Assignment 1

1 Practice: Fast API Implementation

In Module 2 we used the **spacy** library to implement word embedding functionality. Add the embedding functionality to the API implemented as part of Module 1 setup. Docker deployment is optional, but if you want to implement it, you can reference Module 3 Class Activity. Commit your code to GitHub.

2 Theory: Rules of Probability

Notation

- $A \cap B$ (intersection): "A AND B" - both events occur simultaneously
  Set interpretation: elements that belong to both set A and set B
- $A \cup B$ (union): "A OR B" - at least one of the events occurs
  Set interpretation: elements that belong to either set A or set B (or both)
- A' or A^c (complement): "NOT A" - event A does not occur
  Set interpretation: elements that do not belong to set A
- $P(A|B)$ (conditional): "probability of A given B" - probability of A when we know B occurred

Probability Rules

- Independence: $P(A \cap B) = P(A) \times P(B)$ if A and B are independent
- Addition Rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- Conditional Probability: $P(A|B) = \frac{P(A \cap B)}{P(B)}$
- Bayes' Rule: $P(A|B) = \frac{P(B|A) \times P(A)}{P(B)}$
- Law of Total Probability: $P(B) = P(B|A) \times P(A) + P(B|A') \times P(A')$
  Equivalently: $P(B) = P(B \cap A) + P(B \cap A')$

Expected Value and Variance

- Expected Value: $E[X] = \sum x \cdot P(X = x)$
- Variance: $\mathrm{Var}(X) = E[X^2] - (E[X])^2$
- Linearity of Expectation: $E[aX + b] = aE[X] + b$

Information Theory

- Entropy: $H(X) = -\sum P(X = x) \log P(X = x)$
- Cross Entropy: $H(p, q) = -\sum p(X = x) \log q(X = x)$

3 Problems

Note: Please show your work for all calculations. You may either write out the mathematical steps or provide Python code with results.

Question 1

Events A and B are independent, with $P(A) = 0.4$ and $P(B) = 0.3$. Calculate:

(a) $P(A \cap B)$

(b) $P(A \cup B)$

Question 2

Suppose $P(A) = 0.5$, $P(B) = 0.4$, and $P(A|B) = 0.7$. Are events A and B independent? Explain clearly.

Question 3

Use Bayes' Rule to solve the following: Given $P(A) = 0.6$, $P(B|A) = 0.5$, and $P(B|A') = 0.2$, calculate $P(A|B)$.

Question 4

A medical test for a disease is 95% accurate for detecting the disease when a person actually has it (true positive rate), and it correctly gives a negative result for 90% of the healthy people (true negative rate). If the probability of having the disease is 2%, calculate the probability that a person who tests positive actually has the disease. (Use Bayes' Rule).

Question 5

A discrete random variable X represents exam scores with the following probability distribution:

X	85	90	95	100
$P(X = x)$	0.375	0.375	0.125	0.125

(a) Calculate the expected value $E[X]$.

(b) Calculate the variance $\text{Var}(X) = E[X^2] - (E[X])^2$.

(c) The following sample was drawn from this distribution: $\{85, 90, 85, 95, 90, 85, 100, 90\}$. Calculate the sample mean and compare it with the expected value from part (a).

Question 6

A communication system transmits one of four possible messages with the following probabilities:

Message	A	B	C	D
Probability	0.4	0.3	0.2	0.1

(a) Calculate the entropy $H(X)$ of this message distribution in bits (use $\log_2$).

(b) If we could redesign the system to have equal probabilities for all messages, what would be the new entropy?

1. a) $P(A \cup B) = P(A) \cdot P(B)$

$0.4 \cdot 0.3 = 0.12$

b) $P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$= 0.4 + 0.3 - 0.12$

$= 0.58$

2. A, B are indep if $P(A \cup B) = P(A) \cdot P(B)$

$P(A | B) = \frac{P(A \cap B)}{P(B)}$

$P(A \cap B) = P(A | B) \cdot P(B) = 0.7 \cdot 0.4 = 0.28$

$P(A) \cdot P(B) = 0.2$

$P(A \cap B) = 0.28$

they are not indep

3. $P(A | B) = \frac{P(B | A) \cdot P(A)}{P(B)}$

$P(B) = P(B | A) \cdot P(A) + P(B | A') \cdot P(A')$

$P(A') = 1 - P(A) = 0.4$

$P(B) = 0.3 + 0.08 = 0.38$

$P(A | B) = \frac{0.5 \cdot 0.6}{0.38} = \frac{0.3}{0.38} \approx 0.789$

4. $P(D)=0.02 \quad P(D')=0.98$

$P(TP|D)=0.95 \quad P(FP|D')=0.05$

$P(TN|D)=0.9 \quad P(FN|D')=0.1$

$$P(D|TP)=\frac{P(TP|D)\cdot P(D)}{P(TP)}=\frac{0.95\cdot 0.02}{0.117}=0.162$$

$P(TP)=P(TP|D)\cdot P(D)+P(FN|D')\cdot P(D')$

$=(0.95\cdot 0.02)+(0.1\cdot 0.98)=0.117$

5 a) $E(X)=\sum X\cdot P(X)$

$=0.375\cdot 85+0.375\cdot 90+0.125\cdot 95$
$+0.125\cdot 100$

$=90$

b) $Var(X)=E(X^2)-(E(X))^2$

$E(X^2)=\sum X^2\cdot P(X)=8125$

$=85^2\cdot 0.375+90^2\cdot 0.375+95^2\cdot 0.125$
$+100^2\cdot 0.125$

$(E(X))^2 = 90^2 = 8100$

$Var(X) = 8125 - 8100 = 25$

c). $\bar{X} = \frac{\sum x_i}{n} \quad \frac{(85\cdot 3)+(90\cdot 3)+(95\cdot 1)+(100\cdot 1)}{8}$

$= 90$ Sample mean $= E(X)$.

6a). $H(X) = -\sum_{i=1}^{n} p(x_i)\log_2(p(x_i))$

$= -[0.4\log_2(0.4) + 0.3\log_2(0.3) + 0.2\log_2(0.2)$

$+ 0.1\log_2(0.1)]$

$= 1.846.$

b). $p_i = 0.25$

$H(X') = -\sum_{i=1}^{4} 0.25\log_2(0.25) = -4\cdot(0.25\log_2(0.25))$

$= -4\cdot(0.25\cdot\log_2(1/4) = -2)$

$= 4\cdot 0.25\cdot 2 = 2$