by the investor up to time t

To ensure consistency, we need to specify what trading strategies are allowed in the market. Two rules impose in the market, self-financing condition and exclusion of doubling strategies.

Self-Financing Condition

Self-Financing Condition:

$$\sum_{i=0}^{d} \phi_i(t) S_i(t) = \sum_{i=0}^{d} \phi_i(0) S_i(0) + \sum_{i=0}^{d} \int_0^t \phi_i(u) dS_i(u) - C(t), \quad 0 \le t \le T$$

Changes in wealth $=$ trading gains – consumption, no outside cash flows.

A portfolio process is defined as $\pi(\cdot) = (\pi_1(\cdot), \ldots, \pi_d(\cdot))$, where $\pi_i(t) = \phi_i(t) S_i(t)$ means that total amount of money invested in the ith risky asset. The total wealth $X(t)$ is equal to:

$$X(t) = \sum_{i=0}^{d} \phi_i(t) S_i(t) = \sum_{i=1}^{d} \pi_i(t) + \phi_0(t) S_0(t)$$

Both the wealth process $X(\cdot)$ and the portfolio $\pi(\cdot)$ can clearly take both positive and negative values.

Self-Financing Condition

For a given initial capital x and a portfolio process $\pi(\cdot)$, the self-financing condition is translated to:

$$X(t) = X(0) + \sum_{i=0}^{d} \int_0^t \phi_i(u) dS_i(u) - C(t)$$

In terms of the differential form, we have:

$$dX(t) = \sum_{i=0}^{d} \phi_i(t) dS_i(t) - dC(t)$$

Express in terms of portfolio π and plug in dynamics of assets:

$$\begin{aligned} dX(t) &= \frac{X(t) - \sum_{i=1}^{d} \pi_i(t)}{S_0(t)} dS_0(t) + \sum_{i=1}^{d} \frac{\pi_i(t)}{S_i(t)} dS_i(t) - dC(t) \\ &= X(t) r(t) dt + \sum_{i=1}^{d} \pi_i(t) \left[(b_i(t) - r(t)) dt + \sum_{j=1}^{d} \sigma_{ij}(t) dW_j(t) \right] - dC(t) \end{aligned}$$

Self-Financing Condition

By vector notation, Brownian motion with drift and Sharpe ratio, we can rewrite as:

$$dX(t) = X(t)r(t)dt + \pi^\top(t)\sigma(t)dW_0(t) - dC(t), \quad X(0) = x$$

Since $d\gamma(t) \cdot dX(t) = 0$, by Itô formula:

$$\begin{aligned} d(\gamma(t)X(t)) &= \gamma(t)dX(t) - r(t)\gamma(t)X(t)dt \\ &= \gamma(t)\pi^\top(t)\sigma(t)dW_0(t) - \gamma(t)dC(t) \end{aligned}$$

Therefore, we have the wealth equation:

$$\gamma(t)X(t) = x - \int_0^t \gamma(s)dC(s) + \int_0^t \gamma(s)\pi^\top(s)\sigma(s)dW_0(s)$$

For a triple (x, π, C), if there exists a unique strong solution $X(\cdot)$, it's called the wealth process. For the stochastic integral to be well-defined:

$$\int_0^T \|\pi(t)\|^2 dt < \infty$$

Risk-Neutral Probability via Girsanov Theorem

Definition of Risk-Neutral Measure:

$$\mathbb{P}^0(A) := \mathbb{E}[Z(T)1_A], \quad A \in \mathcal{F}_T$$

$Z(T)$ is the exponential martingale. By Girsanov's Theorem, $W_0(t) = W(t) + \int_0^t \theta(s)ds$ is a standard Brownian motion under $\mathbb{P}^0$.
Thus rewirte the wealth process:

$$\begin{aligned} N_0(t) &= \gamma(t)X(t) + \int_0^t \gamma(s)dC(s) \\ &= x + \int_0^t \gamma(s)\pi^\top(s)\sigma(s)dW_0(s) \end{aligned}$$

A continuous $\mathbb{P}^0$-local martingale.

Risk-Neutral Probability via Girsanov Theorem

Stock dynamics under risk-neutral measure:

$$dS_i(t) = S_i(t)\left[b_i(t)dt - \sum_{j=1}^{d}\sigma_{ij}(t)\theta_j(t)dt + \sum_{j=1}^{d}\sigma_{ij}(t)dW_0^{(j)}(t)\right]$$

Since $b(t) - \sigma(t)\theta(t) = r(t)\mathbf{1}$:

$$dS_i(t) = S_i(t)\left[r(t)dt + \sum_{j=1}^{d}\sigma_{ij}(t)dW_0^{(j)}(t)\right], \quad i = 1, \ldots, d$$

Since $dr(t) \cdot dS_i(t) = 0$, apply Itô:

$$\begin{aligned} d(\gamma(t)S_i(t)) &= S_i(t)dr(t) + \gamma(t)dS_i(t) \\ &= \gamma(t)S_i(t)\sum_{j=1}^{d}\sigma_{ij}(t)dW_0^{(j)}(t) \end{aligned}$$

Hence, the discounted stock processes $\gamma(\cdot)S_i(\cdot)$ are local martingales. This also confirms our intuition that every asset $S_i(t)$ should have a growth rate $r(t)$ in the risk neutral world.

Doubling Strategies and Admissibility

Doubling Strategy: Double investment after each loss, leads to arbitrarily large wealth at T. Requires wealth process $X(t)$ unbounded from below. Need to exclude, because creates arbitrage opportunities and violates no-arbitrage principle.

A uniform boundedness condition is needed to prevent the doubling strategy. Wealth must satisfy:

$$X^{x,\pi,C}(t) \geq -\Lambda, \quad 0 \leq t \leq T$$

With $\mathbb{E}^0[\Lambda^p] < \infty$ for some $p > 1$.

In summary, the admissibility on (π, C) essentially requires the portfolio to be self-financing and not to be a doubling strategy.

Some Properties

(i) Supermartingale Property:

If (π, C) is admissible, $N_0(t)$ is bound below. $N_0(t)$ is a $\mathbb{P}^0$-supermartingale. Consequently:

$$\mathbb{E}^0 \left[\gamma(T)X(T) + \int_0^T \gamma(t)dC(t) \right] \leq x$$

Expected discounted terminal wealth and consumption less than or equal to initial wealth to ensures no arbitrage.

(ii) Scaling Property:

Wealth dynamics are linear in (x, π, C). For any $a \neq 0$:

$$X^{ax, a\pi, aC}(t) = a \cdot X^{x,\pi,C}(t)$$

In particular, $a > 0$ outcome is scaled wealt and $a = -1$ is mirror strategy with wealth is $-X^{x,\pi,C}(t)$. The intuition is that the model is homogeneous of degree 1 in wealth.

Dominant Opportunities and Dominance-Free Interval

Consider European Contingent Claim (ECC), payoff at maturity is $\psi(T) \geq 0$. For example the Call option payoff $(S_1(T) - K)^+$. To ensure that option price is finite, we assume that $\mathbb{E}[(\psi(T))^{1+\epsilon}] < \infty, \ \forall \epsilon > 0$.

Price at time 0 is $\psi(0)$. Question: what is the fair value of $\psi(0)$? Price too low $\rightarrow$ buyer arbitrage, price too high $\rightarrow$ seller arbitrage. Therefore, the correct price must lie in an interval that rules out dominance.

The main purpose of this section is to find out what $\psi(0)$ should be in the market $\mathcal{M}$ with the ECC, denoted by (M, ψ) for short, with ψ standing for the pair $(\psi(0), \psi(T))$.

A dominance opportunity exists in market $(\mathcal{M}, \psi)$ if we start with some initial wealth $x \geq 0$ (or $x \leq 0$ depending on position). We take an admissible strategy (π, C) and add a position in the ECC (long if $a = -1$, short if $a = +1$).

A Definition of Dominate Opportunity

Condition:

$$x + a \cdot \psi(0) < 0$$

yet at maturity:

$$\mathbb{P}\{X^{x,\pi,C}(T) + a \cdot \psi(T) \geq 0\} = 1$$

You start with negative initial wealth, but end up with a guaranteed nonnegative payoff. This is a riskless profit strictly better than bond returns.

Moreover, the scaling property will result the unlimited arbitrage. Therefore, dominance opportunities must be excluded in rational, well-behaved markets.

Definition: The admissible price $\psi(0)$ must lie in an dominance-free interval $[a, b]$, such that:

- If $\psi(0) > b$: dominance opportunity exists (price too high)
- If $\psi(0) < a$: dominance opportunity exists (price too low)
- If $a \leq \psi(0) \leq b$: no dominance opportunities exist

Lower and Upper Hedging Classes

Upper Hedging Class $\mathcal{U}$:

$$\mathcal{U} := \{x \geq 0 : \exists(\hat{\pi}, \hat{C}) \in \mathcal{A},\ X^{x,\hat{\pi},\hat{C}}(0) = x,\ X^{x,\hat{\pi},\hat{C}}(T) \geq \psi(T)\ a.s.\}$$

Starting with capital x, you can construct an admissible strategy whose terminal wealth is always at least as large as the claim payoff $\psi(T)$ So, the minimum of $\mathcal{U}$ gives the upper bound for the fair price. $\mathcal{U}$ may be empty (think about quadratic payoff).

Lower Hedging Class $\mathcal{L}$:

$$\mathcal{L} := \{x \geq 0 : \exists(\check{\pi}, \check{C}) \in \mathcal{A},\ X^{x,\check{\pi},\check{C}}(0) = -x,\ X^{x,\check{\pi},\check{C}}(T) \geq -\psi(T)\ a.s.\}$$

With initial wealth $-x$ (i.e. receiving x up front), you can construct a strategy whose terminal wealth is always greater than or equal to $-\psi(T)$. So, the maximum of $\mathcal{L}$ gives the lower bound for the fair price.

Observe that both sets are intervals (connected):

- If $x_1 \in \mathcal{L}$ and $0 \leq y_1 \leq x_1$, then $y_1 \in \mathcal{L}$
- If $x_2 \in \mathcal{U}$ and $y_2 \geq x_2$, then $y_2 \in \mathcal{U}$

Thus it would be interesting to look at the endpoints of the intervals.

Upper and Lower Hedging Prices

Upper bound: $h_{\mathrm{up}} := \inf \mathcal{U}$. Because it's the cheapest initial capital needed to guarantee covering the claim $\psi(T)$. $\inf \mathcal{U}$ is the minimal fair one for the seller.

Lower bound: $h_{\mathrm{low}} := \sup \mathcal{L}$. Because it's the maximum initial amount from which a buyer can still hedge against the debt of paying $\psi(T)$. So $\sup \mathcal{L}$ is the highest "safe" price for the buyer.

The intuition suggests that the lower dominate price cannot be bigger than the upper hedging price.

Let $u_0 := \mathbb{E}^0[\gamma(T)\psi(T)]$. Show that $u_0 < \infty$. In fact, we can show a strong result that

$$\mathbb{E}^0[\{\gamma(T)\psi(T)\}^a] < \infty, \quad 1 < a < 1 + \epsilon.$$

Proof:
Note that $u_0 = \mathbb{E}^0[\gamma(T)\psi(T)] = \mathbb{E}[\gamma(T)\psi(T)Z(T)]$. Let $c < \infty$ be a fixed constant such that $\|\theta(t)\| \le c$, $\gamma(T) \le c$. Also let $p = 1 + \epsilon$, $1/p + 1/q = 1$. We have:

$$
\begin{aligned}
&\mathbb{E}\left[e^{-q\int_0^T \theta^\top(s)dW(s) - \frac{1}{2}q\int_0^T \|\theta(s)\|^2 ds}\right] \\
&= \mathbb{E}\left[e^{-q\int_0^T \theta^\top(s)dW(s) - \frac{1}{2}q^2\int_0^T \|\theta(s)\|^2 ds} \cdot e^{\frac{1}{2}(q^2-q)\int_0^T \|\theta(s)\|^2 ds}\right] \\
&\le \mathbb{E}\left[e^{-q\int_0^T \theta^\top(s)dW(s) - \frac{1}{2}q^2\int_0^T \|\theta(s)\|^2 ds}\right] e^{\frac{1}{2}(q^2-q)c^2T} \text{(Typo: c)} \\
&\le e^{q(q-1)c^2T/2}
\end{aligned}
$$

where the last inequality holds because inside of $\mathbb{E}(\cdot)$ is a martingale. Therefore, by the Hölder inequality:

Upper and Lower Hedging Prices

$$
\begin{aligned}
u_0 &\leq c\mathbb{E}(\psi(T)Z(T)) \\
&\leq c(\mathbb{E}(\psi(T))^p)^{1/p} \cdot (\mathbb{E}(Z(T))^q)^{1/q} \\
&= c(\mathbb{E}(\psi(T))^p)^{1/p} \cdot \left(\mathbb{E}\left[e^{-q\int_0^T \theta^\top(s)dW(s) - \frac{1}{2}q\int_0^T \|\theta(s)\|^2 ds}\right]\right)^{1/q} \\
&\leq c(\mathbb{E}(\psi(T))^p)^{1/p} \cdot e^{(q-1)c^2T/2} < \infty
\end{aligned}
$$

For $\mathbb{E}^0[\{\gamma(T)\psi(T)\}^a]$. Follow the same steps, change the measure and choose Hölder exponents, $p = \frac{1+\varepsilon}{a} > 1, \quad q = \frac{p}{p-1}$. Then:

$$
\mathbb{E}[\psi^a Z] \leq (\mathbb{E}[\psi^{ap}])^{1/p} (\mathbb{E}[Z^q])^{1/q} = \left(\mathbb{E}[\psi^{1+\varepsilon}]\right)^{1/p} \cdot (\mathbb{E}[Z^q])^{1/q}
$$

The first factor is finite by assumption; the second is bounded exactly as in previous part. Thus:

$$
\mathbb{E}^0[\{\gamma(T)\psi(T)\}^a] \leq c^a \left(\mathbb{E}[\psi(T)^{1+\varepsilon}]\right)^{1/p} \exp\left\{\frac{1}{2}(q-1)c^2T\right\} < \infty, \quad \forall 1 < a < 1+\varepsilon
$$

An Inequality for the Upper and Lower Prices

At any time $t \in [0, T]$, we have $0 \le h_{\text{low}} \le u_0 \le h_{\text{up}} \;\; a.s.$, where $u_0 = \mathbb{E}^0[\gamma(T)\psi(T)]$.

For upper bound: If $\mathcal{U}$ is empty, $h_{\text{up}} = +\infty$ and inequality is trivial. If $\mathcal{U} \neq \emptyset$, then by definition of $\mathcal{U}$, for every $x \in \mathcal{U}$, there exists some admissible portfolio $(\hat{\pi}, \hat{C})$ with:

$$X^{\hat{\pi},\hat{C}}(0) = x, \quad X^{\hat{\pi},\hat{C}}(T) \ge \psi(T)$$

Apply the supermartingale property to the discounted wealth process:

$$x \ge \mathbb{E}^0 \left[\gamma(T) X^{\hat{\pi},\hat{C}}(T) + \int_0^T \gamma(s) d\hat{C}(s) \right]$$

Since $X^{\hat{\pi},\hat{C}}(T) \ge \psi(T)$, we get $x \ge \mathbb{E}^0[\gamma(T)\psi(T)]$. This mean that any initial capital in $\mathcal{U}$ must be at least u_0, $u_0 \le h_{\text{up}}$.

An Inequality for the Upper and Lower Prices

For lower bound: Similarly, we can show that $0 \leq h_{\text{low}} \leq u_0$. Indeed, since the set $\mathcal{L}$ contains $x = 0$, it is nonempty. For any $x \geq 0$ in this set, again by the supermartingale property, almost surely.

$$\begin{aligned} -x &\geq \mathbb{E}^0 \left[\gamma(T) X^{\check{\pi},\check{C}}(T) + \int_0^T \gamma(s) d\hat{C}(s) \right] \\ &\geq \mathbb{E}^0 \left[\gamma(T)(-\psi(T)) + \int_0^T \exp\left(-\int_0^s r(u)du \right) d\hat{C}(s) \right] \\ &\geq \mathbb{E}^0 \left[\gamma(T)(-\psi(T)) \right] \end{aligned}$$

By the definition of $\mathcal{L}$. Hence, $x \leq \mathbb{E}^0 \left[\gamma(T)\psi(T) \right]$ and $0 \leq h_{\text{low}} \leq u_0$, almost surely.

Dominance Outside the Interval

Claim: for any ECC price $\psi(0) > h_{\text{up}}$, there exists a dominant opportunity in (M, ψ); similarly for any ECC price $\psi(0) < h_{\text{low}}$.

Suppose that $\psi(0) > h_{\text{up}}$. Then for any $x_1 \in (h_{\text{up}}, \psi(0))$ we know that $x_1 \in \mathcal{U}$, since h_{up} is the left endpoint of a connected interval $\mathcal{U}$. By the definition of $\mathcal{U}$, there exists a $(\hat{\pi}, \hat{C}) \in \mathcal{A}$ such that:

$$X^{\hat{\pi},\hat{C}}(0) - \psi(0) = x_1 - \psi(0) < 0$$

because $x_1 < \psi(0)$, and

$$X^{\hat{\pi},\hat{C}}(T) - \psi(T) \geq \psi(T) - \psi(T) = 0$$

Hence the definition of the dominance is satisfied with $a = -1$.

Dominance Outside the Interval

Suppose $\psi(0) < h_{\text{low}}$. Then for any $x_1 \in (\psi(0), h_{\text{low}})$ we know that $x_1 \in \mathcal{L}$, since h_{low} is the right endpoint of a connected interval $\mathcal{L}$. By definition of $\sup \mathcal{L}$, there exists an admissible strategy $(\check{\pi}, \check{C})$ such that:

$$X^{\hat{\pi},\hat{C}}(0) = -x_1, \quad X^{\hat{\pi},\hat{C}}(T) \geq -\psi(T)$$

Consider:

$$X^{\hat{\pi},\hat{C}}(0) + \psi(0) = -x_1 + \psi(0) < 0$$

because $x_1 > \psi(0)$, and

$$X^{\hat{\pi},\hat{C}}(T) + \psi(T) \geq -\psi(T) + \psi(T) = 0$$

Hence the definition of the dominance is satisfied with $a = 1$.

No dominance within the Interval

Show that for any $\psi(0) \in [h_{\text{low}}, h_{\text{up}}]$ there is no dominant opportunity in $(\mathcal{M}, \psi)$.
Proof:
Suppose there is a dominant opportunity with $\psi(0) \in [h_{\text{low}}, h_{\text{up}}]$.
Case 1: The dominant opportunity satisfies the definition with $a = -1$. In this case, there exist an initial wealth $x \in [0, \infty)$ and a pair $(\pi_1, C_1) \in \mathcal{A}$, such that:

$$x - \psi(0) = X^{\pi_1, C_1}(0) - \psi(0) < 0$$

whence $x < \psi(0)$, and

$$X^{\pi_1, C_1}(T) - \psi(T) \geq 0, \text{ a.s.}$$

From the definition of $\mathcal{U}$ we know that $x \in \mathcal{U}$, where $x \geq h_{\text{up}}$, by the definition of h_{up}.
Therefore, $h_{\text{up}} \leq x < \psi(0)$; a contradiction, since by assumption $h_{\text{up}} \geq \psi(0)$.

No dominance within the Interval

Case 2: