MF921 Topics in Dynamic Asset Pricing

Week 8

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Chapter 9 Dynamic Money Management Problem

Other Results Related to Utility Maximization

Infinite-Time Horizon HJB Equation

We have two assets:

$$\begin{cases} \frac{dB(t)}{B(t)} = r\,dt, & \text{(risk-free bond)} \\ \frac{dS(t)}{S(t)} = \mu\,dt + \sigma\,dW(t), & \text{(risky stock)} \end{cases}$$

Let $\pi(t)$ be the amount of money invested in the stocks, and $X(t)$ be the wealth. Then we have

$$dX(t) = rX(t)\,dt + \pi(t)\sigma[dW(t) + \theta\,dt] - c(t)\,dt,$$

where

$$\theta = \frac{\mu - r}{\sigma}.$$

and $c(t)$ is the instantaneous consumption process.
A pair (π, c) is admissible if:

$$X(t) \geq 0,\ \forall t \geq 0.$$

This ensures the investor never goes bankrupt or uses arbitrage doubling strategies. Sometimes the constraint is relaxed to: $X(t)$ bounded below by some random variable with finite expectation (technical condition).

Infinite-Time Horizon HJB Equation

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Infinite-Time Horizon HJB Equation

The investor maximizes expected discounted utility of consumption:

$$V(t,x) = \sup_{(\pi,c)\in A} \mathbb{E}\left[\int_t^\infty e^{-\rho(s-t)} U_1(c(s))\,ds \,\middle|\, X(t)=x\right].$$

- $U_1(\cdot)$: instantaneous utility (e.g. CRRA or log)
- $\rho > 0$: subjective discount rate
- $V(t,x)$: value function (max expected utility from time t given wealth x)

Because the horizon is infinite and coefficients are time-homogeneous, the problem is stationary.
Hence:

$$V(t,x) = V(0,x) = V(x),$$

so the HJB becomes an ordinary differential equation (ODE) in x only — not a PDE in (t,x). Only the current wealth x matters, not the calendar time t.

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Derivation of the HJB Equation

Now imagine applying a temporary control (π, c) only during $[t, t+dt]$, and then switching to the optimal policy afterwards. Then we must have

$$\mathbb{E}\left[\int_t^\infty e^{-\rho(s-t)} U_1(c(s))ds \middle| X(t)=x\right] = \mathbb{E}\left[\int_t^{t+dt} e^{-\rho(s-t)} U_1(c(s))ds \middle| X(t)=x\right]$$
$$+\mathbb{E}\left[\mathbb{E}\left\{\int_{t+dt}^\infty e^{-\rho(s-t)} U_1(c(s))ds \middle| X(t+dt)\right\}\middle| X(t)=x\right].$$

For the inner expectation of the second term, we can factor out the constant discount term $e^{-\rho dt}$

$$\mathbb{E}\left\{\int_{t+dt}^\infty e^{-\rho(s-t)} U_1(c(s))ds \middle| X(t+dt)\right\} = e^{-\rho dt}\mathbb{E}\left\{\int_{t+dt}^\infty e^{-\rho(s-t-dt)} U_1(c(s))ds \middle| X(t+dt)\right\}$$
$$= e^{-\rho dt} V(X(t+dt)),$$

where the last equality follows from the optimality of the policy after time $t + dt$.

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where the last equality follows from the optimality of the policy after time $t+dt$.

Derivation of the HJB Equation

Since $V(x)$ is related to the optimal strategies, and the above strategy is only a particular strategy, we must have

$$V(x) \geq \mathbb{E}\left[\int_t^{t+dt} e^{-\rho(s-t)} U_1(c(s))ds + e^{-\rho dt} V(X(t+dt)) \middle| X(t) = x\right],$$

in particular,

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But any optimal strategy can be decomposed to a particular strategy from t to $t+dt$ and optimal strategy after $t+dt$, which implies

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Derivation of the HJB Equation

For the first term in the above equation

$$\mathbb{E}\left[\int_t^{t+dt} e^{-\rho(s-t)} U_1(c(s))ds \middle| X(t) = x\right] = U_1(c(t))dt + o(dt),$$

because as $\Delta t \to 0$

$$\frac{1}{\Delta t}\int_t^{t+\Delta t} e^{-\rho(s-t)} U_1(c(s))ds \to U_1(c(t)).$$

For the second term

$$e^{-\rho dt} V(X(t+dt)) = (1-\rho dt)V(X(t+dt)) + o(dt).$$

Thus, we have

$$V(x) = \sup_{(\pi,c)\in A} \{U_1(c(t))dt + (1-\rho dt)\mathbb{E}[V(X(t+dt))|X(t)=x]\} + o(dt),$$

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or, adding $\rho dt \cdot V(x)$ to both left and right sides leads to

$$\rho dt \cdot V(x) + o(dt) = \sup_{(\pi,c)\in A} \{U_1(c(t))dt + (1-\rho dt)\{\mathbb{E}[V(X(t+dt))|X(t)=x]\}\} - V(x) + \rho dt \cdot V(x)$$

$$= \sup_{(\pi,c)\in A} U_1(c(t))dt + (1-\rho dt)\mathbb{E}[V(X(t+dt)) - V(x)|X(t)=x].$$

Now the only unknown is $\mathbb{E}[V(X(t+dt)) - V(x)|X(t)=x]$.
Recall:

$$dX(t) = [rX(t) + \pi(t)\sigma\theta - c(t)]dt + \pi(t)\sigma dW(t).$$

Then by Itô's formula:

$$dV(X(t)) = \frac{\partial V(X(t))}{\partial X(t)} dX(t) + \frac{1}{2}\frac{\partial^2 V(X(t))}{\partial (X(t))^2}(dX(t))^2.$$

Take expectations conditional on $X(t) = x$:

$$\mathbb{E}[dV(X(t))|X(t)=x] = \left[\frac{\partial V}{\partial x}(rx + \theta\sigma\pi - c) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}\pi^2\sigma^2\right] dt.$$

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Substitute this back into the earlier equation

$$\rho dt\, V(x) + o(dt) = \sup_{(\pi,c)} \left\{ U_1(c)dt + (1-\rho dt)\left[\frac{\partial V}{\partial x}(rx + \theta\sigma\pi - c) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}\pi^2\sigma^2\right] dt \right\}.$$

Divide both sides by dt and let $dt \to 0$:

$$\rho V(x) = \sup_{(\pi,c)} \left\{ U_1(c) + \frac{\partial V}{\partial x}(rx + \theta\sigma\pi - c) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}\pi^2\sigma^2 \right\}.$$

In other words, we have an HJB equation

$$-\rho V(x) + \max_{(\pi,c)} \left\{ U_1(c) + \frac{\partial V}{\partial x}(rx + \theta\sigma\pi - c) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}\pi^2\sigma^2 \right\} = 0,$$

which is an ordinary differential equation.

Derivation of the HJB Equation

Integrating from t to $t+dt$:

$$\mathbb{E}[V(X(t+dt)) - V(x)|X(t)=x] = \mathbb{E}\left[\int_t^{t+dt}\left\{\frac{\partial V}{\partial x}(rX(s)+\theta\sigma\pi - c) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}\pi^2\sigma^2\right\}ds \,\middle|\, X(t)=x\right]$$

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Derivation of the HJB Equation

Define a generic (non-maximized) value function

$$\tilde{V}(x) = \mathbb{E}\left[\int_0^\infty e^{-\rho s} U_1(c(s))\, ds \,\middle|\, X(0) = x\right],$$

where $(c(s), \pi(s))$ are some given controls. So $\tilde{V}$ just measures how good this particular policy is. If we later maximize over all admissible strategies, we recover the true value function:

$$V(x) = \max_{\pi, c} \tilde{V}(x).$$

The Feynman-Kac theorem says: If you have a process $X(t)$ satisfying the SDE

$$dX_t = [rX_t + \pi\sigma\theta - c]\, dt + \pi\sigma\, dW_t,$$

then the expectation

$$\tilde{V}(x) = \mathbb{E}_x\left[\int_0^\infty e^{-\rho s} U_1(c(s))\, ds\right]$$

solves the following linear partial differential equation (or ODE here, since there's only x):

$$-\rho\tilde{V}(x) + U_1(c) + \frac{\partial \tilde{V}}{\partial x}(rx + \theta\sigma\pi - c) + \frac{1}{2}\frac{\partial^2 \tilde{V}}{\partial x^2}\pi^2\sigma^2 = 0.$$

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$\tilde{V}(x)$ as the expected total discounted utility if you keep using the same policy forever. Then, since we're in an infinite-horizon stationary setup, the rate of change of the "expected value" must balance:

$$\text{discounting } (-\rho V) = \text{instantaneous utility} + \text{expected infinitesimal change in value due to wealth dynamics}$$

Now, among all possible controls, the optimal one maximizes this expression at every point x:

$$V(x) = \max_{\pi, c} \tilde{V}(x).$$

So we simply replace $\tilde{V}$ with V and take the maximum inside the equation:

$$\max_{\pi, c} \left\{ -\rho V(x) + U_1(c) + V'(x)(rx + \theta\sigma\pi - c) + \frac{1}{2} V''(x)\pi^2\sigma^2 \right\} = 0.$$

There is no terminal condition, as the time horizon is infinite. This is exactly the Hamilton-Jacobi-Bellman equation we derived earlier using dynamic programming.

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Explicit Solutions of the Infinite-Horizon HJB Equation

Suppose $U_1(x) = x^{1-\gamma}/(1-\gamma), \gamma > 0$. Then the optimization problem becomes

$$V(x) = \sup_{(\pi,c)\in A} \mathbb{E}\left[\int_0^\infty e^{-\rho s}\frac{1}{1-\gamma}(c(s))^{1-\gamma}ds \middle| X(0) = x\right].$$

The HJB equation becomes

$$-\rho V(x) + \max_{(\pi,c)}\left\{\frac{1}{1-\gamma}c^{1-\gamma} + \frac{\partial V}{\partial x}(rx + \theta\sigma\pi - c) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}\pi^2\sigma^2\right\} = 0.$$

Now suppose that we increase the initial wealth by a times, i.e. considering $V(ax)$. Since the wealth equation

$$d(aX(t)) = r(t)aX(t)dt + \pi(t)\sigma(t)[dW(t) + \theta(t)dt] - c(t)dt,$$

hold if π is changed to $a\pi$, and c to ac, and the resulting wealth equation will change from $X(t)$ to $aX(t)$.
In other words, we speculate from first equation that

$$V(ax) = a^{1-\gamma}V(x).$$

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Therefore, we have for some constant g, to be determined later, that

$$V(x) = \frac{g}{1-\gamma} x^{1-\gamma}.$$

Now with this form of $V(x)$ we get

$$\frac{\partial V}{\partial x} = g x^{-\gamma}, \quad \frac{\partial^2 V}{\partial x^2} = g(-\gamma) x^{-\gamma-1},$$

and the HJB equation becomes

$$-\rho g \frac{x^{1-\gamma}}{1-\gamma} + \max_{(\pi,c)} \left\{ \frac{c^{1-\gamma}}{1-\gamma} + g x^{-\gamma}(rx + \theta\sigma\pi - c) - \frac{1}{2} g\gamma x^{-\gamma-1} \pi^2 \sigma^2 \right\} = 0.$$

The first-order condition implies that the maximum is achieved at

$$g x^{-\gamma} \theta\sigma - g\gamma x^{-\gamma-1} \pi\sigma^2 = 0,$$

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Explicit Solutions of the Infinite-Horizon HJB Equation

In other words

$$\pi^* = \frac{x\theta}{\gamma\sigma} = \frac{\mu - r}{\gamma\sigma^2}x, \quad c^* = \{g\}^{-\frac{1}{\gamma}} \cdot x,$$

resulting in a maximal value given by

$$\frac{\gamma}{1-\gamma}\{g\}^{\frac{\gamma-1}{\gamma}} \cdot x^{1-\gamma} + gx^{1-\gamma}r + \frac{1}{2}gx^{1-\gamma}\frac{\theta^2}{\gamma}.$$

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Fortunately, our speculation of the separated solution works out, as the term $x^{1-\gamma}$ cancels out in the above equation, leading to an equation

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or

$$g = \left\{\frac{\rho}{\gamma} - \frac{1-\gamma}{\gamma}\left\{r + \frac{\theta^2}{2\gamma}\right\}\right\}^{-\gamma}.$$

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In summary

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Note that to make sure that $c^*(x) \geq 0$, we need to impose that

$$g \geq 0.$$

This leads to a requirement that

$$\rho \geq (1-\gamma) \left\{ r + \frac{\theta^2}{2\gamma} \right\},$$

unlike in the finite-horizon case where we always have $g(t) \geq 0$. The above constraint is trivially true if $\gamma > 1$, as the right side is negative.

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Explicit Solutions of the Infinite-Horizon HJB Equation

This constraint on the parameter is linked to the transversality condition, which requires that the solution for the infinite horizon problem does not explode to infinity. Indeed, if

$$\rho < (1-\gamma)\left\{r + \frac{\theta^2}{2\gamma}\right\},$$

then h and $g(t,T)$ from the last chapter about the HJB for finite horizon problem are given by

$$h = \frac{(1-\gamma)r}{\gamma} + \frac{(1-\gamma)\theta^2}{2\gamma^2} - \frac{\rho}{\gamma} > 0,$$

$$g(t,T) = \left[\left(1+\frac{1}{h}\right)\exp\{(T-t)h\} - \frac{1}{h}\right]^{\gamma} \to \infty, \quad T \to \infty, \quad \forall t,$$

and the optimal solution for the finite horizon problem goes to infinity,

$$V^*(t,T,x) = g(t,T)\frac{x^{1-\gamma}}{1-\gamma} \to \infty, \quad T \to \infty.$$

That means your expected utility is infinite, the investor could get unbounded lifetime happiness by postponing consumption and letting wealth compound forever. Therefore, to make the infinite-horizon solution well-behaved, we must impose the opposite condition for ρ.

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A Duality Approach to Portfolio Optimization

One risk-free asset (bank account) $B(t)$

$$\frac{dB(t)}{B(t)} = r(t)\,dt,$$

d risky assets (stocks) $S_i(t)$

$$\frac{dS_i(t)}{S_i(t)} = \mu_i(t)\,dt + \sigma_i(t)\,dW(t), \quad i = 1, \ldots, d.$$

Note here all the coefficients could be random, adapted processes.

We assume the investor has a utility function $U(\cdot)$ is strictly increasing and strictly concave, and $U'(0) = +\infty$ and $U'(\infty) = 0$. Examples of this include $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$, with $0 < \gamma < 1.$, and $U(x) = \ln(x)$.

At time 0, the investor chooses a consumption and portfolio strategy $(c(t), \pi(t))$ to maximize expected utility from both consumption and terminal wealth:

$$V(x) = \sup_{(\pi,c)\in\mathcal{A}} \mathbb{E}\left[\int_0^T U_1(c(t))\,dt + U_2(X^\pi(T))\right].$$

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and a starting wealth $X_0 = x > 0$.

The value function $V(t, x)$ satisfies an HJB of the form

$$0 = \sup_{\pi, c} \left\{ U_1(c) + V_t + (r_t x + \pi'(\mu_t - r_t 1) - c) V_x + \frac{1}{2} V_{xx} \pi' \sigma \sigma' \pi \right\},$$

which is difficult to solve when coefficients or dimensions are non-constant. So we look for an alternative formulation that's linear.

Utility maximization problems are concave, and concave optimization can be equivalently studied through convex duality. Instead of maximizing over controls (c, π), we can minimize over a dual variable Y that represents the state-price density.

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A Duality Approach to Portfolio Optimization

In the primal problem we choose consumption and portfolio to affects wealth X_t. In the dual problem we choose a price system Y_t that prices all attainable wealth processes. Every feasible wealth process satisfies the budget constraint

$$\mathbb{E}\left[Y_T X_T + \int_0^T Y_t c_t \, dt\right] \leq x,$$

or every state-price density Y_t (a nonnegative martingale satisfying $Y_0 = 1$ and $Y_t B_t^{-1} S_t$ is a martingale).

Introduce a positive multiplier $y > 0$ for this budget constraint. Define the Lagrangian:

$$\mathcal{L}((\pi, c), y) = \mathbb{E}\left[\int_0^T U_1(c_t)\, dt + U_2(X_T)\right] - y\left(\mathbb{E}\left[\int_0^T Y_t c_t \, dt + Y_T X_T\right] - x\right).$$

Now the original constrained maximization

$$\max_{(\pi,c)} \mathbb{E}[\cdots] \quad \text{s.t. budget constraint}$$

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followed by a minimization over $y > 0$:

$$V(x) = \inf_{y>0} \max_{(\pi,c)} \mathcal{L}((\pi,c),y).$$

Inside the expectation, the Lagrangian separates in c_t and X_T:

$$\mathcal{L}((\pi,c),y) = \mathbb{E}\left[\int_0^T (U_1(c_t) - yY_t c_t)\, dt + (U_2(X_T) - yY_T X_T)\right] + xy.$$

Each term has the generic structure $U(x) - yx$.

For each utility U, define its convex conjugate:

$$\tilde{U}(y) = \max_{x>0}[U(x) - xy].$$

Then

$$U(x) = \min_{y>0}[\tilde{U}(y) + xy],$$

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Substituting this definition into the Lagrangian,

$$\max_{x>0}[U(x) - yx] = \tilde{U}(y),$$

we can carry out the maximization over c_t and X_T immediately:

$$\max_{c_t>0}[U_1(c_t) - yY_t c_t] = \tilde{U}_1(yY_t), \quad \max_{X_T>0}[U_2(X_T) - yY_T X_T] = \tilde{U}_2(yY_T).$$

After optimizing out c_t and X_T, the value becomes

$$V(x) = \inf_{y>0}\left\{\mathbb{E}\left[\int_0^T \tilde{U}_1(yY_t)\,dt + \tilde{U}_2(yY_T)\right] + xy\right\}.$$

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A Duality Approach to Portfolio Optimization

Once you have the optimal y^* and Y_t^*, you can recover the original (primal) variables through the first-order condition

$$U'(c_t^*) = y^* Y_t^*, \quad U'(X_T^*) = y^* Y_T^*.$$

Equivalently,

$$c_t^* = I_1(y^* Y_t^*), \quad X_T^* = I_2(y^* Y_T^*),$$

where $I_i = (U_i')^{-1}$ are the inverse marginal-utility functions.

Review of Option Pricing

Let the state-price process $\xi(t)$ be

$$\xi(t) = \exp\left\{-\int_0^t r(s)ds\right\} Z(t) > 0,$$

where

$$Z(t) := \exp\left\{-\int_0^t \theta^T(s)dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds\right\},$$

$$\theta(t) := \sigma^{-1}(t)[b(t) - r(t)\mathbf{1}],$$

and $\mathbf{1} = (1, 1, \ldots, 1)^\top$.

In other words, $\xi(t)$ is the multiplication of the discount factor and risk-neutral Radon-Nikodym derivative, i.e., the risk-neutral measure is defined as

$$P^0(A) := \mathbb{E}[Z(T)I_A], \quad A \in \mathcal{F}_T.$$

Then by the Itô formula, $\xi(t)$ is the unique solution of the stochastic differential equation

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$$Z(t) := \exp\left\{-\int_0^t \theta^T(s)dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds\right\},$$
$$\theta(t) := \sigma^{-1}(t)[b(t) - r(t)\mathbf{1}],$$

and $\mathbf{1} = (1, 1, \ldots, 1)^\top$.
In other words, $\xi(t)$ is the multiplication of the discount factor and risk-neutral Radon-Nikodym derivative, i.e., the risk-neutral measure is defined as

$$P^0(A) := \mathbb{E}[Z(T)I_A], \quad A \in \mathcal{F}_T.$$

Then by the Itô formula, $\xi(t)$ is the unique solution of the stochastic differential equation

$$\frac{d\xi(t)}{\xi(t)} = -r(t)dt - \theta(t)dW(t), \quad \xi(0) = 1.$$

Review of Option Pricing

The price of a European option with a final payoff $\Psi(T)$ at time T and continuous dividend payment $\psi(t)$ paid continuously to the holder of the option between time 0 and T is given

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$$= \mathbb{E}\left[\exp\left\{-\int_0^T r(s)ds\right\}\Psi(T)Z(T)\right] + \mathbb{E}^0\left[\int_0^T \exp\left\{-\int_0^t r(s)ds\right\}\psi(t)dt\right]$$

$$= \mathbb{E}[\xi(T)\Psi(T)] + \mathbb{E}^0\left[\int_0^T \exp\left\{-\int_0^t r(s)ds\right\}\psi(t)dt\right].$$

Next, we shall show that the second expectation above is

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Indeed, by Fubini's theorem which is applicable because the integrand is nonnegative, we have

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Thus, in terms of the state-price process, the price of a European option with a final payoff $\Psi(T)$ at time T and continuous dividend payment $\psi(t)$ paid continuously to the holder of the option between time 0 and T is given

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In fact, $\xi(t) = 1/S^*(t)$ where $S^*(t)$ is the optimal wealth that puts a fraction θ/σ in stock and $1-\theta/\sigma$ in the money market account, which is exactly the optimal wealth for Kelly criterion.

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Review of Option Pricing

It is easy to check that

$$d(\xi(t)X(t)) = -c(t)\xi(t)dt + \left\{-\xi(t)X(t)\theta(t) + \xi(t)\pi(t)^\top\sigma(t)\right\} dW(t).$$

Therefore

$$\xi(t)X(t) + \int_0^t c(s)\xi(s)ds$$

is a stochastic integral, whence a super-martingale (because it is nonnegative, as $X(t) \geq 0$ and $\xi(t) \geq 0$),

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Upper bound

By the definition $\tilde{U}$ we have $U(x) \leq \tilde{U}(a) + xa$, and therefore,

$$U_1(c(t)) \leq \tilde{U}_1(y\xi(t)) + c(t)y\xi(t)$$

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Hence

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T U_1(c(t))dt + U_2(X(T))\right] \\
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&= \mathbb{E}\left[\int_0^T \tilde{U}_1(y\xi(t))dt + \tilde{U}_2(y\xi(T))\right] + y\mathbb{E}\left[\int_0^T c(t)\xi(t)dt + X(T)\xi(T)\right] \\
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If we define

$$\tilde{V}(y) := \mathbb{E}\left[\int_0^T \tilde{U}_1(y\xi(t))dt + \tilde{U}_2(y\xi(T))\right],$$

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Thus, taking the maximum of the trading and consumption strategies yields

$$V(x) \leq \tilde{V}(y) + xy.$$

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A Related European Option that Attains the Upper Bound

Consider a European option with a final payoff $I_2(y\xi(T))$ and continuous dividend payment $I_1(y\xi(t))$ paid continuously to the holder of the option between time 0 and T. The price of such an option is given by $A(y)$,

$$A(y) := \mathbb{E}\left[\int_0^T \xi(t)I_1(y\xi(t))dt + \xi(T)I_2(y\xi(T))\right].$$

Now choose y^* such that

$$A(y^*) = x.$$

In other words, the price of the option is precisely the initial wealth available to the investor.

Then we claim the optimal consumption strategy is just to consume the continuous dividend amount

$$c^*(t) = I_1(y^*\xi(t)),$$

and the optimal investment portfolio is nothing but the replicating portfolio $\pi^*(t)$ for the option $A(y^*)$, i.e. the replicating portfolio for the wealth process $X^*(t)$,

$$X^*(t) = \frac{1}{\xi(t)}\mathbb{E}\left[\int_t^T \xi(t)I_1(y^*\xi(t))dt + \xi(T)I_2(y^*\xi(T))\middle|\mathcal{F}_t\right].$$

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Note that
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We want to show that
$$\min_y\{\tilde{V}(y) + xy\} = \tilde{V}(y^*) + xy^*. \quad (*)$$

This is a critical step that provides a link between option pricing and utility maximization because y^* comes from matching the initial option price to x, while the left side is from utility maximization. Note that the argmin of $\tilde{V}(y) + xy$ is achieved at
$$\tilde{V}'(y) + x = 0,$$

or
$$\frac{d}{dy}\mathbb{E}\left[\int_0^T \tilde{U}_1(y\xi(t))dt + \tilde{U}_2(y\xi(T))\right] = -x.$$

Assuming the interchangeability of the derivative and expectation, we have
$$\mathbb{E}\left[\int_0^T \xi(t)\tilde{U}_1'(y\xi(t))dt + \xi(T)\tilde{U}_2'(y\xi(T))\right] = -x.$$

In other words,
$$\mathbb{E}\left[\int_0^T \xi(t)I_1(y\xi(t))dt + \xi(T)I_2(y\xi(T))\right] = x,$$

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Using $c^*(t)$ and $\pi^*(t)$ from the option pricing above, we have

$$\mathbb{E}\left[\int_0^T U_1(c^*(t))dt + U_2(X^*(T))\right] = \mathbb{E}\left[\int_0^T U_1(I_1(y^*\xi(t)))dt + U_2(I_2(y^*\xi(T)))\right]$$

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where the second to the last equality follows from $\tilde{U}(y) = U(I(y)) - yI(y)$, and the last equality follows from $A(y^*) = x$.

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The replicating (hedging) strategy of the option with

- Price $A(y^*) = x$,
- Final payoff $I_2(y^*\xi(T)) = X^*(T)$,
- Continuous dividend $I_1(y^*\xi(t)) = c^*(t)$,

is exactly the same as the optimal consumption strategy (π^*, c^*) for the investor with initial wealth x.

Because

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the maximum achievable utility.

So holding that European option and consuming its continuous dividends is equivalent to implementing the dynamic trading strategy that maximizes utility. The two problems coincide in value and structure.

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Remark:
(i) The duality approach points to an interesting connection between option pricing and utility maximization. This can be used to solve the utility maximization problem via Monte Carlo simulation. In particular, we can use Monte Carlo simulation to compute the related option price, which is the same as the maximal value of the utility maximization.

(ii) The dual approach also suggests the role of financial institutions. A small investor might not have the resources or flexibility to continuously trade and rebalance to follow the optimal $\pi^*(t)$. A financial institution, however, can create and hedge such an option efficiently. Therefore, the investor can simply buy the contingent claim that replicates the optimal wealth and consumption stream. In this way, the investor achieves the same utility$V(x)$without managing the portfolio himself.

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