$$x_0 \longrightarrow x_1 \cdots \longrightarrow x_{t-1} \xrightarrow{q(x_t|x_{t-1})} x_t \longrightarrow \cdots \longrightarrow x_T$$

$$\cancel{q(x_t|x_{t-1}) = \mathcal{N}(\sqrt{\alpha_t}\, x_{t-1}, \sqrt{1-\alpha_t})}$$

$$x_t = \sqrt{\alpha_t}\, x_{t-1} + \sqrt{1-\alpha_t}\, \varepsilon$$

$$q(x_t|x_{t-1}) = \mathcal{N}(\sqrt{\alpha_t}\, x_{t-1}, (1-\alpha_t) I)$$

$$q(x_T) = \int q(x_0)\, q(x_1|x_0)\, q(x_2|x_1) \cdots q(x_T|x_{T-1})\, dx_{0:T}$$

如果用真实 $q(x_{t-1}|x_t) = \dfrac{q(x_t|x_{t-1})\, q(x_{t-1})}{q(x_t)}$ 进行采样

则逆过程的

$$p(x_0) = \int q(x_T) \cdot \prod_{t=T}^{1} q(x_{t-1}|x_t)\, dx_{1:T}$$

$$= \int q(x_T) \cdot \prod_{t=T}^{1} \frac{q(x_t|x_{t-1})\, q(x_{t-1})}{q(x_t)}\, dx_{1:T}$$

$$= \int q(x_0) \cdot \prod_{t=1}^{T} q(x_t|x_{t-1})\, dx_{1:T} = q(x_0)$$

能成功还原 $q(x_0)$

~~为了完成这个目标，有两种方式 ①有~~

~~方法一： $p_\theta(x_{t-1}|x_t)$ 拟合~~

也就是说如果有网络能够拟合 $\cancel{q(x_t|x_{t-1})}$ $q(x_{t-1}|x_t)$

我们能完成完美采样

定义网络 $p_\theta(x_{t-1}|x_t)$. 如何优化？

极大似然估计 $p_\theta(x_0) = \int p_\theta(x_{0:T})\, dx_{1:T}$

$$= \int p_\theta(x_0|x_{1:T})\, p_\theta(x_{1:T})\, dx_{1:T}$$

$$= \frac{1}{N} \sum_{i=1}^{N} p_\theta(x_0|x^i_{1:T})$$

① 采样非常困难.

② 采样效率低下，x_0 与 $x_{1:T}$ 无关联.

采用重要性采样

$$p_\theta(x_0)=\int p_\theta(x_{0:T})/q(x_{1:T}|x_0)\cdot q(x_{1:T}|x_0)\,dx_{1:T}$$

好处：① 引入 $x_{1:T}$ 与 x_0 的关联，提高网络取采样到有意义区域

② 提高采取样本速度，正向过程中的 $x_{1:T}$ 计算方便

上式利用蒙特卡洛方法可以计算为

$$p_\theta(x_0)\approx\frac{1}{N}\sum_{i=1}^{N}\frac{p_\theta(x_{0:T})}{q(x_{1:T}|x_0)}$$

问题：采样估计存在偏差，上述目标可能不稳定。

利用不等式进行放缩，最大化下界

$$\log p_\theta(x_0)=\log\int\frac{p_\theta(x_{0:T})}{q(x_{1:T}|x_0)}\cdot q(x_{1:T}|x_0)\,dx_{1:T}$$

$$\geq\int q(x_{1:T}|x_0)\log\frac{p_\theta(x_{0:T})}{q(x_{1:T}|x_0)}\,dx_{1:T}$$

$$=E_{x_{1:T}\sim q}\left[\log\frac{p_\theta(x_{0:T})}{q(x_{1:T}|x_0)}\right]$$

$$p_\theta(x_{0:T})=\cancel{p_\theta(x_T)}\,p_\theta(x_T)\cdot\prod_{t=T}^{1}p_\theta(x_{t-1}|x_t)$$

$$q(x_{1:T}|x_0)=\underbrace{\prod_{t=1}^{T}q(x_t|x_{t-1})}_{①}=\underbrace{q(x_1|x_0)\prod_{t=2}^{T}q(x_t|x_{t-1},x_0)}_{②}$$

如果按式①展开

$$\log p_\theta(x_0)\geq E_{x_{1:T}\sim q}\left[\log\frac{p_\theta(x_T)\prod_{t=T}^{1}p_\theta(x_{t-1}|x_t)}{\prod_{t=1}^{T}q(x_t|x_{t-1})}\right]$$

负的KL

$$=E_{x_{1:T}\sim q}\left[\log\frac{p_\theta(x_T)\prod_{t=T}^{1}p_\theta(x_{t-1}|x_t)}{\prod_{t=1}^{T}\frac{q(x_{t-1}|x_t)\,q(x_t)}{q(x_{t-1})}}\right]$$

与E结合成E[KL]

不确定在

令 $p_\theta(x_{t-1}|x_0)$

拟合 $q(x_{t-1}|x_0)$

$$=E_{x_{1:T}\sim q}\left[\log\frac{p_\theta(x_T)}{q(x_T)}+\sum_{t=1}^{T}\log\frac{p_\theta(x_{t-1}|x_t)}{q(x_{t-1}|x_t)}+\log q(x_0)\right]$$

②式展开:

$$\log p_\theta(x_0) \geq E_{x_{1:T}\sim q}\left[\log \frac{p_\theta(x_T)\prod_{t=T}^{1} p_\theta(x_{t-1}|x_t)}{q(x_1|x_0)\prod_{t=2}^{T} q(x_t|x_{t-1},x_0)}\right]$$

$$= E_{x_{1:T}\sim q}\left[\log \frac{p_\theta(x_T)\prod_{t=T}^{1} p_\theta(x_{t-1}|x_t)}{q(x_1|x_0)\prod_{t=2}^{T} \frac{q(x_{t-1}|x_t,x_0)\,q(x_t|x_0)}{q(x_{t-1}|x_0)}}\right]$$

$$= E_{x_{1:T}\sim q}\left[\log \frac{p_\theta(x_T)\prod_{t=T}^{1} p_\theta(x_{t-1}|x_t)\cdot \cancel{q(x_1|x_0)}}{\cancel{q(x_1|x_0)}\,q(x_T|x_0)\prod_{t=2}^{T} q(x_{t-1}|x_t,x_0)}\right]$$

$$= E_{x_{1:T}\sim q}\left[\log \frac{p_\theta(x_T)}{q(x_T|x_0)} + \sum_{t=2}^{T}\log \frac{p_\theta(x_{t-1}|x_t)}{q(x_{t-1}|x_t,x_0)} + \log p_\theta(x_0|x_1)\right]$$

$\cancel{= E_{x_{1:T}\sim q}[}$

$$E_{x_0}[\log p_\theta(x_0)] \geq E_q\left[-KL(q(x_T|x_0)\,||\,p_\theta(x_T)) - \sum_{t=2}^{T} KL(q(x_{t-1}|x_t,x_0)\,||\,p_\theta(x_{t-1}|x_t)) + \log p_\theta(x_0|x_1)\right]$$

$$q(x_{t-1}|x_t,x_0) = \frac{q(x_t|x_{t-1},x_0)\,q(x_{t-1}|x_0)}{q(x_t|x_0)}$$

$$\propto \exp\left[-\frac{(x_t-\sqrt{\alpha_t}\,x_{t-1})^2}{2(1-\alpha_t)} - \frac{(x_{t-1}-\sqrt{\bar\alpha_{t-1}}\,x_0)^2}{2(1-\bar\alpha_{t-1})} + \frac{(x_t-\sqrt{\bar\alpha_t}\,x_0)^2}{2(1-\bar\alpha_t)}\right]$$

快速凑平方.

$$\frac{\partial \log q(x_{t-1}|x_t,x_0)}{\partial x_{t-1}} = \frac{\sqrt{\alpha_t}(x_t-\sqrt{\alpha_t}\,x_{t-1})}{1-\alpha_t} - \frac{x_{t-1}-\sqrt{\bar\alpha_{t-1}}\,x_0}{1-\bar\alpha_{t-1}} = 0$$

$$x_T = \frac{\sqrt{\alpha_t}(1-\bar\alpha_{t-1})x_t + (1-\alpha_t)\sqrt{\bar\alpha_{t-1}}\,x_0}{1-\bar\alpha_t} \rightarrow x_0 = \frac{x_t-\sqrt{1-\bar\alpha_t}\,\varepsilon}{\sqrt{\bar\alpha_t}}$$

$$= \frac{\alpha_t(1-\bar\alpha_{t-1})x_t + (1-\alpha_t)(x_t-\sqrt{1-\bar\alpha_t}\,\varepsilon)}{\sqrt{\alpha_t}\,(1-\bar\alpha_t)}$$

$$= \frac{1}{\sqrt{\alpha_t}}\left(x_t - \frac{(1-\alpha_t)}{\sqrt{1-\bar\alpha_t}}\varepsilon\right)$$

~~求~~ 求方差

$$\frac{\partial^2 \log}{\partial x_{t-1}} = -\frac{\alpha_t}{1-\alpha_t} - \frac{1}{1-\bar{\alpha}_{t-1}} = \frac{-\alpha_t + \bar{\alpha}_t - 1 + \alpha_t}{(1-\alpha_t)(1-\bar{\alpha}_{t-1})} = 2a.$$

$$-\frac{1}{2a} = \frac{1-\bar{\alpha}_{t-1}}{1-\bar{\alpha}_t}(1-\alpha_t)$$

$$-\frac{(x-u)^2}{2\cdot(-\frac{1}{2a})}$$

求得 $u = \frac{1}{\sqrt{\alpha_t}}\left[x_t - \frac{(1-\alpha_t)}{\sqrt{1-\bar{\alpha}_t}}\varepsilon\right]$ $\left(ps: u = \frac{1}{\sqrt{\alpha_t}}(x_t - \frac{\beta_t}{\sqrt{1-\bar{\alpha}_t}}\varepsilon)\right)$

$$\sigma = \frac{1-\bar{\alpha}_{t-1}}{1-\bar{\alpha}_t}(1-\alpha_t)$$

定 $p_\theta(x_{t-1}|x_t) = \mathcal{N}(x_{t-1}; u_\theta(x_t,t), \sigma(x_t,t))$

$$u_\theta(x_t,t) = \frac{1}{\sqrt{\alpha_t}}\left(x_t - \frac{\beta_t}{\sqrt{1-\bar{\alpha}_t}}\varepsilon_\theta(x_t,t)\right).$$

$$\sigma_\theta(x_t,t) = \sigma$$

则 ~~KL(p(~~ $KL(q(x_{t-1}|x_t,x_0)\,\|\,p_\theta(x_{t-1}|x_t))$

$$= \frac{1}{2\sigma}\|u_\theta - u\|^2$$

$$= \frac{\beta_t^2}{2\sigma\alpha_t(1-\bar{\alpha}_t)}\|\varepsilon_\theta - \varepsilon\|^2$$ 权重一般是递减的

简化 loss $= E_{x_0,\varepsilon,t}\|\varepsilon_\theta(\sqrt{\bar{\alpha}_t}x_0 + \sqrt{1-\bar{\alpha}_t}\varepsilon, t) - \varepsilon\|^2$

$\downarrow$

简化loss在 $t\to 0$ 处约束变小

$t\to 1$ 处约束变大.

随机过程 $\{X(t), t\in T\}$ 可以看作是时间序列上的一系列随机变量：

$B(t)$ or $W(t)$ 布朗运动或维纳过程.

① $B(0)=0$.

② ~~独立~~独立增量：$t_1>S_1$，$t_2>S_2$ 且 $[t_1, S_1]$，$[t_2, S_2]$ 没有交集

则 $B(t_1)-B(S_1)$ 与 $B(t_2)-B(S_2)$ 独立

③ 正态分布增量．$t>S$．，$B(t)-B(S)\sim N(0, t-S)$

④ 无记忆性：~~增量~~增量 $\Delta B=B(t)-B(S)$ 只与 $t-S$ 有关 与 $B(S)$ $B(t)$ 值无关

随机微分方程（SDE）.

$$dX=\underbrace{f(X,t)dt}_{\text{确定性}}+\underbrace{g(X,t)dB}_{\text{随机噪声}}.$$

$dB\sim N(0,dt)$，$(dB)^2=dt$ (ps: $E(dB^2)=dt$)

看一个例子．$dX=g(t)dB$.

易解得 $X=X_0+\underbrace{\int_0^t g(u)dB}_{\text{有什么性质？}}$.

$I(t)=\int_0^t g(u)dB$

① 由于 dB 每个时间 t 都相互独立．$I(t)$ 可以看作相互独立的高斯分布（正态）的叠加 可知 $I(t)\sim N(u,\sigma^2)$.

$$u=E(I(t))=E\left(\int_0^t g(u)dB\right)=\int_0^t E(g(u)dB)=\int_0^t 0=0$$

$$\sigma^2=D(I(t))=E(I^2(t))-[E(I(t))]^2=E(I^2(t))=E\left(\int_0^t g(u)dB\int_0^t g(v)dB\right)$$

$$E(I^2(t)) = E\left(\int_0^t g(u)dB(u)\int_0^t g(v)dB(v)\right) = E\left(\int_0^t\int_0^t g(u)g(v)dB(u)dB(v)\right)$$

$$= \int_0^t \int_{u=v} E(g(u)g(v)dB(u)dB(v)) + \int_{u\neq v} E(g(u)g(v)dB(u)dB(v))$$

$$= \int_0^t g^2(u)dt.$$

$\sigma^2 = \int_0^t g^2(u)dt.$ (~~换~~ u时刻 $dB \sim N(0,dt)$.

则 $g(u)dB \sim N(0, g^2(u)dt)$.

$\int S =$ 方差相加则 $D(I(t)) = \int_0^t g^2(u)dt$)

再看一个ODE：

$$dx = f(t)x\,dt.$$

不难解得： $x = x_0 \exp(\int_0^t f(u)du)$.

如何理解 $\exp(\int_0^t f(u)du)$ 这一项？

$$dx = x_{t+dt} - x_t = f(t)x_t dt$$

$$x_{t+dt} = (1+f(t)dt)x_t.$$

~~$x_t = (1+f(t))$~~

$$x_{dt} = (1+f(0)dt)x_0$$

$$x_{2dt} = (1+f(dt)dt)x_{dt} = (1+f(dt)dt)(1+f(0)dt)x_0.$$

$$x_t = (1+f(t-dt)dt)\cdots\cdots(1+f(0)dt)x_0.$$

$$\overset{taylor}{=} \exp(f(t-dt)dt)\cdots\exp(f(0)dt)$$

$$= x_0\exp\left(\sum_k f(kdt)dt\right) \overset{k\to\infty}{=} x_0\exp\left(\int_0^t f(u)du\right)$$

再看一个 SDE： $dx = f(t)x\,dt + g(t)\,dB$.

ODE: $dx = f(t)x\,dt$ 解为 $x = x_0 \exp(\int_0^t f(u)du)$

构造 $\mu(t) = \exp(\int_0^t -f(u)du)$

$$\mu(t)dx - \mu(t)f(t)x\,dt = g(t)\mu(t)dB$$

$$\Downarrow$$

$$d(\mu(t)x) = g(t)\mu(t)dB$$

$$d(\mu(t)x) = d\mu(t)\cdot x + \mu(t)dx$$

$$= \mu(t)\cdot(-f(t))x\cdot dt + \mu(t)dx.$$

$$\mu(t)x = \int_0^t g(u)\mu(u)dB + \mu(0)x_0.$$

$\cancel{x =}$ $\cancel{\frac{\mu(u)}{\mu(t)} = \frac{\exp(\int_0^u -f(s)ds)}{\exp(\int_0^t -f(s)ds)}}$

$$\frac{\mu(s)}{\mu(t)} = \frac{\exp(\int_0^s -f(u)du)}{\exp(\int_0^t -f(u)du)} = \exp(\int_s^t f(u)du).$$

$$x = \underbrace{\overbrace{x_0 \exp(\int_0^t f(u)du)}^{\text{衰减项}}}_{\text{确定性项}} + \underbrace{\int_0^t \overbrace{\exp(\int_u^t f(s)ds)}^{\text{衰减项}} g(u)\,dB}_{\text{噪声项}}$$

显然将 $dx = f(t)x\,dt + g(t)dB$ 按照前面 ODE 的写法

可以写成 $\cancel{x = x_0 \exp}$

$$x = x_0 \prod_{u=0}^{t}(1+f(u)du) + \sum_{u=0}^{t}\left(\prod_{s=u}^{t}(1+f(s)ds)\right)g(u)dB.$$

~~Euler~~ taylor

$$= x_0 \exp(\sum_{u=0}^{t} f(u)du) + \sum_{u=0}^{t} \exp(\sum_{s=u}^{t} f(s)ds)\,g(u)dB$$

$$= x_0 \exp(\int_0^t f(u)du) + \int_0^t \exp(\int_u^t f(s)ds)\,g(u)dB$$

回归正题.

$$dx = f(x,t)dt + g(t)dB.$$

则 $x_{t+dt} = x_t + f(x_t,t)dt + g(t)dB.$

$$p(x_{t+dt}|x_t) = \mathcal{N}(x_{t+dt};\ x_t + f(x_t,t)dt,\ g^2(t)dt)$$

按照~~前面~~反向加Bayes，用真实的 $p(x_t|x_{t+dt})$ 逆采样可得 $p(x_0)$

$$p(x_t|x_{t+dt}) = \frac{p(x_{t+dt}|x_t)\,p(x_t)}{p(x_{t+dt})}$$

$$\propto \exp\left(-\frac{[x_{t+dt}-(x_t+f(x_t,t)dt)]^2}{2g^2(t)dt} + \log p(x_t) - \log p(x_{t+dt})\right)$$

$$\log p(x_t) - \log p(x_{t+dt}) = -d\log p(x_t)$$

$$= -\left(\nabla_x \log p(x_t)\,dx + \frac{\partial \log p(x)}{\partial t}dt\right)$$

$$= (x_t - x_{t+dt})\nabla_x \log p(x_t) + O(t).$$

$$\propto \exp\left(-\frac{(x_{t+dt}-(x_t+f(x_t,t)dt))^2}{2g^2(t)dt} + (x_t - x_{t+dt})\nabla_x \log p(x_t)\right)$$

快速配方法：$\dfrac{\partial \log p(x_t|x_{t+dt})}{\partial x_t} = \dfrac{x_{t+dt}-(x_t+f(x_t,t)dt)}{g^2(t)dt} + \nabla_x \log p(x_t) = 0$

$$\Rightarrow x_t = x_{t+dt} - [f(x_t,t) - g^2(t)\nabla_x \log p(x)]dt.$$

取 $\Delta t = -dt$，一阶近似 $h(t+dt)dt = h(t)dt$；

$$\frac{\partial^2 \log p}{\partial x^2} = -\frac{1}{g^2(t)dt} = 2a$$

$$\sigma^2 = -\frac{1}{2a} = g^2(t)dB$$

有 $dx = [f(x,t) - g^2(t)\nabla_x \log p(x)]dt + g(t)dB.$

即前向 $dx = f(x,t)dt + g(t)dB.$

对应逆向 $dx = [f(x,t) - g^2(t)\nabla_x \log p(x)]dt + g(t)dB.$

NCSN:

$$x_{i+1} = x_i + \sqrt{\sigma^2(i+1) - \sigma^2(i)}\,\varepsilon \quad , i = 1, \cdots, N$$

取 ~~$\Delta t = \frac{1}{N}$~~, $t = \frac{i}{N}$, $\Delta t = \frac{1}{N}$.

则 $x_{t+\Delta t} = x_t + \sqrt{\sigma^2(t+\Delta t) - \sigma^2(t)}\,\varepsilon$

$$= x_t + \sqrt{\frac{d[\sigma^2(t)]}{dt}\Delta t}\;\varepsilon$$

$$= x_t + \sqrt{\frac{d[\sigma^2(t)]}{dt}}\;\Delta B$$

$$\Rightarrow dx = \sqrt{\frac{d[\sigma^2(t)]}{dt}}\,dB.$$

解得. $x = x_0 + \int_0^t \frac{d[\sigma^2(u)]}{du}\,du$

$= x_0 + \sigma^2(t)$. 符合原来离散形式.

DDPM: $x_i = \sqrt{1-\beta_i}\,x_{i-1} + \sqrt{\beta_i}\,\varepsilon$. , $i = 1, \cdots, N$

$dt = \frac{1}{N}$

~~$\sqrt{\beta(t)} = \sqrt{\beta_i}$~~ $\boxed{\beta_i = \beta(t)dt}$

取 $\beta(t) = N\beta_{i/N}$, $\Delta t = \frac{1}{N}$

则 $x_t = \sqrt{1-\beta(t)\Delta t}\,x_{t-\Delta t} + \sqrt{\beta(t)\Delta t}\,\varepsilon$

$$\approx \left[1 - \frac{1}{2}\beta(t)\Delta t\right] x_{t-\Delta t} + \sqrt{\beta(t)}\,\Delta B$$

$h(x) = \sqrt{1-x}$

$h(x) = h(0) + \frac{h'(0)}{1!}(x-0)$

$= 1 - \frac{1}{2}x$

$$\therefore\ dx = -\frac{1}{2}\beta(t)x\,dt + \sqrt{\beta(t)}\,dB$$

解得 $x = x_0 \exp\left(\int_0^t -\frac{1}{2}\beta(u)du\right) + \int_0^t \exp\left(\int_u^t -\frac{1}{2}\beta(s)ds\right)\sqrt{\beta(u)}\,dB$

$= x_0 \exp\left(\sum_{u=1}^{t} -\frac{1}{2}\beta(u)du\right) + \sum_{u=1}^{t} \exp\left(\sum_{s=u+1}^{t} -\frac{1}{2}\beta(s)ds\right)\sqrt{\beta(u)}\,dB$

$= x_0 \prod_{u=1}^{t} \exp\left(-\frac{1}{2}\beta(u)du\right) + \sum_{u=1}^{t}\left(\prod_{s=u+1}^{t} \exp\left(-\frac{1}{2}\beta(s)ds\right)\right)\sqrt{\beta(u)}\,dB$

$= x_0 \prod_{u=1}^{t}\left(1 - \frac{1}{2}\beta(u)du\right) + \sum_{u=1}^{t}\left(\prod_{s=u+1}^{t}\left(1 - \frac{1}{2}\beta(s)ds\right)\right)\sqrt{\beta(u)}\,dB.$

$= x_0 \prod_{u=1}^{t}\sqrt{1-\beta(u)du} + \sum_{u=1}^{t}\left(\prod_{s=u+1}^{t}\sqrt{1-\beta(s)ds}\right)\sqrt{\beta(u)}\,dB$

$$= X_0 \cdot \sqrt{\bar{\alpha}_t} + \sum_{u=1}^{t} \left(\prod_{s=u+1}^{t} \sqrt{\alpha_s} \right) \sqrt{\beta(u)}\, dB$$

$$= X_0 \sqrt{\bar{\alpha}_t} + \underbrace{\sum_{u=1}^{t} \left(\prod_{s=u+1}^{t} \sqrt{\alpha_t} \right) \sqrt{\beta_u}\, \varepsilon_u}_{\text{合并高斯噪声}} \quad (坐标稍微对齐一下)$$

$$u=0. \quad \sigma^2 = \alpha_2 \cdots \alpha_t (1-\alpha_1) + \alpha_3 \cdots \alpha_t (1-\alpha_2) + \cdots + 1-\alpha_t.$$

$$= \cancel{\alpha_2 \cdots \alpha_t} - \alpha_1 \cdots \alpha_t + \cancel{\alpha_3 \cdots \alpha_t} - \cancel{\alpha_2 \cdots \alpha_t} + \cdots + 1 - \cancel{\alpha_t}.$$

$$= 1 - \alpha_1 \cdots \alpha_t = 1 - \bar{\alpha}_t$$

$$= \sqrt{\bar{\alpha}_t}\, x_0 + \sqrt{1-\bar{\alpha}_t}\, \bar{\varepsilon}_t \quad 符合原式.$$

$dx = -\frac{1}{2}\beta(t) x\, dt + \sqrt{\beta(t)}\, dB \longrightarrow$ 逆向为 $dx = [-\frac{1}{2}\beta(t) x - \beta(t) \nabla \log p(x)]\, dt + \sqrt{\beta(t)}\, dB$

$$x_{t-dt} = x_t + \frac{1}{2}\beta(t)\, dt\, x_t + \beta(t)\, dt\, \nabla \log p(x) + \sqrt{\beta(t)}\, dB$$

$$= (1 + \frac{1}{2}\beta(t)\, dt)\, x_t + \beta(t)\, dt\, \nabla \log p(x) + \sqrt{\beta(t)}\, dB.$$

$$= \frac{1}{\sqrt{1-\beta(t)\, dt}} (x_t + \beta(t)\, dt \cdot \sqrt{1-\beta(t)\, dt}\, \nabla \log p(x)] + \text{noise}.$$

$$= \frac{1}{\sqrt{1-\beta(t)\, dt}} (x_t + \beta(t)\, dt\, (1 - \frac{1}{2}\beta(t)\, dt)\, \text{Score}] + \text{noise}.$$

$$= \frac{1}{\sqrt{1-\beta(t)\, dt}} (x_t + \beta(t)\, dt\, \text{Score}) + \text{noise}.$$

$$= \frac{1}{\sqrt{\alpha_t}} (x_t + \beta_t\, \text{Score}) + \sqrt{\beta(t)}\, \varepsilon.$$

Score: $\nabla_{x_t}\log p(x_t)$

$$= \frac{1}{p(x_t)}\nabla p(x_t) = \frac{1}{p(x_t)}\nabla_{x_t}\int p(x_t|x_0)\cdot p(x_0)dx_0$$

$$= \frac{1}{p(x_t)}\int \nabla_{x_t} p(x_t|x_0)\, p(x_0)dx_0$$

$$= \frac{1}{p(x_t)}\int[\nabla_{x_t}\log p(x_t|x_0)]\cdot p(x_t|x_0)p(x_0)dx_0$$

$$= \frac{1}{p(x_t)}\int[\nabla_{x_t}\left(\frac{-(x_t-\mu)^2}{2\sigma^2}\right)p(x_t,x_0)dx_0$$

$$= \frac{1}{p(x_t)}\int -\frac{\varepsilon}{\sigma}p(x_t,x_0)dx_0$$

$$= -\frac{\varepsilon}{\sigma} = -\frac{\bar{\varepsilon}}{\sqrt{1-\bar{\alpha}_t}}$$

$$x_{t-1} = \frac{1}{\sqrt{\alpha_t}}\left(x_t - \beta_t\frac{\bar{\varepsilon}}{\sqrt{1-\bar{\alpha}_t}}\right)+\sqrt{\beta_t}\,\varepsilon$$

~~prop~~ ~~pro~~ flo

概率流 ODE：

$dx = f(x,t)dt + g(t)dB \rightarrow$ 逆向 $dx=[f-g^2\text{score}]dt + \text{noise}$

F-K 方程

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial(f(x,t)p(x,t))}{\partial x}+\frac{1}{2}g^2\frac{\partial^2 p(x,t)}{\partial x^2}.$$

等价于 $dx = (f(x,t)-\frac{1}{2}(g^2-\sigma^2)\nabla\log p(x))dt + \sigma dB$

$\forall\ \sigma(t) < g(t)$

反向：$dx = [f(x,t)-\frac{1}{2}(g^2+\sigma^2)\nabla\log p(x)]dt+\sigma dB$

$\sigma\to 0.$

得 ODE：$dx = [f(x,t)-\frac{1}{2}g^2\nabla\log p(x)]dt$