A very quickly growing function is defined by

$$A_k(j) \begin{cases} j+1 & \text{if } k=0 \\ A_{k-1}^{(j+1)}(j) & \text{if } k\geq 1 \end{cases}$$ for $k\geq 0$ and $j\geq 1$
$k, j \in \mathbb{Z}$

Specifically, $A_{k-1}^{0}(j) = j$

i) Use induction to prove $A_0^{(i)}(j) = j+i$ and derive $A_1(j)$

base case $A_0^{(0)}(j) = j = j+0$

Assume that $A_0^{(i-1)}(j) = j+(i-1)$, then

$A_0^{(i)}(j) = A_0(A_0^{(i-1)}(j)) = (j+(i-1))+1 = j+i$

因為 $A_0^{(i)}(j) = j+1$ 可以推得

$A_1(j) = A_0^{(j+1)}(j) = 2j+1$ for any integer $j\geq 1$

ii) Use induction to prove $A_1^{(i)}(j) = 2^i(j+1)-1$ and derive $A_2(j)$

base case $A_1^{(0)}(j) = 2^0(j+1)-1 = j$

Assume that $A_1^{(i-1)}(j) = 2^{i-1}(j+1)-1$, then

$$A_1^{(i)}(j) = A_1(A_1^{(i-1)}(j)) = A_1(2^{i-1}(j+1)-1)$$

$$= 2\left[2^{i-1}(j+1)-1\right]+1 = 2^{i}(j+1)-1$$

因為 $A_1^{(i)}(j) = 2^{i}(j+1)-1$ 可推得

$A_2(j) = A_1^{(j+1)}(j) = 2^{j+1}(j+1)-1$ for any integer $j \geq 1$

$A_1(1) = 2\times 1+1 = 3$ , $A_2(1) = 2^2 \times 2 - 1 = 7$

$$A_3(1) = A_2^{(2)}(1) = A_2(A_2(1)) = A_2(7)$$
$$= 2^{7+1}(7+1)-1 = 2^{11}-1 = 2047$$

$$A_4(1) = A_3^{(2)}(1) = A_3(A_3(1)) = A_3(2047)$$
$$= A_2^{2048}(2047)$$

$$\gg A_2(2047) = 2^{2048} \times 2048 - 1$$

$$> 2^{2048} = 16^{512} > 10^{512} > 10^{80}$$

Hence, we can conclude that $A_4(1)$ is much larger than $10^{80}$