Lemma 1

Let x be any node in Fibonacci heap, and suppose that $x.degree = k$. Let $y_1, y_2, \cdots, y_k$ denote the children of x in the order in which they were linked to x, from the earliest to the latest. Then, $y_1.degree \geq 0$ and $y_i.degree \geq i-2$ for $i = 2, 3, 4, \cdots k$

pf: Obviously $y_1.degree \geq 0$

For $i \geq 2$, we note that when y_i was linked to x, all of $y_1, y_2 \cdots, y_{i-1}$ were children of x, and so we must have $x.degree \geq i-1$. Because node y_i was linked to x (by CONSOLIDATE) only if $x.degree = y_i.degree$, we also had $y_i.degree \geq i-1$ at that time.

Since then, node y_i has lost at most one child, because it would have been cut from x (by CASCADING-CUT) if it had lost two children. We conclude that $y_i.degree \geq i-2$.

Fibonacci number is defined by

$$F_k = \begin{cases} 0 & \text{if } k=0 \\ 1 & \text{if } k=1 \\ F_{k-1} + F_{k-2} & \text{if } k \geq 2 \end{cases}$$

Lemma 2

For all integers $k \geq 0$

$$F_{k+2} = 1 + \sum_{i=0}^{k} F_i$$

The proof is by induction

Base case : when $k=0$,

$$1 + \sum_{i=0}^{0} F_k = 1 + F_0 = 1 = F_2$$

Assume that $F_{k+1} = 1 + \sum_{i=0}^{k-1} F_i$ and we have

$$\begin{aligned} F_{k+2} &= F_k + F_{k+1} \\ &= F_k + \left(1 + \sum_{i=0}^{k-1} F_i\right) \\ &= 1 + \sum_{i=0}^{k} F_i \end{aligned}$$

Lemma 3

For all integers $k \geq 0$, the $(k+2)$ Fibonacci number satisfies $F_{k+2} \geq \phi^k$

The proof is by induction of k.

Base case:

when $k=0$ $\quad F_2 = 1 = \phi^0$

when $k=1$ $\quad F_3 = 2 = 1.619 > \phi^1$

Inductive step: for $k \geq 2$,

we assume that $F_{i+2} > \phi^i$ for $i = 0, 1, \cdots, k-1$

Recall that ϕ is the positive root of $x^2 = x+1$

$(\therefore \phi^2 = \phi + 1)$

$$
\begin{aligned}
F_{k+2} &= F_{k+1} + F_k \\
&\geq \phi^{k-1} + \phi^{k-2} \quad \text{by induction} \\
&= \phi^{k-2}(\phi + 1) \\
&= \phi^{k-2}\phi^2 \\
&= \phi^k
\end{aligned}
$$

Lemma 4

Let x be any node in Fibonacci heap, and let $k = x.degree$. Then $size(x) \geq F_{k+2} \geq \phi^k$

where $\phi = (1+\sqrt{5})/2$

proof

Let S_k denote the minimum possible size of any node of degree k in any Fibonacci heap. Trivially, $S_0 = 1$ and $S_1 = 2$.

The number S_k is at most $size(x)$, because adding children to a node cannot decrease the node's size, the value of S_k increase monotonically with k.

Consider some node z, in any Fibonacci heap, such that $z.degree = k$ and $size(z) = S_k$.

Because $S_k \leq size(x)$, we compute a lower bound on $size(x)$ by computing a lower bound on S_k

As Lemma 1, let $y_1, y_2, \ldots, y_k$ denote the children of z in order to in which they linked to z.

To bound S_k, we count one for z itself and one for the first child y_1, giving

$$\begin{aligned} size(x) &\geq S_k \\ &\geq 2 + \sum_{i=2}^{k} S_{y_i.degree} \\ &\geq 2 + \sum_{i=2}^{k} S_{i-2} \qquad (\text{from Lemma 1 } y_i.degree \geq i-2) \end{aligned}$$

Now, we show by induction on k that $S_k \geq F_{k+2}$ for all nonnegative k.

Base case

for $k=0$, $S_0 = 1 = 1 = F_2$

for $k=1$, $S_1 = 2 = 2 = F_3$

Inductive step, we assume that $k \geq 2$ and that $S_i \geq F_{i+2}$ for $i = 0, 1, \ldots, k-1$

$$\begin{aligned} S_k &\geq 2 + \sum_{i=2}^{k} S_{i-2} \geq 2 + \sum_{i=2}^{k} F_i \\ &= 1 + \sum_{k=0}^{k} F_i \\ &= F_{k+2} \qquad (\text{by Lemma 2}) \\ &\geq \phi^k \qquad (\text{by Lemma 3}) \end{aligned}$$

Hence we have $size(x) \geq S_k \geq F_{k+2} \geq \phi^k$

Corollary

The maximum degree $D(n)$ of any node in an n-node Fibonacci heap is $O(\lg n)$

pf:

Let x be any node in an n-node Fibonacci heap, and let $k = x.degree$.

By Lemma 4, we have $n \geq size(x) \geq \phi^k$.

Taking base-ϕ logarithm gives us $k \leq \log_\phi n$

(because k is an integer, $k \leq \lfloor \log_\phi n \rfloor$)

Thus, the maximum degree $D(n)$ of any node is $O(\lg n)$