1.1 应力与应变

本文以 CFRC 悬臂板为研究对象，将 MFC 粘贴于 CFRC 板的根部作为致动器和传感器，CFRC 悬臂板的结构如图 1 所示。图一（a）中的蓝色区域为 CFRC-MFC 悬臂层合板的固定约束区域，图一（b）为 CFRC 板的碳纤维材料铺层示意图。

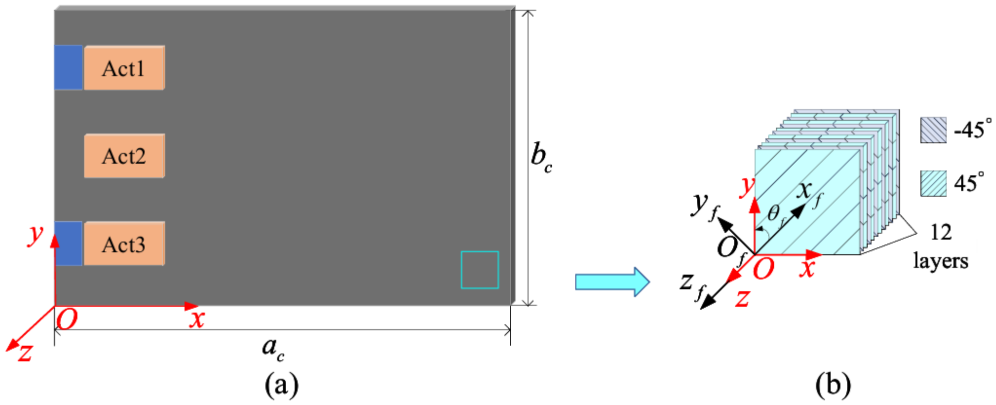

Fig1 CFRC-MFC 悬臂板示意图:(a) 层合板结构;（b）铺层示意图

CFRC 板的单层碳纤维材料在纤维坐标系 $O_f - x_f y_f z_f$ 中描述的线性本构关系如下:

$$\boldsymbol{\sigma}_c^f = \begin{bmatrix} \sigma_{x_f x_f}^c \\ \sigma_{y_f y_f}^c \\ \tau_{x_f y_f}^c \end{bmatrix} = \begin{bmatrix} C_{x_f x_f} & C_{x_f y_f} & 0 \\ C_{y_f x_f} & C_{y_f y_f} & 0 \\ 0 & 0 & G_{x_f y_f}^c \end{bmatrix} \begin{bmatrix} \varepsilon_{x_f x_f}^c \\ \varepsilon_{y_f y_f}^c \\ \gamma_{x_f y_f}^c \end{bmatrix} = \mathbf{D}_c^f \boldsymbol{\varepsilon}_c^f \qquad (1)$$

其中，$\boldsymbol{\sigma}_c^f$，$\boldsymbol{\varepsilon}_c^f$ 和 $\mathbf{D}_c^f$ 分别为单层碳纤维材料的应力矢量，应变矢量和弹性系数矩阵。弹性系数表达式如下:

$$C_{x_f x_f} = \frac{E_1^c}{1-\nu_{12}^c \nu_{21}^c}, C_{y_f y_f} = \frac{E_2^c}{1-\nu_{12}^c \nu_{21}^c}, \quad C_{y_f x_f} = \frac{\nu_{21}^c E_1^c}{1-\nu_{12}^c \nu_{21}^c}, \quad C_{x_f y_f} = \frac{\nu_{12}^c E_2^c}{1-\nu_{12}^c \nu_{21}^c}, G_{x_f y_f}^c = G_{12}^c \qquad (2)$$

其中，E_1^c、E_2^c 分别为碳纤维材料在纤维轴向（xf）和纤维横向(yf)的杨氏模量，G_{12}^c 为剪切模量，ν_{12}^c、ν_{21}^c 为泊松比，1 代指 x_f，2 代指 y_f，$\nu_{21}^c = \nu_{12}^c E_2^c / E_1^c$。

而 MFC 的粘贴位置和结构如图 2 所示。图 2（a）和(b)分别为 CFRC-MFC 悬臂层合板的侧视图和俯视图，展示了 MFC 传感器和致动器的粘贴位置；图 2（c）和(d)分别为 d33 type MFC 和 d31 type MFC 的结构示意图。

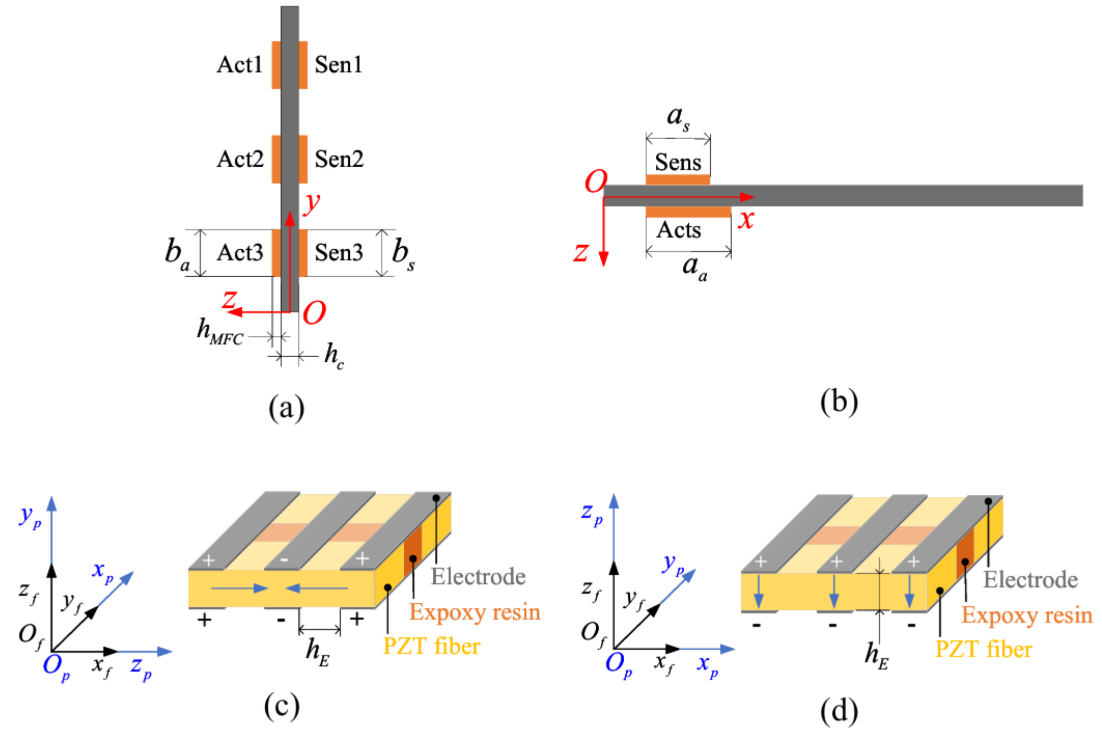

Fig2 MFC 的粘贴位置和结构示意图:(a) 侧视图;（b）俯视图;（c）d33 type MFC;(d)d31 type MFC.

MFC 在纤维坐标系 $O_f - x_f y_f z_f$ 中描述的压电本构关系如下:

$$
\begin{aligned}
\boldsymbol{\sigma}_p^f &= \begin{bmatrix} \sigma^p_{x_f x_f} \\ \sigma^p_{y_f y_f} \\ \tau^p_{x_f y_f} \end{bmatrix} = \begin{bmatrix} P_{x_f x_f} & P_{x_f y_f} & 0 \\ P_{y_f x_f} & P_{y_f y_f} & 0 \\ 0 & 0 & G^p_{x_f y_f} \end{bmatrix} \begin{bmatrix} \varepsilon^p_{x_f x_f} \\ \varepsilon^p_{y_f y_f} \\ \gamma^p_{x_f y_f} \end{bmatrix} - \begin{bmatrix} e_{z_p x_f} \\ e_{z_p y_f} \\ 0 \end{bmatrix} E_{z_p} = \mathbf{D}_p^f \boldsymbol{\varepsilon}_p^f - \mathbf{e}^f E_{z_p} \\
D_{z_p} &= \begin{bmatrix} e_{z_p x_f} & e_{z_p y_f} & 0 \end{bmatrix} \begin{bmatrix} \varepsilon^p_{x_f x_f} \\ \varepsilon^p_{y_f y_f} \\ \gamma^p_{x_f y_f} \end{bmatrix} + \varsigma_{z_p z_p} E_{z_p} = (\mathbf{e}^f)^{\mathrm{T}} \varepsilon_p^f + \varsigma_{z_p z_p} E_{z_p}
\end{aligned} \quad (3)
$$

其中，$\boldsymbol{\sigma}_p^f$、$\boldsymbol{\varepsilon}_p^f$、$\mathbf{D}_p^f$ 为 MFC 的应力矢量、应变矢量和弹性系数矩阵，下标 $p = a, s$，分别代表致动器和传感器。弹性系数表达式如下:

$$
P_{x_f x_f} = \frac{E_1^p}{1 - \nu_{12}^p \nu_{21}^p}, P_{y_f y_f} = \frac{E_2^p}{1 - \nu_{12}^p \nu_{21}^p}, P_{y_f x_f} = \frac{\nu_{21}^p E_1^p}{1 - \nu_{12}^p \nu_{21}^p}, P_{x_f y_f} = \frac{\nu_{12}^p E_2^p}{1 - \nu_{12}^p \nu_{21}^p}, G^p_{x_f y_f} = G^p_{12} \quad (4)
$$

其中，E_1^{p} 和 E_2^{p} 分别是 MFC 纤维主轴方向和纤维横向上的杨氏模量，G_{12}^{p} 是 MFC 的剪切模量，ν_{12}^{p}、ν_{21}^{p} 是 MFC 泊松比。而 $\mathbf{e}^f$、$\varsigma_{z_p z_p}$、E_{z_p}、D_{z_p} 分别为 MFC 的压电应力常数、介电常数、极化方向的电场强度和电位移矢量 。值得注意的是，由图 2（c），(d)可知，d33 型 MFC 和 d31 型 MFC 的极化坐标系并不一致，因此本文采用极化坐标系与纤维坐标系相结合的表示方法，获得了 MFC 压电本构方程的统一形式。此外，压电应力常数 $\mathbf{e}^f$ 各分量和电场强度 E_{z_p} 的表达式如下：

$$e_{z_p x_f} = d_{z_p x_f} P_{x_f x_f} + d_{z_p y_f} P_{y_f x_f},$$
$$e_{z_p y_f} = d_{z_p x_f} P_{x_f y_f} + d_{z_p y_f} P_{y_f y_f}, \quad (5)$$
$$E_{z_p} = \frac{\phi_{z_p}}{h_E}$$

其中，$d_{z_p x_f}$ 和 $d_{z_p y_f}$ 为压电应变常数 $\mathbf{d}^f$ 的分量，ϕ_{z_p} 为极化方向上的外加电压，h_E 为叉状电极正负极间的距离。smartstruture 公司发布的 mfc 的压电参数 d33 等的描述方式是基于极化坐标系 $O_p - x_p y_p z_p$ 的，在本文中 d33 型（P1 型）MFC，$d_{z_p x_f} = d_{33}$，而对于 d31 型（P2 型）MFC，$d_{z_p x_f} = d_{31}$。

根据复合材料力学，在纤维坐标系 $O_f - x_f y_f z_f$ 内描述的应力矢量和应变矢量与在整体坐标系 $O - xyz$ 内描述的应力矢量和应变矢量间的转换关系如下：

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \left. \begin{bmatrix} \mathrm{c}^2 & \mathrm{s}^2 & -2\mathrm{sc} \\ \mathrm{s}^2 & \mathrm{c}^2 & 2\mathrm{sc} \\ \mathrm{sc} & -\mathrm{sc} & \mathrm{c}^2 - \mathrm{s}^2 \end{bmatrix} \right|_{\theta=\theta_f} \begin{bmatrix} \sigma_{x_f x_f} \\ \sigma_{y_f y_f} \\ \tau_{y_f y_f} \end{bmatrix} = \mathbf{T}_\sigma \boldsymbol{\sigma}^f \quad (6)$$

$$\boldsymbol{\varepsilon}^f = \begin{bmatrix} \varepsilon_{x_f x_f} \\ \varepsilon_{y_f y_f} \\ \gamma_{x_f y_f} \end{bmatrix} = \left. \begin{bmatrix} \mathrm{c}^2 & \mathrm{s}^2 & \mathrm{sc} \\ \mathrm{s}^2 & \mathrm{c}^2 & -\mathrm{sc} \\ -2\mathrm{sc} & 2\mathrm{sc} & \mathrm{c}^2 - \mathrm{s}^2 \end{bmatrix} \right|_{\theta=\theta_f} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \mathbf{T}_\varepsilon \boldsymbol{\varepsilon} \quad (7)$$

其中，$c = \cos(\theta)$，$s = \sin(\theta)$，$\theta = \theta_f$ 为纤维坐标系与整体坐标系间的偏转角度。结合式(6),(7)和式(1)可得在整体坐标系 $O - xyz$ 中描述的 CFRC 板单层

碳纤维材料的线性本构关系和 MFC 压电贴片的压电本构关系如下：

$$\boldsymbol{\sigma}_c^k = \mathbf{T}_\sigma^k \mathbf{D}_c^f \mathbf{T}_\varepsilon^k \boldsymbol{\varepsilon}_c^k = \bar{\mathbf{D}}_c^k \boldsymbol{\varepsilon}_c^k \quad (8)$$

$$\boldsymbol{\sigma}_p = \mathbf{T}_\sigma^p \mathbf{D}_p^f \mathbf{T}_\varepsilon^p \boldsymbol{\varepsilon}_p - \mathbf{T}_\sigma^p \mathbf{e}^f E_{z_p} = \bar{\mathbf{D}}_p \boldsymbol{\varepsilon}_p - \bar{\mathbf{e}} E_{z_p}$$

$$D_{z_p} = \left(\left(\mathbf{T}_\varepsilon^p\right)^{\mathrm{T}} \mathbf{e}^f\right)^{\mathrm{T}} \varepsilon_p + \varsigma_{z_p z_p} E_{z_p} = \left(\bar{\mathbf{e}}\right)^{\mathrm{T}} \boldsymbol{\varepsilon}_p + \varsigma_{z_p z_p} E_{z_p} \quad (9)$$

其中，$\boldsymbol{\sigma}_c^k$ 和 $\boldsymbol{\varepsilon}_c^k$ 分别为第 k 层碳纤维材料在整体坐标系 $O-xyz$ 内的应力矢量和应变矢量，$\boldsymbol{\sigma}_p$ 和 $\boldsymbol{\varepsilon}_p$ 分别为 MFC 压电片在整体坐标系 $O-xyz$ 内的应力矢量和应变矢量。$\mathbf{T}_\sigma^k = \mathbf{T}_\sigma\big|_{\theta=\theta_f^k}$，$\mathbf{T}_\varepsilon^k = \mathbf{T}_\varepsilon\big|_{\theta=\theta_f^k}$，$\mathbf{T}_\sigma^p = \mathbf{T}_\sigma\big|_{\theta=\theta_f^p}$，$\mathbf{T}_\varepsilon^p = \mathbf{T}_\varepsilon\big|_{\theta=\theta_f^p}$，$\theta_f^k$ 为第 k 层碳纤维材料层的铺层角度，θ_f^p 为 MFC 传感器（p=s）或致动器（p = a）的粘贴角度。

接下来，本文基于经典板理论，结合 ANCF 理论建立ＣＦＲＣ-MFC 层合板在整体坐标系 $O-xyz$ 的应变—位移关系：

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \frac{1}{2}\left(\left(\tilde{\mathbf{r}}_{,x}\right)^{\mathrm{T}} \tilde{\mathbf{r}}_{,x} - 1\right) \\ \frac{1}{2}\left(\left(\tilde{\mathbf{r}}_{,y}\right)^{\mathrm{T}} \tilde{\mathbf{r}}_{,y} - 1\right) \\ \left(\tilde{\mathbf{r}}_{,x}\right)^{\mathrm{T}} \tilde{\mathbf{r}}_{,y} \end{bmatrix} \quad (10)$$

$\tilde{\mathbf{r}}$ 为板单元内任一点的位置矢量

$$\tilde{\mathbf{r}} = \mathbf{r} + z\mathbf{n}_0 \quad (11)$$

$\mathbf{r}$ 和 $\mathbf{n}_0$ 分别为中面上任意点的位置矢量和单位法向量。

$$\mathbf{n}_0 = \frac{\mathbf{n}}{\bar{n}}, \mathbf{n} = \mathbf{r}_{,x} \times \mathbf{r}_{,y}, \bar{n} = \sqrt{\mathbf{n}^{\mathrm{T}}\mathbf{n}} \quad (12)$$

$\mathbf{n}$ 为板中面上任意点的法向量，其具体表达式详见于附录 A，$\bar{n}$ 为 $\mathbf{n}$ 的模长。

将式(12)代入式(11)，忽略 z^2 的高阶项，可推得：

$$\boldsymbol{\varepsilon}=\begin{bmatrix}\frac{1}{2}\left(\left(\mathbf{r}_{,x}\right)^{\mathrm{T}}\mathbf{r}_{,x}-1\right)\\ \frac{1}{2}\left(\left(\mathbf{r}_{,y}\right)^{\mathrm{T}}\mathbf{r}_{,y}-1\right)\\ \left(\mathbf{r}_{,x}\right)^{\mathrm{T}}\mathbf{r}_{,y}\end{bmatrix}+z\begin{bmatrix}\left(\mathbf{r}_{,x}\right)^{\mathrm{T}}\frac{\partial\mathbf{n}_0}{\partial x}\\ \left(\mathbf{r}_{,y}\right)^{\mathrm{T}}\left(\frac{\partial\mathbf{n}_0}{\partial y}\right)\\ \left(\mathbf{r}_{,x}\right)^{\mathrm{T}}\frac{\partial\mathbf{n}_0}{\partial y}+\left(\frac{\partial\mathbf{n}_0}{\partial x}\right)^{\mathrm{T}}\mathbf{r}_{,y}\end{bmatrix}\tag{13}$$

由于$\left(\mathbf{r}_{,x}\right)^{\mathrm{T}}\mathbf{n}_0=0,\left(\mathbf{r}_{,y}\right)^{\mathrm{T}}\mathbf{n}_0=0$，对等式两边分别对 x,y 求导可得:

$$\begin{aligned}&\frac{\partial\left(\left(\mathbf{r}_{,x}\right)^{\mathrm{T}}\mathbf{n}_0\right)}{\partial x}=\left(\mathbf{n}_0\right)^{\mathrm{T}}\frac{\partial\left(\mathbf{r}_{,x}\right)}{\partial x}+\left(\mathbf{r}_{,x}\right)^{\mathrm{T}}\frac{\partial\left(\mathbf{n}_0\right)}{\partial x},\frac{\partial\left(\left(\mathbf{r}_{,x}\right)^{\mathrm{T}}\mathbf{n}_0\right)}{\partial y}=\left(\mathbf{n}_0\right)^{\mathrm{T}}\frac{\partial\left(\mathbf{r}_{,x}\right)}{\partial y}+\left(\mathbf{r}_{,x}\right)^{\mathrm{T}}\frac{\partial\left(\mathbf{n}_0\right)}{\partial y}\\&\frac{\partial\left(\left(\mathbf{r}_{,y}\right)^{\mathrm{T}}\mathbf{n}_0\right)}{\partial x}=\left(\mathbf{n}_0\right)^{\mathrm{T}}\frac{\partial\left(\mathbf{r}_{,y}\right)}{\partial x}+\left(\mathbf{r}_{,y}\right)^{\mathrm{T}}\frac{\partial\left(\mathbf{n}_0\right)}{\partial x},\frac{\partial\left(\left(\mathbf{r}_{,y}\right)^{\mathrm{T}}\mathbf{n}_0\right)}{\partial y}=\left(\mathbf{n}_0\right)^{\mathrm{T}}\frac{\partial\left(\mathbf{r}_{,y}\right)}{\partial y}+\left(\mathbf{r}_{,y}\right)^{\mathrm{T}}\frac{\partial\left(\mathbf{n}_0\right)}{\partial y}\end{aligned}\tag{14}$$

式中的等式左端都是 0，于是推得:

$$\begin{aligned}&\left(\mathbf{r}_{,x}\right)^{\mathrm{T}}\frac{\partial\left(\mathbf{n}_0\right)}{\partial x}=-\left(\mathbf{n}_0\right)^{\mathrm{T}}\mathbf{r}_{,xx},\left(\mathbf{r}_{,x}\right)^{\mathrm{T}}\frac{\partial\left(\mathbf{n}_0\right)}{\partial y}=-\left(\mathbf{n}_0\right)^{\mathrm{T}}\mathbf{r}_{,xy}\\&\left(\mathbf{r}_{,y}\right)^{\mathrm{T}}\frac{\partial\left(\mathbf{n}_0\right)}{\partial x}=-\left(\mathbf{n}_0\right)^{\mathrm{T}}\mathbf{r}_{,yx},\left(\mathbf{r}_{,y}\right)^{\mathrm{T}}\frac{\partial\left(\mathbf{n}_0\right)}{\partial y}=-\left(\mathbf{n}_0\right)^{\mathrm{T}}\mathbf{r}_{,yy}\end{aligned}\tag{15}$$

将式(15)代入式(13)，于是推得:

$$\boldsymbol{\varepsilon}=\begin{bmatrix}\frac{1}{2}\left(\left(\mathbf{r}_{,x}\right)^{\mathrm{T}}\mathbf{r}_{,x}-1\right)\\ \frac{1}{2}\left(\left(\mathbf{r}_{,y}\right)^{\mathrm{T}}\mathbf{r}_{,y}-1\right)\\ \left(\mathbf{r}_{,x}\right)^{\mathrm{T}}\mathbf{r}_{,y}\end{bmatrix}-z\begin{bmatrix}\left(\mathbf{r}_{,xx}\right)^{\mathrm{T}}\mathbf{n}_0\\ \left(\mathbf{r}_{,yy}\right)^{\mathrm{T}}\mathbf{n}_0\\ 2\left(\mathbf{r}_{,xy}\right)^{\mathrm{T}}\mathbf{n}_0\end{bmatrix}=\boldsymbol{\varepsilon}_m-z\boldsymbol{\kappa}\tag{16}$$

其中，$\boldsymbol{\varepsilon}_m$ 和 $\boldsymbol{\kappa}$ 分别为板中面的膜应变和弯曲曲率。

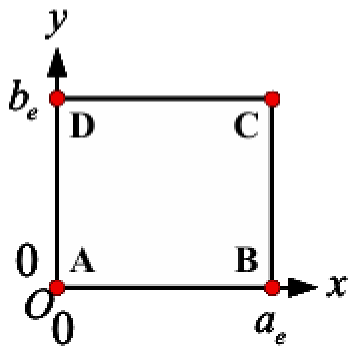

Fig3 单元节点示意图

采用有限元方法划分网格，单元内节点和局部坐标系如图 3 所示，其中a_e和b_e分别为单元长度和宽度。采用形函数对单元内中面上任意点的位置矢量$\mathbf{r}$进行离散可得：

$$\mathbf{r} = \mathbf{S}\mathbf{q}_e \qquad (17)$$

其中，$\mathbf{q}_e$为四节点单元位移矢量，表达式如下：

$$\mathbf{q}_e = \left[\left(\mathbf{q}_e^1\right)^{\mathrm{T}}, \left(\mathbf{q}_e^2\right)^{\mathrm{T}}, \left(\mathbf{q}_e^3\right)^{\mathrm{T}}, \left(\mathbf{q}_e^4\right)^{\mathrm{T}} \right]^{\mathrm{T}} \qquad (18)$$

$$\mathbf{q}_e^i = \left[\mathbf{r}^{\mathrm{T}}, \left(\mathbf{r}_{,x}\right)^{\mathrm{T}}, \left(\mathbf{r}_{,y}\right)^{\mathrm{T}} \right]^{\mathrm{T}}, i = 1,2,3,4 \qquad (19)$$

其中，$\mathbf{q}_e^i$为单元内第i个节点的位移矢量，单元位移矢量与整体位移矢量间的关系为：

$$\mathbf{q}_e = \mathbf{B}_e\mathbf{q} \qquad (20)$$

其中，$\mathbf{B}_e$为单元 e 的布尔矩阵。

$\mathbf{S}$为单元的形函数，表达式如下：

$$\mathbf{S} = \left[S_1 I_{3\times3} \quad S_2 I_{3\times3} \quad \mathrm{L} \quad S_{12} I_{3\times3} \right] \qquad (21)$$

$$
\begin{aligned}
&S_1=-(\xi-1)(\eta-1)\left(2\eta^2-\eta+2\xi^2-\xi-1\right)\\
&S_2=-a_e\xi(\xi-1)^2(\eta-1)\\
&S_3=-b_e\eta(\xi-1)(\eta-1)^2\\
&S_4=\xi(\eta-1)\left(2\eta^2-\eta+2\xi^2-3\xi\right)\\
&S_5=-a_e\xi^2(\xi-1)(\eta-1)\\
&S_6=b_e\xi\eta(\eta-1)^2\\
&S_7=-\xi\eta\left(2\eta^2-3\eta+2\xi^2-3\xi+1\right)\\
&S_8=a_e\xi^2\eta(\xi-1)\\
&S_9=b_e\xi\eta^2(\eta-1)\\
&S_{10}=\eta(\xi-1)\left(2\eta^2-3\eta+2\xi^2-\xi\right)\\
&S_{11}=a_e\xi\eta(\xi-1)^2\\
&S_{12}=-b_e\eta^2(\xi-1)(\eta-1)
\end{aligned}
\tag{22}
$$

其中， a_e和b_e分别代表未变形状态下板单元的长度和宽度；$I_{3\times3}$是一个3×3的单位矩阵。$\xi=\dfrac{x}{a_e},\eta=\dfrac{y}{b_e}$。$(x,y)$为未变形时中面上任意点在板单元坐标系内的坐标。形函数顺序对应的单元节点顺序为 $\mathrm{A}(0,0)$，$\mathrm{B}(a_e,0)$，$\mathrm{C}(a_e,b_e)$，$\mathrm{D}(0,b_e)$，这决定了单元节点坐标在整体坐标中的对应位置。

将式(17)代入式(16)中，获得面内膜应变$\boldsymbol{\varepsilon}_m$和弯曲曲率$\boldsymbol{\kappa}$的表达式如下：

$$
\boldsymbol{\varepsilon}_m=\frac{1}{2}\begin{bmatrix}(\mathbf{r}_{,x})^{\mathrm T}\mathbf{r}_{,x}-1\\(\mathbf{r}_{,y})^{\mathrm T}\mathbf{r}_{,y}-1\\2(\mathbf{r}_{,x})^{\mathrm T}\mathbf{r}_{,y}\end{bmatrix}=\frac{1}{2}\begin{bmatrix}(\mathbf{S}_{,x}\mathbf{q}_e)^{\mathrm T}\mathbf{S}_{,x}\mathbf{q}_e\\(\mathbf{S}_{,y}\mathbf{q}_e)^{\mathrm T}\mathbf{S}_{,y}\mathbf{q}_e\\2(\mathbf{S}_{,x}\mathbf{q}_e)^{\mathrm T}\mathbf{S}_{,y}\mathbf{q}_e\end{bmatrix}-\frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix}
\tag{23}
$$

$$\boldsymbol{\kappa}=\begin{bmatrix}\dfrac{\left(\mathbf{r}_{,xx}\right)^{\mathrm{T}}\mathbf{n}}{\bar{n}}\\ \dfrac{\left(\mathbf{r}_{,yy}\right)^{\mathrm{T}}\mathbf{n}}{\bar{n}}\\ \dfrac{\left(\mathbf{r}_{,xy}\right)^{\mathrm{T}}\mathbf{n}}{\bar{n}}\end{bmatrix}=\begin{bmatrix}\left(\mathbf{S}_{,xx}\mathbf{q}_{e}\right)^{\mathrm{T}}\\ \left(\mathbf{S}_{,yy}\mathbf{q}_{e}\right)^{\mathrm{T}}\\ \left(2\mathbf{S}_{,xy}\mathbf{q}_{e}\right)^{\mathrm{T}}\end{bmatrix}\frac{\mathbf{n}}{\bar{n}} \qquad (24)$$

综上所述，本小节建立了 CFRC-MFC 层合板在整体坐标系 $O-xyz$ 内的应力-应变关系和应变-位移关系，接下来在下一小节将基于 Hamilton 原理建立 CFRC-MFC 层合板的动力学方程。

1.2 ANCF 动力学方程推导

首先，Hamilton 原理是常用的动力学建模方法，其表达式如下：

$$\delta\int_{t_1}^{t_2}L\mathrm{d}t=0 \qquad (25)$$

其中，L 为拉格朗日量，其表达式如下：

$$L=\left(T_c+T_p\right)-\left(U_c+U_p\right)+\left(H_F-H_p\right) \qquad (26)$$

其中，T_c、T_p、U_c、U_p、H_F 和 H_p 分别为 CFRC 层合板的动能、MFC 压电片的动能、CFRC 层合板的应变能、MFC 压电片的应变能、集中力所做功和 MFC 压电片外加电压所做的功，$p=a,s$。

对式（25）进行变分推导可得拉格朗日方程如下：

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\mathbf{q}}}\right)+\frac{\partial L}{\partial\mathbf{q}}=0 \qquad (27)$$

上式中的 $\mathbf{q}$ 代表独立变量，实际上包含 $\mathbf{q}$（整体位移矢量）、ϕ_a（MFC 致动器电压）和 ϕ_s（MFC 传感器电压），分别获得如下表达式：

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial L}{\partial \dot{\mathbf{q}}}\right)+\dfrac{\partial L}{\partial \mathbf{q}}=0 \\ -\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial L}{\partial \dot{\phi}_a}\right)+\dfrac{\partial L}{\partial \phi_a}=0 \\ -\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial L}{\partial \dot{\phi}_s}\right)+\dfrac{\partial L}{\partial \phi_s}=0 \end{cases} \tag{28}$$

将式(26)代入式(27)可得

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial T_c}{\partial \dot{\mathbf{q}}}+\dfrac{\partial T_p}{\partial \dot{\mathbf{q}}}\right)+\left(-\dfrac{\partial U_c}{\partial \mathbf{q}}-\dfrac{\partial U_p}{\partial \mathbf{q}}+\dfrac{\partial H_F}{\partial \mathbf{q}}-\dfrac{\partial H_p}{\partial \mathbf{q}}\right)=0 \\ -\dfrac{\partial U_a}{\partial \phi_a}-\dfrac{\partial H_a}{\partial \phi_a}=0 \\ \dfrac{\partial U_s}{\partial \phi_a}=0 \end{cases} \tag{29}$$

由式（29）可知推导动力学方程需要能量方程及其对独立变量的导数，CFRC 层合板的动能和应变能表达式如下：

$$T_c=\frac{1}{2}\iiint_{V_c}\rho_c \dot{\mathbf{r}}^{\mathrm{T}}\dot{\mathbf{r}}\mathrm{d}V=\frac{1}{2}\left(\dot{\mathbf{q}}\right)^{\mathrm{T}}\sum_e^{N_e^c}\left(\mathbf{B}_e\right)^{\mathrm{T}}\iint_{S_e}I_{c1}\left(\mathbf{S}\right)^{\mathrm{T}}\mathbf{S}\mathrm{d}x\mathrm{d}y\mathbf{B}_e\dot{\mathbf{q}}=\frac{1}{2}\left(\dot{\mathbf{q}}\right)^{\mathrm{T}}\mathbf{M}_c\dot{\mathbf{q}} \tag{30}$$

$$\begin{aligned} U_c&=\frac{1}{2}\iiint_{V_c}\left(\boldsymbol{\varepsilon}_c\right)^{\mathrm{T}}\boldsymbol{\sigma}_c\mathrm{d}V=\frac{1}{2}\sum_e^{N_e^c}\iint_{S_e}\sum_{k=1}^{N_f}\int_{h_b^k}^{h_u^k}\left(\boldsymbol{\varepsilon}_c^k\right)^{\mathrm{T}}\bar{\mathbf{D}}_c^k\boldsymbol{\varepsilon}_c^k\mathrm{d}z\mathrm{d}x\mathrm{d}y \\ &=\frac{1}{2}\sum_e^{N_e^c}\iint_{S_e}\left(\boldsymbol{\varepsilon}_m\right)^{\mathrm{T}}\mathbf{D}_{c1}\boldsymbol{\varepsilon}_m+2\left(\boldsymbol{\kappa}\right)^{\mathrm{T}}\mathbf{D}_{c2}\boldsymbol{\varepsilon}_m+\left(\boldsymbol{\kappa}\right)^{\mathrm{T}}\mathbf{D}_{c3}\boldsymbol{\kappa}\mathrm{d}x\mathrm{d}y \end{aligned} \tag{31}$$

其中，N_e^c是 CFRC 板的单元数量，N_f为碳纤维材料层数，ρ_c为复合材料密度，$\mathbf{D}_{c1}$、$\mathbf{D}_{c2}$、$\mathbf{D}_{c3}$和I_{c1}等刚度和惯性系数矩阵的具体内容见于附录 A。 k 表示碳纤维材料层的编号，h_u^k、h_b^k分别为第 k 层碳纤维材料的上界和下界。将式(30),(31)代入式(29)的求导过程可以获得：

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T_c}{\partial \dot{\mathbf{q}}}\right)=\left(\ddot{\mathbf{q}}\right)^{\mathrm{T}}\mathbf{M}_c \tag{32}$$

$$
\begin{aligned}
\frac{\partial U_c}{\partial \mathbf{q}} &= \frac{1}{2}\sum_{e}^{N_e^c}\iint_{S_e}\left(\frac{\partial\left(\boldsymbol{\varepsilon}_m\right)^{\mathrm{T}}\mathbf{D}_{c1}\boldsymbol{\varepsilon}_m}{\partial \mathbf{q}_e}+2\frac{\partial\left(\boldsymbol{\kappa}\right)^{\mathrm{T}}\mathbf{D}_{c2}\boldsymbol{\varepsilon}_m}{\partial \mathbf{q}_e}+\frac{\partial\left(\boldsymbol{\kappa}\right)^{\mathrm{T}}\mathbf{D}_{c3}\boldsymbol{\kappa}}{\partial \mathbf{q}_e}\right)\frac{\partial \mathbf{q}_e}{\partial \mathbf{q}}\mathrm{d}x\mathrm{d}y \\
&= \sum_{e}^{N_e^c}\iint_{S_e}\left(\left(\boldsymbol{\varepsilon}_m\right)^{\mathrm{T}}\mathbf{D}_{c1}\frac{\partial \boldsymbol{\varepsilon}_m}{\partial \mathbf{q}_e}+\left(\boldsymbol{\kappa}\right)^{\mathrm{T}}\mathbf{D}_{c2}\frac{\partial \boldsymbol{\varepsilon}_m}{\partial \mathbf{q}_e}+\left(\boldsymbol{\varepsilon}_m\right)^{\mathrm{T}}\mathbf{D}_{c2}\frac{\partial \boldsymbol{\kappa}}{\partial \mathbf{q}_e}+\left(\boldsymbol{\kappa}\right)^{\mathrm{T}}\mathbf{D}_{c3}\frac{\partial \boldsymbol{\kappa}}{\partial \mathbf{q}_e}\right)\mathbf{B}_e\mathrm{d}x\mathrm{d}y \\
&= \sum_{e}^{N_e^c}\iint_{S_e}\left(\left(\boldsymbol{\varepsilon}_m\right)^{\mathrm{T}}\mathbf{D}_{c1}\boldsymbol{\varepsilon}_{m,q_e}+\left(\boldsymbol{\kappa}\right)^{\mathrm{T}}\mathbf{D}_{c2}\boldsymbol{\varepsilon}_{m,q_e}+\left(\boldsymbol{\varepsilon}_m\right)^{\mathrm{T}}\mathbf{D}_{c2}\boldsymbol{\kappa}_{,q_e}+\left(\boldsymbol{\kappa}\right)^{\mathrm{T}}\mathbf{D}_{c3}\boldsymbol{\kappa}_{,q_e}\right)\mathbf{B}_e\mathrm{d}x\mathrm{d}y \\
&= \left(\mathbf{F}_c^{ela}\right)^{\mathrm{T}}
\end{aligned}
\tag{33}
$$

其中，$\mathbf{M}_c$ 为 CFRC 层合板的质量阵，$\mathbf{F}_c^{ela}$ 为 CFRC 层合板的非线性弹性力矢量，$\boldsymbol{\varepsilon}_{m,q_e}$ 和 $\boldsymbol{\kappa}_{,q_e}$ 的表达式见于附录 A。

MFC 压电片的动能和应变能的表达式如下：

$$
T_p = \frac{1}{2}\iiint_{V_p}\rho_p \dot{\mathbf{r}}^{\mathrm{T}}\dot{\mathbf{r}}\mathrm{d}V = \frac{1}{2}\left(\dot{\mathbf{q}}\right)^{\mathrm{T}}\sum_{e}^{N_e^p}\left(\mathbf{B}_e\right)^{\mathrm{T}}\iint_{S_e}I_{p1}\left(\mathbf{S}\right)^{\mathrm{T}}\mathbf{S}\mathrm{d}x\mathrm{d}y\mathbf{B}_e\dot{\mathbf{q}} = \frac{1}{2}\left(\dot{\mathbf{q}}\right)^{\mathrm{T}}\mathbf{M}_p\dot{\mathbf{q}} \tag{34}
$$

$$
\begin{aligned}
U_p &= \frac{1}{2}\iiint_{V_p}\left(\boldsymbol{\varepsilon}_p\right)^{\mathrm{T}}\boldsymbol{\sigma}_p\mathrm{d}V - \frac{1}{2}\iiint_{V_p}D_{z_p}E_{z_p}\mathrm{d}V \\
&= \frac{1}{2}\sum_{e}^{N_e^p}\iint_{S_e}\left(\boldsymbol{\varepsilon}_m\right)^{\mathrm{T}}\mathbf{D}_{p1}\boldsymbol{\varepsilon}_m+2\left(\boldsymbol{\kappa}\right)^{\mathrm{T}}\mathbf{D}_{p2}\boldsymbol{\varepsilon}_m+\left(\boldsymbol{\kappa}\right)^{\mathrm{T}}\mathbf{D}_{p3}\boldsymbol{\kappa}\mathrm{d}x\mathrm{d}y \\
&- \sum_{e}^{N_e^p}\iint_{S_e}\phi_p\left(\mathbf{e}_{p1}\right)^{\mathrm{T}}\frac{1}{h_E}\boldsymbol{\varepsilon}_m+\phi_p\left(\mathbf{e}_{p2}\right)^{\mathrm{T}}\frac{1}{h_E}\boldsymbol{\kappa}\mathrm{d}x\mathrm{d}y \\
&- \frac{1}{2}\sum_{e}^{N_e^p}\iint_{S_e}\varsigma_{z_pz_p}^{1}\left(\frac{\phi_p}{h_E}\right)^2\mathrm{d}x\mathrm{d}y
\end{aligned}
\tag{35}
$$

其中，N_e^p 是单个 MFC 传感器（$p=s$）或致动器（$p=a$）的单元数量，ρ_p 为 MFC 的密度。I_{p1}、$\mathbf{D}_{p1}$、$\mathbf{D}_{p2}$、$\mathbf{D}_{p3}$、$\mathbf{e}_{p1}$、$\mathbf{e}_{p2}$、$\varsigma_{z_pz_p}^{1}$ 等参数矩阵可以通过在厚度方向上进行积分运算获得，具体内容见于附录 A。将式(34),(35)代入式(29)的求导过程可以获得：

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T_p}{\partial \dot{\mathbf{q}}}\right) = \left(\ddot{\mathbf{q}}\right)^{\mathrm{T}}\mathbf{M}_p \tag{36}
$$

$$
\begin{aligned}
\frac{\partial U_p}{\partial \mathbf{q}} &= \frac{1}{2}\sum_e^{N_e^p}\iint_{S_e}\left(\frac{\partial(\boldsymbol{\varepsilon}_m)^{\mathrm{T}}\mathbf{D}_{p1}\boldsymbol{\varepsilon}_m}{\partial \mathbf{q}_e}+2\frac{\partial(\boldsymbol{\kappa})^{\mathrm{T}}\mathbf{D}_{p2}\boldsymbol{\varepsilon}_m}{\partial \mathbf{q}_e}+\frac{\partial(\boldsymbol{\kappa})^{\mathrm{T}}\mathbf{D}_{p3}\boldsymbol{\kappa}}{\partial \mathbf{q}_e}\right)\frac{\partial \mathbf{q}_e}{\partial \mathbf{q}}\mathrm{d}x\mathrm{d}y \\
&-\sum_e^{N_e^p}\iint_{S_e}\left(\phi_p(\mathbf{e}_{p1})^{\mathrm{T}}\frac{1}{h_E}\frac{\partial \boldsymbol{\varepsilon}_m}{\partial \mathbf{q}_e}+\phi_p(\mathbf{e}_{p2})^{\mathrm{T}}\frac{1}{h_E}\frac{\partial \boldsymbol{\kappa}}{\partial \mathbf{q}_e}\right)\frac{\partial \mathbf{q}_e}{\partial \mathbf{q}}\mathrm{d}x\mathrm{d}y \qquad (37) \\
&= \sum_e^{N_e^p}\iint_{S_e}\left((\boldsymbol{\varepsilon}_m)^{\mathrm{T}}\mathbf{D}_{p1}\boldsymbol{\varepsilon}_{m,q_e}+(\boldsymbol{\kappa})^{\mathrm{T}}\mathbf{D}_{p2}\boldsymbol{\varepsilon}_{m,q_e}+(\boldsymbol{\varepsilon}_m)^{\mathrm{T}}\mathbf{D}_{p2}\boldsymbol{\kappa}_{,q_e}+(\boldsymbol{\kappa})^{\mathrm{T}}\mathbf{D}_{p3}\boldsymbol{\kappa}_{,q_e}\right)\mathbf{B}_e\mathrm{d}x\mathrm{d}y \\
&-\sum_e^{N_e^p}\iint_{S_e}\left(\phi_p(\mathbf{e}_{p1})^{\mathrm{T}}\frac{1}{h_E}\boldsymbol{\varepsilon}_{m,q_e}+\phi_p(\mathbf{e}_{p2})^{\mathrm{T}}\frac{1}{h_E}\boldsymbol{\kappa}_{,q_e}\right)\mathbf{B}_e\mathrm{d}x\mathrm{d}y \\
&= \left(\mathbf{F}_p^{ela}\right)^{\mathrm{T}}-\mathbf{K}_{cp}\phi_p
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial U_p}{\partial \phi_p} &= -\sum_e^{N_e^p}\iint_{S_e}(\mathbf{e}_{p1})^{\mathrm{T}}\frac{1}{h_E}\boldsymbol{\varepsilon}_m+(\mathbf{e}_{p2})^{\mathrm{T}}\frac{1}{h_E}\boldsymbol{\kappa}\mathrm{d}x\mathrm{d}y \\
&-\sum_e^{N_e^p}\iint_{S_e}\zeta^1_{z_p z_p}\left(\frac{1}{h_E}\right)^2\phi_p\mathrm{d}x\mathrm{d}y \qquad (38) \\
&= -K_{pc}-K_{pp}\phi_p
\end{aligned}
$$

其中，$\mathbf{M}_p$ 为 MFC 压电片的质量阵，$\mathbf{F}_p^{ela}$ 为 MFC 压电片的非线性弹性力矢量，而 $\mathbf{K}_{cp}$ 和 K_{pc} 分别是 MFC 压电片的机电耦合系数矩阵和等效机电耦合系数，K_{pp} 是 MFC 压电片的等效介电系数。

集中力所做的功表达式如下：

$$
H_F=\mathbf{F}\mathbf{r}(x_F,y_F,t)=\mathbf{F}\mathbf{S}_F\mathbf{q} \qquad (39)
$$

其中，$\mathbf{F}=\begin{bmatrix}0 & 0 & F_{ext}\end{bmatrix}$ 为外部集中力，F_{ext} 为集中力幅值。$\mathbf{S}_F=\mathbf{S}(x_F,y_F)$，$(x_F,y_F)$ 为集中力施加位置所在单元内局部坐标系下的坐标。等式两边都对 $\mathbf{q}$ 求导获得：

$$
\frac{\partial H_F}{\partial \mathbf{q}}=\mathbf{F}\mathbf{S}_F \qquad (40)
$$

在 MFC 致动器外部施加的电压所做的功的表达式如下：

$$
H_p=\frac{1}{2}\left(\frac{h_E}{h_p}\right)^2K_{aa}(\phi_a)^2 \qquad (41)
$$

等式两边都对 ϕ_a 求导获得：

$$\frac{\partial H_p}{\partial \phi_a} = \left(\frac{h_E}{h_p}\right)^2 K_{aa}\phi_a^c \qquad (42)$$

在实际控制中，式（42）右边的ϕ_a是指前一时间步内保持的$\phi_a\left(t-\mathrm{d}t\right)$，即在电压$\phi_a$变化的微时间段 dt 内，认为$\dfrac{\partial H_p}{\partial \phi_a}$不变。或可以将式（42）右边的$\phi_a$理解为是施加的控制电压$\phi_a^c$，施加了之后就变成了节点上的独立变量$\phi_a$，并在控制电压再次施加前随应变变化而产生变化$\mathrm{d}\phi_a$。

将式（32)、(33)、(36)、(37)、(38)、(40)、(42）代入式（29）推得如下非线性动力学方程组：

$$\left(\mathbf{M}_c + \mathbf{M}_a + \mathbf{M}_s\right)\ddot{\mathbf{q}} + \mathbf{F}_c^{ela} + \mathbf{F}_a^{ela} + \mathbf{F}_s^{ela} - \left(\mathbf{K}_{ca}\right)^{\mathrm{T}}\phi_a - \left(\mathbf{K}_{cs}\right)^{\mathrm{T}}\phi_s - \left(\mathbf{FS}_F\right)^{\mathrm{T}} = 0$$
（43）

$$K_{ac} + K_{aa}\phi_a - \left(\frac{h_E}{h_p}\right)^2 K_{aa}\phi_a^c = 0 \qquad (44)$$

$$K_{sc} + K_{ss}\phi_s = 0 \qquad (45)$$

将式（44)、(45）代入式(43),并引入阻尼矩阵可得 MFC-CFRC 层合板的整体动力学方程：

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{F}^{ela} = \left(\mathbf{FS}_F\right)^{\mathrm{T}} + \mathbf{K}_{con}\phi_a^c \qquad (46)$$

式中，$\mathbf{F}^{ela}$为 CFRC-MFC 层合板的非线性弹性力列矢量，$\mathbf{K}_{con}$为 MFC 致动器控制力输入矩阵，$\mathbf{C}$是采用瑞丽阻尼模型建立的阻尼矩阵，其表达式如下：

$$\mathbf{C} = c_0\mathbf{M} + c_1 \left.\frac{\partial \mathbf{F}^{ela}}{\partial \mathbf{q}}\right|_{q=q_0} \qquad (47)$$

其中，c_0和c_1是瑞丽阻尼参数，表达式如下：

$$c_0 = 2\left(\frac{\xi_2}{\omega_2} - \frac{\xi_1}{\omega_1}\right) \Big/ \left(\frac{1}{\left(\omega_2\right)^2} - \frac{1}{\left(\omega_1\right)^2}\right), c_1 = 2\left(\xi_2\omega_2 - \xi_1\omega_1\right) \Big/ \left(\left(\omega_2\right)^2 - \left(\omega_1\right)^2\right) \qquad (48)$$

其中，ω_1、ω_2、ξ_1、ξ_2分别为 CFRC-MFC 层合板的前两阶自然频率及对应的模态阻尼比。将式（45）整理可得 MFC 传感器输出方程：

$$\phi_s = \mathbf{K}_{out}\mathbf{q} \qquad (49)$$

其中， $\mathbf{K}_{out}$为 MFC 传感器电压输出矩阵。此外，式（46）和（49）中的各个矩阵详见附录 B。

CFRC 悬臂板的约束区域如图 1 所示，边界约束方程如下：

$$\Phi(\mathbf{q},t) = 0 \qquad (50)$$

采用拉格朗日乘子$\boldsymbol{\lambda}$将边界约束方程引入动力学方程可得：

$$\begin{aligned} &\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} + \left(\Phi_{,q}\right)^{\mathrm{T}}\boldsymbol{\lambda} = \left(\mathbf{F}\mathbf{S}_F\right)^{\mathrm{T}} + \mathbf{K}_{con}\phi_a^c \\ &\Phi(\mathbf{q},t) = 0 \end{aligned} \qquad (51)$$

其中，$\Phi_{,q} = \mathbf{G}$。

1.3 数值求解算法-基于加速度

$$\begin{aligned} &\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{F}^{ela} + \left(\Phi_{,\ddot{q}}\right)^{\mathrm{T}}\boldsymbol{\lambda} = \left(\mathbf{F}\mathbf{S}_F\right)^{\mathrm{T}} + \mathbf{K}_{con}\phi_a^c \\ &\Phi(\ddot{\mathbf{q}},t) = 0 \end{aligned} \qquad (52)$$

对于基于加速度的求解，令$\Phi(\ddot{\mathbf{q}},t) = \mathbf{G}\ddot{\mathbf{q}}$，$\Phi_{,\ddot{q}} = \mathbf{G}$，矩阵$\mathbf{G}$用于标记受约束的自由度。

采用 newmark-beta 法基于加速度求解

其中，$\dot{\mathbf{q}}_{i+1}$和$\mathbf{q}_{i+1}$为第 i+1 个时间步的预测值，他们的表达式如下：

$$\begin{aligned} &\mathbf{q}_{i+1} = \mathbf{q}_i + \Delta t\dot{\mathbf{q}}_i + \frac{\Delta t^2}{2}\left(\left(1-2\beta\right)\ddot{\mathbf{q}}_i + 2\beta\ddot{\mathbf{q}}_{i+1}\right) \\ &\dot{\mathbf{q}}_{i+1} = \dot{\mathbf{q}}_i + \Delta t\left(\left(1-\gamma\right)\ddot{\mathbf{q}}_i + \gamma\ddot{\mathbf{q}}_{i+1}\right) \end{aligned} \qquad (53)$$

其中,当$\gamma \geq \dfrac{1}{2}, \beta \geq \dfrac{\gamma}{2}$时，求解过程是无条件稳定的。第 i+1 个时间步的动力学方程残差$\mathbf{R}_d$和约束方程残差$\mathbf{R}_c$如下:

$$\begin{aligned}
&\mathbf{R}_d\left(\ddot{\mathbf{q}}_{i+1}, \boldsymbol{\lambda}_{i+1}\right)=\mathbf{M}\ddot{\mathbf{q}}_{i+1}+\mathbf{C}\dot{\mathbf{q}}_{i+1}+\mathbf{F}^{ela}\left(\mathbf{q}_{i+1}\right)+\mathbf{G}^{\mathrm{T}}\boldsymbol{\lambda}_{i+1}-\left(\mathbf{F}\left(t_{i+1}\right)\mathbf{S}_F\right)^{\mathrm{T}} \\
&\mathbf{R}_c\left(\ddot{\mathbf{q}}_{i+1}, \boldsymbol{\lambda}_{i+1}\right)=\mathbf{G}\ddot{\mathbf{q}}_{i+1}
\end{aligned} \qquad (54)$$

根据式（53），式中只有$\ddot{\mathbf{q}}_{i+1}$，$\boldsymbol{\lambda}_{i+1}$是未知独立变量。

接下来采用牛顿-拉斐逊迭代法迭代求解$\ddot{\mathbf{q}}_{i+1}$和$\boldsymbol{\lambda}_{i+1}$，我们定义$\left(q_{i+1}^k, \dot{q}_{i+1}^k, \ddot{q}_{i+1}^k, \lambda_{i+1}^k\right)$是在第 k 次迭代中获得的$\left(q_{i+1}, \dot{q}_{i+1}, \ddot{q}_{i+1}, \lambda_{i+1}\right)$的近似解，那么第 k+1 次迭代的残差方程为:

$$\begin{aligned}
&\mathbf{R}_d\left(\ddot{\mathbf{q}}_{i+1}^{k+1}, \boldsymbol{\lambda}_{i+1}^{k+1}\right) \approx \mathbf{R}_d\left(\ddot{\mathbf{q}}_{i+1}^{k}, \boldsymbol{\lambda}_{i+1}^{k}\right)+\left.\frac{\partial \mathbf{R}_d}{\partial \ddot{\mathbf{q}}}\right|_{\ddot{\mathbf{q}}=\ddot{\mathbf{q}}_{i+1}^{k}} \Delta \ddot{\mathbf{q}}_{i+1}^{k}+\left.\frac{\partial \mathbf{R}_d}{\partial \boldsymbol{\lambda}}\right|_{\lambda=\lambda_{i+1}^{k}} \Delta \boldsymbol{\lambda}_{i+1}^{k} \\
&\mathbf{R}_c\left(\ddot{\mathbf{q}}_{i+1}^{k+1}, \boldsymbol{\lambda}_{i+1}^{k+1}\right) \approx \mathbf{R}_c\left(\ddot{\mathbf{q}}_{i+1}^{k}, \boldsymbol{\lambda}_{i+1}^{k}\right)+\left.\frac{\partial \mathbf{R}_c}{\partial \ddot{\mathbf{q}}}\right|_{\ddot{\mathbf{q}}=\ddot{\mathbf{q}}_{i+1}^{k}} \Delta \ddot{\mathbf{q}}_{i+1}^{k}+\left.\frac{\partial \mathbf{R}_c}{\partial \boldsymbol{\lambda}}\right|_{\lambda=\lambda_{i+1}^{k}} \Delta \boldsymbol{\lambda}_{i+1}^{k}
\end{aligned} \qquad (55)$$

迭代目标是使得第 k+1 次迭代的残差趋近于零，因此假设第 k+1 次迭代的残差为零，式（55）可写成下式:

$$\mathbf{S}^k\begin{bmatrix}\Delta \ddot{\mathbf{q}}_{i+1}^{k} \\ \Delta \boldsymbol{\lambda}_{i+1}^{k}\end{bmatrix}=-\mathbf{R}^k \qquad (56)$$

$\mathbf{S}^k$和$\mathbf{R}^k$具体表达式见于附录 C。

于是

$$\begin{bmatrix}\Delta \ddot{\mathbf{q}}_{i+1}^{k} \\ \Delta \boldsymbol{\lambda}_{i+1}^{k}\end{bmatrix}=-\left(\mathbf{S}^k\right)^{-1}\mathbf{R}^k \qquad (57)$$

$$\begin{bmatrix}\ddot{\mathbf{q}}_{i+1}^{k+1} \\ \boldsymbol{\lambda}_{i+1}^{k+1}\end{bmatrix}=\begin{bmatrix}\ddot{\mathbf{q}}_{i+1}^{k} \\ \boldsymbol{\lambda}_{i+1}^{k}\end{bmatrix}+\begin{bmatrix}\Delta \ddot{\mathbf{q}}_{i+1}^{k} \\ \Delta \boldsymbol{\lambda}_{i+1}^{k}\end{bmatrix} \qquad (58)$$

将近似解$\ddot{\mathbf{q}}_{i+1}^{k+1}$代入式（53）获得$\mathbf{q}_{i+1}^{k+1}$、$\dot{\mathbf{q}}_{i+1}^{k+1}$，并重新计算残差$\mathbf{R}^{k+1}$若是残差满足$\left\|\mathbf{R}^{k+1}\right\| \leq \Xi$，则认为此次迭代的近似解满足精度要求，进入下一时间步，直到求解结束。采用上述流程，即可获得一段时间内的非线性振动响应。

值得注意的是，每个时间步的迭代求解过程都要更新$\mathbf{q}_{i+1}^{k}$对应的$\mathbf{F}^{ela}\left(\mathbf{q}_{i+1}\right)$和$\dfrac{\partial \mathbf{F}^{ela}}{\partial \mathbf{q}}$，为了进一步提高计算效率，本文采用了高斯-勒让德数值积分算法，以实现上述迭代过程中对非线性弹性力及其雅可比矩阵的快速更新。以非线性弹性力的高斯数值积分公式为例，其表达式如下：

$$
\begin{aligned}
&\mathbf{F}^{ela}\left(\mathbf{q}_{i+1}^{k}\right)=\int_{0}^{b_e}\int_{0}^{a_e}\mathbf{f}^{ela}\left(x,y,\mathbf{q}_{i+1}^{k}\right)\mathrm{d}x\mathrm{d}y \\
&=\frac{a_e b_e}{2}\int_{-1}^{1}\int_{-1}^{1}\mathbf{f}^{ela}\left(\frac{a_e}{2}\xi+\frac{a_e}{2},\frac{b_e}{2}\eta+\frac{b_e}{2},\mathbf{q}_{i+1}^{k}\right)\mathrm{d}\xi\mathrm{d}\eta \\
&=\frac{a_e b_e}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}w_{\xi_i}w_{\eta_j}\mathbf{f}^{ela}\left(\frac{a_e}{2}\xi_i+\frac{a_e}{2},\frac{b_e}{2}\eta_j+\frac{b_e}{2},\mathbf{q}_{i+1}^{k}\right)
\end{aligned}
\tag{59}
$$

式中，$\mathbf{f}^{ela}\left(x,y,\mathbf{q}_{i+1}^{k}\right)=\left(\mathbf{B}_e\right)^{\mathrm{T}}\left(\left(\boldsymbol{\varepsilon}_{m,q_e}\right)^{\mathrm{T}}\mathbf{D}_{c1}\boldsymbol{\varepsilon}_m+\left(\boldsymbol{\varepsilon}_{m,q_e}\right)^{\mathrm{T}}\mathbf{D}_{c2}\boldsymbol{\kappa}+\left(\boldsymbol{\kappa}_{,q_e}\right)^{\mathrm{T}}\mathbf{D}_{c2}\boldsymbol{\varepsilon}_m+\left(\boldsymbol{\kappa}_{,q_e}\right)^{\mathrm{T}}\mathbf{D}_{c3}\boldsymbol{\kappa}\right)$，$\xi_i$、$\eta_j$、$w_{\xi_i}$、$w_{\eta_j}$（$i=1,\cdots,m$；$j=1,\cdots,n$）分别为$x$、$y$方向的高斯点及其对应的权重因子，$m$、$n$分别$x$、$y$方向的高斯点数量。$\xi$、$\eta$为积分区间转换后的积分坐标。

$\mathbf{f}^{ela}\left(x,y,\mathbf{q}_{i+1}^{k}\right)=\left(\mathbf{B}_e\right)^{\mathrm{T}}\left(\left(\boldsymbol{\varepsilon}_{m,q_e}\right)^{\mathrm{T}}\mathbf{D}_{c1}\boldsymbol{\varepsilon}_m+\left(\boldsymbol{\varepsilon}_{m,q_e}\right)^{\mathrm{T}}\mathbf{D}_{c2}\boldsymbol{\kappa}+\left(\boldsymbol{\kappa}_{,q_e}\right)^{\mathrm{T}}\mathbf{D}_{c2}\boldsymbol{\varepsilon}_m+\left(\boldsymbol{\kappa}_{,q_e}\right)^{\mathrm{T}}\mathbf{D}_{c3}\boldsymbol{\kappa}\right)$为非线性弹性力积分前的矩阵函数。高斯点的选取是高斯数值积分的重点，构造高斯点的方法有很多，其中最常用的是使用 Legendre 多项式函数构造高斯点。

首先通过迭代方法构造 Legendre 多项式函数：

$$
\begin{aligned}
&L_0(x)=1\\
&L_1(x)=x\\
&L_n(x)=\frac{2n-1}{n}xL_{n-1}(x)-\frac{n-1}{n}L_{n-2}(x)\quad(n\geq 2)
\end{aligned}
\tag{60}
$$

使用 matlab 求解上述 Legendre 多项式$L_n(x)$的零点x_k作为高斯点。n为 Legendre 多项式的阶数，同时也是零点个数。根据求解的精度需求可以选择合适高斯点数，从而确定 Legendre 多项式的阶数。

高斯点对应的权重因子的计算公式如下：

$$
w_k=\frac{2}{\left(1-(x_k)^2\right)\left(\left.\frac{\partial L_n(x)}{\partial x}\right|_{x=x_k}\right)^2}\tag{61}
$$

附录 A

法向量表达式如下

$$
\begin{aligned}
\mathbf{n}=\mathbf{r}_{,x}\times\mathbf{r}_{,y}&=\begin{bmatrix}0&-\mathbf{r}_{3,x}&\mathbf{r}_{2,x}\\\mathbf{r}_{3,x}&0&-\mathbf{r}_{1,x}\\-\mathbf{r}_{2,x}&\mathbf{r}_{1,x}&0\end{bmatrix}\begin{bmatrix}\mathbf{r}_{1,y}\\\mathbf{r}_{2,y}\\\mathbf{r}_{3,y}\end{bmatrix}\\
&=\begin{bmatrix}\mathbf{r}_{2,x}\mathbf{r}_{3,y}-\mathbf{r}_{3,x}\mathbf{r}_{2,y}\\\mathbf{r}_{3,x}\mathbf{r}_{1,y}-\mathbf{r}_{1,x}\mathbf{r}_{3,y}\\\mathbf{r}_{1,x}\mathbf{r}_{2,y}-\mathbf{r}_{2,x}\mathbf{r}_{1,y}\end{bmatrix}=\begin{bmatrix}(\mathbf{S}_{2,x}\mathbf{q}_e)^{\mathrm{T}}\mathbf{S}_{3,y}\mathbf{q}_e-(\mathbf{S}_{3,x}\mathbf{q}_e)^{\mathrm{T}}\mathbf{S}_{2,y}\mathbf{q}_e\\(\mathbf{S}_{3,x}\mathbf{q}_e)^{\mathrm{T}}\mathbf{S}_{1,y}\mathbf{q}_e-(\mathbf{S}_{1,x}\mathbf{q}_e)^{\mathrm{T}}\mathbf{S}_{3,y}\mathbf{q}_e\\(\mathbf{S}_{1,x}\mathbf{q}_e)^{\mathrm{T}}\mathbf{S}_{2,y}\mathbf{q}_e-(\mathbf{S}_{2,x}\mathbf{q}_e)^{\mathrm{T}}\mathbf{S}_{1,y}\mathbf{q}_e\end{bmatrix}
\end{aligned}
\tag{A.1}
$$

$\boldsymbol{\varepsilon}_{m,q_e}$和$\boldsymbol{\kappa}_{,q_e}$的表达式如下

$$
\boldsymbol{\varepsilon}_{m,q_e}=\begin{bmatrix}(\mathbf{S}_{,x}\mathbf{q}_e)^{\mathrm{T}}\mathbf{S}_{,x}\\(\mathbf{S}_{,y}\mathbf{q}_e)^{\mathrm{T}}\mathbf{S}_{,y}\\(\mathbf{S}_{,x}\mathbf{q}_e)^{\mathrm{T}}\mathbf{S}_{,y}+(\mathbf{S}_{,y}\mathbf{q}_e)^{\mathrm{T}}\mathbf{S}_{,x}\end{bmatrix}\tag{A.2}
$$

$$
\boldsymbol{\kappa}_{,q_e}=\frac{\partial \boldsymbol{\kappa}}{\partial \mathbf{q}_e}=\begin{bmatrix} \dfrac{\dfrac{\partial\left(\left(\mathbf{r}_{,xx}\right)^{\mathrm{T}}\mathbf{n}\right)}{\partial \mathbf{q}_e}\bar{n}-\left(\mathbf{r}_{,xx}\right)^{\mathrm{T}}\mathbf{n}\dfrac{\partial(\bar{n})}{\partial \mathbf{q}_e}}{\bar{n}^2} \\ \dfrac{\dfrac{\partial\left(\left(\mathbf{r}_{,yy}\right)^{\mathrm{T}}\mathbf{n}\right)}{\partial \mathbf{q}_e}\bar{n}-\left(\mathbf{r}_{,yy}\right)^{\mathrm{T}}\mathbf{n}\dfrac{\partial(\bar{n})}{\partial \mathbf{q}_e}}{\bar{n}^2} \\ \dfrac{\dfrac{\partial\left(\left(\mathbf{r}_{,xy}\right)^{\mathrm{T}}\mathbf{n}\right)}{\partial \mathbf{q}_e}\bar{n}-\left(\mathbf{r}_{,xy}\right)^{\mathrm{T}}\mathbf{n}\dfrac{\partial(\bar{n})}{\partial \mathbf{q}_e}}{\bar{n}^2} \end{bmatrix}=\begin{bmatrix} \dfrac{\left(\mathbf{n}^{\mathrm{T}}\dfrac{\partial\left(\mathbf{r}_{,xx}\right)}{\partial \mathbf{q}_e}+\left(\mathbf{r}_{,xx}\right)^{\mathrm{T}}\dfrac{\partial(\mathbf{n})}{\partial \mathbf{q}_e}\right)\bar{n}-\left(\mathbf{r}_{,xx}\right)^{\mathrm{T}}\mathbf{n}\dfrac{\partial(\bar{n})}{\partial \mathbf{q}_e}}{\bar{n}^2} \\ \dfrac{\left(\mathbf{n}^{\mathrm{T}}\dfrac{\partial\left(\mathbf{r}_{,yy}\right)}{\partial \mathbf{q}_e}+\left(\mathbf{r}_{,yy}\right)^{\mathrm{T}}\dfrac{\partial(\mathbf{n})}{\partial \mathbf{q}_e}\right)\bar{n}-\left(\mathbf{r}_{,yy}\right)^{\mathrm{T}}\mathbf{n}\dfrac{\partial(\bar{n})}{\partial \mathbf{q}_e}}{\bar{n}^2} \\ \dfrac{\left(\mathbf{n}^{\mathrm{T}}\dfrac{\partial\left(\mathbf{r}_{,xy}\right)}{\partial \mathbf{q}_e}+\left(\mathbf{r}_{,xy}\right)^{\mathrm{T}}\dfrac{\partial(\mathbf{n})}{\partial \mathbf{q}_e}\right)\bar{n}-\left(\mathbf{r}_{,xy}\right)^{\mathrm{T}}\mathbf{n}\dfrac{\partial(\bar{n})}{\partial \mathbf{q}_e}}{\bar{n}^2} \end{bmatrix}
$$

$$
=\begin{bmatrix} \dfrac{\left(\mathbf{n}^{\mathrm{T}}\mathbf{S}_{,xx}+\left(\mathbf{r}_{,xx}\right)^{\mathrm{T}}\mathbf{n}_{,q_e}\right)\bar{n}-\left(\mathbf{r}_{,xx}\right)^{\mathrm{T}}\mathbf{n}\bar{n}_{,q_e}}{\bar{n}^2} \\ \dfrac{\left(\mathbf{n}^{\mathrm{T}}\mathbf{S}_{,yy}+\left(\mathbf{r}_{,yy}\right)^{\mathrm{T}}\mathbf{n}_{,q_e}\right)\bar{n}-\left(\mathbf{r}_{,yy}\right)^{\mathrm{T}}\mathbf{n}\bar{n}_{,q_e}}{\bar{n}^2} \\ \dfrac{\left(\mathbf{n}^{\mathrm{T}}2\mathbf{S}_{,xy}+\left(\mathbf{r}_{,xy}\right)^{\mathrm{T}}\mathbf{n}_{,q_e}\right)\bar{n}-\left(\mathbf{r}_{,xy}\right)^{\mathrm{T}}\mathbf{n}\bar{n}_{,q_e}}{\bar{n}^2} \end{bmatrix}=\begin{bmatrix} \dfrac{\mathbf{n}^{\mathrm{T}}\mathbf{S}_{,xx}}{\bar{n}} \\ \dfrac{\mathbf{n}^{\mathrm{T}}\mathbf{S}_{,yy}}{\bar{n}} \\ \dfrac{\mathbf{n}^{\mathrm{T}}2\mathbf{S}_{,xy}}{\bar{n}} \end{bmatrix}+\begin{bmatrix} \dfrac{\left(\mathbf{r}_{,xx}\right)^{\mathrm{T}}\mathbf{n}_{,q_e}}{\bar{n}} \\ \dfrac{\left(\mathbf{r}_{,yy}\right)^{\mathrm{T}}\mathbf{n}_{,q_e}}{\bar{n}} \\ \dfrac{\left(\mathbf{r}_{,xy}\right)^{\mathrm{T}}\mathbf{n}_{,q_e}}{\bar{n}} \end{bmatrix}-\boldsymbol{\kappa}\frac{\bar{n}_{,q_e}}{\bar{n}} \qquad \text{(A.3)}
$$

其中

$$
\begin{aligned}
\mathbf{r}_{,xx} &= \mathbf{S}_{,xx}\mathbf{q}_e \\
\mathbf{r}_{,yy} &= \mathbf{S}_{,yy}\mathbf{q}_e \qquad \text{(A.4)} \\
\mathbf{r}_{,xy} &= 2\mathbf{S}_{,xy}\mathbf{q}_e
\end{aligned}
$$

$$
\mathbf{n}_{,q_e}=\frac{\partial \mathbf{n}}{\partial \mathbf{q}_e}=\begin{bmatrix} \left(\mathbf{S}_{2,x}\mathbf{q}_e\right)^{\mathrm{T}}\mathbf{S}_{3,y}+\left(\mathbf{S}_{3,y}\mathbf{q}_e\right)^{\mathrm{T}}\mathbf{S}_{2,x}-\left(\mathbf{S}_{3,x}\mathbf{q}_e\right)^{\mathrm{T}}\mathbf{S}_{2,y}-\left(\mathbf{S}_{2,y}\mathbf{q}_e\right)^{\mathrm{T}}\mathbf{S}_{3,x} \\ \left(\mathbf{S}_{3,x}\mathbf{q}_e\right)^{\mathrm{T}}\mathbf{S}_{1,y}+\left(\mathbf{S}_{1,y}\mathbf{q}_e\right)^{\mathrm{T}}\mathbf{S}_{3,x}-\left(\mathbf{S}_{1,x}\mathbf{q}_e\right)^{\mathrm{T}}\mathbf{S}_{3,y}-\left(\mathbf{S}_{3,y}\mathbf{q}_e\right)^{\mathrm{T}}\mathbf{S}_{1,x} \\ \left(\mathbf{S}_{1,x}\mathbf{q}_e\right)^{\mathrm{T}}\mathbf{S}_{2,y}+\left(\mathbf{S}_{2,y}\mathbf{q}_e\right)^{\mathrm{T}}\mathbf{S}_{1,x}-\left(\mathbf{S}_{2,x}\mathbf{q}_e\right)^{\mathrm{T}}\mathbf{S}_{1,y}-\left(\mathbf{S}_{1,y}\mathbf{q}_e\right)^{\mathrm{T}}\mathbf{S}_{2,x} \end{bmatrix} \qquad \text{(A.5)}
$$

$$
\bar{n}_{,q_e}=\frac{\partial \bar{n}}{\partial \mathbf{q}_e}=\frac{\partial \bar{n}}{\partial \mathbf{n}}\frac{\partial \mathbf{n}}{\partial \mathbf{q}_e}=\frac{\mathbf{n}^{\mathrm{T}}}{\bar{n}}\mathbf{n}_{,q_e} \qquad \text{(A.6)}
$$

惯性、刚度和压电等参数矩阵如下：

$$
I_{c1}=\rho_c\left(h_u-h_b\right) \qquad \text{(A.7)}
$$

$$
\mathbf{D}_{c1}=\sum_{k=1}^{N_f}\left(h_u^k-h_b^k\right)\bar{\mathbf{D}}_c^k
$$

$$
\mathbf{D}_{c2}=\sum_{k=1}^{N_f}\left(\frac{\left(h_u^k\right)^2}{2}-\frac{\left(h_b^k\right)^2}{2}\right)\bar{\mathbf{D}}_c^k
$$

$$\mathbf{D}_{c3}=\sum_{k=1}^{N_f}\left(\frac{\left(h_u^k\right)^3}{3}-\frac{\left(h_b^k\right)^3}{3}\right)\bar{\mathbf{D}}_c^k$$

$$I_{p1}=\rho_p\left(h_p^u-h_p^b\right)$$

$$\mathbf{D}_{p1}=\left(h_p^u-h_p^b\right)\bar{\mathbf{D}}_p$$

$$\mathbf{D}_{p2}=\left(\frac{\left(h_p^u\right)^2}{2}-\frac{\left(h_p^b\right)^2}{2}\right)\bar{\mathbf{D}}_p$$

$$\mathbf{D}_{p3}=\left(\frac{\left(h_p^u\right)^3}{3}-\frac{\left(h_p^b\right)^3}{3}\right)\bar{\mathbf{D}}_p$$

$$\mathbf{e}_{p1}=\left(h_p^u-h_p^b\right)\bar{\mathbf{e}}$$

$$\mathbf{e}_{p2}=\left(\frac{\left(h_p^u\right)^2}{2}-\frac{\left(h_p^b\right)^2}{2}\right)\bar{\mathbf{e}}$$

$$\varsigma_{33}^1=\left(h_p^u-h_p^b\right)\varsigma_{33}$$

附录 B

$$\left(\mathbf{M}_c+\mathbf{M}_a+\mathbf{M}_s\right)\ddot{\mathbf{q}}+\mathbf{F}_c^{ela}+\mathbf{F}_a^{ela}+\mathbf{F}_s^{ela}-\left(\mathbf{K}_{ca}\right)^{\mathrm{T}}\phi_a-\left(\mathbf{K}_{cs}\right)^{\mathrm{T}}\phi_s-\left(\mathbf{FS}_F\right)^{\mathrm{T}}=0$$

(B.1)

$$K_{ac}+K_{aa}\phi_a-\left(\frac{h_E}{h_p}\right)^2K_{aa}\phi_a^c=0 \qquad \text{(B.2)}$$

$$K_{sc}+K_{ss}\phi_s=0 \qquad \text{(B.3)}$$

由（B.2）和（B.3）可得

$$\phi_a=\left(K_{aa}\right)^{-1}\left(\left(\frac{h_E}{h_p}\right)^2K_{aa}\phi_a^c-K_{ac}\right) \qquad \text{(B.4)}$$

$$\phi_s = \left(K_{ss}\right)^{-1}\left(-K_{sc}\right) \qquad \text{(B.5)}$$

将（B.4）和（B.5）代入式（B.1）

$$\begin{aligned}
&\left(\mathbf{M}_c + \mathbf{M}_a + \mathbf{M}_s\right)\ddot{\phi} \\
&+\mathbf{F}_c^{ela} + \mathbf{F}_a^{ela} + \mathbf{F}_s^{ela} \\
&-\left(\mathbf{K}_{ca}\right)^{\mathrm{T}}\left(K_{aa}\right)^{-1}\left(\left(\frac{h_E}{h_p}\right)^2 K_{aa}\phi_a^c - K_{ac}\right) \\
&-\left(\mathbf{K}_{cs}\right)^{\mathrm{T}}\left(K_{ss}\right)^{-1}\left(-K_{sc}\right) \\
&=\left(\mathbf{F}\mathbf{S}_F\right)^{\mathrm{T}}
\end{aligned} \qquad \text{(B.6)}$$

进一步推得

$$\begin{aligned}
&\left(\mathbf{M}_c + \mathbf{M}_a + \mathbf{M}_s\right)\ddot{\phi} \\
&+\mathbf{F}_c^{ela} + \mathbf{F}_a^{ela} + \mathbf{F}_s^{ela} + \left(\mathbf{K}_{ca}\right)^{\mathrm{T}}\left(K_{aa}\right)^{-1}K_{ac} + \left(\mathbf{K}_{cs}\right)^{\mathrm{T}}\left(K_{ss}\right)^{-1}K_{sc} \\
&=\left(\mathbf{F}\mathbf{S}_F\right)^{\mathrm{T}} + \left(\mathbf{K}_{ca}\right)^{\mathrm{T}}\left(\frac{h_E}{h_p}\right)^2\phi_a^c
\end{aligned} \qquad \text{(B.7)}$$

因此

$$\mathbf{M} = \mathbf{M}_c + \mathbf{M}_a + \mathbf{M}_s \qquad \text{(B.8)}$$

$$\mathbf{F}^{ela} = \mathbf{F}_c^{ela} + \mathbf{F}_a^{ela} + \mathbf{F}_s^{ela} + \left(\mathbf{K}_{ca}\right)^{\mathrm{T}}\left(K_{aa}\right)^{-1}K_{ac} + \left(\mathbf{K}_{cs}\right)^{\mathrm{T}}\left(K_{ss}\right)^{-1}K_{sc} \qquad \text{(B.9)}$$

$$\mathbf{K}_{con} = \left(\mathbf{K}_{ca}\right)^{\mathrm{T}}\left(\frac{h_E}{h_p}\right)^2 \qquad \text{(B.10)}$$

其中

$$\mathbf{M}_c = \sum_{e}^{N_e^c}\iint_{S_e}\left(\mathbf{B}_e\right)^{\mathrm{T}}\left(I_{c1}\left(\mathbf{S}\right)^{\mathrm{T}}\mathbf{S}\right)\mathbf{B}_e\,\mathrm{d}x\mathrm{d}y \qquad \text{(B.11)}$$

$$\mathbf{M}_p = \sum_{e}^{N_e^p}\iint_{S_e}\left(\mathbf{B}_e\right)^{\mathrm{T}}\left(I_{p1}\left(\mathbf{S}\right)^{\mathrm{T}}\mathbf{S}\right)\mathbf{B}_e\,\mathrm{d}x\mathrm{d}y \qquad \text{(B.12)}$$

$$\mathbf{F}_c^{ela} = \sum_{e}^{N_e^c}\iint_{S_e}\left(\mathbf{B}_e\right)^{\mathrm{T}}\left(\left(\boldsymbol{\varepsilon}_{m,q_e}\right)^{\mathrm{T}}\mathbf{D}_{c1}\boldsymbol{\varepsilon}_m + \left(\boldsymbol{\varepsilon}_{m,q_e}\right)^{\mathrm{T}}\mathbf{D}_{c2}\boldsymbol{\kappa} + \left(\boldsymbol{\kappa}_{,q_e}\right)^{\mathrm{T}}\mathbf{D}_{c2}\boldsymbol{\varepsilon}_m + \left(\boldsymbol{\kappa}_{,q_e}\right)^{\mathrm{T}}\mathbf{D}_{c3}\boldsymbol{\kappa}\right)\mathrm{d}x\mathrm{d}y \qquad \text{(B.13)}$$

$$\mathbf{F}_p^{ela} = \sum_{e}^{N_e^p}\iint_{S_e}\left(\mathbf{B}_e\right)^{\mathrm{T}}\left(\left(\boldsymbol{\varepsilon}_{m,q_e}\right)^{\mathrm{T}}\mathbf{D}_{p1}\boldsymbol{\varepsilon}_m + \left(\boldsymbol{\varepsilon}_{m,q_e}\right)^{\mathrm{T}}\mathbf{D}_{p2}\boldsymbol{\kappa} + \left(\boldsymbol{\kappa}_{,q_e}\right)^{\mathrm{T}}\mathbf{D}_{p2}\boldsymbol{\varepsilon}_m + \left(\boldsymbol{\kappa}_{,q_e}\right)^{\mathrm{T}}\mathbf{D}_{p3}\boldsymbol{\kappa}\right)\mathrm{d}x\mathrm{d}y \qquad \text{(B.14)}$$

$$\mathbf{K}_{cp}=\sum_{e}^{N_e^p}\iint_{S_e}\left(\left(\mathbf{e}_{p1}\right)^{\mathrm{T}}\frac{1}{h_E}\boldsymbol{\varepsilon}_{m,q_e}+\left(\mathbf{e}_{p2}\right)^{\mathrm{T}}\frac{1}{h_E}\boldsymbol{\kappa}_{,q_e}\right)\mathbf{B}_e\mathrm{d}x\mathrm{d}y \qquad \text{(B.15)}$$

$$K_{pc}=\sum_{e}^{N_e^p}\iint_{S_e}\left(\mathbf{e}_{p1}\right)^{\mathrm{T}}\frac{1}{h_E}\boldsymbol{\varepsilon}_m+\left(\mathbf{e}_{p2}\right)^{\mathrm{T}}\frac{1}{h_E}\boldsymbol{\kappa}\mathrm{d}x\mathrm{d}y \qquad \text{(B.16)}$$

$$K_{pp}=\sum_{e}^{N_e^p}\iint_{S_e}\zeta^1_{z_p z_p}\left(\frac{1}{h_E}\right)^2\phi_p\mathrm{d}x\mathrm{d}y \qquad \text{(B.17)}$$

附录 C

$$\mathbf{S}^k=\begin{bmatrix}\left.\dfrac{\partial\mathbf{R}_d}{\partial\ddot{\mathbf{q}}}\right|_{\ddot{\mathbf{q}}=\ddot{\mathbf{q}}_{i+1}^k} & \left.\dfrac{\partial\mathbf{R}_d}{\partial\boldsymbol{\lambda}}\right|_{\lambda=\lambda_{i+1}^k}\\ \left.\dfrac{\partial\mathbf{R}_c}{\partial\ddot{\mathbf{q}}}\right|_{\ddot{\mathbf{q}}=\ddot{\mathbf{q}}_{i+1}^k} & \left.\dfrac{\partial\mathbf{R}_c}{\partial\boldsymbol{\lambda}}\right|_{\lambda=\lambda_{i+1}^k}\end{bmatrix}=\begin{bmatrix}\left.\dfrac{\partial\mathbf{R}_d}{\partial\ddot{\mathbf{q}}}\right|_{\ddot{\mathbf{q}}=\ddot{\mathbf{q}}_{i+1}^k} & \mathbf{G}^{\mathrm{T}}\\ \mathbf{G} & 0\end{bmatrix} \qquad \text{(C.1)}$$

$$\mathbf{R}^k=\begin{bmatrix}\mathbf{R}_d\left(\ddot{\mathbf{q}}_{i+1}^k,\boldsymbol{\lambda}_{i+1}^k\right)\\ \mathbf{R}_c\left(\ddot{\mathbf{q}}_{i+1}^k,\boldsymbol{\lambda}_{i+1}^k\right)\end{bmatrix}$$

$$\frac{\partial\mathbf{R}_d}{\partial\ddot{\mathbf{q}}}=\mathbf{M}+\mathbf{C}\frac{\partial\dot{\mathbf{q}}}{\partial\ddot{\mathbf{q}}}+\frac{\partial\mathbf{F}^{ela}}{\partial\mathbf{q}}\frac{\partial\mathbf{q}}{\partial\ddot{\mathbf{q}}} \qquad \text{(C.2)}$$

根据式（53），其中 $\mathbf{q}_i$， $\dot{\mathbf{q}}_i$ 和 $\ddot{\mathbf{q}}_i$ 在 i+1 时刻都是已知的，由此可得

$$\frac{\partial\dot{\mathbf{q}}}{\partial\ddot{\mathbf{q}}}=\Delta t\gamma,\frac{\partial\mathbf{q}}{\partial\ddot{\mathbf{q}}}=\frac{\Delta t^2}{2}2\beta \quad \text{(C.3)}$$

$$
\begin{aligned}
\frac{\partial \mathbf{F}^{ela}}{\partial \mathbf{q}} &= \frac{\partial\left(\mathbf{F}_c^{ela}+\mathbf{F}_a^{ela}+\mathbf{F}_s^{ela}+\left(\mathbf{K}_{ca}\right)^{\mathrm{T}}\left(K_{aa}\right)^{-1}K_{ac}+\left(\mathbf{K}_{cs}\right)^{\mathrm{T}}\left(K_{ss}\right)^{-1}K_{sc}\right)}{\partial \mathbf{q}} \\
&= \frac{\partial \mathbf{F}_c^{ela}}{\partial \mathbf{q}}+\frac{\partial \mathbf{F}_a^{ela}}{\partial \mathbf{q}}+\frac{\partial \mathbf{F}_s^{ela}}{\partial \mathbf{q}}+\frac{\partial\left(\left(\mathbf{K}_{ca}\right)^{\mathrm{T}}\left(K_{aa}\right)^{-1}K_{ac}\right)}{\partial \mathbf{q}}+\frac{\partial\left(\left(\mathbf{K}_{cs}\right)^{\mathrm{T}}\left(K_{ss}\right)^{-1}K_{sc}\right)}{\partial \mathbf{q}} \\
&= \frac{\partial \mathbf{F}_c^{ela}}{\partial \mathbf{q}}+\frac{\partial \mathbf{F}_a^{ela}}{\partial \mathbf{q}}+\frac{\partial \mathbf{F}_s^{ela}}{\partial \mathbf{q}} \\
&\quad +\left(\mathbf{K}_{ca}\right)^{\mathrm{T}}\left(K_{aa}\right)^{-1}\frac{\partial\left(K_{ac}\right)}{\partial \mathbf{q}}+\left(K_{aa}\right)^{-1}K_{ac}\frac{\partial\left(\mathbf{K}_{ca}\right)^{\mathrm{T}}}{\partial \mathbf{q}} \\
&\quad +\left(\mathbf{K}_{cs}\right)^{\mathrm{T}}\left(K_{ss}\right)^{-1}\frac{\partial\left(K_{sc}\right)}{\partial \mathbf{q}}+\left(K_{ss}\right)^{-1}K_{sc}\frac{\partial\left(\mathbf{K}_{cs}\right)^{\mathrm{T}}}{\partial \mathbf{q}}
\end{aligned}
\tag{C.4}
$$

其中

$$
\begin{aligned}
\frac{\partial \mathbf{F}_c^{ela}}{\partial \mathbf{q}} &= \frac{\partial\left(\sum_e^{N_e^c}\iint_{S_e}\left(\mathbf{B}_e\right)^{\mathrm{T}}\left(\left(\boldsymbol{\varepsilon}_{m,q_e}\right)^{\mathrm{T}}\mathbf{D}_{c1}\boldsymbol{\varepsilon}_m+\left(\boldsymbol{\varepsilon}_{m,q_e}\right)^{\mathrm{T}}\mathbf{D}_{c2}\boldsymbol{\kappa}+\left(\boldsymbol{\kappa}_{,q_e}\right)^{\mathrm{T}}\mathbf{D}_{c2}\boldsymbol{\varepsilon}_m+\left(\boldsymbol{\kappa}_{,q_e}\right)^{\mathrm{T}}\mathbf{D}_{c3}\boldsymbol{\kappa}\right)\mathrm{d}x\mathrm{d}y\right)}{\partial \mathbf{q}} \\
&= \sum_e^{N_e^c}\iint_{S_e}\left(\mathbf{B}_e\right)^{\mathrm{T}}\left(\frac{\partial\left(\boldsymbol{\varepsilon}_{m,q_e}\right)^{\mathrm{T}}\mathbf{D}_{c1}\boldsymbol{\varepsilon}_m}{\partial \mathbf{q}_e}+\frac{\partial\left(\boldsymbol{\varepsilon}_{m,q_e}\right)^{\mathrm{T}}\mathbf{D}_{c2}\boldsymbol{\kappa}}{\partial \mathbf{q}_e}+\frac{\partial\left(\boldsymbol{\kappa}_{,q_e}\right)^{\mathrm{T}}\mathbf{D}_{c2}\boldsymbol{\varepsilon}_m}{\partial \mathbf{q}_e}+\frac{\partial\left(\boldsymbol{\kappa}_{,q_e}\right)^{\mathrm{T}}\mathbf{D}_{c3}\boldsymbol{\kappa}}{\partial \mathbf{q}_e}\right)\frac{\partial \mathbf{q}_e}{\partial \mathbf{q}}\mathrm{d}x\mathrm{d}y \\
&= \sum_e^{N_e^c}\iint_{S_e}\left(\mathbf{B}_e\right)^{\mathrm{T}}\left(\begin{gathered}
\left(\boldsymbol{\varepsilon}_{m,q_e}\right)^{\mathrm{T}}\mathbf{D}_{c1}\boldsymbol{\varepsilon}_{m,q_e}+\mathbf{D}_{c1}(1,:)\boldsymbol{\varepsilon}_m\boldsymbol{\varepsilon}_{m1,q_eq_e}+\mathbf{D}_{c1}(2,:)\boldsymbol{\varepsilon}_m\boldsymbol{\varepsilon}_{m2,q_eq_e}+\mathbf{D}_{c1}(3,:)\boldsymbol{\varepsilon}_m\boldsymbol{\varepsilon}_{m3,q_eq_e} \\
+\left(\boldsymbol{\varepsilon}_{m,q_e}\right)^{\mathrm{T}}\mathbf{D}_{c2}\boldsymbol{\kappa}_{,q_e}+\mathbf{D}_{c2}(1,:)\boldsymbol{\kappa}\boldsymbol{\varepsilon}_{m1,q_eq_e}+\mathbf{D}_{c2}(2,:)\boldsymbol{\kappa}\boldsymbol{\varepsilon}_{m2,q_eq_e}+\mathbf{D}_{c2}(3,:)\boldsymbol{\kappa}\boldsymbol{\varepsilon}_{m3,q_eq_e} \\
+\left(\boldsymbol{\kappa}_{,q_e}\right)^{\mathrm{T}}\mathbf{D}_{c2}\boldsymbol{\varepsilon}_{m,q_e}+\mathbf{D}_{c2}(1,:)\boldsymbol{\varepsilon}_m\boldsymbol{\kappa}_{1,q_eq_e}+\mathbf{D}_{c2}(2,:)\boldsymbol{\varepsilon}_m\boldsymbol{\kappa}_{2,q_eq_e}+\mathbf{D}_{c2}(3,:)\boldsymbol{\varepsilon}_m\boldsymbol{\kappa}_{3,q_eq_e} \\
\left(\boldsymbol{\kappa}_{,q_e}\right)^{\mathrm{T}}\mathbf{D}_{c3}\boldsymbol{\kappa}_{,q_e}+\mathbf{D}_{c3}(1,:)\boldsymbol{\kappa}\boldsymbol{\kappa}_{1,q_eq_e}+\mathbf{D}_{c3}(2,:)\boldsymbol{\kappa}\boldsymbol{\kappa}_{2,q_eq_e}+\mathbf{D}_{c3}(3,:)\boldsymbol{\kappa}\boldsymbol{\kappa}_{3,q_eq_e}
\end{gathered}\right)\mathbf{B}_e\mathrm{d}x\mathrm{d}y
\end{aligned}
\tag{C.5}
$$

同理

$$
\frac{\partial \mathbf{F}_p^{ela}}{\partial \mathbf{q}} = \sum_e^{N_e^p}\iint_{S_e}\left(\mathbf{B}_e\right)^{\mathrm{T}}\left(\begin{gathered}
\left(\boldsymbol{\varepsilon}_{m,q_e}\right)^{\mathrm{T}}\mathbf{D}_{p1}\boldsymbol{\varepsilon}_{m,q_e}+\mathbf{D}_{p1}(1,:)\boldsymbol{\varepsilon}_m\boldsymbol{\varepsilon}_{m1,q_eq_e}+\mathbf{D}_{p1}(2,:)\boldsymbol{\varepsilon}_m\boldsymbol{\varepsilon}_{m2,q_eq_e}+\mathbf{D}_{p1}(3,:)\boldsymbol{\varepsilon}_m\boldsymbol{\varepsilon}_{m3,q_eq_e} \\
+\left(\boldsymbol{\varepsilon}_{m,q_e}\right)^{\mathrm{T}}\mathbf{D}_{p2}\boldsymbol{\kappa}_{,q_e}+\mathbf{D}_{p2}(1,:)\boldsymbol{\kappa}\boldsymbol{\varepsilon}_{m1,q_eq_e}+\mathbf{D}_{p2}(2,:)\boldsymbol{\kappa}\boldsymbol{\varepsilon}_{m2,q_eq_e}+\mathbf{D}_{p2}(3,:)\boldsymbol{\kappa}\boldsymbol{\varepsilon}_{m3,q_eq_e} \\
+\left(\boldsymbol{\kappa}_{,q_e}\right)^{\mathrm{T}}\mathbf{D}_{p2}\boldsymbol{\varepsilon}_{m,q_e}+\mathbf{D}_{p2}(1,:)\boldsymbol{\varepsilon}_m\boldsymbol{\kappa}_{1,q_eq_e}+\mathbf{D}_{p2}(2,:)\boldsymbol{\varepsilon}_m\boldsymbol{\kappa}_{2,q_eq_e}+\mathbf{D}_{p2}(3,:)\boldsymbol{\varepsilon}_m\boldsymbol{\kappa}_{3,q_eq_e} \\
\left(\boldsymbol{\kappa}_{,q_e}\right)^{\mathrm{T}}\mathbf{D}_{p3}\boldsymbol{\kappa}_{,q_e}+\mathbf{D}_{p3}(1,:)\boldsymbol{\kappa}\boldsymbol{\kappa}_{1,q_eq_e}+\mathbf{D}_{p3}(2,:)\boldsymbol{\kappa}\boldsymbol{\kappa}_{2,q_eq_e}+\mathbf{D}_{p3}(3,:)\boldsymbol{\kappa}\boldsymbol{\kappa}_{3,q_eq_e}
\end{gathered}\right)\mathbf{B}_e\mathrm{d}x\mathrm{d}y
\tag{C.6}
$$

$$
\begin{aligned}
\frac{\partial\left(K_{pc}\right)}{\partial \mathbf{q}} &= \frac{\partial\left(\sum_e^{N_e^p}\iint_{S_e}\left(\mathbf{e}_{p1}\right)^{\mathrm{T}}\frac{1}{h_E}\boldsymbol{\varepsilon}_m+\left(\mathbf{e}_{p2}\right)^{\mathrm{T}}\frac{1}{h_E}\boldsymbol{\kappa}\mathrm{d}x\mathrm{d}y\right)}{\partial \mathbf{q}} \\
&= \sum_e^{N_e^p}\iint_{S_e}\left(\left(\mathbf{e}_{p1}\right)^{\mathrm{T}}\frac{1}{h_E}\boldsymbol{\varepsilon}_{m,q_e}+\left(\mathbf{e}_{p2}\right)^{\mathrm{T}}\frac{1}{h_E}\boldsymbol{\kappa}_{,q_e}\right)\mathbf{B}_e\mathrm{d}x\mathrm{d}y
\end{aligned}
\tag{C.7}
$$

$$\frac{\partial\left(\mathbf{K}_{cp}\right)^{\mathrm{T}}}{\partial\mathbf{q}}=\frac{\partial\sum_{e}^{N_e^p}\iint_{S_e}\left(\mathbf{B}_e\right)^{\mathrm{T}}\left(\left(\boldsymbol{\varepsilon}_{m,q_e}\right)^{\mathrm{T}}\frac{1}{h_E}\mathbf{e}_{p1}+\left(\boldsymbol{\kappa}_{,q_e}\right)^{\mathrm{T}}\frac{1}{h_E}\mathbf{e}_{p2}\right)\mathrm{d}x\mathrm{d}y}{\partial\mathbf{q}}$$

$$=\sum_{e}^{N_e^p}\iint_{S_e}\left(\mathbf{B}_e\right)^{\mathrm{T}}\left(\begin{array}{l}\frac{1}{h_E}\mathbf{e}_{p1}(1)\boldsymbol{\varepsilon}_{m1,q_eq_e}+\frac{1}{h_E}\mathbf{e}_{p1}(2)\boldsymbol{\varepsilon}_{m2,q_eq_e}+\frac{1}{h_E}\mathbf{e}_{p1}(3)\boldsymbol{\varepsilon}_{m3,q_eq_e}\\+\frac{1}{h_E}\mathbf{e}_{p2}(1)\boldsymbol{\kappa}_{1,q_eq_e}+\frac{1}{h_E}\mathbf{e}_{p2}(2)\boldsymbol{\kappa}_{2,q_eq_e}+\frac{1}{h_E}\mathbf{e}_{p2}(3)\boldsymbol{\kappa}_{3,q_eq_e}\end{array}\right)\mathbf{B}_e\mathrm{d}x\mathrm{d}y \tag{C.8}$$

其中

$$\begin{aligned}\boldsymbol{\varepsilon}_{m1,q_eq_e}&=\left(\mathbf{S}_{,x}\right)^{\mathrm{T}}\mathbf{S}_{,x}\\\boldsymbol{\varepsilon}_{m2,q_eq_e}&=\left(\mathbf{S}_{,y}\right)^{\mathrm{T}}\mathbf{S}_{,y}\\\boldsymbol{\varepsilon}_{m3,q_eq_e}&=\left(\mathbf{S}_{,x}\right)^{\mathrm{T}}\mathbf{S}_{,y}+\left(\mathbf{S}_{,y}\right)^{\mathrm{T}}\mathbf{S}_{,x}\end{aligned} \tag{C.9}$$

$$\begin{aligned}\boldsymbol{\kappa}_{1,q_eq_e}&=\frac{\partial\boldsymbol{\kappa}_{1,q_e}}{\partial\mathbf{q}_e}=\frac{\partial}{\partial\mathbf{q}_e}\left(\frac{\mathbf{n}^{\mathrm{T}}\mathbf{S}_{,xx}}{\bar{n}}+\frac{\left(\mathbf{r}_{,xx}\right)^{\mathrm{T}}\mathbf{n}_{,q_e}}{\bar{n}}-\boldsymbol{\kappa}_1\frac{\bar{n}_{,q_e}}{\bar{n}}\right)\\&=\frac{\left(\mathbf{S}_{,xx}\right)^{\mathrm{T}}\mathbf{n}_{,q_e}\bar{n}-\left(\mathbf{n}^{\mathrm{T}}\mathbf{S}_{,xx}\right)^{\mathrm{T}}\bar{n}_{,q_e}}{\bar{n}^2}\\&+\frac{\left(\left(\mathbf{n}_{,q_e}\right)^{\mathrm{T}}\mathbf{S}_{,xx}+\mathbf{r}_{1,xx}\mathbf{n}_{1,q_eq_e}+\mathbf{r}_{2,xx}\mathbf{n}_{2,q_eq_e}+\mathbf{r}_{3,xx}\mathbf{n}_{3,q_eq_e}\right)\bar{n}-\left(\left(\mathbf{r}_{,xx}\right)^{\mathrm{T}}\mathbf{n}_{,q_e}\right)^{\mathrm{T}}\bar{n}_{,q_e}}{\bar{n}^2}\\&-\boldsymbol{\kappa}_1\frac{\bar{n}_{,q_eq_e}\bar{n}-\left(\bar{n}_{,q_e}\right)^{\mathrm{T}}\bar{n}_{,q_e}}{\bar{n}^2}-\left(\frac{\bar{n}_{,q_e}}{\bar{n}}\right)^{\mathrm{T}}\boldsymbol{\kappa}_{1,q_e}\end{aligned} \tag{C.10}$$

$$\begin{aligned}\boldsymbol{\kappa}_{2,q_eq_e}&=\frac{\partial\boldsymbol{\kappa}_{2,q_e}}{\partial\mathbf{q}_e}=\frac{\partial}{\partial\mathbf{q}_e}\left(\frac{\mathbf{n}^{\mathrm{T}}\mathbf{S}_{,yy}}{\bar{n}}+\frac{\left(\mathbf{r}_{,yy}\right)^{\mathrm{T}}\mathbf{n}_{,q_e}}{\bar{n}}-\boldsymbol{\kappa}_1\frac{\bar{n}_{,q_e}}{\bar{n}}\right)\\&=\frac{\left(\mathbf{S}_{,yy}\right)^{\mathrm{T}}\mathbf{n}_{,q_e}\bar{n}-\left(\mathbf{n}^{\mathrm{T}}\mathbf{S}_{,yy}\right)^{\mathrm{T}}\bar{n}_{,q_e}}{\bar{n}^2}\\&+\frac{\left(\left(\mathbf{n}_{,q_e}\right)^{\mathrm{T}}\mathbf{S}_{,yy}+\mathbf{r}_{1,yy}\mathbf{n}_{1,q_eq_e}+\mathbf{r}_{2,yy}\mathbf{n}_{2,q_eq_e}+\mathbf{r}_{3,yy}\mathbf{n}_{3,q_eq_e}\right)\bar{n}-\left(\left(\mathbf{r}_{,yy}\right)^{\mathrm{T}}\mathbf{n}_{,q_e}\right)^{\mathrm{T}}\bar{n}_{,q_e}}{\bar{n}^2}\\&-\boldsymbol{\kappa}_2\frac{\bar{n}_{,q_eq_e}\bar{n}-\left(\bar{n}_{,q_e}\right)^{\mathrm{T}}\bar{n}_{,q_e}}{\bar{n}^2}-\left(\frac{\bar{n}_{,q_e}}{\bar{n}}\right)^{\mathrm{T}}\boldsymbol{\kappa}_{2,q_e}\end{aligned} \tag{C.11}$$

$$
\begin{aligned}
\boldsymbol{\kappa}_{3,q_e q_e} &= \frac{\partial \boldsymbol{\kappa}_{3,q_e}}{\partial \mathbf{q}_e} = \frac{\partial}{\partial \mathbf{q}_e}\left(\frac{\mathbf{n}^{\mathrm{T}} 2\mathbf{S}_{,xy}}{\bar{n}} + \frac{\left(\mathbf{r}_{,xy}\right)^{\mathrm{T}} \mathbf{n}_{,q_e}}{\bar{n}} - \boldsymbol{\kappa}_1 \frac{\bar{n}_{,q_e}}{\bar{n}} \right) \\
&= \frac{\left(2\mathbf{S}_{,xy}\right)^{\mathrm{T}} \mathbf{n}_{,q_e} \bar{n} - \left(\mathbf{n}^{\mathrm{T}} 2\mathbf{S}_{,xy}\right)^{\mathrm{T}} \bar{n}_{,q_e}}{\bar{n}^2} \\
&+ \frac{\left(\left(\mathbf{n}_{,q_e}\right)^{\mathrm{T}} 2\mathbf{S}_{,xy} + \mathbf{r}_{1,xy}\mathbf{n}_{1,q_e q_e} + \mathbf{r}_{2,xy}\mathbf{n}_{2,q_e q_e} + \mathbf{r}_{3,xy}\mathbf{n}_{3,q_e q_e}\right)\bar{n} - \left(\left(\mathbf{r}_{,xy}\right)^{\mathrm{T}} \mathbf{n}_{,q_e}\right)^{\mathrm{T}} \bar{n}_{,q_e}}{\bar{n}^2} \\
&- \boldsymbol{\kappa}_3 \frac{\bar{n}_{,q_e q_e} \bar{n} - \left(\bar{n}_{,q_e}\right)^{\mathrm{T}} \bar{n}_{,q_e}}{\bar{n}^2} - \left(\frac{\bar{n}_{,q_e}}{\bar{n}}\right)^{\mathrm{T}} \boldsymbol{\kappa}_{3,q_e}
\end{aligned}
\tag{C.12}
$$

其中，

$$
\bar{n}_{,q_e q_e} = \frac{\partial \bar{n}_{,q_e}}{\partial \mathbf{q}_e} = \frac{\left(\left(\mathbf{n}_{,q_e}\right)^{\mathrm{T}} \mathbf{n}_{,q_e} + \mathbf{n}_1\mathbf{n}_{1,q_e q_e} + \mathbf{n}_2\mathbf{n}_{2,q_e q_e} + \mathbf{n}_3\mathbf{n}_{3,q_e q_e}\right)\bar{n} - \left(\mathbf{n}^{\mathrm{T}}\mathbf{n}_{,q_e}\right)^{\mathrm{T}} \bar{n}_{,q_e}}{\bar{n}^2}
\tag{C.13}
$$

$$
\begin{aligned}
\mathbf{n}_{1,q_e q_e} &= \frac{\partial \mathbf{n}_{1,q_e}}{\partial \mathbf{q}_e} = \left(\mathbf{S}_{2,x}\right)^{\mathrm{T}} \mathbf{S}_{3,y} + \left(\mathbf{S}_{3,y}\right)^{\mathrm{T}} \mathbf{S}_{2,x} - \left(\mathbf{S}_{3,x}\right)^{\mathrm{T}} \mathbf{S}_{2,y} - \left(\mathbf{S}_{2,y}\right)^{\mathrm{T}} \mathbf{S}_{3,x} \\
\mathbf{n}_{2,q_e q_e} &= \frac{\partial \mathbf{n}_{2,q_e}}{\partial \mathbf{q}_e} = \left(\mathbf{S}_{3,x}\right)^{\mathrm{T}} \mathbf{S}_{1,y} + \left(\mathbf{S}_{1,y}\right)^{\mathrm{T}} \mathbf{S}_{3,x} - \left(\mathbf{S}_{1,x}\right)^{\mathrm{T}} \mathbf{S}_{3,y} - \left(\mathbf{S}_{3,y}\right)^{\mathrm{T}} \mathbf{S}_{1,x} \\
\mathbf{n}_{3,q_e q_e} &= \frac{\partial \mathbf{n}_{3,q_e}}{\partial \mathbf{q}_e} = \left(\mathbf{S}_{1,x}\right)^{\mathrm{T}} \mathbf{S}_{2,y} + \left(\mathbf{S}_{2,y}\right)^{\mathrm{T}} \mathbf{S}_{1,x} - \left(\mathbf{S}_{2,x}\right)^{\mathrm{T}} \mathbf{S}_{1,y} - \left(\mathbf{S}_{1,y}\right)^{\mathrm{T}} \mathbf{S}_{2,x}
\end{aligned}
\tag{C.14}
$$