Preservation of Superdistributivity between Triangular Norms under Transformations

Xiya Lu[0009−0008−8684−0890], Gang Li[0000−0003−4048−3978]

Department of Mathematics and Artificial Intelligence, Qilu University of Technology (Shandong Academy of Sciences), Jinan 250353
2794040906@qq.com, sduligang@163.com

Abstract. The study of superdistributivity for triangular norms is of great significance in fuzzy logic and uncertain reasoning. By analyzing distributivity inequalities, it not only compensates for the limitations of traditional logical operations but also optimizes model construction in fields such as decision analysis and artificial intelligence. This paper investigates the preservation of superdistributivity between triangular norms under some transformations, such as functional transformations and additive generator transformations.

Keywords: triangular norm· superdistributivity · triangular subnorm · additive generator.

1 Introduction

The origin of triangular norms(t-norms) can be traced back to 1942, when Karl Menger first introduced the concept of triangular functions while constructing statistical metric spaces [1]. After many years of development, they play an important role across various domains including fuzzy systems, fuzzy control and information aggregation [2–4]. In particular, the algebraic properties of triangular norms provide a rigorous mathematical framework for constructing fuzzy rules and integrating heterogeneous data sources. However, when t-norms undergo specific transformations to meet application requirements, some algebraic properties may be compromised. This paper discusses the preservation of superdistributivity in triangular norms transformations. Distributivity is a crucial property that governs the relationship between operators. In classical logic, for instance, the distributive law of multiplication over addition forms the cornerstone of algebraic systems. Similarly, in fuzzy logic, studying distributivity between t-norms facilitates the construction of more sophisticated logical systems. In recent years, the superdistributivity and subdistributivity were also discussed in [11, 12]. This paper specifically investigates the preservation of superdistributivity for t-norm under some transformations. It is well known that the superdistributivity (subdistributivity) between two operators preserves under the order isomorphism [11]. Hence, the investigation of superdistributivity preservation for t-norm (t-conorm) under some transformations carries significant theoretical importance.

The main theoretical advancement of this paper consists in a complete characterization of superdistributivity preservation for transformed t-norms. First, based on triangular subnorms, we prove that the superdistributivity property remains valid when they are transformed to triangular norms. Second, we analyze the superdistributivity of triangular norms under a transformation based on a monotonic function and t-norms. Furthermore, we investigate the superdistributivity of continuous Archimedean triangular norms under additive generator transformations base on Minkowski inequality and prove that superdistributivity is preserved under specific conditions for certain parametric families.

After this introduction, we give some fundamental preliminaries relevant to the content discussed in this paper in section 2. We discusses the preservation of superdistributivity under transformations, supported by concrete examples in section 3. Finally, we summarize the work with a brief summary of the main contributions.

2 Preliminaries

In this section, we provide foundational definitions and properties related to triangular norms.

Definition 1. *[5–9] The two variable function $T : [0,1]^2 \to [0,1]$ is t-norm, for all $l, m, n \in [0,1]$, which complies with the axioms $(T1) - (T4)$:*
$(T1)$ $T(l,m) = T(m,l)$.
$(T2)$ $T(T(l,m), n) = T(l, T(m,n))$.
$(T3)$ $T(l,m) \le T(m,n)$ *whenever* $m \le n$.
$(T4)$ $T(l,1) = l$.

Let T be a t-norm. As a direct consequence, $T(l,m) \le \min(l,m)$ for all $(l,m) \in [0,1]^2$. T is strictly monotone if $T(l,m) < T(l,n)$ for all $l > 0$ and $m < n$; T is called Archimedean if there exists some natural number $a \in \mathbb{N}$, $l_T^{(a)} = T(\underbrace{l, ..., l}_{a}) < m$ whenever $(l,m) \in]0,1[^2$; It is called strict if T satisfies both the continuity property and strictly monotone; T is called nilpotent if it is continuous and there exists some positive integer a with $l_T^{(a)} = 0$ for each $l \in]0,1[$.

It is well established that every continuous Archimedean t-norm is classified as strict or nilpotent.For the corresponding concepts and properties of t-conorms, the readers are advised to refer to the monographs [5].

Below we introduce two basic t-norms, which are extensively discussed in the work.

$$\begin{aligned} Lukasiewicz\ \ t-norm\ \ & T_L(l,m) = \max(l+m-1, 0), \\ Product\ \ t-norm\ \ & T_P(l,m) = l \cdot m \end{aligned}$$

In the subsequent discussion, we present several construction methods of t-norms.

Definition 2. *[5] Let $\psi : [u, v] \to [w, z]$ with $[u, v], [w, z] \subseteq [-\infty, \infty]$ be non-increasing and non-constant. The pseudo-inverse $\psi^{(-1)} : [w, z] \to [u, v]$ is characterized by*

$$\psi^{(-1)}(\beta) = \sup\{\alpha \in [u, v] : \psi(\alpha) > \beta\}$$

Moreover, $\psi^{(-1)} = \psi^{-1}$ if and only if ψ is a bijection, ψ^{-1} is the inverse function of ψ. If ψ is strictly monotone, then $\psi^{(-1)} \circ f = \mathrm{id}_{[u,v]}$.

Definition 3. *[5, 10] The function $\Phi : [0, 1]^2 \to [0, 1]$ is called a t-subnorm if it fulfills the axioms $(T1) - (T3)$, and $\Phi(l, m) \leq \min(l, m)$ for all $l, m, n \in [0, 1]$.*

Proposition 1. *[5, 10] If A is a t-subnorm, then the two variable function $T_{[A]} : [0, 1]^2 \to [0, 1]$ is characterized by*

$$T_{[A]}(l, m) = \begin{cases} A(l, m) & (l, m) \in [0, 1[^2, \\ \min(l, m) & \textit{otherwise}, \end{cases}$$

is a triangular norm.

Theorem 1. *[5] Given a non-decreasing function $\eta : [0, 1] \to [0, 1]$ and a t-norm T satisfying:*

- **(i)** *$T(\eta(l), \eta(m)) \in Ran(\eta) \cup [0, f(0^+)]$, for all $l, m \in [0, 1[$;*
- **(ii)** *$\eta \circ \eta^{(-1)}(T(\eta(l), \eta(m))) = T(\eta(l), \eta(m))$ whenever $(l, m) \in [0, 1]^2$ with $T(\eta(l), \eta(m)) \in Ran(\eta)$.*

The function $T_{[\eta]} : [0, 1]^2 \to [0, 1]$ given by

$$T_{[\eta]}(l, m) = \begin{cases} \eta^{(-1)}(T(\eta(l), \eta(m))) & (l, m) \in [0, 1[^2, \\ \min(l, m) & \textit{otherwise}. \end{cases} \tag{1}$$

is a t-norm.

Definition 4. *[5] Let the function $h : [0, 1] \to [0, \infty]$ is called additive generator of a t-norm T if it is strictly decreasing, vanishes at 1 and $\lim_{l \to 0^+} h(l) = h(0)$, satisfying for any $(l, m) \in [0, 1]^2$ we get*

$$\begin{aligned} h(l) + h(m) &\in Ran(h) \cup [h(0), \infty], \\ T(l, m) &= h^{(-1)}(h(l) + h(m)). \end{aligned}$$

Example 1. [5] We give now some additive generators $h : [0, 1] \to [0, \infty]$ that satisfy the conditions in Definition 4, and characterize the corresponding t-norms they induce.

- **(i)** For the function $h(l) = 1 - l$, which yields T_L.
- **(ii)** If we put for each $\mu \in]0, 1]$

$$h_\mu(l) = \begin{cases} -\log(\mu l) & l \in [0, 1[, \\ 0 & l = 1, \end{cases} \tag{2}$$

then the expression (2)induces

$$T_\mu(l,m) = \begin{cases} \mu lm & (l,m) \in [0,1[^2, \\ \min(l,m) & \text{otherwise.} \end{cases} \tag{3}$$

Lemma 1. *With $h : [0,1] \to [0,\infty]$ as additive generator of t-norm T.*

(i) *If T be a continuous Archimedean, then for every $\mu \in]0,\infty[$, the function $t^\mu : [0,1] \to [0,\infty]$ given by*

$$h^\mu(l) = (h(l))^\mu \tag{4}$$

acts as an additive generator for a continuous Archimedean t-norm $T^{(\mu)}$.

(ii) *If T be a strict t-norm, then for every $\mu \in]0,\infty[$, the function $h_\mu : [0,1] \to [0,\infty]$ given by*

$$h_\mu(l) = h((h)^{-1}(\mu h(l))) \tag{5}$$

acts as an additive generator for a continuous Archimedean t-norm $T_{(\mu)}$.

Lemma 1 introduces an approach to constructing new t-norms by applying different transformations to a given additive generator.

Definition 5. *[11] Let $F, G : [0,1]^2 \to [0,1]$ be two operations.*

(i) *F is left subdistributive with respect to G if*

$$F(l, G(m,n)) \le G(F(l,m), F(l,n)) \quad \textit{for any} \;\; l,m,n \in [0,1].$$

(ii) *F is left superdistributive with respect to G if*

$$F(l, G(m,n)) \ge G(F(l,m), F(l,n)) \quad \textit{for any} \;\; l,m,n \in [0,1].$$

Similarly, the right subdistributivity and superdistributivity can be defined. It is obvious that the left subdistributivity (superdistributivity) is equivalent to right subdistributivity (superdistributivity) if the two operations F, G are commutative.It is easy to see that the order isomorphism preserves both left and right subdistributivity and superdistributivity between t-norms (or t-subnorms) (see Theorem 4.8 in [11]).

3 Superdistributivity under Transformation of t-norm Operators

This section deals with the preservation of superdistributivity between t-norms under the transformations.

Proposition 2. *Let $J, K : [0,1]^2 \to [0,1]$ be two t-subnorms and $T_{[J]}, T_{[K]}$ be the corresponding t-norms. If J is superdistributive over K, then $T_{[J]}$ is superdistributive over $T_{[K]}$.*

Proof. Suppose J is superdistributive over K. By Proposition 1, $T_{[J]}$ and $T_{[K]}$ are given by

$$T_{[J]}(l,m)=\begin{cases} J(l,m) & (l,m)\in[0,1[^2,\\ \min(l,m) & \text{otherwise,}\end{cases}$$
$$T_{[K]}(l,m)=\begin{cases} K(l,m) & (l,m)\in[0,1[^2,\\ \min(l,m) & \text{otherwise.}\end{cases}$$

Clearly, $T_{[J]}$ is superdistributive over $T_{[K]}$ for all $l,m,n\in[0,1[$. In the following, we prove that the superdistributivity holds for at lest one of l,m,n equals 1.

(i) If $l=1$, then $T_{[J]}(l,T_{[K]}(m,n))=T_{[K]}(m,n)=T_{[K]}(T_{[J]}(l,m),T_{[K]}(l,n))$.
(ii) If $m=1$, then

$$T_{[J]}(l,T_{[K]}(m,n))=T_{[J]}(l,n),$$
$$T_{[K]}(T_{[J]}(l,m),T_{[J]}(l,n))=T_{[K]}(l,T_{[J]}(l,n)).$$

By Definition 1, we have $T_{[J]}(l,n)\geq T_{[K]}(l,T_{[J]}(l,n))$ and $T_{[J]}(l,T_{[K]}(m,n))\geq T_{[K]}(T_{[J]}(l,m),T_{[J]}(l,n))$.
(iii) If $n=1$, the proof follows the analogous approach to case (ii).

These imply, $T_{[J]}$ is superdistributive over $T_{[K]}$.

Example 2. Assume t-subnorms J,K defined by $J(l,m)=0$, $K(l,n)=\frac{1}{2}xy$ for all $(l,m)\in[0,1]^2$. Clearly, J is superdistributive over K due to $J(l,K(m,n))=K(J(l,m),J(l,n))=0$ for all $(l,m,n)\in[0,1]^3$. Their corresponding t-norms are defined by, respectively:

$$T_{[J]}(l,m)=\begin{cases} 0 & (l,m)\in[0,1[^2,\\ \min(l,m) & \text{otherwise,}\end{cases}$$
$$T_{[B]}(l,m)=\begin{cases} \frac{1}{2}lm & (l,m)\in[0,1[^2,\\ \min(l,m) & \text{otherwise.}\end{cases}$$

It is obvious that $T_{[J]}$ is superdistributive over $T_{[K]}$.

Proposition 3. *Let $\eta:[0,1]\to[0,1]$ be a non-decreasing function, T_1 and T_2 be t-norms for which the requirements of Theorem 1. If T_1 is superdistributive over T_2, then $T_{1[\eta]}$ is superdistributive over $T_{2[\eta]}$, where $T_{1[\eta]}$, $T_{2[\eta]}$ are the t-norms from the expression (1).*

Proof. By Eq. (1), $T_{1[\eta]}$ and $T_{2[\eta]}$ are given by

$$T_{1[\eta]}(l,m)=\begin{cases} \eta^{(-1)}(T_1(\eta(l),\eta(m))) & (l,m)\in[0,1[^2,\\ \min(l,m) & \text{otherwise,}\end{cases}$$
$$T_{2[\eta]}(l,m)=\begin{cases} \eta^{(-1)}(T_2(\eta(l),\eta(m))) & (l,m)\in[0,1[^2,\\ \min(l,m) & \text{otherwise.}\end{cases}$$

Clearly, when one of l, m, or n equals 1, $T_{1[\eta]}$ is superdistributive over $T_{2[\eta]}$. Consider $l, m, n \in [0, 1[$. If $T_2(\eta(m), \eta(n)) \in Ran(\eta)$ then

$$\begin{aligned} T_{1[\eta]}(l, T_{2[\eta]}(m, n)) &= \eta^{(-1)}(T_1(\eta(l), \eta(\eta^{(-1)}(T_2(\eta(m), \eta(n)))))) \\ &= \eta^{(-1)}(T_1(\eta(l), T_2(\eta(m), \eta(n)))) \\ T_{2[\eta]}(T_{1[\eta]}(l, m), T_{1[\eta]}(l, n)) &= \eta^{(-1)}(T_2(\eta(\eta^{(-1)}(T_1(\eta(l), \eta(m)))), \eta(\eta^{(-1)}(T_1(\eta(l), \eta(n)))))) \\ &= \eta^{(-1)}(T_2(T_1(\eta(l), \eta(m)), T_1(\eta(l), \eta(n)))) \end{aligned}$$

T_1 is superdistributive over T_2, and function $\eta^{(-1)}$ is non-decreasing, this implies

$$T_{1[\eta]}(l, T_{2[\eta]}(m, n)) \geq T_{2[\eta]}(T_{1[\eta]}(l, m), T_{1[\eta]}(l, n)).$$

Besides, if $T_2(\eta(m), \eta(n)) \leq \eta(0^+)$ then

$$T_{1[\eta]}(l, T_{2[\eta]}(m, n)) = 0 = T_{2[\eta]}(T_{1[\eta]}(l, m), T_{1[\eta]}(l, n)).$$

Therefore, $T_{1[\eta]}$ is superdistributive over $T_{2[\eta]}$.

Example 3. Consider T_P and T_L. Obviously, T_P is superdistributive over T_L (see Table 3 in [11]). Given an unary function $\eta : [0, 1] \to [0, 1], \eta(l) = cl$, where parameter $c \in]0, 1[$. It is obvious that η is a bijection from $[0, 1] \to [0, c]$, and it is strictly increasing. Thus, the construction approach presented in Theorem 1 holds for arbitrary t-norms.

$$T_{P[\eta]}(l, m) = \begin{cases} \frac{1}{c} \cdot lm & (l, m) \in [0, 1[^2, \\ \min(l, m) & \text{otherwise}, \end{cases}$$

$$T_{L[\eta]}(l, m) = \begin{cases} \frac{1}{c} \cdot \max(l + m - 1, 0) & (l, m) \in [0, 1[^2, \\ \min(l, m) & \text{otherwise}. \end{cases}$$

Trivially, $T_{P[\eta]}$ is superdistributive over $T_{L[\eta]}$, which one of l, m, n equals 1. When $l, m, n \in [0, 1[$, we get $T_{P[\eta]}(l, T_{L[\eta]}(m, n)) = \frac{1}{c^2} \max(lm + ln - l, 0)$ and $T_{L[\eta]}(T_{P[\eta]}(l, m), T_{P[\eta]}(l, n)) = \frac{1}{c^2} \max(lm + ln - 1, 0)$. Therefore, $T_{P[\eta]}$ is superdistributive over $T_{L[\eta]}$.

Proposition 4. *For some t-norm T, its additive generator is $h : [0, 1] \to [0, \infty]$, let $r, s \in [0, \infty]$. If $r \geq s$, then $T^{(r)}$ is superdistributive over $T^{(s)}$.*

Proof. By Definition 4 we obtain

$$T^{(\mu)}(l, m) = h^{(-1)}(h^{\mu}(l) + h^{\mu}(m))^{\frac{1}{\mu}}.$$

Therefore,

$$\begin{aligned} T^{(r)}(l, T^{(s)}(m, n)) &= h^{(-1)}\left[h^r(l) + h^r(T^{(s)}(m, n))\right]^{\frac{1}{r}} \\ &= h^{(-1)}\left[h^r(l) + (h^s(m) + h^s(n))^{\frac{r}{s}}\right]^{\frac{1}{r}} \end{aligned} \quad (6)$$

$$\begin{aligned} & T^{(s)}(T^{(r)}(l,m),T^{(a)}(l,n)) \\ &= h^{(-1)}\left[h^s(T^{(r)}(l,m))+h^s(T^{(r)}(l,n))\right]^{\frac{1}{s}} \\ &= h^{(-1)}\left[(h^r(l)+h^r(m))^{\frac{s}{r}}+(h^r(l)+h^r(n))^{\frac{s}{r}}\right]^{\frac{1}{s}} \end{aligned} \tag{7}$$

By the Minkowski's inequality in [13], we have

$$\begin{aligned} (h^r(l)+h^r(m))^{\frac{s}{r}}+(h^r(l)+h^r(n))^{\frac{s}{r}} &\geq ((h^s(l)+h^s(l))^{\frac{r}{s}}+(h^s(m)+h^s(n))^{\frac{r}{s}})^{\frac{s}{r}} \\ &\geq (h^r(l)+(h^s(m)+h^s(n))^{\frac{r}{s}})^{\frac{s}{r}} \end{aligned} \tag{8}$$

Hence,

$$\left((h^r(l)+h^r(m))^{\frac{s}{r}}+(h^r(l)+h^r(n))^{\frac{s}{r}}\right)^{\frac{1}{s}} \geq (h^r(l)+(h^s(m)+h^s(n))^{\frac{r}{s}})^{\frac{1}{r}} \tag{9}$$

As $h^{(-1)}$ is strictly decreasing, we conclude,

$$T^{(r)}(l,T^{(s)}(m,n)) \geq T^{(s)}(T^{(r)}(l,m),T^{(r)}(l,n)).$$

That is, $T^{(r)}$ is superdistributive over $T^{(s)}$.

Example 4. By Example 1, we know that $h(l)=1-l$ acts as an additive generator for T_L. Now let's consider the t-norms family $T_L^{(\mu)}$, which is obtained through additive generator of T_L from Lemma 1. By Eq. (4), we have $h^\mu(l)=(1-l)^\mu$ and $T_L^{(\mu)}(l,m)=\max(1-((1-l)^\mu+(1-m)^\mu)^{\frac{1}{\mu}},0),\quad \mu\in]0,+\infty[$. By simple computation, we can see that $T_L^{(r)}$ is superdistributive over $T_L^{(s)}$ for $r\geq s$.

Example 5. Now we consider the superdistributivity of the t-norms family $T_P^{(\mu)}$, which is obtained by applying the method from Lemma 1 to the additive generator of T_P. For T_P, its additive generator is $h(l)=-\log l$, then $h^\mu(l)=(-\log l)^\mu$ is additive generator of $T_P^{(\mu)}$. By Definition 4,

$$T_P^{(\mu)}(l,m)=\exp(-\left[(-\log l)^\mu+(-\log m)^\mu\right]^{\frac{1}{\mu}}).$$

By the computation, $T_P^{(r)}$ is superdistibutive over $T_P^{(s)}$ for $r\geq s$.

Remark 1. Note that $h_\mu(l)=1-l^\mu$ is the additive generator of t-norm $T_{L(\mu)}$ by Eq. (5). $T_{L(\mu)}(l,m)=\max((l^\mu+m^\mu-1)^{\frac{1}{\mu}},0)$ by Definition 4. If $T_{L(\mu)}(l,m)$, $T_{L(\mu)}(l,n)$, $T_{L(\mu)}(m,n)\geq 0$, then for $r,s\in]0,+\infty[$ with $r\geq s$,

$$T_{L(r)}(l,T_{L(s)}(m,n))=(l^r+(m^s+n^s-1)^{\frac{r}{s}})^{\frac{1}{r}},$$

$$T_{L(s)}(T_{L(r)}(l,m),T_{L(r)}(l,n))=((l^r+m^r-1)^{\frac{s}{r}}+(l^r+n^r-1)^{\frac{s}{r}}-1)^{\frac{1}{s}}.$$

It is impossible to obtain the superdistributivity of $T_{L(r)}$ and $T_{L(s)}$. In fact, if $r\geq s, T_{L(s)}$ is superdistributive over $T_{L(r)}$ (see Theorem 5.14 in [11]).

4 Conclusion

In the paper, we analyze the preservation of superdistributivity of t-norms under some transformations which include the transformation between t-norms and t-subnorms, T and T_η through the general function η, and those related to the additive generators. Due to the duality of t-conorms and t-norms, these conclusions can be easily generalized to t-conorms. For example, assume that $J, K : [0,1]^2 \to [0,1]$ are two t-supernorms [5] and $S_{[J]}, S_{[K]}$ are the corresponding t-conorms. If J is subdistributive over K, then $S_{[J]}$ is subdistributive over $S_{[K]}$.

Acknowledgements This work is supported by National Nature Science Foundation of China under Grant 61977040 and Natural Science Foundation of Shandong Province under Grant ZR2019MF055.

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