Manifolds of Hilbert space homeomorphic to sphere in Finite von Neumann algebra

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ABSTRACT. Enter abstract here

1. INTRODUCTION

Let $\mathcal{H}$ be a Hilbert space, $Proj(\mathcal{H})$ be the set of self-adjoint projections and $Uint(\mathcal{H})$ be the set of unitary operators. We will routinely identify a closed subspace with its associated orthogonal projection in $B(\mathcal{H})$. For a set $\mathcal{L}$ of orthogonal projections in $Proj(\mathcal{H})$, we denote by $Alg\mathcal{L}$ the set of all bounded linear operators on $\mathcal{H}$ leaving each element in $\mathcal{L}$ invariant. Then $Alg\mathcal{L}$ is an unital weak-operator closed subalgebra of $B(\mathcal{H})$. Similarly, for a subset $\mathcal{S}$ of $B(\mathcal{H})$, let $Lat\mathcal{S}$ be the set of invariant projections for every operators in $\mathcal{S}$. Then $Lat\mathcal{S}$ is a strong-operator closed lattice of projections. A subalgebra $\mathcal{A}$ of $B(\mathcal{H})$ is said to be reflexive if $\mathcal{A} = Alg\mathcal{L}at\mathcal{A}$, similarly a lattice $\mathcal{L}$ of projections is reflexive if $\mathcal{L} = LatAlg\mathcal{L}$.

2. HILBERT FIELDS OVER SPHERE

Suppose $Q(\infty)$, $Q(0)$ and $Q(-1)$ are three projections in a finite von Neumann algebra $\mathfrak{A}$, and $Q(\infty)$, $Q(0)$ and $Q(-1)$ are in general position, i.e., the intersection of any two is zero and the join of any two is I, then $\mathfrak{A} \cong Q(\infty)\mathfrak{A}Q(\infty) \otimes M_2(\mathbb{C})$ [15, Proposition 2.4]. Moreover we can write $Q(\infty)$, $Q(0)$ and $Q(-1)$ in terms of 2×2 operator matrices (with respect to the canonical matrix units in $I \otimes M_2(\mathbb{C})$) as follows:

$$
\begin{aligned}
&Q(\infty) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, Q(0) = \begin{pmatrix} H_1 & \sqrt{H_1(I-H_1)} \\ \sqrt{H_1(I-H_1)} & I - H_1 \end{pmatrix}, \\
&Q(-1) = \begin{pmatrix} H_2 & \sqrt{H_2(I-H_2)}V \\ V^*\sqrt{H_2(I-H_2)} & V^*(I-H_2)V \end{pmatrix},
\end{aligned} \tag{1}
$$

where H_i is a contractive positive operator in $Q(\infty)\mathfrak{A}Q(\infty)$ such that $Ker(I - H_i) = 0, i = 1, 2$, and V is a unitary operator in $Q(\infty)\mathfrak{A}Q(\infty)$.

In order to describe the invariant subspace lattice $\mathcal{L}at(\mathcal{A}lg(\{Q(\infty), Q(0), Q(-1)\}))$, unbounded operator will be used. Let $\mathfrak{A}$ be a von Neumann algebra, and $\widetilde{\mathfrak{A}}$ be the set of closed, densely defined operators affiliated with $\mathfrak{A}$. When $\mathfrak{A}$ is finite, the family of operators affiliated with $\mathfrak{A}$ has remarkable properties, that is the operators in $\widetilde{\mathfrak{A}}$ admits algebraic operations of addition and multiplication. In another word, $\widetilde{\mathfrak{A}}$ is an unital * algebra (cf. [23]), and the elements in $\widetilde{\mathfrak{A}}$ can be manipulated as if they were bounded operators. In the rest of the paper, we will repeatedly make use this fact without mentioning it explicitly. For a elegant treatment of this subject, we refer readers to [24].

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Theorem 2.1 ([15], Theorem 2.1). *With the above notation and assumptions, we have $\mathcal{L}at(\mathcal{A}lg(\{Q(\infty), Q(0), Q(-1)\})) \setminus \{0, I\}$ endowed with the strong operator topology is homeomorphic to $\mathbb{C} \cup \{\infty\} (\cong S^2)$ with the homeomorphism given by*

$$\rho[Q(\infty), Q(0), Q(-1)](z) = \begin{pmatrix} K_z & \sqrt{K_z(I-K_z)}U_z \\ U_z^*\sqrt{K_z(I-K_z)} & U_z^*(I-K_z)U_z \end{pmatrix}, \qquad \forall z \in \mathbb{C},$$

and $\rho[Q(\infty), Q(0), Q(-1)](\infty) = Q(\infty)$, where K_z and U_z are uniquely determined by the following polar decomposition:

$$\begin{aligned} \sqrt{K_z(I-K_z)^{-1}}U_z &= (1+z)\sqrt{H_1(I-H_1)^{-1}} - z\sqrt{H_2(I-H_2)^{-1}}V \\ &= zS + \sqrt{H_1(I-H_1)^{-1}} \qquad (S = \sqrt{H_1(I-H_1)^{-1}} - \sqrt{H_2(I-H_2)^{-1}}V). \end{aligned} \tag{2}$$

Moreover, the reflexive lattice $\mathcal{L}at(\mathcal{A}lg(\{Q(\infty), Q(0), Q(-1)\}))$ can be determined by arbitrary three nontrivial projections (not 0 or I) in it.

Remark 2.1. *Since $\sqrt{K_0(I-K_0)^{-1}}U_0 = 0 \times S + \sqrt{H_1(I-H_1)^{-1}}$ implies $K_0 = H_0$ and $U_0 = I$, we have $\rho[Q(\infty), Q(0), Q(-1)](0) = Q(0)$. Similarly, $\rho[Q(\infty), Q(0), Q(-1)](-1) = Q(-1)$. Therefore, we will also use $Q(z)$ to denote $\rho[Q(\infty), Q(0), Q(-1)](z)$ throughout the rest of this paper.*

Definition 2.1 (Definition 2.1. (i) in [4]). *A topological subspace manifold in $B(\mathcal{H})$ of dimension n is a set $\mathcal{M} \subset Proj(\mathcal{H})$, considered with the relative strong operator topology, which is locally homeomorphic to $\mathbb{R}^n$.*

Given any there projections $Q(\infty), Q(0), Q(-1)$ in a finite von Neumann algebra, we have $\mathcal{L}at(\mathcal{A}lg(\{Q(\infty), Q(0), Q(-1)\})) \setminus \{0, I\}$ is a topological subspace manifold of dimension 2 if $Q(\infty), Q(0), Q(-1)$ are in general position by Theorem 2.1.

Example 2.1 (Tautological line bundle over $\mathbb{CP}^1$). *Let*

$$Q(\infty) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, Q(0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, Q(-1) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

then

$$Q(z) = \frac{1}{1+|z|^2}\begin{pmatrix} |z|^2 & z \\ \overline{z} & 1 \end{pmatrix}.$$

Note that the map $\phi : \mathbb{CP}^1 \to S^2$ given by $\phi([z_1, z_2]) = z_1/z_2$ is a homeomorphism. We have

$$\phi^*(Q)([z_1, z_2]) = \frac{1}{|z_1|^2 + |z_2|^2}\begin{pmatrix} |z_1|^2 & z_1\overline{z_2} \\ \overline{z_1}z_2 & |z_2|^2 \end{pmatrix}$$

is a line bundle over $\mathbb{CP}^1$. It is actually the tautological line bundle over $\mathbb{CP}^1$.

In this paper will study the subgroup of automorphisms of $\mathfrak{A}$ which leaves $Lat(\mathcal{A}lg(\{Q(\infty), Q(0), Q(-1)\}))$ invariant. The rest of this paper is organized as follows. Next, we point out that the above result naturally induce a coordinate chart of the reflexive lattice generated by a double triangle lattice of projections (exclude 0 and I) in a finite von Neumann algebra. In section 3, we prove that the transition maps between the charts are Möbius transformations. We then study the $\mathcal{L}$-invariant subgroup of automorphisms of a von Neumann algebra $\mathfrak{A}$, where $\mathcal{L}$ is a reflexive subspace lattice contained in $\mathfrak{A}$. In particular, we show if $\mathfrak{A}$ is a finite factor or finite dimensional, and $\mathcal{L}$ is generated by a double triangle lattice of projections in $\mathfrak{A}$, then the $\mathcal{L}$-invariant automorphism group is homeomorphic to a closed subgroup of $SO(3)$(Corollary **??**). In section 4, we compute the $\mathcal{L}$-invariant automorphism group when $\mathfrak{A}$ is the interpolated free group factor $L_{F_{\frac{3}{2}}}$, and $\mathcal{L}$ is determined by the three free projection that generate $L_{F_{\frac{3}{2}}}$. In the appendix, we give a detailed proof of the following fact: If $\mathcal{L}$ is a reflexive lattice in a von Neumann algebra $\mathfrak{A}$, and φ is a *-isomorphism of $\mathfrak{A}$, then $\varphi(\mathcal{L})$ is also reflexive.

Since any three projections in $\mathcal{L}at(\mathcal{A}lg(\{Q(\infty), Q(0), Q(-1)\})) \setminus \{0, I\}$ are in the general position, form Theorem Theorem 2.1, we have the following corollary.

Corollary 2.1. *With the notations in the Theorem 2.1 and the Remark 2.1, suppose $Q(z_1)$, $Q(z_2)$ and $Q(z_3)$ are three nontrivial projections in $\mathcal{L}at(\mathcal{A}lg(\{Q(\infty), Q(0), Q(-1)\}))$, then there is a homeomorphism $\rho[Q(z_1), Q(z_2), Q(z_3)]$ form S^2 onto $\mathcal{L}at(\mathcal{A}lg(\{Q(\infty), Q(0), Q(-1)\})) \setminus \{0, I\}$ such that $\rho[Q(z_1), Q(z_2), Q(z_3)](z)$ is determined by the following relation:*

(3)
$$\begin{aligned}(I - Q(z_1)\rho[Q(z_1), Q(z_2), Q(z_3)](z)Q_{z_1})^{-1}[Q(z_1)\rho[Q(z_1), Q(z_2), Q(z_3)](z)(I - Q(z_1))] \\ = (1+z)(I - Q(z_1)Q(z_2)Q(z_1))^{-1}[Q(z_1)Q(z_2)(I - Q(z_1))] \\ - z(I - Q(z_1)Q(z_3)Q(z_1))^{-1}[Q(z_1)Q(z_3)(I - Q(z_1))].\end{aligned}$$

By (3), $\rho[Q(z_1), Q(z_2), Q(z_3)](\infty) = Q(z_1)$, $\rho[Q(z_1), Q(z_2), Q(z_3)](0) = Q(z_2)$ and $\rho[Q(z_1), Q(z_2), Q(z_3)](-1) = Q(z_3)$.

The inverse of the homeomorphism $\rho[Q(z_1), Q(z_2), Q(z_3)]$ in the Corollary 2.1 actually gives a coordinate chart of $(\mathcal{L}at(\mathcal{A}lg(\{Q(\infty), Q(0), Q(-1)\})) \setminus \{0, I, Q(z_1)\}$, $\rho[Q(z_1), Q(z_2), Q(z_3)]^{-1})$ of $\mathcal{L}at(\mathcal{A}lg(\{Q(\infty), Q(0), Q(-1)\})) \setminus \{0, I\}$. So $\mathcal{L}at(\mathcal{A}lg(\{Q(\infty), Q(0), Q(-1)\})) \setminus \{0, I\}$ is a 2-dimensional (topological) manifold with atlas $\{\rho[Q(z_1), Q(z_2), Q(z_3)]^{-1} | z_1, z_2, z_3 \in \mathbb{C} \cup \{\infty\}\}$. In the next section, we will determine the transition maps between the charts in this atlas.

3. Determined by two idempotent

Let $Q(\infty)$, $Q(0)$ and $Q(-1)$ be trace half projections in a finite von Neumann algebra $\mathfrak{A}$. If $Q(\infty) \wedge Q(i) = 0$, $i = 0, -1$, then we have there exists a two closed idempotents S_1 and S_2 such that S_1 and S_2 are affiliated with $\mathfrak{A}$ and $\mathcal{A}lg(Q(\infty), Q(0), Q(-1)) = \{T | TS_i \subset S_i T, i = 1, 2\}$ by Lemma 2.3 in [16].

Now assume that T_1 and T_2 be two closed operator affiliated with a finite von Neumann algebra $\mathfrak{A}$. Consider two idempotents in $\mathfrak{A} \otimes M_2(\mathbb{C})$:

$$S_1 = \begin{pmatrix} I & T_1 \\ 0 & 0 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} I & T_2 \\ 0 & 0 \end{pmatrix}.$$

Let $Q(\infty) = Ran(S_1)$, $Q(0) = Ker(S_1)$ and $Q(-1) = Ker(S_2)$ be three trace half projections. Note that

$$\mathcal{A}lg(Q(\infty), Q(0), Q(-1)) = \{T | TS_i \subset S_i T, i = 1, 2\}.$$

By conjugating a unitary, we could assume that

$$T_1 - T_2 = \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}$$

where $K \geq 0$ and K has a closed inverse. We also write T_1 as $\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$.

It is not hard to check that $\mathcal{A}lg(Q(\infty), Q(0), Q(-1))$ contains the following elements

$$\begin{pmatrix} A_1 & 0 & A_1T_{11} - T_{11}K^{-1}A_1K & A_1T_{12} \\ 0 & 0 & -T_{21}K^{-1}A_1K & 0 \\ 0 & 0 & K^{-1}A_1K & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_2 & A_2T_{21} & A_2T_{22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & A_3 & A_3T_{21} & A_3T_{22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & -T_{12}D_1 & 0 \\ 0 & 0 & -T_{22}D_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & D_1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & -T_{12}D_2 \\ 0 & 0 & 0 & -T_{22}D_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_2 \end{pmatrix}$$

If $Q \in \mathcal{L}at(\mathcal{A}lg(Q(\infty), Q(0), Q(-1)))$ such that $Q \wedge Q(\infty) = 0$ and $Q \wedge Q(\infty) = I$, then there is a idempotent

$$S = \begin{pmatrix} I & T_1 + S_1 \\ 0 & 0 \end{pmatrix}$$

such that $SA = AS$ for any $A \in \mathcal{A}lg(Q(\infty), Q(0), Q(-1))$. This implies that

$$\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = 0 \text{ and } \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} = 0.$$

Therefore S_{12}, S_{21} and S_{22} are all equals 0. Since

$$\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} S_{11} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} S_{11} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} K^{-1}A_1K & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{for all } A_1 \in Ran(K)\mathfrak{A}Ran(K).$$

This implies that $S = zK$ for some $z \in \mathbb{C}$. We will denote this projection by $Q(z)$.

If $Q \wedge Q(\infty) = 0$ and $Q \vee Q(\infty) \neq I$, then

$$Q \vee Q(\infty) = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix} = E.$$

Therefore, there must exist a $\beta = (\xi_1, \xi_2, \xi_3, \xi_4)^T \in Q\mathcal{H}$ such that $\xi_4 \neq 0$. This implies that

$$\{ \begin{pmatrix} -T_{12}\xi \\ -T_{22}\xi \\ 0 \\ \xi \end{pmatrix} |\xi\} \subset Q\mathcal{H}.$$

Consider the trace of Q, we know that

$$\{ \begin{pmatrix} -T_{12}\xi \\ -T_{22}\xi \\ 0 \\ \xi \end{pmatrix} |\xi\} = Q\mathcal{H}.$$

It is not hard to check that

$$Q(z_1) \wedge Q(z_2) = \{ \begin{pmatrix} -T_{12}\xi \\ -T_{22}\xi \\ 0 \\ \xi \end{pmatrix} |\xi\}$$

for any z_1 and $z_2 \in \mathbb{C}$.

If $Q \wedge Q(\infty) \neq 0$, then it is easy to see that

$$Q \wedge Q(\infty) = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = F.$$

Since $E \leq Q \vee Q(\infty)$, $Q(z_1) \wedge Q(z_2) \leq Q$. If $Q \vee Q(\infty) = E$, then $\tau(Q) = \frac{1}{2}$. Hence

$$Q = \{ \begin{pmatrix} \xi_1 \\ -T_{22}\xi \\ 0 \\ \xi \end{pmatrix} |\xi_1, \xi\} = F \vee Q(z), \qquad z \in \mathbb{C}.$$

The last possibility is $Q \vee Q(\infty) = I$. Then there must exists a vector $(\xi_1, \xi_2, \xi_3, \xi_4)^T \in Q$ such that $\xi_3 \neq 0$. Note that $\tau(Q) = \frac{1}{2} + \tau(F)$ and

$$\{ \begin{pmatrix} 0 \\ -T_{21}\xi \\ \xi \\ 0 \end{pmatrix} |\xi\} \subset Q\mathcal{H}.$$

Then it is not hard to see that

$$\{ \begin{pmatrix} \xi_1 \\ -T_{21}\xi_2 - T_{22}\xi_3 \\ \xi_2 \\ \xi_3 \end{pmatrix} |\xi_1, \xi_2, \xi_3\} = Q = Q(z_1) \vee Q(z_2)$$

for any z_1 and $z_2 \in C$.

Remark 3.1. *$E = Q(\infty) \vee (Q(0) \wedge Q(-1))$ and $F = (Q(0) \vee Q(-1)) \wedge Q(\infty)$.*

4. Homotopy property of the bundle

Theorem 4.1. *Let $Q(\infty)$, $Q(0)$, $Q(-1)$ and $P(\infty)$, $P(0)$, $P(-1)$ be trace half projections in a finite factor $\mathfrak{A}$. If $Q(i) \wedge Q(j) = 0 = P(i) \wedge P(j)$ and $Q(i) \vee Q(j) = I = P(i) \vee P(j)$, $i \neq j$ and $i, j \in \{0, -1\}$, then there exist continuous paths $P(i, t)$ of trace half projections such that $P(i, 0) = P(i)$ and $P(i, 1) = Q(i)$ where $i = \infty, 0, -1$. Furthermore, we can request that $P(\infty, t)$, $P(0, t)$ and $P(-1, t)$ are in general position and these maps can be extended to be a continuous map form $S^1 \times [0, t]$ into the set of trace half projections such that $P(z, 0) = P(z)$ and $P(z, 1) = Q(z)$.*

Proof. We could assume that $Q(\infty) = \left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $Q(0) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right)$, $Q(-1) = \left(\begin{smallmatrix} \frac{1}{2}I & \frac{1}{2}I \\ \frac{1}{2}I & \frac{1}{2}I \end{smallmatrix}\right)$. Let W be a unitary such that $W^*P(\infty)W = Q(\infty)$. Since the set of unitaries $U(\mathfrak{A})$ is connected, there exists a path $W(t)$ in $U(\mathfrak{A})$ such that $W(0) = I$ and $W(1) = W$. By considering $W(t)^*P(i)W(t)$, we may assume that $Q(\infty) = P(\infty)$.

Now assume that

$$P(0) = \begin{pmatrix} K_1 & \sqrt{K_1(I-K_1)}V_1 \\ V_1^*\sqrt{K_1(I-K_1)} & V_1^*(I-K_1)V_1 \end{pmatrix}, \quad P(-1) = \begin{pmatrix} K_2 & \sqrt{K_2(I-K_2)}V_2 \\ V_2^*\sqrt{K_2(I-K_2)} & V_2^*(I-K_2)V_2 \end{pmatrix}.$$

Then $P(0)\mathcal{H} = \{(\sqrt{\frac{K_1}{I-K_1}}V_1\xi, \xi)\}$ and $P(-1)\mathcal{H} = \{(\sqrt{\frac{K_2}{I-K_2}}V_2\xi, \xi)\}$. Let

$$P(\infty, t) = P(\infty),\, P(0,t)\mathcal{H} = \{(1-t)\sqrt{\frac{K_1}{I-K_1}}V_1\xi, \xi)\} \text{ and}$$
$$P(-1,t)\mathcal{H} = \{(\sqrt{\frac{K_2}{I-K_2}}V_2 - t\sqrt{\frac{K_1}{I-K_1}}V_1)\xi, \xi)\}.$$

Let

$$P(z,t) = \begin{pmatrix} K(z,t) & \sqrt{K(z,t)(I-K(z,t))}V(z,t) \\ V(z,t)^*\sqrt{K(z,t)(I-K(z,t))} & V(z,t)^*(I-K(z,t))V(z,t) \end{pmatrix},$$

where

$$\sqrt{\frac{K(z,t)}{I-K(z,t)}}V(z,t) = zS + (1-t)\sqrt{\frac{K_1}{I-K_1}}V_1, \quad S = \sqrt{\frac{K_1}{I-K_1}}V_1 - \sqrt{\frac{K_2}{I-K_2}}V_2.$$

We will show that $P(z,t)$ is continuous.

$$\begin{aligned}
&\|P(z,t_1) - P(z,t_2)\|_2^2 = 1 - 2\tau(P(z,t_1)P(z,t_2)) \\
&= tr(K(z,t_1)(K(z,t_1) - K(z,t_2))) \\
&+ tr(V(z,t_1)^*(I-K(z,t_1))V(z,t_1)(V(z,t_2)^*K(z,t_2)V(z,t_2) - V(z,t_1)^*K(z,t_1)V(z,t_1))) \\
&+ tr(\sqrt{K(z,t_1)(I-K(z,t_1))}V(z,t_1)(V(z,t_1)^*\sqrt{K(z,t_1)(I-K(z,t_1))} - V(z,t_2)^*\sqrt{K(z,t_2)(I-K(z,t_2))}) \\
&+ tr(V(z,t_1)^*\sqrt{K(z,t_1)(I-K(z,t_1))}(\sqrt{K(z,t_1)(I-K(z,t_1)))}V(z,t_1) - \sqrt{K(z,t_2)(I-K(z,t_2))}V(z,t_2)).
\end{aligned}$$

Let

$$F(z,t) = \frac{K(z,t)}{I-K(z,t)} = |z|^2SS^* + (1-t)(zSV_1^*\sqrt{\frac{K_1}{I-K_1}} + \bar{z}\sqrt{\frac{K_1}{I-K_1}}V_1S^*) + (1-t)^2\frac{K_1}{I-K_1}.$$

Note that

$$\begin{aligned}
K(z,t_1) - K(z,t_2) &= (I+F(z,t_2))^{-1}(F(z,t_1) - F(z,t_2))(I+F(z,t_1))^{-1} \\
&= (t_2 - t_1)(I+F(z,t_2))^{-1}(zSV_1^*\sqrt{\frac{K_1}{I-K_1}} + \bar{z}\sqrt{\frac{K_1}{I-K_1}}V_1S^* \\
&\qquad + (2-t_1-t_2)\frac{K_1}{I-K_1})(I+F(z,t_1))^{-1}.
\end{aligned}$$

For any $\varepsilon > 0$, let E be a projection such that $\tau(E) \geq 1-\varepsilon$ and

$$\|SV_1^*\sqrt{\frac{K_1}{I-K_1}}E\| \leq C, \|\sqrt{\frac{K_1}{I-K_1}}V_1S^*E\| \leq C, \|\frac{K_1}{I-K_1}E\| \leq C, \|\sqrt{\frac{K_1}{I-K_1}}E\| \leq C,$$

where C is a constant determined by E. Let $E_1(z,t)$ be the projection such that $(I-F(z,t))^{-1}E_1(z,t) = E(I-F(z,t))^{-1}E_1(z,t)$. Then

$$\begin{aligned}
|tr(K(z,t_1)(K(z,t_1) - K(z,t_2)))| &\leq |tr(K(z,t_1)(K(z,t_1) - K(z,t_2)E_1(z,t_1)))| \\
&\qquad + |tr(K(z,t_1)(K(z,t_1) - K(z,t_2)(I-E_1(z,t_1))))| \\
&\leq |t_2 - t_1|(2|z|+2)C + 2\varepsilon.
\end{aligned}$$

Now consider

$$(\sqrt{K(z,t_1)(I-K(z,t_1))})V(z,t_1)-\sqrt{K(z,t_2)(I-K(z,t_2))}V(z,t_2))V(z,t_1)^*\sqrt{K(z,t_1)(I-K(z,t_1))}$$

$$=(K(z,t_2)-K(z,t_1))K(z,t_1)+(t_2-t_1)(1-K(z,t_2))\sqrt{\frac{K_1}{I-K_1}}V_1V(z,t_1)^*\sqrt{K(z,t_1)(I-K(z,t_1))}.$$

Let $E_2(z,t)$ be a projection such that $tr(E_2(z,t))\geq 1-\varepsilon$ and

$$\|\sqrt{\frac{K_1}{I-K_1}}V_1V(z,t)^*\sqrt{K(z,t)(I-K(z,t))}E_2(z,t)\|\leq C.$$

Then

$$|tr((\sqrt{K(z,t_1)(I-K(z,t_1))})V(z,t_1)-\sqrt{K(z,t_2)(I-K(z,t_2))}V(z,t_2))V(z,t_1)^*\sqrt{K(z,t_1)(I-K(z,t_1))})|$$
$$\leq|t_2-t_1|(2|z|+3)C+4\varepsilon.$$

The similar argument will give us similar inequlities

$$|tr(V(z,t_1)^*(I-K(z,t_1))V(z,t_1)(V(z,t_2)^*K(z,t_2)V(z,t_2)-V(z,t_1)^*K(z,t_1)V(z,t_1)))|$$
$$\leq|t_2-t_1|(2|z|+2)C+2\varepsilon$$

and

$$|tr(\sqrt{K(z,t_1)(I-K(z,t_1))}V(z,t_1)(V(z,t_1)^*\sqrt{K(z,t_1)(I-K(z,t_1))}-V(z,t_2)^*\sqrt{K(z,t_2)(I-K(z,t_2))})|$$
$$\leq|t_2-t_1|(2|z|+3)C+4\varepsilon.$$

This implies that $P(z,t)$ is continuous at all point except (∞,t). Argue exactly as in the proof of Proposition 2.2 in [15], we can show that $P(z,t)$ is also continuous at (∞,t).

Note that $P(0,1)=Q(0,1)$. Thus we now assume that $P(0)=Q(0)$ and

$$P(-1)=\begin{pmatrix}K_2 & \sqrt{K_2(I-K_2)}V_2\\ V_2^*\sqrt{K_2(I-K_2)} & V_2^*(I-K_2)V_2\end{pmatrix}.$$

By considering the path

$$P(-1,t)\mathcal{H}=\{(\sqrt{\frac{K_2}{I-K_2}}V(t)\xi,\xi)\},$$

where $V(t)$ is a continuous map form $[0,1]$ into the set of unitaries of $P(\infty)\mathfrak{A}P(\infty)$ such that $V(0)=V_2$ and $V(1)=I$. We may assume that $V_2=I$. Now

$$P(-1,t)\mathcal{H}=\{(\frac{t}{2}I+(1-t)\sqrt{\frac{K_2}{I-K_2}})\xi,\xi)\},$$

connect $P(-1)$ and $Q(-1)$. □

Lemma 4.1. *With the same notations as in Theorem 2.1, there is a dense subset of $\mathcal{H}$ for any fix z such that $\frac{\partial Q(z)\xi}{\partial z}$ and $\frac{\partial Q(z)\xi}{\partial \bar{z}}$ exist for any ξ in the subset. Actually,*

$$\frac{\partial Q(z)}{\partial z}=\begin{pmatrix}(I-K_z) & 0\\ 0 & U_z^*\sqrt{K_z(I-K_z)}\end{pmatrix}\times\begin{pmatrix}S & S\\ -S & -S\end{pmatrix}\times\begin{pmatrix}U_z^*\sqrt{K_z(I-K_z)} & 0\\ 0 & U_z^*(I-K_z)U_z\end{pmatrix},$$

and $\frac{\partial Q(z)\xi}{\partial \bar{z}}=(\frac{\partial Q(z)\xi}{\partial z})^*$.

If S is bounded, then the map $(x,y) \to Q(x+iy)$ is C^∞. Next we will show that for any map $\mathbb{R}^2 \to Proj(\mathcal{H})$ defined by Theorem 2.1, there exists a C^∞ map approximate it.

Lemma 4.2. *Let P and Q be two trace half projections in a finite von Neumann algebra $\mathfrak{A}$. If $\|P-Q\|_2 \le \varepsilon$, then there exists a unitary $U \in \mathfrak{A}$ such that $U^*PU = Q$ and $\|U-I\|_2 \le \sqrt{2}\varepsilon$.*

Proof. Let $E = P \wedge Q$, $F = P \vee Q$. If $E \neq 0$, then $(F-E)P \wedge (F-E)Q = 0$. If there exists a $U_1 \in (F-E)\mathfrak{A}(F-E)$ such that $U_1^*(F-E)PU_1 = (F-E)Q$ and $\|U_1 - I\|_2 \le \sqrt{2}\varepsilon$. Then $U = E + U_1 + (I-F)$ will satisfies the conditions in the lemma.

Now, assume that $P \wedge Q = 0$, $P \vee Q = I$ and

$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} H & \sqrt{H(I-H)} \\ \sqrt{H(I-H)} & I-H \end{pmatrix},$$

where $H \ge 0$. Since $\|P-Q\|_2 \le \varepsilon$, $tr(I-H) \le \varepsilon^2$, where tr is the trace on $P\mathfrak{A}P$. Let

$$U = \begin{pmatrix} \sqrt{H} & \sqrt{I-H} \\ -\sqrt{I-H} & \sqrt{H} \end{pmatrix}.$$

It is clear that $U^*PU = Q$. Since $1-\varepsilon^2 \le tr(H) \le tr(\sqrt{H}) \le 1$, $\|U-I\|^2 = 2 - 2tr(\sqrt{H}) \le 2\varepsilon^2$. $\square$

Corollary 4.1. *Let $P(\infty)$, $P(0)$, $P(-1)$ and $Q(\infty)$ be trace half projections in a finite von Neumann algebra $\mathfrak{A}$. If $\|Q(\infty) - P(\infty)\|_2 \le \varepsilon$, and $P(\infty)$, $P(0)$, $P(-1)$ are in general position, then there exist trace half projections $Q(0)$ and $Q(-1)$ such that $Q(\infty)$, $Q(0)$, $Q(-1)$ are in general position and*

$$\|Q(z) - P(z)\| \le 2\sqrt{2}\varepsilon, \qquad \forall z \in \mathbb{C} \cup \{\infty\},$$

where $Q(z)$ and $P(z)$ are the projections determined by $P(\infty)$, $P(0)$, $P(-1)$ and $Q(\infty)$, $Q(0)$, $Q(-1)$ as in Theorem 2.1.

Proof. By Lemma 4.2, we have a unitary U in $\mathfrak{A}$ such that $U^*P(\infty)U = Q(\infty)$ and $\|U-I\|_2 \le \sqrt{2}\varepsilon$. It is clear that $Q(z) = U^*P(z)U$ satisfy the conditions. $\square$

Lemma 4.3. *Let $\mathfrak{A}$ be a finite von Neumann algebra and τ be a faithful normal trace on $\mathfrak{A}$. Suppose that P and Q are two trace half projections in $\mathfrak{A}$ such that $P \wedge Q = 0$ and $P \vee Q = I$. For any $\varepsilon > 0$, there exists a $\delta > 0$ such that if E and F are two projections in $\mathfrak{A}$ satisfying $\|E-P\|_2 \le \delta$ and $\|F-Q\|_2 \le \delta$, then $\tau(E \wedge F) \le \varepsilon$.*

Proof. Since $P \wedge Q = 0$, there exists a $n \in \mathbb{N}$ such that $\tau((PQ)^n) \le \frac{\varepsilon}{2}$. We claim that $\delta = \frac{\varepsilon}{4n}$ satisfies the condition. Indeed, if $\|E-P\|_2 \le \frac{\varepsilon}{4n}$ and $\|F-Q\|_2 \le \frac{\varepsilon}{4n}$, then

$$\begin{aligned}
|\tau(E \wedge F)| &\le |\tau((EF)^n)| \\
&\le \tau((PQ)^n) + \sum_{i=0}^{n-1} |\tau((PQ)^{n-i-1}(EF)^i E(F-Q))| + |\tau(Q(PQ)^{n-i-1}(EF)^i(E-P))| \\
&\le \tau((PQ)^n) + \sum_{i=0}^{n-1} \|F-Q\|_2 + \|E-P\|_2 \le \varepsilon.
\end{aligned}$$

$\square$

The following proposition is immediate from the lemma above.

Proposition 4.1. *Let* $Q(\infty) = \left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)$ *and*

$$Q(0) = \begin{pmatrix} H_1 & \sqrt{H_1(I-H_1)} \\ \sqrt{H_1(I-H_1)} & I-H_1 \end{pmatrix}, Q(-1) = \begin{pmatrix} H_2 & \sqrt{H_2(I-H_2)}U_2 \\ U_2^*\sqrt{H_2(I-H_2)} & U_2^*(I-H_2)U_2 \end{pmatrix}$$

be trace half projections in a finite von Neumann algebras. Suppose that $Q(\infty)$, $Q(0)$ *and* $Q(-1)$ *are in general position. Then for any* ε, *there exists two trace half projections*

$$P(0) = \begin{pmatrix} K_1 & \sqrt{K_1(I-K_1)}V_1 \\ V_1^*\sqrt{K_1(I-K_1)} & V_1^*(I-K_1)V_1 \end{pmatrix}, P(-1) = \begin{pmatrix} K_2 & \sqrt{K_2(I-K_2)}V_2 \\ V_2^*\sqrt{K_2(I-K_2)} & V_2^*(I-K_2)V_2 \end{pmatrix}$$

in $\mathfrak{A}$ *such that*

(1) $\|P(0) - Q(0)\| \le \varepsilon$ *and* $\|P(-1) - Q(-1)\| \le \varepsilon$;
(2) $\tau(Ran(\sqrt{\frac{H_1}{I-H_1}} - \sqrt{\frac{K_1}{I-K_1}}V_1)) \le \varepsilon$ *and* $\tau(Ran(\sqrt{\frac{H_2}{I-H_2}}U_2 - \sqrt{\frac{K_2}{I-K_2}}V_2)) \le \varepsilon$;
(3) $\|I - V_1\| \le \varepsilon$ *and* $\|U_2 - V_2\| \le \varepsilon$;
(4) $P(0) \wedge Q(\infty) = 0$ *and* $P(-1) \wedge Q(\infty) = 0$;
(5) $0 < \|K_1\| < 1$ *and* $0 < \|K_2\| < 1$;
(6) $P(0) \wedge P(-1) = 0$.

Proof. Let

$$h(x) = \begin{cases} \delta & x \in [0,\delta) \\ x & x \in [\delta, 1-\delta] \\ 1-\delta & x \in (1-\delta, 1] \end{cases}.$$

For δ small enough, Lemma 4.3 implies that

$$P(0) = \begin{pmatrix} K_1 & \sqrt{K_1(I-K_1)} \\ \sqrt{K_1(I-K_1)} & I-K_1 \end{pmatrix} \text{ and } F = \begin{pmatrix} \widetilde{H_2} & \sqrt{\widetilde{H_2}(I-\widetilde{H_2})}U_2 \\ U_2^*\sqrt{\widetilde{H_2}(I-\widetilde{H_2})} & U_2^*(I-\widetilde{H_2})U_2 \end{pmatrix},$$

satisfy $\tau(P(0) \wedge F) \le \frac{\varepsilon}{4}$, where $K_1 = h(H_1)$ and $\widetilde{H_2} = h(H_2)$. It is clear that $\sqrt{\frac{K_1}{I-K_1}}$ and $\sqrt{\frac{\widetilde{H_2}}{I-\widetilde{H_2}}}\widetilde{U}_2$ are bounded. Let $S = \sqrt{\frac{K_1}{I-K_1}} - \sqrt{\frac{\widetilde{H_2}}{I-\widetilde{H_2}}}\widetilde{U}_2$ and V be the partial isometry from $KerS$ onto $(RanS)^\perp$ (V exists, because $\mathfrak{A}$ is finite). We have $\tau(Ran(V)) \le \frac{\varepsilon}{4}$ and $Ker(S + zV) = \{0\}$ for any $z \neq 0$. Let

$$P(-1) = \begin{pmatrix} K_2 & \sqrt{K_2(I-K_2)}V_2 \\ K_2^*\sqrt{K_2(I-K_2)} & K_2^*(I-K_2)V_2 \end{pmatrix},$$

where

$$\sqrt{\frac{K_2}{I-K_2}}V_2 = \sqrt{\frac{\widetilde{H_2}}{I-\widetilde{H_2}}}\widetilde{U}_2 + zV.$$

If z is small enough, we have $P(-1)$ satisfies all the conditions in the proposition. □

Lemma 4.4. *Let* $Q(\infty) = \left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)$ *and*

$$Q(0) = \begin{pmatrix} H_1 & \sqrt{H_1(I-H_1)} \\ \sqrt{H_1(I-H_1)} & I-H_1 \end{pmatrix}, \quad Q(-1) = \begin{pmatrix} H_2 & \sqrt{H_2(I-H_2)}U_2 \\ U_2^*\sqrt{H_2(I-H_2)} & U_2^*(I-H_2)U_2 \end{pmatrix},$$

$$P(0) = \begin{pmatrix} K_1 & \sqrt{K_1(I-K_1)}V_1 \\ V_1^*\sqrt{K_1(I-K_1)} & V_1^*(I-K_1)V_1 \end{pmatrix}, \quad P(-1) = \begin{pmatrix} K_2 & \sqrt{K_2(I-K_2)}V_2 \\ V_2^*\sqrt{K_2(I-K_2)} & V_2^*(I-K_2)V_2 \end{pmatrix}$$

be trace half projections in a finite von Neumann algebra $\mathfrak{A}$. Suppose that $Q(\infty)$, $Q(0)$ and $Q(-1)$ are in general position and $Q(\infty)$, $P(0)$ and $P(-1)$ are in general position. If $\tau(Ran(\sqrt{\frac{H_1}{I-H_1}} - \sqrt{\frac{K_1}{I-K_1}}V_1)) \leq \varepsilon$ and $\tau(Ran(\sqrt{\frac{H_2}{I-H_2}}U_2 - \sqrt{\frac{K_2}{I-K_2}}V_2)) \leq \varepsilon$, then

$$\|Q(z) - P(z)\|_2 \leq 12\varepsilon \qquad \text{for any } z \in \mathbb{C}.$$

Proof. Let $F_1 = ker(\sqrt{\frac{H_1}{I-H_1}} - \sqrt{\frac{K_1}{I-K_1}}V_1)$, $F_2 = ker(\sqrt{\frac{H_2}{I-H_2}}U_2 - \sqrt{\frac{K_2}{I-K_2}}V_2)$ and $E = F_1 \wedge F_2$. Then $\tau(E) \geq 1 - 2\varepsilon$. Note that

$$\begin{aligned}\|Q(z) - P(z)\|_2^2 = 1 - 2\tau(Q(z)P(z))&\\ &= tr(H_z(H_z - K_z)) + tr(U_z^*(I - H_z)U_z(V_z^*K_zV_z - U_z^*H_zU_z))\\ &+ tr(\sqrt{H_z(I - H_z)}U_z(U_z^*\sqrt{H_z(I - H_z)} - V_z^*\sqrt{K_z(I - K_z)})\\ &+ tr(U_z^*\sqrt{H_z(I - H_z)}(\sqrt{H_z(I - H_z)}U_z - \sqrt{K_z(I - K_z)}V_z),\end{aligned}$$

where tr is the trace on $Q(\infty)\mathfrak{A}Q(\infty)$. Let

$$F(z) = \frac{H_z}{I - H_z} = \left((1+z)\sqrt{\frac{H_1}{I - H_z}} - z\sqrt{\frac{H_2}{I - H_2}}U_2\right)\left((1+\bar{z})\sqrt{\frac{H_1}{I - H_z}} - \bar{z}U_2^*\sqrt{\frac{H_2}{I - H_2}}\right)$$

$$G(z) = \frac{K_z}{I - K_z} = \left((1+z)\sqrt{\frac{K_1}{I - K_z}}V_1 - z\sqrt{\frac{K_2}{I - K_2}}V_2\right)\left((1+\bar{z})V_1^*\sqrt{\frac{K_1}{I - K_z}} - \bar{z}V_2^*\sqrt{\frac{K_2}{I - K_2}}\right)$$

Note that $EF(z)E = EG(z)E$ and $Ran(I - H_z) = Ran(I - K_z) = I$. Let E_1 and E_2 be the projections such that $(I + F(z))^{-1}E_1 = E(I + F(z))^{-1}E_1$ and $E_2(I + G(z))^{-1} = E_2(I + G(z))^{-1}E$. Note that $tr(E_1) = tr(E_2) = tr(E) \geq 1 - \varepsilon$. Then

$$\begin{aligned}tr(H_z(H_z - K_z)) &= tr(H_z(I - E_2)(H_z - K_z)) + tr(H_zE_2(H_z - K_z)(I - E_1))\\ &+ tr(H_zE_2(H_z - K_z)E_1).\end{aligned}$$

Note that

$$\begin{aligned}E_2(H_z - K_z)E_1 &= E_2((I + G(z))^{-1} - (I + F(z))^{-1})E_1\\ &= E_2(I + G(z))^{-1}E(F(z) - G(z))E(I + F(z))^{-1}E_1 = 0\end{aligned}$$

Therefore, $|tr(H_z(H_z - K_z))| \leq 2\varepsilon$. Similarly, we can show that

$$|tr(U_z^*(I - H_z)U_z(V_z^*K_zV_z - U_z^*H_zU_z))| \leq 2\varepsilon.$$

Let $E_3 = Ran(U_z^*\sqrt{H_z(I - H_z)})$ and $E_4 = E \wedge E_3$. Then $tr(E_4) \geq tr(E_3) - 2\varepsilon$. Then it is not hard to see that there exist a projection E_5 such that $tr(E_5) \geq 1 - 2\varepsilon$ and $U_z^*\sqrt{H_z(I - H_z)E_5} = E_4U_z^*\sqrt{H_z(I - H_z)}E_5$. Hence

$$\begin{aligned}&tr((\sqrt{H_z(I - H_z)}U_z - \sqrt{K_z(I - K_z)}V_z)U_z^*\sqrt{H_z(I - H_z)})\\ &= tr((\sqrt{H_z(I - H_z)}U_z - \sqrt{K_z(I - K_z)}V_z)U_z^*\sqrt{H_z(I - H_z)}E_5)\\ &+ tr((\sqrt{H_z(I - H_z)}U_z - \sqrt{K_z(I - K_z)}V_z)U_z^*\sqrt{H_z(I - H_z)}(1 - E_5)).\end{aligned}$$

By the definition of E_5, we have

$$\begin{aligned}&tr((\sqrt{H_z(I - H_z)}U_z - \sqrt{K_z(I - K_z)}V_z)U_z^*\sqrt{H_z(I - H_z)}E_5)\\ &= tr((K_z - H_z)\sqrt{\frac{H_z}{I - H_z}}U_z)U_z^*\sqrt{H_z(I - H_z)}E_5)\\ &= tr((K_z - H_z)H_zE_5).\end{aligned}$$

Therefore
$$|tr((\sqrt{H_z(I-H_z)}U_z - \sqrt{K_z(I-K_z)}V_z)U_z^*\sqrt{H_z(I-H_z)})| \leq 4\varepsilon.$$
Hence
$$\|Q(z)-P(z)\|_2 \leq 12\varepsilon.$$
□

Let $dQ(z) = \frac{\partial Q(z)}{\partial z}dz + \frac{\partial Q(z)}{\partial \bar{z}}d\bar{z}$.

Lemma 4.5. *With the notations as in Theorem 2.1, we have*
$$Q(z)(dQ(z)) = (dQ(z))(I-Q(z)) \text{ and } (dQ(z))Q(z) = (I-Q(z))(dQ(z)).$$

Proposition 4.2. *With the notations as in Theorem 2.1, we have*
$$\bigtriangledown_Q^2(z) \equiv Q(z)(dQ(z)\wedge dQ(z)) = (dQ(z))(I-Q(z))(dQ(z)) = (dQ(z)\wedge dQ(z))Q(z).$$

Remark 4.1. *Let*
$$W(z) = \begin{pmatrix} \sqrt{K_z} & \sqrt{I-K_z} \\ V_z^*\sqrt{I-K_z}V_z & -V_z^*\sqrt{K_z} \end{pmatrix}.$$
We have
$$W(z)^*(dQ(z))^2W(z) = \begin{pmatrix} \sqrt{I-K_z}U_zS^*(I-K_z)SU_z^*\sqrt{I-K_z} & 0 \\ 0 & \sqrt{I-K_z}SU_z^*(I-K_z)U_zS\sqrt{I-K_z} \end{pmatrix} dz\wedge d\bar{z},$$
$$W(z)^* \bigtriangledown_Q^2 (z)W(z) = \begin{pmatrix} \sqrt{I-K_z}U_zS^*(I-K_z)SU_z^*\sqrt{I-K_z} & 0 \\ 0 & 0 \end{pmatrix} dz\wedge d\bar{z}.$$

5. Geodesic of the bundle

Let $\mathfrak{A}$ be a finite von Neumann algebra. If $S \in \mathcal{I} = \{X^2 = I : X \in \mathfrak{A}\}$, then
$$S = \begin{pmatrix} I & 2B \\ 0 & -I \end{pmatrix} = \begin{pmatrix} I & -B \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} I & B \\ 0 & I \end{pmatrix},$$
and $T = \frac{S+I}{2}$ is an idempotent. Then the tangent space to S is given by
$$\begin{aligned} T_S &= \{X \in \mathfrak{A} : XS + SX = 0\} \\ &= \{\begin{pmatrix} I & -B \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & X_{12} \\ X_{21} & 0 \end{pmatrix}\begin{pmatrix} I & B \\ 0 & I \end{pmatrix} = \begin{pmatrix} -BX_{21} & -BX_{21}B + X_{12} \\ X_{21} & X_{21}B \end{pmatrix}\}. \end{aligned}$$
Let
$$E_S(X) = e^{\frac{XS}{2}}Se^{\frac{-XS}{2}}.$$
E_S will be called the exponential map. Note that $E_S(X) \in \mathcal{I}$ and $E_S(X) = e^{XS}S$, if $X \in T_S$.

It is obvious that T_S has a complex structure. Therefore S is a complex submanifold of $\mathfrak{A}$.

Let $\mathcal{R} = \{H^* = H : H \in \mathcal{I}\}$. Then $\mathcal{R}$ is only a (real) analytic submanifold of $\mathfrak{A}$, since
$$T_H = \{\begin{pmatrix} 0 & X_{12} \\ X_{12}^* & 0 \end{pmatrix}\}$$
is not a complex linear space.

Let $S \in \mathcal{I}$ and π_S be the projection of $\mathfrak{A}$ onto T_S given by
$$\pi_S(X) = \frac{1}{2}(X - SXS).$$
Recall the following definition.

Definition 5.1. *Suppose that E_1 and E_2 are Banach spaces, and that U is an open subset of E_1. A continuous map $f : U \to E_2$ is said to be differentiable at the point $x_0 \in U$ if there exists a continuous linear map $T : E_1 \to E_2$ such that*

$$\lim_{\|h\|\to 0} \frac{\|f(x_0+h) - f(x_0) - T(h)\|}{\|h\|} = 0.$$

T is called the derivative of f at x_0 and written as $Df(x_0)$. Note that

$$Df(x_0)h = \lim_{t\to 0} \frac{f(x_0+th) - f(x_0)}{t}.$$

Definition 5.2. *Let $T(\mathcal{I})$ be the tangent bundles over $\mathcal{I}$ and $D(\mathcal{I})$ be the set of all vector fields over $\mathcal{I}$, i.e. a smooth map form $\mathcal{I}$ into $T(\mathcal{I})$. Given X and Y in $D(\mathcal{I})$, let*

$$D(X,Y)(S) = \pi_S(DY(S)(X(S))) = \pi_S(\frac{d}{dt}Y(c(t))|_{t=0}),$$

where $c : [-1,1] \to \mathcal{I}$ be a smooth curve such that $c(0) = S$ and $c'(0) = X(S)$.

It is easy to see that $D(\cdot,\cdot)$ satisfies the standard axioms of a connection. Following the standard terminology, we say that a smooth curve g is a geodesic if

$$D(g',g') = \pi_{g(t)}(\frac{d^2}{dt^2}(g(t))) = 0.$$

Lemma 5.1 (Lemma 2.9 in [26]). *Given $S \in \mathcal{I}$ and $X \in T_S$, there exists a unique total geodesic $g = E_S(tX)$ such that $g(0) = S$ and $g'(0) = X$.*

Example 5.1. *Let $S = 2P - I$ and $T = 2Q - I$, where P and Q are two projections in $\mathfrak{A}$. If $\|S - T\| = 2\|P - Q\| < 2$, then $P \sim Q$ in $\mathfrak{A}$. Suppose that*

$$\begin{aligned} S &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \\ T &= \begin{pmatrix} 2H - I & 2\sqrt{H(I-H)} \\ 2\sqrt{H(I-H)} & I - 2H \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{H} & \sqrt{I-H} \\ \sqrt{I-H} & -\sqrt{H} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} \sqrt{H} & \sqrt{I-H} \\ \sqrt{I-H} & -\sqrt{H} \end{pmatrix}. \end{aligned}$$

Let

$$\begin{aligned} X &= \frac{\pi i}{2} \begin{pmatrix} I - \sqrt{H} & -\sqrt{I-H} \\ -\sqrt{I-H} & I + \sqrt{H} \end{pmatrix} \\ W &= \begin{pmatrix} \sqrt{H} & \sqrt{I-H} \\ \sqrt{I-H} & -\sqrt{H} \end{pmatrix} = e^X. \end{aligned}$$

Then the geodesic connect S and T is $g(t) = e^{tX}Se^{-tX}$. And

$$g'(0) = i\pi \begin{pmatrix} 0 & \sqrt{I-H} \\ -\sqrt{I-H} & 0 \end{pmatrix}.$$

Definition 5.3. *Given a smooth curve $c : [a,b] \to \mathcal{R}$, we define the length $L(c)$ of c by*

$$L(c) = \int_a^b \|c'(t)\|_2 dt.$$

Lemma 5.2. *The $\|\cdot\|_2$ give the geodesic distance on $\mathcal{R}$.*

6. Conclusion

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