Description of relative motion using Eulerian pole

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The following content is excerpted and synthesized from the following sources:

- Pichon, X. L., Francheteau, J. & Bonnin, J. Plate Tectonics; Developments in Geotectonics 6; Hardcover – January 1, 1973. Page 28-29.
- Cox, A., and Hart, R.B. (1986) Plate tectonics: How it works. Blackwell Scientific Publications, Palo Alto. DOI: 10.4236/ojapps.2015.54016. Page 145-156.
- Transformations between ECEF and ENU coordinates, Navipedia, ESA

General matrix description

In the 60's, plate motions analysis on a sphere have revived among Earth scientists [1, 3] a theorem proved by Leonhard Euler in 1776 and also called the fixed-point theorem. The theorem states that, if a rigid body is turned about one of its points taken fixed, the displacement of this body from one given position to another is equivalent to a rotation about some fixed axis going through the fixed point. The center of the sphere can be considered part of any rigid body constrained to move at its surface. Thus, in the case of rigid plates constrained to move at the surface of a sphere, the motions will be only rotations. There are no possible translations, but only rotations. This, often, has not been understood and authors still speak about a combination of a rotation and a translation for Arabia with respect to Africa for example. Actually, they mean the combination of rotations with near and far poles, which of course, is still equivalent to another rotation [4].

An angular velocity behaves as vectors, the infinitesimal rotation can be entirely described by a vector

$$\boldsymbol{\Omega} = \omega \hat{\boldsymbol{k}} \tag{1}$$

where $\hat{\boldsymbol{k}}$ is a unit vector along the rotation axis and ω is a scalar value of the angular velocity. The sign convention is chosen such that $\hat{\boldsymbol{k}}$ is positive when pointing outward from the center of the sphere, and the rotation is clockwise when looking at from the center of the sphere. The rotation axis pierces the surface of the Earth at two points called poles of rotation, or Eulerian poles, or simply, Euler poles.

It is convenient to decompose the Eulerian poles into its Cartesian coordinates ω_x, ω_y, ω_z. The positive x- and y-axis are corresponding to 0° and 90°E in the equatorial plane, and the z-axis is the line joining the center of the Earth to the North Pole. Such that,

$$\omega \hat{\boldsymbol{k}} = \omega \cdot [k_x, k_y, k_z] = [\omega_x, \omega_y, \omega_z]$$

with

$$k_x = cos\lambda_0 \; cos\phi_0, \; k_y = cos\lambda_0 \; sin\phi_0, \; k_z = sin\lambda_0$$

where λ_0 and ϕ_0 are the latitude and the longitude of the Euler pole (the piercing points of the rotation axis). Thus, the Euler rotation can be expressed in the Cartesian coordinates

$$\omega_x = \omega \cdot cos\lambda_0 \; cos\phi_0, \; \omega_y = \omega \cdot cos\lambda_0 \; sin\phi_0, \; \omega_z = \omega \cdot sin\lambda_0$$

If we know the instantaneous velocity vector ${}_A\boldsymbol{\omega}_B$ between plates A and B and ${}_B\boldsymbol{\Omega}_C$ between plates B and C the instantaneous velocity vector between plates A and C is obtained by vector addition:

$${}_A\boldsymbol{\Omega}_C =_A \boldsymbol{\Omega}_B +_B \boldsymbol{\Omega}_C \tag{2}$$

and its Cartesian components are simply the algebraic sum of the corresponding Cartesian components of the two vectors.

The linear velocity $\boldsymbol{v}$ between the plates at point A on the Earth's surface is:

$$\boldsymbol{V} = \boldsymbol{\Omega} \times \boldsymbol{r}_i \tag{3}$$

where $\boldsymbol{r}_i$ is the vector joining the center of the Earth to A, with a radius R.

$$\boldsymbol{r}_i = [cos\lambda \; cos\phi, \; cos\lambda \; sin\phi, \; sin\lambda]$$

Since this is a cross product, the linear velocity has a modulus, $\omega R sin\psi$, where ψ is the angular distance between the Euler pole and A. Thus, the value of the linear velocity will be zero at the two Euler poles of rotation, be maximum at the Euler equator, and will vary as the sine of the angular distance to the Euler pole. The trajectory of any point will be on a small circle having $\boldsymbol{\Omega}$ for the axis.

Finally, one can convert the linear velocity $\boldsymbol{V}$ from global Cartesian coordinates (x, y, z) (or Earth Centred Earth Fixed, ECEF) to a new vector $\boldsymbol{V}_{local} = (V_e, V_n, V_u)$ in the local ENU (east, north, up) Cartesian coordinates. This coordinate conversion mathematically is simply a matrix rotation operation [2]:

$$\boldsymbol{V}_{local} = \boldsymbol{T}\boldsymbol{V} \tag{4}$$

The local Cartesian components, $\boldsymbol{V}_{local} = (V_e, V_n, V_u)$, at some point A on a perfect sphere will depend upon the coordinates (λ, ϕ) of A, which are encapsulated in this 3×3 rotation matrix $\boldsymbol{T}$. How to get this $\boldsymbol{T}$? Recall that in Cartesian space, rotate counter-clockwise with an angle, θ, about the x-axis, y-axis, and z-axis, can be expressed respectively as:

$$\boldsymbol{R}_1[\theta] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & sin\theta \\ 0 & -sin\theta & cos\theta \end{bmatrix}, \; \boldsymbol{R}_2[\theta] = \begin{bmatrix} cos\theta & 0 & -sin\theta \\ 0 & 1 & 0 \\ sin\theta & 0 & cos\theta \end{bmatrix}, \; \boldsymbol{R}_3[\theta] = \begin{bmatrix} cos\theta & sin\theta & 0 \\ -sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To convert from global Cartesian to local ENU coordinates we take two steps of rotations:

1. Rotate counter-clockwise about the z-axis $\frac{\pi}{2} + \phi$ to align the x-axis with the local east-axis. That is a rotation operation $\boldsymbol{R}_3[\frac{\pi}{2} + \phi]$
2. Rotate counter-clockwise about the x-axis $\frac{\pi}{2} - \lambda$ to align the z-axis with the local up-axis. That is a rotation operation $\boldsymbol{R}_1[\frac{\pi}{2} - \lambda]$

Thus, we can rewrite (4) as:

$$\boldsymbol{V}_{local} = \boldsymbol{R}_1[\frac{\pi}{2} - \lambda]\boldsymbol{R}_3[\frac{\pi}{2} + \phi]\boldsymbol{V} = \boldsymbol{T}\boldsymbol{V}$$

$$T = \boldsymbol{R}_1[\frac{\pi}{2} - \lambda]\boldsymbol{R}_3[\frac{\pi}{2} + \phi]$$

$$= \begin{bmatrix} -sin\phi & cos\phi & 0 \\ -sin\lambda \; cos\phi & -sin\lambda \; sin\phi & cos\lambda \\ cos\lambda \; cos\phi & cos\lambda \; sin\phi & sin\lambda \end{bmatrix}$$

The inverse conversion from local ENU back to a global Cartesian coordinates is then done by $\boldsymbol{T}^{-1}$, which is also a trivial derivation. See Transformations between ECEF and ENU coordinates, [5] if needed.

Detailed algebraic of the three-component surface velocity

The previous section is already self explanatory and sufficient for understanding the math. This section includes the tedious algebraic of all the terms in the matrices and provide a simpler description for coding implementation.

According to the previous section, we know that the local velocity at a point A with latitude and longitude (λ, ϕ) is:

$$\boldsymbol{V}_{local} = \begin{bmatrix} V_e \\ V_n \\ V_u \end{bmatrix} = \boldsymbol{TV} = \begin{bmatrix} -sin\phi & cos\phi & 0 \\ -sin\lambda\ cos\phi & -sin\lambda\ sin\phi & cos\lambda \\ cos\lambda\ cos\phi & cos\lambda\ sin\phi & sin\lambda \end{bmatrix} \Big(\boldsymbol{\Omega} \times \boldsymbol{r_i} \Big)$$

$$= \begin{bmatrix} -sin\phi & cos\phi & 0 \\ -sin\lambda\ cos\phi & -sin\lambda\ sin\phi & cos\lambda \\ cos\lambda\ cos\phi & cos\lambda\ sin\phi & sin\lambda \end{bmatrix} \left(\omega[k_x, k_y, k_z] \times R \begin{bmatrix} cos\lambda\ cos\phi \\ cos\lambda\ sin\phi \\ sin\lambda \end{bmatrix}^T \right)$$

$$= \begin{bmatrix} -sin\phi & cos\phi & 0 \\ -sin\lambda\ cos\phi & -sin\lambda\ sin\phi & cos\lambda \\ cos\lambda\ cos\phi & cos\lambda\ sin\phi & sin\lambda \end{bmatrix} \left(\omega \begin{bmatrix} cos\lambda_0\ cos\phi_0 \\ cos\lambda_0\ sin\phi_0 \\ sin\lambda_0 \end{bmatrix}^T \times R \begin{bmatrix} cos\lambda\ cos\phi \\ cos\lambda\ sin\phi \\ sin\lambda \end{bmatrix}^T \right)$$

We can do the algebraic in details and further simplify to get the following:

$$\begin{cases} V_e = R \cdot \omega \cdot cos\lambda(sin\lambda_0 - cos\lambda_0 \cdot tan\lambda \cdot cos(\phi_0 - \phi)) \\ V_n = R \cdot \omega \cdot cos\lambda \cdot sin(\phi_0 - \phi) \\ V_u = 0 \end{cases} \tag{5}$$

Further notes

- Absolute velocity in local ENU:
$$\left| \boldsymbol{V}_{local} \right| = \sqrt{V_e^2 + V_n^2}$$
- Direction counter-clockwise from east:
$$\arctan\left(\frac{V_e}{V_n}\right)$$
- Given Cartesian rotation vector $[\omega_x, \omega_y, \omega_z]$, find the Euler pole (λ_0, ϕ_0) and the scalar rotation rate, ω:
$$\lambda_0 = \arctan\left(\frac{\omega_z}{\sqrt{\omega_x^2 + \omega_y^2}}\right)$$
$$\phi_0 = \arctan\left(\frac{\omega_y}{\omega_x}\right)$$
$$\omega = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}$$

References

[1] Edward Bullard, J. E. Everett, and A. Gilbert Smith. "The Fit of the Continents around the Atlantic". In: *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences* 258.1088 (1965), pp. 41–51. ISSN: 0080-4614. URL: https://www.jstor.org/stable/73331 (visited on 07/25/2022).

[2] Allan Cox and R. B. Hart. *Plate Tectonics: How It Works*. John Wiley & Sons, 1986. 419 pp. ISBN: 978-1-4443-1421-2.

[3] D. P. McKenzie and R. L. Parker. "The North Pacific: an Example of Tectonics on a Sphere". In: *Nature* 216.5122 (Dec. 1967). ZSCC:00852, pp. 1276–1280. ISSN: 0028-0836, 1476-4687. DOI: 10.1038/2161276a0. URL: https://www.nature.com/articles/2161276a0 (visited on 07/25/2022).

[4] Xavier Le Pichon, Jean Francheteau, and Jean Bonnin. *Plate Tectonics*. ZSCC:00597 Google-Books-ID: lxTgBAAAQBAJ. Elsevier, Jan. 1, 1973. 315 pp. ISBN: 978-1-4832-5727-3.

[5] J Sanz Subirana, JM Juan Zornoza, and M Hernández-Pajares. "Transformations between ECEF and ENU coordinates". In: (2011). URL: https://gssc.esa.int/navipedia/index.php/Transformations_between_ECEF_and_ENU_coordinates (visited on 07/24/2022).