ST/ NUMERICAL MODELING
CS559 Fall 2020
Oregon State University

Homework 1

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This homework answers the problem set sequentially:

1.4 **Find the coefficient vectors α and β for the trapezoidal method. Find also the stability region of the trapezoidal method**.

Solution:

(1) As we all know, the *the trapezoidal formula* can be expressed as:

$$x_{n+1} - x_n \approx \frac{h}{2}\Big(f\big(t_n, x_n\big) + f\big(t_{n+1}, x_{n+1}\big)\Big) \tag{1}$$

And when we consider the approximation $v_{n+r} \approx x(t_{n+r})$, (1) can be expressed as:

$$v_{n+1} - v_n = \frac{h}{2}\Big(f\big(t_n, v_n\big) + f\big(t_{n+1}, v_{n+1}\big)\Big) \tag{2}$$

However, we can use LMM form to create the*trapezoidal formula* as shown below:

$$\sum_{j=0}^{r} \alpha_j v_{n+j} = h \sum_{j=0}^{r} \beta(f\big(t_{j+1}, v_{j+1}\big) \tag{3}$$

Comparing the coefficients between (3) and (2), we can get that: $\alpha_0 = -1, \alpha_1 = 1, \alpha_{j>1} = 0$ and $\beta_0 = 0.5, \beta_1 = 0.5, \beta_{j>1} = 0$.

Thus because the *trapezoidal formula* is 1-step method, so the coefficient vectors α and β are:

$$\alpha = \begin{bmatrix}\alpha_0\\ \alpha_1\end{bmatrix} = \begin{bmatrix}-1\\ 1\end{bmatrix}, \beta = \begin{bmatrix}\beta_0\\ \beta_1\end{bmatrix} = \begin{bmatrix}0.5\\ 0.5\end{bmatrix}$$

Furthermore, if we cite the idea of the *trapezoidal formula*, we can get the *r-step trapezoidal formula* with approximation:

$$v_{n+r} - v_n = \frac{h}{2}\Big(f\big(t_n, v_n\big) + 2f\big(t_{n+1}, v_n 1\big) + ... + 2f\big(t_{n+r-1}, v_{n+r-1}\big) + f\big(t_{n+r}, v_{n+r}\big)\Big) \tag{4}$$

Formula (4) is based on transitivity of trapezoid calculation, which can be seen in the figure below:

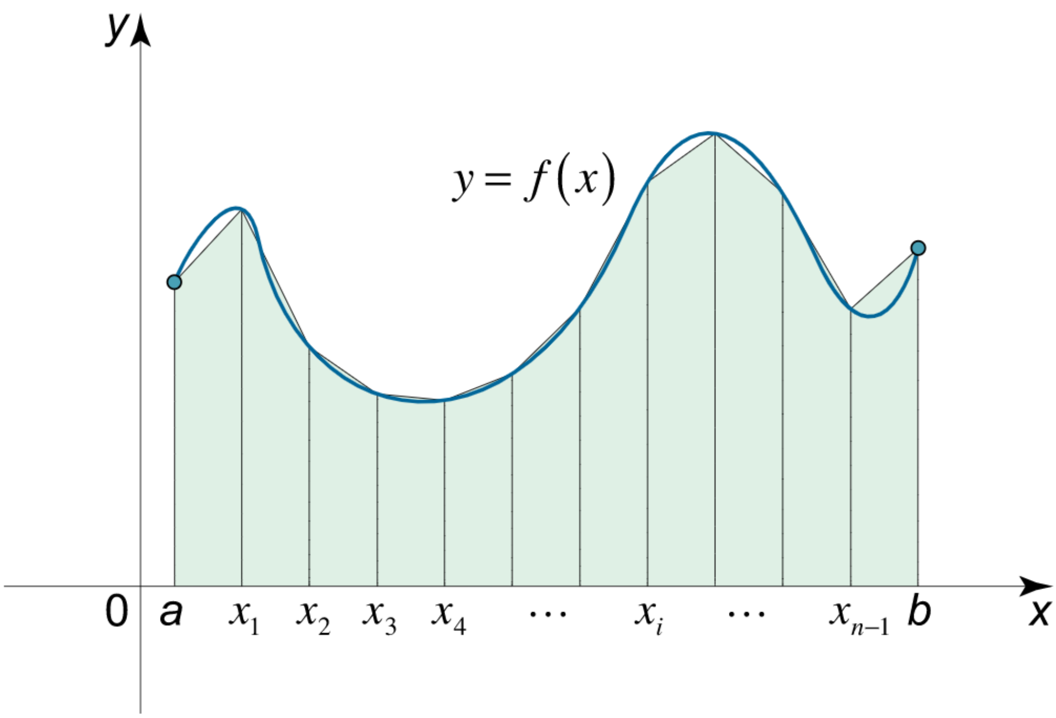

Figure 1: Trapezoids combination between points **a** and **b**

Thus, comparing the coefficients between (4) and (2), we can get the coefficient vectors α and β of *r-step trapezoidal formula* as below:

$$\alpha = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \dots \\ \alpha_{r-1} \\ \alpha_r \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ \dots \\ 0 \\ 1 \end{bmatrix}, \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \dots \\ \beta_r - 1 \\ \beta_r \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \\ \dots \\ 1 \\ 0.5 \end{bmatrix}$$

(2) When talking about the trapezoidal method, with time step h and $\lambda \in \mathbb{C}, Re(\alpha) \leq 0$, we have, for $v_j \approx x(t_j)$,

$$v_{j+1} = v_j + \frac{h}{2}\lambda\big(v_j + v_{j+1}\big) \tag{5}$$

Formula (5) can be expressed as:

$$v_{j+1} = \left(\frac{2 + h\lambda}{2 - h\lambda}\right) v_j \tag{6}$$

After doing recursion, we can get:

$$v_{j+1} = \left(\frac{2 + h\lambda}{2 - h\lambda}\right)^{j+1} v_j \xlongequal[v_0 = x_0]{h\lambda = z} \left(\frac{2 + z}{2 - z}\right)^{j+1} v_j \tag{7}$$

Thus, to guarantee that the solution remains bounded as $j \to \infty$, we require that:

$$\left|\frac{2+z}{2-z}\right| \le 1 \; or, equivalently, \; \left|2+z\right| \le \left|2-z\right| \tag{8}$$

Because Z is the complex number, we can think it is a point in the complex plane. So the absolute value $|$2+z$|$means the distance between (x,y) and (-2,0), which we call it d_1 (x is the coordinate of real axis and y is the coordinate of the imaginary axis). Similarly, $|$2-z$|$means the distance between (x,y) and (2,0), which we call it d_2. Thus, the inequation means the lines value satisfy $d_1 \le d_2$. In the complex plane, the stability region can be shown as the green area below:

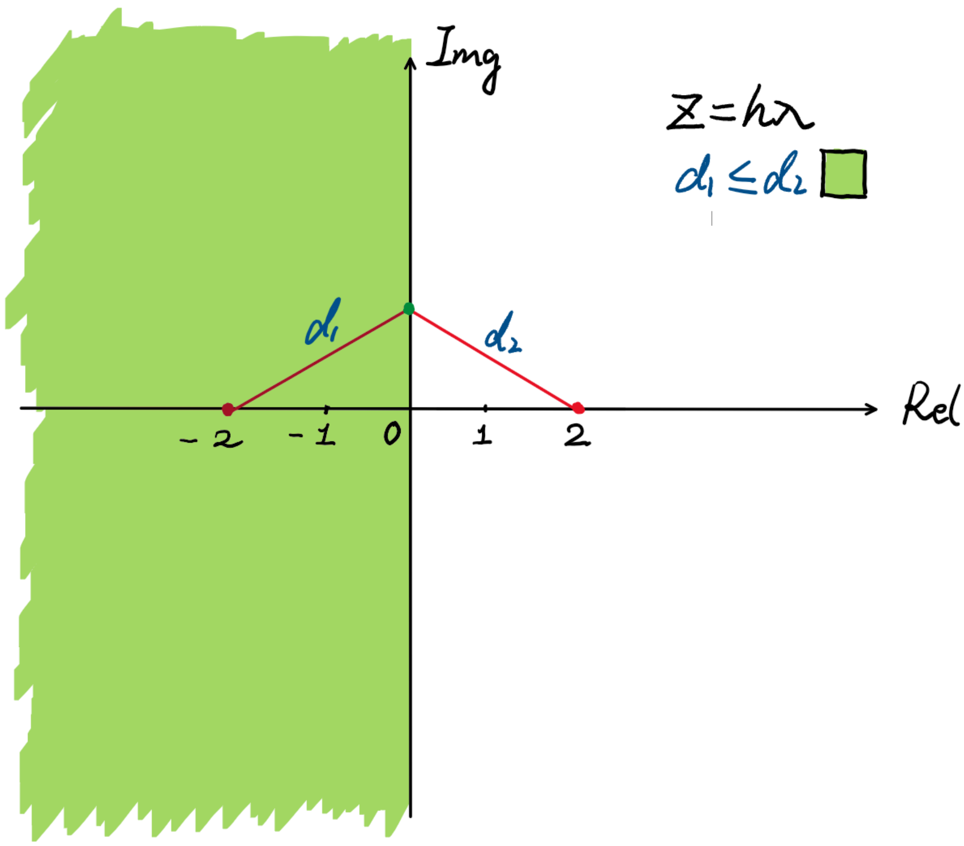

Figure 2: The stability region of trapezoidal method

1.9 **Verify that the trapezoidal rule has order of accuracy 2,while the Euler methods have order of accuracy 1?**

Solution:

(1) Firstly, we need to get it clear how to calculate the *local truncation error* :

$$\tau_{n+1}(h) = \frac{Solution_{Real} - Solution_{numerical}}{h} = \frac{x(t_{n+1}) - v_{n+1}}{h} \tag{9}$$

Then, for the real solution $x(t_{n+1})$, we can get the solution expression by using:

$$\frac{dx}{dt} = \lambda x,\ x(t) = x_t \tag{10}$$

If we integrate both sides of equation (10), setting the upper bound of the integral is t_{j+1} and the lower bound is t_j, we will get the equation calculations as below:

$$\int_{t_j}^{t_{j+1}} \frac{1}{x} dx = \int_{t_j}^{t_{j+1}} \lambda dt$$

$$\Rightarrow \ln\left(\frac{x_{j+1}}{x_j}\right) = \lambda(t_{j+1} - t_j)$$

$$\Rightarrow x_{j+1} = x_j \exp(\lambda h) = x_j \exp z \tag{11}$$

Right now, we can quote the expression from (6) and we can get:

$$v_{j+1} = \left(\frac{2+z}{2-z}\right) v_j \tag{12}$$

Now we do subtraction between (11) and (12), with approximation $v_{j+1} \approx x_{j+1}$, we can get:

$$\begin{aligned}
\tau_{j+1} = \frac{x_{j+1} - v_{j+1}}{h} &= \left(x_j \exp z - \left(\frac{2+z}{2-z}\right) v_j\right) \frac{1}{h} \\
&\approx \left(\exp z - \left(\frac{2+z}{2-z}\right)\right) \frac{v_j}{h} \\
&\overset{tylor}{=\!=} \left[\left(1 + z + \frac{1}{2!} z^2 + \frac{1}{3!} z^3 + O(h^4)\right) - \left(\frac{2+z}{2-z}\right)\right] \frac{v_j}{h} \\
&\overset{tylor}{=\!=} \left[\left(1 + z + \frac{1}{2!} z^2 + \frac{1}{3!} z^3 + O(h^4)\right) - \left(1 + z + \frac{1}{2} z^2 \right.\right. \\
&\qquad \left.\left. + \frac{1}{4} z^3 + O(h^4)\right)\right] \frac{v_j}{h} \\
&= \left(-\frac{1}{6} z^3 + O(h^4)\right) \frac{v_j}{h} \\
&= \left(-\frac{1}{6} h^2 \lambda^3 + O(h^3)\right) v_j
\end{aligned} \tag{13}$$

Thus, $\tau_{j+1}(h) = O(h^2)$, which means the *trapezoidal rule* has order of accuracy 2.

(2) Then, we could imitate the calculation of the *local truncation error* of the *trapezoidal method*, to calculate the *local truncation error* of the *Euler method* (we only think about the *Backward Euler method*, because the *Backward Euler method* error we have learnt before):

$$\begin{aligned}\tau_{j+1} = \frac{x_{j+1} - v_{j+1}}{h} &\overset{tylor}{=\!=\!=} \left[\left(1 + z + \frac{1}{2!}z^2 + O(h^3)\right) - \left(1 + z + z^2 + O(h^3)\right)\right]\frac{v_j}{h} \\ &= \left(-\frac{1}{2}z^2 + O(h^3)\right)\frac{v_j}{h} \\ &= \left(-\frac{1}{2}h\lambda^2 + O(h^2)\right)v_j \end{aligned} \tag{14}$$

Thus, $\tau_{j+1}(h) = O(h)$, which means the *Backward Euler method* has the order of accuracy 1.

In conclusion, combining the *Forward Euler method* which also has the order of accuracy 1, the *Euler method rule* has the order of accuracy 1.

Matlab-Code

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None.
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