The Modularity Condition of T-uninorms over Semi-t-operators

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Abstract. The modularity equation involving aggregation functions holds significant value in fuzzy sets theory. This paper investigates the modularity condition of T-uninorms over semi-t-operators under no additional assumptions. Moreover, when the underlying pseudo-t-norm of the semi-t-operator is commutative, a universal framework for the structures of modular T-uninorm and semi-t-operator pairs is presented.

Keywords: aggregation functions, modularity, T-uninorms, semi-t-operators

1 Introduction

Aggregation functions serve as essential tools across multiple disciplines such as fuzzy logic, decision theory, approximate reasoning and image processing [2, 3, 5, 10]. Aggregation functions possessing annihilator elements play a pivotal role in modeling rational decision-making processes. These functions are most notably connected to Arrow's impossibility theorem (see [5] for details), while also playing a significant role in the analysis of aggregation functions operating on a bipolar scale (see [2, 3] for details).

T-uninorm is a prominent class of aggregation functions with annihilator. And the function is built upon nullnorms and conjunctive uninorms. At the same time, by relaxing the commutativity constraint of t-operators, semi-t-operators generalize both t-operators and nullnorms .

In the research of aggregation functions, in addition to characterizing internal structure and exploring their algebraic properties, equations about aggregation functions are always helpful in the domain of fuzzy set theory and fuzzy logic [1]. In particular, the modularity condition has become one of the core issues of research in recent years. Not only is modularity strongly correlated to the distributivity condition, but it also can be regarded as a restricted general associative equation. As is widely known, the former is in most cases necessary in fuzzy connectives, and the later is also important in fuzzy set theory. Based on this, many scholars have conducted lots of research on this new topic.

Mas et al. [8] characterized the modularity condition involving t-operators and uninorms belonging to the $U_{\max}$ and $U_{\min}$. Zhang et al. [13] established the solution of modularity equation involving uni-nullnorm(null-uninorm) and overlap (grouping) function. The modularity condition between uninorm pairs has been studied by Su et al. [9], in which no fewer than one uninorm belongs to the generic classes.

In the previous study, the modularity of semi-t-operators and semi-uninorms under the assumption that both of them have continuous underlying functions has been studied by Zhan et al. [11, 12]. Then, Zhao and Liu [14] investigates the modulairity condition between T-uninorms and semi-t-operators with continuous underlying functions, besides, the structures of modular T-uninorms and semi-t-operators pairs are obtained through the assumption that T-uninorms own the underlying uninorm that is locally internal on the boundary. This paper investigates the more general case that the functions are without any further assumptions. And on this basis, we make further characterization of their structure when the underlying pseudo-t-norm of semi-t-operator is commutative. The new perspective offers a fresh look at the modularity equation, presenting several solutions that were previously overlooked. Consequently, we believe that this research supplements existing studies of the modularity equation.

This paper is structured into four sections. Section 2 briefly reviews basic definitions and existing results used in this paper. The modular T-uninorms and semi-t-operators pairs is characterized in Section 3, while Section 4 outlines concluding remarks.

2 Preliminaries

Some fundamental concepts and well-known results required later are reviewed in this section.

Definition 1. *[6] Let A be a binary function on $[0,1]$. Then A is called an aggregation function if it is non-decreasing in each variable, and 0 and 1 is the idempotent element of A, respectively.*

Definition 2. *[9] Let $e_G \in [0,1]$, and let G be a binary aggregation function on $[0,1]$. Then G is called a uninorm if it fulfills associativity, commutativity, and has a neutral element e_G.*

When $e_G = 1$, a uninorm becomes a *t-norm* and when $e_G = 0$, it is a *t-conorm*. For any uninorm G it holds that $G(1,0) \in \{0,1\}$. And we say G is *conjuctive* when $G(1,0) = 0$. Otherwise, G is *disjunctive*. We define $\mathcal{G}_{e_G}$ as the class of uninorms with neutral element e_G, $\mathcal{G}^c$ and $\mathcal{G}^d$ as the class of conjunctive uninorms and disjunctive uninorms, respectively.

Definition 3. *[7] Let $h \in]0,1]$, and P be a binary aggregation function on $[0,1]$. Then P is called a T-uninorm if it fulfills associativity, commutativity, has an annihilator element h, and*

– *$P(1,\cdot)$ exhibits continuity, $P(0,\cdot)$ is discontinuous.*
– *$P(e_P,\cdot)$ is continuous and $P(e_P,0)=0$, whereas $e_P \in]0,1[$ is the idempotent element of P.*

The structural properties of T-uninorms can be characterized as follows.

Theorem 1. *[7] Let $h \in]0,1]$, and P be a binary aggregation function on $[0,1]$. Then P is a T-uninorm with an annihilator element h if and only if*

$$P(\beta,\gamma)=\begin{cases} hU_P(\frac{\beta}{h},\frac{\gamma}{h}) & \text{if } 0\le\beta,\gamma\le h,\\ h+(1-h)T_P(\frac{\beta-h}{1-h},\frac{\gamma-h}{1-h}) & \text{if } h\le\beta,\gamma\le 1,\\ h & \text{otherwise,}\end{cases} \tag{1}$$

where $T_P\in\mathcal{G}_1$, and $U_P\in\mathcal{G}^d_{e_G}$ with $e_G=\frac{e_P}{h}$.

Denote by $\mathcal{P}^T_{e_P,h}$ the class of all T-uninorms with $0<e_P<h\le 1$, which have annihilator element h, the underlying functions $U_P\in\mathcal{G}^d_{e_G}$ and $T_P\in\mathcal{G}_1$, respectively. In particular, if $h=1$, a T-uninorm becomes a disjunctive uninorm.

Definition 4. *[4] A function $T:[0,1]^2\to[0,1]$ is called a* pseudo-t-norm *if it is associative, increasing in each variable and has a neutral element 1.*

A function $S:[0,1]^2\to[0,1]$ is called a pseudo-t-conorm *if it is associative, increasing in each variable and has a neutral element 0.*

Definition 5. *[4] An aggregation function $Q:[0,1]^2\to[0,1]$ is classified as a* semi-t-operator *if it is associative, increasing in each variable, and Q_0,Q_1,Q^0,Q^1 are continuous, where $Q_0=Q(0,\beta),Q_1=Q(1,\beta),Q^0=Q(\beta,0)$ and $Q^1=Q(\beta,1)$ for any $\beta\in[0,1]$.*

Let $\mathcal{Q}_{s,t}$ denote the set of semi-t-operators Q such that $Q(0,1)=s$ and $Q(1,0)=t$.

Theorem 2. *[4] The function $Q\in\mathcal{Q}_{s,t}$ if and only if*

– *when $s\le t$,*

$$Q(\beta,\gamma)=\begin{cases} sS_Q\left(\frac{\beta}{s},\frac{\gamma}{s}\right) & \text{if } 0\le\beta,\gamma\le s,\\ t+(1-t)T_Q\left(\frac{\beta-t}{1-t},\frac{\gamma-t}{1-t}\right) & \text{if } t\le\beta,\gamma\le 1,\\ s & \text{if } 0\le\beta\le s\le\gamma\le 1,\\ t & \text{if } 0\le\gamma\le t\le\beta\le 1,\\ \beta & \text{otherwise,}\end{cases} \tag{2}$$

– *when $s\ge t$,*

$$Q(\beta,\gamma)=\begin{cases} tS_Q\left(\frac{\beta}{t},\frac{\gamma}{t}\right) & \text{if } 0\le\beta,\gamma\le t,\\ s+(1-s)T_Q\left(\frac{\beta-s}{1-s},\frac{\gamma-s}{1-s}\right) & \text{if } s\le\beta,\gamma\le 1,\\ s & \text{if } 0\le\beta\le s\le\gamma\le 1,\\ t & \text{if } 0\le\gamma\le t\le\beta\le 1,\\ \gamma & \text{otherwise,}\end{cases} \tag{3}$$

where T_Q is a pseudo-t-norm, and S_Q is a pseudo-t-conorm.

Definition 6. *[8] Let $P, Q : [0,1]^2 \to [0,1]$ be two functions. Define that P is modular over Q if for any $0 \leq \beta, \gamma, \theta \leq 1$ with $\theta \leq \beta$, the following equation holds,*

$$P(\beta,\, Q(\gamma,\, \theta)) = Q(P(\beta,\, \gamma),\, \theta)\,. \tag{4}$$

Lemma 1. *[14] If $P \in \mathcal{P}^T_{e_P,h}$ is modular over $Q \in \mathcal{Q}_{s,t}$, then $h \leq t$.*

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Commutativity simplifies the analysis, allowing consideration of only the modularity equation (4). We extend the investigation for the modularity of $P \in \mathcal{P}^T_{e_P,h}$ over $Q \in \mathcal{Q}_{s,t}$.

Lemma 2. *If $h \leq t$, $P \in \mathcal{P}^T_{e_P,h}$ is modular over $Q \in \mathcal{Q}_{s,t}$, then:*

(i) $P(\beta,\gamma) = \min(\beta,\gamma)$ *for any* $h \leq \min(\beta,\gamma) \leq t \leq \max(\beta,\gamma) \leq 1$.
(ii) $P(\beta,\gamma) = P(\gamma,\beta) = Q(\beta,\gamma)$ *for any* $t \leq \beta,\gamma \leq 1$ *with* $\gamma \leq \beta$.

Proof. Suppose that P and Q satisfy Equation (4).

(i) Setting $\beta \in [h,t]$, $\theta = 0$ and $\gamma = 1$ into Equation (4), then $h \leq P(\beta,\gamma) \leq t$, we obtain $P(\beta, Q(1,0)) = P(\beta,t)$, and $Q(P(\beta,1),0) = Q(\beta,0) = \beta$, implying that item (i) holds, for the reason that P is monotonous and commutative, 1 is the neutral element of T_P.

(ii) Substituting $\beta,\theta \in [t,1]$ and $\gamma = 1$ with $\theta \leq \beta$ into Equation (4). We then obtain $P(\beta, Q(1,\theta)) = P(\beta,\theta)$ and $Q(P(\beta,1),\theta) = Q(\beta,\theta)$, implying that item (ii) holds, for the reason that P is commutative. □

Depending on Lemma 2 and the relationship between h, e_P, s and t, we divide our work into the following four subcases: 1. $s < e_P < h \leq t$; 2. $e_P \leq s \leq h \leq t$; 3. $h < s \leq t$; 4. $h \leq t < s$.

3.1 $s < e_P < h \leq t$

Lemma 3. *If $s < e_P < h \leq t$, $P \in \mathcal{P}^T_{e_P,h}$ is modular over $Q \in \mathcal{Q}_{s,t}$, then*

(i) $P(\beta,\gamma) = \min(\beta,\gamma)$ *for any* $(\beta,\gamma) \in [0,e_P] \times [0,s] \cup [0,s] \times [0,e_P]$.
(ii) $S_Q = \max$.
(iii) for any $\beta \in [e_P,h]$, we have either $P(\beta,\gamma) = \gamma$ for all $0 \leq \gamma \leq s$ or $P(\beta,\gamma) = P(\beta,0)$ for all $0 \leq \gamma \leq s$.

Proof. Suppose that P and Q satisfy Equation (4).

(i) Setting $\beta \in [0, e_P]$, $\theta \in [0, s]$ with $\theta \le \beta$ and setting $\gamma = 0$ into Equation (4), we obtain

$$P(\beta, \theta) = P(\beta, Q(0, \theta)) = Q(P(\beta, 0), \theta) = Q(0, \theta) = \theta\,.$$

implying that for any $(\beta, \gamma) \in [0, e_P] \times [0, s] \cup [0, s] \times [0, e_P]$, $P(\beta, \gamma) = \min(\beta, \gamma)$ by the commutativity of P.

(ii) Substituting $\beta = \theta \in [0, s]$ and setting $\gamma = e_P$ into Equation (4), we then obtain

$$\beta = P(\beta, e_P) = P(\beta, Q(e_P, \beta)) = Q(P(\beta, e_P), \beta) = Q(\beta, \beta)\,,$$

implying that for any $0 \le \beta, \gamma \le s$, $Q(\beta, \gamma) = \max(\beta, \gamma)$.

(iii) Substituting $\beta \in [e_P, h]$, $\theta \in [0, s]$ and setting $\gamma = 0$ into Equation (4). Then we obtain

$$P(\beta, \theta) = P(\beta, Q(0, \theta)) = Q(P(\beta, 0), \theta) = \max\{P(\beta, 0), \theta\}\,. \tag{5}$$

If $P(\beta, \theta) > P(\beta, 0)$, then equation (5) yields $P(\beta, \theta) = \theta$. Thus, either $P(\beta, \theta) = P(\beta, 0)$ or $P(\beta, \theta) = \theta$ holds. Let $x_P \in [e_P, h]$ and

$$\alpha = \sup\{m \in [0, s] \mid P(\beta_P, m) = P(\beta_P, 0)\}.$$

Below, we show that $\alpha = 0$ or $\alpha = s$. Otherwise, suppose that $\alpha \in]0, s[$. Then $P(\beta_P, \theta) = P(\beta_P, 0)$ if $0 \le \theta < \alpha$, and $P(\beta_P, \theta) = \theta$ if $\alpha < \theta \le s$. The associativity of P implies that for $y \in]\alpha, s[$ and $z \in]0, \alpha[$, we have

$$P(\beta_P, 0) = P(\beta_P, \theta) = P(\beta_P, P(\gamma, \theta)) = P(P(\beta_P, \gamma), \theta) = P(\gamma, \theta) = \theta\,,$$

contradicting the fact that $P(\beta_P, 0)$ is a constant. Bringing all finding above together, we deduce that for any given $\beta \in [e_P, h]$, we have either $P(\beta, \gamma) = \gamma$ for all $0 \le \gamma \le s$ or $P(\beta, \gamma) = P(\beta, 0)$ for all $0 \le \gamma \le s$. □

Under the additional assumption that the underlying function T_Q of Q is commutative, the subsequent theorem holds.

Theorem 3. *Let $s < e_P < h \le t$, $Q \in \mathcal{Q}_{s,t}$ with T_Q is commutative and $P \in \mathcal{P}^T_{e_P,h}$. Then P and Q satisfy Equation (4) if and only if:*

(i) $P(\beta, \gamma) = \min(\beta, \gamma)$ *for any* $(\beta, \gamma) \in [h, t] \times [t, 1] \cup [t, 1] \times [h, t]$.
(ii) $P(\beta, \gamma) = \min(\beta, \gamma)$ *for any* $(\beta, \gamma) \in [0, e_P] \times [0, s] \cup [0, s] \times [0, e_P]$.
(iii) $S_Q = \max$.
(iv) *for any* $\beta \in [e_P, h]$, *either* $P(\beta, \gamma) = \gamma$ *for all* $0 \le \gamma \le s$ *or* $P(\beta, \gamma) = P(\beta, 0)$ *for all* $0 \le \gamma \le s$.
(v) $P(\beta, \gamma) = Q(\beta, \gamma)$ *for any* $(\beta, \gamma) \in [t, 1]^2$.

Proof. Suppose that P and Q satisfy Equation (4). Items (i)-(v) can be obviously obtained by Lemma 2, Lemma 3 together with the commutativity of T_Q.

Reciprocally, a direct computation shows that P is modular over Q. □

Example 1. Let $s < e_P < h \leq t$ and U'_P be any uninorm. Considering P_1, Q_1 be binary aggregation functions on $[0,1]$ defined by

$$P_1(\beta,\gamma) = \begin{cases} s + (h-s)U'_P(\frac{\beta-s}{h-s}, \frac{\gamma-s}{h-s}) & \text{if } s \leq \beta, \gamma \leq h, \\ h & \text{if } 0 \leq \min(\beta,\gamma) \leq h \leq \max(\beta,\gamma) \leq 1, \\ \min(\beta,\gamma) & \text{otherwise,} \end{cases}$$

$$Q_1(\beta,\gamma) = \begin{cases} \max(\beta,\gamma) & \text{if } 0 \leq \beta, \gamma \leq s, \\ \min(\beta,\gamma) & \text{if } t \leq \beta, \gamma \leq 1, \\ s & \text{if } 0 \leq \beta \leq s \leq \gamma \leq 1, \\ t & \text{if } 0 \leq \gamma \leq t \leq \beta \leq 1, \\ \beta & \text{otherwise.} \end{cases}$$

Theorem 3 indicates that P_1 is modular over Q_1.

3.2 $e_P \leq s \leq h \leq t$

Lemma 4. *If $e_P \leq s \leq h \leq t$, $P \in \mathcal{P}^T_{e_P,h}$ is modular over $Q \in \mathcal{Q}_{s,t}$, then $P(\beta,\gamma) = \max(\beta,\gamma)$ for any $0 \leq \min(\beta,\gamma) \leq s < \max(\beta,\gamma) \leq h$.*

Theorem 4. *Let $e_P \leq s \leq h \leq t$, $Q \in \mathcal{Q}_{s,t}$ with T_Q is commutative and $P \in \mathcal{P}^T_{e_P,h}$. Then P is modular over Q if and only if:*

(i)

$$P(\beta,\gamma) = \begin{cases} sU'_P(\frac{\beta}{s}, \frac{\gamma}{s}) & \text{if } 0 \leq \beta, \gamma \leq s, \\ \max(\beta,\gamma) & \text{if } 0 \leq \min(\beta,\gamma) \leq s < \max(\beta,\gamma) \leq h, \\ s + (h-s)S'_P(\frac{\beta-s}{h-s}, \frac{\gamma-s}{h-s}) & \text{if } s \leq \beta, \gamma \leq h, \\ h + (t-h)T'_P(\frac{\beta-h}{t-h}, \frac{\gamma-h}{t-h}) & \text{if } h \leq \beta, \gamma \leq t, \\ \min(\beta,\gamma) & \text{if } h \leq \min(\beta,\gamma) \leq t \leq \max(\beta,\gamma) \leq 1, \\ t + (1-t)T_Q(\frac{\beta-t}{1-t}, \frac{\gamma-t}{1-t}) & \text{if } t \leq \beta, \gamma \leq 1, \\ h & \text{otherwise,} \end{cases}$$

$$Q(\beta,\gamma) = \begin{cases} sS_Q(\frac{\beta}{s}, \frac{\gamma}{s}) & \text{if } 0 \leq \beta, \gamma \leq s, \\ t + (1-t)T_Q(\frac{\beta-t}{1-t}, \frac{\gamma-t}{1-t}) & \text{if } t \leq \beta, \gamma \leq 1, \\ s & \text{if } 0 \leq \beta \leq s \leq \gamma \leq 1, \\ t & \text{if } 0 \leq \gamma \leq t \leq \beta \leq 1, \\ \beta & \text{otherwise,} \end{cases}$$

where $T'_P, T_Q \in \mathcal{G}_1$, S_Q is a pseudo-t-conorm, $S'_P \in \mathcal{G}_0$ and $U'_P \in \mathcal{G}_{e_G}$ with $e_G = \frac{e_P}{s}$.

(ii) U'_P is modular over S_Q.

3.3 $h < s \leq t$

Lemma 5. *If $h < s \leq t$, $P \in \mathcal{P}^T_{e_P,h}$ is modular over $Q \in \mathcal{Q}_{s,t}$, then*

(i) $P(\beta,\gamma) = \min(\beta,\gamma)$ for any $(\beta,\gamma) \in [h,1] \times [h,s] \cup [h,s] \times [h,1]$.

(ii) $Q(\beta,\gamma)=\max(\beta,\gamma)$ *for any* $(\beta,\gamma)\in[h,s]\times[0,s]\cup[0,s]\times[h,s]$.

Theorem 5. *Let* $h<s\leq t$, $Q\in\mathcal{Q}_{s,t}$ *with* T_Q *is commutative and* $P\in\mathcal{P}^T_{e_P,h}$. *Then* P *is modular over* Q *if and only if:*

(i)

$$P(\beta,\gamma)=\begin{cases} hU_P(\frac{\beta}{h},\frac{\gamma}{h}) & \text{if } 0\leq\beta,\gamma\leq h,\\ s+(t-s)T'_P(\frac{\beta-s}{t-s},\frac{\gamma-s}{t-s}) & \text{if } s\leq\beta,\gamma\leq t,\\ t+(1-t)T_Q(\frac{\beta-t}{1-t},\frac{\gamma-t}{1-t}) & \text{if } t\leq\beta,\gamma\leq 1,\\ h & \text{if } 0\leq\min(\beta,\gamma)\leq h\leq\max(\beta,\gamma)\leq 1,\\ \min(\beta,\gamma) & \text{otherwise,}\end{cases}$$

$$Q(\beta,\gamma)=\begin{cases} hS'_Q(\frac{\beta}{h},\frac{\gamma}{h}) & \text{if } 0\leq\beta,\gamma\leq h,\\ s & \text{if } 0\leq\beta\leq s\leq\gamma\leq 1,\\ \beta & \text{if } s\leq\beta\leq t,\ 0\leq\gamma\leq 1,\\ t+(1-t)T_Q(\frac{\beta-t}{1-t},\frac{\gamma-t}{1-t}) & \text{if } t\leq\beta,\gamma\leq 1,\\ t & \text{if } 0\leq\gamma\leq t\leq\beta\leq 1,\\ \max(\beta,\gamma) & \text{otherwise,}\end{cases}$$

where $T'_P,T_Q\in\mathcal{G}_1$, S'_Q *is a pseudo-t-conorm, and* $U_P\in\mathcal{G}^d_{e_G}$ *with* $e_G=\frac{e_P}{h}$.

(ii) U_P *is modular over* S'_Q.

3.4 $h\leq t<s$

Lemma 6. *If* $h\leq t<s$, $P\in\mathcal{P}^T_{e_P,h}$ *is modular over* $Q\in\mathcal{Q}_{s,t}$, *then*

(i) $P(\beta,\gamma)=\min(\beta,\gamma)$ *for any* $(\beta,\gamma)\in[h,1]\times[h,s]\cup[h,s]\times[h,1]$.

(ii) $Q(\beta,\gamma)=\max(\beta,\gamma)$ *for any* $(\beta,\gamma)\in[h,t]\times[0,t]\cup[0,t]\times[h,t]$.

Theorem 6. *Let* $h\leq t<s$, $Q\in\mathcal{Q}_{s,t}$ *with* T_M *is commutative and* $P\in\mathcal{P}^T_{e_P,h}$. *Then* P *is modular over* Q *if and only if:*

(i)

$$P(\beta,\gamma)=\begin{cases} hU_P(\frac{\beta}{h},\frac{\gamma}{h}) & \text{if } 0\leq\beta,\gamma\leq h,\\ h & \text{if } 0\leq\min(\beta,\gamma)\leq h\leq\max(\beta,\gamma)\leq 1,\\ s+(1-s)T_Q(\frac{\beta-s}{1-s},\frac{\gamma-s}{1-s}) & \text{if } s\leq\beta,\gamma\leq 1,\\ \min(\beta,\gamma) & \text{otherwise,}\end{cases}$$

$$Q(\beta,\gamma)=\begin{cases} hS'_Q(\frac{\beta}{h},\frac{\gamma}{h}) & \text{if } 0\leq\beta,\gamma\leq h,\\ s & \text{if } 0\leq\beta\leq s\leq\gamma\leq 1,\\ \gamma & \text{if } 0\leq\beta\leq 1,\ t\leq\gamma\leq s,\\ s+(1-s)T_Q(\frac{\beta-s}{1-s},\frac{\gamma-s}{1-s}) & \text{if } s\leq\beta,\gamma\leq 1,\\ t & \text{if } 0\leq\gamma\leq t\leq\beta\leq 1,\\ \max(\beta,\gamma) & \text{otherwise,}\end{cases}$$

where $T_Q\in\mathcal{G}_1$, S'_Q *is a pseudo-t-conorm, and* $U_P\in\mathcal{G}^d_{e_G}$ *with* $e_G=\frac{e_P}{h}$.

(ii) U_P *is modular over* S'_Q.

4 Conclusions

This work studied the modularity condition of T-uninorms over semi-t-operators in most general settings. We establish complete necessary and sufficient conditions that modularity equation holds with an additional assumption that the underlying pseudo-t-norm of semi-t-operators satisfies commutativity. Some new solutions to the modularity equation, which had not been included in earlier studies, were obtained by removing the continuity assumptions.

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