On Residual Co-implications Derived from q-Rung Orthopair Fuzzy t-Conorms

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Abstract. In this paper, we consider q-rung orthopair fuzzy (q-ROF) residual co-implications induced by q-ROF t-conorms. The expression corresponding to representable q-ROF t-conorms is provided explicitly. Moreover, the q-ROF residual implications and co-implications are employed to construct inclusion measures between q-ROF sets.

Keywords: Inclusion measures · Residual implications · Residual co-implications · q-rung orthopair fuzzy t-conorms.

1 Introduction

Fuzzy implications [1] serve as a fundamental mechanism for modeling IF-THEN rules under uncertainty and vagueness. They quantify the truth values of fuzzy propositions, which makes them essential for processing human-like and linguistic rules. Over the past several decades, fuzzy implications have been widely used in approximate reasoning, decision-making, control systems, etc.

There are many classes of fuzzy implications with different properties to meet different requirements. Residual implications [2] are an important type of fuzzy implications obtained by the residuation property. De Baets [6] proposed residual co-implications, the dual connectives of residual implications, on a bounded ordered set. Ruiz and Torrens [12] studied the residual co-implications defined from idempotent uninorms. Wang and Fang [15] introduced the residual co-implications of left and right uninorms on a complete lattice. Su et al. [13,14] investigated the residual co-implications generated from pseudo-uninorms. Dai et al. [5] presented fuzzy co-implications on the poset of closed intervals.

Cornelis et al. [3] introduced some intuitionistic fuzzy implications, residual implications included, in intuitionistic fuzzy set environments. Hu and Wong [10] presented the interval-valued fuzzy residual implications and co-implications. Recently, Xie et al. [16] defined two kinds of q-ROF residual implications derived from q-ROF t-norms and q-ROF overlap functions. Unfortunately, the q-ROF residual co-implications are once again neglected in the existing study. In this paper, we will fill this gap and propose residual co-implications within the q-ROF framework.

Fuzzy implications are suggested for building inclusion measures in the fuzzy set community. Grzegorzewski and Mrówka [9] proposed a subsethood measure of intuitionistic fuzzy sets based on the Hamming distance. The resulted subsethood is a real number in $[0, 1]$, which may lead to anomalies in some cases. Cornelis and Kerre [4] presented intuitionistic fuzzy inclusion measures whose results are intuitionistic fuzzy values. Following this line of thought, by using q-ROF residual implications and co-implications, two types of q-ROF inclusion degrees are developed in this paper.

In Sect. 2, we review basic concepts of t-(co)norms, residual (co-)implications, q-ROF t-(co)norms and q-ROF residual implications. In Sect. 3, we introduce residual co-implications on q-ROF values (q-ROFVs) and provide the explicit expression for representable q-ROF t-conorms. In Sect. 4, we propose two inclusion measures between q-ROF sets (q-ROFSs) by the use of q-ROF residual implications and co-implications. In Sect. 5, we conclude this paper.

2 Preliminaries

In this section, we mainly recall some well-known fuzzy connectives and some operations on q-ROFVs.

An ordered pair $\langle \mu, \nu \rangle$ is called a q-ROFV [8,17] if

$$0 \leq \mu, \nu \leq 1 \text{ and } 0 \leq \mu^q + \nu^q \leq 1, \tag{1}$$

where $q \geq 1$. Throughout this paper, the set of all q-ROFVs is denoted by Δ. The order relation in Δ is defined as

$$\langle \mu_1, \nu_1 \rangle \preceq \langle \mu_2, \nu_2 \rangle \text{ iff } \mu_1 \leq \mu_2, \nu_1 \geq \nu_2. \tag{2}$$

Obviously, the least element and the greatest element in Δ associated with relation $\preceq$ are $\langle 0, 1 \rangle$ and $\langle 1, 0 \rangle$, respectively.

Similar to t-(co)norms on $[0, 1]$ [11], one can define t-(co)norms on q-ROFVs with respect to order $\preceq$. In this paper, we focus on representable [7] q-ROF t-norms $\mathcal{T}$ and t-conorms $\mathcal{S}$. Concretely, there exists a pair of t-norm T and t-conorm S such that for all $\mathbb{x} = \langle \mu_1, \nu_1 \rangle, \mathbb{y} = \langle \mu_2, \nu_2 \rangle \in \Delta$,

$$\mathcal{T}(\mathbb{x}, \mathbb{y}) = \left\langle \sqrt[q]{T\left(\mu_1^q, \mu_2^q\right)}, \sqrt[q]{S\left(\nu_1^q, \nu_2^q\right)} \right\rangle, \tag{3}$$

$$\mathcal{S}(\mathbb{x}, \mathbb{y}) = \left\langle \sqrt[q]{S\left(\mu_1^q, \mu_2^q\right)}, \sqrt[q]{T\left(\nu_1^q, \nu_2^q\right)} \right\rangle, \tag{4}$$

which are shortly denoted as $\mathcal{T} = (T, S)$ and $\mathcal{S} = (S, T)$, respectively. In what follows, we always assume that T is a left-continuous t-norm, and its dual t-conorm S is right-continuous. In addition, the standard q-ROF negation [17] is $\mathcal{N}(\langle \mu, \nu \rangle) = \langle \nu, \mu \rangle$ for all $\langle \mu, \nu \rangle \in \Delta$.

Example 1. The representable q-ROF t-(co)norms produced by some popular t-(co)norms are as follows: for $\mathbb{x} = \langle \mu_1, \nu_1 \rangle, \mathbb{y} = \langle \mu_2, \nu_2 \rangle \in \Delta$,

$$\mathcal{T}_{\mathcal{M}}(\mathbb{x}_1, \mathbb{x}_2) = \langle \mu_1 \wedge \mu_2, \nu_1 \vee \nu_2 \rangle,$$

$$\begin{aligned}
\mathcal{S}_{\mathcal{M}}(\mathbb{x}_1, \mathbb{x}_2) &= \langle \mu_1 \vee \mu_2, \nu_1 \wedge \nu_2 \rangle. \\
\mathcal{T}_{\mathcal{P}}(\mathbb{x}_1, \mathbb{x}_2) &= \left\langle \mu_1\mu_2, \sqrt[q]{\nu_1^q + \nu_2^q - \nu_1^q\nu_2^q} \right\rangle, \\
\mathcal{S}_{\mathcal{P}}(\mathbb{x}_1, \mathbb{x}_2) &= \left\langle \sqrt[q]{\mu_1^q + \mu_2^q - \mu_2^q\mu_1^q}, \nu_1\nu_2 \right\rangle. \\
\mathcal{T}_{\mathcal{L}}(\mathbb{x}_1, \mathbb{x}_2) &= \left\langle \sqrt[q]{\max\left(\mu_1^q + \mu_2^q - 1, 0\right)}, \sqrt[q]{\min\left(\nu_1^q + \nu_2^q, 1\right)} \right\rangle, \\
\mathcal{S}_{\mathcal{L}}(\mathbb{x}_1, \mathbb{x}_2) &= \left\langle \sqrt[q]{\min\left(\mu_1^q + \mu_2^q, 1\right)}, \sqrt[q]{\max\left(\nu_1^q + \nu_2^q - 1, 0\right)} \right\rangle.
\end{aligned}$$

Definition 1. *[16] Let $\mathcal{T}$ be a q-ROF t-norm. The residual implication derived from $\mathcal{T}$ is as follows: for $\mathbb{x}, \mathbb{y} \in \Delta$,*

$$\Theta_{\mathcal{T}}(\mathbb{x}, \mathbb{y}) = \sup\{\mathbb{z} \in \Delta : \mathcal{T}(\mathbb{x}, \mathbb{z}) \preceq \mathbb{y}\}. \tag{5}$$

Theorem 1. *[16] Let $\mathbb{x} = \langle \mu_1, \nu_1 \rangle$, $\mathbb{y} = \langle \mu_2, \nu_2 \rangle$ be two q-ROFVs and $\mathcal{T}$ be a representable q-ROF t-norm determined by (T, S). Then we have*

$$\Theta_{\mathcal{T}}(\mathbb{x}, \mathbb{y}) = \left\langle \sqrt[q]{\theta_T\left(\mu_1^q, \mu_2^q\right) \bigwedge \theta_T\left(1 - \nu_1^q, 1 - \nu_2^q\right)}, \sqrt[q]{\psi_S\left(\nu_1^q, \nu_2^q\right)} \right\rangle, \tag{6}$$

where θ_T and ψ_S are residual implication and residual co-implication derived respectively from T and S. These two operators are defined as: for $x, y \in [0, 1]$,

$$\begin{aligned}
\theta_T(x, y) &= \sup\{z \in [0, 1] : T(x, z) \leq y\}, \\
\psi_S(x, y) &= \inf\{z \in [0, 1] : S(x, z) \geq y\}.
\end{aligned}$$

3 Residual Co-implications on q-ROFVs

In this section, the q-ROF residual co-implications are generated by q-ROF t-conorms and the expression is given for the representable case.

Definition 2. *Let $\mathcal{S}$ be a q-ROF t-conorm. The mapping $\Psi_{\mathcal{S}} : \Delta^2 \to \Delta$ is defined by: $\mathbb{x}, \mathbb{y} \in \Delta$,*

$$\Psi_{\mathcal{S}}(\mathbb{x}, \mathbb{y}) = \inf\{\mathbb{z} \in \Delta : \mathcal{S}(\mathbb{x}, \mathbb{z}) \succeq \mathbb{y}\}. \tag{7}$$

By the monotonicity of $\mathcal{S}$, clearly, we have $\Psi_{\mathcal{S}}(\mathbb{x}, \mathbb{y})$ decreases as $\mathbb{x}$ increases, while increases as $\mathbb{y}$ increases. For some specific q-ROFVs, we have $\Psi_{\mathcal{S}}(\langle 0, 1 \rangle, \langle 0, 1 \rangle) = \langle 0, 1 \rangle$, $\Psi_{\mathcal{S}}(\langle 1, 0 \rangle, \langle 1, 0 \rangle) = \langle 0, 1 \rangle$ and $\Psi_{\mathcal{S}}(\langle 0, 1 \rangle, \langle 1, 0 \rangle) = \langle 1, 0 \rangle$. Therefore, operation $\Psi_{\mathcal{S}}$ is indeed a residual co-implication. Furthermore, we have $\Psi_{\mathcal{S}}(\langle 1, 0 \rangle, \mathbb{x}) = \Psi_{\mathcal{S}}(\mathbb{x}, \langle 0, 1 \rangle) = \langle 0, 1 \rangle$ for all $\mathbb{x} \in \Delta$.

Theorem 2. *Let $\mathbb{x} = \langle \mu_1, \nu_1 \rangle$, $\mathbb{y} = \langle \mu_2, \nu_2 \rangle$ be two q-ROFVs and $\mathcal{S} = (S, T)$ be a representable q-ROF t-conorm. Then, the q-ROF residual co-implication $\Psi_{\mathcal{S}}(\mathbb{x}, \mathbb{y})$ can be expressed as*

$$\Psi_{\mathcal{S}}(\mathbb{x}, \mathbb{y}) = \left\langle \sqrt[q]{\psi_S\left(\mu_1^q, \mu_2^q\right)}, \sqrt[q]{\theta_T\left(\nu_1^q, \nu_2^q\right) \bigwedge \theta_T\left(1 - \mu_1^q, 1 - \mu_2^q\right)} \right\rangle. \tag{8}$$

Proof. From Definition 2, it follows that

$$\begin{aligned}\Psi_{\mathcal{S}}(\mathbb{x},\mathbb{y}) &= \inf\{\mathbb{z}\in\Delta : \mathcal{S}(\mathbb{x},\mathbb{z})\succeq\mathbb{y}\}\\ &= \inf\{\mathbb{z}=\langle\mu,\nu\rangle\in\Delta : S\big(\mu_1^q,\mu^q\big)\geq\mu_2^q, T\big(\nu_1^q,\nu^q\big)\leq\nu_2^q\}\\ &= \inf\{(\mu,\nu)\in[0,1]^2 : \mu^q\geq\psi_S\big(\mu_1^q,\mu_2^q\big), \nu^q\leq\theta_T\big(\nu_1^q,\nu_2^q\big), \mu^q+\nu^q\leq 1\}\\ &= \Big\langle \sqrt[q]{\psi_S\big(\mu_1^q,\mu_2^q\big)}, \sqrt[q]{\theta_T\big(\nu_1^q,\nu_2^q\big)\bigwedge\theta_T\big(1-\mu_1^q,1-\mu_2^q\big)}\Big\rangle.\end{aligned}$$

For a better understanding of the proof, the feasible regions for two cases and their respective minimal element are displayed in Fig. 1. It can be seen that Eq. (8) cannot be written incorrectly as:

$$\Psi_{\mathcal{S}}(\mathbb{x},\mathbb{y}) = \Big\langle \sqrt[q]{\psi_S\big(\mu_1^q,\mu_2^q\big)}, \sqrt[q]{\theta_T\big(\nu_1^q,\nu_2^q\big)}\Big\rangle.$$

The second component of non-membership in Eq. (8) can ensure the result is always a q-ROFV.

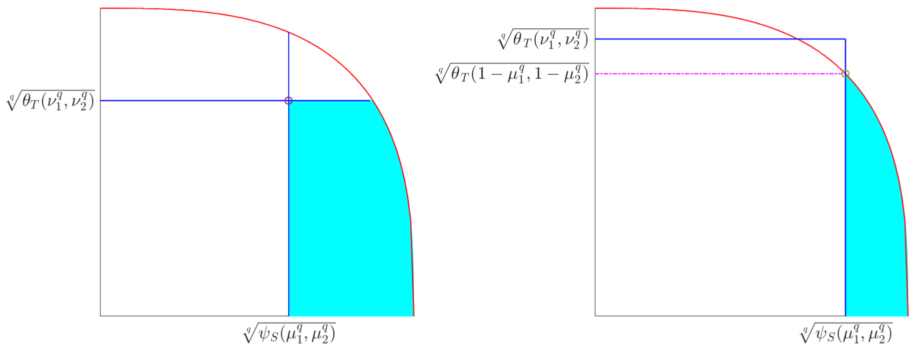

Fig. 1. Two cases of feasible regions and minimal elements in the proof of Theorem 2.

Example 2. For nilpotent minimum $T_{\mathrm{n}M}$ and nilpotent maximum $S_{\mathrm{n}M}$: $T_{\mathrm{n}M}(x,y) = \begin{cases}0, & \text{if } x+y\leq 1,\\ x\wedge y, & \text{otherwise,}\end{cases}$ and $S_{\mathrm{n}M}(x,y) = \begin{cases}1, & \text{if } x+y\geq 1,\\ x\vee y, & \text{otherwise,}\end{cases}$ their residual implication and co-implication are

$$\theta_{T_{\mathrm{n}M}}(x,y) = \begin{cases}1, & \text{if } x\leq y,\\ \max(1-x,y), & \text{if } x>y,\end{cases}$$

$$\psi_{S_{\mathrm{n}M}}(x,y) = \begin{cases}\min(1-x,y), & \text{if } x<y,\\ 0, & \text{if } x\geq y.\end{cases}$$

If one construct q-ROF t-conorm $\mathcal{S}_{\mathrm{n}\mathcal{M}}$ by $T_{\mathrm{n}M}$ and $S_{\mathrm{n}M}$, i.e., $\mathcal{S}_{\mathrm{n}\mathcal{M}} = (S_{\mathrm{n}M}, T_{\mathrm{n}M})$, its residual co-implication $\Psi_{\mathcal{S}_{\mathrm{n}\mathcal{M}}}$ is computed as follows.

- If $\mu_1 < \mu_2, \nu_1 \leq \max(\mu_2, \nu_2)$, then either $\mu_2 \geq \max(\nu_1, \nu_2)$ or $\nu_2 \geq \max(\mu_2, \nu_1)$ holds. By $\mu_1 < \mu_2$, we always have $\psi^{1/q}_{S_{\mathrm{n}M}}\left(\mu_1^q, \mu_2^q\right) = \min\left(\sqrt[q]{1-\mu_1^q}, \mu_2\right)$ and $\theta^{1/q}_{T_{\mathrm{n}M}}\left(1-\mu_1^q, 1-\mu_2^q\right) = \max\left(\mu_1, \sqrt[q]{1-\mu_2^q}\right)$. Moreover, if $\mu_2 \geq \max(\nu_1, \nu_2)$, it follows that $\theta^{1/q}_{T_{\mathrm{n}M}}\left(\nu_1^q, \nu_2^q\right) = 1$ if $\nu_1 \leq \nu_2$. Otherwise, $\theta^{1/q}_{T_{\mathrm{n}M}}\left(\nu_1^q, \nu_2^q\right) = \max\left(\sqrt[q]{1-\nu_1^q}, \nu_2\right)$. Clearly, we have $\mu_1 \leq \sqrt[q]{1-\nu_1^q}$ and $\sqrt[q]{1-\mu_2^q} \leq \sqrt[q]{1-\nu_1^q}$, and hence $\max\left(\mu_1, \sqrt[q]{1-\mu_2^q}\right) \leq \sqrt[q]{1-\nu_1^q} \leq \max\left(\sqrt[q]{1-\nu_1^q}, \nu_2\right)$. While if $\nu_2 \geq \max(\mu_2, \nu_1)$, then $\theta^{1/q}_{T_{\mathrm{n}M}}\left(\nu_1^q, \nu_2^q\right) = 1$. Therefore, in all subcases, we have $\theta^{1/q}_{T_{\mathrm{n}M}}\left(\nu_1^q, \nu_2^q\right) \geq \theta^{1/q}_{T_{\mathrm{n}M}}\left(1-\mu_1^q, 1-\mu_2^q\right)$.
  Thus, $\Psi_{\mathcal{S}_{\mathrm{n}\mathcal{M}}}(\mathbb{x}, \mathbb{y}) = \left\langle \min\left(\sqrt[q]{1-\mu_1^q}, \mu_2\right), \max\left(\mu_1, \sqrt[q]{1-\mu_2^q}\right)\right\rangle$.
- If $\mu_1 < \mu_2, \nu_1 > \max(\mu_2, \nu_2)$, then $\psi^{1/q}_{S_{\mathrm{n}M}}\left(\mu_1^q, \mu_2^q\right) = \min\left(\sqrt[q]{1-\mu_1^q}, \mu_2\right)$, $\theta^{1/q}_{T_{\mathrm{n}M}}\left(\nu_1^q, \nu_2^q\right) = \max\left(\sqrt[q]{1-\nu_1^q}, \nu_2\right)$, $\theta^{1/q}_{T_{\mathrm{n}M}}\left(1-\mu_1^q, 1-\mu_2^q\right) = \max\left(\mu_1, \sqrt[q]{1-\mu_2^q}\right)$. By the conditions $\nu_1 > \mu_2$ and $\langle \mu_2, \nu_2 \rangle \in \varDelta$, we have $\sqrt[q]{1-\nu_1^q} \leq \sqrt[q]{1-\mu_2^q}$ and $\nu_2 \leq \sqrt[q]{1-\mu_2^q}$. Therefore, it follows that $\max\left(\sqrt[q]{1-\nu_1^q}, \nu_2\right) \leq \sqrt[q]{1-\mu_2^q} \leq \max\left(\mu_1, \sqrt[q]{1-\mu_2^q}\right)$.
  Thus, $\Psi_{\mathcal{S}_{\mathrm{n}\mathcal{M}}}(\mathbb{x}, \mathbb{y}) = \left\langle \min\left(\sqrt[q]{1-\mu_1^q}, \mu_2\right), \max\left(\sqrt[q]{1-\nu_1^q}, \nu_2\right)\right\rangle$.
- If $\mu_1 \geq \mu_2, \nu_1 \leq \nu_2$, then $\psi^{1/q}_{S_{\mathrm{n}M}}\left(\mu_1^q, \mu_2^q\right) = 0$, $\theta^{1/q}_{T_{\mathrm{n}M}}\left(\nu_1^q, \nu_2^q\right) = \theta^{1/q}_{T_{\mathrm{n}M}}\left(1-\mu_1^q, 1-\mu_2^q\right) = 1$. Thus, $\Psi_{\mathcal{S}_{\mathrm{n}\mathcal{M}}}(\mathbb{x}, \mathbb{y}) = \langle 0, 1 \rangle$.
- If $\mu_1 \geq \mu_2, \nu_1 > \nu_2$, then $\psi^{1/q}_{S_{\mathrm{n}M}}\left(\mu_1^q, \mu_2^q\right) = 0$, $\theta^{1/q}_{T_{\mathrm{n}M}}\left(\nu_1^q, \nu_2^q\right) = \max\left(\sqrt[q]{1-\nu_1^q}, \nu_2\right)$, $\theta^{1/q}_{T_{\mathrm{n}M}}\left(1-\mu_1^q, 1-\mu_2^q\right) = 1$. Thus, $\Psi_{\mathcal{S}_{\mathrm{n}\mathcal{M}}}(\mathbb{x}, \mathbb{y}) = \left\langle 0, \max\left(\sqrt[q]{1-\nu_1^q}, \nu_2\right)\right\rangle$.

Now we can summarize the above cases and conclude that

$$
\begin{aligned}
&\Psi_{\mathcal{S}_{\mathrm{n}\mathcal{M}}}(\mathbb{x}_1, \mathbb{x}_2) \\
&= \begin{cases}
\left\langle \min\left(\sqrt[q]{1-\mu_1^q}, \mu_2\right), \max\left(\mu_1, \sqrt[q]{1-\mu_2^q}\right)\right\rangle, & \text{if } \mu_1 < \mu_2, \nu_1 \leq \max(\mu_2, \nu_2), \\
\left\langle \min\left(\sqrt[q]{1-\mu_1^q}, \mu_2\right), \max\left(\sqrt[q]{1-\nu_1^q}, \nu_2\right)\right\rangle, & \text{if } \mu_1 < \mu_2, \nu_1 > \max(\mu_2, \nu_2), \\
\langle 0, 1 \rangle, & \text{if } \mu_1 \geq \mu_2, \nu_1 \leq \nu_2, \\
\left\langle 0, \max\left(\sqrt[q]{1-\nu_1^q}, \nu_2\right)\right\rangle, & \text{if } \mu_1 \geq \mu_2, \nu_1 > \nu_2.
\end{cases}
\end{aligned}
$$

Similarly, the residual co-implications derived from q-ROF t-conorms listed in Example 1 are as follows:

$$
\Psi_{\mathcal{S}_{\mathcal{M}}}(\mathbb{x}_1, \mathbb{x}_2) = \begin{cases}
\left\langle \mu_2, \sqrt[q]{1-\mu_2^q}\right\rangle, & \text{if } \mu_1 < \mu_2, \nu_1 \leq \nu_2, \\
\langle \mu_2, \nu_2 \rangle, & \text{if } \mu_1 < \mu_2, \nu_1 > \nu_2, \\
\langle 0, 1 \rangle, & \text{if } \mu_1 \geq \mu_2, \nu_1 \leq \nu_2, \\
\langle 0, \nu_2 \rangle, & \text{if } \mu_1 \geq \mu_2, \nu_1 > \nu_2.
\end{cases}
$$

$$
\Psi_{\mathcal{S}_{\mathcal{P}}}(\mathbb{x}_1, \mathbb{x}_2) = \begin{cases}
\left\langle \sqrt[q]{\frac{\mu_2^q-\mu_1^q}{1-\mu_1^q}}, \frac{\nu_2}{\nu_1}\right\rangle, & \text{if } \mu_1 < \mu_2, \frac{1-\mu_1^q}{\nu_1^q} < \frac{1-\mu_2^q}{\nu_2^q}, \\
\left\langle \sqrt[q]{\frac{\mu_2^q-\mu_1^q}{1-\mu_1^q}}, \sqrt[q]{\frac{1-\mu_2^q}{1-\mu_1^q}}\right\rangle, & \text{if } \mu_1 < \mu_2, \frac{1-\mu_1^q}{\nu_1^q} \geq \frac{1-\mu_2^q}{\nu_2^q}, \\
\langle 0, 1 \rangle, & \text{if } \mu_1 \geq \mu_2, \nu_1 \leq \nu_2, \\
\left\langle 0, \frac{\nu_2}{\nu_1}\right\rangle, & \text{if } \mu_1 \geq \mu_2, \nu_1 > \nu_2.
\end{cases}
$$

$$\Psi_{\mathcal{S}_{\mathcal{L}}}(\mathbb{x}_1, \mathbb{x}_2) = \begin{cases} \left\langle \sqrt[q]{\mu_2^q - \mu_1^q}, \sqrt[q]{1 + \mu_1^q - \mu_2^q} \right\rangle, & \text{if } \mu_1 < \mu_2, \mu_1^q + \nu_1^q \leq \mu_2^q + \nu_2^q, \\ \left\langle \sqrt[q]{\mu_2^q - \mu_1^q}, \sqrt[q]{1 - \nu_1^q + \nu_2^q} \right\rangle, & \text{if } \mu_1 < \mu_2, \mu_1^q + \nu_1^q > \mu_2^q + \nu_2^q, \\ \langle 0, 1 \rangle, & \text{if } \mu_1 \geq \mu_2, \nu_1 \leq \nu_2, \\ \left\langle 0, \sqrt[q]{1 - \nu_1^q + \nu_2^q} \right\rangle, & \text{if } \mu_1 \geq \mu_2, \nu_1 > \nu_2. \end{cases}$$

4 Inclusion Degrees of q-ROFSs

In this section, based on the q-ROF residual (co-)implications, two graded inclusion degrees for q-ROFSs are developed.

The set of all q-ROFSs on X is denoted by q-ROFS(X), and a q-ROFS A is represented by $A = \sum_{x \in X} \langle \mu_A(x), \nu_A(x) \rangle / x$. For $A, B \in q$-ROFS(X), we say A is included in B [17], denoted by $A \subseteq B$, iff $A(x) \preceq B(x)$, $\forall x \in X$.

Definition 3. *Let $\Theta_{\mathcal{T}}$ and $\Psi_{\mathcal{S}}$ be q-ROF residual implication and co-implication and X be the universe of discourse. For two q-ROFSs A and B, the two types of degrees to which A is included in B are defined as*

$$\mathrm{Inc}^{(1)}(A, B) = \inf_{x \in X} \Theta_{\mathcal{T}}(A(x), B(x)), \tag{9}$$

$$\mathrm{Inc}^{(2)}(A, B) = \inf_{x \in X} \mathcal{N}(\Psi_{\mathcal{S}}(B(x), A(x))). \tag{10}$$

The following theorem shows that $\mathrm{Inc}^{(1)}(\cdot, \cdot)$ and $\mathrm{Inc}^{(2)}(\cdot, \cdot)$ both satisfy the axiomatic definition of inclusion measures [18].

Theorem 3. *Let $\mathrm{Inc}^{(i)}(\cdot, \cdot)$, $i = 1, 2$ be two measures of q-ROFSs. Then the following statements hold: for all $A, B, C \in q$-ROFS(X),*

(1) $\langle 0, 1 \rangle \preceq \mathrm{Inc}^{(i)}(A, B) \preceq \langle 1, 0 \rangle$.
(2) $\mathrm{Inc}^{(i)}(A, B) = \langle 1, 0 \rangle$ *iff* $A \subseteq B$.
(3) *If* $A \subseteq B$, *then* $\mathrm{Inc}^{(i)}(A, C) \succeq \mathrm{Inc}^{(i)}(B, C)$ *and* $\mathrm{Inc}^{(i)}(C, A) \preceq \mathrm{Inc}^{(i)}(C, B)$.

Example 3. Take $X = \{x_1, x_2, x_3, x_4\}$ and $A = \langle 0.6, 0.7 \rangle / x_1 + \langle 0.2, 0.8 \rangle / x_2 + \langle 0.1, 0.4 \rangle / x_3 + \langle 0.5, 0.6 \rangle / x_4$ and $B = \langle 0.4, 0.5 \rangle / x_1 + \langle 0.3, 0.9 \rangle / x_2 + \langle 0.7, 0.6 \rangle / x_3 + \langle 0.4, 0.2 \rangle / x_4$. They both can be regarded as q-ROFSs with $q = 2$. Then, based on different q-ROF residual (co-)implications, four possible inclusion measures between A and B are collected in Table 1.

Table 1. Inclusion degrees of A and B based on different q-ROF logical operations.

Inclusion Degrees	$\Theta_{\mathcal{T}_{\mathcal{M}}}/\Psi_{\mathcal{S}_{\mathcal{M}}}$	$\Theta_{\mathcal{T}_{\mathcal{P}}}/\Psi_{\mathcal{S}_{\mathcal{P}}}$	$\Theta_{\mathcal{T}_{\mathcal{L}}}/\Psi_{\mathcal{S}_{\mathcal{L}}}$	$\Theta_{\mathcal{S}_{\mathrm{n}\mathcal{M}}}/\Psi_{\mathcal{S}_{\mathrm{n}\mathcal{M}}}$
$\mathrm{Inc}^{(1)}(A, B)$	$\langle 0.4, 0.9 \rangle$	$\langle 0.6667, 0.6872 \rangle$	$\langle 0.8944, 0.4472 \rangle$	$\langle 0.8, 0.6 \rangle$
$\mathrm{Inc}^{(2)}(A, B)$	$\langle 0.4, 0.6 \rangle$	$\langle 0.6667, 0.4880 \rangle$	$\langle 0.8944, 0.4472 \rangle$	$\langle 0.8, 0.6 \rangle$
$\mathrm{Inc}^{(1)}(B, A)$	$\langle 0.1, 0.7 \rangle$	$\langle 0.1429, 0.5774 \rangle$	$\langle 0.7211, 0.5657 \rangle$	$\langle 0.7141, 0.7 \rangle$
$\mathrm{Inc}^{(2)}(B, A)$	$\langle 0.2, 0.7 \rangle$	$\langle 0.3333, 0.6963 \rangle$	$\langle 0.7211, 0.6928 \rangle$	$\langle 0.7141, 0.7 \rangle$

It can be seen from Table 1 that there is no static relation between $\mathrm{Inc}^{(1)}(\cdot,\cdot)$ and $\mathrm{Inc}^{(2)}(\cdot,\cdot)$. For example, $\mathrm{Inc}^{(1)}(A,B)$ is less than $\mathrm{Inc}^{(2)}(A,B)$ by $\Theta_{\mathcal{T}_\mathcal{M}}/\Psi_{\mathcal{S}_\mathcal{M}}$; $\mathrm{Inc}^{(1)}(B,A)$ is greater than $\mathrm{Inc}^{(2)}(B,A)$ by $\Theta_{\mathcal{T}_\mathcal{L}}/\Psi_{\mathcal{S}_\mathcal{L}}$; $\mathrm{Inc}^{(1)}(A,B)$ equals to $\mathrm{Inc}^{(2)}(A,B)$ by $\Theta_{\mathcal{S}_{\mathrm{n}\mathcal{M}}}/\Psi_{\mathcal{S}_{\mathrm{n}\mathcal{M}}}$; $\mathrm{Inc}^{(1)}(B,A)$ and $\mathrm{Inc}^{(2)}(B,A)$ are incomparable by $\Theta_{\mathcal{T}_\mathcal{P}}/\Psi_{\mathcal{S}_\mathcal{P}}$.

Example 4. Take $A=\sum_{x\in[0,\infty)}\left\langle\frac{1}{1+x^2},\frac{x}{1+x^2}\right\rangle/x$ and $B=\sum_{x\in[0,\infty)}\left\langle\frac{1}{2},\frac{1}{2}\right\rangle/x$. Obviously, they are both q-ROFSs on $[0,\infty)$ with $q=2$. Then, for $\Psi_\mathcal{S}\in\{\Psi_{\mathcal{S}_\mathcal{M}},\Psi_{\mathcal{S}_\mathcal{P}},\Psi_{\mathcal{S}_\mathcal{L}},\Psi_{\mathcal{S}_{\mathrm{n}\mathcal{M}}}\}$, we have

$$\mathrm{Inc}^{(2)}(A,B)=\inf_{x\in[0,1]\cup(1,\infty)}\mathcal{N}\big(\Psi_\mathcal{S}\big(\left\langle\tfrac{1}{2},\tfrac{1}{2}\right\rangle,\left\langle\tfrac{1}{1+x^2},\tfrac{x}{1+x^2}\right\rangle\big)\big)$$

$$=\begin{cases}\inf\limits_{x\in[0,1)}\left\langle\frac{x}{1+x^2},\frac{1}{1+x^2}\right\rangle\bigwedge\inf\limits_{x\in[1,\infty)}\left\langle\frac{x}{1+x^2},0\right\rangle, & \text{if } \Psi_\mathcal{S}=\Psi_{\mathcal{S}_\mathcal{M}},\\ \inf\limits_{x\in[0,1)}\left\langle\sqrt{\frac{4((1+x^2)^2-1)}{3(1+x^2)^2}},\sqrt{\frac{4-(1+x^2)^2}{3(1+x^2)^2}}\right\rangle\bigwedge\inf\limits_{x\in[1,\infty)}\left\langle\frac{2x}{1+x^2},0\right\rangle, & \text{if } \Psi_\mathcal{S}=\Psi_{\mathcal{S}_\mathcal{P}},\\ \inf\limits_{x\in[0,1)}\left\langle\sqrt{\frac{5(1+x^2)^2-4}{4(1+x^2)^2}},\sqrt{\frac{4-(1+x^2)^2}{4(1+x^2)^2}}\right\rangle\bigwedge & \text{if } \Psi_\mathcal{S}=\Psi_{\mathcal{S}_\mathcal{L}},\\ \quad\inf\limits_{x\in[1,\infty)}\left\langle\sqrt{\frac{3(1+x^2)^2+4x^2}{4(1+x^2)^2}},0\right\rangle, & \\ \inf\limits_{x\in[0,1)}\left\langle\max\left(\sqrt{\frac{(1+x^2)^2-1}{(1+x^2)^2}},\frac{1}{2}\right),\min\left(\frac{1}{1+x^2},\frac{\sqrt{3}}{2}\right)\right\rangle\bigwedge & \text{if } \Psi_\mathcal{S}=\Psi_{\mathcal{S}_{\mathrm{n}\mathcal{M}}},\\ \quad\inf\limits_{x\in[1,\infty)}\left\langle\frac{\sqrt{3}}{2},0\right\rangle, & \end{cases}$$

$$=\begin{cases}\langle 0,1\rangle\wedge\langle 0,0\rangle, & \text{if } \Psi_\mathcal{S}=\Psi_{\mathcal{S}_\mathcal{M}} \text{ or } \Psi_\mathcal{S}=\Psi_{\mathcal{S}_\mathcal{P}},\\ \left\langle\frac{1}{2},\frac{\sqrt{3}}{2}\right\rangle\bigwedge\left\langle\frac{\sqrt{3}}{2},0\right\rangle, & \text{if } \Psi_\mathcal{S}=\Psi_{\mathcal{S}_\mathcal{L}} \text{ or } \Psi_\mathcal{S}=\Psi_{\mathcal{S}_{\mathrm{n}\mathcal{M}}},\end{cases}$$

$$=\begin{cases}\langle 0,1\rangle, & \text{if } \Psi_\mathcal{S}=\Psi_{\mathcal{S}_\mathcal{M}} \text{ or } \Psi_\mathcal{S}=\Psi_{\mathcal{S}_\mathcal{P}},\\ \left\langle\frac{1}{2},\frac{\sqrt{3}}{2}\right\rangle, & \text{if } \Psi_\mathcal{S}=\Psi_{\mathcal{S}_\mathcal{L}} \text{ or } \Psi_\mathcal{S}=\Psi_{\mathcal{S}_{\mathrm{n}\mathcal{M}}}.\end{cases}$$

Analogously, for $\Theta_\mathcal{T}\in\{\Theta_{\mathcal{T}_\mathcal{M}},\Theta_{\mathcal{T}_\mathcal{P}},\Theta_{\mathcal{T}_\mathcal{L}},\Theta_{\mathcal{T}_{\mathrm{n}\mathcal{M}}}\}$, we have

$$\mathrm{Inc}^{(1)}(A,B)=\inf_{x\in[0,1]\cup(1,\infty)}\Theta_\mathcal{T}\big(\left\langle\tfrac{1}{1+x^2},\tfrac{x}{1+x^2}\right\rangle,\left\langle\tfrac{1}{2},\tfrac{1}{2}\right\rangle\big)=\left\langle\tfrac{1}{2},\tfrac{1}{2}\right\rangle.$$

5 Conclusions

In this paper, we have proposed the q-ROF residual co-implications and provided their applications. The main findings are: (1) the q-ROF residual co-implication $\Psi_\mathcal{S}$ derived from a representable q-ROF t-conorm $\mathcal{S}=(S,T)$ has been presented by using ψ_S and θ_T; (2) two inclusion measures of q-ROFSs have been developed based on q-ROF residual implications and co-implications, respectively.

In further analysis, comparisons with other inclusion degrees are required. And applications of q-ROF residual (co-)implications to approximate reasoning deserve a deeper study to explore their potential benefits.

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