Rendering Equation

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0. Let’s review two concepts we have talked before....

1. Radiance (L)

2. Irradiance (E)

Which are usually used to measure the spatial and angular (solid angle) properties of light

Illumination Terminology

https://www.cs.princeton.edu/courses/archive/fall00/cs426/lectures/light/sld049.htm

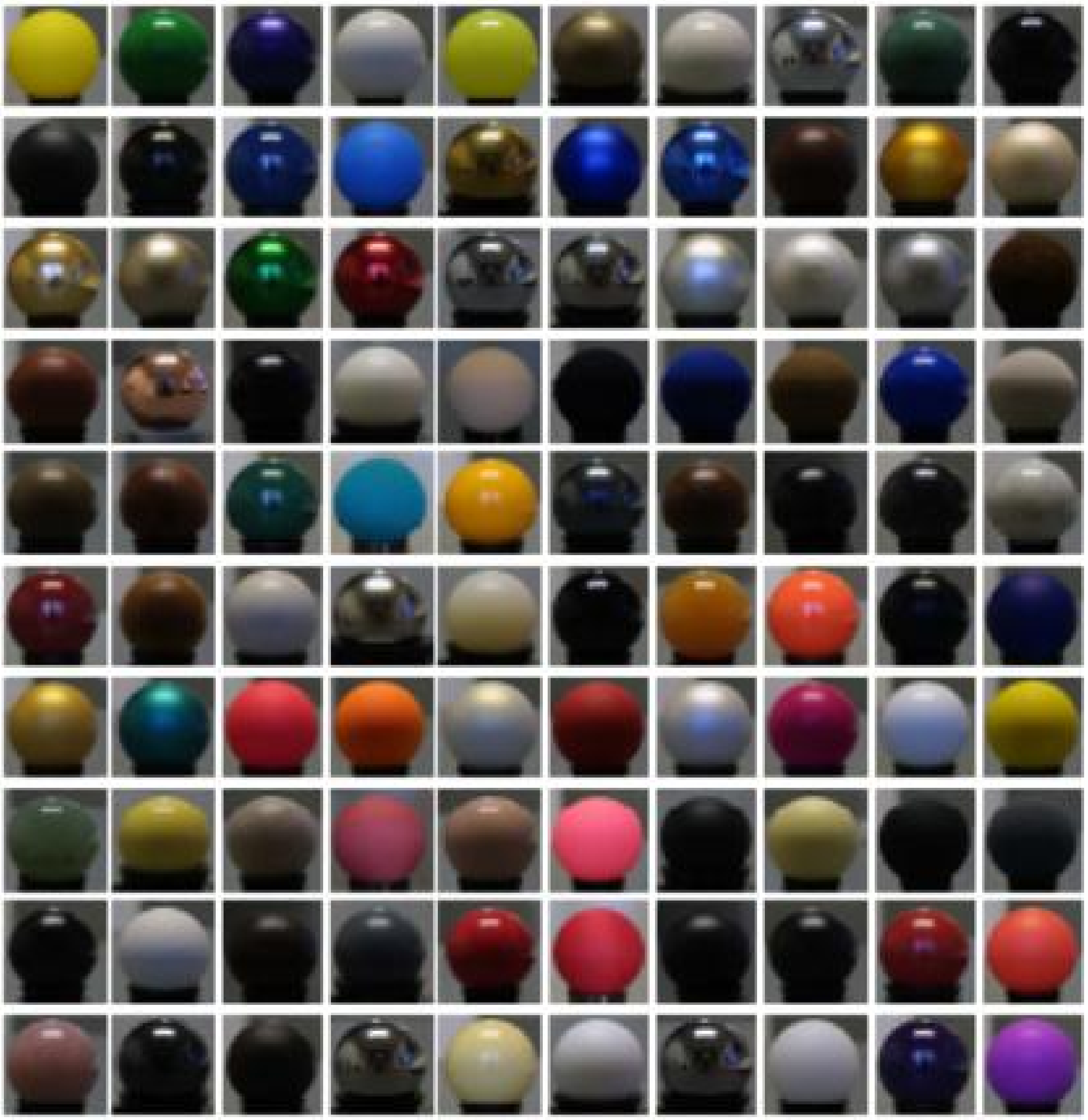
- Radiant power [flux] (Φ)
  - Rate at which light energy is transmitted (in Watts).
- Radiant Intensity (I)
  - Power radiated onto a unit solid angle in direction (in Watts/sr)
    - » e.g.: energy distribution of a light source (inverse square law)
- $\boxed{\text{Radiance (L)}}$
  - Radiant intensity per unit projected surface area (in Watts/m^2sr)
    - » e.g.: light carried by a single ray (no inverse square law)
- $\boxed{\text{Irradiance (E)}}$
  - Incident flux density on a locally planar area (in Watts/m^2)
    - » e.g.: light hitting a surface along a
- Radiosity (B)
  - Exitant flux density from a locally planar area (in Watts/ m^2)

1. Reflection Functions: **BRDF**

Bi-Directional Reflectance Distribution Function

- Based on incident light and view direction
- Relates incoming light to outgoing light energy
- A framework for different materials

Materials as BRDFs



https://cdfg.csail.mit.edu
/wojciech/brdfdatabase

1. Reflection Functions: **BRDF**

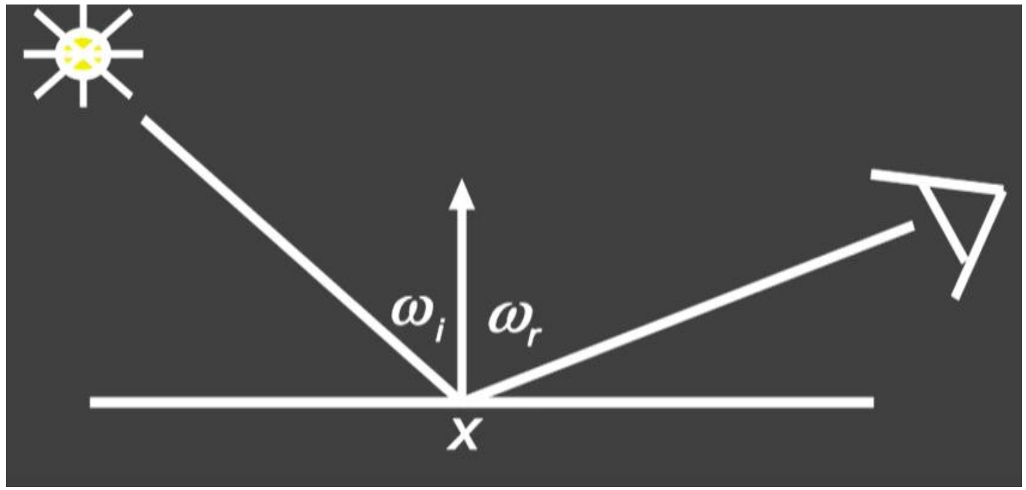
Definition: A constant of proportionality: reflects **Radiance** (watts/m^2sr) (<u>outgoing light</u>) from a surface is proportional to the incident **Irradiance** (watts/m^2) (<u>incoming light</u>)

$$f(\omega_i, \omega_r) = \frac{\boxed{L_r(\omega_r)}}{\boxed{L_i(\omega_i)\cos\theta_i d\omega_i}}$$

$$L_r(\omega_r) = L_i(\omega_i) f(\omega_i, \omega_r) \cos\theta_i d\omega_i$$

where **L** is radiance, or power per unit solid-angle-in-the-direction-of-a-ray per unit projected-area-perpendicular-to-the-ray, **E** is irradiance, or power per unit surface area, and $\boldsymbol{\theta_i}$ is the angle between $\mathbf{w_i}$ and the surface normal, **n**.

2. Reflection Equation

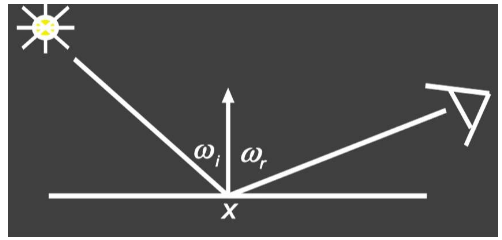


$$L_r(x,\omega_r) = L_e(x,\omega_r) + L_i(x,\omega_i) f(x,\omega_i,\omega_r)(\omega_i \bullet n)$$

Reflected Light (Output Image)	Emission	Incident Light (from light source)	BRDF

https://www.youtube.com/watch?v=GwxMTKVjhu8

2. Reflection Equation



The light source energy from the surface. For the most of the term, object’s surface has no light so $\mathbf{L_e}$ is 0 in this case

To get a Radiance , which unit is Watt/m2sr

$$L_r(x,\omega_r) = L_e(x,\omega_r) + \boxed{L_i(x,\omega_i) f(x,\omega_i,\omega_r)(\omega_i \bullet n)}$$

Reflected Light (Output Image)	Emission	Incident Light (from light source)	BRDF

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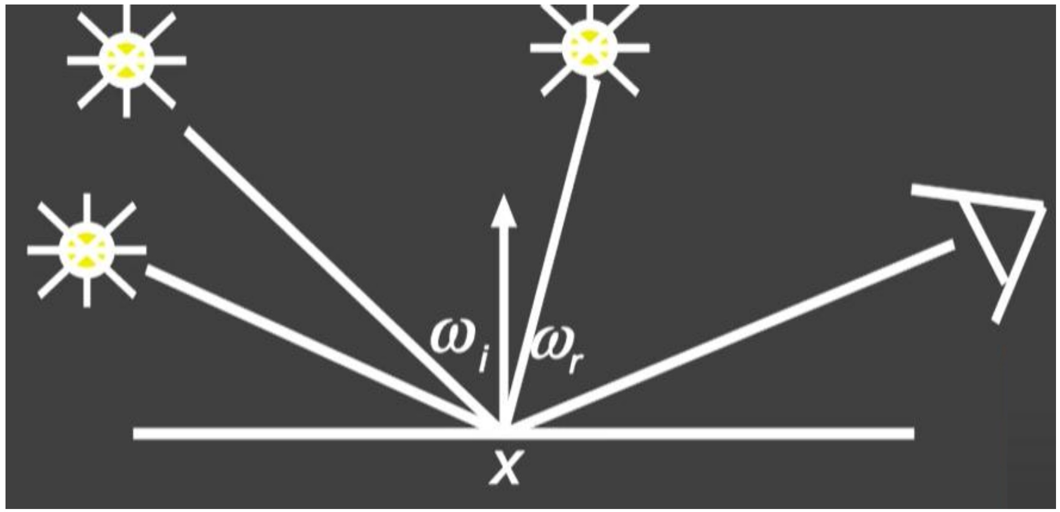
2. Reflection Equation (Multiple light source)



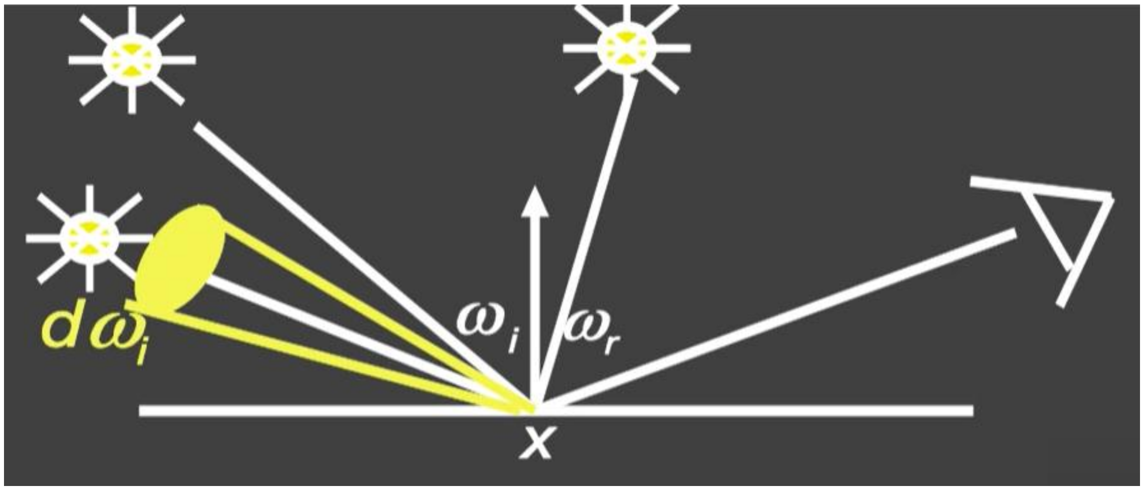
Sum over all light sources

$$L_r(x,\omega_r) = L_e(x,\omega_r) + \sum L_i(x,\omega_i) f(x,\omega_i,\omega_r)(\omega_i \bullet n)$$

Reflected Light (Output Image)	Emission	Incident Light (from light source)	BRDF	Cosine of Incident angle

https://www.youtube.com/watch?v=GwxMTKVjhu8

2. Reflection Equation (Intergral format, compared it with Rendering Equation, it do not consider the hemisphere for incident light)



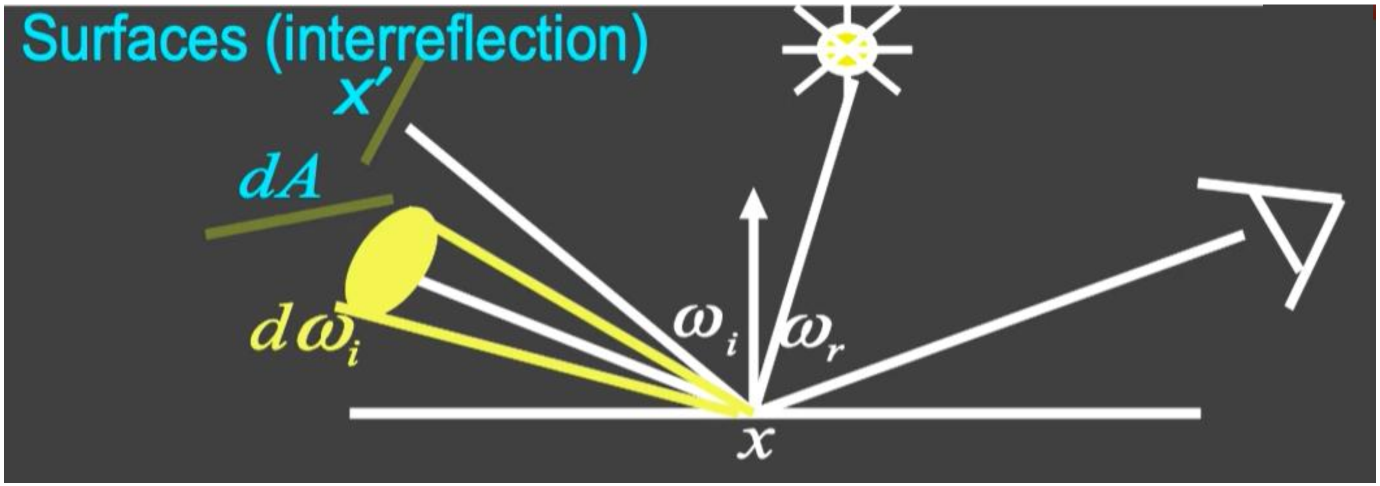
Replace sum with integral

$$L_r(x,\omega_r) = L_e(x,\omega_r) + \int_{\Omega} L_i(x,\omega_i) f(x,\omega_i,\omega_r) \cos\theta_i \, d\omega_i$$

Reflected Light (Output Image)	Emission	Incident Light (from light source)	BRDF	Cosine of Incident angle

https://www.youtube.com/watch?v=GwxMTKVjhu8

3. Rendering Equation



$$L_r(x,\omega_r) = L_e(x,\omega_r) + \int_{\Omega} L_r(x',-\omega_i) f(x,\omega_i,\omega_r) \cos\theta_i \, d\omega_i$$

Reflected Light (Output Image)	Emission	Reflected Light	BRDF	Cosine of Incident angle
UNKNOWN	KNOWN	UNKNOWN	KNOWN	KNOWN

https://www.youtube.com/watch?v=GwxMTKVjhu8

3. Rendering Equation

How to solve this equation as both left side and right side are unknow?

$$L_r(x,\omega_r) = L_e(x,\omega_r) + \int_{\Omega} L_r(x',-\omega_i) f(x,\omega_i,\omega_r) \cos\theta_i \, d\omega_i$$

Reflected Light (Output Image)	Emission	Reflected Light	BRDF	Cosine of Incident angle
UNKNOWN	KNOWN	UNKNOWN	KNOWN	KNOWN

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3. Rendering Equation

$$L_r(x,\omega_r) = L_e(x,\omega_r) + \int_{\Omega} L_r(x',-\omega_i) f(x,\omega_i,\omega_r) \cos\theta_i d\omega_i$$

Reflected Light (Output Image)	Emission	Reflected Light	BRDF	Cosine of Incident angle
UNKNOWN	KNOWN	UNKNOWN	KNOWN	KNOWN

This format looks close to the Fredholm Intergral Equation of second kind

Equation of the second kind [edit]

An inhomogeneous Fredholm equation of the second kind is given as

$$\varphi(t) = f(t) + \lambda \int_a^b K(t,s)\varphi(s)\,\mathrm{d}s.$$

Given the kernel $K(t,s)$, and the function $f(t)$, the problem is typically to find the function $\varphi(t)$.

https://www.youtube.com/watch?v=GwxMTKVjhu8

3. Rendering Equation

$$L_r(x,\omega_r) = L_e(x,\omega_r) + \int_{\Omega} L_r(x',-\omega_i) f(x,\omega_i,\omega_r) \cos\theta_i d\omega_i$$

Reflected Light (Output Image)	Emission	Reflected Light	BRDF	Cosine of Incident angle
UNKNOWN	KNOWN	UNKNOWN	KNOWN	KNOWN

This format looks close to the Fredholm Intergral Equation of second kind

$$l(u) = e(u) + \int l(v)\, K(u,v) dv$$

Let use mathematical canonical form to express it!

https://www.youtube.com/watch?v=GwxMTKVjhu8

Linear Operator Theory

- Linear operators act on functions like matrices act on vectors or discrete representations

$$h(u) = (M \circ f)(u)$$

M is a linear operator.
f and h are functions of u

- Basic linearity relations hold

a and b are scalars
f and g are functions

$$M \circ (af + bg) = a(M \circ f) + b(M \circ g)$$

- Examples include integration and differentiation

$$(K \circ f)(u) = \int k(u,v) f(v)\,dv$$

$$(D \circ f)(u) = \frac{\partial f}{\partial u}(u)$$

https://www.youtube.com/watch?v=GwxMTKVjhu8

3. Rendering Equation

So we can treat it as a **Equation** between **vectors** and **matrix...** **K** is the matrix. **L** and **E** are vectors.

$$l(u) = e(u) + \int l(v) \boxed{K(u,v)dv}$$

Kernel of equation
Light Transport Operator

$$L = E + KL$$

Too hard for analytic solution. We use the numerical **Monte Carlo** methods to approximate set of all paths of light in scene.

https://www.youtube.com/watch?v=GwxMTKVjhu8

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3. Rendering Equation

Too hard for analytic solution. We use the numerical **Monte Carlo** methods to approximate set of all paths of light in scene.

$$L = E + KL$$
$$IL - KL = E$$
$$(I - K)L = E$$
$$L = (I - K)^{-1}E$$

Think about Tylor Expansion:

When variable x satisfies: $|x| < 1$,

we have: $(1 - x)^{-1} = (1 + x + x^2 + x^3 + x^4 + ...)$

Binomial Theorem

$$L = (I + K + K^2 + K^3 + ...)E$$
$$L = \boxed{E} + KE + K^2E + K^3E + ...$$

Note! In the **Reflaction Equation**, the emission can be 0. However, in this **Rendering Equation**, the emission E can not be 0, or the whole scene will be rendered into DARK!

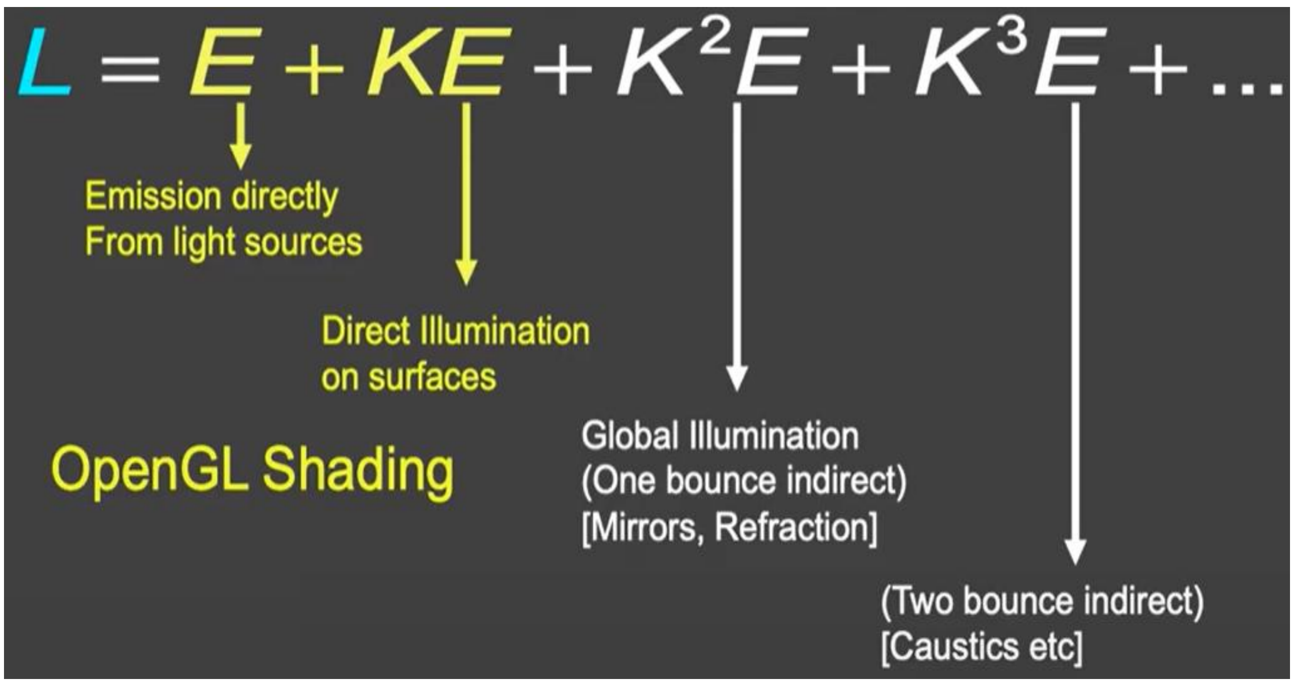
3. Rendering Equation

Binomial Theorem

$$L = (I + K + K^2 + K^3 + \ldots)E$$

$$L = E + \boxed{K}E + K^2E + K^3E + \ldots$$

The order of **K** corresponding to the bounce time of light



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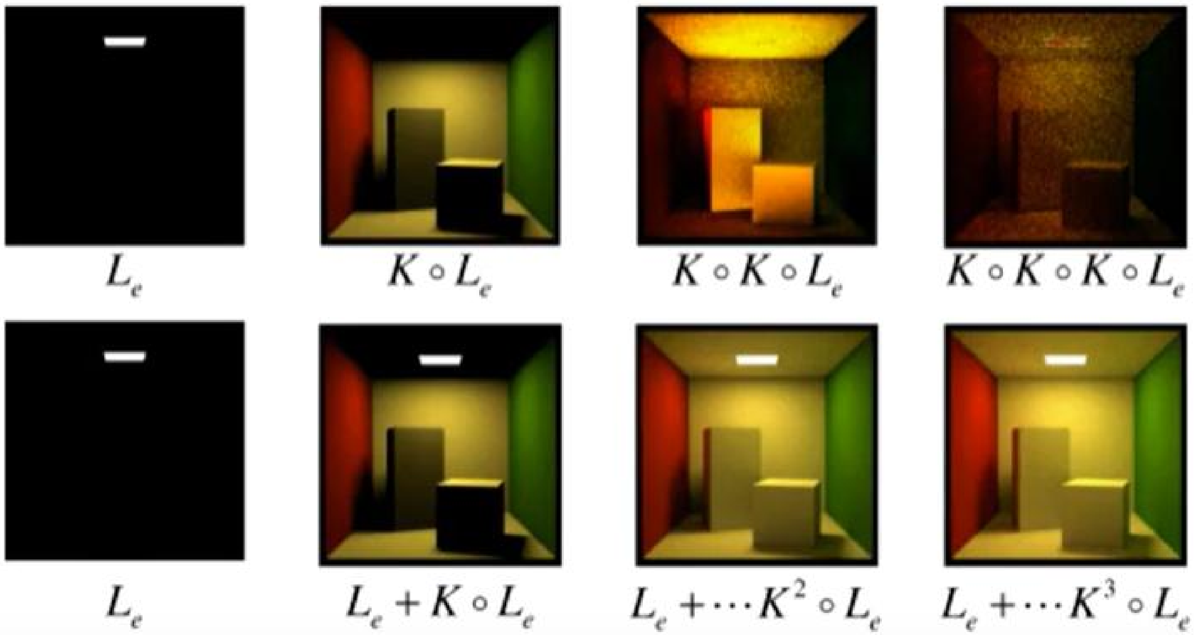
3. Rendering Equation

$$L = E + KE + K^2E + K^3E + \ldots$$

Successive Approximation



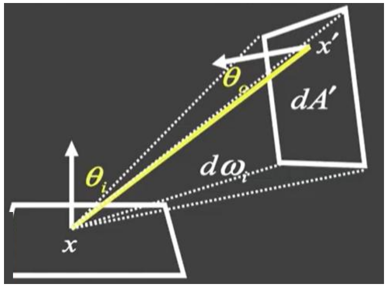
CS348B Lecture 13

Pat Hanrahan, Spring 2009

https://www.youtube.com/watch?v=GwxMTKVjhu8

3. Rendering Equation (change variable to surface)

$$L_r(x,\omega_r) = L_e(x,\omega_r) + \int_{\Omega} L_r(x',-\omega_i) f(x,\omega_i,\omega_r) \cos\theta_i \, d\omega_i$$



$$d\omega_i = \frac{dA' \cos\theta_o}{|x-x'|^2}$$

Integral over angles sometimes insufficient. Write integral in terms of surface radiance only (change of variables)

$$L_r(x,\omega_r) = L_e(x,\omega_r) + \int_{\text{all } x' \text{ visible to } x} L_r(x',-\omega_i) f(x,\omega_i,\omega_r) \boxed{\frac{\cos\theta_i \cos\theta_o}{|x-x'|^2}} \, dA'$$

Domain integral awkward.
Introduce binary visibility function V

$$G(x,x') = G(x',x) = \frac{\cos\theta_i \cos\theta_o}{|x-x'|^2}$$

Final Formula:

$$L_r(x,\omega_r) = L_e(x,\omega_r) + \int_{\text{all surfaces } x'} L_r(x',-\omega_i) f(x,\omega_i,\omega_r) G(x,x') V(x,x') \, dA'$$

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