

MF921 Topics in Dynamic Asset Pricing

Stochastic Analysis & Stochastic Calculus in Quantitative Finance

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Week 2

Option Pricing via the Change of Numeraire Argument

Change of Numeraire: Motivation and Key Idea

In option pricing, we usually price under the risk-neutral measure using the money market account $B(t) = e^{rt}$ as the numeraire. But sometimes payoffs become simpler if we change the unit of measurement (the numeraire). Instead of measuring in “dollars,” measure in “shares of stock”.

The key idea is :

- Pick any strictly positive traded asset $N(t)$ as the numeraire.
- Then define a new probability measure $\tilde{\mathbb{P}}$ such that $\frac{S(t)}{N(t)}$ is a martingale under $\tilde{\mathbb{P}}$. No-arbitrage is preserved.

We first look at the details how this work (Radon Nikodym derivative & Girsanov Theorem) and then apply the scheme to price different type of options.

Change of Numeraire

Given $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}^*)$ with d -dim Brownian W :

- Money market account (baseline numeraire): $dB_t = B_t r_t dt$
- Traded asset $S(t)$: $dS_t = S_t(r_t dt + \sigma_t dW_t)$, $\frac{S_t}{B_t}$ is a martingale.
- Derivative pricing rule: for payoff X_T at maturity T , $V_0 = \mathbb{E}^{\mathbb{P}^*} \left[\frac{X_T}{B_T} \right]$

Our goal is to pick another strictly positive traded asset $N(t)$ and define a new measure $\tilde{\mathbb{P}}$ such that $\frac{S(t)}{N(t)}$ is a martingale for every traded asset $S(t)$.

Change of Numeraire Con.

Observe $\frac{S(t)}{B(t)}$ is a martingale under \mathbb{P}^* . We want $\frac{S(t)}{N(t)}$ to be a martingale under $\tilde{\mathbb{P}}$.

Define $\tilde{\mathbb{P}}$ via the Radon–Nikodym derivative with respect to \mathbb{P}^* :

$$\left. \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^*} \right|_{\mathcal{F}_T} = Z_T := \frac{N(T)/B(T)}{N(0)/B(0)}$$

By construction, $\frac{N(T)}{B(T)}$ is a martingale under \mathbb{P}^* , $Z_T > 0$ and $\mathbb{E}^{\mathbb{P}^*}[Z_T] = 1$ and take any payoff X_T :

$$V(0) = N(0) \mathbb{E}^{\tilde{\mathbb{P}}} \left[\frac{X_T}{N(T)} \right] = N(0) \mathbb{E}^{\mathbb{P}^*} \left[\frac{X_T}{N(T)} Z_T \right] = \mathbb{E}^{\mathbb{P}^*} \left[\frac{X_T}{B(T)} \right]$$

So the choice of Radon–Nikodym derivative guarantees the prices are consistent under both measures and no arbitrage is preserved.

Change of Numeraire Con.

What is the $dS(t)$ looks like under measure $\tilde{\mathbb{P}}$?

Note: Under Q , we have
$$\begin{cases} dS(t) = r(t)S(t) dt + \sigma(t)S(t) dW(t) \\ dN(t) = r(t)N(t) dt + \gamma(t)N(t) dW(t) \end{cases}$$

Denote $\hat{N}_t = \frac{N_t}{B_t}$, apply Itô we get $\frac{d\hat{N}_t}{\hat{N}_t} = \gamma_t dW_t$, $\hat{N}_t = \hat{N}_0 e^{\left(\int_0^t \gamma_s \cdot dW_s - \frac{1}{2} \int_0^t \|\gamma_s\|^2 ds\right)}$.

Observe that $Z_t = \frac{\hat{N}_t}{\hat{N}_0} = e^{\left(\int_0^t \gamma_s \cdot dW_s - \frac{1}{2} \int_0^t \|\gamma_s\|^2 ds\right)}$

Girsanov's theorem says: if we define a new measure $\tilde{\mathbb{P}}$ via this Z_t , then the process

$$\tilde{W}(t) = W(t) - \int_0^t \gamma_s dt$$

is a Brownian motion under $\tilde{\mathbb{P}}$. Substitute into $dS(t)$ to get the $\tilde{\mathbb{P}}$ dynamics:

$$dS(t) = S(t) \left[(r(t) + \sigma(t) \cdot \gamma(t)) dt + \sigma(t) \cdot d\tilde{W}(t) \right]$$

$$S(t) = S_0 \exp \left(\int_0^t \left(r(s) + \sigma(s) \cdot \gamma(s) - \frac{1}{2} \|\sigma(s)\|^2 \right) ds + \int_0^t \sigma(s) \cdot d\tilde{W}(s) \right)$$

Black-Scholes Formula

Given r, σ are constant, we have $S(T) = S(0) \exp \left\{ \left(r - \frac{1}{2} \sigma^2 \right) T + \sigma W(T) \right\}$.

The no-arbitrage price for the call option:

$$\begin{aligned}\psi_c(0) &= \mathbb{E}^{\mathbb{P}^*} (e^{-rT} (S(T) - K)^+) \\ &= \mathbb{E}^{\mathbb{P}^*} (e^{-rT} (S(T) - K) I(S(T) \geq K)) \\ &= \mathbb{E}^{\mathbb{P}^*} (e^{-rT} S(T) I(S(T) \geq K)) - K e^{-rT} \mathbb{P}^*(S(T) \geq K) \\ &= I - K e^{-rT} \cdot II\end{aligned}$$

For II :

$$\begin{aligned}II = \mathbb{P}^*(S(T) \geq K) &= 1 - \Phi \left(\frac{\log(K/S(0)) - (r - \frac{1}{2} \sigma^2) T}{\sigma \sqrt{T}} \right) \\ &= \Phi \left(\frac{\log(S(0)/K) + (r - \frac{1}{2} \sigma^2) T}{\sigma \sqrt{T}} \right)\end{aligned}$$

Note: Φ is the CDF of the standard normal distribution.

Black-Scholes Formula Con.

For I , we apply the change of numeraire and use stock itself as numeraire. Then based on the early definition we have $\left. \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^*} \right|_{\mathcal{F}_T} = Z_T := e^{-rT} \frac{S(T)}{S(0)}$ and $\gamma_t = \sigma$. Therefore, under $\tilde{\mathbb{P}}$ we have the following dynamics of $S(t)$:

$$\frac{dS_t}{S_t} = rdt + \sigma^2 dt + \sigma d\tilde{W}_t, \quad S(t) = S(0) \exp \left\{ (r + \sigma^2/2)t + \sigma \tilde{W}_t \right\}$$

Then we can rewrite I :

$$\begin{aligned} I &= S(0) \mathbb{E}^{\mathbb{P}^*} \left(e^{-rT} \frac{S(T)}{S(0)} I(S(T) \geq K) \right) = S(0) \mathbb{E}^{\tilde{\mathbb{P}}} (I(S(T) \geq K)) \\ &= S(0) \tilde{\mathbb{P}}(S(T) \geq K) \\ &= S(0) \Phi \left(\frac{\log(S(0)/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right). \end{aligned}$$

Putting together, we have the price of the call option is given by:

$$I - Ke^{-rT} \cdot II = S(0)\Phi(d_+) - Ke^{-rT}\Phi(d_-)$$

$$\text{where } d_{\pm} = \frac{\log(S(0)/K) + (r \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

One Dimensional Barrier Options

Barrier options are path-dependent derivatives whose payoff is activated (knock-in) or extinguished (knock-out) if the underlying asset crosses a pre-specified barrier. They extend vanilla calls/puts by adding a barrier condition.

We first study continuously monitored barriers and derive Merton's closed-form pricing formulas (1973) for single-barrier options.

Given $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}^*)$ with 1-dim Brownian W . The Market setting following:

$$dB(t) = B(t)r dt, \quad dS(t) = rS(t)dt + \sigma S(t)dW(t)$$

A continuously monitored barrier option has payoff = vanilla option payoff \times indicator of the barrier condition. For example:

- Up-and-out call:

$$V_0 = \mathbb{E}^{\mathbb{P}^*} \left[e^{-rT} (S(T) - K)^+ I \left\{ \max_{0 \leq t \leq T} S(t) \leq H \right\} \right], \quad H > S(0)$$

- Down-and-in put:

$$V_0 = \mathbb{E}^{\mathbb{P}^*} \left[e^{-rT} (K - S(T))^+ I \left\{ \min_{0 \leq t \leq T} S(t) \leq H \right\} \right], \quad H < S(0)$$

Study the case of the down-and-in call option (DAIC) with strike K , barrier $H < S(0)$:

$$\text{DAIC} = e^{-rT} \mathbb{E}^{\mathbb{P}^*} \left[(S(T) - K)^+ I \left\{ \min_{0 \leq t \leq T} S(t) \leq H \right\} \right]$$

One Dimensional Barrier Options Con.

For notation simplicity, denote a drifted Brownian motion:

$$W_{\mu,\sigma}(t) = \mu t + \sigma W(t), \quad M_t = \max_{0 \leq s \leq t} W_{\mu,\sigma}(s).$$

Some useful results from the reflection principle for a Brownian motion with a drift:

(i) When $x \leq y$, $y > 0$, $\sigma > 0$:

- $P(W_{\mu,\sigma}(t) \leq x, M_t \geq y) = e^{2\mu y/\sigma^2} \Phi\left(\frac{x-2y-\mu t}{\sigma\sqrt{t}}\right)$
- $P(W_{\mu,\sigma}(t) \leq x, M_t \leq y) = \Phi\left(\frac{x-\mu t}{\sigma\sqrt{t}}\right) - e^{2\mu y/\sigma^2} \Phi\left(\frac{x-2y-\mu t}{\sigma\sqrt{t}}\right)$

(ii) When $x \geq y > 0$, $\sigma > 0$:

- $P(W_{\mu,\sigma}(t) \leq x, M_t \leq y) = P(M_t \leq y) = \Phi\left(\frac{y-\mu t}{\sigma\sqrt{t}}\right) - e^{2\mu y/\sigma^2} \Phi\left(\frac{-y-\mu t}{\sigma\sqrt{t}}\right)$
- $P(W_{\mu,\sigma}(t) \leq x, M_t \geq y) = P(W_{\mu,\sigma}(t) \leq x) - P(W_{\mu,\sigma}(t) \leq x, M_t \leq y) = \Phi\left(\frac{x-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{y-\mu t}{\sigma\sqrt{t}}\right) + e^{2\mu y/\sigma^2} \Phi\left(\frac{-y-\mu t}{\sigma\sqrt{t}}\right)$

(iii) When $x \geq y$, $y < 0$, $\sigma > 0$:

- $P\left(W_{\mu,\sigma}(t) \geq x, \min_{0 \leq s \leq t} W_{\mu,\sigma}(s) \leq y\right) = e^{2\mu y/\sigma^2} \Phi\left(\frac{-x+2y+\mu t}{\sigma\sqrt{t}}\right)$

One Dimensional Barrier Options Con.

Back to the valuation of DAIC:

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}^*} \left[e^{-rT} (S(T) - K)^+ I \left(\min_{0 \leq t \leq T} S(t) \leq H \right) \right] \\ &= \mathbb{E}^{\mathbb{P}^*} \left[e^{-rT} (S(T) - K) I \left(S(T) \geq K, \min_{0 \leq t \leq T} S(t) \leq H \right) \right] \\ &= \mathbb{E}^{\mathbb{P}^*} \left[e^{-rT} S(T) I \left(S(T) \geq K, \min_{0 \leq t \leq T} S(t) \leq H \right) \right] \\ &\quad - K e^{-rT} P^* \left(S(T) \geq K, \min_{0 \leq t \leq T} S(t) \leq H \right) \\ &= I - K e^{-rT} \cdot II \end{aligned}$$

For II:

$$\begin{aligned} II &= P^* \left(S(T) \geq K, \min_{0 \leq t \leq T} S(t) \leq H \right) \\ &= P \left\{ W_{r - \frac{\sigma^2}{2}, \sigma}(T) \geq \log(K/S(0)), \min_{0 \leq t \leq T} W_{r - \frac{\sigma^2}{2}, \sigma}(t) \leq \log(H/S(0)) \right\} \\ &= \exp \left\{ \frac{2(r - \sigma^2/2)}{\sigma^2} \log(H/S(0)) \right\} \cdot \Phi \left(\frac{2 \log(H/S(0)) - \log(K/S(0)) + (r - \sigma^2/2)T}{\sigma \sqrt{T}} \right) \end{aligned}$$

One Dimensional Barrier Options Con.

For I, by changing of numeraire we can get:

$$\begin{aligned} I &= S(0) \mathbb{E}^{\mathbb{P}^*} \left(e^{-rT} \frac{S(T)}{S(0)} \cdot I \left\{ S(T) \geq K, \min_{0 \leq t \leq T} S(t) \leq H \right\} \right) \\ &= S(0) \tilde{P} \left(S(T) \geq K, \min_{0 \leq t \leq T} S(t) \leq H \right) \\ &= S(0) P \left\{ W_{r+\frac{\sigma^2}{2}, \sigma}(T) \geq \log(K/S(0)), \min_{0 \leq t \leq T} W_{r+\frac{\sigma^2}{2}, \sigma}(t) \leq \log(H/S(0)) \right\} \\ &= S(0) \cdot (H/S(0))^{\frac{2r}{\sigma^2}+1} \Phi \left(\frac{\log(\{H^2/S(0)\}/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \right) \\ &= (H/S(0))^{\frac{2r}{\sigma^2}-1} (H^2/S(0)) \Phi \left(\frac{\log(\{H^2/S(0)\}/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \right) \end{aligned}$$

Putting the two terms together, we get $I - Ke^{-rT} \cdot II = (H/S(0))^{\frac{2r}{\sigma^2}-1} \text{BSC}(H^2/S(0))$.
Where $\text{BSC}(x)$ is the Black-Scholes formula for a call option with the initial stock price being x :

$$\text{BSC}(x) = x\Phi(d_+) - Ke^{-rT}\Phi(d_-) \text{ with } d_{\pm} = \frac{\log(x/K) + (r \pm \sigma^2/2)T}{\sigma\sqrt{T}}$$

Exchange Options

Given $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}^*)$ with 2-dim independent Brownian, $W_1(t)$ and $W_2(t)$. We have two traded assets $S_1(t)$ and $S_2(t)$ with the following dynamics:

$$\begin{aligned}\frac{dS_1(t)}{S_1(t)} &= rdt + \sigma_1 dW_1(t) \\ \frac{dS_2(t)}{S_2(t)} &= rdt + \sigma_2 \left\{ \rho dW_1(t) + \sqrt{1 - \rho^2} dW_2(t) \right\}\end{aligned}$$

The exchange option gives the holder the right, but not the obligation, to exchange asset S_2 for asset S_1 at maturity T . The price of this option as following:

$$\begin{aligned}u(0) &= \mathbb{E}^{\mathbb{P}^*} \left[e^{-rT} (S_1(T) - S_2(T))^+ \right] \\ &= S_2(0) \mathbb{E}^{\mathbb{P}^*} \left[\frac{e^{-rT} S_2(T)}{S_2(0)} \left(\frac{S_1(T)}{S_2(T)} - 1 \right)^+ \right] \\ &= S_2(0) \mathbb{E}^{\tilde{\mathbb{P}}} \left[\left(\frac{S_1(T)}{S_2(T)} - 1 \right)^+ \right] \\ &= S_2(0) \mathbb{E}^{\tilde{\mathbb{P}}} \left[(F(T) - 1)^+ \right]\end{aligned}$$

Exchange Options Con.

Apply Itô, we have the Radon–Nikodym derivative for numeraire:

$$\left. \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^*} \right|_{\mathcal{F}_T} = Z_T^2 := \frac{e^{-rT} S_2(T)}{S_2(0)} = \exp \left[\sigma_2 \left\{ \rho W_1(T) + \sqrt{1 - \rho^2} W_2(T) \right\} - \frac{T}{2} \sigma_2^2 \right]$$
$$\left. \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^*} \right|_{\mathcal{F}_T} = Z_T^1 := \frac{e^{-rT} S_1(T)}{S_1(0)} = \frac{e^{-rT} S_1(T)}{S_1(0)} = \exp \left(\sigma_1 W_1(T) - \frac{1}{2} \sigma_1^2 T \right)$$

By Girsanov theorem, under new measure $\tilde{\mathbb{P}}$:

$$\tilde{W}_1(t) = W_1(t) - \rho \sigma_2 t, \quad \tilde{W}_2(t) = W_2(t) - \sigma_2 \sqrt{1 - \rho^2} t$$

Apply Itô, we can get $d \ln S_1$, $d \ln S_2$:

$$\begin{aligned} d \ln F(t) &= d \ln S_1(t) - d \ln S_2(t) \\ &= \left[-\frac{1}{2} \sigma_1^2 - \frac{1}{2} \sigma_2^2 + \rho \sigma_1 \sigma_2 \right] dt + (\sigma_1 - \rho \sigma_2) d\tilde{W}_1 - \sigma_2 \sqrt{1 - \rho^2} d\tilde{W}_2. \end{aligned}$$

Apply Itô to $g(x) = e^x$ with $x = \ln F(t)$:

$$\frac{dF_t}{F_t} = d(\ln F_t) + \frac{1}{2} d \langle \ln F \rangle_t = (\sigma_1 - \rho \sigma_2) d\tilde{W}_{1t} - \sigma_2 \sqrt{1 - \rho^2} d\tilde{W}_{2t}$$

Exchange Options Con.

Denote $\sigma = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$, $\tilde{W}(t) := \frac{1}{\sigma} \left\{ (\sigma_1 - \rho\sigma_2)\tilde{W}_1(t) - \sigma_2\sqrt{1-\rho^2}\tilde{W}_2(t) \right\}$

Observe that \tilde{W} is a standard Brownian motion under $\tilde{\mathbb{P}}$. We have $\frac{dF(t)}{F(t)} = \sigma d\tilde{W}(t)$, observe that $F_T = F_0 \exp\left(-\frac{1}{2}\sigma^2 T + \sigma\sqrt{T}Z\right)$, $Z \sim N(0, 1)$ under $\tilde{\mathbb{P}}$. Similarly, we have $F_T = F_0 \exp\left(\frac{1}{2}\sigma^2 T + \sigma\sqrt{T}Z\right)$, $Z \sim N(0, 1)$ under $\hat{\mathbb{P}}$.

Then we can rewrite $u(0)$:

$$\begin{aligned} u(0) &= S_2(0)\mathbb{E}^{\tilde{\mathbb{P}}}[(F(T) - 1)^+] \\ &= S_2(0)\mathbb{E}^{\tilde{\mathbb{P}}}[(F(T) - 1)I(F(T) > 1)] \\ &= S_2(0)\left[\mathbb{E}^{\tilde{\mathbb{P}}}[F_T I\{F_T > 1\}] - \tilde{\mathbb{P}}(F_T > 1)\right] \\ &= S_2(0)\left[\mathbb{E}^{\mathbb{P}^*}\left[\frac{e^{-rT}S_2(T)}{S_2(0)}\frac{S_1(T)}{S_2(T)}I\{F_T > 1\}\right] - \tilde{\mathbb{P}}(F_T > 1)\right] \\ &= S_2(0)\left[\frac{1}{S_2(0)}\mathbb{E}^{\mathbb{P}^*}\left[\frac{e^{-rT}S_1(T)}{S_1(0)}S_1(0)I\{F_T > 1\}\right] - \tilde{\mathbb{P}}(F_T > 1)\right] \end{aligned}$$

Exchange Options Con.

$$\begin{aligned} &= S_2(0) \left[\frac{S_1(0)}{S_2(0)} \mathbb{E}^{\hat{\mathbb{P}}} [I\{F_T > 1\}] - \tilde{\mathbb{P}}(F_T > 1) \right] \\ &= S_1(0) \hat{\mathbb{P}}[I\{F_T > 1\}] - S_2(0) \tilde{\mathbb{P}}(F_T > 1) \\ &= S_1(0) \Phi(d_+) - S_2(0) \Phi(d_-) \end{aligned}$$

Where:

$$d_{\pm} = \frac{\log(F(0)) \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = \frac{\log(S_1(0)/S_2(0)) \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}.$$

- (i) If the second asset is cash, or $S_2(t) = Ke^{-r(T-t)}$, then the formula degenerates to the Black-Scholes formula.
- (ii) The hedging strategy is given by long $\Phi(d_+)$ shares of the first asset and short $\Phi(d_-)$ shares of the second asset.

Two-Dimensional Barrier Options

Suppose we have two Wiener processes, $X(t)$ and $Y(t)$, governed by the following dynamics

$$dX(t) = \mu_1 dt + \sigma_1 dW_1(t), \quad X(0) = 0, \quad \sigma_1 > 0,$$

$$dY(t) = \mu_2 dt + \sigma_2 \left\{ \rho dW_1(t) + \sqrt{1 - \rho^2} dW_2(t) \right\}, \quad Y(0) = 0, \quad \sigma_2 > 0,$$

where the W_1 and W_2 are two independent standard Brownian motions.

For $b > 0$, consider the first passage time of the process $Y(t)$:

$$\tau_b^Y = \inf\{t \geq 0 : Y(t) = b > 0\}.$$

We shall prove that the joint distribution between $X(T)$ and the first passage time of $Y(t)$ is given by:

$$\begin{aligned} P(X(T) < a, \tau_b^Y > T) &= P\left(X(T) < a, \max_{0 \leq t \leq T} Y(t) < b\right) \\ &= \Phi_2\left(\frac{a - \mu_1 T}{\sigma_1 \sqrt{T}}, \frac{b - \mu_2 T}{\sigma_2 \sqrt{T}}; \rho\right) - e^{2\mu_2 b / \sigma_2^2} \Phi_2\left(\frac{a - \mu_1 T - 2\rho b \sigma_1 / \sigma_2}{\sigma_1 \sqrt{T}}, \frac{-b - \mu_2 T}{\sigma_2 \sqrt{T}}; \rho\right) \end{aligned}$$

where $b > 0$ and $\Phi_2(x, y; \rho)$ denotes the bivariate normal distribution given by

$$\Phi_2(x, y; \rho) = P(Z_1 \leq x, Z_2 \leq y),$$

with Z_1 and Z_2 being two standard normal random variables with correlation ρ .

Two-Dimensional Barrier Options Con.

Remark:

- Above equation holds for both $a \geq b$ and $a \leq b$, as long as $b > 0$. That's more general than the 1D reflection principle formulas which needed to be split into separate cases depending on $a \leq b$ or $a \geq b$.
- when $\rho = 1$, $\mu_1 = \mu_2 = \mu$, $\sigma_1 = \sigma_2 = \sigma$, the two dimensional case reduces to the one-dimensional case, as it becomes:

$$\begin{aligned} & P \left(X(T) < a, \max_{0 \leq t \leq T} X(t) < b \right) \\ &= \Phi_2 \left(\frac{a - \mu T}{\sigma \sqrt{T}}, \frac{b - \mu T}{\sigma \sqrt{T}}; 1 \right) - e^{2\mu b / \sigma^2} \Phi_2 \left(\frac{a - \mu T - 2b}{\sigma \sqrt{T}}, \frac{-b - \mu T}{\sigma \sqrt{T}}; 1 \right) \\ &= P \left\{ Z \leq \frac{a - \mu T}{\sigma \sqrt{T}}, Z \leq \frac{b - \mu T}{\sigma \sqrt{T}} \right\} - e^{2\mu b / \sigma^2} P \left\{ Z \leq \frac{a - \mu T - 2b}{\sigma \sqrt{T}}, Z \leq \frac{-b - \mu T}{\sigma \sqrt{T}} \right\} \\ &= P \left\{ Z \leq \min \left\{ \frac{a - \mu T}{\sigma \sqrt{T}}, \frac{b - \mu T}{\sigma \sqrt{T}} \right\} \right\} - e^{2\mu b / \sigma^2} P \left\{ Z \leq \min \left\{ \frac{a - \mu T - 2b}{\sigma \sqrt{T}}, \frac{-b - \mu T}{\sigma \sqrt{T}} \right\} \right\} \end{aligned}$$

Which incorporates two cases in one dimensional case.

Two-Dimensional Barrier Options Con.

Next we proof the formula of the joint distribution between $X(T)$ and the first passage time of $Y(t)$:

[Proof]

Consider the case of $\sigma_1 = \sigma_2 = 1$. Define a new process $V(t)$ to decouple X and Y :

$$V(t) := X(t) - \rho Y(t)$$

First check independence between V and Y :

$$\begin{aligned} dV(t)dY(t) &= (dX(t) - \rho dY(t))dY(t) \\ &= \left((1 - \rho^2)dW_1 - \rho\sqrt{1 - \rho^2}dW_2 \right) \cdot \left(\rho dW_1 + \sqrt{1 - \rho^2}dW_2 \right) \\ &= (1 - \rho^2)\rho(dW_1)^2 - \rho(1 - \rho^2)(dW_2)^2 \\ &= (1 - \rho^2)\rho dt - (1 - \rho^2)\rho dt = 0 \end{aligned}$$

Since $V(T) = X(T) - \rho Y(T)$, it is Gaussian. Its mean is:

$$\mathbb{E}[V(T)] = \mu_1 T - \rho \mu_2 T$$

Two-Dimensional Barrier Options Con.

Its variance is:

$$\begin{aligned}\text{Var}(V(T)) &= \text{Var}(X(T)) + \rho^2 \text{Var}(Y(T)) - 2\rho \text{Cov}(X(T), Y(T)) \\ &= T + \rho^2 T - 2\rho^2 T = (1 - \rho^2)T\end{aligned}$$

Thus:

$$V(T) \sim N((\mu_1 - \rho\mu_2)T, (1 - \rho^2)T).$$

Incidentally, the same logic applying to two standard normal random variables with correlation ρ also leads to a representation for the bivariate normal distribution:

$$\Phi_2(\alpha, \beta; \rho) = \int_{z_2=-\infty}^{\beta} \int_{z_1=-\infty}^{\alpha} \frac{1}{\sqrt{1-\rho^2}} \varphi\left(\frac{z_1 - \rho z_2}{\sqrt{1-\rho^2}}\right) \varphi(z_2) dz_1 dz_2.$$

Where $\varphi(\cdot)$ is the standard normal density function, $\varphi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)$.

Two-Dimensional Barrier Options Con.

Now, in terms of $V(T)$, we can rewrite $P(X(T) < a, \tau_b^Y > T)$ as:

$$\begin{aligned} &P(X(T) < a, \tau_b^Y > T) \\ &= \int_{x=-\infty}^a \int_{y=-\infty}^b P(X(T) \in dx, Y(T) \in dy, \tau_b^Y > T) \end{aligned}$$

Note: the transformation is linear with determinant 1 and the independence of V and Y

$$= \int_{x=-\infty}^a \int_{y=-\infty}^b P(V(T) \in dx - \rho dy) P(Y(T) \in dy, \tau_b^Y > T)$$

There are two terms inside the integrand. For the first term since $V(T)$ has a normal distribution with mean $\mu_1 T - \rho \mu_2 T$ and variance $(1 - \rho^2)T$, we have:

$$P(V(T) \in dx - \rho dy) = \frac{1}{\sqrt{(1 - \rho^2)T}} \varphi \left(\frac{x - \rho y - \mu_1 T + \rho \mu_2 T}{\sqrt{(1 - \rho^2)T}} \right),$$

Two-Dimensional Barrier Options Con.

For the second term, recall the early result in one-dim, When $x \leq y$, $y > 0, \sigma > 0$:

$$P(W_{\mu,\sigma}(t) \leq x, M_t \leq y) = \Phi\left(\frac{x - \mu t}{\sigma\sqrt{t}}\right) - e^{2\mu y/\sigma^2} \Phi\left(\frac{x - 2y - \mu t}{\sigma\sqrt{t}}\right)$$

We have for all $y < b, b > 0$:

$$P(Y(T) \leq y, \tau_b^Y > T) = \Phi\left(\frac{y - \mu_2 T}{\sqrt{T}}\right) - e^{2\mu_2 b} \Phi\left(\frac{y - 2b - \mu_2 T}{\sqrt{T}}\right).$$

Differentiating the above equation yields:

$$P(Y(T) \in dy, \tau_b^Y > T) = \frac{1}{\sqrt{T}} \varphi\left(\frac{y - \mu_2 T}{\sqrt{T}}\right) - \frac{1}{\sqrt{T}} e^{2\mu_2 b} \varphi\left(\frac{y - 2b - \mu_2 T}{\sqrt{T}}\right).$$

Plugging the above two terms into:

$$\int_{x=-\infty}^a \int_{y=-\infty}^b P(V(T) \in dx - \rho dy) P(Y(T) \in dy, \tau_b^Y > T)$$

Two-Dimensional Barrier Options Con.

$$P(X(T) < a, \tau_b^Y > T) = I - II$$

where:

$$I = \int_{x=-\infty}^a \int_{y=-\infty}^b \frac{1}{\sqrt{(1-\rho^2)T}} \varphi\left(\frac{x - \rho y - \mu_1 T + \rho \mu_2 T}{\sqrt{(1-\rho^2)T}}\right) \frac{1}{\sqrt{T}} \varphi\left(\frac{y - \mu_2 T}{\sqrt{T}}\right) dy dx,$$

$$II = e^{2\mu_2 b} \int_{x=-\infty}^a \int_{y=-\infty}^b \frac{1}{\sqrt{(1-\rho^2)T}} \varphi\left(\frac{x - \rho y - \mu_1 T + \rho \mu_2 T}{\sqrt{(1-\rho^2)T}}\right) \frac{1}{\sqrt{T}} \varphi\left(\frac{y - 2b - \mu_2 T}{\sqrt{T}}\right) dy dx,$$

and $\varphi(x) = e^{-x^2/2}/\sqrt{2\pi}$ is the standard normal density function.

With $\tilde{x} = \frac{x - \mu_1 T}{\sqrt{T}}$, $\tilde{y} = \frac{y - \mu_2 T}{\sqrt{T}}$, Then:

$$dx = \sqrt{T} d\tilde{x}, \quad dy = \sqrt{T} d\tilde{y}$$

$$x \leq a \iff \tilde{x} \leq \frac{a - \mu_1 T}{\sqrt{T}}, \quad y \leq b \iff \tilde{y} \leq \frac{b - \mu_2 T}{\sqrt{T}}$$

Two-Dimensional Barrier Options Con.

we have

$$\begin{aligned} I &= \int_{x=-\infty}^a \int_{y=-\infty}^b \frac{1}{\sqrt{(1-\rho^2)T}} \varphi\left(\frac{\tilde{x} - \rho\tilde{y}}{\sqrt{(1-\rho^2)}}\right) \frac{1}{\sqrt{T}} \varphi(\tilde{y}) dy dx \\ &= \int_{-\infty}^{\frac{a-\mu_1 T}{\sqrt{T}}} \int_{-\infty}^{\frac{b-\mu_2 T}{\sqrt{T}}} \frac{1}{\sqrt{(1-\rho^2)}} \varphi\left(\frac{\tilde{x} - \rho\tilde{y}}{\sqrt{(1-\rho^2)}}\right) \varphi(\tilde{y}) d\tilde{y} d\tilde{x} \end{aligned}$$

By the conditional-Gaussian factorization, this integrand is exactly the joint pdf of a standard bivariate normal (Z_1, Z_2) with correlation ρ :

$$f_{Z_1, Z_2}(\tilde{x}, \tilde{y}) = \frac{1}{\sqrt{1-\rho^2}} \varphi\left(\frac{\tilde{x} - \rho\tilde{y}}{\sqrt{1-\rho^2}}\right) \varphi(\tilde{y})$$

Hence the double integral is, by definition,

$$I = \Phi_2\left(\frac{a-\mu_1 T}{\sqrt{T}}, \frac{b-\mu_2 T}{\sqrt{T}}; \rho\right)$$

Two-Dimensional Barrier Options Con.

Similarly, with

$$\hat{x} = \frac{x - \mu_1 T - 2\rho b}{\sqrt{T}}, \quad \hat{y} = \frac{y - 2b - \mu_2 T}{\sqrt{T}}$$

simplifying the term II yields

$$\begin{aligned} II &= \int_{x=-\infty}^a \int_{y=-\infty}^b \frac{1}{\sqrt{(1-\rho^2)T}} \varphi\left(\frac{\hat{x} - \rho\hat{y}}{\sqrt{(1-\rho^2)}}\right) \frac{1}{\sqrt{T}} e^{2\mu_2 b} \varphi(\hat{y}) dy dx \\ &= \int_{-\infty}^{\frac{a - \mu_1 T - 2\rho b}{\sqrt{T}}} \int_{-\infty}^{\frac{-b - \mu_2 T}{\sqrt{T}}} \frac{1}{\sqrt{(1-\rho^2)}} \varphi\left(\frac{\hat{x} - \rho\hat{y}}{\sqrt{(1-\rho^2)}}\right) e^{2\mu_2 b} \varphi(\hat{y}) dy dx \\ &= e^{2\mu_2 b} \Phi_2\left(\frac{a - \mu_1 T - 2\rho b}{\sqrt{T}}, \frac{-b - \mu_2 T}{\sqrt{T}}; \rho\right), \end{aligned}$$

from which the result follows. The general case can be reduced to this particular case by letting:

$$\begin{aligned} \tilde{X}(t) &= X(t)/\sigma_1, \quad \tilde{Y}(t) = Y(t)/\sigma_2, \\ \tilde{b} &= b/\sigma_2, \quad \tilde{a} = a/\sigma_1, \quad \tilde{\mu}_1 = \mu_1/\sigma_1, \quad \tilde{\mu}_2 = \mu_2/\sigma_2. \end{aligned}$$

Two-Dimensional Barrier Options Con.

Given the joint distribution between $X(T)$ and the first passage time of $Y(t)$ by :

$$\begin{aligned} P(X(T) < a, \tau_b^Y > T) &= P\left(X(T) < a, \max_{0 \leq t \leq T} Y(t) < b\right) \\ &= \Phi_2\left(\frac{a - \mu_1 T}{\sigma_1 \sqrt{T}}, \frac{b - \mu_2 T}{\sigma_2 \sqrt{T}}; \rho\right) - e^{\frac{2\mu_2 b}{\sigma_2^2}} \Phi_2\left(\frac{a - \mu_1 T - 2\rho b \sigma_1 / \sigma_2}{\sigma_1 \sqrt{T}}, \frac{-b - \mu_2 T}{\sigma_2 \sqrt{T}}; \rho\right) \end{aligned}$$

Remark:

(i) Using the facts that $P(X(T) > a, \tau_b^Y > T) = P(-X(T) < -a, \tau_b^Y > T)$, that the correlation between $-X(t)$ and $Y(t)$ is $-\rho$, we can show that for $b > 0$ (the following equation will use for next example to price of an up-and-out option):

$$\begin{aligned} P(X(T) > a, \tau_b^Y > T) &= P\left(-X(T) < -a, \max_{0 \leq t \leq T} Y(t) < b\right) \\ &= \Phi_2\left(-\frac{a - \mu_1 T}{\sigma_1 \sqrt{T}}, \frac{b - \mu_2 T}{\sigma_2 \sqrt{T}}; -\rho\right) - e^{\frac{2\mu_2 b}{\sigma_2^2}} \Phi_2\left(-\frac{a - \mu_1 T - 2\rho b \sigma_1 / \sigma_2}{\sigma_1 \sqrt{T}}, \frac{-b - \mu_2 T}{\sigma_2 \sqrt{T}}; -\rho\right) \end{aligned}$$

(ii) Using the fact that $P(X(T) < a, \tau_{-b}^Y > T) = P(X(T) < a, \tau_b^{-Y} > T)$, that the correlation between $X(t)$ and $-Y(t)$ is $-\rho$, we can show that for $b > 0$:

$$\begin{aligned} P(X(T) < a, \tau_{-b}^Y > T) &= P\left(X(T) < a, \min_{0 \leq t \leq T} Y(t) > -b\right) = P\left(X(T) < a, \max_{0 \leq t \leq T} (-Y(t)) < b\right) \\ &= \Phi_2\left(\frac{a - \mu_1 T}{\sigma_1 \sqrt{T}}, \frac{b + \mu_2 T}{\sigma_2 \sqrt{T}}; -\rho\right) - e^{\frac{2\mu_2 b}{\sigma_2^2}} \Phi_2\left(\frac{a - \mu_1 T + 2\rho b \sigma_1 / \sigma_2}{\sigma_1 \sqrt{T}}, \frac{-b + \mu_2 T}{\sigma_2 \sqrt{T}}; -\rho\right) \end{aligned}$$

Two-Dimensional Barrier Options Con.

Let's calculate the price of an up-and-out call option, we have the following set up:

$$U_0 = e^{-rT} \mathbb{E}^{\mathbb{P}^*} \left[(S_1(T) - K)^+ I \left\{ \max_{0 \leq t \leq T} S_2(t) \leq H \right\} \right], \quad S_i(t) = S_i(0)e^{X_i(t)}, X_1 = X, X_2 = Y$$

Under the risk-neutral measure \mathbb{P}^* ,

$$dX(t) = \mu_1 dt + \sigma_1 dW_1(t), \quad \mu_1 = r - \frac{1}{2}\sigma_1^2,$$
$$dY(t) = \mu_2 dt + \sigma_2 \left\{ \rho dW_1(t) + \sqrt{1 - \rho^2} dW_2(t) \right\}, \quad \mu_2 = r - \frac{1}{2}\sigma_2^2,$$

with W_1, W_2 independent.

Write the barrier level in log space

$$b := \log \frac{H}{S_2(0)}, \quad \text{and } a := \log \frac{K}{S_1(0)}$$

Then we have:

$$\{\max S_2(t) \leq H\} = \{\max S_2(0)e^{Y(t)} \leq H\} = \left\{ \max_{0 \leq t \leq T} Y(t) \leq b \right\}$$

$$\{S_1(T) > K\} = \{S_1(0)e^{X(t)} > K\} = \{X(T) > a\}$$

Two-Dimensional Barrier Options Con.

$$\begin{aligned}
 U_0 &= e^{-rT} \mathbb{E}^{\mathbb{P}^*} \left[(S_1(T) - K)^+ I \left\{ \max_{0 \leq t \leq T} S_2(t) \leq H \right\} \right] \\
 &= e^{-rT} \mathbb{E}^{\mathbb{P}^*} \left[(S_1(T) - K) I \left\{ S_1(T) > K, \max_{0 \leq t \leq T} S_2(t) \leq H \right\} \right] \\
 &= e^{-rT} \mathbb{E}^{\mathbb{P}^*} \left[S_1(T) I \left\{ X(T) > a, \max_{0 \leq t \leq T} Y(t) \leq b \right\} \right] - K e^{-rT} \mathbb{P}^*(X(T) > a, \max_{0 \leq t \leq T} Y(t) \leq b) \\
 &= I - K e^{-rT} \cdot II
 \end{aligned}$$

Apply Itô, we have the Radon–Nikodym derivative for numeraire:

$$\left. \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^*} \right|_{\mathcal{F}_T} = Z_T := \frac{e^{-rT} S_1(T)}{S_1(0)} = \exp \left[\sigma_1 W_1(T) - \frac{1}{2} \sigma_1^2 T \right]$$

By Girsanov theorem, under new measure $\tilde{\mathbb{P}}$:

$$\tilde{W}_1(t) = W_1(t) - \sigma_1 t, \quad \tilde{W}_2(t) = W_2(t)$$

And the drifts of X and Y under $\tilde{\mathbb{P}}$ become:

$$\mu_1^{(1)} = \mu_1 + \sigma_1^2 = r + \frac{1}{2} \sigma_1^2, \quad \mu_2^{(1)} = \mu_2 + \rho \sigma_1 \sigma_2 = r - \frac{1}{2} \sigma_2^2 + \rho \sigma_1 \sigma_2$$

Two-Dimensional Barrier Options Con.

For I, using change of numeraire and the formula in remark (i) :

$$\begin{aligned} I &= \mathbb{E}^{\mathbb{P}^*} \left[e^{-rT} S_1(T) I \left\{ X(T) > a, \max_{0 \leq t \leq T} Y(t) \leq b \right\} \right] \\ &= \mathbb{E}^{\mathbb{P}^*} \left[\frac{e^{-rT} S_1(T)}{S_1(0)} S_1(0) I \left\{ X(T) > a, \max_{0 \leq t \leq T} Y(t) \leq b \right\} \right] \\ &= S_1(0) \tilde{\mathbb{P}} \left\{ X(T) > a, \max_{0 \leq t \leq T} Y(t) \leq b \right\} \\ &= S_1(0) \left[\Phi_2 \left(-\frac{a - \mu_1^{(1)} T}{\sigma_1 \sqrt{T}}, \frac{b - \mu_2^{(2)} T}{\sigma_2 \sqrt{T}}; -\rho \right) - e^{\frac{2\mu_2^{(2)} b}{\sigma_2^2}} \Phi_2 \left(-\frac{a - \mu_1^{(1)} T - 2\rho b \sigma_1 / \sigma_2}{\sigma_1 \sqrt{T}}, \frac{-b - \mu_2^{(2)} T}{\sigma_2 \sqrt{T}}; -\rho \right) \right] \end{aligned}$$

For II, apply the formula in remark (i) directly:

$$\begin{aligned} II &= \mathbb{P}^* (X(T) > a, \max_{0 \leq t \leq T} Y(t) \leq b) \\ &= \Phi_2 \left(-\frac{a - \mu_1 T}{\sigma_1 \sqrt{T}}, \frac{b - \mu_2 T}{\sigma_2 \sqrt{T}}; -\rho \right) - e^{\frac{2\mu_2 b}{\sigma_2^2}} \Phi_2 \left(-\frac{a - \mu_1 T - 2\rho b \sigma_1 / \sigma_2}{\sigma_1 \sqrt{T}}, \frac{-b - \mu_2 T}{\sigma_2 \sqrt{T}}; -\rho \right) \end{aligned}$$

Introduction to Stochastic Calculus for Jump Processes

Counting Processes

A counting process $N(t)$: tracks the number of events up to time t . We have the following Key properties of any counting process:

- $N(t) \geq 0$
- Takes integer values
- Non-decreasing ($N(s) \leq N(t)$ if $s < t$)
- Increment $N(t) - N(s)$ counts events in $(s, t]$

To make analysis tractable, the following assumptions are typically imposed:

- Independent increments: the number of events occurring in disjoint time intervals is statistically independent
- Stationary increments: distribution of increments depends only on interval length, not location.

These two properties are also underpin the definition of Brownian motion.

Two Equivalent Definitions of Poisson Processes

First definition(I) of a Poisson process:

Counting process $\{N(t), t \geq 0\}$ with rate $\lambda > 0$ such that :

- 1 $N(0) = 0$
- 2 The process exhibits independent and stationary increments
- 3 For each $t \geq 0$, the random variable $N(t)$ follows a Poisson distribution:

$$\mathbb{P}[N(t) = n] = e^{-\lambda t} \frac{(\lambda t)^n}{n!} \quad \text{for } n = 0, 1, 2, \dots, \quad \mathbb{E}[N(t)] = \lambda t$$

Second definition(II) of a Poisson process:

Counting process $\{N(t), t \geq 0\}$ with rate $\lambda > 0$ such that :

- 1 $N(0) = 0$
- 2 The process exhibits independent and stationary increments
- 3 $\mathbb{P}[N(h) = 1] = \lambda h + o(h)$
- 4 $\mathbb{P}[N(h) \geq 2] = o(h)$

Two Equivalent Definitions of Poisson Processes Con.

We show the equivalence of the two definitions:

Proof:

(i) $I \Rightarrow II$

For small h , use Taylor expansion of the exponential:

$$\mathbb{P}(N(h) = 1) = e^{-\lambda h}(\lambda h) = (1 - \lambda h + \frac{1}{2}(\lambda h)^2 + o(h^2))(\lambda h) = \lambda h + o(h)$$

Similarly:

$$\mathbb{P}(N(h) \geq 2) = 1 - \mathbb{P}(0) - \mathbb{P}(1) = 1 - \left(1 - \lambda h + \frac{1}{2}(\lambda h)^2 + o(h^2)\right) - (\lambda h + o(h)) = o(h)$$

(ii) $II \Rightarrow I$ (Some Intuition)

Partition $(0, t]$ into m small subintervals of length $h = t/m$ and define increments

$X_i = N(ih) - N((i-1)h)$:

$$N(t) = \sum_{i=1}^m X_i$$

Let $m \rightarrow \infty$, by the small-interval conditions:

$$\mathbb{P}\{X_i = 1\} = \lambda h + o(h), \quad \mathbb{P}\{X_i = 0\} = 1 - \lambda h + o(h), \quad \mathbb{P}\{X_i \geq 2\} = o(h)$$

Two Equivalent Definitions of Poisson Processes Con.

So each X_i behaves like a Bernoulli(λh). due to the independent increments property, the X_i are mutually independent. It follows that $N(t)$ is approximately binomial with parameters m and $p = \lambda h$.

As $m \rightarrow \infty$ and $p \rightarrow 0$, the classical Poisson approximation to the binomial distribution implies that $N(t)$ converges in distribution to a Poisson random variable with rate

$$mp = m\lambda h = m\lambda \frac{t}{m} = \lambda t,$$

which precisely corresponds to the distribution given in equation of the requirement three in the first definition.

Remark: Powerful tool for modeling infrequent extreme events. In financial contexts, poisson processes can capture market shocks and discontinuities missed by continuous-path models. Important for pricing derivatives sensitive to jump risk.

Interarrival and Waiting Times

Interarrival time T_n is the time between the $(n-1)$ st and n th event. Waiting time S_n is the time of the n th event:

$$S_n = \sum_{i=1}^n T_i$$

Key properties:

- $S_n \leq t \iff N(t) \geq n$. (n th event occurs by time $t \iff$ at least n arrivals by t).
- Alternative representation of counting process:

$$N(t) = \max\{n : S_n \leq t\} = \min\{n : S_{n+1} > t\}.$$

The Third Definition of Poisson Processes

Suppose the counting process $N(t)$ satisfies the second definition of a Poisson process. Demonstrate that the interarrival times $T_n, n \geq 1$, are independent exponential random variables with rate λ . Consequently, the expected value of the first interarrival time is $\mathbb{E}[T_1] = 1/\lambda$.

[Proof]

Note that since $T_1 > t$ means that there is no event before time t . Therefore, we have

$$P(T_1 > t) = P(N(t) = 0) = e^{-\lambda t}.$$

Furthermore,

$$P(T_2 > t | T_1 = s) = P(\text{no events in } (s, s+t] | T_1 = s) = P(\text{no events in } (s, s+t]) = e^{-\lambda t},$$

Thus, conditioning on T_1 we have

$$P(T_2 > t) = \int_0^\infty P(T_2 > t | T_1 = s) f_{T_1}(s) ds = \int_0^\infty e^{-\lambda t} f_{T_1}(s) ds = e^{-\lambda t}$$

Therefore, T_2 has an exponential distribution with same rate λ and T_1 and T_2 are independent. Repeating the same argument, we can show T_3, T_4, \dots .

The Third Definition of Poisson Processes Con.

Since the sum of independent and identically distributed exponential random variables follows a gamma distribution:

$$S_n = \sum_{i=1}^n T_i \sim \text{Gamma}(n, \lambda), \quad f_{S_n}(t) = \frac{\lambda e^{-\lambda t} (\lambda t)^{n-1}}{\Gamma(n)}$$

Third equivalent definition(III), poisson process can be defined via arrival times:

$$N(t) = \max\{n : S_n \leq t\} = \min\{n : S_{n+1} > t\} \quad (*)$$

With $T_i \stackrel{iid}{\sim} \text{Exp}(\lambda)$.

Note: If $N(t)$ satisfies Definition II, then T_i are exponential $\Rightarrow (*)$. Conversely, if we build $N(t)$ from i.i.d. exponential interarrivals via $(*)$, then $N(t)$ has Poisson(λt) law \Rightarrow Definition I.

Moreover, a Poisson process can be expressed as $N(t) = M(t) - 1$, where $M(t)$ corresponds to a special case of a rst passage time process, defined as:

$$M(t) = \min\{n > 0 : S_n > t\}, \quad S_0 = 0, \quad t > 0$$