首先定義 load factor 及 potential function

load factor $\alpha(T) = \dfrac{T.num}{T.size}$

$$\phi(T) \begin{cases} 2\,T.num - T.size & \text{if } \alpha(T) \geq \frac{1}{2} \\ T.size/2 - T.num & \text{if } \alpha(T) < \frac{1}{2} \end{cases}$$

以下分為 8 種情形討論

i th operation

insert
- $\alpha_{i-1} \geq \frac{1}{2}$
  - no expansion (1)
  - expansion (2)
- $\alpha_{i-1} < \frac{1}{2}$
  - $\alpha_i < \frac{1}{2}$ (3)
  - $\alpha_i \geq \frac{1}{2}$ (4)

delete
- $\alpha_{i-1} < \frac{1}{2}$
  - no contraction (5)
  - contraction (6)
- $\alpha_{i-1} \geq \frac{1}{2}$
  - $\alpha_i \geq \frac{1}{2}$ (7)
  - $\alpha_i < \frac{1}{2}$ (8)

Let amortized cost be $\hat{c}_i$, actual cost be c_i

α_i denote the load factor after i th operation case (1) $\alpha_{i-1} \geq \frac{1}{2}$ and ith TABLE-INSERT does Not trigger an expansion

$$
\begin{aligned}
\hat{c}_i &= c_i + \phi_i - \phi_{i-1} \\
&= 1 + (2\ num_i - T.size_i) - (2\ num_{i-1} - T.size_{i-1}) \\
&= 1 + (2\ num_i - \cancel{T.size_i}) - (2(num_i - 1) - \cancel{T.size_i}) \\
&= 3
\end{aligned}
$$

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case (2) $\alpha_{i-1} \geq \frac{1}{2}$ and i th TABLE-INSERT does trigger an expansion

$num_i = num_{i-1} + 1$ $\quad\quad$ $T.size_i = 2\,T.size_{i-1}$

$(num_{i-1} = T.size_{i-1})$

$$
\begin{aligned}
\hat{c}_i &= c_i + \phi_i - \phi_{i-1} \\
&= num_i + (2\,num_i - T.size_i) - (2\,num_{i-1} - T.size_{i-1}) \\
&= (num_{i-1} + 1) + (2(num_{i-1}+1) - 2\,num_{i-1}) - (2\,num_{i-1} - num_{i-1}) \\
&= 3
\end{aligned}
$$

case (3) $\alpha_{i-1} < \frac{1}{2}$ and i th TABLE-INSERT does Not trigger and $\alpha_i < \frac{1}{2}$

$T.size_i = T.size_{i-1}$

$$
\begin{aligned}
\hat{c}_i &= c_i + \phi_i - \phi_{i-1} \\
&= 1 + (T.size_i/2 - num_i) - (T.size_{i-1}/2 - num_{i-1}) \\
&= 1 + (T.size_i/2 - num_i) - (T.size_i/2 - (num_i - 1)) \\
&= 0
\end{aligned}
$$

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case (4) If $\alpha_{i-1} < \frac{1}{2}$ but $\alpha_i \geq \frac{1}{2}$

$$
\begin{aligned}
\hat{c}_i &= c_i + \phi_i - \phi_{i-1} \\
&= 1 + (2\,num_i - T.size_i) - (T.size_{i-1}/2 - num_{i-1}) \\
&= 1 + \big(2(num_{i-1} + 1) - T.size_{i-1}\big) - (T.size_{i-1}/2 - num_{i-1}) \\
&= 3\,num_{i-1} - \tfrac{3}{2} T.size_{i-1} + 3 \\
&= 3\,\alpha_{i-1} \times T.size_{i-1} - \tfrac{3}{2} T.size_{i-1} + 3 \\
&< \tfrac{3}{2} T.size_{i-1} - \tfrac{3}{2} T.size_{i-1} + 3 = 3
\end{aligned}
$$

Thus, the amortized cost of a TABLE-INSERT is at most 3.

case (5) $\alpha_{i-1} < \frac{1}{2}$ and i^{th} TABLE-DELETE and does Not cause contraction

$$\hat{c}_i = c_i + \phi_i - \phi_{i-1}$$

$$= 1 + (T.size_i/2 - num_i) - (T.size_{i-1}/2 - num_{i-1})$$

$$= 1 + (T.size_i/2 - num_i) - (T.size_i/2 - (num_i + 1))$$

$$= 2$$

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case (6) $\alpha_{i-1} < \frac{1}{2}$ and i^{th} TABLE-DELETE and trigger a contraction

$$\frac{T.size_{i-1}}{4} = num_{i-1}$$

$$\hat{c}_i = c_i + \phi_i - \phi_{i-1}$$

$$= num_i + 1 + (T.size_i/2 - num_i) - (T.size_{i-1}/2 - num_{i-1})$$

$$= num_{i-1} + \left(num_{i-1} - (num_{i-1} - 1)\right) - \left(2num_{i-1} - num_{i-1}\right)$$

$$= 1$$

$\because$ delete and move num_i item $\therefore c_i = num_i + 1$

case (7) $\alpha_{i-1} \geq \frac{1}{2}$ and $\alpha_i \geq \frac{1}{2}$

$$\hat{c_i} = c_i + \phi_i - \phi_{i-1}$$
$$= 1 + (2num_i - T.size_i) - (2num_{i-1} - T.size_{i-1})$$

$$= 1 + (2num_i - T.size_i) - \left(2(num_i + 1) - T.size_i\right)$$
$$= -1$$

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case (8) $\alpha_{i-1} \geq \frac{1}{2}$ but $\alpha_i < \frac{1}{2}$

$$\hat{c_i} = c_i + \phi_i - \phi_{i-1}$$
$$= 1 + (T.size_i/2 - num_i) - (2num_{i-1} - T.size_{i-1})$$
$$= 1 + \left(T.size_{i-1}/2 - (num_{i-1} - 1)\right) - (2num_{i-1} - T.size_{i-1})$$

$$= 2 + \frac{3}{2} T.size_{i-1} - 3num_{i-1}$$

$$= 2 + \frac{3}{2} T.size_{i-1} - 3 T.size_{i-1} \times \alpha_{i-1}$$
$$\leq 2 + \frac{3}{2} T.size_{i-1} - \frac{3}{2} T.size_{i-1} = 2$$

Hence, insertion and deletion operations are constant in amortized time.