Universal First-Order Theory of Relative Nearness for All Metric Spaces

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Abstract. In the present paper, we solve an open problem mentioned in [3], namely a complete description of the universal first-order theory of relative nearness relation, which is a ternary relation $N(w, v, u)$ which means that $d(w, v) < d(w, u)$, where $d(w, v)$ is the distance between w and v. Our axiomatization makes use of a variation of the Farkas' Lemma [6] on the solvability conditions of finite linear equation systems.

Keywords: Spatial logic · metric space · relative nearness

1 Introduction

Logics for reasoning about spatial features are growing more attentions (see [2] for a comprehensive reference). Spatial logics provide formal tools to express and reason about geometric structures such as topological spaces, metric spaces, etc. In the study of spatial logic, the ones about distance in a metric space are studied extensively [1, 5, 10–13, 15, 20]. Our focus in this paper is relative nearness, which has been studied in [4, 9, 16, 17].

Relative nearness relation is a ternary relation $N(w, v, u)$ which means that $d(w, v) < d(w, u)$, where $d(w, v)$ is the distance between w and v. Typically, we require that d is a metric function on a metric space (W, d) satisfying certain conditions. In [3], the authors describe some properties that the relative nearness relation satisfies, but the complete description of the universal first-order theory of relative nearness relation is left as an open problem.

In our paper, we solve this open problem by using a variant of Farkas' Lemma [6] to characterize the metrizable finite models of the form $\mathcal{M} = (W, N)$ where W is a non-empty finite set and N is a ternary relation on W. Farkas' Lemma is a celebrated result in linear algebra and optimization theory, which is about the solvability conditions of finite linear equation (or inequation) systems. It has been used in the study of modal logic extended with the "most" modality M where $\mathsf{M}\varphi$ means that there are strictly more successors satisfying φ than successors satisfying $\neg\varphi$ [7] and monadic first-order logic without equality extended with the generalized unary quantifier $\mathsf{M}x$ where $\mathsf{M}x\varphi(x)$ means that there are more objects in the domain satisfying $\varphi(x)$ than objects satisfying $\neg\varphi(x)$ [8].

Our contributions are both technical and methodological: we not only provide a complete description of the universal first-order theory of relative nearness relation by characterizing the metrizable finite models of the form $\mathcal{M} = (W, N)$ as

described above, but also make use of tools from linear algebra and optimization theory, namely the Farkas' Lemma.
The structure of the paper is organized as follows: Section 2 gives all the necessary preliminaries on metric spaces and relative nearness relation. Section 3 gives the axiomatization of the universal first-order theory of relative nearness relation using a variant of Farkas' Lemma. Section 4 gives the conclusions of the paper.

2 Preliminaries

In the present section, we gives some preliminaries on metric space and relative nearness relation. For references, see [3, 19].

Definition 1 (Metric Space). *A metric space is a pair (W, d) where W is a non-empty set and $d : W \times W \to \mathbb{R}$ is a metric (or distance function) on W such that for any $w, v, u \in W$:*

- $d(w, v) \geq 0$. *(Non-negativity)*
- $d(w, v) = 0$ *iff* $w = v$.
- $d(w, v) = d(v, w)$. *(Symmetry)*
- $d(w, v) + d(v, u) \geq d(w, u)$. *(Triangle inequality)*

Definition 2 (Relative Nearness Relation). *Given any metric space (W, d), the relative nearness relation N_d induced by d is a ternary relation on W such that $N_d(x, y, z)$ iff $d(x, y) < d(x, z)$.*

In what follows, we use $N_d(x, y, z)$ to indicate that $(x, y, z) \in N_d$ and $\sim N_d(x, y, z)$ to indicate that $(x, y, z) \in W^3 - N_d$. We use $\#(x, y, z)$ to indicate that x, y, z are all distinct.
We have the following properties of the relative nearness relation induced by a metric space:

Theorem 1. *For any metric space (W, d), the relative nearness relation N_d induced by d satisfies the following properties:*

1. *For any $x, y \in W$, $\sim N_d(x, y, y)$ holds.*
2. *For any $x, y \in W$ such that $x \neq y$, $N_d(x, x, y)$ holds.*
3. *For any $x, y \in W$, $\sim N_d(x, y, x)$ holds.*
4. *For any $x_1, \ldots, x_m, y_1, \ldots, y_m, z_1, \ldots, z_m, u_1, \ldots, u_n, v_1, \ldots, v_n, w_1, \ldots, w_n \in W$ such that $m \geq 1$, $n \geq 0$ and $(\{x_1, y_1\}, \ldots, \{x_m, y_m\}, \{u_1, w_1\}, \ldots, \{u_n, w_n\})$ and $(\{x_1, z_1\}, \ldots, \{x_m, z_m\}, \{u_1, v_1\}, \ldots, \{u_n, v_n\})$ are permutations of each other, if $\#(x_i, y_i, z_i)$ and $N_d(x_i, y_i, z_i)$ for all $1 \leq i \leq m$ and $\#(u_j, v_j, w_j)$ for all $1 \leq j \leq n$, then $N_d(u_j, v_j, w_j)$ holds for some $1 \leq j \leq n$.*

Proof. 1. For any $x, y \in W$, since $d(x, y) \geq d(x, y)$, we have $\sim N_d(x, y, y)$.
2. For any $x, y \in W$ such that $x \neq y$, since $d(x, x) = 0 < d(x, y)$, we have $N_d(x, x, y)$.

3. For any $x, y \in W$, since $d(x,y) \geq 0 = d(x,x)$, we have $\sim N_d(x,y,x)$.
4. For any $x_1,\dots,x_m,y_1,\dots,y_m,z_1,\dots,z_m,u_1,\dots,u_n,v_1,\dots,v_n,w_1,\dots,w_n \in W$ such that
   - $m \geq 1$,
   - $n \geq 0$,
   - and $(\{x_1,y_1\},\dots,\{x_m,y_m\},\{u_1,w_1\},\dots,\{u_n,w_n\})$ and $(\{x_1,z_1\}, \dots, \{x_m,z_m\}, \{u_1,v_1\},\dots,\{u_n,v_n\})$ are permutations of each other,
   - $\#(x_i,y_i,z_i)$ for all $1 \leq i \leq m$,
   - $N_d(x_i,y_i,z_i)$ for all $1 \leq i \leq m$,
   - $\#(u_j,v_j,w_j)$ for all $1 \leq j \leq n$,
   - $\sim N_d(u_j,v_j,w_j)$ holds for all $1 \leq j \leq n$,

   we have that $d(x_i,y_i) < d(x_i,z_i)$ for all $1 \leq i \leq m$ and $d(u_j,w_j) \leq d(u_j,v_j)$ for all $1 \leq j \leq n$. By adding them up, we have that the left-handside is the same as the right-handside, but the inequality is strict, a contradiction. Therefore the property holds.

Corollary 1. *1. For any $x,y,z,w \in W$, $N_d(x,y,z)$ and $N_d(x,z,w)$ imply $N_d(x,y,w)$.*
2. For any $x,y,z \in W$, $N_d(x,y,z)$ implies $\sim N_d(x,z,y)$.
3. For any $x,y,z,w \in W$, $N_d(x,y,z)$ implies $N_d(x,y,w)$ or $N_d(x,w,z)$.

Proof. 1. If some of x,y,z,w are equal, then it follows from Theorem 1(1)-(3). If $\#(x,y,z,w)$, then it follows from Theorem 1(4), by taking $(x_1,y_1,z_1) = (x,y,z), (x_2,y_2,z_2) = (x,z,w), (u_1,v_1,w_1) = (x,y,w)$. Indeed, $(\{x,y\},\{x,z\},\{x,w\})$ and $(\{x,z\},\{x,w\},\{x,y\})$ are permutations of each other.
2. If some of x,y,z are equal, then it follows from Theorem 1(1)-(3). If $\#(x,y,z)$, then it follows from Theorem 1(4), by taking $(x_1,y_1,z_1) = (x,y,z)$, $(x_2,y_2,z_2) = (x,z,y)$ and $n = 0$. Indeed, $(\{x,y\},\{x,z\})$ and $(\{x,z\},\{x,y\})$ are permutations of each other. Indeed, what we get is a contradiction from $N_d(x,y,z)$ and $N_d(x,z,y)$.
3. If some of x,y,z,w are equal, then it follows from Theorem 1(1)-(3). If $\#(x,y,z,w)$, then it follows from Theorem 1(4), by taking $(x_1,y_1,z_1) = (x,y,z), (u_1,v_1,w_1) = (x,y,w), (u_2,v_2,w_2) = (x,w,z)$. Indeed, $(\{x,y\},\{x,w\},\{x,z\})$ and $(\{x,z\},\{x,y\},\{x,w\})$ are permutations of each other.

3 The Axiomatization

In what follows, we will axiomatize the universal first-order theory of relative nearness relation using a variant of Farkas' Lemma.

3.1 The Axioms of Relative Nearness

The first-order language we consider only contains one non-logical symbol, which is a ternary predicate symbol N corresponding to the relative nearness relation. We define quantifier-free formulas as follows:

$$\varphi ::= x = y \mid Nxyz \mid \bot \mid \varphi \to \varphi$$

where x, y, z are individual variables. Then we define universal formulas as follows:

$$\theta ::= \varphi \mid \forall x\theta$$

where x is an individual variable. First-order formulas are defined as follows:

$$\psi ::= x = y \mid Nxyz \mid \bot \mid \psi \to \psi \mid \forall x\psi$$

$\neg, \wedge, \vee, \leftrightarrow, \exists x$ as well as all standard notions in first-order logic are defined as usual. We use $\forall\vec{x}$ to denote the quantifiers $\forall x_1 \ldots \forall x_n$, where $\vec{x}$ is the sequence $(x_1, \ldots, x_n)$.
For the semantics,

- A model $\mathcal{M} = (W, N^{\mathcal{M}})$ is such that W is a non-empty set and $N^{\mathcal{M}}$ is a ternary relation on W. We often omit the superscript of N when it is clear from the context.
- A model $\mathcal{M} = (W, N^{\mathcal{M}})$ is metrizable iff there is a metric $d : W \times W \to \mathbb{R}$ on W such that $N^{\mathcal{M}} = N_d$.
- Assignments, satisfaction relation, truth of sentences and validity are defined as usual.
- For any set Φ of formulas and any formula φ, we say that φ semantically follows from Φ (notation: $\Phi \vDash \varphi$), if for any model $\mathcal{M}$ and any assignment s on $\mathcal{M}$, if $\mathcal{M}, s \vDash \Phi$, then $\mathcal{M}, s \vDash \varphi$.

Definition 3 (Universal First-Order Theory). *We consider the universal first-order theory of a set Φ of sentences as $\mathsf{Th}_\forall(\Phi) := \{\varphi \mid \Phi \vDash \varphi$ and φ is a universal sentence$\}$. We say that Φ is the set of axioms of $\mathsf{Th}_\forall(\Phi)$ if Φ is also a set of universal sentences.*

Consider the set Φ_M of all universal sentences that are true on all metrizable models. We will prove that $\Phi_M = \mathsf{Th}_\forall(\Psi)$, where Ψ consists of the following universal sentences:

Definition 4 (Axioms of Relative Nearness). *Ψ consists of the following universal sentences:*

1. $\forall x \forall y \neg Nxyy$.
2. $\forall x \forall y (x \neq y \to Nxxy)$.
3. $\forall x \forall y \neg Nxyx$.
4. $\forall\vec{x}\forall\vec{y}\forall\vec{z}\forall\vec{u}\forall\vec{v}\forall\vec{w}(\bigwedge \#(x_i, y_i, z_i) \wedge \bigwedge \#(u_i, v_i, w_i) \wedge Nx_1y_1z_1 \wedge \ldots \wedge Nx_my_mz_m \wedge \neg Nu_1v_1w_1 \wedge \ldots \wedge \neg Nu_nv_nw_n \to \bot)$, *where the lists* $(\{x_1, y_1\}, \ldots, \{x_m, y_m\}, \{u_1, w_1\}, \ldots, \{u_n, w_n\})$ *and* $(\{x_1, z_1\}, \ldots, \{x_m, z_m\}, \{u_1, v_1\}, \ldots, \{u_n, v_n\})$ *are permutations of each other and* $m \geq 1, n \geq 0$.

3.2 Completeness

In this section, we will show that $\Phi_M = \mathsf{Th}_\forall(\Psi)$, i.e. Ψ axiomatizes the universal first-order theory of relative nearness.

Before proving the theorem, we first prove the following variations of Farkas' Lemma, which is about solvability of finite linear inequation systems. We use $\mathbb{N}$ and $\mathbb{Q}$ to denote the set of natural numbers and the set of rational numbers, respectively. We use $A^{n\times m}$ to denote an $n\times m$ matrix of elements in A and A^T to denote the transposition of the matrix A. We use $\mathbf{a}\leq\mathbf{b}$ to indicate that every coordinate of $\mathbf{a}$ is smaller than or equal to the corresponding coordinate in $\mathbf{b}$, $\mathbf{a}<\mathbf{b}$ to indicate that every coordinate of $\mathbf{a}$ is strictly smaller than the corresponding coordinate in $\mathbf{b}$, and $\mathbf{a}=\mathbf{b}$ to indicate that the two vectors are the same. We use $\mathbf{0}$ to denote a vector whose all coordinates are 0 and sometimes we use superscripts to indicate its dimension.

Lemma 1 (A Variation of Farkas' Lemma). *Consider a finite linear inequation system* $\{\mathbf{AX}\geq\mathbf{0},\mathbf{CX}>\mathbf{0}\}$*, where* $\mathbf{A}\in\mathbb{Q}^{n_1\times m}$*,* $\mathbf{C}\in\mathbb{Q}^{n_2\times m}$ *and* $\mathbf{X}=(x_1,\ldots,x_m)^T$ *is a vector of variables,* $n_1,n_2,m\geq 1$*.*
The system has no non-negative real solution iff there are $\mathbf{b}\in\mathbb{N}^{n_1\times 1}$ *and* $\mathbf{0}^{n_2\times 1}\neq\mathbf{d}\in\mathbb{N}^{n_2\times 1}$ *such that* $\mathbf{b}^T\mathbf{A}+\mathbf{d}^T\mathbf{C}\leq\mathbf{0}^{1\times m}$*, i.e. by adding up the rows of* $\mathbf{A}$ *and* $\mathbf{C}$ *(where each row can be used many times or not used at all), we can get a horizontal vector of non-positive numbers, such that the rows in* $\mathbf{C}$ *are used at least once.*

Proof. We prove by induction on m.

- For the case $m=1$, we essentially have a system $\{\mathbf{AX}\geq\mathbf{0},\mathbf{CX}>\mathbf{0}\}$ of the form $\{a_1x\geq 0,\ldots,a_{n_1}x\geq 0,c_1x>0,\ldots,c_{n_2}x>0\}$.
  - If the system has a non-negative real solution $\mathbf{X}=x$ and there are $\mathbf{b}\in\mathbb{N}^{n_1\times 1}$ and $\mathbf{0}^{n_2\times 1}\neq\mathbf{d}\in\mathbb{N}^{n_2\times 1}$ such that $\mathbf{b}^T\mathbf{A}+\mathbf{d}^T\mathbf{C}\leq\mathbf{0}$, then $\{a_1x\geq 0,\ldots,a_{n_1}x\geq 0,c_1x>0,\ldots,c_{n_2}x>0\}$ has a non-negative real solution. Apparently $x\neq 0$, otherwise from $c_1x>0$ we get $0>0$, a contradiction. Therefore $x>0$, and thus $a_1,\ldots,a_{n_1}\geq 0,c_1,\ldots,c_{n_2}>0$. Then since $\mathbf{b}\in\mathbb{N}^{n_1\times 1}$ and $\mathbf{0}^{n_2\times 1}\neq\mathbf{d}\in\mathbb{N}^{n_2\times 1}$, we have that $(\Sigma_{i=1}^{n_1}b_ia_i)+(\Sigma_{j=1}^{n_2}d_jc_j)>0$, a contradiction to $\mathbf{b}^T\mathbf{A}+\mathbf{d}^T\mathbf{C}\leq\mathbf{0}$.
  - If the system has no non-negative real solution, then $\{a_1x\geq 0,\ldots,a_{n_1}x\geq 0,c_1x>0,\ldots,c_{n_2}x>0\}$ has no such solution.
    - If there is a $c_j\leq 0$, then consider $\mathbf{b}=(b_1,\ldots,b_{n_1})^T$ and $\mathbf{d}=(d_1,\ldots,d_{n_2})^T$ such that all b_i's and d_k's are 0 except that $d_j=1$, then $(\Sigma_{i=1}^{n_1}b_ia_i)+(\Sigma_{j=1}^{n_2}d_jc_j)=c_jd_j\leq 0$, i.e. $\mathbf{b}^T\mathbf{A}+\mathbf{d}^T\mathbf{C}\leq\mathbf{0}$.
    - If $c_j>0$ for all $j=1,\ldots,n_2$, then the inequation system is equivalent to $\{a_1x\geq 0,\ldots,a_{n_1}x\geq 0,x>0\}$. If $a_i\geq 0$ for all $i=1,\ldots,n_1$, then the system has a solution $x=1$, a contradiction. Therefore there is an a_i such that $a_i<0$. Then for $a_i<0$ and $c_j>0$, since $a_i,c_j\in\mathbb{Q}$, there are $b_i,d_j\in\mathbb{N}$ such that $b_ia_i+d_jc_j=0$ and $d_j>0$. By taking all b_t's (where $t\neq i$) and d_k's (where $k\neq j$) to be 0, we get $\mathbf{b}\in\mathbb{N}^{n_1\times 1}$ and $\mathbf{0}^{n_2\times 1}\neq\mathbf{d}\in\mathbb{N}^{n_2\times 1}$ such that $\mathbf{b}^T\mathbf{A}+\mathbf{d}^T\mathbf{C}\leq\mathbf{0}$.
- Suppose we have proved the lemma for all $m\leq k$. For $m=k+1$, we essentially have a system $\{\mathbf{AX}\geq\mathbf{0},\mathbf{CX}>\mathbf{0}\}$.
  - If the system has a non-negative real solution $\mathbf{X}=\mathbf{x}$ and there are $\mathbf{b}\in\mathbb{N}^{n_1\times 1}$ and $\mathbf{0}^{n_2\times 1}\neq\mathbf{d}\in\mathbb{N}^{n_2\times 1}$ such that $\mathbf{b}^T\mathbf{A}+\mathbf{d}^T\mathbf{C}\leq\mathbf{0}^{1\times m}$,

then $\mathbf{b}^T\mathbf{Ax} + \mathbf{d}^T\mathbf{Cx} \leq 0$. However, from $\mathbf{Ax} \geq \mathbf{0}^{n_1\times 1}$, $\mathbf{Cx} > \mathbf{0}^{n_2\times 1}$, $\mathbf{b} \geq \mathbf{0}^{n_1\times 1}$, $\mathbf{d} \geq \mathbf{0}^{n_2\times 1}$ and $\mathbf{d} \neq \mathbf{0}^{n_2\times 1}$ we get that $\mathbf{b}^T\mathbf{Ax} \geq 0$ and $\mathbf{d}^T\mathbf{Cx} > 0$, a contradiction to $\mathbf{b}^T\mathbf{Ax} + \mathbf{d}^T\mathbf{Cx} \leq 0$.

- If the system has no non-negative real solution, denote
  - $\mathbf{X} = (x_1, \ldots, x_{k+1})^T$,
  - $\mathbf{A} = (a_{i,j})_{n_1\times(k+1)}$,
  - $\mathbf{C} = (c_{i,j})_{n_1\times(k+1)}$,
  - $(\mathbf{AX})_i = \Sigma_{j=1}^{k+1} a_{i,j}x_j$ as the i-th row of $\mathbf{AX}$,
  - $(\mathbf{CX})_\ell = \Sigma_{j=1}^{k+1} c_{\ell,j}x_j$ as the ℓ-th row of $\mathbf{CX}$,

  then the system $\{\Sigma_{j=1}^{k+1} a_{1,j}x_j \geq 0, \ldots, \Sigma_{j=1}^{k+1} a_{n_1,j}x_j \geq 0, \Sigma_{j=1}^{k+1} c_{1,j}x_j > 0, \ldots, \Sigma_{j=1}^{k+1} c_{n_2,j}x_j > 0\}$ has no non-negative real solution. Now we divide the lines into the following types:
  - $\Sigma_{j=1}^{k} a_{i,j}x_j + a_{i,k+1}x_{k+1} \geq 0$, where $a_{i,k+1} > 0$.
  - $\Sigma_{j=1}^{k} a_{i,j}x_j + a_{i,k+1}x_{k+1} \geq 0$, where $a_{i,k+1} = 0$.
  - $\Sigma_{j=1}^{k} a_{i,j}x_j + a_{i,k+1}x_{k+1} \geq 0$, where $a_{i,k+1} < 0$.
  - $\Sigma_{j=1}^{k} c_{\ell,j}x_j + c_{\ell,k+1}x_{k+1} > 0$, where $c_{\ell,k+1} > 0$.
  - $\Sigma_{j=1}^{k} c_{\ell,j}x_j + c_{\ell,k+1}x_{k+1} > 0$, where $c_{\ell,k+1} = 0$.
  - $\Sigma_{j=1}^{k} c_{\ell,j}x_j + c_{\ell,k+1}x_{k+1} > 0$, where $c_{\ell,k+1} < 0$.

  We denote the six sets of indices of i's and ℓ's as $I_1, I_2, I_3, L_1, L_2, L_3$ respectively, then we have that each type of inequation can be rewritten as follows:
  - $x_{k+1} \geq -\Sigma_{j=1}^{k}(a_{i,j}/a_{i,k+1})x_j$, where $i \in I_1$.
  - $\Sigma_{j=1}^{k} a_{i,j}x_j \geq 0$, where $i \in I_2$.
  - $x_{k+1} \leq -\Sigma_{j=1}^{k}(a_{i,j}/a_{i,k+1})x_j$, where $i \in I_3$.
  - $x_{k+1} > -\Sigma_{j=1}^{k}(c_{\ell,j}/c_{\ell,k+1})x_j$, where $\ell \in L_1$.
  - $\Sigma_{j=1}^{k} c_{\ell,j}x_j > 0$, where $\ell \in L_2$.
  - $x_{k+1} < -\Sigma_{j=1}^{k}(c_{\ell,j}/c_{\ell,k+1})x_j$, where $\ell \in L_3$.

  Now, the system has no non-negative real solution iff the following system has no non-negative real solutions:

$$\begin{cases} -\Sigma_{j=1}^{k}(a_{i,j}/a_{i,k+1})x_j \leq -\Sigma_{j=1}^{k}(a_{i',j}/a_{i',k+1})x_j \text{ where } i \in I_1 \text{ and } i' \in I_3 \\ -\Sigma_{j=1}^{k}(a_{i,j}/a_{i,k+1})x_j < -\Sigma_{j=1}^{k}(c_{\ell,j}/c_{\ell,k+1})x_j \text{ where } i \in I_1 \text{ and } \ell \in L_3 \\ -\Sigma_{j=1}^{k}(c_{\ell,j}/c_{\ell,k+1})x_j < -\Sigma_{j=1}^{k}(a_{i,j}/a_{i,k+1})x_j \text{ where } \ell \in L_1 \text{ and } i \in I_3 \\ -\Sigma_{j=1}^{k}(c_{\ell,j}/c_{\ell,k+1})x_j < -\Sigma_{j=1}^{k}(c_{\ell',j}/c_{\ell',k+1})x_j \text{ where } \ell \in L_1 \text{ and } \ell' \in L_3 \\ 0 \leq -\Sigma_{j=1}^{k}(a_{i,j}/a_{i,k+1})x_j \text{ where } i \in I_3 \\ 0 < -\Sigma_{j=1}^{k}(c_{\ell,j}/c_{\ell,k+1})x_j \text{ where } \ell \in L_3 \\ \Sigma_{j=1}^{k} a_{i,j}x_j \geq 0 \text{ where } i \in I_2 \\ \Sigma_{j=1}^{k} c_{\ell,j}x_j > 0 \text{ where } \ell \in L_2 \end{cases}$$

  The basic idea of this equivalence is as follows: each inequation determines the value of x_{k+1} as an interval. The system has non-negative real solutions iff the intersection of all intervals together with $[0, +\infty)$ is non-empty and the resulting system with one less variable has non-negative real solutions. Now we can re-organize the inequation system as follows:

$$\begin{cases} \Sigma_{j=1}^{k}(a_{i,k+1}a_{i',j} - a_{i',k+1}a_{i,j})x_j \geq 0 \text{ where } i \in I_1 \text{ and } i' \in I_3 \\ \Sigma_{j=1}^{k}(a_{i,k+1}c_{\ell,j} - c_{\ell,k+1}a_{i,j})x_j > 0 \text{ where } i \in I_1 \text{ and } \ell \in L_3 \\ \Sigma_{j=1}^{k}(c_{\ell,k+1}a_{i,j} - a_{i,k+1}c_{\ell,j})x_j > 0 \text{ where } \ell \in L_1 \text{ and } i \in I_3 \\ \Sigma_{j=1}^{k}(c_{\ell,k+1}c_{\ell',j} - c_{\ell',k+1}c_{\ell,j})x_j > 0 \text{ where } \ell \in L_1 \text{ and } \ell' \in L_3 \\ \Sigma_{j=1}^{k}a_{i,j}x_j \geq 0 \text{ where } i \in I_2 \cup I_3 \\ \Sigma_{j=1}^{k}c_{\ell,j}x_j > 0 \text{ where } \ell \in L_2 \cup L_3 \end{cases}$$

By induction hypothesis, this inequation system has no non-negative real solution iff we can use the left-handside of lines in the system above (where each line can also be used many times or not used at all) to add up to some linear combination of $x_1, \ldots, x_k$ such that all coefficients for $x_1, \ldots, x_k$ in this combination are non-positive and the strict inequations are used at least once. Then

$$\begin{aligned} &\Sigma_{j=1}^{k}(a_{i,k+1}a_{i',j} - a_{i',k+1}a_{i,j})x_j \\ &= a_{i,k+1}\Sigma_{j=1}^{k}a_{i',j}x_j - a_{i',k+1}\Sigma_{j=1}^{k}a_{i,j}x_j \\ &= a_{i,k+1}\Sigma_{j=1}^{k+1}a_{i',j}x_j - a_{i',k+1}\Sigma_{j=1}^{k+1}a_{i,j}x_j \\ &= a_{i,k+1}(\mathbf{AX})_{i'} - a_{i',k+1}(\mathbf{AX})_i, \end{aligned}$$

and similarly for other lines. Therefore, each type of inequations of the system can be rewritten as follows:

1. $a_{i,k+1}(\mathbf{AX})_{i'} - a_{i',k+1}(\mathbf{AX})_i \geq 0$, where $i \in I_1$ and $i' \in I_3$.
2. $a_{i,k+1}(\mathbf{CX})_\ell - c_{\ell,k+1}(\mathbf{AX})_i > 0$, where $i \in I_1$ and $\ell \in L_3$.
3. $c_{\ell,k+1}(\mathbf{AX})_i - a_{i,k+1}(\mathbf{CX})_\ell > 0$, where $\ell \in L_1$ and $i \in I_3$.
4. $c_{\ell,k+1}(\mathbf{CX})_{\ell'} - c_{\ell',k+1}(\mathbf{CX})_\ell > 0$, where $\ell \in L_1$ and $\ell' \in L_3$.
5. $(\mathbf{AX})_i - a_{i,k+1}x_{k+1} \geq 0$, where $i \in I_2 \cup I_3$.
6. $(\mathbf{CX})_\ell - c_{\ell,k+1}x_{k+1} > 0$, where $\ell \in L_2 \cup L_3$.

This inequation system has no non-negative real solution iff we can use the left-handside of lines in the system above (where each line can also be used many times or not used at all) to add up to some linear combination of $x_1, \ldots, x_k$ (notice that x_{k+1} always has coefficient 0) such that all coefficients for $x_1, \ldots, x_k$ in this combination are non-positive and the strict inequations are used at least once. This is essentially a linear combination of $(\mathbf{AX})_i$'s, $(\mathbf{CX})_\ell$'s and x_{k+1} such that the coefficients of $(\mathbf{AX})_i$'s, $(\mathbf{CX})_\ell$'s and x_{k+1} are all non-negative and the strict inequations are used at least once.

Apparently, all coefficients for $(\mathbf{AX})_i$'s, $(\mathbf{CX})_\ell$'s are positive in all inequations.

* If no inequations of the form $(\mathbf{AX})_i - a_{i,k+1}x_{k+1} \geq 0$ (for $i \in I_3$) or $(\mathbf{CX})_\ell - c_{\ell,k+1}x_{k+1} > 0$ (for $\ell \in L_3$) are used, then we have a linear combination of $(\mathbf{AX})_i$'s and $(\mathbf{CX})_\ell$'s such that the coefficients of $(\mathbf{AX})_i$'s and $(\mathbf{CX})_\ell$'s are all non-negative and the strict inequations are used at least once. Since strict inequations of type 2-4 have positive coefficients of $(\mathbf{CX})_\ell$'s, we have non-negative $\mathbf{b} \in \mathbb{Q}^{n_1 \times 1}$ and $\mathbf{0}^{n_2 \times 1} \neq \mathbf{d} \in \mathbb{Q}^{n_2 \times 1}$ such that $\mathbf{b}^T\mathbf{A} + \mathbf{d}^T\mathbf{C} \leq \mathbf{0}^{1 \times m}$. By appropriate scaling, we can make sure that all coordinates of $\mathbf{b}$ and $\mathbf{d}$ are in $\mathbb{N}$.
* If some inequations of the form $(\mathbf{AX})_i - a_{i,k+1}x_{k+1} \geq 0$ (for $i \in I_3$) or $(\mathbf{CX})_\ell - c_{\ell,k+1}x_{k+1} > 0$ (for $\ell \in L_3$) are used, then we have a linear combination of $(\mathbf{AX})_i$'s, $(\mathbf{CX})_\ell$'s and x_{k+1} such that:

- The coefficients of $(\mathbf{AX})_i$'s, $(\mathbf{CX})_\ell$'s and x_{k+1} (regarded as if they are variables) are all non-negative.
- All coefficients of $x_1, \ldots, x_k$ in this combination are non-positive.
- The strict inequations of types 2,3,4,6 (for $\ell \in L_3$) are used at least once.

Then by erasing all $-a_{i,k+1}x_{k+1}$'s (for $i \in I_3$) and all $-c_{\ell,k+1}x_{k+1}$'s (for $\ell \in L_3$) in the system and use the same way to add them up, then we have a linear combination of $(\mathbf{AX})_i$'s and $(\mathbf{CX})_\ell$'s such that:

- The coefficients of $(\mathbf{AX})_i$'s and $(\mathbf{CX})_\ell$'s are all non-negative.
- All coefficients for $x_1, \ldots, x_k, x_{k+1}$ in this combination are non-positive.
- The strict inequations of types 2,3,4,6 (for $\ell \in L_3$) are used at least once.

Since strict inequations of type 2,3,4,6 have positive coefficients of $(\mathbf{CX})_\ell$'s, we have non-negative $\mathbf{b} \in \mathbb{Q}^{n_1 \times 1}$ and $\mathbf{0}^{n_2 \times 1} \neq \mathbf{d} \in \mathbb{Q}^{n_2 \times 1}$ such that $\mathbf{b}^T\mathbf{A} + \mathbf{d}^T\mathbf{C} \leq \mathbf{0}^{1 \times m}$. By appropriate scaling, we can make sure that all coordinates of $\mathbf{b}$ and $\mathbf{d}$ are in $\mathbb{N}$.

Therefore, in any case, we have that there are $\mathbf{b} \in \mathbb{N}^{n_1 \times 1}$ and $\mathbf{0}^{n_2 \times 1} \neq \mathbf{d} \in \mathbb{N}^{n_2 \times 1}$ such that $\mathbf{b}^T\mathbf{A} + \mathbf{d}^T\mathbf{C} \leq \mathbf{0}$.

Remark 1. Intuitively, Farkas' Lemma is a kind of soundness and completeness result in disguise. If we regard the rows of the finite linear inequation system as logical axioms, and the linear combination of the rows as a derivation rule, and the result of the linear combinations as theorems, then the lemma essentially states that the system is not satisfiable iff there is a linear combination of the "axioms" which can derive a contradiction, i.e. the system is not consistent.

We give a concrete example to illustrate how to determine whether a linear inequation system has non-negative real solutions:

Example 1. Let us consider the following linear inequation system:

$$\begin{cases} -x_1 + x_2 + x_3 > 0 \\ x_1 - x_2 + x_3 > 0 \\ x_1 + x_2 - x_3 > 0 \end{cases}$$

To check whether this system has non-negative real solutions, we first rewrite the inequation system as follows:

$$\begin{cases} x_2 > x_1 - x_3 \\ x_1 + x_3 > x_2 \\ x_2 > x_3 - x_1 \end{cases}$$

Then we transform the system above into the following system, which has non-negative real solutions iff the previous system has non-negative real solutions. Basically, we eliminate all occurrences of x_2 by concatenating the inequalities where x_2 occurs on the left of $>$ and the inequalities where x_2 occurs on the right of $>$:

$$\begin{cases} x_1 + x_3 > x_1 - x_3 \\ x_1 + x_3 > x_3 - x_1 \end{cases}$$

Now we rewrite the system as follows:

$$\begin{cases} 2x_3 > 0 \\ 2x_1 > 0 \end{cases}$$

This system obviously has non-negative real solutions.

Now we have the following characterization result:

Theorem 2. *Given any finite model $\mathcal{M} = (W, N)$, $\mathcal{M}$ is metrizable iff $\mathcal{M}$ makes Ψ true in Definition 4.*

Proof. When $W = \{w\}$, $\mathcal{M}$ is metrizable by the metric function $d(w, w) = 0$. When $W = \{w, v\}$ has two elements, $\mathcal{M}$ is metrizable by the metric function d such that $d(w, w) = d(v, v) = 0$ and $d(w, v) = d(v, w) = 1$. In both cases the axioms in Ψ are true in $\mathcal{M}$. Therefore, it suffices to consider the situation where $|W| \geq 3$.

The left-to-right direction follows from Theorem 1. For the other direction, consider variables $x_{w,v}$ for all $w, v \in W$ which is intended to refer to the value of $d(w, v)$.

Then $\mathcal{M}$ is metrizable iff the following inequation system has non-negative real solutions:

$$\begin{cases} x_{w,v} > 0 \text{ (for all } w, v \in W \text{ such that } w \neq v) \\ x_{w,w} = 0 \text{ (for all } w \in W) \\ x_{w,v} = x_{v,w} \text{ (for all } w, v \in W) \\ x_{w,v} + x_{v,u} \geq x_{w,u} \text{ (for all } w, v, u \in W), \\ x_{w,v} < x_{w,u} \text{ (for all } (w, v, u) \in N) \\ x_{w,v} \geq x_{w,u} \text{ (for all } (w, v, u) \in W^3 - N) \end{cases}$$

Since Axioms 1-3 in Definition 4 describe all the conditions that should be met for tuples (x, y, z) that contain repeated elements, therefore once Axioms 1-3 are true, it suffices to consider all the tuples (x, y, z) such that $\#(x, y, z)$.

By taking $x_{w,w} := 0$ for all $w \in W$, this inequation system has non-negative real solutions iff the following system has non-negative real solutions (for the last three lines, we also eliminate all the inequations with repeated elements; for the last two lines, they are guaranteed by Axioms 1-3):

$$\begin{cases} x_{w,v} > 0 \text{ (for all } w, v \in W \text{ such that } w \neq v) \\ x_{w,v} = x_{v,w} \text{ (for all } w, v \in W \text{ such that } w \neq v) \\ x_{w,v} + x_{v,u} \geq x_{w,u} \text{ (for all } w, v, u \in W \text{ such that } \#(w, v, u)), \\ x_{w,v} < x_{w,u} \text{ (for all } (w, v, u) \in N \text{ such that } \#(w, v, u)) \\ x_{w,v} \geq x_{w,u} \text{ (for all } (w, v, u) \in W^3 - N \text{ such that } \#(w, v, u)) \end{cases}$$

Since $x_{w,v} = x_{v,w}$ for all $w, v \in W$ such that $w \neq v$, we can change the notation for variables such that the variable $x_{w,v}$ becomes $x_{\{w,v\}}$, then the previous inequation system has non-negative real solutions iff the following inequation system has non-negative real solutions:

$$\begin{cases} x_{\{w,v\}} > 0 \text{ (for all } w, v \in W \text{ such that } w \neq v) \\ x_{\{w,v\}} + x_{\{v,u\}} - x_{\{w,u\}} \geq 0 \text{ (for all } w, v, u \in W \text{ such that } \#(w, v, u)). \\ x_{\{w,v\}} < x_{\{w,u\}} \text{ (for all } (w, v, u) \in N \text{ such that } \#(w, v, u)) \\ x_{\{w,v\}} \geq x_{\{w,u\}} \text{ (for all } (w, v, u) \in W^3 - N \text{ such that } \#(w, v, u)) \end{cases}$$

Let us re-order the inequalities as follows:

$$\begin{cases} x_{\{w,v\}} + x_{\{v,u\}} - x_{\{w,u\}} \geq 0 \text{ (for all } w,v,u \in W \text{ such that } \#(w,v,u)\text{)}. \\ x_{\{w,v\}} - x_{\{w,u\}} \geq 0 \text{ (for all } (w,v,u) \in W^3 - N \text{ such that } \#(w,v,u)\text{)} \\ x_{\{w,v\}} > 0 \text{ (for all } w,v \in W \text{ such that } w \neq v\text{)} \\ x_{\{w,u\}} - x_{\{w,v\}} > 0 \text{ (for all } (w,v,u) \in N \text{ such that } \#(w,v,u)\text{)} \end{cases}$$

Let us denote the inequation system as $\{\mathbf{BX} \geq \mathbf{0}, \mathbf{CX} \geq \mathbf{0}, \mathbf{DX} > \mathbf{0}, \mathbf{EX} > \mathbf{0}\}$, where $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$ correspond to the four types of inequations. By Lemma 1, this system has no non-negative real solutions iff there are non-negative natural number vertial vectors $\mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}$ such that $\mathbf{b}^T\mathbf{B} + \mathbf{c}^T\mathbf{C} + \mathbf{d}^T\mathbf{D} + \mathbf{e}^T\mathbf{E} \leq \mathbf{0}$ and $(\mathbf{d}^T, \mathbf{e}^T)$ contains at least one positive number.

Now let us denote $\Sigma\mathbf{b}$, $\Sigma\mathbf{c}$, $\Sigma\mathbf{d}$, $\Sigma\mathbf{e}$ as the sums of all numbers occurring in the vectors $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, respectively.

Suppose $\mathcal{M}$ makes Ψ true but is not metrizable. Then in the inequation system above, there are $\mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}$ satisfying the conditions above.

Then summing up all numbers in $\mathbf{b}^T\mathbf{B} + \mathbf{c}^T\mathbf{C} + \mathbf{d}^T\mathbf{D} + \mathbf{e}^T\mathbf{E}$, we have $\Sigma\mathbf{b} + \Sigma\mathbf{d}$, which should be non-positive. Since $\mathbf{b}, \mathbf{d}$ only contain natural numbers, we have that $\mathbf{b}$ and $\mathbf{d}$ are both zero vectors, so $\mathbf{c}^T\mathbf{C} + \mathbf{e}^T\mathbf{E} \leq \mathbf{0}$.

Since $\mathbf{d}$ is a zero vector and $(\mathbf{d}^T, \mathbf{e}^T)$ contains at least one positive number, it must be in $\mathbf{e}$. Therefore, from $\mathbf{c}^T\mathbf{C} + \mathbf{e}^T\mathbf{E} \leq \mathbf{0}$ we have that there are $x_{\{x_i,z_i\}} - x_{\{x_i,y_i\}}$ with $(x_i, y_i, z_i) \in N$ and $i = 1, \ldots, m$ $(m \geq 1)$ and $x_{\{u_j,v_j\}} - x_{\{u_j,w_j\}}$ with $(u_j, v_j, w_j) \notin N$ and $j = 1, \ldots, n$ $(n \geq 0)$ such that in their sum, all variables have non-positive coefficients. Since the sum of all the coefficients are 0, we have that it is only possible that for each variable, the sum of its coefficients is 0.

Now consider a variable $x_{w,v}$. If its coefficient in the sum is 0, then among all the (x_i, y_i, z_i)'s and (u_j, v_j, w_j)'s, $x_{w,v}$ occurs as $x_{\{x_i,y_i\}}$ or $x_{\{u_j,w_j\}}$ for the same number of times as $x_{\{x_i,z_i\}}$ or $x_{\{u_j,v_j\}}$. Then the lists $(\{x_1, y_1\}, \ldots, \{x_m, y_m\}, \{u_1, w_1\}, \ldots, \{u_n, w_n\})$ and $(\{x_1, z_1\}, \ldots, \{x_m, z_m\}, \{u_1, v_1\}, \ldots, \{u_n, v_n\})$ are permutations of each other. Then by the definition of the inequation system, we have that $x_{\{u_j,v_j\}} - x_{\{u_j,w_j\}}$ with $(u_j, v_j, w_j) \notin N$ and $j = 1, \ldots, m$ $(m \geq 1)$, and $x_{\{u_j,v_j\}} - x_{\{u_j,w_j\}}$ with $(u_j, v_j, w_j) \notin N$ and $j = 1, \ldots, n$ $(n \geq 0)$. Then by Axiom 4 in Definition 4 we get a contradiction. Therefore $\mathcal{M}$ is metrizable.

Theorem 3. *The universal first-order theory Φ_M of relative nearness relation is* $\mathsf{Th}_\forall(\Psi)$.

Proof. It is easy to check $\mathsf{Th}_\forall(\Psi) \subseteq \Phi_M$ by Theorem 1. For the other direction, consider a universal sentence $\forall\vec{x}\varphi(\vec{x})$ where φ is quantifier-free. If $\forall\vec{x}\varphi(\vec{x}) \notin \mathsf{Th}_\forall(\Psi)$, then $\Psi \cup \{\exists\vec{x}\neg\varphi(\vec{x})\}$ is satisfiable, so for $\exists\vec{x}\neg\varphi(\vec{x})$, take a finite vector of constants $\vec{c}$ which has the same length of $\vec{x}$ and consider formula $\neg\varphi(\vec{c})$. Then $\Psi \cup \{\neg\varphi(\vec{c})\}$ is satisfiable on a model $\mathcal{M} = (W, N)$. Take the submodel of $\mathcal{M}$ whose domain consists of the interpretations of all constants in $\vec{c}$, then we get a finite model $\mathcal{M}'$ such that $\neg\varphi(\vec{c})$ is still satisfiable, and therefore $\exists\vec{x}\neg\varphi(\vec{x})$ is also satisfiable on $\mathcal{M}'$. Since $\mathcal{M}'$ is a submodel of $\mathcal{M}$ and Ψ is a set of universal sentences, we have that $\mathcal{M}' \vDash \Psi$. By Theorem 2, $\mathcal{M}'$ is metrizable. So $\{\exists\vec{x}\neg\varphi(\vec{x})\}$

is satisfiable in a metrizable finite model and $\forall\vec{x}\varphi(\vec{x})$ is falsifiable there. So $\forall\vec{x}\varphi(\vec{x}) \notin \Phi_M$.

3.3 Further Simplifications of the Axioms

As we can see from Axiom 4 in 4, the presentation of this axiom schema is rather complicated. In what follows, we will simplify this axiom schema to make the axiomatization simpler, which is mainly for theoretic purposes, but they will also be useful in the intuitive understanding of the axioms.

Definition 5 (The First Simplification). *Θ consists of the following universal sentences:*

1. $\forall x \forall y \neg Nxyy$.
2. $\forall x \forall y (x \neq y \to Nxxy)$.
3. $\forall x \forall y \neg Nxyx$.
4. $\forall\vec{x}\forall\vec{y}\forall\vec{z}\forall\vec{u}\forall\vec{v}\forall\vec{w}(Nx_1y_1z_1 \wedge \ldots \wedge Nx_my_mz_m \wedge \neg Nu_1v_1w_1 \wedge \ldots \wedge \neg Nu_nv_nw_n \to \bot)$, *where the lists* $(\{x_1, y_1\}, \ldots, \{x_m, y_m\}, \{u_1, w_1\}, \ldots, \{u_n, w_n\})$ *and* $(\{x_1, z_1\}, \ldots, \{x_m, z_m\}, \{u_1, v_1\}, \ldots, \{u_n, v_n\})$ *are permutations of each other and* $m \geq 1, n \geq 0$.

The idea behind this simplification is that even if we do not have the requirements that x_i, y_i, z_i $(i = 1, \ldots, m)$ are pairwise different and similarly for u_j, v_j, w_j $(j = 1, \ldots, n)$, the axiomatization is equivalent to the version with the requirements. We have the following theorem:

Theorem 4. *Given any finite model $\mathcal{M} = (W, N)$, the following are equivalent:*

1. *$\mathcal{M}$ is metrizable.*
2. *$\mathcal{M}$ makes Ψ true in Definition 4.*
3. *$\mathcal{M}$ makes Θ true in Definition 5.*

Proof.
- The equivalence between 1 and 2 is from Theorem 2.
- For 1⇒3, suppose that $\mathcal{M}$ is metrizable but $\mathcal{M}$ does not make all axioms in Θ true. Then from 1⇔2, $\mathcal{M}$ makes the first three axioms true, so $\mathcal{M}$ falsifies some instance of Axiom 4, so there are $x_i, y_i, z_i \in W$ $(i = 1, \ldots, m)$ and $u_j, v_j, w_j \in W$ $(j = 1, \ldots, n)$ such that
  - $m \geq 1, n \geq 0$.
  - $N(x_i, y_i, z_i)$ for $i = 1, \ldots, m$.
  - $\sim N(u_j, v_j, w_j)$ for $j = 1, \ldots, n$.
  - $(\{x_1, y_1\}, \ldots, \{x_m, y_m\}, \{u_1, w_1\}, \ldots, \{u_n, w_n\})$ and $(\{x_1, z_1\}, \ldots, \{x_m, z_m\}, \{u_1, v_1\}, \ldots, \{u_n, v_n\})$ are permutations of each other.

  Then consider the metric d for $\mathcal{M}$, then
  - $d(x_i, y_i) < d(x_i, z_i)$ for $i = 1, \ldots, m$.
  - $d(u_j, w_j) \leq d(u_j, v_j)$ for $j = 1, \ldots, n$.

  Adding all the inequalities up, we will have a sum of distances smaller than the same sum, a contradiction. So $\mathcal{M} \vDash \Theta$.

– For 3⇒2, suppose that $\mathcal{M} \vDash \Theta$, then since every instance of the Axiom 4 in Ψ is a semantic consequence of the corresponding instance (the one without the pairwise difference statements) of Axiom 4 in Θ, we have that $\mathcal{M} \vDash \Psi$.

We can further simplify Axiom 4 such that we do not need to use the permutation condition. In what follows, we use $N_{\leq}xyz$ to represent $\neg Nxzy$ and $Mxyz$ to represent either $Nxyz$ or $N_{\leq}xyz$.

Definition 6 (The Second Simplification). *Δ consists of the following universal sentences:*

1. $\forall x \forall y \neg Nxyy$.
2. $\forall x \forall y (x \neq y \to Nxxy)$.
3. $\forall x \forall y \neg Nxyx$.
4. $\forall x_1 \ldots \forall x_n (Nx_1y_1z_1 \wedge Mx_2y_2z_2 \wedge \ldots \wedge Mx_ny_nz_n \to \bot)$, *where* $n \geq 1$ *and* $\{x_2, y_2\} = \{x_1, z_1\}$, $\{x_3, y_3\} = \{x_2, z_2\}$, $\{x_n, y_n\} = \{x_{n-1}, z_{n-1}\}$, $\{x_1, y_1\} = \{x_n, z_n\}$.

Intuitively, in Axiom 4 of Δ, we have inequalities of the form $d(x_1, y_1) < d(x_1, z_1), d(x_2, y_2) < (\leq) d(x_2, z_2), \ldots, d(x_{n-1}, y_{n-1}) < (\leq) d(x_{n-1}, z_{n-1}), d(x_n, y_n) < (\leq) d(x_n, z_n)$, and by adding them up, we have a strict inequality between two identical sums, a contradiction.

Theorem 5. *Given any finite model $\mathcal{M} = (W, N)$, the following are equivalent:*

1. *$\mathcal{M}$ is metrizable.*
2. *$\mathcal{M}$ makes Ψ true in Definition 4.*
3. *$\mathcal{M}$ makes Θ true in Definition 5.*
4. *$\mathcal{M}$ makes Δ true in Definition 6.*

Proof. – The equivalence between 1,2,3 is from Theorem 5.
– 3⇒4 follows from that Δ is a special case of Θ.
– For 4⇒3, suppose that the antecedent holds for some $\vec{x}, \vec{y}, \vec{z}, \vec{u}, \vec{v}, \vec{w}$ such that $(\{x_1, y_1\}, \ldots, \{x_m, y_m\}, \{u_1, w_1\}, \ldots, \{u_n, w_n\})$ and $(\{x_1, z_1\}, \ldots, \{x_m, z_m\}, \{u_1, v_1\}, \ldots, \{u_n, v_n\})$ are permutations of each other. By the property of permutation, we can decompose the permutation into one or more disjoint cycles. Take the cycle such that N occurs positively in the antecedent (i.e. some x_i, y_i, z_i occur in the cycle), then we have a cycle of the form as described in the antecedent of Axiom 4 in Δ, so we get a contradiction.

4 Conclusions and Future Works

In the present paper, we give an axiomatization of the universal theory of relative nearness relation, using a variant of Farkas' Lemma. This result shows the usefulness of methods for linear algebra and optimization theory in problems in logics where certain numerical informations are involved. In addition, the result of the present paper will be useful in the study of modal logic, hybrid logic and conditional logic involving semantics based on metric spaces.
For future work, we mention the following topics:

- The first one is about the finite axiomatizability of the universal first-order theory of relative nearness. In this direction, either we find a finite axiomatization of the theory, or we can use the Ehrenfeucht-Fraisse game method to show that the universal first-order theory of relative nearness is not finitely axiomatizable.
- The second one is to consider the axiomatization of the modal logic and hybrid logic of relative nearness (see [2, 3]) where the following binary modality $\langle N \rangle$ is used:

$$(W, N, V), w \Vdash \langle N \rangle(\varphi, \psi) \text{ iff there exists } v, u \in W \text{ such that } Nwvu \text{ and } (W, N, V), v \Vdash \varphi \text{ and } (W, N, V), u \Vdash \psi.$$

- Another topic is about conditional logic. In Lewis' account of conditional logic (see [14]), a formula $\varphi > \psi$ is true at w if among all φ-worlds, all the ones nearest to w make ψ true. Here the concept of "nearest" can be understood in terms of a metric space (see [18] for more discussions on this). It would be interesting to see if we can axiomatize conditional logic with respect to semantics based on metric spaces.

Acknowledgement. The research of the author is supported by Shandong Provincial Natural Science Foundation, China (project number: ZR2023QF021) and Taishan Young Scholars Program of the Government of Shandong Province, China (No.tsqn201909151).

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