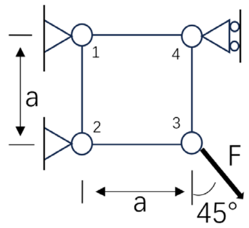
题：分析弹性正方形结构（如下图）在受力时各个节点的位移、反作用力和积分点 2 上的应力/应变大小。



其中 a=1mm；E=210N/mm2 ；v=0.25 ；F=10N ；n1（0，1），n2（0，0），n3（1，0），n4（1，1）

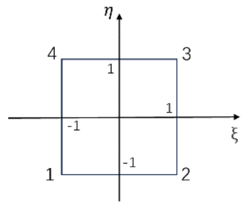
在 matlab 中定义变量 Xi 和 Eta
采用线性平面应变单元，[N]如下：

$$N1(\xi,\eta)=\frac{1}{4}(1-\xi)(1-\eta)$$

$$N2(\xi,\eta)=\frac{1}{4}(1+\xi)(1-\eta)$$

$$N3(\xi,\eta)=\frac{1}{4}(1+\xi)(1+\eta)$$

$$N4(\xi,\eta)=\frac{1}{4}(1-\xi)(1+\eta)$$



对单元的积分采用高斯二次积分。先计算 shape function 对于单元局部坐标的偏微分：

$$\frac{\partial N1(0,0)}{\partial\xi}=\begin{bmatrix}\frac{\partial N1(0,0)}{\partial\xi} & \frac{\partial N1(0,0)}{\partial\eta}\\ \frac{\partial N2(0,0)}{\partial\xi} & \frac{\partial N2(0,0)}{\partial\eta}\\ \frac{\partial N3(0,0)}{\partial\xi} & \frac{\partial N3(0,0)}{\partial\eta}\\ \frac{\partial N4(0,0)}{\partial\xi} & \frac{\partial N4(0,0)}{\partial\eta}\end{bmatrix}=\frac{1}{4}\begin{bmatrix}(-1)(1-\eta)|_{(0,0)} & (-1)(1-\xi)|_{(0,0)}\\ (1)(1-\eta)|_{(0,0)} & (-1)(1+\xi)|_{(0,0)}\\ (1)(1+\eta)|_{(0,0)} & (1)(1+\xi)|_{(0,0)}\\ (-1)(1+\eta)|_{(0,0)} & (1)(1-\xi)|_{(0,0)}\end{bmatrix}=\frac{1}{4}\begin{bmatrix}-1 & -1\\ 1 & -1\\ 1 & 1\\ -1 & 1\end{bmatrix}$$

雅可比矩阵为：

$$J=\begin{bmatrix}\frac{\partial X}{\partial\xi} & \frac{\partial X}{\partial\eta}\\ \frac{\partial Y}{\partial\xi} & \frac{\partial Y}{\partial\eta}\end{bmatrix}=\begin{bmatrix}XI\frac{\partial NI}{\partial\xi} & XI\frac{\partial NI}{\partial\eta}\\ YI\frac{\partial NI}{\partial\xi} & YI\frac{\partial NI}{\partial\eta}\end{bmatrix}=\begin{bmatrix}X1 & X2 & X3 & X4\\ Y1 & Y2 & Y3 & Y4\end{bmatrix}\begin{bmatrix}\frac{\partial N1}{\partial\xi} & \frac{\partial N1}{\partial\eta}\\ \frac{\partial N2}{\partial\xi} & \frac{\partial N2}{\partial\eta}\\ \frac{\partial N3}{\partial\xi} & \frac{\partial N3}{\partial\eta}\\ \frac{\partial N4}{\partial\xi} & \frac{\partial N4}{\partial\eta}\end{bmatrix}$$

其中

$$\frac{\partial X}{\partial\xi}=X\frac{\partial N}{\partial\xi}=\begin{bmatrix}X1 & X2 & X3 & X4\\ Y1 & Y2 & Y3 & Y4\end{bmatrix}\frac{1}{4}\begin{bmatrix}-1 & -1\\ 1 & -1\\ 1 & 1\\ -1 & 1\end{bmatrix}=\begin{bmatrix}0 & 0.5\\ -0.5 & 0\end{bmatrix}$$

雅可比的特征值和逆矩阵可以计算得到：

$$\det[J]=\det\left[\frac{\partial X}{\partial \xi}\right]=0.25$$

$$(\frac{\partial X}{\partial \xi})^{-1}=\begin{bmatrix}0 & -2\\ 2 & 0\end{bmatrix}$$

将[N]对空间坐标求导，得到应变关系矩阵[B]

$$\frac{\partial N}{\partial \mathrm{X}}=\frac{\partial N}{\partial \xi}(\frac{\partial X}{\partial \xi})^{-1}=\begin{bmatrix}-0.5 & 0.5\\ -0.5 & -0.5\\ 0.5 & -0.5\\ 0.5 & 0.5\end{bmatrix}$$

$$B_{I=}\begin{bmatrix}\frac{\partial N_I(0,0)}{\partial \mathrm{x}} & 0\\ 0 & \frac{\partial N_I(0,0)}{\partial y}\\ \frac{\partial N_I(0,0)}{\partial \mathrm{y}} & \frac{\partial N_I(0,0)}{\partial x}\end{bmatrix}$$

$$B=\left[\begin{array}{cc|cc|cc|cc}-0.5 & 0 & -0.5 & 0 & 0.5 & 0 & 0.5 & 0\\ 0 & 0.5 & 0 & -0.5 & 0 & -0.5 & 0 & 0.5\\ 0.5 & -0.5 & -0.5 & -0.5 & -0.5 & 0.5 & 0.5 & 0.5\end{array}\right]$$

材料弹性矩阵为：

$$C=\begin{bmatrix}\lambda+2\mu & \lambda & 0\\ \lambda & \lambda+2\mu & 0\\ 0 & 0 & \mu\end{bmatrix}$$

$$\lambda=\frac{vE}{(1+v)(1-2v)}=84N/mm^2$$

$$\mu=G=\frac{E}{2(1+v)}=84N/mm^2$$

$$C=\begin{bmatrix}252 & 84 & 0\\ 84 & 252 & 0\\ 0 & 0 & 84\end{bmatrix}N/mm^2$$

单元的刚度[K]与材料弹性矩阵[C]和应变关系矩阵[B]关系为：

$$k=\int_{\Omega}B^TCBdV=\int_0^1\int_0^1 B^TCBdxdy=\begin{bmatrix}84 & -42 & 42 & 0 & -84 & 42 & -42 & 0\\ -42 & 84 & 0 & -42 & 42 & -84 & 0 & 42\\ 42 & 0 & 84 & 42 & -42 & 0 & -84 & -42\\ 0 & -42 & 42 & 84 & 0 & 42 & -42 & -84\\ -84 & 42 & -42 & 0 & 84 & -42 & 42 & 0\\ 42 & -84 & 0 & 42 & -42 & 84 & 0 & -42\\ -42 & 0 & -84 & -42 & 42 & 0 & 84 & 42\\ 0 & 42 & -42 & -84 & 0 & -42 & 42 & 84\end{bmatrix}$$

静力学状态下

$$ku=f$$

位移为 0，消掉对应位置的刚度矩阵，可得：

$$\begin{bmatrix} 84 & -42 & 0 \\ -42 & 84 & -42 \\ 0 & -42 & 84 \end{bmatrix}\begin{bmatrix} u_{x3} \\ u_{y3} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} 10\sqrt{2}/2 \\ -10\sqrt{2}/2 \\ 0 \end{bmatrix}$$

计算位移:

$$u = k^{-1}f$$

$$\begin{bmatrix} u_{x3} \\ u_{y3} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} 0.04209 \\ -0.08418 \\ -0.04209 \end{bmatrix} mm$$

代入位移向量中，计算外力向量:

$$\begin{bmatrix} 84 & -42 & 42 & 0 & -84 & 42 & -42 & 0 \\ -42 & 84 & 0 & -42 & 42 & -84 & 0 & 42 \\ 42 & 0 & 84 & 42 & -42 & 0 & -84 & -42 \\ 0 & -42 & 42 & 84 & 0 & 42 & -42 & -84 \\ -84 & 42 & -42 & 0 & 84 & -42 & 42 & 0 \\ 42 & -84 & 0 & 42 & -42 & 84 & 0 & -42 \\ -42 & 0 & -84 & -42 & 42 & 0 & 84 & 42 \\ 0 & 42 & -42 & -84 & 0 & -42 & 42 & 84 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_{x3} \\ u_{y3} \\ 0 \\ u_{y4} \end{bmatrix} = \begin{bmatrix} R_{x1} \\ R_{y1} \\ R_{x2} \\ R_{y1} \\ F cos\dfrac{\Pi}{4} \\ -F cos\dfrac{\Pi}{4} \\ R_{x1} \\ 0 \end{bmatrix}$$

Rx1=-10$\sqrt{2}$/2N

Ry1=-10$\sqrt{2}$/2N

Rx2=-10$\sqrt{2}$/2N

Ry2=-10$\sqrt{2}$/2N

应变和应力为:

$$\varepsilon(\xi,\eta) = \varepsilon(0,0) \approx \sum_{I=1}^{NNODE} B_I(0,0)\cdot u_I = \begin{bmatrix} 0.021045 \\ 0.021045 \\ -0.08418 \end{bmatrix}$$

$$\sigma(\xi,\eta) = C\varepsilon(\xi,\eta) = \begin{bmatrix} 10\sqrt{2}/2 \\ 10\sqrt{2}/2 \\ -10\sqrt{2}/2 \end{bmatrix} N/mm^2$$