Response Performance of Fuzzy Systems Derived From the Symmetric Quintuple Implicational Method

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Abstract. Based on the symmetric quintuple implicational (SQI) method proposed by us, in this study, we propose a new fuzzy system based on the SQI method (SQI fuzzy system for short) and conduct strict theoretical and practical research on its response performance. To begin with, in order to address the limitations of traditional fuzzy reasoning methods, we innovatively introduce the SQI method into the fuzzy system. This advancement optimizes the similarity calculation between the rule base and the input, and generalizes the fuzzy implication operator at the same time. In addition, the SQI fuzzy system is systematically constructed. For multi-rule scenarios, we formalize its implementation framework by adopting a singleton fuzzifier and a centroid defuzzifier to ensure compatibility with the actual operating environment. Finally, the response performance of the system under the multi-rule configuration of R-implication operators and other classical operators is analyzed.

Keywords: The CRI method, the Triple I method, symmetric quintuple implicational method, fuzzy system

1 Introduction

Since Zadeh introduced the concepts of fuzzy sets and fuzzy logic [1]-[4], the research on fuzzy reasoning algorithms has been very extensive. The performance of fuzzy control system [5]-[8] fundamentally depends on these reasoning mechanisms. However, the traditional combinatorial reasoning rule (CRI) shows significant limitations, because it only depends on a single fuzzy implication operator and combination operation, thus deviating from the semantic core of logical implication. In order to solve these shortcomings and support the logical basis of CRI, Wang [7] proposed the triple I algorithm, which extends the reasoning process to include triple meanings. Subsequently, Wang [9] established the optimal solution of general fuzzy reasoning within this framework. Zhang and Yang [10] further promoted the field by introducing the triple I algorithm based on generalized roots into the normative fuzzy logic system, while Pei [11] developed the logic framework of the triple I algorithm based on monotone norms.For the Fuzzy Modus Ponens (FMP) problem, the triple I algorithm yields the minimal solution B^* such that

$$(A(x) \to B(y)) \to (A^*(x) \to B^*(y)) \quad (1)$$

gets the maximum value for any $x \in X, y \in Y$. In addition, the solution A^* of the FMT problem is the maximum fuzzy set such that (1) obtains the maximum value.

Despite its improved logical rigor, the triple I method does not account for the similarity between inputs and rule antecedents (e.g., A^* and A), which can result in ineffective approximations in certain cases. To overcome this limitation, Zhou et al. [12] introduced the quintuple implication principle (QIP) algorithm. For the FMP problem, QIP identifies the minimal B^* such that

$$(A(x) \to B(y)) \to \big((A^*(x) \to A(x)) \to (A^*(x) \to B^*(y))\big) \quad (2)$$

Attains the maximum value for any $x \in X, y \in Y$. The solution A^* of the FMT problem is the one making

$$(A(x) \to B(y)) \to \big((B(y) \to B^*(y)) \to (A(x) \to A^*(x))\big). \quad (3)$$

Get the minimum fuzzy set that makes what achieve the maximum value.

Building on these developments, we [13] proposed the symmetric triple I method, which generalizes both the triple I and CRI approaches. This method revises expression (1) to

$$(A(x) \to_1 B(y)) \to_2 (A^*(x) \to_1 B^*(y)), \quad (4)$$

where $\to_1$ and $\to_2$ denote distinct implication operators. In [13], we provided its definition, a unified solution, and an analysis of its reductivity. As a generalized approach, this method facilitates the development of more optimized fuzzy systems.

Subsequently, we [14] extended the sustainment measure and introduced a novel symmetric implication method that encompasses both symmetric implication algorithms and the triple I method as special cases. Further advancing this work, we [15] proposed the symmetric quintuple implicational (SQI) method, which generalizes expression (4) to

$$\big(A(x) \to_1 B(y)\big) \to_2 \Big(\big(A^*(x) \to_1 A(x)\big) \to_2 \big(A^*(x) \to_1 B^*(y)\big)\Big). \quad (5)$$

Building upon these foundations, this study proposes a new fuzzy system based on the SQI method, termed the SQI fuzzy system. We conduct a comprehensive analysis of its establishment and response performance, employing R-implication and other classical operators.

2 Preliminaries

Definition 1. [16] Let a binary operation T $[0,1]^2 \to [0,1]$ satisfy the following conditions ($a, b, c \in [0,1]$): (i) $T(a,b) = T(b,a)$; (ii) $T\big(a, T(b,c)\big) = T(T(a,b), c)$; (iii) $T(1,a) = a$; (iv) if$a < b$, that $T(a,c) < T(b,c)$. Then T is called an $t-$norm on $[0,1]$. If t additionally satisfies $T(a, \vee \{x_i | i \in P\}) = \vee \{T(a \otimes x_i)\}(a, x_i \in [0,1], P \neq \phi)$, then T is termed a left-continuous t −norm.

Definition 2. [17] Let I be a mapping satisfying $[0,1]^2 \to [0,1]$:

$$I(0,0) = I(0,1) = I(1,1) = 1,\ I(1,0) = 0. \tag{6}$$

Then I is defined as an implication operator on $[0,1]$.

Definition 3. [16] Let T and I be two mappings $[0,1]^2 \to [0,1]$. The pair (T,I) is defined as an adjoint pair if the following adjoint condition holds $(a, b, c \in [0,1])$:

$$T(a, b) \leq c \ \textit{ if and only if } \ a \leq I(b, c). \tag{7}$$

Definition 4. [18] The double residual operator is defined as $(a, b \in [0,1])$

$$a \leftrightarrow b = I(a, b) \wedge I(b, a). \tag{8}$$

Definition 5. [17] A function I: $[0,1]^2 \to [0,1]$ is called an R-implication operator if there exists a left-continuous t −norm T satisfying:

$$I(a, b) = \vee \{y \in [0,1] | T(a, y) \leq b\},\ a, b \in [0,1]. \tag{9}$$

and

$$T(a, b) = \wedge \{x \in [0,1] | b \leq I(a, x)\},\ a, b \in [0,1]. \tag{10}$$

Lemma 1. [16] Let T be a left-continuous $t-$ norm on $[0,1]$, and I be derived from Equation (9). If (T,I) forms an adjoint pair, then for all $a, b, c, x_i \in [0,1]$ and any non-empty index set P, I satisfies: (C1) Non-decreasing in b: $b_1 \leq b_2 \Rightarrow I(a, b_1) \leq I(a, b_2)$; (C2) Right-continuous in b; (C3) Non-increasing in a: $a_1 \leq a_2 \Rightarrow I(a_2, b) \leq I(a_1, b)$; (C4) $a \leq b \Rightarrow I(a, b) = 1$; (C5) $I(1, a) = a$; (C6) $a \leq I(b, c) \Leftrightarrow b \leq I(a, c)$; (C7) $I\big(a, I(b, c)\big) = I\big(b, I(a, c)\big)$; (C8) $inf\{I(a, x_i) | i \in P\} = I(a, inf\{x_i | i \in P\})$; (C9) $inf\{I(x_i, b) | i \in P\} = I(sup\{x_i | i \in P\}, b)$.
Where $a, b, c, x_i \in [0,1]$ and P is non-empty.

Lemma 2. [16]. Let I be an R-implication operator and (T, I) form an adjoint pair. If the biimplication operator $\leftrightarrow$ is related to I and derived from Equation (8), then $\leftrightarrow$ satisfies the following properties: (C10) $a \leftrightarrow a = 1$; (C11) $a \leftrightarrow b \leq a \leftrightarrow T\big(b, I(b, a)\big)$; (C12) $T\big((a \leftrightarrow b), (b \leftrightarrow c)\big) \leq a \leftrightarrow c$; (C13) $(a \leftrightarrow b) \wedge (c \leftrightarrow d) \leq (a \wedge c) \leftrightarrow (b \wedge d)$; (C14) $(a \leftrightarrow b) \vee (c \leftrightarrow d) \leq (a \vee c) \leftrightarrow (b \vee d)$; (C15) $T\big((a \leftrightarrow b), (c \leftrightarrow d)\big) \leq T(a, c) \leftrightarrow T(b, d)$.

Definition 6. Let Z be a non-empty family of sets. A function C: $Z \to [0,1]$ is called a fuzzy set on Z. The collection of all such fuzzy subsets over ZZZ is denoted by $F(Z)$.

Definition 7. Let $F(Z)$ denote the family of all fuzzy subsets on Z. We define the fuzzy partial order $\leq_F$ as follows: $A \leq_F B \Leftrightarrow A(z_0) \leq_F B(z_0)$ for all $\forall A, B \in F(Z)$.

Lemma 3. The pair $< F(Z), \leq_F >$ forms a complete lattice.

3 The Proposed SQI Fuzzy Systems

3.1 Fuzzy rules

In practical applications, multi-rule fuzzy systems are commonly utilized. In this section, we will explore how to incorporate the newly proposed algorithm into multi-rule systems for handling multiple fuzzy rules. Specifically, the Fuzzy Modus Ponens (FMP) is reformulated into a more complex form:

FMP: Given n rules $A_i \to B_i$ and input A^*, compute the output B^*. (11)

Here $A_i, A^* \in F(x),\ B_i, B^* \in F(y)\ \ (i = 1,2,\dots,n)$.

Let NS be a system with n fuzzy rules $A_i \to B_i$, where each rule $A_i \to B_i$ is represented by the following fuzzy relation:

$$R_i(x,y) = I\big(A_i(x), B_i(y)\big). \tag{12}$$

For multi-rule NS systems, there are generally two strategies.

The first is FITA (First Infer Then Aggregate). That is, we initially convert the fuzzy relation R_i into a fuzzy inference rule $I\big(A_i(x), B_i(y)\big)$, then perform compositional inference between A^* and R_i to obtain corresponding results $B_i^{'}$, and finally aggregate $B_i^{'}$ to acquire the ultimate result B^*.

The second method is FATI (First Aggregate Then Infer). That is, we first transform the multiple fuzzy rules of NS into a single fuzzy relation R_i, then directly perform compositional inference between A^* and R_i to derive the final result B^*.

3.2 Construction of Fuzzy Systems

A complete fuzzy system additionally includes a fuzzifier and a defuzzifier. The conventional approach typically employs a singleton fuzzifier and centroid defuzzifier method. Specifically, the process is as follows [19]:

(i) Map x^* is mapped to a singleton fuzzy set:

$$A^*(x) = \begin{cases} 1, & x = x^* \\ 0, & x \neq x^* \end{cases} \quad \text{(denoted as } A^*_{x^*}\text{);}$$

(ii) We perform fuzzy inference using the SQI method to obtain the fuzzy set $B^*(y)$;

(iii) We apply the centroid defuzzification method: $y^* = \frac{\int_y yB^*(y)dy}{\int_y B^*(y)dy}$.

Thus, for each input x^*, a unique $y^* = f(x^*)$ can be obtained. This constructs a Single-Input Single-Output (SISO) fuzzy system based on the SQI method (abbreviated as the SQI fuzzy system). Here, $y = f(x)$ is called the response function of this fuzzy system, and the performance of the response function is referred to as response performance.

Definition 8. Let Z be an arbitrary non-empty set, and $C = \{C_i\}_{(1 \le i \le n)}$ be a family of normal fuzzy sets on Z, where each C_i has exactly one peak point Z_i (i.e., $C_i(Z_i) = 1$). If the following condition holds that $(\forall z \in Z)(\sum_{i=1}^{n} C_i(z) = 1)$ then C is defined as a fuzzy partition on Z. Here, C_i are called primitives of C, and C is termed a primitive family of Z.

Remark 1. From Definition 8, it directly follows that $(\forall i, j)(i \ne j \Rightarrow z_i \ne z_j)$ and the Kronecker property holds for $C_i(z_j) = \delta_{ij}$, where: $\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \ne j \end{cases}$.

The following assumptions are given: Let $\mathrm{A} = \{A_i\}_{(1 \le i \le n)}$ and $\mathrm{B} = \{B_i\}_{(1 \le i \le n)}$ be fuzzy partitions on X and Y respectively, where X and Y are real-number intervals that $X = [a, b], Y = [c, d]$, with peak points ordered as $a < x_1 < x_2 < \cdots < x_n < b, c < y_1 < y_2 < \cdots < y_n < d$, where x_i, y_i are the peak points of A_i, B_i respectively. Additionally, A_i, B_i are assumed to be integrable. These assumptions are readily satisfiable in practical applications.

Let $h_1 = y_1 - c, h_i = y_i - y_{i-1}\hat{q} = 2,3,\cdots,n)$, and $h = \underset{1 \le i \le n}{max}\{h_i\}$. Considering that A and B are fuzzy partitions, they satisfy the Kronecker property:

$$A_i(x_j) = \delta_{ij} = B_i(y_j).$$

By the definition of definite integrals, for the centroid defuzzifier, we have:

$$y^* = \frac{\int_I yB^*(y)dy}{\int_I B^*(y)dy} \approx \frac{\sum_{i=1}^{n} y_i B^*(y_i) h_i}{\sum_{i=1}^{n} B^*(y_i) h_i}. \tag{13}$$

4 Response Performance of the Proposed SQI Fuzzy Systems

Proposition 2. ([15]) Let $\rightarrow_1$ be an R-implication operator and $\otimes_1$ its adjoint left-continuous t-norm. In fuzzy systems employing the SQI method, the FITA solution of MinP-SQI is given by:

$$B^*(y) = \vee_{i=1}^{n} \bigvee_{x \in X} \{A^*(x) \otimes_1 ((A^*(x) \rightarrow_1 A_i(x)) \otimes_2 (A_i(x) \rightarrow_1 B_i(y)))\}, y \in Y.$$

Proposition 3. ([15]) Let $\rightarrow_1$ be an R-implication operator and $\otimes_1$ its adjoint left-continuous t-norm. In fuzzy systems employing the SQI method, the FATI solution of MinP-SQI is given by:

$$B^*(y) = \vee_{i=1}^{n} \bigvee_{x \in X} \{A^*(x) \otimes_1 ((A^*(x) \rightarrow_1 A_i(x)) \otimes_2 R(x, y))\}, y \in Y.$$

where R is obtained using the SQI method in the fuzzy system from (12).

Theorem 1. Let $\rightarrow_1$ be an R-implication operator. There exists $A^* = \{A_i^*\}$ $(1 \le i \le n)$ such that the FITA solution of the SQI method for the SQI fuzzy system approximates a univariate piecewise interpolation function, where: $\psi(x) = \sum_{i=1}^{n} A_i^*(x) y_i$, where A^* forms a fuzzy partition on X. Furthermore, when $\{y_i\}$ $(1 \le i \le n)$ are equidistant, A^* degenerates to $\psi(x) = \sum_{i=1}^{n} A_i^*(x) y_i$.

Proof: According to Proposition 2, in fuzzy systems employing the SQI method, the FITA solution of MinP-SQI is:

$$B^*(y) = \vee_{i=1}^{n} \vee_{x\in X}\{A^*(x)\otimes_1((A^*(x)\to_1 A_i(x))\otimes_2(A_i(x)\to_1 B_i(y)))\}, y \in Y.$$

where $\otimes_1$ is the adjoint left-continuous t-norm of $\to_1$.

Given that $B_k(y_i) = \vartheta_{ki}$ and the implication $\to_2$ satisfies (6), the following can be derived from (13):

$$y^* \approx \frac{\sum_{i=1}^{n} h_i[\vee_{k=1}^{n}(A_k{}^*(x)\otimes_1((A_k{}^*(x)\to_1 A_k(x))\otimes_2(A_k(x)\to_1 B_k(y_i)))]y_i}{\sum_{i=1}^{n} h_i[\vee_{k=1}^{n}(A_k{}^*(x)\otimes_1((A_k{}^*(x)\to_1 A_k(x))\otimes_2(A_k(x)\to_1 B_k(y_i)))]}$$

$$= \frac{\sum_{i=1}^{n} h_i A_i(x^*)y_i}{\sum_{i=1}^{n} h_i A_i(x^*)}. \tag{14}$$

Noticing that $A = \{A_i\}_{(1\le i\le n)}$ is a fuzzy partition on X, there exists an $i \in (1 \le i \le n)$ such that $A^*(x^*) > 0$. Consequently, $\sum_{i=1}^{n} h_i A_i(x^*) > 0$ and Equation (14) is well-defined.

Let $A_i^*(x^*) = h_i A_i(x^*)/\sum_{i=1}^{n} h_i A_i(x^*)$, then we have $y^* \approx \sum_{i=1}^{n} h_i A_i^*(x^*)$, and define: $A^* = \{A^*\}_{1\le i\le n}$, $\psi(x) = \sum_{i=1}^{n} A_i^*(x)\, y_i$. Noting that $A_k(x_i) = \vartheta_{ki}$, we have:

$$\psi(x_i) = \sum_{k=1}^{n} A_k^*(x_i)\, y_k = \frac{\sum_{k=1}^{n} h_k A_k(x_i) y_k}{\sum_{k=1}^{n} h_k A_k(x_i)} = y_i.$$

Therefore, $\psi(x)$ is a piecewise univariate interpolation function.

Moreover, the equality

$$\sum_{i=1}^{n} A_i^*(x) = h_i A_i(x^*)/\sum_{i=1}^{n} h_i A_i(x) = 1$$

is valid $(x \in X)$, Therefore, A^* forms a fuzzy partition on X.

Finally, when $\{y_i\}\,(1 \le i \le n)$ are equidistant (i.e., $(\forall i)(h_i = h)$), it clearly follows that $A_i^* = A_i$, $A^* = A$ and consequently one has: $\psi(x) = \sum_{i=1}^{n} A_i(x)\, y_i$.

Q.E.D.

According to Proposition 3, we can prove Theorem 2 using a similar method.

Theorem 2. Let $\to_1$ be an R-implication operator. There exists $A^* = \{A_i^*\}\,(1 \le i \le n)$ such that the FATI solution of the SQI method for the SQI fuzzy system approximates a univariate piecewise interpolation function, where: $\psi(x) = \sum_{i=1}^{n} A_i^*(x) y_i$, where A^* forms a fuzzy partition on X. Furthermore, when $\{y_i\}(1 \le i \le n)$ are equidistant, A^* degenerates to A (i.e., $\psi(x) = \sum_{i=1}^{n} A_i^*(x) y_i$).

Similarly, when we assume that $\to_1$ is the I_R-implication ($I_R(x, y) = 1 - \mathrm{x} + \mathrm{xy}$) and I_{KD}-implication ($I_{KD}(x, y) = (1 - x) \vee y$) respectively, we can use this method to prove that their FITA and FATI solutions approximate a univariate piecewise fitting function for SQI fuzzy systems.

Remark 2. Here we elucidate the meaning of "approximation" in the aforementioned theorem. For centroidal fuzzy cases, it is evident that the core idea of "approximation" lies in (13). Based on the definition of definite integrals, the following interpretation can be derived (considering the uniform distribution of $\{q_1, y_1, y_2, \cdots, y_n, q_2\}$). First, if n is large, then $\sum_{i=1}^{n} y_i B^*(y_i)\, h_i / \sum_{i=1}^{n} B^*(y_i) h_i$ becomes a better approximation of $\int_Y \; y\, B^*(y) dy / \int_Y \; B^*\,(y) dy$.

Furthermore, for any ε > 0, there exists N such that for all n>N:

$$\left| \int_Y \; y\, B^*(y) dy / \int_Y \; B^*\,(y) dy - \sum_{i=1}^{n} y_i\, B^*(y_i) h_i / \sum_{i=1}^{n} B^*\,(y_i) h_i \right| < \varepsilon.$$

By the aforementioned theorem, the fuzzy system constructed using the SQI method is a universal approximator, and thus should be prioritized in practical applications.

5 Conclusions

This paper proposes a novel fuzzy system based on the SQI method, with rigorous theoretical and practical investigations into its response characteristics.

To address the limitations of conventional fuzzy inference methods, we introduce the SQI method into fuzzy systems, optimizing similarity computation between rule bases and inputs while generalizing fuzzy implication operators. Subsequently, we systematically construct the SQI fuzzy system using a singleton fuzzifier and centroid defuzzifier, formalizing a multi-rule implementation framework compatible with real-world applications. Finally, we analyze the system's response performance under R-implication operators and other classical operators in multi-rule configurations.

In the future, we will attempt to combine fuzzy clustering [20]-[22] with the SQI fuzzy system to establish a new clustering-reasoning mode.

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