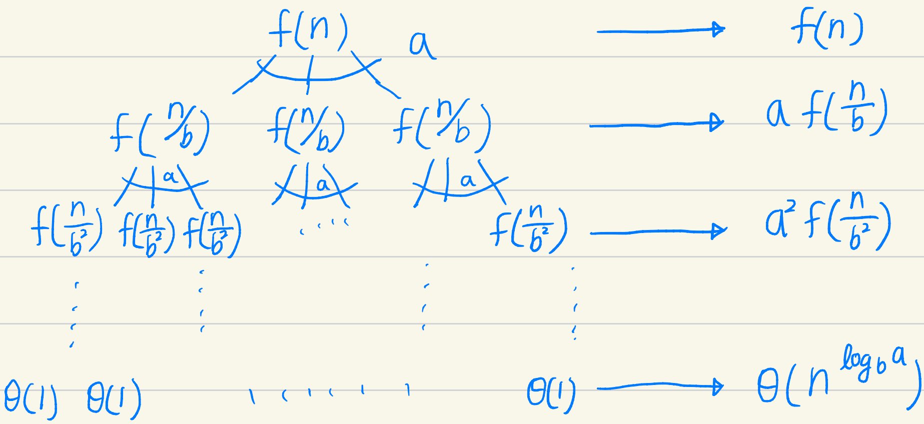
首先，用 recursion tree 的方式 求遞迴關係



在 depth j 時，有 a^j 個 nodes，而每一個 node 的 cost 為 $f(\frac{n}{b^j})$

leaf node 位於 depth $= \log_b n$ $(\because n = b^{\log_b n})$，且全部共有 $a^{\log_b n} = n^{\log_b a}$ 個 leaf nodes，因此 total cost of leaf node $= \Theta(n^{\log_b a})$

而 total cost of internal nodes $= \sum_{j=0}^{\log_b n-1} a^j f(\frac{n}{b^j})$

令 $g(n) = \sum_{j=0}^{\log_b n-1} a^j f(\frac{n}{b^j})$

則 $\boxed{T(n) = \Theta(n^{\log_b a}) + g(n)}$

Case 1 $\quad f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$

$\because f(n) = O(n^{\log_b a - \varepsilon}) \quad \therefore f(\frac{n}{b^j}) = O((\frac{n}{b^j})^{\log_b a - \varepsilon})$

$$\Rightarrow g(n) = O\left(\sum_{j=0}^{\log_b n - 1} a^j \left(\frac{n}{b^j}\right)^{\log_b a - \varepsilon}\right)$$

相加後仍被 O-notation 限制

$$\sum_{j=0}^{\log_b n - 1} a^j \left(\frac{n}{b^j}\right)^{\log_b a - \varepsilon} = n^{\log_b a - \varepsilon} \sum_{j=0}^{\log_b n - 1} \left(\frac{a}{b^{\log_b a - \varepsilon}}\right)^j$$

$$b^{\log_b a - \varepsilon} = b^{\log_b a} / b^{\varepsilon} = \frac{a}{b^{\varepsilon}}$$

$$= n^{\log_b a - \varepsilon} \sum_{j=0}^{\log_b n - 1} (b^{\varepsilon})^j$$

等比級數和

$$= n^{\log_b a - \varepsilon}\left(\frac{b^{\varepsilon \log_b n} - 1}{b^{\varepsilon} - 1}\right)$$

$$= n^{\log_b a - \varepsilon}\left(\frac{n^{\varepsilon} - 1}{b^{\varepsilon} - 1}\right)$$

因為 b, ε 都是常數 可將 $n^{\log_b a - \varepsilon}\left(\frac{n^{\varepsilon} - 1}{b^{\varepsilon} - 1}\right)$ 寫為 $n^{\log_b a - \varepsilon} O(n^{\varepsilon})$

$\therefore g(n) = O(n^{\log_b a})$

又 $T(n) = \Theta(n^{\log_b a}) + g(n)$

$$= \Theta(n^{\log_b a}) + O(n^{\log_b a}) = \Theta(n^{\log_b a})$$

得出 $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$，

則 $T(n) = \Theta(n^{\log_b a})$

Case 2　設 $f(n)=\Theta(n^{\log_b a}) \Rightarrow f(\frac{n}{b^j}) = \Theta((\frac{n}{b^j})^{\log_b a})$

代入 $g(n)$ 得　$\Theta(\sum_{j=0}^{\log_b n-1} a^j (\frac{n}{b^j})^{\log_b a})$

相加仍被 Θ- notation 限制

$$\sum_{j=0}^{\log_b n-1} a^j \left(\frac{n}{b^j}\right)^{\log_b a} = n^{\log_b a} \sum_{j=0}^{\log_b n-1} \left(\frac{a}{b^{\log_b a}}\right)^j$$

$$= n^{\log_b a} \sum_{j=0}^{\log_b n-1} 1 = n^{\log_b a} \log_b n$$

因此　$g(n) = \Theta(n^{\log_b a} \log_b n) = \Theta(n^{\log_b a} \lg n)$

又　$T(n) = \Theta(n^{\log_b a}) + g(n)$

$$= \Theta(n^{\log_b a}) + \Theta(n^{\log_b a} \lg n)$$

$$= \Theta(n^{\log_b a} \lg n)$$

得出　$f(n) = \Theta(n^{\log_b a})$ 時，$T(n) = \Theta(n^{\log_b a} \lg n)$

Case 3

若 $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$

$a f(\frac{n}{b}) \le c f(n)$ for some constant $c < 1$

$a f(\frac{n}{b}) \le c f(n)$ 迭代 j 次得 $a^j f(\frac{n}{b^j}) \le c^j f(n)$

$$g(n) = \sum_{j=0}^{\log_b n - 1} a^j f\left(\frac{n}{b^j}\right) \le \sum_{j=0}^{\log_b n - 1} c^j f(n) + O(1)$$

$$\le f(n) \sum_{j=0}^{\infty} c^j + O(1)$$

無窮等比級數

$$= f(n) \left(\frac{1}{1-c}\right) + O(1) = O(f(n))_{(1)} \quad (\because c \text{ is a constant})$$

$$g(n) = \sum_{j=0}^{\log_b n - 1} a^j f\left(\frac{n}{b^j}\right) \ge f(n) = \Omega(n^{\log_b a + \varepsilon}) = \Omega(n^{\log_b a})_{(2)}$$

由 (1) (2) 可得 $g(n) = \Theta(f(n))$

又 $T(n) = \Theta(n^{\log_b a}) + g(n) = \Theta(n^{\log_b a}) + \Theta(f(n)) = \Theta(f(n))$

$(\because f(n) = \Omega(n^{\log_b a + \varepsilon}))$

得出 $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$ 且

$a f(\frac{n}{b}) \le c f(n)$ for some constant $c < 1$ 時，

$$T(n) = \Theta(f(n))$$