# Logics for Kleene Algebra with Modal Operators: Kripke-completeness and FMP*

Yiheng Wang[1] (✉) and Yudi Ding[2]

[1] Institute of Logic and Cognition, Sun Yat-sen University, Guangzhou 510275, China
[2] Department of Philosophy, Xiamen University, Xiamen 361005, China
ianwang747@gmail.com (Y.W.); dingyudi0923@gmail.com (Y.D.)

**Abstract.** This paper studies two kinds of logics for Kleene algebra with modal operators inspired by two different research traditions: positive modal logic and algebraic studies of rough set theory. We show the Kripke-completeness of some logics in the former approach using the canonical model method. We also prove the finite model property of some logics in the other using an algebraic proof-theoretical method.

**Keywords:** Kleene algebra with modal operators; Kripke-completeness; finite model property; algebraic rough set theory; positive modal logic

## 1 Introduction

A Kleene algebra[3] is a structure $(A, \wedge, \vee, \sim, 0, 1)$ where $(A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and $\sim$ is a Kleene negation: satisfying for any $a, b \in A$, (K1) $\sim(a \vee b) = \sim a \wedge \sim b$, (K2) $\sim\sim a = a$, (K3) $\sim 0 = 1$, and (K4) $a \wedge \sim a \leq b \vee \sim b$. If we drop (K4), one obtains a quasi-Boolean algebra (also known as De Morgan algebra, cf. [3,16]). Various algebraic and logical works have been conducted on this structure (cf. [11,12,6,14,5]). Kleene algebra can be regarded as the algebraic basis for the theory of inexact classes (cf. [6]). Kumar and Banerjee [12] also introduced rough set-theoretical semantics and representation for this structure. If modal operators $\Box, \Diamond$ are added, one obtains Kleene algebra with modal operators. However, there are two different ways of adding modal operators: one comes from the tradition of positive modal logic (cf. [9]); the other comes from the algebraic studies of rough set theory (cf. [17]). This paper will study two kinds of logics for Kleene algebra with modal operators arising from these two different but related approaches.

The positive modal logic $\mathsf{K}_+$ is $\mathsf{K}$ without negation [9]. Its algebraic counterpart, the positive modal algebra, is a bounded distributive lattice satisfying (P1)-(P3) in Definition 1. (P3)'s interaction axioms are essential for relational

* Both authors contributed equally to this work and should be considered as co-first authors.

[3] There is another "Kleene algebra" in the study of regular expressions and program semantics (cf. [19]). However, this paper uses this terminology to denote a particular extension of quasi-Boolean algebra, following the study tradition of [12,4,5,1].

semantics. Consequently, the first modal extension of Kleene algebra enriches it with $\Box, \Diamond$ satisfying (P1)-(P3), defining a modal Kleene algebra (Definition 1). In algebra studies of rough set theory, modal extensions of quasi-Boolean algebras are used to model structures abstracted from applying rough set to knowledge representation [17]. The approximation space algebra induced from Pawlak's rough set yields the concept of topological quasi-Boolean algebra [2]: a quasi-Boolean algebra with $\Box, \Diamond$ satisfying (P1), (P2), (T) ($\Box a \leq a \leq \Diamond a$), and (4) ($\Box a \leq \Box\Box a$, $\Diamond\Diamond a \leq \Diamond a$). Dropping (T) and (4) gives a semi-topological quasi-Boolean algebra (cf. [18, Definition 2.3]). Since Kleene algebras extend quasi-Boolean algebras, their modal extension adds $\Box, \Diamond$ satisfying only (P1) and (P2), defining a semi-topological Kleene algebra (section 3).

The contributions of this paper are twofold. On one hand, We establish relational semantics for modal Kleene algebra logics via compatibility frames (based on Dunn's perp semantics [8]), adding order-relation constraints (cf. Definition 3 (R1,R2)) to existing frame conditions for (K2)–(K4) [10,12]. Discrete duality is shown for $\mathsf{KMK}$ (cf. Theorem 1). Then we use the canonical model method to prove completeness for $\mathsf{KMK}$ and some extensions (cf. Theorem 3). On the other hand, we prove the finite model property (FMP) for logics of semi-topological Kleene algebra and its extensions (cf. Theorem 4), constructing finite counter-algebras via an algebraic proof-theoretical method (cf. e.g. [13]). We should clarify that $\Box$ and $\Diamond$ are not De Morgan duals in (1) (due to perp semantics), but are treated as duals in (2) since relational semantics are unavailable.

## 2 Logics for Modal Kleene Algebras

In this section, we introduce sequential logics for modal Kleene algebras. We then establish its relational semantics with discrete duality. By using the canonical model method, we obtain the Kripke-completeness for some logics.

**Definition 1.** *An algebra structure* $\mathbf{A} = (A, \wedge, \vee, \sim, \Box, \Diamond, 0, 1)$ *is a* modal Kleene algebra *(MKA) if* $(A, \wedge, \vee, \sim, 0, 1)$ *is a Kleene algebra and* $\Box, \Diamond$ *are unary operations satisfying the followings: for any* $a, b \in A$, (P1) $\Box 1 = 1$ *and* $\Diamond 0 = 0$, (P2) $\Box(a \wedge b) = \Box a \wedge \Box b$ *and* $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$, (P3) $\Box a \wedge \Diamond b \leq \Diamond(a \wedge b)$ *and* $\Box(a \vee b) \leq \Box a \vee \Diamond b$. *Let* $\mathbf{MKA}$ *be the variety of all MKAs.*

The *set of formulas* $\mathscr{F}$ is defined as follows: $\mathscr{F} \ni \alpha ::= p \mid \bot \mid (\alpha_1 \wedge \alpha_2) \mid {\sim}\alpha \mid \Box\alpha \mid \Diamond\alpha$ where $p \in \mathbf{Var}$, a countable set of variables. Let $\top := {\sim}\bot$ and $\alpha_1 \vee \alpha_2 := {\sim}({\sim}\alpha_1 \wedge {\sim}\alpha_2)$. For $n \geq 0$ and $\ddagger \in \{\Box, \Diamond, \sim\}$, let $\ddagger^n\alpha$ be the formula defined by $\ddagger^0\alpha = \alpha$ and $\ddagger^{n+1}\alpha = \ddagger\ddagger^n\alpha$. A *basic sequent* is an expression of the form $\alpha \Rightarrow \beta$ where $\alpha, \beta \in \mathscr{F}$. Let $s, t$ etc. denote any basic sequents, and $\mathcal{BS}$ denote the *set of all basic sequents*.

**Definition 2.** *A* modal Kleene sequential logic *(MKL) is a set* $L$ *of basic sequents satisfying the following conditions:*
(1) *$L$ contains all instances of the following axiom schemata:*

(ID) $\alpha \Rightarrow \alpha$ (⊤) $\alpha \Rightarrow \top$ (⊥) $\bot \Rightarrow \alpha$ (DIS) $\alpha\wedge(\beta\vee\gamma) \Rightarrow (\alpha\wedge\beta)\vee(\alpha\wedge\gamma)$ (KL) $\alpha\wedge{\sim}\alpha \Rightarrow \beta\vee{\sim}\beta$

(DN1) $\alpha \Rightarrow {\sim}{\sim}\alpha$ (DN2) ${\sim}{\sim}\alpha \Rightarrow \alpha$ (K$_\Box$) $\Box\alpha \wedge \Box\beta \Rightarrow \Box(\alpha\wedge\beta)$ (K$_\Diamond$) $\Diamond(\alpha\vee\beta) \Rightarrow \Diamond\alpha\vee\Diamond\beta$
(N$_\Box$) $\top \Rightarrow \Box\top$ (N$_\Diamond$) $\Diamond\bot \Rightarrow \bot$ (I$_\Box$) $\Box\alpha \wedge \Diamond\beta \Rightarrow \Diamond(\alpha\wedge\beta)$ (I$_\Diamond$) $\Box(\alpha\vee\beta) \Rightarrow \Box\alpha \vee \Diamond\beta$
(2) *$L$ is closed under the following rules: for $i \in \{1,2\}$,*

$$\frac{\alpha_i \Rightarrow \beta}{\alpha_1 \wedge \alpha_2 \Rightarrow \beta}(\wedge\mathrm{L}) \quad \frac{\alpha \Rightarrow \beta_1 \quad \alpha \Rightarrow \beta_2}{\alpha \Rightarrow \beta_1 \wedge \beta_2}(\wedge\mathrm{R}) \quad \frac{\alpha \Rightarrow \beta}{{\sim}\beta \Rightarrow {\sim}\alpha}(\mathrm{CP}) \quad \frac{\alpha \Rightarrow \chi \quad \chi \Rightarrow \beta}{\alpha \Rightarrow \beta}(\mathrm{CUT})$$

$$\frac{{\sim}\alpha_1 \Rightarrow \beta \quad {\sim}\alpha_2 \Rightarrow \beta}{{\sim}(\alpha_1 \wedge \alpha_2) \Rightarrow \beta}({\sim}\wedge\mathrm{L}) \quad \frac{\alpha \Rightarrow {\sim}\beta_i}{\alpha \Rightarrow {\sim}(\beta_1 \wedge \beta_2)}({\sim}\wedge\mathrm{R}) \quad \frac{\alpha \Rightarrow \beta}{\Box\alpha \Rightarrow \Box\beta}(\mathrm{M}_\Box) \quad \frac{\alpha \Rightarrow \beta}{\Diamond\alpha \Rightarrow \Diamond\beta}(\mathrm{M}_\Diamond)$$

(3) *$L$ is closed under substitution: if $\alpha \Rightarrow \beta \in L$, then $\alpha^\sigma \Rightarrow \beta^\sigma \in L$ for any $\sigma$.*

A basic sequent $s$ is *derivable* in $L$ (written $\vdash_L s$) if $s \in L$. We write $\alpha \Leftrightarrow \beta$ if $\vdash_L \alpha \Rightarrow \beta$ and $\vdash_L \beta \Rightarrow \alpha$. Let $\mathsf{KMK}$ be the smallest MKL. A MKL $L$ is *consistent* if $L \neq \mathcal{BS}$.

**Lemma 1.** *For every MKL L, the following hold: (1) $\vdash {\sim}(\alpha\wedge\beta) \Leftrightarrow {\sim}\alpha\vee{\sim}\beta$ and $\vdash {\sim}(\alpha\vee\beta) \Leftrightarrow {\sim}\alpha\wedge{\sim}\beta$. (2) $\vdash \alpha \Leftrightarrow {\sim}{\sim}\alpha$ and $\vdash \alpha\vee(\beta\wedge\gamma) \Leftrightarrow (\alpha\vee\beta)\wedge(\alpha\vee\gamma)$. (3) If $\vdash \alpha \Rightarrow \beta$, then $\vdash \Diamond\alpha \Rightarrow \Diamond\beta$ and $\vdash \Box\alpha \Rightarrow \Box\beta$. (4) $\vdash \Box\top \Leftrightarrow \top$ and $\vdash \Diamond\bot \Leftrightarrow \bot$. (5) $\vdash \Box(\alpha\wedge\beta) \Leftrightarrow \Box\alpha \wedge \Box\beta$ and $\vdash \Diamond(\alpha\vee\beta) \Leftrightarrow \Diamond\alpha \vee \Diamond\beta$.*

Let $\mathbf{A}$ be a MKA. An *assignment* in $\mathbf{A}$ is a homomorphism $\theta : \mathscr{F} \to A$. A basic sequent $\alpha \Rightarrow \beta$ is *true* in $\mathbf{A}$ (written $\mathbf{A}, \theta \models \alpha \Rightarrow \beta$) for an assignment $\theta$ if $\theta(\alpha) \leq \theta(\beta)$ in $\mathbf{A}$. A basic sequent $s$ is *valid* in $\mathbf{A}$ (written $\mathbf{A} \models s$) if it is true for all assignment. A basic sequent $s$ is *valid* in a class of algebras (written $\mathbf{C} \models s$) if it is valid in every algebra of that class. Now we turn to relational semantics and Kripke-completeness. For any binary relations $R$ and $S$, the *composition* of $R$ and $S$ is $S \circ R$ s.t. $(x, y) \in S \circ R$ iff there is a $z$, $xRz$ and $zSy$.

**Definition 3.** *A* modal Kleene frame *is a 4-tuple $\mathfrak{F} = (W, C, \leq, R)$ where $(W, \leq)$ is a non-empty partially ordered set, and $C, R$ are binary relations on $W$ satisfying the following conditions:* (C1)$\forall w\forall u\forall w'\forall u'((w' \leq w \wedge u' \leq u \wedge wCu) \to w'Cu')$, (C2)$\forall w\forall u(wCu \to uCw)$, (C3)$\forall w\exists u(wCu \wedge \forall v(uCv \to v \leq w)$, (C4)$\forall w(wCw \vee \forall u(wCu \to u \leq w))$, (R1)$(R\circ \leq) \subseteq (\leq \circ R)$, *and* (R2)$(\leq \circ R^{-1}) \subseteq (R^{-1}\circ \leq)$.

For any $w \in W$, let $R(w) = \{u : wRu\}$ and $C(w) = \{u : wCu\}$, respectively. A *compatibility frame* is a triple $(W, \leq, C)$ satisfying (C1) (cf. [8]) Note that by (C3), $C(w) \neq \varnothing$ for any $w$. A valuation in $\mathfrak{F} = (W, C, \leq, R)$ is a function $V : Var \to UP(2^W)$ where $UP(2^W)$ is the set of all upward-closed subsets of $W$ w.r.t. $\leq$. A *modal Kleene model* is a tuple $\mathfrak{M} = (\mathfrak{F}, V)$ where $\mathfrak{F}$ is a modal Kleene frame and $V$ is a valuation. Clearly, $V$ satisfies the *hereditary condition*: If $w \in V(p)$ and $w \leq u$, then $u \in V(p)$, for any $w, u \in W$. Homomorphically, $V$ can be extended to $\mathscr{F}$ and the *truth set* for any formula is defined as follows: for any $w, u \in W$, $V(\bot) = \varnothing, V(\alpha\wedge\beta) = V(\alpha) \cap V(\beta), V({\sim}\alpha) = {\sim}_C V(\alpha), V(\Box\alpha) = \Box_R V(\alpha), V(\Diamond\alpha) = \Diamond_R V(\alpha)$ where ${\sim}_C, \Box_R.\Diamond_R$ are defined as follows: ${\sim}_C(X) = \{w \in W : C(w) \cap X = \varnothing\}, \Box_R(X) = \{w \in W : R(w) \subseteq X\}, \Diamond_R(X) = \{w \in W : R(w) \cap X \neq \varnothing\}$. For any MKL-frame $\mathfrak{F}$, let $\mathfrak{F}^+ = (UP(2^W), \cap, \cup, {\sim}_C, \Box_R, \Diamond_R, \varnothing, W)$. Let $\mathfrak{M} = (\mathfrak{F}, V)$ be a modal Kleene model. A formula $\alpha$ is *true at $w$ in $M$* (written $\mathfrak{M}, w \models \alpha$) if $w \in V(\alpha)$. For any basic sequent $\alpha \Rightarrow \beta$, it is *true at $w$ in $\mathfrak{M}$* if $\mathfrak{M}, w \models \alpha$ implies $\mathfrak{M}, w \models \beta$ i.e. $w \notin V(\alpha)$ or $w \in V(\beta)$. Definitions related to validity shall follow naturally (cf. e.g. [9]). The *sequential theory of* $\mathcal{K}$ is defined as the set $\mathsf{Th}(\mathcal{K}) = \{s \in \mathcal{BS} : \mathcal{K} \models s\}$. Let $\mathsf{Fr}(\mathcal{T}) = \{\mathfrak{F} : \mathfrak{F} \models \mathcal{T}\}$. A MKL $L$ is *Kripke-complete* if $L = \mathsf{Th}(\mathsf{Fr}(L))$. Now we turn to the discrete duality. For any MKA $\mathbf{A}$, let $PF(A)$ ($PI(A)$) be the set of all prime filters (prime ideals) of $\mathbf{A}$. For any subset $B \subseteq A$, let $\ddagger^{-1}(B) = \{a : \ddagger a \in B\}$ and $\sim(B) = \{\sim a \in A : a \in B\}$ where $\ddagger \in \{\Box, \Diamond, \sim\}$. For any $F, G \in PF(A)$, let $(F, G) \in C_A$ iff $\sim^{-1}(F) \subseteq \overline{G}$ and $(F, G) \in R_A$ iff $\Box^{-1}(F) \subseteq G \subseteq \Diamond^{-1}(F)$. Let $\mathbf{A}_+ = (PF(A), C_A, \subseteq, R_A)$.

**Lemma 2.** *Let $(A, \wedge, \vee, \sim, \Box, \Diamond, 0, 1)$ be a MKA and $F \in PF(A)$. Then (1) $\Box a \in F$ iff for all $G \in PF(A)$, $FR_AG$ implies $a \in G$; (2) $\Diamond a \in F$ iff there exists $G \in PF(A)$, $FR_AG$ and $a \in G$.*

*Proof.* For (1), the left-to-right part is trivial. Conversely, suppose $\Box a \notin F$. Let $I$ be the ideal generated by $\{a\} \cup \overline{\Diamond^{-1}(F)}$. Suppose $c \in \Box^{-1}(F) \cap I$, then $\Box c \in F$. Then $c \leq a \vee b$ for some $b \in \overline{\Diamond^{-1}(F)}$ or $c = a$ or $c \in \overline{\Diamond^{-1}(F)}$. For $c \leq a \vee b$, clearly $\Box c \leq \Box(a \vee b) \leq \Box a \vee \Diamond b$. Since $\Box a, \Diamond b \notin F$, $\Box a \vee \Diamond b \notin F$. Then $\Box c \notin F$, which is a contradiction. For $c = a$, then $\Box a \in F$, which is a contradiction. For $c \in \overline{\Diamond^{-1}(F)}$, then $\Diamond c \notin F$ and $\Box c \in F$. Then $\Box a \vee \Diamond c \notin F$, $\Box(a \vee c) \leq \Box a \vee \Diamond c$, and $\Box(a \vee c) \notin F$. Then $\Box a \vee \Box c \notin F$. However, since $\Box c \leq \Box a \vee \Box c$ and $\Box c \in F$, $\Box a \vee \Box c \in F$, which results in a contradiction. Thus $\Box^{-1}(F) \cap I = \varnothing$. By the Prime Filter Theorem for Distributive Lattice (cf. [7, pp. 235-236]), there is a $G \supseteq \Box^{-1}(F)$ s.t. $G \cap I = \varnothing$. Then for any $b \notin \Diamond^{-1}(F)$, $a \vee b \notin G$ and thus $a \notin G$. Therefore, there is a $G \in PF(A)$ s.t. $\Box^{-1}(F) \subseteq G \subseteq \Diamond^{-1}(F)$ and $a \notin G$. For (2), it can be treated similarly as (1) by $\Box a \wedge \Diamond b \leq \Diamond(a \wedge b)$.

**Corollary 1.** *(1) If $\Box^{-1}(F) \subseteq G$, then there is a $H \in PF(A)$ s.t. $FR_AH$ and $H \subseteq G$; (2) If $G \subseteq \Diamond^{-1}(F)$, then there is a $H \in PF(A)$ s.t. $FR_AH$ and $G \subseteq H$.*

**Lemma 3.** *The following hold: (1) $\mathfrak{F}^+$ is a MKA; (2) $\mathbf{A}_+$ is a MKL-frame.*

**Theorem 1 (Discrete Duality).** *(1) $h$ is an embedding from $\mathbf{A}$ to $(\mathbf{A}_+)^+$ where $h(a) = \{F \in PF(A) : a \in F\}$. (2) $k$ is an embedding from $\mathfrak{F}$ to $(\mathfrak{F}^+)_+$ where $k(w) = \{X \in UP(2^W) : w \in X\}$.*

Now we employ the canonical model method to show Kripke-completeness for $\mathsf{KMK}$ and some of its common modal extensions. For any MKL $L$, a set of formulas $\Gamma$ is a *theory* of $L$ if $\Gamma = \{\alpha \in \mathscr{F} : \Gamma \vdash_L \alpha\}$. $\Gamma$ is *consistent* if $\perp \notin \Gamma$. $\Gamma$ is a *prime theory* if it is a consistent theory and $\alpha \vee \beta \in \Gamma$ implies $\alpha \in \Gamma$ or $\beta \in \Gamma$. Let $PT(L)$ denote the set of all prime theories of $L$. Let $\ddagger^{-1}(\Gamma) = \{\alpha : \ddagger\alpha \in \Gamma\}$ where $\ddagger \in \{\Box, \Diamond, \sim\}$. A *dual theory* is a set of formulas $\Sigma = \{\alpha : \vdash \alpha \Rightarrow \bigvee \Delta\}$ where $\Delta \subseteq \Sigma$, a finite subset of $\Sigma$. Let $L$ be a MKL, the *canonical model* of $L$ is a 5-tuple $\mathfrak{M}^L = (W^L, \subseteq, C^L, R^L, V^L)$ defined as follows: for any $\Gamma, \Theta \in PT(L)$, (1) $W^L = PT(L)$, (2) $\Gamma C^L \Theta$ iff $\sim^{-1}(\Gamma) \subseteq \overline{\Theta}$, (3) $\Gamma R^L \Theta$ iff $\Box^{-1}(\Gamma) \subseteq \Theta \subseteq \Diamond^{-1}(\Gamma)$, and (4) $V^L(p) = \{\Gamma \in W^L : p \in \Gamma\}$ for any $p \in \mathbf{Var}$. The 4-tuple $\mathfrak{F}^L = (W^L, \subseteq, C^L, R^L)$ is the *canonical frame* of $L$.

**Lemma 4.** *For any $\Gamma \in PT(L)$, the followings hold: (1) $\Box\alpha \in \Gamma$ iff $\alpha \in \Theta$ for all $\Theta \in R^L(\Gamma)$; (2) $\Diamond\alpha \in \Gamma$ iff $\alpha \in \Theta$ for some $\Theta \in R^L(\Gamma)$; (3) If $\Box^{-1}(\Gamma) \subseteq \Theta$, then there is a $\Sigma \in PT(L)$ s.t. $\Gamma R^L \Sigma$ and $\Sigma \subseteq \Theta$; (4) If $\Theta \subseteq \Diamond^{-1}(\Gamma)$, then there is a $\Sigma \in PT(L)$ s.t. $\Gamma R^L \Sigma$ and $\Theta \subseteq \Sigma$.*

**Lemma 5.** *For any MKL $L$, the canonical frame $\mathfrak{F}^L$ is a MKL-frame.*

**Lemma 6 (Truth Lemma).** *For any formula $\alpha \in \mathscr{F}$, $\mathfrak{M}^L, \Gamma \models \alpha$ iff $\alpha \in \Gamma$.*

**Theorem 2.** *For any $\alpha \Rightarrow \beta$ and any MKL $L$, $\alpha \Rightarrow \beta \in L$ iff $\mathfrak{M}^L \models \alpha \Rightarrow \beta$.*

An MKL $L$ is *canonical* if $\mathfrak{F}^L \models L$. A set of basic sequents $\mathcal{S}$ is *canonical* if for any MKL $L$, $\mathcal{S} \subseteq L$ implies $L$ is canonical. Consider the following modal axioms: $(\mathrm{D}_\Box)$ $\top \Rightarrow {\sim}\Box\bot$; $(\mathrm{D}_\Diamond)$ $\top \Rightarrow \Diamond\top$; $(\mathrm{D}_\Box)$ $\Box p \Rightarrow p$; $(\mathrm{T}_\Diamond)$ $p \Rightarrow \Diamond p$; $(4_\Box)$ $\Box p \Rightarrow \Box\Box p$; $(4_\Diamond)$ $\Diamond\Diamond p \Rightarrow \Diamond p$; $(\mathrm{B}_\Box)$ $p \Rightarrow \Box\Diamond p$; $(\mathrm{B}_\Diamond)$ $\Diamond\Box p \Rightarrow p$; $(5_\Box)$ $\Diamond p \Rightarrow \Box\Diamond p$; $(5_\Diamond)$ $\Diamond\Box p \Rightarrow \Box p$. Further let $\mathsf{KMD} = \mathsf{KMK} \oplus (\mathrm{D}_\Box) \oplus (\mathrm{D}_\Diamond)$; $\mathsf{KMT} = \mathsf{KMK} \oplus (\mathrm{T}_\Box) \oplus (\mathrm{T}_\Diamond)$; $\mathsf{KM4} = \mathsf{KMK} \oplus (4_\Box) \oplus (4_\Diamond)$; $\mathsf{KMB} = \mathsf{KMK} \oplus (\mathrm{B}_\Box) \oplus (\mathrm{B}_\Diamond)$; $\mathsf{KM5} = \mathsf{KMK} \oplus (5_\Box) \oplus (5_\Diamond)$.

**Lemma 7.** *For any $\mathfrak{F} = (W, \leq, C, R)$, the followings hold: for any $w, u, v \in W$[4], (1) $\mathfrak{F} \models (\mathrm{D}_\Box) \wedge (\mathrm{D}_\Diamond)$ iff $R(w) \neq \varnothing$; (2) $\mathfrak{F} \models (\mathrm{T}_\Box) \wedge (\mathrm{T}_\Diamond)$ iff $wRw$; (3) $\mathfrak{F} \models (4_\Box) \wedge (4_\Diamond)$ iff $wRu$ and $uRv$ implies $wRv$; (4) $\mathfrak{F} \models (\mathrm{B}_\Box) \wedge (\mathrm{B}_\Diamond)$ iff $wRu$ implies $uRw$; (5) $\mathfrak{F} \models (5_\Box) \wedge (5_\Diamond)$ iff $wRu$ and $wRv$ implies $uRv$ or $vRu$.*

**Theorem 3.** $\mathsf{KMK}$, $\mathsf{KMD}$, $\mathsf{KMT}$, $\mathsf{KM4}$, $\mathsf{KMB}$, *and* $\mathsf{KM5}$ *are Kripke-complete.*

## 3 FMP for Logics of Semi-Topological Kleene Algebras

In this section, we move from the modal Kleene algebra to the semi-topological Kleene algebra. A *semi-topological Kleene algebra* (STKA) $(A, \wedge, \vee, \sim, \Diamond, 0, 1)$ is obtained by dropping (P3) of MKA defined in Definition 1. Let $\Box a := {\sim}\Diamond{\sim}a$ in any STKA and **STKA** be the variety of all STKAs. Let $\mathscr{F}' \ni \alpha ::= p \mid \bot \mid (\alpha_1 \wedge \alpha_2) \mid (\alpha_1 \vee \alpha_2) \mid {\sim}\alpha \mid \Diamond\alpha$ be the new language. Let $\top := {\sim}\bot$ and $\Box\alpha := {\sim}\Diamond{\sim}\alpha$. The smallest sequential logic for **STKA**, denoted by $\mathsf{SK}$, is obtained from $\mathsf{MKM}$ by deleting $(\mathrm{I}_\Box)$ and $(\mathrm{I}_\Diamond)$[5]. Now we introduce a calculus equivalent to $\mathsf{SK}$ w.r.t. derivability but in a different style. We define the set of all formula structures $\mathscr{S}$ as: $\mathscr{S} \ni \circ^n(\alpha) ::= \alpha \mid \circ^n(\alpha)$ where $\alpha \in \mathscr{F}'$ and $n \geq 0$. Let $\circ^n(\alpha) = \circ(\circ^{n-1}(\alpha))$ and $\circ^0(\alpha) = \alpha$. A *sequent* is an expression of the form $\circ^n(\alpha) \Rightarrow \beta$ where $\circ^n(\alpha) \in \mathscr{S}$ and $\beta \in \mathscr{F}'$.

**Definition 4.** *The sequent calculus* $\mathsf{GSK}$ *for* **STKA** *consists of the following axioms and rules: for $n, m \geq 0$ and $i = \{1, 2\}$,*

$$(\mathrm{ID})\ \alpha \Rightarrow \alpha \quad (\mathrm{DIS})\ \alpha \wedge (\beta \vee \gamma) \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma) \quad (\mathrm{KL})\ \alpha \wedge {\sim}\alpha \Rightarrow \beta \vee {\sim}\beta$$

$$(\top)\ \circ^n(\alpha) \Rightarrow \top \quad (\bot)\ \circ^n(\bot) \Rightarrow \beta \quad (\mathrm{DN1})\ \alpha \Rightarrow \neg\neg\alpha \quad (\mathrm{DN2})\ \neg\neg\alpha \Rightarrow \alpha$$

[4] Here $\mathfrak{F} \models (\mathrm{D}_\Box) \wedge (\mathrm{D}_\Diamond)$ means $\mathfrak{F} \models (\mathrm{D}_\Box)$ and $\mathfrak{F} \models (\mathrm{D}_\Diamond)$, and same for others.

[5] Since $\Box\alpha := {\sim}\Diamond{\sim}\alpha$, $(\mathrm{K}_\Box)$, $(\mathrm{N}_\Box)$, and $(\mathrm{M}_\Box)$ are actually redundant in $\mathsf{SK}$.

$$\frac{\alpha \Rightarrow \beta}{\sim\beta \Rightarrow \sim\alpha}(\mathrm{CP}) \quad \frac{\circ^n(\alpha_i) \Rightarrow \beta}{\circ^n(\alpha_1 \wedge \alpha_2) \Rightarrow \beta}(\wedge\Rightarrow) \quad \frac{\circ^n(\alpha) \Rightarrow \beta_1 \quad \circ^n(\alpha) \Rightarrow \beta_2}{\circ^n(\alpha) \Rightarrow \beta_1 \wedge \beta_2}(\Rightarrow\wedge) \quad \frac{\circ^n(\alpha_1) \Rightarrow \beta \quad \circ^n(\alpha_2) \Rightarrow \beta}{\circ^n(\alpha_1 \vee \alpha_2) \Rightarrow \beta}(\vee\Rightarrow)$$

$$\frac{\circ^n(\alpha) \Rightarrow \beta_i}{\circ^n(\alpha) \Rightarrow \beta_1 \vee \beta_2}(\Rightarrow\vee) \quad \frac{\circ^{n+1}(\alpha) \Rightarrow \beta}{\circ^n(\Diamond\alpha) \Rightarrow \beta}(\Diamond\Rightarrow) \quad \frac{\circ^n(\alpha) \Rightarrow \beta}{\circ^{n+1}(\alpha) \Rightarrow \Diamond\beta}(\Rightarrow\Diamond) \quad \frac{\circ^n(\alpha) \Rightarrow \beta \quad \circ^m(\beta) \Rightarrow \gamma}{\circ^{m+n}(\alpha) \Rightarrow \gamma}(\mathrm{Cut})$$

The notations related to GSK are similar to those in Definition 2. Observe that Lemma 1 also holds for GSK.

**Lemma 8.** $\vdash_{\mathsf{GSK}} \alpha \Rightarrow \beta$ *iff* $\vdash_{\mathsf{SK}} \alpha \Rightarrow \beta$.

Let $\mathbf{A} = (A, \sim, \Diamond)$ be a STKA. Similar to algebraic semantics for MKL, an assignment is a function $\theta : \mathscr{F}' \to \mathbf{A}$. A sequent $\vdash \circ^n(\alpha) \Rightarrow \beta$ is *true* in $\mathbf{A}$ with $\theta$ if $\Diamond^n\theta(\alpha) \leq \theta(\beta)$. The validity is defined accordingly. Observe that by an easy induction on the height of derivation, $\vdash \circ^n(\alpha) \Rightarrow \beta$ iff $\vdash \Diamond^n(\alpha) \Rightarrow \beta$. Let GSS4 be the S4-extensions of GSK by adding following rules: for $n \geq 0$,

$$\frac{\circ^{n+1}(\top) \Rightarrow \beta}{\circ^n(\top) \Rightarrow \beta}(\mathrm{D}) \quad \frac{\circ^{n+1}(\alpha) \Rightarrow \beta}{\circ^n(\alpha) \Rightarrow \beta}(\mathrm{T}) \quad \frac{\circ^{n+1}(\alpha) \Rightarrow \beta}{\circ^{n+2}(\alpha) \Rightarrow \beta}(4)$$

Let GSD, GST and GS4 be the D, T, and 4-extensions of GSK, respectively, obtained by adding the corresponding rule to GSK. We call the strongest algebra i.e. STKA satisfying $1 \leq \Diamond 1$, $a \leq \Diamond a$, and $\Diamond\Diamond a \leq \Diamond a$ the *topological Kleene algebra* (TKA).

**Lemma 9.** *(1)* $\vdash_{\mathsf{GSD}} \top \Rightarrow \Diamond\top$*; (2)* $\vdash_{\mathsf{GST}} \alpha \Rightarrow \Diamond\alpha$*; (3)* $\vdash_{\mathsf{GS4}} \Diamond\Diamond\alpha \Rightarrow \Diamond\alpha$.

*Proof.* We take (3) as an example. By $(\Rightarrow\Diamond)$ on (ID), $\vdash \circ(\alpha) \Rightarrow \Diamond\alpha$. By rule (4), $\vdash \circ^2(\alpha) \Rightarrow \Diamond\alpha$. By $(\Diamond\Rightarrow)$ twice, $\vdash \Diamond\Diamond\alpha \Rightarrow \Diamond\alpha$.

Now we show the FMP for GSK and its extensions with any combinations of D, T, and 4. Our proof strategy is to show that FMP holds for the strongest logic i.e. GSS4, then other weaker variants' proof can be obtained modularly by simply ignoring the irrelevant cases throughout the proof. We should show that if $\nvdash_{\mathsf{GSS4}} \circ^n(\alpha) \Rightarrow \beta$, then there is a finite TKA $\mathbf{A}$ s.t. $\mathbf{A} \not\models \circ^n(\alpha) \Rightarrow \beta$. For any set of formulas $\mathscr{T}$, a formula structure $\circ^n(\alpha)$ is a $\mathscr{T}$-formula structure if $\alpha \in \mathscr{T}$. Let $\mathscr{S}(\mathscr{T})$ be the set of all $\mathscr{T}$-formula structures. A sequent $\circ^n(\alpha) \Rightarrow \beta$ is a $\mathscr{T}$-sequent if $\alpha, \beta \in \mathscr{T}$. Let $\vdash \circ^n(\alpha) \Rightarrow_{\mathscr{T}} \beta$ denote the derivation of $\vdash \circ^n(\alpha) \Rightarrow \beta$ s.t. all sequents appearing in it are $\mathscr{T}$-sequents. For any set of formulas $\mathscr{T}$, let $c(\mathscr{T})$ denote the closure of $\mathscr{T}$ under $\{\bot, \wedge, \vee, \sim\}$ and taking subformulas. Throughout this section, let $\mathscr{T} = c(\mathscr{T})$.

**Lemma 10 (Interpolation).** *In* GSS4*, for any* $n \geq 1, m \geq 0$*, if* $\vdash \circ^{m+n}(\alpha) \Rightarrow_{\mathscr{T}} \beta$*, then there is a* $\gamma \in \mathscr{T}$ *s.t.* $\vdash \circ^n(\alpha) \Rightarrow_{\mathscr{T}} \gamma$, $\vdash \circ^m(\gamma) \Rightarrow_{\mathscr{T}} \beta$*, and* $\vdash \circ(\gamma) \Rightarrow_{\mathscr{T}} \gamma$.

*Proof.* Proofs are similar to [15, Lemma 5].

A formula $\alpha \in \mathscr{F}'$ is called a *letter* if $\alpha \in \mathbf{Var} \cup \{\top, \bot\}$ or $\alpha = \Diamond\beta$ for some $\beta \in \mathscr{F}'$. Let *le* be the set of all letters. A formula $\alpha$ is called a *literal* if $\alpha \in le$ or $\alpha = \sim\beta$ for some $\beta \in le$. Let *li* be the set of all literals. A formula is in *disjunctive normal form* (DNF) if it is of the form $\bigvee_{i \leq n} \bigwedge_{j \leq k} \varphi_{ij}$ where each $\varphi_{ij} \in li$. Let $\mathscr{T}_{li} \subseteq \mathscr{T}$ be the set of all literals in $\mathscr{T}$. $\mathscr{T}$ is *finitely based* if $\mathscr{T}_{li}$ is finite. Clearly, there is a DNF formula $\beta \in \mathscr{T}$ s.t. $\alpha$ is equivalent to $\beta$. Apart from the repetition, there exists a unique formula in DNF equivalent to $\alpha$, which is denoted by $DNF_{\mathscr{T}}(\alpha)$. Let $DNF(\mathscr{T}) = \{DNF_{\mathscr{T}}(\alpha) \mid \alpha \in \mathscr{T}\}$. If $\mathscr{T}_{li}$ is finite, then $DNF(\mathscr{T})$ is finite. Observe that by Lemma 10, if $\vdash \circ^{n+m}(\alpha) \Rightarrow_{\mathscr{T}} \beta$, then there is a $\gamma \in DNF(\mathscr{T})$ s.t. $\vdash \circ^n(\alpha) \Rightarrow_{\mathscr{T}} \gamma$ and $\vdash \circ^m(\gamma) \Rightarrow_{\mathscr{T}} \beta$. Now we define an order on $\mathscr{S}(\mathscr{T})$: for any $\circ^n(\alpha), \circ^m(\beta) \in \mathscr{S}(\mathscr{T})$, $\circ^k(-)$, and $\varphi \in \mathscr{T}$:$\circ^n(\alpha) \trianglelefteq \circ^m(\beta)$ iff $\vdash \circ^k(\circ^m(\beta)) \Rightarrow_{\mathscr{T}} \varphi$ implies $\vdash \circ^k(\circ^n(\alpha)) \Rightarrow_{\mathscr{T}} \varphi$. Observe that $\trianglelefteq$ is a preorder i.e. reflexive and transitive. Let $\circ^n(\alpha) \approx_{\mathscr{T}} \circ^m(\beta)$ be $\circ^n(\alpha) \trianglelefteq \circ^m(\beta)$ and $\circ^m(\beta) \trianglelefteq \circ^n(\alpha)$. Clearly $\approx_{\mathscr{T}}$ is an equivalence relation. Let $[\alpha]_{\mathscr{T}} = \{\circ^m(\beta) \mid \circ^m(\beta) \approx_{\mathscr{T}} \alpha \;\&\; \circ^m(\beta) \in \mathscr{S}(\mathscr{T})\}$ and $[\mathscr{T}] = \{[\alpha]_{\mathscr{T}} \mid \alpha \in \mathscr{T})\}$. Since $[\alpha] = [DNF_{\mathscr{T}}(\alpha)]$ and the number of $[DNF_{\mathscr{T}}(\alpha)]$ is finite, $[\mathscr{T}]$ is finite.

**Lemma 11.** *For any $\circ(\alpha) \in \mathscr{S}(\mathscr{T})$, there is a $\beta \in DNF(\mathscr{T})$ s.t. $\circ(\alpha) \approx \beta$.*

Now we construct a quotient algebra. Let $\mathscr{T}$ be finitely based. The quotient algebra of $[\mathscr{T}]$ is a structure $\mathbf{Q} = ([\mathscr{T}], \wedge^\star, \vee^\star, \sim^\star, \Diamond^\star, \bot^\star, \top^\star)$ where operations $\bot^\star, \wedge^\star, \vee^\star, \sim^\star, \Diamond^\star$ in $[\mathscr{T}]$ are defined as: $\bot^\star = [\bot]_{\mathscr{T}}, \sim^\star[\alpha] = [\sim\alpha], [\alpha] \wedge^\star [\beta] = [\alpha \wedge \beta], [\alpha] \vee^\star [\beta] = [\alpha \vee \beta], \Diamond^\star[\alpha] = [\gamma]$ *s.t.* $\gamma \approx \circ(\alpha)$. where $\gamma = DNF_{\mathscr{T}}(\circ(\alpha))$. Let $[\alpha] \leq^\star [\beta]$ be defined as $[\alpha] \wedge^\star [\beta] = [\alpha]$ and $\top^* = \sim^* \bot^*$. By Lemma 11, $[\gamma]$ exists and is unique. Observe that the above operations are well-defined.

**Lemma 12.** *The followings are equivalent: (1) $\alpha \trianglelefteq \beta$; (2) $\vdash \alpha \Rightarrow_{\mathscr{T}} \beta$; (3) $[\alpha] \leq^\star [\beta]$.*

**Lemma 13.** *The following conditions hold for $\mathbf{Q}$: for any $[\alpha], [\beta], [\gamma] \in [\mathscr{T}]$,*

(1) $[\gamma] \leq^\star [\alpha] \wedge^\star [\beta]$ *iff* $[\gamma] \leq^\star [\alpha]$ *and* $[\gamma] \leq^\star [\beta]$.
(2) $[\alpha] \vee^\star [\beta] \leq^\star [\gamma]$ *iff* $[\alpha] \leq^\star [\gamma]$ *and* $[\beta] \leq^\star [\gamma]$.
(3) $[\alpha] \wedge^\star ([\beta] \vee^\star [\gamma]) = ([\alpha] \wedge^\star [\beta]) \vee^\star ([\alpha] \wedge^\star [\gamma])$.
(4) $\bot^\star \leq^\star [\alpha] \leq^\star \top^\star$, $\sim^\star\sim^\star[\alpha] = [\alpha]$, *and* $\sim^\star \bot^\star = \top^\star$.
(5) *If* $[\alpha] \leq^\star [\beta]$, *then* $\Diamond^\star[\alpha] \leq^\star \Diamond^\star[\beta]$.
(6) $\sim^\star([\alpha] \vee^\star [\beta]) = \sim^\star[\alpha] \wedge^\star \sim^\star[\beta]$ *and* $[\alpha] \wedge^\star \sim^\star[\alpha] \leq^\star [\beta] \vee^\star \sim^\star[\beta]$.
(7) $\Diamond^\star \bot^\star = \bot^\star$.
(8) $\Diamond^\star([\alpha] \vee [\beta]) = \Diamond^\star[\alpha] \vee \Diamond^\star[\beta]$.
(9) $[\alpha] \leq^\star \Diamond^\star[\alpha]$.
(10) $\top^\star \leq^\star \Diamond^\star \top^\star$.
(11) $\Diamond^\star \Diamond^\star[\alpha] \leq^\star \Diamond^\star[\alpha]$.

*Proof.* Items (1)-(4) and (6) are shown easily by Lemma 12 and Lemma 9. For (5), suppose $[\alpha] \leq^\star [\beta]$. Let $\Diamond^\star[\alpha] = [\gamma_1]$ and $\Diamond^\star[\beta] = [\gamma_2]$ s.t. $\circ(\alpha) \approx \gamma_1$ and $\circ(\beta) \approx \gamma_2$. By Lemma 12, $\vdash \alpha \Rightarrow_{\mathscr{T}} \beta$ and $\vdash \circ(\beta) \Rightarrow_{\mathscr{T}} \gamma_2$. By (Cut), $\vdash \circ(\alpha) \Rightarrow_{\mathscr{T}} \gamma_2$. Therefore, $\Diamond^\star[\alpha] \leq^\star \Diamond^\star[\beta]$. For (7), suppose $\Diamond^\star[\bot] = [\gamma]$ s.t. $\circ(\bot) \approx \gamma$. By $(\bot)$, $\vdash \circ(\bot) \Rightarrow_{\mathscr{T}} \bot$. Then $\vdash \gamma \Rightarrow_{\mathscr{T}} \bot$ and by Lemma 12, $\Diamond^\star \bot^\star \leq^\star \bot^\star$. The other direction is easy to check. For (8), suppose $\Diamond^\star[\alpha] = [\theta_1]$, $\Diamond^\star[\beta] = [\theta_2]$, and $\Diamond^\star[\alpha \vee \beta] = [\theta_3]$ s.t. $\circ(\alpha) \approx \theta_1$, $\circ(\beta) \approx \theta_2$, and $\circ(\alpha \vee \beta) \approx \theta_3$. By Lemma 12, (i) $\vdash \circ(\alpha) \Rightarrow_{\mathscr{T}} \theta_1$, (ii) $\vdash \circ(\beta) \Rightarrow_{\mathscr{T}} \theta_2$, and (iii) $\vdash \circ(\alpha \vee \beta) \Rightarrow_{\mathscr{T}} \theta_3$. By $(\vee\Rightarrow)$ and $(\Rightarrow\vee)$ on (i) and (ii), $\vdash \circ(\alpha \vee \beta) \Rightarrow_{\mathscr{T}} \theta_1 \vee \theta_2$. Thus $\circ(\alpha \vee \beta) \trianglelefteq \theta_1 \vee \theta_2$. Since $\theta_3 \trianglelefteq \circ(\alpha \vee \beta)$ and $\trianglelefteq$ is transitive, $\theta_3 \trianglelefteq \theta_1 \vee \theta_2$. By Lemma 12, $\Diamond^\star([\alpha] \vee [\beta]) \leq^\star \Diamond^\star[\alpha] \vee \Diamond^\star[\beta]$. Conversely, by item (6) on $[\alpha] \leq^\star [\alpha] \vee [\beta]$, $\Diamond^\star[\alpha] \leq^\star \Diamond^\star([\alpha] \vee [\beta])$. Similarly one has $\Diamond^\star[\beta] \leq^\star \Diamond^\star([\alpha] \vee [\beta])$. By item (2), $\Diamond^\star[\alpha] \vee^\star \Diamond^\star[\beta] \leq^\star \Diamond^\star([\alpha] \vee [\beta])$. For (9), suppose $\Diamond^\star[\alpha] = [\gamma]$ s.t. $\circ(\alpha) \approx \gamma$. By Lemma 12, $\vdash \circ(\alpha) \Rightarrow_{\mathscr{T}} \gamma$. By (T), $\vdash \alpha \Rightarrow_{\mathscr{T}} \gamma$. By Lemma 12, $[\alpha] \leq^\star \Diamond^\star[\alpha]$. The proof of (10) is similar. For (11), suppose $\Diamond^\star[\alpha] = [\gamma_1]$ and $\Diamond^\star[\gamma_1] = [\gamma_2]$ s.t. $\circ(\alpha) \approx \gamma_1$ and $\circ(\gamma_1) \approx \gamma_2$. By Lemma 12, $\vdash \circ(\alpha) \Rightarrow_{\mathscr{T}} \gamma_1$. By the rule (4), $\vdash \circ^2(\alpha) \Rightarrow_{\mathscr{T}} \gamma_1$. Since $\circ(\alpha) \approx \gamma_1$ and $\circ(\gamma_1) \approx \gamma_2$, $\vdash \gamma_2 \Rightarrow_{\mathscr{T}} \gamma_1$. By Lemma 12, $\Diamond^\star \Diamond^\star[\alpha] \leq^\star \Diamond^\star[\alpha]$.

**Corollary 2.** *If $\mathscr{T}$ is finitely based,* $\mathbf{Q}$ *is a finite topological Kleene algebra.*

**Lemma 14.** *If $\Diamond\alpha \in \mathscr{T}$, then $\Diamond^\star[\alpha] = [\Diamond\alpha]$.*

The sequential logic of $\mathsf{GSS4}$ is denoted as $\mathsf{SS4}$, others are defined similarly.

**Lemma 15.** *If $\nvdash_{\mathsf{GSS4}} \circ^n(\alpha) \Rightarrow \beta$, then there is a finite topological Kleene algebra* $\mathbf{Q}$ *s.t.* $\mathbf{Q} \not\models \circ^n(\alpha) \Rightarrow \beta$.

**Theorem 4.** $\mathsf{SK}$*,* $\mathsf{SKD}$*,* $\mathsf{SKT}$*,* $\mathsf{SK4}$*,* $\mathsf{SKD4}$*, and* $\mathsf{SS4}$ *have FMP and are decidable.*

## References

1. Balbes, R., Dwinger, P.: Distributive Lattices. University of Missouri Press (1977)
2. Banerjee, M., Chakraborty, M.K.: Rough sets through algebraic logic. Fundamenta Informaticae **28**(3-4), 211–221 (1996)
3. Białynicki-Birula, A., Rasiowa, H.: On the representation of quasi-Boolean algebras. Bulletin of the Polish Academy of Sciences **Cl. III 5**(3), 259–261 (1957)
4. Cignoli, R.: Boolean elements in łukasiewicz algebras i. In: Proceedings of the Japan Academy. pp. 670–675 (1965)
5. Cignoli, R.: The class of Kleene algebras satisfying an interpolation property and Nelson algebras. Algebra universalis **23**, 262–292 (1986)
6. Cleave, J.P.: Quasi-Boolean algebras, empirical continuity and three-valued logic. Mathematical Logic Quarterly **22**(1), 481–500 (1976)
7. Davey, B.A., Priestley, H.A.: Introduction to Lattices and Order. Cambridge University Press (2002)
8. Dunn, J.M.: Star and perp: Two treatments of negation. Philosophical Perspectives **7**, 331–357 (1993)
9. Dunn, J.M.: Positive modal logic. Studia Logica **55**(2), 301–317 (1995)
10. Dunn, J.M., Zhou, C.: Negation in the context of gaggle theory. Studia Logica **80**, 235–264 (2005)
11. Kalman, J.A.: Lattices with involution. Transactions of the American Mathematical Society **87**(2), 485–491 (1958)
12. Kumar, A., Banerjee, M.: Kleene algebras and logic: Boolean and rough set representations, 3-valued, rough set and perp semantics. Studia Logica **105**(3), 439–469 (2017)
13. Lin, Z.: Non-associative Lambek calculus with modalities: interpolation, complexity and FEP. Logic Journal of the IGPL **22**(3), 494–512 (2014)
14. Muškardin, V.: Representation theorem for finite quasi-Boolean algebras. Annales scientifiques de l'Université de Clermont. Mathématiques **60**(13), 117–128 (1976)
15. Peng, Y., Wang, Y.: On the finite model property of non-normal modal logics. In: International Workshop on Logic, Rationality and Interaction. pp. 207–221. Springer (2023)
16. Rasiowa, H.: An Algebraic Approach to Non-classical Logics. North-Holland Publishing Company, Amsterdam (1974)
17. Saha, A., Sen, J., Chakraborty, M.K.: Algebraic structures in the vicinity of pre-rough algebra and their logics I. Information Sciences **282**, 296–320 (2014)
18. Sardar, M.R., Chakraborty, M.K.: Rough set models of some abstract algebras close to pre-rough algebra. Information Sciences **621**, 104–118 (2023)
19. Shannon, C.E., McCarthy, J.: Automata Studies. Princeton University Press (1956)