ST/ NUMERICAL MODELING
CS559 Fall 2020
Oregon State University

Homework 2

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This homework answers the problem set sequentially:

2.1 **Find analytic solution of the logistic growth model:**

$$X' = rX\left(1 - \frac{X}{K}\right).$$

Solution:

we can change the expression of the equation as below:

$$\begin{aligned}\frac{dX(t)}{dt} &= rX(t)\left(1 - \frac{X(t)}{K}\right)\\ &= rX(t) - \frac{r}{K}X(t)^2 \qquad (1)\end{aligned}$$

According to the classification of the ODE, (1) can be approximately classified as the non-homogeneous *Bernoulli differential equation* with the following format, when we ignore "y" of the left-second par of (2):

$$\frac{dy}{dx} + P(x)y = Q(x)y^n, (n \neq 0, 1) \qquad (2)$$

Thus, we can doing below steps to change (1) into linear equation:

(1) Firstly, we divide both sides of (1) by $X(t)^2$, then we get:

$$X^{-2}\frac{dX}{dt} = rX^{-1} - \frac{r}{K} \qquad (3)$$

(2) We set a new variable $Z = X^{-1}$, then we get:

$$\frac{dZ}{dt} = \frac{d}{dt}(X^{-1}) = -X^{-2}\frac{dX}{dt}$$

Multiply both sides of equation by -1, we get:

$$-\frac{dZ}{dt} = X^{-2}\frac{dX}{dt} \qquad (4)$$

(3) Then, we combine the right side of (3) and left side of (4), we get:

$$-\frac{dZ}{dt} = rX^{-1} - \frac{r}{K}$$

$$\Leftrightarrow \frac{dZ}{dt} = -rZ + \frac{r}{K} \tag{5}$$

(4) Now the equation (5) is the *First order differential equation*, which can be integrated as below:

$$\int \frac{1}{\Big(-rZ + \frac{r}{K}\Big)} dZ = \int 1 dt$$

$$\Rightarrow -\frac{1}{r} \cdot \ln(-rZ + \frac{r}{K}) = t + C$$

$$\Rightarrow \ln(-rZ + \frac{r}{K}) = -r(t + C)$$

$$\Rightarrow -rZ + \frac{r}{K} = e^{-r(t+C)}$$

$$\Rightarrow -rZ + \frac{r}{K} = Ce^{-rt} \tag{6}$$

(5) As we have $Z_0 = {X_0}^{-1} = X(t_0)^{-1} = X(t = 0)^{-1}$, we will have $e^{-rt} = 1$. Then we can get constant C:

$$C = -rZ_0 + \frac{r}{K} \tag{7}$$

(6) We put C back to (6) and will get:

$$-rZ + \frac{r}{K} = \Big(-rZ_0 + \frac{r}{K}\Big)e^{-rt}$$

$$\Rightarrow -Z + \frac{1}{K} = \Big(-Z_0 + \frac{1}{K}\Big)e^{-rt}$$

$$\Rightarrow -\frac{1}{X} + \frac{1}{K} = \Big(-\frac{1}{X_0} + \frac{1}{K}\Big)e^{-rt}$$

$$\Rightarrow -\frac{1}{X} + \frac{1}{K} = \Big(-\frac{1}{X_0} + \frac{1}{K}\Big)e^{-rt}$$

$$\Rightarrow X = \frac{KX_0}{X_0 + (K - X_0)e^{-rt}} \tag{8}$$

Thus, the analytic solution of the logistic growth model is $X = \dfrac{KX_0}{X_0 - (K - X_0)e^{-rt}}$.

2.3 **If the fishing quota of the previous problem is based on the size of the fish population at some earlier time, the model becomes:**

$$X'(t) = rX(t)\left(1 - \frac{X(t)}{K}\right) - h_0 X(t-\tau),$$

where τ is the time lag. Test this model numerically, simulating the behavior of the fish population for different values of the lag parameter.

Solution:

To solve this simulation task with time lag problem, we need to use DDE function in MATLAB. The grammar of the DDE can be expressed as below:

$$\mathbf{sol} = \mathbf{ddesd}(\mathbf{ddefun}, \mathbf{lags}, \mathbf{history}, \mathbf{time\_span})$$

If we set the $r = 1, h_0 = 0.4, k = 200$, we can get the picture as shown below:

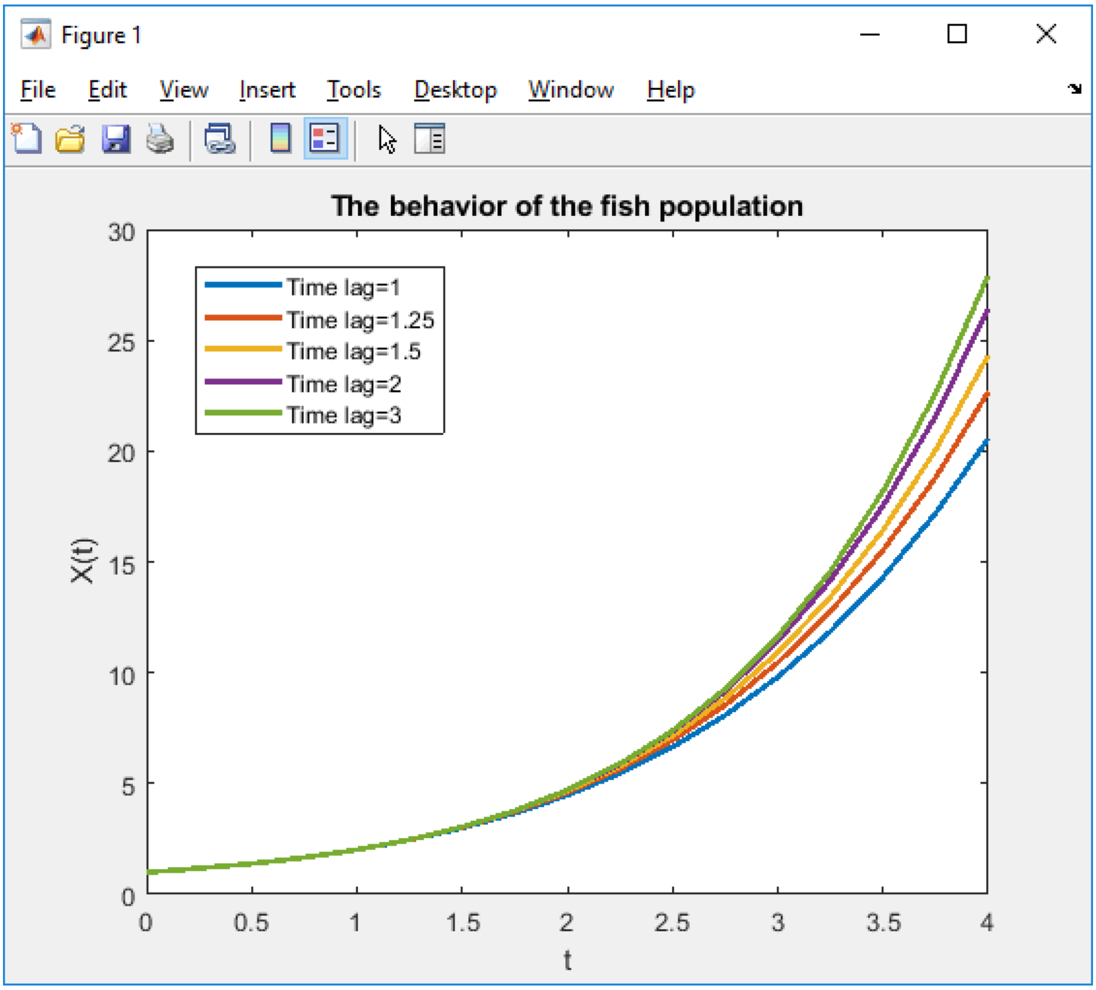

Figure 1: The behavior of the fish population

Matlab-Code

Method 1:

```
r= 1;  % growth rate
h0=0.4;
k=200;% carrying capacity

ddefun=@(t,x,z)[(r*x(1)*(1-x(1)/k)-h0*z(1,1))
                (r*x(2)*(1-x(2)/k)-h0*z(2,2))
                (r*x(3)*(1-x(3)/k)-h0*z(3,3))
                (r*x(4)*(1-x(4)/k)-h0*z(4,4))
                (r*x(5)*(1-x(5)/k)-h0*z(5,5))];
lags=[1,1.25,1.5,2,3];
history=@(t)ones(5,1);
tspan=[0,4];

sol=dde23(ddefun,lags,history,tspan);

plot(sol.x,sol.y,'LineWidth',2); ylim([0, 30]);
title('The behavior of the fish population');xlabel('t');ylabel('X(t)');

legend('Time lag=1','Time lag=1.25','Time lag=1.5','Time lag=2','Time lag=3');
```

Method 2:

```
lags=[1,1.25,1.5,2,3];tspan=[0,4];
sol=dde23(@ddefun,lags,@history,tspan);
plot(sol.x,sol.y,'LineWidth',2); ylim([0, 30]);
title('The behavior of the fish population');xlabel('t');ylabel('X(t)');
legend('Time lag=1','Time lag=1.25','Time lag=1.5','Time lag=2','Time lag=3');

function dxdt = ddefun(t,x,Z)
r= 1;  % growth rate
h0=0.4;
k=200; % carrying capacity
xlag1 = Z(:,1);
xlag2 = Z(:,2);
xlag3 = Z(:,3);
xlag4 = Z(:,4);
xlag5 = Z(:,5);
dxdt=[(r*x(1)*(1-x(1)/k)-h0*xlag1(1))
      (r*x(2)*(1-x(2)/k)-h0*xlag2(2))
      (r*x(3)*(1-x(3)/k)-h0*xlag3(3))
      (r*x(4)*(1-x(4)/k)-h0*xlag4(4))
      (r*x(5)*(1-x(5)/k)-h0*xlag5(5))];
end

function s = history(t)
```

```
s = ones(5,1);
end
```