Improving the Agent's Formalization of Relevance: An Epistemic Logic Grounded in Possible Knowledge Bases

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Abstract. When faced with complex epistemic-combinatorial situations, agents struggle to formally differentiate between relational patterns, creating gaps in formal models. To address this, we introduce strictly relevant operators($(\phi \mid \psi)$, $(\phi \parallel \psi)$, $(\phi \nmid \psi)$) and construct an Epistemic Logic based on Possible Knowledge Bases (EL_{PKB}). Among these, these new operators require not only the absence of counterexample situations but also every truth cases must exist, ensuring a precise representation.To this end, we introduce a non-Kripke model that incorporates PKBs to define the semantics. In this context, a PKB refers to the knowledge combinations that an agent might possess in a given state. Then we explore the correspondence between PKBs and Global Modal Logic models within the proposed framework, while also providing variants of RN for EL_{PKB}. Finally, we also explore the cognitive attributes of agents, particularly focusing on two distinct levels of facticity and positive introspection within agent cognition. This capability to identify logical relevance is also a critical step toward enabling AI to approach human-level cognition.

Keywords: Logical relevance between propositions · Strictly relevant sufficient condition · Possible Knowledge Bases · Epistemic logic.

1 Introduction

Current generative AI systems fundamentally rely on the correlation degrees between tokens of large language models[3], which excel in question-answering tasks. However, when confronted with cognitively relevant scenarios beyond their trained memory, these AI systems struggle with performing cognitive combinatorial judgments on the relation of relevance between uncertain propositions [2, 3].

[3] They are automatically extractable language traces that reflect human language habits, and have the context-sensitive statistical property [1].

This capability represents a key step toward enabling AI to approach human-level cognition. It directly contributes to the field of AI cognitive modeling and facilitates research on explainable frameworks for AI uncertainty reasoning. Therefore, the first task is to capture the logical structures (knowledge-driven) that humans employ to address such problems.

In the real world, information is characterized by its complexity and diversity. From the epistemic perspective of the agent, there exists information that remains uncertain, driven by intellectual curiosity, we seek to deepen our understanding of these uncertainties. One pathway to achieve this is by determining the logical relevance among uncertain propositions within the agent' s cognitive system.

The characterization of logical relevance between propositions has always been a central concern of formal logic. For example, the formalization of sufficient conditions pays particular attention to implication relations, such as classical material implication ($p \to q$) primarily focuses on the truth-value in conditional statements[4, 5], but this approach encounters paradoxes of implication. Strict implication ($\Box(p \to q)$) introduces a necessity modal operator requiring $p \to q$ to hold in all possible worlds [6, 7]. Relevance logic emphasizes the requirement of content-relatedness between premises and conclusions (e.g., shared propositional variables)[8], some research also introduced epistemic contexts [9, 10]. Modal dependence logic [11] provides another method to characterize propositional relation by integrating dependence logic [12, 13] into Kripke models. It employs $=(p, q)$ to denote dependency and introduces $p_1 \perp_q p_2$ for independence [14], thereby enhancing its expressiveness.

Gerhard Lakemeyer(1997) explored relevance from an epistemological perspective, conducting a series of analytical reflections that integrated epistemic logic[15].In recent years, some studies have also focused on characterizing 'propositional relations' from a cognitive perspective. Xuefeng Wen (2024) discussed the influence of indicative terms and epistemic modalities on conditional reasoning, proposing a novel context-sensitive semantics [16]. David Makinson (2009) proposed that the relation of relevance was considered modulo the choice of a background belief set[17]. Jarmużek, T.(2020) formulated a logic based on relating semantics to evaluate both the extensional conditions and sufficiently relevant intensional conditions of propositional formulas[18].

Within relevant cognitive contexts, agents can already distinguish different types of propositional relationships (e.g., sufficient condition relations, equivalence relations, and irrelevance) to some extent using existing logical tools. However, when faced with complex epistemic combinations, agents struggle to formally distinguish different relational patterns, creating gaps in formal models.

We now present two intuitive examples to illustrate the existing issues and the stricter propositional relationships we aim to capture within epistemic contexts.

1.1 Example 1:The Barber Shop

In 1894, Lewis Carroll proposed a celebrated logical paradox[19]. Three barbers, Allen, Brown, and Carr, operate a barber shop. (1) The shop is always open, which means at least one barber is present. (2) After Allen developed a fever, he became anxious and never leaves the shop unless Brown accompanies him when he goes out.

Now we use a to denote "Allen is in the shop", b to denote "Brown is in the shop", c to denote "Carr is in the shop". We can derive two key pieces of information:(1)$a \vee b \vee c$ (2) $\neg a \rightarrow \neg b$. According to Modus Tollens[4], we now have the following inference.

(α) If Carr is out then if Allen is out then Brown is in.

(β) It is not the case that if Allen is out then Brown is in.

(γ) Therefore, it is not the case that Carr is out.

Based on (1) and (2), we can conclude that both (α) $\neg c \rightarrow (\neg a \rightarrow b)$ and (β) $\neg(\neg a \rightarrow b)$ are true. However, (γ) is clearly false because when Brown is in the shop and Carr is out, both (1) (2) regarding the barber shop are satisfied.

Why do we get different results? The critical issue lies in the different number of truth-value cases required for $(\neg a \rightarrow b)$ in (α) and (β).

In (α), when Carr is absent (i.e.,$\neg c$), the truth-values of a and b in $(\neg a \rightarrow b)$ are restricted to two possibilities: $a \wedge b$ and $a \wedge \neg b$. This satisfies the notice (1) and (2). However, (β) aims to express that the situation"Allen went out alone without Brown"is impossible, that is, we only negate $\neg a \wedge b$. However, $(\neg a \rightarrow b)$ can hold in other truth situations.

Therefore, (β) is incorrect. If we want to retain the original meaning, we must introduce a "weaker"formulation than (β)[5] , thereby avoiding the structure of Modus Tollens. Consequently, we no longer get the contradictory result (γ).

1.2 Example 2: The Candy House

This is an original example we developed. On the first day, the child entered the Candy House, where there was a candy box (denoted as Box 1) on the table. The shopkeeper described its properties: "This candy box contains only two types of candies: milk candies(m) and fruit candies(f). When you press the button on top, one of three outcomes will occur randomly[6] :①*both m and f* ②*no candies* or ③*only f*. The case of ④*only m* is impossible."

The shopkeeper then asked a question to the child: "Let's denote proposition p as 'milk candy appears when you press the button'[7], and proposition q as 'fruit candy appears when you press the button'. Based on this information, what relation do you perceive between p and q?" The child replied: "First, I think that the situations $p \wedge q$, $\neg p \wedge \neg q$, and $\neg p \wedge q$ are all possible, while $p \wedge \neg q$ is impossible. From this, I conclude that p is a sufficient condition for q."

The shopkeeper nodded and invited the child to return the next day.

The next day, the child returned to find two identical-looking candy boxes on the table. The shopkeeper explained: "In addition to yesterday's candy box

[4] Modus Tollens means form $\phi \rightarrow \psi$ and $\neg\psi$, we can get $\neg\phi$.

[5] i.e., a stronger formal expression than $(\neg a \rightarrow b)$.

[6] It is guaranteed that all possible cases occur.

[7] $\neg p$ as 'milk candy does not appear when you press the button'.

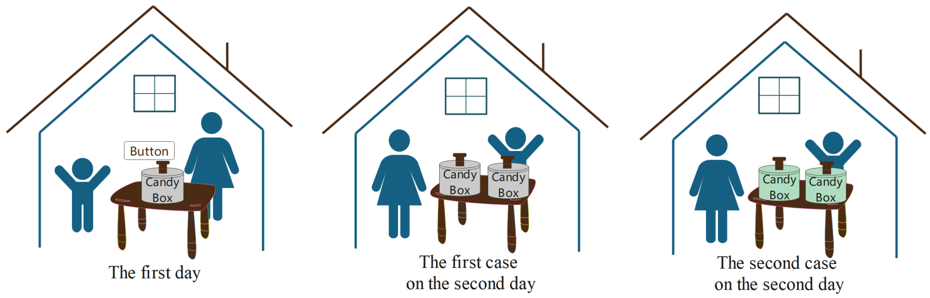

Fig. 1. The three scenarios in the Candy House

(Box 1), we've added another box (Box 2). The new box also contains only two types of candies, but differs in that pressing the button will randomly result in one of three outcomes: ①*both m and f* ②*no candies* or ④*only m*. The case of ③*only f* is impossible."

The shopkeeper asked the child to randomly stand in front of one of the boxes and questioned: "You don't know which box you're facing. Under this uncertainty, what relation do you perceive between p and q?" The child responded: "If I'm facing yesterday's box (Box 1), analysis shows that p could be a sufficient condition for q. Similarly, if I'm facing the new box (Box 2), then q could be a sufficient condition for p. Therefore, both $p \to q$ and $q \to p$ are possible."

The shopkeeper replied: "While 'both $p \to q$ and $q \to p$ are possible' describes your cognition in the above situation, it could also apply to other situation–for instance, if there were two boxes, one box ③*only appears f* after pressing the button and the other ④*only m*. Although the formal representation remains the same, your judgment of the relationship between p and q in this situation is different from before. Do we have a stricter formulation to respond this?"

The child thought for a moment and answered: "I'll revert to the initial analysis. Since I might be facing Box 1, I consider $p \wedge q$, $\neg p \wedge \neg q$, and $\neg p \wedge q$ as possible. If it's Box 2, then $p \wedge \neg q$ is also possible. Combining both possibilities, the outcomes $p \wedge q$, $\neg p \wedge \neg q$, $\neg p \wedge q$, and $p \wedge \neg q$ are all possible." The shopkeeper noted: "This is indeed stricter, but could it also apply to other situations?"

The shopkeeper then led the child to another table with two identical boxes, and explained: "These new boxes also contain only two types of candies. One box (Box3) will, upon pressing the button, randomly result in one of the two outcomes: either ①*both m and f*, or ②*no candies*; the other box (Box 4) will randomly result in one of the four outcomes: ①*both m and f* ②*no candies* ③*only f*, or④*only m*. " The shopkeeper again asked about the relation between p and q. The child replied: "Analyzing each box separately and then combining the possibilities, the outcomes $p \wedge q$, $\neg p \wedge \neg q$, $\neg p \wedge q$, and $p \wedge \neg q$ remain possible."

The epistemic judgment of the relation between propositions p and q remains same when confronted with two distinct candy box combinations (1+2 vs. 3+4). Therefore, we need a strictly relationship to describe the difference between (Box 1,Box,2) and (Box 3,Box,4).

1.3 Research Objectives

This paper focuses on distinguishing the logical relevance between propositions from an epistemic perspective. By introducing strictly relevant operators, we construct an "Epistemic Logic based on possible knowledge bases (EL_{PKB})" to characterize agents' epistemic differentiation of the logical relevance, thereby effectively addressing the above complex combination in cognition.

Building upon the above examples: Candy Box 1 formally captures the intuition that "when pressing the button,the appearance of a milk candy (p) strictly relevantly guarantees the appearance of fruit candy (q), while the converse does not hold." This indicates that p is a ***strictly relevant sufficient condition*** for q. Similarly, Candy Box 2 expresses that q is a strictly relevant sufficient condition for p. Candy Box 3 shows that "pressing the button always results in both milk (p) and fruit candies (q) appearing at the same time," which corresponds to p ***strictly relevant equivalent*** to q. Candy Box 4 illustrates that "the appearance of milk (p) and fruit candies (q) are no connection when pressing the button," meaning p and q are ***strictly irrelevant*** in this context.

The concept of strict relevance here involves three key aspects:

First, these strictly relevant operators require not only the absence of counterexample situation but also that every truth situations must exist, ensuring a precise representation. Crucially, in complex cognitive situations, these operators can capture distinct combinations of propositional relations, which distinguishes them from material implication and strict implication. And such semantics can avoid the "one-to-many" [8] problems in the Candy House example.

Second, this semantics cannot be captured by Kripke possible-world models, necessitating the introduction of the ***Possible Knowledge Bases*** in Section 2 of this paper. In this context, a PKB refers to the knowledge combinations that an agent might possess in a given state. When defining the PKBs, we first established atomic cases, then defined cases for K (representing knowledge operator), and finally formalized cases for the strict relevance operators. This process is also an inherent manifestation of our logical relevance.

Third, it is important to note that when an agent considers p to be a strictly relevant sufficient condition for q, this requires that both p and q are propositions whose truth-values are epistemically uncertain to the agent[9]. For example, consider the statement: "If 1+1=3, then x>3". Since the antecedent "1+1=3" is cognitively necessarily false, the strictly relevant sufficient condition proposed above should not apply here[10].

The paper considers that an agent's epistemic judgment of propositional relevance inherently constitutes a manifestation of cognitive capacity. Additionally,

[8] This refers to the example where the child's epistemic judgment of propositional relevance between p and q yields identical conclusions across different candy box combinations (e.g., 1+2, 3+4).

[9] Laplace (1820) posited that the laws of nature are deterministic, and randomness arises solely from our ignorance of underlying factors[20].

[10] This concept excludes propositions that are tautologies and contradictions, thereby avoiding paradoxes of material implication.

this paper distinguishes the cognitive levels of agents through cognitive attributes (such as "facticity", "positive self-reflectivity" and so on) , thereby enriching the modeling of higher-order cognitive abilities[21]. In Section 4, this hierarchical distinction is mapped through knowledge bases defined under different definitions, enabling the differentiation of agents with distinct degrees of higher-order cognitive capabilities. Meanwhile, this also provides significant insights into enhancing the cognitive capabilities of AI agents.

The present paper is structured as follows. Section 2 provides the language, models and semantics of EL_{PKB}, Section 3 discusses some properties of Possible Knowledge Bases, and section 4 elaborate on model properties and corresponding definitions.

2 Language, Models and Semantics

Definition 1 (Language $\mathcal{L}$). *The formulas ϕ of Epistemic Logic grounded in Possible Knowledge Bases (EL_{PKB}) are given by the rule*

$$\phi ::= \bot \ \mid \ p \mid \phi \to \phi \mid \phi|\psi \mid \phi||\psi \mid \phi \nmid \psi \mid K\phi,$$

where p is a propositional variable.

Extending the standard symbolism of basic epistemic modal logic, we formalize propositional relevance through three novel operators:$\phi|\psi$ (strictly relevant sufficient condition), $\phi||\psi$ (strictly relevant equivalence), and $\phi \nmid \psi$ (strict irrelevance). The epistemic constructions should be interpreted as: $K(\phi|\psi)$ denotes that ϕ being a strictly relevant sufficient condition for ψ is the agent's knowledge; $K(\phi||\psi)$ denotes that ϕ and ψ are being strictly relevant equivalence is the agent's knowledge; $K(\phi \nmid \psi)$ denotes that ϕ and ψ are being strict irrelevance is the agent's knowledge.

In the Candy House example, the agent's knowledge is partitioned based on different possible combinations. Therefore, when constructing the model, we allow an agent to possess distinct possible knowledge bases at the same state and determine their ultimate knowledge by evaluating each knowledge base they hold.

We define the model of EL_{PKB} as follows:

Definition 2 (Models). *$\mathcal{M} = (W, V, B, F)$, W is a non-empty set , regarded as a set of possible worlds. V is an assignment on W(from a set of propositional letters Φ to $\mathcal{P}(W)$), B is a collection of formula sets(will be defined later), regarded as the set of possible knowledge bases. F is a function from W to $\mathcal{P}(B)$.*

In contrast to conventional epistemic models, this novel frame eliminates epistemic accessibility relations defined over state points. Instead, B is regarded as a set of possible knowledge bases, and through the function F, the possible knowledge base possessed by the agent at the state point is given. Next, we define the conditions of the possible knowledge base B :

Generally speaking, a possible knowledge base is regarded as a combination of some cognitive states of the agent, needs to satisfy the following properties:

1 Maximality and closure property of propositional formulas: maximality requires that for any proposition and its negation at least one must be epistemically possible within the possible knowledge base. Closure property ensures that the possible knowledge base is closed under Modus Ponens[11] .
2 Higher-order knowledge: we assume that each agent has the ability of higher-order cognition.
3 Relevant condition: with the help of the relevant operator, agents can recognize the relevance of different propositions.

Definition 3. *To ensure those properties are satisfied in possible knowledge base, the following conditions are required:*

For each $s \in B$, s is a set of $\mathcal{L}$-formulas, meeting the following conditions :
1. Maximality and closure property of propositional formulas:[12]

1.1 $\perp \notin s$.
1.2 for any formula ϕ. if $\neg\phi \notin s$, then $\phi \in s$.
1.3 $\phi_1 \wedge \phi_2 \wedge ... \wedge \phi_n \in s$ if and only if with $\phi_{a_1} \wedge \phi_{a_2} \wedge ... \wedge \phi_{a_n} \in s$, where $a_1, a_2, ..., a_n$ is a permutation of $1, 2, ..., n$.
1.4 Let ψ be such a formula, in the form of replacing $\neg\neg\alpha$ anywhere in ψ with α, then $\phi \in s$ if and only if $\psi \in s$.
1.5 If $\phi \in s$, then $\phi \wedge \psi \in s$ or $\phi \wedge \neg\psi \in s$.
1.6 $\phi \wedge \neg(\psi_1 \wedge \psi_2) \in s$ if and only if $\phi \wedge \neg\psi_1 \in s$ or $\phi \wedge \neg\psi_2 \in s$.
1.7 If $\phi \wedge \psi \in s$, then $\phi \neq \neg\psi$ and $\phi \in s$, $\psi \in s$ ("=" means the equivalence about string).

2.Higher-order knowledge:

2.1 If $\neg K\neg\phi \in s$, then $\phi \in s$.
2.2 If $K\phi_1 \wedge ... \wedge K\phi_n \wedge \neg K\neg\psi \in s$, then $\phi_1 \wedge ... \wedge \phi_n \wedge \psi \in s$.
2.3 Let α, β be propositional formulas, let ϕ be formula contains K. Let α^p_ϕ and β^p_ϕ be results of replacing p by ϕ. For any formula set s, if $\alpha \in s$ implies $\beta \in s$, then $\alpha^p_\phi \in s$ implies $\beta^p_\phi \in s$.[13]

3.Relevant conditions:

3.1 $\neg(\phi|\psi) \in s$ iff $(\phi|\psi) \notin s$.
3.2 $\neg(\phi||\psi) \in s$ iff $(\phi||\psi) \notin s$.
3.3 $\neg(\phi \nmid \psi) \in s$ iff $(\phi \nmid \psi) \notin s$.
3.4 $(\phi|\psi) \in s$ iff $\phi \wedge \psi \in s, \neg\phi \wedge \psi \in s, \neg\phi \wedge \neg\psi \in s, \phi \wedge \neg\psi \notin s$.
3.5 $(\phi||\psi) \in s$ iff $\phi \wedge \psi \in s, \neg\phi \wedge \psi \notin s, \neg\phi \wedge \neg\psi \in s, \phi \wedge \neg\psi \notin s$.
3.6 $(\phi \nmid \psi) \in s$ iff $\phi \wedge \psi \in s, \neg\phi \wedge \psi \in s, \neg\phi \wedge \neg\psi \in s, \phi \wedge \neg\psi \in s$.

[11] Modus Ponens means form ϕ and $\phi \to \psi$, we can get ψ.
[12] Define $\phi \vee \psi := \neg(\neg\phi \wedge \neg\psi)$, and $\phi \to \psi := \neg(\phi \wedge \neg\psi)$.
[13] This condition can be seen as the preservation of monotonicity under substitution in higher-order knowledge.

3.7 Let α, β be propositional formulas, ϕ be formula contains $K, |, ||, \nmid$. Let α^p_ϕ and β^p_ϕ be results of replacing p by ϕ. For any formula set s , if $\alpha \in s$ implies $\beta \in s$, then $\alpha^p_\phi \in s$ implies $\beta^p_\phi \in s$.[14]

Definition 4 (semantics). *For each $\phi \in \mathcal{L}$, define ϕ is satisfied in $\mathcal{M}$ at state $u \in W$ as follows:*

- $\mathcal{M}, u \vDash \bot$ *never.*
- $\mathcal{M}, u \vDash p$ *iff* $u \in V(p)$.
- $\mathcal{M}, u \vDash \neg\phi$ *iff* $u \nvDash \phi$.
- $\mathcal{M}, u \vDash \phi \to \psi$ *iff* $u \vDash \phi$ *imples* $u \vDash \psi$.
- $\mathcal{M}, u \vDash K\phi$ *iff for every* $s \in F(u)$, $\neg\phi \notin s$.
- $\mathcal{M}, u \vDash (\phi|\psi)$ *never,* $\mathcal{M}, u \vDash (\phi||\psi)$ *never,* $\mathcal{M}, u \vDash (\phi \nmid \psi)$ *never.* [15]

Within this semantic framework, an agent's knowledge must always be evaluated against the set of possible knowledge bases available at the given state. Furthermore, since propositional relevance is restricted to estimate unknown propositions, any judgment regarding propositional relevance at a state point is inherently invalid.

Furthermore, strict irrelevance between propositions can be defined through strictly relevant sufficient conditions and strictly relevant equivalence:

Fact 1 *For every model $\mathcal{M} = (W, V, B, F)$, let $s \in B$, then $(\phi \nmid \psi) \in s$ if and only if $\neg(\phi|\psi) \wedge \neg(\neg\phi|\psi) \wedge \neg(\phi|\neg\psi) \wedge \neg(\neg\phi|\neg\psi) \wedge \neg(\phi||\psi) \wedge \neg(\neg\phi||\psi) \in s$ and $\phi \in s$, $\neg\phi \in s$, $\psi \in s$, $\neg\psi \in s$.*

Proof. From left to right: by ***Definition 3.3*** , $(\phi \nmid \psi) \in s$ if and only if $\phi \wedge \psi \in s, \neg\phi \wedge \psi \in s, \neg\phi \wedge \neg\psi \in s, \phi \wedge \neg\psi \in s$. Since $\phi \wedge \psi \in s$, then $(\phi|\neg\psi) \notin s$ and $(\neg\phi||\psi) \notin s$; since $\neg\phi \wedge \psi \in s$, then $(\neg\phi|\neg\psi) \notin s$ and $(\phi||\psi) \notin s$; since $\neg\phi \wedge \neg\psi \in s$, then $(\neg\phi|\psi) \notin s$; since $\phi \wedge \neg\psi \in s$, then $(\phi|\psi) \notin s$. Therefore, $\neg(\phi|\psi) \wedge \neg(\neg\phi|\psi) \wedge \neg(\phi|\neg\psi) \wedge \neg(\neg\phi|\neg\psi) \wedge \neg(\phi||\psi) \wedge \neg(\neg\phi||\psi) \in s$. And $\phi \wedge \psi \in s$ implies $\phi \in s$ and $\psi \in s$, $\neg\phi \wedge \neg\psi \in s$ implies $\neg\phi \in s$ and $\neg\psi \in s$, then $(\phi \nmid \psi) \in s$ entails $\phi \in s$, $\neg\phi \in s$, $\psi \in s$, $\neg\psi \in s$.

From right to left: by ***Definition 3.3***, if $(\phi \nmid \psi) \notin s$:

Assume $\phi \wedge \psi \notin s$, as $\phi \in s$ and $\psi \in s$, then $\phi \wedge \neg\psi \in s$ and $\neg\phi \wedge \psi \in s$. Since $\neg(\phi|\psi) \in s$, then $\neg\phi \wedge \neg\psi \in s$. That implies $\phi|\neg\psi \in s$, a contradiction.

The other cases are same as above.

Fact 2 *For every model $\mathcal{M} = (W, V, B, F)$, let $u \in W$,then:*

(1) If $\mathcal{M}, u \vDash K(\phi \nmid \psi)$, then $\mathcal{M}, u \vDash K(\neg(\phi|\psi) \wedge \neg(\neg\phi|\psi) \wedge \neg(\phi|\neg\psi) \wedge \neg(\neg\phi|\neg\psi) \wedge \neg(\phi||\psi) \wedge \neg(\neg\phi||\psi))$.

(2) If $\mathcal{M}, u \vDash K(\neg(\phi|\psi) \wedge \neg(\neg\phi|\psi) \wedge \neg(\phi|\neg\psi) \wedge \neg(\neg\phi|\neg\psi) \wedge \neg(\phi||\psi) \wedge \neg(\neg\phi||\psi)) \wedge K\neg K\neg\phi \wedge K\neg K\phi \wedge K\neg K\neg\psi \wedge K\neg K\psi$, then $\mathcal{M}, u \vDash K(\phi \nmid \psi)$.

[14] This condition can be regard as the preservation of monotonicity under substitution in higher-order knowledge and relevant conditions.

[15] At a specific state, ϕ and ψ both have certain values, this situation contradicts our desired intuition, hence $\mathcal{M}, u \vDash (\phi|\psi)$ is always false.

Proof. (1): Suppose $\mathcal{M}, u \vDash K(\phi \nmid \psi)$, that is for every $s \in F(u)$: $\neg(\phi|\psi) \wedge \neg(\neg\phi|\psi) \wedge \neg(\phi|\neg\psi) \wedge \neg(\neg\phi|\neg\psi) \wedge \neg(\phi||\psi) \wedge \neg(\neg\phi||\psi) \in s$, then for every $s \in F(u)$: $(\phi|\psi) \notin s$, $(\neg\phi|\psi) \notin s$, $(\phi|\neg\psi) \notin s$, $(\neg\phi|\neg\psi) \notin s$, $(\phi||\psi) \notin s$, $(\neg\phi|\psi) \notin s$. That implies for every $s \in F(u)$: $(\phi|\psi) \vee (\neg\phi|\psi) \vee (\phi|\neg\psi) \vee (\neg\phi|\neg\psi) \vee (\phi||\psi) \vee (\neg\phi||\psi) \notin s$, then $\mathcal{M}, u \vDash K(\neg(\phi|\psi) \wedge \neg(\neg\phi|\psi) \wedge \neg(\phi|\neg\psi) \wedge \neg(\neg\phi|\neg\psi) \wedge \neg(\phi||\psi) \wedge \neg(\neg\phi||\psi))$.

(2): Suppose $\mathcal{M}, u \vDash K(\neg(\phi|\psi) \wedge \neg(\neg\phi|\psi) \wedge \neg(\phi|\neg\psi) \wedge \neg(\neg\phi|\neg\psi) \wedge \neg(\phi||\psi) \wedge \neg(\neg\phi||\psi)) \wedge K\neg K\neg\phi \wedge K\neg K\phi \wedge K\neg K\neg\psi \wedge K\neg K\psi$, that implies for every $s \in F(u)$: $\neg(\phi|\psi) \wedge \neg(\neg\phi|\psi) \wedge \neg(\phi|\neg\psi) \wedge \neg(\neg\phi|\neg\psi) \wedge \neg(\phi||\psi) \wedge \neg(\neg\phi||\psi) \in s$ and $\phi \in s$, $\neg\phi \in s$, $\psi \in s$, $\neg\psi \in s$. By **Fact 1**, for every $s \in F(u)$:$(\phi \nmid \psi) \in s$, that is $\mathcal{M}, u \vDash K(\phi \nmid \psi)$.

Bcak to the Barber Shop example, (α) $\neg c \to (\neg a \to b)$ can retain its original or be formalized as $\neg c|(\neg a \to b)$. However, it cannot be formalized as $\neg c \to (\neg a|b)$, because when $\neg c$ holds, the truth values only allow $a \wedge b$ and $a \wedge \neg b$, whereas $(\neg a|b)$ also requires $\neg a \wedge b$.

(β) only aims to negate $\neg a \wedge b$, but $(\neg a \to b)$ holds in multiple truth situations, so it cannot be negated. (β) can be formalized as $\neg(\neg a|b)$, because $(\neg a|b)$ requires every truth cases must exist. Once $\neg a \wedge b$ is negated, $(\neg a|b)$ is negated.

Thus, based on EL_{PKB} formalization of the example above, we find that it does not be Modus Tollens. Note that $a|(b|c)$ always fails to hold here, as the internal $(b|c)$ is always either true or false, while $a|\top$ and $a|\bot$ never hold.

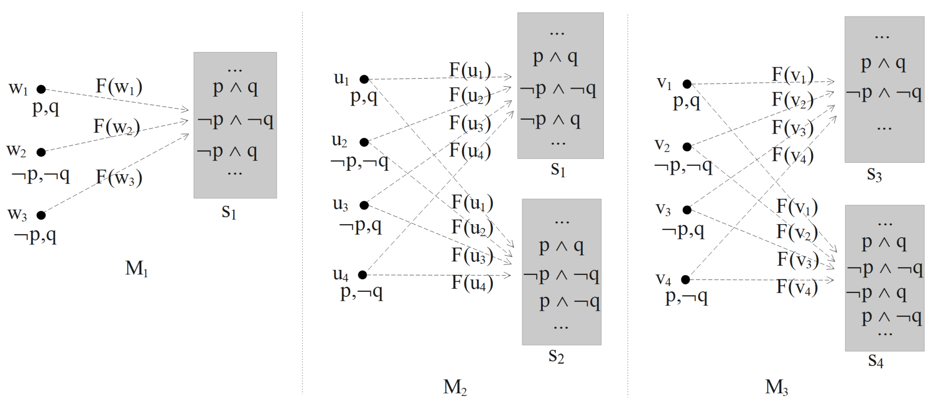

Fig. 2. The semantic model of the Candy House example

Back to the Candy House example. In fig.2, models M_1, M_2 and M_3 respond to the characteristics of agents' cognitive states under the three scenarios in fig.1. States w_1, w_2, w_3, u_1, v_1 etc., represent actual-world situations. For instance, in world w_3, milk candy does not appear when the button is pressed, while fruit candy does. PKBs s_1, s_2, s_3, and s_4 represent the knowledge combinations an agent might possess in a given state, each corresponding to the properties of Candy Box 1, 2, 3, and 4, respectively. From fig.2, we obtain the following results:

In the case of the Box1, the relationship between the two cognitively uncertain propositions p and q is formalized as $(p|q)$. The situation in Box 2 should be $(q|p)$, the situation in Box 3 should be $(p||q)$, the situation in Box 4 should be $(p \nmid q)$. Meanwhile, the cognition of agent on the first day should be $K(p|q)$. The first case on the second day should be $\neg K\neg(p|q) \wedge \neg K\neg(q|p) \wedge K((p|q) \vee (q|p))$. The second case should be $\neg K\neg(p||q) \wedge \neg K\neg(p \nmid q) \wedge K((p||q) \wedge (p \nmid q))$.

For example, based on M_1 in Fig.2, let us first examine possible world w_1: For every $s \in F(w_1)$, specifically s_1, the following holds: $p \wedge q \in s_1$, $\neg p \wedge \neg q \in s_1$, $\neg p \wedge q \in s_1$, $p \wedge \neg q \notin s_1$. By ***Definition 3.3.4***, this means $(p|q) \in s_1$. By ***Definition 3.3.1***, it follows that $\neg(p|q) \notin s_1$. Applying ***Definition 4***, we derive $M_1, w_1 \models K(p|q)$. Similarly, we obtain $M_1, w_2 \models K(p|q)$ and $M_1, w_3 \models K(p|q)$. Thus, the agent's epistemic judgment in this scenario is $K(p|q)$.

3 Some Properties of Possible Knowledge Bases

Fact 3 *Let s be a set of formulas defined above, then:*

(1) $\neg(K(\phi \to \psi) \to (K\phi \to K\psi)) \notin s$.
(2) $\neg(K(\phi \wedge \psi) \leftrightarrow (K\phi \wedge K\psi)) \notin s$.
(3) $\neg(\neg K\neg(\phi \vee \psi) \leftrightarrow (\neg K\neg\phi \vee \neg K\neg\psi)) \notin s$.

Proof. We will prove (1), and (2)(3) are same.

If there is s such that $\neg(K(\phi \to \psi) \to (K\phi \to K\psi)) \in s$ holds, then $K(\phi \to \psi) \wedge \neg(K\phi \to K\psi) \in s$. That is $K(\phi \to \psi) \wedge K\phi \wedge \neg K\psi \in s$, by ***Definition3.2.2***, we can see $(\neg\phi \vee \psi) \wedge (\phi \wedge \neg\psi) \in s$, that is $(\phi \wedge \neg\psi) \wedge \neg(\phi \wedge \neg\psi) \in s$, a contradiction with ***Definition 3.1.7***.

By ***Definition 3.2.3***, fix β as $\top$, we can see s can preserve uniform substitution:

Fact 4 *Let α be formula and α^p_ϕ be result of replacing p by ϕ. For any formula set s defined above, if $\alpha \in s$, then $\alpha^p_\phi \in s$.*

This corollary ensures that all fundamental validities of normal modal logic persist in every possible knowledge base, with their existential integrity preserved under uniform substitution.Consequently, we get the following proposition:

Proposition 1. *For any $s \in B$, let s' be a maximal subset of s that contains no formulas in which $|$, $||$, $\nmid$ occur. Then s' is a normal modal logic.(define $\Box$ as K)*

Furthermore,we formally designate s' as the Basic Epistemic Modal Logic (BEML) fragment of s. This framework establishes deeper structural connections between arbitrary possible knowledge bases s and Kripke models: For every $s \in B$ there exists a unique s' and an corresponded Kripke model $\mathcal{N}$.

Corollary 1. *Each $s \in B$ is in bijective correspondence with its designated fragment s'.*

Prior to the construction of corresponding Kripke models, it is imperative to formally define the 'size' of formula set s:

Definition 5. *For every $s \in B$, let s' be the BEML-fragment of s. If there are at least n sets $a_1, ..., a_n$ s.t. $s' \subseteq \bigcup a_i$ and for each a_i: $\phi \in a_i$ if and only if $\neg\phi \notin a_i$.*[16] *Then we define $rank(s') = n$.*[17]

Example 1. Fix $s \in B$ and let $\{p, \neg p, q, \neg q, p \wedge q, \neg p \wedge q, p \wedge \neg q\} \subseteq s'$. Then s' needs to be divided into at least 3 subsets, which are: $\{p, q\}, \{\neg p, q\}, \{p, \neg q\}$.

Next, we can correspond s to a Kripke model.

Definition 6. *Let $s \in B$ and s' be the BEML-fragment of s. Let $rank(s') = n$, and the sets are $a_1, ..., a_n$. Let $\mathcal{N} = (W', R', V')$ be a Kripke model where:*

- *$|W'| = rank(s')$ and $W' = \{u_1, ..., u_n\}$;*
- *R' is the binary relation on W' defined by $R'u_iu_j$ if and only if for all formulas ϕ, $\phi \in a_j$ implies $\neg K\neg\phi \in a_i$;*
- *V' is the valuation defined by $u_i \in V'(p)$ if and only if $p \in a_i$.*

This definition is similar to the way that canonical models are defined in basic modal logic. In the same way, we can also prove the results we want by the corresponding existence lemma and truth lemma.

Lemma 1. *Fix a formula set s, let s' be the BEML-fragment of s. Then*

1. *if $\neg K\neg\phi \in a_i$, then there is $u_j \in W$ such that $R'u_iu_j$ and $\phi \in a_j$;*
2. *$\phi \in a_i$ if and only if $\mathcal{N}, u_i \vDash \phi$.*

Proof. Similar to the proof of existence lemma and truth lemma of basic modal logic.

Proposition 2. *(1) $\phi \in s$ if and only if there is $\phi \in a_i$ and $u_i \in W'$ s.t. $\mathcal{N}, u_i \vDash \phi$. (2) $\neg\phi \notin s$ if and only if $\mathcal{N} \vDash \phi$.*

Proof. By ***Lemma 1.2***, (1) is immediate,(2) can be obtained by (1) taking the inverse negative proposition.

Building upon the aforementioned propositions, each BEML fragment of knowledge base s bijectively corresponds to a Kripke model. Intuitively, possible knowledge bases can be viewed as epistemic state composites, over which propositional relevance operators act upon such composites.

To extend this framework to strictly relevant sufficient condition operators, we must employ global modal logic (alternatively termed universal modal logic). This enhanced system introduces dual global modalities E and U, Specifically:

$$\mathcal{M}, w \vDash E\phi \text{ if and only if } \exists u \in W : \mathcal{M}, u \vDash \phi \ ;$$

[16] Avoid the notion of consistency since there are no syntactic definitions involved here.
[17] As the countability of the propositional variables set Φ , $rank(s') \leqslant \aleph_1$.

$$\mathcal{M}, w \vDash U\phi \text{ if and only if } \forall u \in W : \mathcal{M}, u \vDash \phi.$$

The axiom system, completeness and model definability of global modal logic can be referred to some references[22–24].

Proposition 3. *For every $s \in B$, let $\mathcal{N}$ be the Kripke model generated by s'. Then $(\phi|\psi) \in s$ if and only if $E(\phi \wedge \psi) \wedge E(\neg\phi \wedge \psi) \wedge E(\neg\phi \wedge \neg\psi) \wedge \neg E(\phi \wedge \neg\psi)$ is satisfied in $\mathcal{N}$. $(\phi||\psi) \in s$ if and only if $E(\phi \wedge \psi) \wedge E(\neg\phi \wedge \neg\psi) \wedge \neg E(\neg\phi \wedge \psi) \wedge \neg E(\phi \wedge \neg\psi)$ is satisfied in $\mathcal{N}$. $(\phi \nmid \psi) \in s$ if and only if $E(\phi \wedge \psi) \wedge E(\neg\phi \wedge \neg\psi) \wedge E(\neg\phi \wedge \psi) \wedge E(\phi \wedge \neg\psi)$ is satisfied in $\mathcal{N}$.*

Proposition 4. *Let $\mathcal{M} = (W, V, B, F)$ be a model, for every $s \in B$, there is a kripke model $\mathcal{N}_s$, translate $(a|b)$, $(a||b)$, $(a \nmid b)$as above. Then for all formulas ϕ, $\phi \in s$ if and only if ϕ is satisfied in $\mathcal{N}_s$.*

For any $s \in B$, $\mathcal{N}_s$ is defined as a Kripke model generated by s, and $\mathcal{N}_B = \{\mathcal{N}_s | s \in B\}$ is a class of Kripke models generated by M.

4 Model Properties and Corresponding Definitions

Proposition 5. *Here are some formulas that are valid in the model:*

(1) $K\phi \to KK\phi$.
(2) $K(\phi|\psi) \to K(\phi \to \psi)$.
(3) $K\phi \to \neg(K(\phi|\psi) \vee K(\phi||\psi) \vee K(\phi \nmid \psi))$.

Proof. The proof of (1):let $\mathcal{M} = (W, V, B, F)$ be a model, for any state $u \in W$, suppose $\mathcal{M}, u \vDash K\phi$. Then for every $s \in F(u)$, $\neg\phi \notin s$, by ***Definition 3.2.1*** , we have $\neg K\phi \notin s$. Therefore, $\mathcal{M}, u \vDash KK\phi$.

Proposition 5(1) corresponds to the positive introspection property in epistemic logic. In basic epistemic logic, the realization of an agent's positive introspection requires the Kripke model to be defined over a transitive frame class. However, within the possible knowledge base model framework, positive introspection becomes an inherently valid property.

Note that: the necessitation rule (RN) from the Hilbert-style calculus of basic modal logic fails here. While $\mathcal{M} \vDash K\phi \to KK\phi$ holds universally across all EL_{PKB} models, one can construct the countermodel: $\mathcal{M}' \nvDash K(K\phi \to KK\phi)$.

Intuitively, this implies that truths of the world do not necessarily constitute agent's knowledge. In basic modal logic, RN can be interpreted as a form of global semantic consequence :$\phi \vDash^g_{\mathbb{M}} \Box\phi$. Although RN does not hold in EL_{PKB} , we demonstrate that global semantic consequences remain locally preservable:

Proposition 6. *Let $s \in B$ and $\mathcal{N}$ be the generated model by s. For all formulas ϕ and ψ, if $\phi \Vdash^g_{\mathcal{N}} \psi$, then for every $\mathcal{M}$, $\mathcal{M} \vDash K\phi \to K\psi$.*

Proof. Suppose $\phi \Vdash^g_{\mathcal{N}} \psi$ and there is a model $\mathcal{M} = (W, V, B, F)$ such that $\mathcal{M} \vDash K\phi$. Then for any $s \in B$, $\neg\phi \notin s$, by ***Proposition 4*** $\mathcal{N} \vDash \phi$. As $\phi \Vdash^g_{\mathcal{N}} \psi$, we have $\mathcal{N} \vDash \psi$, by ***Proposition 4***, $\neg\psi \notin s$ and $\mathcal{M} \vDash K\psi$. Therefore $\mathcal{M} \vDash K\phi \to K\psi$.

When we introduce more definitions of possible knowledge bases sets B and functions F, we can bring some new properties to EL_{PKB} model:

Proposition 7. *Let $\mathcal{M} = (W, V, B, F)$ be a model, such that for every $s \in B$, if $\phi \wedge \psi \in s$ entails $\phi \wedge \neg K \neg \psi \in s$, then $\mathcal{M} \vDash K(K\phi \to \phi)$.*

Proof. Suppose $u \in W$ and $\mathcal{M}, u \vDash \neg K(K\phi \to \phi)$. Then there is $s \in F(u)$ s.t. $\neg(K\phi \to \phi) \in s$, that is $K\phi \wedge \neg\phi \in s$. Define $\psi_1 := K\phi$ and $\psi_2 := \neg\phi$, then we see $\psi_1 \wedge \psi_2 \in s$, hence $\psi_1 \wedge \neg K \neg \psi_2 \in s$. Therefore $K\phi \wedge \neg K \neg\neg\phi \in s$, by ***Definition 3.2.2***, $\phi \wedge \neg\phi \in s$, a contradiction.

We can also obtain the model properties of the model $\mathcal{N}$ generated by s.

Proposition 8. *Let $\mathcal{M} = (W, V, B, F)$ be a model, such that for every $s \in B$, if $\phi \wedge \psi \in s$ entails $\phi \wedge \neg K \neg \psi \in s$, then $\mathcal{N}_B$ is a class of reflexive models.*

Proof. As $\mathcal{M} \vDash K(K\phi \to \phi)$, then for every $s \in B$, $\neg(K\phi \to \phi) \notin s$. By ***Proposition 4***, that is $\mathcal{N}_s \vDash K\phi \to \phi$. Then, for every state $a \in \mathcal{N}_s$ and every formula ϕ, $\mathcal{N}_B, a \vDash K\phi$ implies $\mathcal{N}_s, a \vDash \phi$, by ***Definition 6***, $(a, a) \in R$. Therefore, $\mathcal{N}_s$ is a reflexive model and $\mathcal{N}_B$ is a class of reflexive models.

Since RN is absent, we can only attain the cognitive recognition of knowledge's facticity—that agents know their knowledge is correct—by corresponding possible knowledge bases with reflexive Kripke models.

Crucially, even with epistemic recognition of knowledge facticity, this does not guarantee the factual grounding of the agent's knowledge ($K\phi \to \phi$). Additional conditionals are required to enforce such factual correspondence.

Notably, while ***Proposition 5*** demonstrates the universal validity of positive introspection within EL_{PKB}, $K(K\phi \to KK\phi)$ is not universal valid.

Next, we give the conditions for $K(K\phi \to KK\phi)$ to be valid and prove it:

Proposition 9. *Let $\mathcal{M} = (W, V, B, F)$ be a model, such that for every $s \in B$, if $\psi \wedge \neg KK \neg\phi \in s$ implies $\psi \wedge \neg K \neg \phi \in s$, then:*

- $\mathcal{M} \vDash K(K\phi \to KK\phi)$;
- *$\mathcal{N}_B$ is a class of transitive models.*

Proof. (1) Suppose $u \in W$ and $\mathcal{M}, u \vDash \neg K(K\phi \to KK\phi)$. Then there is $s \in F(u)$ s.t. $\neg(K\phi \to KK\phi) \in s$, that is $K\phi \wedge \neg KK\phi \in s$. Define $\psi_1 := K\phi$ and $\psi_2 := \neg\phi$, then $\psi_1 \wedge \neg KK \neg\psi_2 \in s$, hence $\psi_1 \wedge \neg K \neg \psi_2 \in s$. That is $K\phi \wedge \neg K \neg\neg\phi$, a contradiction.

(2) As $\mathcal{M} \vDash K(K\phi \to KK\phi)$, then for every $s \in B$, $\neg(K\phi \to KK\phi) \notin s$. By ***Proposition 4***, that is $\mathcal{N}_s \vDash K\phi \to KK\phi$. Then, for every model $\mathcal{N}_s \in \mathcal{N}_B$ and state $a \in \mathcal{N}_s$ and every formula ϕ, we have $\mathcal{N}_s, a \vDash K\phi$ implies $\mathcal{N}_s, a \vDash KK\phi$. Therefore, $\mathcal{N}_s$ is a transitive model and $\mathcal{N}_B$ is a class of transitive models.

In fact, the model enables the expression of numerous different properties, which meaningfully distinguish the cognitive capacities of agents. The table 1 below visually illustrates the relationships between these cognitive capabilities and the properties and definitions of the model.

Table 1. Cognitive properties and model conditions

Conditions	Rules and Formulas	$\mathcal{N}_B$	Cognitive capabilities
$p \in s$ implies $\exists u \in W$ s.t $u \in V(p)$	$\mathcal{M} \vDash p$ entails $\mathcal{M} \vDash Kp$.	/	Propositional generalization
$u \vDash p$ implies $\exists s \in B$ s.t $p \in s$	$\mathcal{M} \vDash Kp \to p$	/	Propositional facticity
$\phi \wedge \psi \in s$ implies $\phi \wedge \neg K \neg \psi \in s$	$\mathcal{M} \vDash K(K\phi \to \phi)$	Reflexive	Cognitive facticity
$\psi \wedge \neg KK\neg\phi \in s$ implies $\psi \wedge \neg K \neg \phi \in s$	$\mathcal{M} \vDash K(K\phi \to KK\phi)$	Transitive	Cognitive positive introspection
$\phi \wedge \psi \in s$ implies $\phi \wedge K\neg K \neg \psi \in s$	$\mathcal{M} \vDash K(\phi \to K\neg K\neg\phi)$	Symmetric	Knowing the negation of true propositions is not knowledge
$\phi \in s$ implies $\neg K \neg \phi \in s$	$\mathcal{M} \vDash KK\phi \to K\phi$	Left-unbounded	No unrecognized state
for every $s \in B$, $K \perp \notin s$	$\mathcal{M} \vDash K\neg K \neg \top$	Right-unbounded	No state without recognizing

5 Future Prospects

One open problem deserving further investigation is the axiomatization of EL_{PKB}. We have demonstrated that the RN principle fails to hold and established a weaker version (Proposition 6). It remains critical to explore how these findings will alter and challenge the axiomatization framework.

Another significant research direction lies in knowledge revision and update mechanisms within EL_{PKB}. The structural properties of possible knowledge bases give rise to two distinct operational modalities for epistemic updates in EL_{PKB}: (1) altering the mapping $F(u)$ at state u, and (2) modifying the constituent possible knowledge bases s. When combined with an agent's varying cognitive capacities, these operations may hold greater research significance or reveal novel theoretical insights into their interplay. In the future, we can also integrate "knowledge revision and update mechanisms within EL_{PKB}" to assist AI systems in enhancing the evaluation of logical relevance within a dynamic cognitive framework and improving reasoning under uncertainty.

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