

MF921 Topics in Dynamic Asset Pricing

Stochastic Analysis & Stochastic Calculus in Quantitative Finance

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Change of Numeraire: Motivation and Key Idea

In option pricing, we usually price under the risk-neutral measure using the money market account $B(t) = e^{rt}$ as the numeraire. But sometimes payoffs become simpler if we change the unit of measurement (the numeraire). Instead of measuring in “dollars,” measure in “shares of stock”.

The key idea is :

- Pick any strictly positive traded asset $N(t)$ as the numeraire.
- Then define a new probability measure $\tilde{\mathbb{P}}$ such that $\frac{S(t)}{N(t)}$ is a martingale under $\tilde{\mathbb{P}}$. No-arbitrage is preserved.

We first look at the details how this work (Radon Nikodym derivative & Girsanov Theorem) and then apply the scheme to price different type of options.

Change of Numeraire

Given $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}^*)$ with d -dim Brownian W :

- Money market account (baseline numeraire): $dB_t = B_t r_t dt$
- Traded asset $S(t)$: $dS_t = S_t(r_t dt + \sigma_t dW_t)$, $\frac{S_t}{B_t}$ is a martingale.
- Derivative pricing rule: for payoff X_T at maturity T , $V_0 = \mathbb{E}^{\mathbb{P}^*} \left[\frac{X_T}{B_T} \right]$

Our goal is to pick another strictly positive traded asset $N(t)$ and define a new measure $\tilde{\mathbb{P}}$ such that $\frac{S(t)}{N(t)}$ is a martingale for every traded asset $S(t)$.

Change of Numeraire Con.

Observe $\frac{S(t)}{B(t)}$ is a martingale under \mathbb{P}^* . We want $\frac{S(t)}{N(t)}$ to be a martingale under $\tilde{\mathbb{P}}$.

Define $\tilde{\mathbb{P}}$ via the Radon–Nikodym derivative with respect to \mathbb{P}^* :

$$\left. \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^*} \right|_{\mathcal{F}_T} = Z_T := \frac{N(T)/B(T)}{N(0)/B(0)}$$

By construction, $\frac{N(T)}{B(T)}$ is a martingale under \mathbb{P}^* , $Z_T > 0$ and $\mathbb{E}^{\mathbb{P}^*}[Z_T] = 1$ and take any payoff X_T :

$$V(0) = N(0) \mathbb{E}^{\tilde{\mathbb{P}}} \left[\frac{X_T}{N(T)} \right] = N(0) \mathbb{E}^{\mathbb{P}^*} \left[\frac{X_T}{N(T)} Z_T \right] = \mathbb{E}^{\mathbb{P}^*} \left[\frac{X_T}{B(T)} \right]$$

So the choice of Radon–Nikodym derivative guarantees the prices are consistent under both measures and no arbitrage is preserved.

Change of Numeraire Con.

What is the $dS(t)$ looks like under measure $\tilde{\mathbb{P}}$?

Note: Under Q , we have
$$\begin{cases} dS(t) = r(t)S(t)dt + \sigma(t)S(t)dW(t) \\ dN(t) = r(t)N(t)dt + \gamma(t)N(t)dW(t) \end{cases}$$

Denote $\hat{N}_t = \frac{N_t}{B_t}$, apply Itô we get $\frac{d\hat{N}_t}{\hat{N}_t} = \gamma_t dW_t$, $\hat{N}_t = \hat{N}_0 e^{\left(\int_0^t \gamma_s \cdot dW_s - \frac{1}{2} \int_0^t \|\gamma_s\|^2 ds\right)}$.

Observe that $Z_t = \frac{\hat{N}_t}{\hat{N}_0} = e^{\left(\int_0^t \gamma_s \cdot dW_s - \frac{1}{2} \int_0^t \|\gamma_s\|^2 ds\right)}$

Girsanov's theorem says: if we define a new measure $\tilde{\mathbb{P}}$ via this Z_t , then the process

$$\tilde{W}(t) = W(t) - \int_0^t \gamma_s dt$$

is a Brownian motion under $\tilde{\mathbb{P}}$. Substitute into $dS(t)$ to get the $\tilde{\mathbb{P}}$ dynamics:

$$dS(t) = S(t) \left[(r(t) + \sigma(t) \cdot \gamma(t)) dt + \sigma(t) \cdot d\tilde{W}(t) \right]$$

$$S(t) = S_0 \exp \left(\int_0^t \left(r(s) + \sigma(s) \cdot \gamma(s) - \frac{1}{2} \|\sigma(s)\|^2 \right) ds + \int_0^t \sigma(s) \cdot d\tilde{W}(s) \right)$$

Black-Scholes Formula

Given r, σ are constant, we have $S(T) = S(0) \exp \left\{ \left(r - \frac{1}{2} \sigma^2 \right) T + \sigma W(T) \right\}$.

The no-arbitrage price for the call option:

$$\begin{aligned}\psi_c(0) &= \mathbb{E}^{\mathbb{P}^*} (e^{-rT} (S(T) - K)^+) \\ &= \mathbb{E}^{\mathbb{P}^*} (e^{-rT} (S(T) - K) I(S(T) \geq K)) \\ &= \mathbb{E}^{\mathbb{P}^*} (e^{-rT} S(T) I(S(T) \geq K)) - K e^{-rT} \mathbb{P}^*(S(T) \geq K) \\ &= I - K e^{-rT} \cdot II\end{aligned}$$

For II :

$$\begin{aligned}II = \mathbb{P}^*(S(T) \geq K) &= 1 - \Phi \left(\frac{\log(K/S(0)) - (r - \frac{1}{2} \sigma^2) T}{\sigma \sqrt{T}} \right) \\ &= \Phi \left(\frac{\log(S(0)/K) + (r - \frac{1}{2} \sigma^2) T}{\sigma \sqrt{T}} \right)\end{aligned}$$

Note: Φ is the CDF of the standard normal distribution.

Black-Scholes Formula Con.

For I , we apply the change of numeraire and use stock itself as numeraire. Then based on the early definition we have $\left. \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^*} \right|_{\mathcal{F}_T} = Z_T := e^{-rT} \frac{S(T)}{S(0)}$ and $\gamma_t = \sigma$. Therefore, under $\tilde{\mathbb{P}}$ we have the following dynamics of $S(t)$:

$$\frac{dS_t}{S_t} = rdt + \sigma^2 dt + \sigma d\tilde{W}_t, \quad S(t) = S(0) \exp \left\{ (r + \sigma^2/2)t + \sigma \tilde{W}_t \right\}$$

Then we can rewrite I :

$$\begin{aligned} I &= S(0) \mathbb{E}^{\mathbb{P}^*} \left(e^{-rT} \frac{S(T)}{S(0)} I(S(T) \geq K) \right) = S(0) \mathbb{E}^{\tilde{\mathbb{P}}} (I(S(T) \geq K)) \\ &= S(0) \tilde{\mathbb{P}}(S(T) \geq K) \\ &= S(0) \Phi \left(\frac{\log(S(0)/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right). \end{aligned}$$

Putting together, we have the price of the call option is given by:

$$I - Ke^{-rT} \cdot II = S(0)\Phi(d_+) - Ke^{-rT}\Phi(d_-)$$

$$\text{where } d_{\pm} = \frac{\log(S(0)/K) + (r \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

One Dimensional Barrier Options

Barrier options are path-dependent derivatives whose payoff is activated (knock-in) or extinguished (knock-out) if the underlying asset crosses a pre-specified barrier. They extend vanilla calls/puts by adding a barrier condition.

We first study continuously monitored barriers and derive Merton's closed-form pricing formulas (1973) for single-barrier options.

Given $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}^*)$ with 1-dim Brownian W . The Market setting following:

$$dB(t) = B(t)r dt, \quad dS(t) = rS(t)dt + \sigma S(t)dW(t)$$

A continuously monitored barrier option has payoff = vanilla option payoff \times indicator of the barrier condition. For example:

- Up-and-out call:

$$V_0 = \mathbb{E}^{\mathbb{P}^*} \left[e^{-rT} (S(T) - K)^+ I \left\{ \max_{0 \leq t \leq T} S(t) \leq H \right\} \right], \quad H > S(0)$$

- Down-and-in put:

$$V_0 = \mathbb{E}^{\mathbb{P}^*} \left[e^{-rT} (K - S(T))^+ I \left\{ \min_{0 \leq t \leq T} S(t) \leq H \right\} \right], \quad H < S(0)$$

Study the case of the down-and-in call option (DAIC) with strike K , barrier $H < S(0)$:

$$\text{DAIC} = e^{-rT} \mathbb{E}^{\mathbb{P}^*} \left[(S(T) - K)^+ I \left\{ \min_{0 \leq t \leq T} S(t) \leq H \right\} \right]$$

One Dimensional Barrier Options Con.

For notation simplicity, denote a drifted Brownian motion:

$$W_{\mu,\sigma}(t) = \mu t + \sigma W(t), \quad M_t = \max_{0 \leq s \leq t} W_{\mu,\sigma}(s).$$

Some useful results from the reflection principle for a Brownian motion with a drift:

(i) When $x \leq y$, $y > 0$, $\sigma > 0$:

- $P(W_{\mu,\sigma}(t) \leq x, M_t \geq y) = e^{2\mu y/\sigma^2} \Phi\left(\frac{x-2y-\mu t}{\sigma\sqrt{t}}\right)$
- $P(W_{\mu,\sigma}(t) \leq x, M_t \leq y) = \Phi\left(\frac{x-\mu t}{\sigma\sqrt{t}}\right) - e^{2\mu y/\sigma^2} \Phi\left(\frac{x-2y-\mu t}{\sigma\sqrt{t}}\right)$

(ii) When $x \geq y > 0$, $\sigma > 0$:

- $P(W_{\mu,\sigma}(t) \leq x, M_t \leq y) = P(M_t \leq y) = \Phi\left(\frac{y-\mu t}{\sigma\sqrt{t}}\right) - e^{2\mu y/\sigma^2} \Phi\left(\frac{-y-\mu t}{\sigma\sqrt{t}}\right)$
- $P(W_{\mu,\sigma}(t) \leq x, M_t \geq y) = P(W_{\mu,\sigma}(t) \leq x) - P(W_{\mu,\sigma}(t) \leq x, M_t \leq y) = \Phi\left(\frac{x-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{y-\mu t}{\sigma\sqrt{t}}\right) + e^{2\mu y/\sigma^2} \Phi\left(\frac{-y-\mu t}{\sigma\sqrt{t}}\right)$

(iii) When $x \geq y$, $y < 0$, $\sigma > 0$:

- $P\left(W_{\mu,\sigma}(t) \geq x, \min_{0 \leq s \leq t} W_{\mu,\sigma}(s) \leq y\right) = e^{2\mu y/\sigma^2} \Phi\left(\frac{-x+2y+\mu t}{\sigma\sqrt{t}}\right)$

One Dimensional Barrier Options Con.

Back to the valuation of DAIC:

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}^*} \left[e^{-rT} (S(T) - K)^+ I \left(\min_{0 \leq t \leq T} S(t) \leq H \right) \right] \\ &= \mathbb{E}^{\mathbb{P}^*} \left[e^{-rT} (S(T) - K) I \left(S(T) \geq K, \min_{0 \leq t \leq T} S(t) \leq H \right) \right] \\ &= \mathbb{E}^{\mathbb{P}^*} \left[e^{-rT} S(T) I \left(S(T) \geq K, \min_{0 \leq t \leq T} S(t) \leq H \right) \right] \\ &\quad - K e^{-rT} P^* \left(S(T) \geq K, \min_{0 \leq t \leq T} S(t) \leq H \right) \\ &= I - K e^{-rT} \cdot II \end{aligned}$$

For II:

$$\begin{aligned} II &= P^* \left(S(T) \geq K, \min_{0 \leq t \leq T} S(t) \leq H \right) \\ &= P \left\{ W_{r - \frac{\sigma^2}{2}, \sigma}(T) \geq \log(K/S(0)), \min_{0 \leq t \leq T} W_{r - \frac{\sigma^2}{2}, \sigma}(t) \leq \log(H/S(0)) \right\} \\ &= \exp \left\{ \frac{2(r - \sigma^2/2)}{\sigma^2} \log(H/S(0)) \right\} \cdot \Phi \left(\frac{2 \log(H/S(0)) - \log(K/S(0)) + (r - \sigma^2/2)T}{\sigma \sqrt{T}} \right) \end{aligned}$$

One Dimensional Barrier Options Con.

For I, by changing of numeraire we can get:

$$\begin{aligned} I &= S(0) \mathbb{E}^{\mathbb{P}^*} \left(e^{-rT} \frac{S(T)}{S(0)} \cdot I \left\{ S(T) \geq K, \min_{0 \leq t \leq T} S(t) \leq H \right\} \right) \\ &= S(0) \tilde{P} \left(S(T) \geq K, \min_{0 \leq t \leq T} S(t) \leq H \right) \\ &= S(0) P \left\{ W_{r+\frac{\sigma^2}{2}, \sigma}(T) \geq \log(K/S(0)), \min_{0 \leq t \leq T} W_{r+\frac{\sigma^2}{2}, \sigma}(t) \leq \log(H/S(0)) \right\} \\ &= S(0) \cdot (H/S(0))^{\frac{2r}{\sigma^2}+1} \Phi \left(\frac{\log(\{H^2/S(0)\}/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \right) \\ &= (H/S(0))^{\frac{2r}{\sigma^2}-1} (H^2/S(0)) \Phi \left(\frac{\log(\{H^2/S(0)\}/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \right) \end{aligned}$$

Putting the two terms together, we get $I - Ke^{-rT} \cdot II = (H/S(0))^{\frac{2r}{\sigma^2}-1} \text{BSC}(H^2/S(0))$.
Where $\text{BSC}(x)$ is the Black-Scholes formula for a call option with the initial stock price being x :

$$\text{BSC}(x) = x\Phi(d_+) - Ke^{-rT}\Phi(d_-) \text{ with } d_{\pm} = \frac{\log(x/K) + (r \pm \sigma^2/2)T}{\sigma\sqrt{T}}$$

Exchange Options

Given $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}^*)$ with 2-dim independent Brownian, $W_1(t)$ and $W_2(t)$. We have two traded assets $S_1(t)$ and $S_2(t)$ with the following dynamics:

$$\begin{aligned}\frac{dS_1(t)}{S_1(t)} &= rdt + \sigma_1 dW_1(t) \\ \frac{dS_2(t)}{S_2(t)} &= rdt + \sigma_2 \left\{ \rho dW_1(t) + \sqrt{1 - \rho^2} dW_2(t) \right\}\end{aligned}$$

The exchange option gives the holder the right, but not the obligation, to exchange asset S_2 for asset S_1 at maturity T . The price of this option as following:

$$\begin{aligned}u(0) &= \mathbb{E}^{\mathbb{P}^*} \left[e^{-rT} (S_1(T) - S_2(T))^+ \right] \\ &= S_2(0) \mathbb{E}^{\mathbb{P}^*} \left[\frac{e^{-rT} S_2(T)}{S_2(0)} \left(\frac{S_1(T)}{S_2(T)} - 1 \right)^+ \right] \\ &= S_2(0) \mathbb{E}^{\tilde{\mathbb{P}}} \left[\left(\frac{S_1(T)}{S_2(T)} - 1 \right)^+ \right] \\ &= S_2(0) \mathbb{E}^{\tilde{\mathbb{P}}} \left[(F(T) - 1)^+ \right]\end{aligned}$$

Exchange Options Con.

Apply Itô, we have the Radon–Nikodym derivative for numeraire:

$$Z_T := \frac{e^{-rT} S_2(T)}{S_2(0)} = \exp \left[\sigma_2 \left\{ \rho W_1(T) + \sqrt{1 - \rho^2} W_2(T) \right\} - \frac{T}{2} \sigma_2^2 \right]$$

By Girsanov theorem, under new measure $\tilde{\mathbb{P}}$:

$$\tilde{W}_1(t) = W_1(t) - \rho \sigma_2 t, \quad \tilde{W}_2(t) = W_2(t) - \sigma_2 \sqrt{1 - \rho^2} t$$

Apply Itô, we can get $d \ln S_1$, $d \ln S_2$:

$$\begin{aligned} d \ln F(t) &= d \ln S_1(t) - d \ln S_2(t) \\ &= \left[-\frac{1}{2} \sigma_1^2 - \frac{1}{2} \sigma_2^2 + \rho \sigma_1 \sigma_2 \right] dt + (\sigma_1 - \rho \sigma_2) d\tilde{W}_1 - \sigma_2 \sqrt{1 - \rho^2} d\tilde{W}_2. \end{aligned}$$

Apply Itô to $g(x) = e^x$ with $x = \ln F(t)$:

$$\frac{dF_t}{F_t} = d(\ln F_t) + \frac{1}{2} d \langle \ln F \rangle_t = (\sigma_1 - \rho \sigma_2) d\tilde{W}_{1t} - \sigma_2 \sqrt{1 - \rho^2} d\tilde{W}_{2t}$$

Exchange Options Con.

Denote $\sigma = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$, $\tilde{W}(t) := \frac{1}{\sigma} \left\{ (\sigma_1 - \rho\sigma_2)\tilde{W}_1(t) - \sigma_2\sqrt{1-\rho^2}\tilde{W}_2(t) \right\}$

Observe that \tilde{W} is a standard Brownian motion under $\tilde{\mathbb{P}}$. We have $\frac{dF(t)}{F(t)} = \sigma d\tilde{W}(t)$,
observe that $F_T = F_0 \exp\left(-\frac{1}{2}\sigma^2 T + \sigma\sqrt{T}Z\right)$, $Z \sim N(0, 1)$.
Then we can rewrite $u(0)$:

$$\begin{aligned} u(0) &= S_2(0)\mathbb{E}^{\tilde{\mathbb{P}}} \left[(F(T) - 1)^+ \right] \\ &= S_2(0)\mathbb{E}^{\tilde{\mathbb{P}}} \left[(F(T) - 1)I(F(T) > 1) \right] \\ &= S_2(0) \left[\mathbb{E}^{\tilde{\mathbb{P}}} [F_T I\{F_T > 1\}] - \tilde{\mathbb{P}}(F_T > 1) \right] \end{aligned}$$