ON THE SPHERE-INVARIANT AUTOMORPHISMS OF FINITE VON NEUMANN ALGEBRAS

LIMING GE AND WEI YUAN

Abstract. In this paper we show that the subgroup of automorphisms of a finite von Neumann algebra $\mathfrak{A}$ that leave $\mathcal{L}$ invariant is isomorphic to a subgroup of Möbius transformations, where $\mathcal{L}$ is a reflexive subspace lattice generated by a double triangle lattice of projections in $\mathfrak{A}$. As a particular case, we show the group is S_3, the symmetric group of 3 elements, when $\mathfrak{A}$ is the interpolated free group factor $L_{F_{\frac{3}{2}}}$, and $\mathcal{L}$ is the reflexive lattice determined by the three free projection that generate $L_{F_{\frac{3}{2}}}$.

1. Introduction and Preliminaries

Let $\mathcal{H}$ be a Hilbert space and $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. For a set $\mathcal{L}$ of orthogonal projections in $B(\mathcal{H})$, we denote by $Alg\mathcal{L}$ the set of all bounded linear operators on $\mathcal{H}$ leaving each element in $\mathcal{L}$ invariant. Then $Alg\mathcal{L}$ is an unital weak-operator closed subalgebra of $B(\mathcal{H})$. Similarly, for a subset $\mathcal{S}$ of $B(\mathcal{H})$, let $Lat\mathcal{S}$ be the set of invariant projections for every operators in $\mathcal{S}$. Then $Lat\mathcal{S}$ is a strong-operator closed lattice of projections. A subalgebra $\mathcal{A}$ of $B(\mathcal{H})$ is said to be reflexive if $\mathcal{A} = Alg\mathrm{Lat}\mathcal{A}$, similarly a lattice $\mathcal{L}$ of projections is reflexive if $\mathcal{L} = LatAlg\mathcal{L}$.

Suppose Q_∞, Q_0 and Q_{-1} are three projections in a finite von Neumann algebra $\mathfrak{A}$, and Q_∞, Q_0 and Q_{-1} are in general position, i.e., the intersection of any two is zero and the join of any two is I, then $\mathfrak{A} \cong Q_\infty \mathfrak{A} Q_\infty \otimes M_2(\mathbb{C})$[14, Proposition 2.4]. Morevoer we can write Q_∞, Q_0 and Q_{-1} in terms of 2×2 operator matrices (with respect to the standard matrix units in $I\otimes M_2(\mathbb{C})$) as follows:

$$\begin{aligned} Q_\infty &= \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, Q_0 = \begin{pmatrix} H_1 & \sqrt{H_1(I-H_1)} \\ \sqrt{H_1(I-H_1)} & I-H_1 \end{pmatrix}, \\ Q_{-1} &= \begin{pmatrix} H_2 & \sqrt{H_2(I-H_2)}V \\ V^*\sqrt{H_2(I-H_2)} & V^*(I-H_2)V \end{pmatrix}, \end{aligned} \tag{1}$$

where H_i is a contractive positive operator in $Q_\infty \mathfrak{A} Q_\infty$ such that $Ker(I-H_i) = 0$, $i = 1, 2$, and V is a unitary operator in $Q_\infty \mathfrak{A} Q_\infty$.

In order to describe the invariant subspace lattice $\mathrm{Lat}(\mathrm{Alg}(\{Q_\infty, Q_0, Q_{-1}\}))$, unbounded operator will be used. Let $\mathfrak{A}$ be a von Neumann algebra, and $\widetilde{\mathfrak{A}}$ be the set of closed, densely defined operators affiliated with $\mathfrak{A}$. When $\mathfrak{A}$ is finite, the family of operators affiliated with $\mathfrak{A}$ has remarkable properties, that is the operators in $\widetilde{\mathfrak{A}}$ admits algebraic operations of addition and multiplication. In another word, $\widetilde{\mathfrak{A}}$ is an unital * algebra (cf. [21]), and the elements in $\widetilde{\mathfrak{A}}$ can be manipulated as if

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they were bounded operators. In the rest of the paper, we will repeatedly make use this fact without mentioning it explicitly. For a elegant treatment of this subject, we refer readers to [22].

Theorem 1.1 ([14], Theorem 2.1)**.** *With the above notation and assumptions, we have* $\mathrm{Lat}(\mathrm{Alg}(\{Q_\infty, Q_0, Q_{-1}\})) \setminus \{0, I\}$ *is homeomorphic to* $\mathbb{C} \cup \{\infty\}(\cong S^2)$ *with the homeomorphism given by* $\rho[Q_\infty, Q_0, Q_{-1}](\infty) = Q_\infty$ *and*

$$\rho[Q_\infty, Q_0, Q_{-1}](z) = \begin{pmatrix} K_z & \sqrt{K_z(I-K_z)}U_z \\ U_z^* \sqrt{K_z(I-K_z)} & U_z^*(I-K_z)U_z \end{pmatrix}, \qquad \forall z \in \mathbb{C},$$

where K_z *and* U_z *are uniquely determined by the following polar decomposition:*

$$\begin{aligned} &\sqrt{K_z(I-K_z)^{-1}}U_z = (1+z)\sqrt{H_1(I-H_1)^{-1}} - z\sqrt{H_2(I-H_2)^{-1}}V \\ &= zS + \sqrt{H_1(I-H_1)^{-1}} \qquad (S = \sqrt{H_1(I-H_1)^{-1}} - \sqrt{H_2(I-H_2)^{-1}}V). \end{aligned} \tag{2}$$

Moreover, the reflexive lattice $\mathrm{Lat}(\mathrm{Alg}(\{Q_\infty, Q_0, Q_{-1}\}))$ *can be determined by arbitrary three nontrivial projections(not* 0 *or* I*) in it.*

Remark 1.1. *Since* $\sqrt{K_0(I-K_0)^{-1}}U_0 = 0 \times S + \sqrt{H_1(I-H_1)^{-1}}$ *implies* $K_0 = H_0$ *and* $U_0 = I$*, we have* $\rho[Q_\infty, Q_0, Q_{-1}](0) = Q_0$*. Similarly,* $\rho[Q_\infty, Q_0, Q_{-1}](-1) = Q_{-1}$*. Therefore, we will also use* Q_z *to denote* $\rho[Q_\infty, Q_0, Q_{-1}](z)$ *throughout the rest of this paper.*

Example 1.1 (Hopf line bundle on the two-sphere)**.** *To get the tautological line bundle over* CP_1*, let*

$$Q_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, Q_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, Q_{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

then $Q_z = \begin{pmatrix} \frac{x^2+y^2}{1+x^2+y^2} & \frac{x+iy}{1+x^2+y^2} \\ \frac{x-iy}{1+x^2+y^2} & \frac{1}{1+x^2+y^2} \end{pmatrix}$*, where* $z = x+iy$*. It is can be shown that it is the line bundle associated to the Hopf fibration* $S^1 \to S^3 \to S^2$*.*

In this paper will study the subgroup of automorphisms of $\mathfrak{A}$ which leaves $Lat(\mathrm{Alg}(\{Q_\infty, Q_0, Q_{-1}\}))$ invariant. The rest of this paper is organized as follows. Next, we point out that the above result naturally induce a coordinate chart of the reflexive lattice generated by a double triangle lattice of projections (exclude 0 and I) in a finite von Neumann algebra. In section 3, we prove that the transition maps between the charts are Möbius transformations. We then study the $\mathcal{L}$-invariant subgroup of automorphisms of a von Neumann algebra $\mathfrak{A}$, where $\mathcal{L}$ is a reflexive subspace lattice contained in $\mathfrak{A}$. In particular, we show if $\mathfrak{A}$ is a finite factor or finite dimensional, and $\mathcal{L}$ is generated by a double triangle lattice of projections in $\mathfrak{A}$, then the $\mathcal{L}$-invariant automorphism group is homeomorphic to a closed subgroup of $SO(3)$(Corollary 2.2). In section 4, we compute the $\mathcal{L}$-invariant automorphism group when $\mathfrak{A}$ is the interpolated free group factor $L_{F_{\frac{3}{2}}}$, and $\mathcal{L}$ is determined by the three free projection that generate $L_{F_{\frac{3}{2}}}$. In the appendix, we give a detailed proof of the following fact: If $\mathcal{L}$ is a reflexive lattice in a von Neumann algebra $\mathfrak{A}$, and φ is a *-isomorphism of $\mathfrak{A}$, then $\varphi(\mathcal{L})$ is also reflexive.

Since any three projections in $\mathrm{Lat}(\mathrm{Alg}(\{Q_\infty, Q_0, Q_{-1}\}))\setminus\{0, I\}$ are in the general position, form Theorem 1.1, we have the following corollary.

Corollary 1.1. *With the notations in the Theorem 1.1 and the Remark 1.1, suppose Q_{z_1}, Q_{z_2} and Q_{z_3} are three nontrivial projections in* $\mathrm{Lat}(\mathrm{Alg}(\{Q_\infty, Q_0, Q_{-1}\}))$, *then there is a homeomorphism $\rho[Q_{z_1}, Q_{z_2}, Q_{z_3}]$ form S^2 onto* $\mathrm{Lat}(\mathrm{Alg}(\{Q_\infty, Q_0, Q_{-1}\}))\setminus\{0, I\}$ *such that $\rho[Q_{z_1}, Q_{z_2}, Q_{z_3}](z)$ is determined by the following relation:*

$$
\begin{aligned}
&(I - Q_{z_1}\rho[Q_{z_1}, Q_{z_2}, Q_{z_3}](z)Q_{z_1})^{-1}[Q_{z_1}\rho[Q_{z_1}, Q_{z_2}, Q_{z_3}](z)(I - Q_{z_1})] \\
&\qquad = (1+z)(I - Q_{z_1}Q_{z_2}Q_{z_1})^{-1}[Q_{z_1}Q_{z_2}(I - Q_{z_1})] \\
&\qquad\quad - z(I - Q_{z_1}Q_{z_3}Q_{z_1})^{-1}[Q_{z_1}Q_{z_3}(I - Q_{z_1})].
\end{aligned} \tag{3}
$$

By (3), $\rho[Q_{z_1}, Q_{z_2}, Q_{z_3}](\infty) = Q_{z_1}$, $\rho[Q_{z_1}, Q_{z_2}, Q_{z_3}](0) = Q_{z_2}$ and $\rho[Q_{z_1}, Q_{z_2}, Q_{z_3}](-1) = Q_{z_3}$.

The inverse of the homeomorphism $\rho[Q_{z_1}, Q_{z_2}, Q_{z_3}]$ in the Corollary 1.1 actually gives a coordinate chart of $(\mathrm{Lat}(\mathrm{Alg}(\{Q_\infty, Q_0, Q_{-1}\}))\setminus\{0, I, Q_{z_1}\}, \rho[Q_{z_1}, Q_{z_2}, Q_{z_3}]^{-1})$ of $\mathrm{Lat}(\mathrm{Alg}(\{Q_\infty, Q_0, Q_{-1}\})) \setminus \{0, I\}$. So $\mathrm{Lat}(\mathrm{Alg}(\{Q_\infty, Q_0, Q_{-1}\})) \setminus \{0, I\}$ is a 2-dimensional (topological) manifold with atlas $\{\rho[Q_{z_1}, Q_{z_2}, Q_{z_3}]^{-1} | z_1, z_2, z_3 \in \mathbb{C} \cup \{\infty\}\}$. In the next section, we will determine the transition maps between the charts in this atlas.

2. The transition maps

A Möbius transformation is a 1-1 map of the Riemann sphere $\widehat{C}$ onto itself such that

$$f(z) = \frac{az+b}{cz+d}, \qquad z \in \widehat{C},$$

where $ad - bc \neq 0$. The set of all Möbius transformations forms a group under the composition called the Möbius group. Möbius group is the automorphism group of the Riemann sphere, and denoted by $Aut(\widehat{\mathbb{C}})$. It is well known that $Aut(\widehat{\mathbb{C}}) \cong PSL(2, \mathbb{C})$. If z_1, z_2, z_3 is a triple of distinct points in $\widehat{\mathbb{C}}$ and let w_1, w_2, w_3 be another such triple, then there is a unique f in $Aut(\widehat{\mathbb{C}})$ such that $f(z_i) = w_i$, $i = 1, 2, 3$.

By corollary 1.1, any three nontrivial projections Q_{z_1}, Q_{z_2} and Q_{z_3} will determine a continuous map $\rho[Q_{z_1}, Q_{z_2}, Q_{z_3}]$ form S^2 onto $\mathrm{Lat}(\mathrm{Alg}(\{Q_\infty, Q_0, Q_{-1}\})) \setminus \{0, I\}$, and $f = \rho[Q_\infty, Q_0, Q_{-1}]^{-1} \circ \rho[Q_{z_1}, Q_{z_2}, Q_{z_3}]$ is a homeomorphism of S^2 satisfying $f(\infty) = z_1$, $f(0) = z_2$, and $f(-1) = z_3$. We will see in the next theorem that f is the unique Möbius transformation determined by the value at ∞, 0 and -1.

Lemma 2.1. *With the notations in the last section, let Q_{z_1}, Q_{z_2} and Q_{z_3} be three nontrivial projections in* $\mathrm{Lat}(\mathrm{Alg}(\{Q_\infty, Q_0, Q_{-1}\}))\setminus\{0, I\}$. *Then $f = \rho[Q_\infty, Q_0, Q_{-1}]^{-1}\circ \rho[Q_{z_1}, Q_{z_2}, Q_{z_3}]$ is the unique Möbius transformation satisfying $f(\infty) = z_1$, $f(0) = z_2$, $f(-1) = z_3$.*

Proof. First assume that none of z_1, z_2 or z_3 is ∞. Since $\rho[Q_{z_1}, Q_{z_2}, Q_{z_3}](z) = Q_{f(z)}$, by (3), we have

$$
\begin{aligned}
&(I - Q_{z_1}Q_{f(z)}Q_{z_1})^{-1}[Q_{z_1}Q_{f(z)}(I - Q_{z_1})] \\
&\qquad = (1+z)(I - Q_{z_1}Q_{z_2}Q_{z_1})^{-1}[Q_{z_1}Q_{z_2}(I - Q_{z_1})] \\
&\qquad\quad - z(I - Q_{z_1}Q_{z_3}Q_{z_1})^{-1}[Q_{z_1}Q_{z_3}(I - Q_{z_1})].
\end{aligned} \tag{4}
$$

Let $W = \begin{pmatrix} \sqrt{K_{z_1}} & \sqrt{I-K_{z_1}}U_{z_1} \\ U_{z_1}^*\sqrt{I-K_{z_1}} & -U_{z_1}^*\sqrt{K_{z_1}}U_{z_1} \end{pmatrix}$. W is a unitary in the von Neumann algebra $\{Q_\infty, Q_0, Q_{-1}\}''$ such that $WQ_{z_1}W = Q_\infty$ and $WQ_\infty W = Q_{z_1}$. Thus, by (4), we have

$$\begin{aligned} &(I - Q_\infty W Q_{f(z)} W Q_\infty)^{-1}[Q_\infty W Q_{f(z)} W (I - Q_\infty)] \\ &\qquad = (1+z)(I - Q_\infty W Q_{z_2} W Q_\infty)^{-1}[Q_\infty W Q_{z_2} W (I - Q_\infty)] \\ &\qquad\quad - z(I - Q_\infty W Q_{z_3} W Q_\infty)^{-1}[Q_\infty W Q_{z_3} W (I - Q_\infty)]. \end{aligned} \tag{5}$$

For any $z \in \mathbb{C}$, direct computation gives that

$$\begin{aligned} (WQ_zW)_{1,1} = (Q_\infty W Q_z W Q_\infty)|_{Q_\infty \mathcal{H}} &= \sqrt{K_{z_1}}K_z\sqrt{K_{z_1}} + \sqrt{I-K_{z_1}}U_{z_1}U_z^*\sqrt{K_z(I-K_z)}\sqrt{K_{z_1}} \\ &+ \sqrt{K_{z_1}}\sqrt{K_z(I-K_z)}U_zU_{z_1}^*\sqrt{I-K_{z_1}} + \sqrt{I-K_{z_1}}U_{z_1}U_z^*(I-K_z)U_zU_{z_1}^*\sqrt{I-K_{z_1}}. \end{aligned}$$

By (2), we have

$$\begin{aligned} I - (WQ_zW)_{1,1} &= \sqrt{K_{z_1}}(I-K_z)\sqrt{K_{z_1}} \\ &\quad - \sqrt{I-K_{z_1}}U_{z_1}U_z^*\sqrt{K_z(I-K_z)}\sqrt{K_{z_1}} \\ &\quad - \sqrt{K_{z_1}}\sqrt{K_z(I-K_z)}U_zU_{z_1}^*\sqrt{I-K_{z_1}} \\ &\quad + \sqrt{I-K_{z_1}}U_{z_1}U_z^*K_zU_zU_{z_1}^*\sqrt{I-K_{z_1}} \\ &= \sqrt{K_{z_1}}(I-K_z)[\sqrt{K_{z_1}(I-K_{z_1})^{-1}}U_{z_1} \\ &\quad - \sqrt{K_z(I-K_z)^{-1}}U_z]U_{z_1}^*\sqrt{I-K_{z_1}} \\ &\quad - \sqrt{I-K_{z_1}}U_{z_1}U_z^*\sqrt{K_z(I-K_z)}[\sqrt{K_{z_1}(I-K_{z_1})^{-1}}U_{z_1} \\ &\quad - \sqrt{K_z(I-K_z)^{-1}}U_z]U_{z_1}^*\sqrt{I-K_{z_1}} \\ &= (z_1 - z)[\sqrt{K_{z_1}}(I-K_z) \\ &\quad - \sqrt{I-K_{z_1}}U_{z_1}U_z^*\sqrt{K_z(I-K_z)}]SU_{z_1}^*\sqrt{I-K_{z_1}} \\ &= (z_1 - z)\sqrt{I-K_{z_1}}U_{z_1}(U_{z_1}^*\sqrt{K_{z_1}(I-K_{z_1})^{-1}} \\ &\quad - U_z^*\sqrt{K_z(I-K_z)^{-1}})(I-K_z)SU_{z_1}^*\sqrt{I-K_{z_1}} \\ &= |z_1 - z|^2\sqrt{I-K_{z_1}}U_{z_1}S^*(I-K_z)SU_{z_1}^*\sqrt{I-K_{z_1}}. \end{aligned} \tag{6}$$

Similarly,

$$\begin{aligned}
(WQ_zW)_{1,2} &= Q_\infty W Q_z W(I-Q_\infty)\\
&= \sqrt{K_{z_1}}K_z\sqrt{I-K_{z_1}}U_{z_1}\\
&\quad + \sqrt{I-K_{z_1}}U_{z_1}U_z^*\sqrt{K_z(I-K_z)}\sqrt{I-K_{z_1}}U_{z_1}\\
&\quad - \sqrt{K_{z_1}}\sqrt{K_z(I-K_z)}U_zU_{z_1}^*\sqrt{K_{z_1}}U_{z_1}\\
&\quad - \sqrt{I-K_{z_1}}U_{z_1}U_z^*(I-K_z)U_zU_{z_1}^*\sqrt{K_{z_1}}U_{z_1}\\
&= -\sqrt{K_{z_1}}(I-K_z)\sqrt{I-K_{z_1}}U_{z_1}\\
&\quad + \sqrt{I-K_{z_1}}U_{z_1}U_z^*\sqrt{K_z(I-K_z)}\sqrt{I-K_{z_1}}U_{z_1}\\
&\quad - \sqrt{K_{z_1}}\sqrt{K_z(I-K_z)}U_zU_{z_1}^*\sqrt{K_{z_1}}U_{z_1}\\
&\quad + \sqrt{I-K_{z_1}}U_{z_1}U_z^*K_zU_zU_{z_1}^*\sqrt{K_{z_1}}U_{z_1}\\
&= \sqrt{I-K_{z_1}}U_{z_1}[U_z^*\sqrt{K_z(I-K_z)^{-1}}\\
&\quad - U_{z_1}^*\sqrt{K_{z_1}(I-K_{z_1}]^{-1}})(I-K_z)\sqrt{I-K_{z_1}}U_{z_1}\\
&\quad + \sqrt{I-K_{z_1}}U_{z_1}[U_z^*\sqrt{K_z(I-K_z)^{-1}}\\
&\quad - U_{z_1}^*\sqrt{K_{z_1}(I-K_{z_1})^{-1}}]\sqrt{K_z(I-K_z)}U_zU_{z_1}^*\sqrt{K_{z_1}}U_{z_1}\\
&= \overline{(z-z_1)}\sqrt{I-K_{z_1}}U_{z_1}S^*[(I-K_z)\sqrt{I-K_{z_1}}U_{z_1}+\sqrt{K_z(I-K_z)}U_zU_{z_1}^*\sqrt{K_{z_1}}U_{z_1}]\\
&= \overline{(z-z_1)}\sqrt{I-K_{z_1}}U_{z_1}S^*(I-K_z)(\sqrt{I-K_{z_1}}U_{z_1}+\sqrt{K_z(I-K_z)^{-1}}U_zU_{z_1}^*\sqrt{K_{z_1}}U_{z_1}).
\end{aligned}$$

Note $\sqrt{K_z(I-K_z)^{-1}}U_z = (z-z_1)S+\sqrt{K_{z_1}(I-K_{z_1})^{-1}}U_{z_1}$, we have

$(WQ_zW)_{1,2} = \overline{(z-z_1)}\sqrt{I-K_{z_1}}U_{z_1}S^*(I-K_z)[(z-z_1)SU_{z_1}^*\sqrt{K_{z_1}}U_{z_1}+\sqrt{(I-K_{z_1})^{-1}}U_{z_1}]$.

This gives us

$$\begin{aligned}
&[I-(WQ_zW)_{1,1}]^{-1}(WQ_zW)_{1,2} =\\
&|z-z_1|^{-2}\sqrt{(I-K_{z_1})^{-1}}U_{z_1}S^{-1}(I-K_z)^{-1}S^{*-1}U_{z_1}^*\sqrt{(I-K_{z_1})^{-1}}\\
&\times\overline{(z-z_1)}\sqrt{I-K_{z_1}}U_{z_1}S^*(I-K_z)[(z-z_1)SU_{z_1}^*\sqrt{K_{z_1}}U_{z_1}+\sqrt{(I-K_{z_1})^{-1}}U_{z_1}]\\
&=\sqrt{K_{z_1}(I-K_{z_1})^{-1}}U_{z_1}+(z-z_1)^{-1}\sqrt{(I-K_{z_1})^{-1}}U_{z_1}S^{-1}\sqrt{(I-K_{z_1})^{-1}}U_{z_1}.
\end{aligned}$$

By (5), we have

$$\begin{aligned}
&\sqrt{K_{z_1}(I-K_{z_1})^{-1}}U_{z_1}+(f(z)-z_1)^{-1}\sqrt{(I-K_{z_1})^{-1}}U_{z_1}S^{-1}\sqrt{(I-K_{z_1})^{-1}}U_{z_1}\\
&=(1+z)[\sqrt{K_{z_1}(I-K_{z_1})^{-1}}U_{z_1}+(z_2-z_1)^{-1}\sqrt{(I-K_{z_1})^{-1}}U_{z_1}S^{-1}\sqrt{(I-K_{z_1})^{-1}}U_{z_1}]\\
&\qquad -z[\sqrt{K_{z_1}(I-K_{z_1})^{-1}}U_{z_1}+(z_3-z_1)^{-1}\sqrt{(I-K_{z_1})^{-1}}U_{z_1}S^{-1}\sqrt{(I-K_{z_1})^{-1}}U_{z_1}],
\end{aligned}$$

which implies that

$$\frac{1}{f(z)-z_1} = \frac{1+z}{z_2-z_1} - \frac{z}{z_3-z_1}.$$

Solve $f(z)$, we have

$$f(z) = \frac{zz_1(z_3-z_2)+z_2(z_3-z_1)}{z(z_3-z_2)+(z_3-z_1)}.$$

If any z_i equals ∞, $i\in\{1,2,3\}$, we can choose three complex number z_1', z_2' and z_3' in $\mathbb{C}\cup\{\infty\}\backslash\{z_1,z_2,z_3\}$. By considering $Q_{z_1'}$, $Q_{z_2'}$ and $Q_{z_3'}$ as Q'_∞, Q'_0 and Q'_{-1}, we have $\rho[Q_{z_1'},Q_{z_2'},Q_{z_3'}]^{-1}\circ\rho[Q_{z_1},Q_{z_2},Q_{z_3}]$ is a Möbius transformation by the above argument. Therefore $\rho[Q_\infty,Q_0,Q_{-1}]^{-1}\circ\rho[Q_{z_1},Q_{z_2},Q_{z_3}] = (\rho[Q_\infty,Q_0,Q_{-1}]^{-1}\circ\rho[Q_{z_1'},Q_{z_2'},Q_{z_3'}])\circ\rho[Q_{z_1'},Q_{z_2'},Q_{z_3'}]^{-1}\circ\rho[Q_{z_1},Q_{z_2},Q_{z_3}]$ is a Möbius transformation, and $\rho[Q_\infty,Q_0,Q_{-1}]^{-1}\circ\rho[Q_{z_1},Q_{z_2},Q_{z_3}](\infty)=z_1$, $\rho[Q_\infty,Q_0,Q_{-1}]^{-1}\circ\rho[Q_{z_1},Q_{z_2},Q_{z_3}](0)=z_2$ and $\rho[Q_\infty,Q_0,Q_{-1}]^{-1}\circ\rho[Q_{z_1},Q_{z_2},Q_{z_3}](-1)=z_3$. □

It is easy to see that $P\wedge Q = 0 (P\vee Q = I)$ if and only if $(I-P)\vee(I-Q) = I((1-P)\wedge(I-Q)=0)$. Thus, there is a homeomorphism $\rho[I-Q_\infty, I-Q_0, I-Q_{-1}]$ form S^2 onto $\mathrm{Lat}(\mathrm{Alg}(\{I-Q_\infty, I-Q_0, I-Q_{-1}\}))$ by Corollary 1.1. Note

$$\begin{aligned}&\mathrm{Lat}(\mathrm{Alg}(\{I-Q_\infty, I-Q_0, I-Q_{-1}\})) =\\ &\qquad I-\mathrm{Lat}(\mathrm{Alg}(\{Q_\infty,Q_0,Q_{-1}\})) = \{I-Q|Q\in\mathrm{Lat}(\mathrm{Alg}(\{Q_\infty,Q_0,Q_{-1}\})),\end{aligned}$$

we have

$$z\to\rho[Q_\infty,Q_0,Q_{-1}]^{-1}(I-\rho[I-Q_\infty,I-Q_0,I-Q_{-1}](z)),\qquad \forall z\in\mathbb{C}\cup\{\infty\},$$

is a continuous map from S^2 onto itself.

Lemma 2.2. *With the above notation,* $\rho[Q_\infty,Q_0,Q_{-1}]^{-1}(I-\rho[I-Q_\infty,I-Q_0,I-Q_{-1}](z)) = \overline{z}(\overline{\infty}=\infty)$.

Proof. We use $\rho(z)$ to denote $\rho[Q_\infty,Q_0,Q_{-1}]^{-1}(I-\rho[I-Q_\infty,I-Q_0,I-Q_{-1}](z))$, for the sake of simplicity. Since $\rho[I-Q_\infty,I-Q_0,I-Q_{-1}](z) = I-Q_{\rho(z)}$, from (3), we have

$$\begin{aligned}&[I-(I-Q_\infty)(I-Q_{\rho(z)})(I-Q_\infty)]^{-1}[(I-Q_\infty)(I-Q_{\rho(z)})Q_\infty]\\ &\qquad = (1+z)[I-(I-Q_\infty)(I-Q_0)(I-Q_\infty)]^{-1}[(I-Q_\infty)(I-Q_{-1})Q_\infty]\\ &\qquad\quad - z[I-(I-Q_\infty)(I-Q_{-1})(I-Q_0)]^{-1}[(I-Q_\infty)(I-Q_{-1})Q_\infty].\end{aligned}$$

By (1), this implies

$$(1+z)\sqrt{H_1(I-H_1)^{-1}} - zV^*\sqrt{H_2(I-H_2)^{-1}} = U_{\rho(z)}^*\sqrt{K_{\rho(z)}(I-K_{\rho(z)})^{-1}}.$$

Thus by (2), we have $\rho(z)=\overline{z}$. □

If a von Nemuann algebra $\mathfrak{A}$ is generated by a set of projections $\mathcal{L}$, then any automorphism of $\mathfrak{A}$ that fixes all projections in $\mathcal{L}$ must be the identity mapping. However, there may exists nontrivial automorphisms that map $\mathcal{L}$ onto itself.

Definition 2.1. *Suppose $\mathcal{L}$ is a set of projections in a von Neumann algebra $\mathfrak{A}$. Let $G[\mathcal{L},\mathfrak{A}] = \{\varphi\in Aut(\mathfrak{A})|\varphi(\mathcal{L})=\mathcal{L}\}$. When $\mathfrak{A}=\mathcal{L}''$, we will omit $\mathfrak{A}$ and write $G[\mathcal{L}]$ for $G[\mathcal{L},\mathfrak{A}]$.*

Let $\mathfrak{A}$ be a von Neumann algebra and $\mathfrak{A}_*$ be the predual of $\mathfrak{A}$, i.e., the set of normal linear functionals on $\mathfrak{A}$. Then the automorphism group $Aut(\mathfrak{A})$ of $\mathfrak{A}$ is a topological group with respect to a natural topology: the topology of pointwise norm convergence in the predual $\mathfrak{A}_*$ of $\mathfrak{A}$. That is $\{\varphi_\alpha\}(\subset Aut(\mathfrak{A}))$ converges towards φ if and only if

$$\lim_\alpha\|\omega\circ\varphi_\alpha-\omega\circ\varphi\|=0,\qquad\forall\omega\in\mathfrak{A}_*.$$

Lemma 2.3. *If $\mathcal{L}$ is a strong-operator (or weak-operator) closed set of projections in a von Neumann algebra $\mathfrak{A}(\subset\mathcal{B}(\mathcal{H}))$, then $G[\mathcal{L},\mathfrak{A}]$ is a closed subgroup of of $Aut(\mathfrak{A})$ when $Aut(\mathfrak{A})$ is endowed with the natural topology.*

Proof. $G[\mathcal{L},\mathfrak{A}]$ obviously is a group, and we only need to show that it is closed.

Let $\{\varphi_\alpha\} \subset G[\mathcal{L},\mathfrak{A}]$. If $\{\varphi_\alpha\}$ converges to $\varphi \in Aut(\mathfrak{A})$, then for any vector state $\omega_{x,x}$ and P in $\mathcal{L}$,

$$\begin{aligned}\|(\varphi_\alpha(P)-\varphi(P))x\|^2 &= \omega_{x,x}(\varphi_\alpha(P)-\varphi(P))^*(\varphi_\alpha(P)-\varphi(P))) \\ &= \omega_{x,x}(\varphi_\alpha(P)+\varphi(P)-\varphi_\alpha(P)\varphi(P)-\varphi(P)\varphi_\alpha(P)) \\ &\to \omega_{x,x}(\varphi(P)+\varphi(P)-\varphi(P)\varphi(P)-\varphi(P)\varphi(P)) = 0.\end{aligned}$$

This implies that $\varphi_\alpha(P)$ tends to $\varphi(P)$ in strong-operator topology and $\varphi(\mathcal{L}) \subset \mathcal{L}$. Because $\{\varphi_\alpha\}^{-1}$ converges to φ^{-1}, by the same argument as before, we have $\varphi^{-1}(\mathcal{L}) \subset \mathcal{L}$. This complete the proof. □

Because any reflexive lattice of projections is strong-operator closed, we have the following corollary.

Corollary 2.1. *Let $\mathcal{L}$ be a reflexive lattice of projections in a von Neumann algebra $\mathfrak{A}$, then $G[\mathcal{L},\mathfrak{A}]$ is a closed subgroup of of $Aut(\mathfrak{A})$ when $Aut(\mathfrak{A})$ is endowed with the natural topology.*

Lemma 2.4. *Let $\varphi \in G[\mathrm{Lat}(\mathrm{Alg}(\{Q_\infty, Q_0, Q_{-1}\})]$, then $\varphi(Q_z) = Q_{\rho_\varphi(z)}$, where $\rho_\varphi = \rho[Q_\infty, Q_0, Q_{-1}]^{-1} \circ \rho[\varphi(Q_\infty), \varphi(Q_0), \varphi(Q_{-1})]$. And $\rho_{\varphi_2\circ\varphi_1} = \rho_{\varphi_2} \circ \rho_{\varphi_1}$, where φ_1, $\varphi_2 \in G[\mathrm{Lat}(\mathrm{Alg}(\{Q_\infty, Q_0, Q_{-1}\})]$.*

Proof. Apply φ on both sides of (3), for $z_1 = \infty$, $z_2 = 0$ and $z_3 = -1$, we obtain

$$\begin{aligned}&(I-\varphi(Q_\infty)\varphi(Q_z)\varphi(Q_\infty))^{-1}[\varphi(Q_\infty)\varphi(Q_z)(I-\varphi(Q_\infty))] \\ &\quad = (1+z)(I-\varphi(Q_\infty)\varphi(Q_0)\varphi(Q_\infty))^{-1}[\varphi(Q_\infty)\varphi(Q_0)(I-\varphi(Q_\infty))] \\ &\qquad - z(I-\varphi(Q_\infty)\varphi(Q_{-1})\varphi(Q_\infty))^{-1}[\varphi(Q_\infty)\varphi(Q_{-1})(I-\varphi(Q_\infty))].\end{aligned}$$

Comparing this equation with (3), it is clear that $\varphi(Q_z) = Q_{\rho_\varphi(z)}$. □

Lemma 2.5. *Suppose $\mathcal{L} = \mathrm{Lat}(\mathrm{Alg}\{P_i | i \in \mathcal{I}\}$ and $\widetilde{\mathcal{L}} = \mathrm{Lat}(\mathrm{Alg}\{Q_i | i \in \mathcal{I}\}$ are two reflexive lattices, where $\mathcal{I}$ be an index set. Let $\mathfrak{A}$ and $\widetilde{\mathfrak{A}}$ be two von Neumann algebras that contain $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ respectively.If φ is a *-isomorphism form $\mathfrak{A}$ onto $\widetilde{\mathfrak{A}}$ such that $\varphi(\{P_i\}_{i\in I}) = \{Q_i\}_{i\in I}$, then $\varphi(\mathcal{L}) = \widetilde{\mathcal{L}}$.*

Proof. $\varphi(\mathcal{L})$ is reflexive by A.2. This implies

$$\varphi(\mathcal{L}) \supseteq \mathrm{Lat}(\mathrm{Alg}(\{Q_i\}_{i\in I})) = \widetilde{\mathcal{L}},$$

since $\varphi(\mathcal{L}) \supseteq \varphi(\{P_i\}_{i\in I}) = \{Q_i\}_{i\in I}$. Similarly, by considering φ^{-1}, we have

$$\varphi^{-1}(\widetilde{\mathcal{L}}) \supseteq \mathcal{L}. \qquad (*)$$

Apply φ to both sides of $(*)$ will give the opposite direction of the inclusion $\widetilde{\mathcal{L}} \supseteq \varphi(\mathcal{L})$. □

Let $d(z_1, z_2)$ be the chordal metric on $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$:

$$d(z_1,z_2) = \begin{cases} \frac{2|z_1-z_2|}{(1+|z_1|^2)^{1/2}(1+|z_2|)^{1/2}}, & z_1, z_2 \in \mathbb{C}; \\ \frac{2}{(1+|z_1|^2)^{1/2}}, & z_2 = \infty. \end{cases} \tag{7}$$

Then $Aut(\widehat{\mathbb{C}})$ is a topological group with respect to the following metric

$$\sigma(f,g) = sup_{z\in\widehat{\mathbb{C}}} d(f(z), g(z)).$$

By Lemma 2.4, $\varphi(\in G[\mathrm{Lat}(\mathrm{Alg}(\{Q_\infty, Q_0, Q_{-1}\})]) \to \rho_\varphi$ is an isomorphic form $G[\mathrm{Lat}(\mathrm{Alg}(\{Q_\infty, Q_0, Q_{-1}\})]$ onto some subgroup of $Aut(\widehat{\mathbb{C}})$. As will be shown by the next lemma, if the von Neumann algebra generated by Q_∞, Q_0 and Q_{-1} is a factor, then the range of this map is closed.

Lemma 2.6. *With the notation in Lemma 2.4, let $\mathcal{G} = G[\mathrm{Lat}(\mathrm{Alg}(\{Q_\infty, Q_0, Q_{-1}\})]$, and $\Phi(\varphi) = \rho_\varphi$ be the Möbius transformations such that $\varphi(Q_z) = Q_{\rho_\varphi(z)}$, $\varphi \in \mathcal{G}$. If the finite von Neuamann algebra $\mathfrak{A}$ generated by Q_∞, Q_0 and Q_{-1} has a faithful norm trace τ that is $\mathcal{G}$-invariant, i.e., $\tau \circ \varphi = \tau$, $\forall \varphi \in \mathcal{G}$, then $\Phi(\mathcal{G})$ is a closed subgroup of $Aut(\widehat{\mathbb{C}})$. Furthermore, Φ^{-1} is continuous from $\Phi(\mathcal{G})$ onto $\mathcal{G}$.*

Proof. Let $\rho_{\varphi_i} = \Phi(\varphi_i)$ be a sequence in $Aut(\widehat{\mathbb{C}})$, where $\varphi_i \in \mathcal{G}$ and $i = 1, 2 \ldots$. We need to show that if ρ_{φ_i} converges to ρ, then there is a $\varphi \in \mathcal{G}$ such that $\Phi(\varphi) = \rho$.

Without loss of generality, we could assume that $\mathfrak{A}$ acts on $L^2(\mathfrak{A}, \tau)$, and Ω is a generating trace vector satisfying $\langle A\Omega, \Omega\rangle = \tau(A)$. Since τ is $\mathcal{G}$-invariant, we have $\|\varphi_i(A - B)\|_2 = \|A - B\|_2$, where $\|.\|_2$ is the trace norm induced by τ, and $A, B \in \mathfrak{A}$. Thus the following mapping

$$\begin{aligned} U_i : P(Q_\infty, Q_0, Q_{-1})\Omega &\to P(\varphi_i(Q_\infty), \varphi_i(Q_0), \varphi_i(Q_{-1}))\Omega \\ &= P(Q_{\rho_{\varphi_i}(\infty)}, Q_{\rho_{\varphi_i}(0)}, Q_{\rho_{\varphi_i}(-1)})\Omega, \qquad i = 1, 2 \ldots, \end{aligned}$$

extends to a unitary operator U_i such that $\varphi_i(A) = U_i A U_i^*$ for $A \in \mathfrak{A}$, where $P(Q_\infty, Q_0, Q_{-1})$ is a polynomial of Q_∞, Q_0 and Q_{-1}. Since ρ_{φ_i} converges to ρ, $Q_{\rho_{\varphi_i}(\infty)}$, $Q_{\rho_{\varphi_i}(0)}$ and $Q_{\rho_{\varphi_i}(-1)}$ converge to $Q_{\rho_\varphi(\infty)}$, $Q_{\rho_\varphi(0)}$ and $Q_{\rho_\varphi(-1)}$ in strong-operator topology. This implies that $P(Q_{\rho_{\varphi_i}(\infty)}, Q_{\rho_{\varphi_i}(0)}, Q_{\rho_{\varphi_i}(-1)})$ converge strongly to $P(Q_{\rho_\varphi(\infty)}, Q_{\rho_\varphi(0)}, Q_{\rho_\varphi(-1)})$, since sums and multiplications of oparators are jointly continuous on the bounded set. Note $\|Q_{\rho_\varphi(z_1)} - Q_{\rho_\varphi(z_2)}\|_2 = \|Q_{z_1} - Q_{z_2}\|_2$, z_1, $z_2 \in \{\infty, 0, -1\}$, and $Q_{\rho_\varphi(\infty)}, Q_{\rho_\varphi(0)}, Q_{\rho_\varphi(-1)}$ also generates $\mathfrak{A}$. Therefore the mapping

$$P(Q_\infty, Q_0, Q_{-1})\Omega \to P(Q_{\rho_\varphi(\infty)}, Q_{\rho_\varphi(0)}, Q_{\rho_\varphi(-1)})\Omega$$

extends to a unitary transformation U of $L^2(\mathfrak{A}, \tau)$ such that $UQ_\infty U^* = Q_{\rho_\varphi(\infty)}$, $UQ_0U^* = Q_{\rho_\varphi(0)}$ and $UQ_{-1}U^* = Q_{\rho_\varphi(-1)}$. Let $\varphi(A) = UAU^*$, $\forall A \in \mathfrak{A}$. We have $\varphi(\mathfrak{A}) = \mathfrak{A}$. By Lemma 2.5, φ is in $\mathcal{G}$.

Because each functional $\omega \in \mathfrak{A}_*$ has the form $\sum_{n=1}^\infty \omega_{x_n, y_n}$, with

$$\sum_{n=1} (\|x_n\|^2 + \|y_n\|^2) < \infty.$$

Then $\omega \circ \varphi_i = \sum_{n=1}^\infty \omega_{U_i^* x_n, U_i^* y_n}$. Since ρ_{φ_i} converges to ρ_φ implies U_i^* tends to U^* in strong-operator topology, $\lim_i \|\omega \circ \varphi_i - \omega \circ \varphi\| = 0$, and Φ^{-1} is continuous. $\square$

The elements in $Aut(\widehat{\mathbb{C}})$ are commonly classified into three types: parabolic, loxodromic and elliptic. The next theorem describes the dynamic behavior when a transformation is iterated.

Theorem 2.1 ([11], Theorem 4.3.10). *Let $g(\neq I)$ be any Möbius transformation. Then*

(1) If g is parabolic with fixed point α. Then for all z in $\widehat{\mathbb{C}}$, we have $g^n(z) \to \alpha$ as $n \to +\infty$. The convergence being uniform on compact subsets of $\mathbb{C} \setminus \{\alpha\}$.

(2) *If g is loxodromic, then the fixed points α and β of g can be labeled so that $g^n(z) \to \alpha$ as $n \to +\infty$(if $z \neq \beta$). The convergence being uniform on compact subsets of $\widehat{\mathbb{C}} \setminus \{\beta\}$.*

(3) *If g is elliptic with fexed points α and β, then g leaves invariant each circle for which α and β are inverse points.*

Lemma 2.7. *With the notation and assumption in Lemma 2.6, for any $\varphi \in G[\mathrm{Lat}(\mathrm{Alg}(\{Q_\infty, Q_0, Q_{-1}\})]$, ρ_φ must be elliptic.*

Proof. Let $\rho_{\varphi^n} = \Phi(\varphi^n)$, where $\varphi \in G[\mathrm{Lat}(\mathrm{Alg}(\{Q_\infty, Q_0, Q_{-1}\}$, $n = 1, 2, \ldots$. If ρ_φ is not elliptic, without lose of generality we may assume that $\lim_{n\to+\infty} d(\rho_n(\infty), \rho_n(0)) = 0$ by Theorem 2.1. This implies

$$\|Q_\infty - Q_0\|_2 = \|\varphi^n(Q_\infty - Q_0)\|_2 = \|Q_{\rho_n(\infty)} - Q_{\rho_n(0)}\|_2 \to 0, \quad n \to +\infty,$$

where $\|\cdot\|_2$ is the 2-norm induced by the $G[\mathrm{Lat}(\mathrm{Alg}(\{Q_\infty, Q_0, Q_{-1}\})]$-invariant faithful norm trace. This contradicts with the fact that $Q_\infty \neq Q_0$. □

A subgroup of $Aut(\widehat{\mathbb{C}})$ contains only elliptic transformations (and I) is conjugate to a subgroup of rotations of the sphere S^2, thus a subgroups of $SU(2)$ (or $SO(3)$). Furthermore, if a closed subgroup contains only elliptic elements, then it must be compact.

Lemma 2.8. *With the notation and assumption in Lemma 2.6, Φ is a homeomorphism between $\mathcal{G}(= G[\mathrm{Lat}(\mathrm{Alg}(\{Q_\infty, Q_0, Q_{-1}\}])$ and $\Phi(\mathcal{G})$ when $\mathcal{G}$ and $\Phi(\mathcal{G})$ are endowed with the natural topology and the topology induce by the metric σ respectively.*

Proof. By the proceeding discussion and Lemma 2.6, $\Phi(\mathcal{G})$ is compact. Since Φ^{-1} is continuous, and any every continuous bijective map form compact space to a Hausdorff space is a homeomorphism, Φ^{-1} is homeomorphism. □

Because any finite factor and finite dimensional von Neumann algebra has a faithful norm trace that is fixed by any automorphism, we have the following corollary.

Corollary 2.2. *With the notation in Lemma 2.6, if the von Neumann algebra generated by Q_∞, Q_0 and Q_{-1} is a finite factor or finite dimensional, then $\mathcal{G} = G[\mathrm{Lat}(\mathrm{Alg}(\{Q_\infty, Q_0, Q_{-1}\})]$ is homeomorphic to a closed subgroup $\Phi(\mathcal{G})$ of $SO(3)$.*

Every finite subgroup of $SO(3)$ is isomorphic to one of the symmetry groups of the regular solids: cyclic groups C_n, dihedral groups D_n, tetrahedral group $T(\approx A_4)$, octahedral group $O(\approx S_4)$, and the icosahedral group $Y(\approx A_5)$. There are also two infinite closed subgroups $C_\infty \approx SO(2)$ generated by an arbitrary rotation around an axis and D_∞ which is generated by C_∞ and a rotation π around an axis orthogonal to the axis of rotation of C_∞.

Proposition 2.1. *Suppose $\{0, Q_\infty, Q_0, Q_{-1}, I\}$ is a double triangle lattice, and Q_∞, Q_0 and Q_{-1} generate a finite von Neumann algebra $\mathfrak{A}$. For each i in a index set $\mathcal{I}$, let z^i_∞, z^i_0 and z^i_{-1} be three elements in $\widehat{\mathbb{C}}$, and $P_k = \sum_i Q_{z^i_k} \otimes E_i$ $(k = \infty, 0, -1)$be a projection in $\mathfrak{A} \otimes l^\infty(\mathcal{I})$, where E_i is the projection onto the space spanned by $e_i = \delta_i(\in l^2(\mathcal{I})$. Furthermore, we assume that $0 \in \mathcal{I}$ and $z^0_\infty = \infty$, $z^0_0 = 0$ and $z^0_{-1} = -1$. Then $\{0, P_\infty, P_0, P_\infty, I\}$ is a double triangle lattice, and any $P_z \in \mathrm{Lat}(\mathrm{Alg}(P_\infty, P_0, P_\infty))$ has the form $P_z = \sum_i Q_{\rho^i(z)}$, where ρ^i is the Möbius transformation satisfying $\rho^i(\infty) = z^i_\infty$, $\rho^i(0) = z^i_0$ and $\rho^i(-1) = z^i_{-1}$ $(\rho^0(z) = z)$.*

Proof. Since $\mathfrak{A} \otimes l^\infty(\mathcal{I})$ is a finite von Neumann algebra, we could apply Theorem 1.1, and proceed as in the proof of Lemma 2.2, we have the result. □

Corollary 2.3. *For any subgroup* G *of Möbius transformations, there is a double triangle lattice* $\{0, P_\infty, P_0, P_{-1}, I\}$ *such that* $\mathrm{G} \subset G[\mathrm{Lat}(\mathrm{Alg}(\{P_\infty, P_0, P_{-1}\})]$.

Proof. Let

$$Q_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, Q_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, Q_{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

From Theorem 1.1, we have

$$Q_z = \begin{pmatrix} \frac{|z|^2}{1+|z|^2} & \frac{|z|e^{i(\pi+\theta)}}{1+|z|^2} \\ \frac{|z|e^{-i(\pi+i\theta)}}{1+|z|^2} & \frac{1}{1+|z|^2} \end{pmatrix}, \qquad z = |z|e^{i\theta}.$$

Let $P_k = \sum_{\rho \in \mathrm{G}} Q_{\rho(k)} \otimes E_\rho (\in M_2(\mathbb{C}) \otimes l^\infty(\mathrm{G}),\ k = \infty, 0, -1$, and $U_\rho = I \otimes R_\rho$, where R_ρ is the unitary in right regular representation R_{G}. Then for $\rho_0 \in \mathrm{G}$, we have

$$U^*_{\rho_0} P_k U_{\rho_0} = \sum_{\rho \in \mathrm{G}} Q_{\rho(\rho_0^{-1}(k))} \otimes E_\rho, \qquad k = \infty, 0, -1,$$

and $U^*_{\rho_0} P_k U_{\rho_0}$ is in $\mathrm{Lat}(\mathrm{Alg}(\{P_\infty, P_0, P_{-1}\}))$ by Proposition 2.1. Thus, $U^*_{\rho_0}\{P_\infty, P_0, P_{-1}\}'' U_{\rho_0} = \{P_\infty, P_0, P_{-1}\}''$ by Lemma 2.5. □

Remark 2.1. *Let P_∞, P_0 and P_{-1} be the projections in the proof of Corollary 2.3. If the group* G *contains a element that is not elliptic, then the von Neumanm algebra $\mathfrak{A}$ generated by P_∞, P_0 and P_{-1} does not have a faithful norm trace that is preserved by any automorphisms of $\mathfrak{A}$, in another word, the center of $\mathfrak{A}$ is infinite dimensional.*

In the next section we will determine $G[\mathrm{Lat}(\mathrm{Alg}(\{Q_\infty, Q_0, Q_{-1}\})]$ when Q_∞, Q_0 and Q_{-1} are three trace half free projections.

3. Automorphism of $L_{F_{\frac{3}{2}}}$ that fixes S^2

We first recall some basic facts and terminology relative to free probability theory. A non-commutative W^*-probability space $(\mathfrak{A}, \tau)$ is a von Neumann algebra $\mathfrak{A}$ with a normal state τ. In particular, we only consider the case when $\mathfrak{A}$ is finite and τ is a faithful norm trace. A family of unital *-subalgebras $\{\mathfrak{A}_l\}_{l \in \mathcal{I}}$ of $\mathfrak{A}$ is called free if $\tau(a_1 a_2 \cdots a_n) = 0$ whenever $a_i \in A_{l_i}$, $l_1 \neq l_2 \neq \cdots \neq l_n$ and $\tau(a_i) = 0$ for all $i = 1, \ldots n$. In particular, if $\mathfrak{A}_l$ is the unital *-algebra generated by $T_l \in \mathfrak{A}$, then $\{T_l\}_{l \in \mathcal{I}}$ is called *-free. Specially, if $\{G_l\}_{l \in \mathcal{I}}$ is a family of groups, then the group von Neumann algebras $\{L_{G_l}\}_{l \in \mathcal{I}}$ is free in L_G, where $G = *_{l \in \mathcal{I}} G_l$ is the group free product of $\{G_l\}$.For a comprehensive treatment on free probability, we refer to [19, 20].

Let $F_{\frac{3}{2}} = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$. $F_{\frac{3}{2}}$ is an i.c.c group so its associated group von Neumann algebra $L_{F_{\frac{3}{2}}}$ is a factor of type II_1 acting on $l^2(F_{\frac{3}{2}})$. Throughout this section we assume $Q_\infty = \frac{I-U_1}{2}$, $Q_0 = \frac{I-U_2}{2}$ and $Q_{-1} = \frac{I-U_3}{2}$ where U_1, U_2 and U_3 are canonical generators for $L_{F_{\frac{3}{2}}}$ corresponding to the generators of $F_{\frac{3}{2}}$ with $U_i^2 = I$. From the above discussion, we have Q_∞, Q_0 and Q_{-1} are three free projections with trace $\frac{1}{2}$, and satisfy all the requirements in the statements of the Theorem 1.1

and the Corollary 2.2. Thus $G[\mathrm{Lat}(\mathrm{Alg}(\{Q_\infty, Q_0, Q_{-1}\}))]$ is isomorphic to a closed subgroup of $SO(3)$.

It is obvious that any permutation of the three free projections Q_∞, Q_0 and Q_{-1} will induce a automorphism of $L_{F_{\frac{3}{2}}}$. More precisely, we have the following lemma.

Lemma 3.1. *With the notation given above, there are two automorphisms φ_1 and φ_2 of $L_{F_{\frac{3}{2}}}$ such that $\varphi_i(Q_z) = Q_{\rho_{\varphi_i}(z)}$ for any $Q_z \in \mathrm{Lat}(\mathrm{Alg}(\{Q_\infty, Q_0, Q_{-1}\}))$, where $\rho_{\varphi_1}(z) = \frac{1}{z}$ and $\rho_{\varphi_2}(z) = -1 - z$.*

Proof. Let φ_1 be the automorphism satisfying $\varphi_1(Q_\infty) = Q_0$, $\varphi_1(Q_0) = Q_\infty$ and $\varphi_1(Q_{-1}) = Q_{-1}$. By Lemma 2.4, $\rho_{\varphi_1}(z) = \frac{1}{z}$. Similarly let φ_2 be the automorphism that switches Q_0 and Q_{-1} and leave Q_∞ fixed, we have $\rho_{\varphi_2}(z) = -1 - z$. □

The *-isomorphisms φ_1 and φ_2 in the last lemma generate a subgroup of $Aut(L_{F_{\frac{3}{2}}})$, which is isomorphic to S_3, the symmetric group of 3 elements. Later it will be shown that $G[\mathrm{Lat}(\mathrm{Alg}(\{Q_\infty, Q_0, Q_{-1}\}))] \approx S_3$. In order to do this, we must dig out more structure information about $\mathrm{Lat}(\mathrm{Alg}(\{Q_\infty, Q_0, Q_{-1}\}))$. In doing so, we shall adopt the notation and definitions developed in [7, 19, 20].

Let $\mathfrak{A}$ be a finite von Neumann algebra. Recall that $\widetilde{\mathfrak{A}}$ is the *-algebra formed by the operators affiliated with $\mathfrak{A}$. Let $T \in \widetilde{\mathfrak{A}}$, by polar decomposition, we have

$$T = U|T| = U \int_0^\infty t dE_{|T|}(t),$$

where $U \in \mathfrak{A}$ is a unitary, and $E_{|T|}$ is the spectral measure for $|T|$ taking values in $\mathfrak{A}$. We may define a probability measure $\mu_{|T|}$ by

$$\mu_{|T|}(B) = \tau(E_{|T|}(B)), (B \in \mathbb{B}),$$

where $\mathbb{B}$ is the σ-algebra of Borel sets in $\mathbb{R}$.

Definition 3.1 (Definition 3.2, [7])**.** *$T \in \widetilde{\mathfrak{A}}$ is said to be $R-$diagonal if there exist a von Neumann algebra $\mathcal{N}$, with a faithful norm trace state ϕ, and *-free elements U and H in $\widetilde{N}$, such that U is Haar unitary (An element $U \in N$ is said to be Haar unitary if it is a unitary and $\phi(U^k) = 0$, $\forall k \in \mathbb{Z} \setminus \{0\}$), $H \geq 0$, and T has the same *-distribution as UH.*

We will use the Proposition 3.11 in [7] which we state here for the convenience of the reader:

Proposition 3.1 (Proposition 3.11, [7])**.** *Let S, $T \in \widetilde{\mathfrak{A}}$ be *-free R-diagonal elements. Then*

$$\widetilde{\mu}_{|S+T|} = \widetilde{\mu}_{|S|} \boxplus \widetilde{\mu}_{|T|}.$$

Here $\widetilde{\mu}_{|S|}(\widetilde{\mu}_{|T|})$ is the symmetrization of $\mu_{|S|}(\mu_{|T|})$, which is defined by

$$\widetilde{\mu}_{|S|}(B) = \frac{1}{2}(\mu_{|S|}(B) + \mu_{|S|}(-B)), \qquad (B \in \mathbb{B}).$$

From now on, let $\mathcal{L} = \mathrm{Lat}(\mathrm{Alg}(\{Q_\infty, Q_0, Q_{-1}\}))$. We will compute the distance form Q_z to Q_∞, $\forall\, Q_z \in \mathcal{L}$. By the discussion in section 1, we may assume that

$$Q_\infty = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, Q_0 = \begin{pmatrix} H_1 & \sqrt{H_1(I-H_1)} \\ \sqrt{H_1(I-H_1)} & I-H_1 \end{pmatrix},$$

$$Q_{-1} = \begin{pmatrix} H_2 & \sqrt{H_2(I-H_2)}V \\ V^*\sqrt{H_2(I-H_2)} & V^*(I-H_2)V \end{pmatrix}.$$

The freeness among Q_∞, Q_0 and Q_{-1} ensures that H_1, H_2 and V are free, H_1 and H_2 have the same distribution as $\cos^2(\frac{\pi}{2}\theta)$ on $[0,1]$ with respect to Lebesgue measure ([19, Exmaple 3.6.7]) and V a Haar unitary element. And $Q_\infty L_{F_{\frac{3}{2}}} Q_\infty$ is generated by H_1, H_2 and V. Let tr be the faithful norm trace on $L_{F_{\frac{3}{2}}}$, then

$$tr\left(\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\right) = \frac{1}{2}(\tau(A_{11}) + \tau(A_{22})).$$

where τ is the faithful norm trace on $Q_\infty L_{F_{\frac{3}{2}}} Q_\infty$.

With the notation in Theorem 1.1, we have

$$\| Q_z - Q_\infty \|_2^2 = 1 - 2tr(Q_z Q_\infty) = 1 - \tau(Q_\infty Q_z Q_\infty|_{Q_\infty l^2_{F_{\frac{3}{2}}}}) = \tau(I - K_z).$$

Therefore we only need to determine the distribution of K_z.

Lemma 3.2. *With the above notation, any Q_z in $\mathcal{L} \setminus \{0, I\}$ can be written as*

$$Q_z = \begin{pmatrix} K_z & \sqrt{K_z(I-K_z)}U_z \\ U_z^*\sqrt{K_z(I-K_z)} & U_z^*(I-K_z)U_z \end{pmatrix}, \qquad \forall z \in \mathbb{C},$$

where K_z and U_z are determined by the polar decomposition:

$$\sqrt{K_z(I-K_z)^{-1}}U_z = (1+z)\sqrt{H_1(I-H_1)^{-1}} - z\sqrt{H_2(I-H_2)^{-1}}V.$$

Furthermore, we have

$$d\mu_{\sqrt{\frac{K_z}{I-K_z}}}(t) = \frac{2}{\pi}\frac{|z| + |z+1|}{t^2 + (|z| + |z+1|)^2} 1_{(0,\infty)}(t)dt.$$

Proof. The first part of the lemma is just a restatement of Theorem 1.1. From the discussion following Proposition 3.1, $H_i (i = 1, 2)$ has the same distribution as $\cos^2 \frac{\pi}{2}\theta$ on $[0,1]$, thus $\sqrt{H_i(I-H_i)^{-1}}$ has the same distribution as $\cot \frac{\pi}{2}\theta$ on $[0,1]$. So we have

$$d\mu_{\sqrt{\frac{H_i}{I-H_i}}}(t) = \frac{2}{\pi}\frac{1}{1+t^2} 1_{(0,\infty)} dt, \qquad i = 1, 2,$$

where dt is the Lebesgue measure on $\mathbb{R}$. Then for any $r \in \mathbb{R}^+$,

$$d\mu_r(t) \overset{\text{def}}{=} d\mu_{(r\sqrt{\frac{H_i}{I-H_i}})}(t) = \frac{2}{\pi}\frac{r}{r^2+t^2} 1_{(0,\infty)} dt.$$

Thus the symmetrization of $\mu_r(t)$ is

$$d\widetilde{\mu_r}(t) = \frac{1}{\pi}\frac{r}{r^2+t^2} 1_{(-\infty,+\infty)} dt.$$

The Cauchy transform of $\widetilde{\mu_r}$ is

$$G_{\widetilde{\mu_r}}(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{1}{\omega - t}\frac{r}{r^2+t^2} dt = \frac{1}{z + ri}. \qquad (Im\omega > 0)$$

Let $F_{\widetilde{\mu_r}}(\omega)=\frac{1}{G_{\widetilde{\mu_r}}(\omega)}=\omega+ri$, and $\varphi_{\widetilde{\mu_r}}(\omega)=F^{-1}_{\widetilde{\mu_r}}(\omega)-\omega=-ri$.

Since H_1, H_2 and V are free and V is a Haar unitary, we may assume that $V=V_2V_1^*$ with V_1 and V_2 Haar unitaries, and $\{H_1,H_2,V_1,V_2\}$ is a free family. Thus $\sqrt{K_z(I-K_z)^{-1}}U_zV_1=(1+z)\sqrt{H_1(I-H_1)^{-1}}V_1-z\sqrt{H_2(I-H_2)^{-1}}V_2$ is a sum of two *-free R-diagonal elements. Form Corollary 5.8 in [8] and Proposition 3.1, we have for any $z\in\mathbb{C}$,

$$\begin{aligned}
\varphi_{\widetilde{\mu_{|z|}}\boxplus\widetilde{\mu_{|z+1|}}}(\omega)&=\varphi_{\widetilde{\mu_{|z|}}}(\omega)+\varphi_{\widetilde{\mu_{|z+1|}}}(\omega)=-i(|z|+|z+1|),\\
F_{\widetilde{\mu_{|z|}}\boxplus\widetilde{\mu_{|z+1|}}}(\omega)&=\omega+i(|z|+|z+1|),\\
G_{\widetilde{\mu_{|z|}}\boxplus\widetilde{\mu_{|z+1|}}}(\omega)&=\frac{1}{\omega+i(|z|+|z+1|)},
\end{aligned}$$

therefore

$$\begin{aligned}
d\widetilde{\mu_{\sqrt{\frac{K_z}{I-K_z}}}}(t)&=d\widetilde{\mu_{|z|}}\boxplus\widetilde{\mu_{|z+1|}}(t)=-\frac{1}{\pi}\lim_{u\to 0+}ImG_{\widetilde{\mu_{|z|}}\boxplus\widetilde{\mu_{|z+1|}}}(t+iu)\\
&=\frac{1}{\pi}\frac{|z|+|z+1|}{t^2+(|z|+|z+1|)^2}1_{(-\infty,+\infty)}dt,
\end{aligned}$$

and

$$d\mu_{\sqrt{\frac{K_z}{I-K_z}}}(t)=\frac{2}{\pi}\frac{|z|+|z+1|}{t^2+(|z|+|z+1|)^2}1_{(0,+\infty)}dt.$$

□

Lemma 3.3. *For any projection $Q_z\in\mathcal{L}\setminus\{0,I\}$, $\parallel Q_z-Q_\infty\parallel_2=\sqrt{\frac{1}{1+|z|+|z+1|}}$, where $\|\cdot\|_2$ is the 2-norm induced by the faithful norm trace tr on $L_{F_{\frac{3}{2}}}$.*

Proof. Let $\Delta_z=|z|+|z+1|$. An easy application of the residue theorem gives that

$$\tau(I-K_z)=\frac{\Delta_z}{2\pi i}\int_{-\infty}^{+\infty}\frac{2i}{(t^2+1)(t^2+\Delta_z^2)}dt=\frac{1}{1+\Delta_z}.$$

Thus

$$\parallel Q_z-Q_\infty\parallel_2^2=\tau(I-K_z)=\frac{1}{1+|z|+|z+1|}.$$

□

Corollary 3.1. *For any $z\in\widehat{\mathbb{C}}$, we have $\parallel Q_z-Q_0\parallel_2=\sqrt{\frac{|z|}{1+|z|+|z+1|}}$, and $\parallel Q_z-Q_{-1}\parallel_2=\sqrt{\frac{|z+1|}{1+|z|+|z+1|}}$.*

Proof. Let φ_2 be the automorphism in Lemma 3.1. Then we have

$$\parallel Q_z-Q_0\parallel_2=\parallel Q_{\frac{1}{z}}-Q_\infty\parallel_2=\sqrt{\frac{|z|}{1+|z|+|z+1|}}.$$

Similarly, the second equation can be proved by considering the automorphism $\varphi_2\circ\varphi_1$. □

Corollary 3.2. *With the notation and assumption in this section, $\|Q_{z_1}-Q_{z_2}\|_2=\|Q_{\overline{z}_1}-Q_{\overline{z}_2}\|_2$, $\forall\ z\in\widehat{\mathbb{C}}$.*

Proof. Since $\{I-Q_\infty, I-Q_0, I-Q_{-1}\}$ is also a free family of trace half projections, the map $Q_i \to I-Q_i$ induces an automorphism φ of $L_{F_{\frac{3}{2}}}$, $i \in \{\infty, 0, -1\}$. Use Lemma 2.2 and argue as in the proof of Lemma 2.4, we have $\varphi(Q_z) = I - Q_{\overline{z}}$. Then the equation in the corollary can be verified as follows:

$$\|Q_{z_1} - Q_{z_2}\|_2 = \|\varphi(Q_{z_1}) - \varphi(Q_{z_2})\|_2 = \|(I-Q_{\overline{z}_1}) - (I-Q_{\overline{z}_2})\|_2 = \|Q_{\overline{z}_1} - Q_{\overline{z}_2}\|_2.$$

□

Before proceeding to the proof of the fact that $G[\mathrm{Lat}(\mathrm{Alg}(\{Q_\infty, Q_0, Q_{-1}\}))] \approx S_3$, we need to learn more about the metric on $\widehat{\mathbb{C}}$ given below:

$$dist(z_1,z_2) = \begin{cases} \sqrt{\frac{2|z_1-z_2|}{(1+|z_1|+|z_1+1|)(1+|z_2|+|z_2+1|)}}, & z_1, z_2 \in \mathbb{C}; \\ \sqrt{\frac{1}{(1+|z_1|+|z_1+1|)}}, & z_2 = \infty. \end{cases} \tag{8}$$

It is not hard to check that $dist$ is a distance function on $\widehat{\mathbb{C}}$.

Lemma 3.4. *With the notation above, dist is a metric on* $\widehat{\mathbb{C}}$.

Proof. To check the subadditivity $dist(z_1,z_3) \le dist(z_1,z_2) + dist(z_2,z_3)$ when z_1, z_2, $z_3 \in \mathbb{C}$, we only need to show

$$|z_1 - z_3|(1+|z_2|+|z_2+1|) \le |z_1-z_2|(1+|z_3|+|z_3+1| + |z_2-z_3|(1+|z_1|+|z_1+1|,$$

which is an immediate consequence of the following inequalities:

$$\begin{aligned} |z_1-z_3| &\le |z_1-z_2| + |z_2-z_3|, \\ |z_2||z_1-z_3| &\le |z_3||z_1-z_2| + |z_1||z_2-z_3|, \\ |z_2+1||z_1-z_3| &\le |z_3+1||z_1-z_2| + |z_1+1||z_2-z_3|. \end{aligned}$$

The other cases can be proved similarly. □

The following lemma is an easy observation of plane geometry.

Lemma 3.5. *Suppose E_1 and E_2 are two ellipses that share the common foci at $(\pm c, 0)$. Let $\frac{x^2}{a_i^2} + \frac{y^2}{a_i^2-c^2} = 1$ be the equation of E_i, $i=1,2$. Then the distance between any two points (x_1,y_1), (x_2,y_2) on E_1 and E_2 respectively takes the maximum value if and only if $(x_1,y_1) = (\pm a_1, 0)$ and $(x_2,y_2) = (\mp a_2, 0)$, and the maximum distance is a_1+a_2.*

Proof. Let $(x_1,y_1) = (a_1\cos\theta_1, b_1\sin\theta_1)$ and $(x_2,y_2) = (a_2\cos\theta_2, b_2\sin\theta_2)$, where $b_i = \sqrt{a_i^2-c^2}$, $\theta_i \in [0,2\pi)$ $(i=1,2)$. Then it is easy to see that the following inequality holds, and the conclusion is thus obvious.

$$\begin{aligned} &(a_2\cos\theta_2 - a_1\cos\theta_1)^2 + (b_2\sin\theta_2 - b_1\sin\theta_1)^2 \\ &= a_2^2\cos^2\theta_2 + b_2^2\sin^2\theta_2 + a_1^2\cos^2\theta_1 + b_1^2\sin^2\theta_1 \\ &\quad - 2a_1a_2\cos\theta_2\cos\theta_1 - 2b_1b_2\sin\theta_2\sin\theta_1 \le (a_1+a_2)^2. \end{aligned}$$

□

Corollary 3.3. *With the above notation, $dist(z_1,z_2) \le \frac{1}{\sqrt{2}}$ and $dist(z_1,z_2) = \frac{1}{\sqrt{2}}$ if and only if*

(i) $z_1 = \infty$ *and* $z_2 \in [-1,0]$ *or*
(ii) $z_1 = 0$ *and* $z_2 \in (-\infty,-1]$ *or*
(iii) $z_1 = -1$ *and* $z_2 \in (0,+\infty)$.

Proof. We need to show

$$4|z_1 - z_2| \leq (1 + |z_1| + |z_1 + 1|)(1 + |z_2| + |z_2 + 1|).$$

Assume z_1 and z_2 are on the ellipses $|z| + |z + 1| = r_1$ and $|z| + |z + 1| = r_2$ respectively, where $r_2 \geq r_1 \geq 1$. By Lemma 3.5, we have

$$LHS = 4|z_1 - z_2| \leq 4|\frac{r_2 - 1}{2} + \frac{r_1 + 1}{2}| = 2(r_1 + r_2).$$

Thus $RHS - LHS \geq (r_1 - 1)(r_2 - 1) \geq 0$. □

If a closed operator T affiliated with a finite von Neumann algebra $\mathfrak{A}$ satisfying

$$\int_0^{+\infty} log^+ t d\mu_{|T|}(t) < \infty, \tag{9}$$

then the Fuglede-Kadison determinant of T

$$\triangle(T) = exp\left(\int_0^{\infty} logtd\mu_{|T|}(t)\right) \in [0, +\infty)$$

is well-defined ([2]), where $log^+(t) = max(0, log(t))$.

Let $L(T) = log(\triangle(T))$. It is proved in [7] that if S, T both satisfy the condition (9), then ST also satisfies (9), and $\triangle(ST) = \triangle(T)\triangle(S)$. Thus $L(ST) = L(S) + L(T)$. This fact will be used in the proof of the next lemma.

Theorem 3.1. $L(I - Q_{z_1}Q_{z_2}Q_{z_1}) = 2log(dist(z_1, z_2))$, z_1, $z_2 \in \mathbb{C}$.

Proof. To avoid confusion, we will use L_1 and L_2 to denote the function L defined in the proceeding discussion on $L_{F_{\frac{3}{2}}}$ and $Q_\infty L_{F_{\frac{3}{2}}} Q_\infty$ respectively. As in the proof of Lemma 2.1, let W be the unitary such that $WQ_{z_1}W = Q_\infty$ and $WQ_\infty W = Q_{z_1}$. Since $L_1(U^*TU) = L_1(T)$ for any unitary U in $L_{F_{\frac{3}{2}}}$, we have $L_1(I - Q_{z_1}Q_{z_2}Q_{z_1}) = L_1(I - Q_\infty WQ_{z_2}WQ_\infty)$. From (6),

$$I - Q_\infty WQ_{z_2}WQ_\infty = \begin{pmatrix} |z_1 - z_2|^2\sqrt{I - K_{z_1}}U_{z_1}S^*(I - K_{z_2})SU_{z_1}^*\sqrt{I - K_{z_1}} & 0 \\ 0 & I \end{pmatrix}.$$

Note $L_2(V) = 0$ for any unitary V, we have

$$\begin{aligned} L_1(I - Q_\infty WQ_{z_2}WQ_\infty) &= \frac{1}{2}(L_2(|z_1 - z_2|^2\sqrt{I - K_{z_1}}U_{z_1}S^*(I - K_{z_2})SU_{z_1}^*\sqrt{I - K_{z_1}}) + L_2(I)) \\ &= \frac{1}{2}(2L_2(|z_1 - z_2|) + L_2(S^*) + L_2(S) + L_2(I - K_{z_1}) + L_2(I - K_{z_2})). \end{aligned}$$

Recall $d\mu_{\sqrt{\frac{K_z}{I-K_z}}}(t) = \frac{2}{\pi}\frac{\Delta_z}{t^2+\Delta_z^2}1_{(0,+\infty)}dt$, where $\Delta_z = |z| + |z + 1|$. Therefore

$$L_2(I - K_{z_i}) = -\frac{2}{\pi}\int_0^{+\infty} log(t^2 + 1)\frac{\Delta_{z_i}}{t^2 + \Delta_{z_i}^2}dt = -2log(\Delta_{z_i} + 1).$$

Proceed as in the proof of Lemma 3.2, we have $d_{\mu_{|S|}}(t) = \frac{4}{\pi}\frac{1}{t^2+4}1_{(0,+\infty)}(t)dt$, thus

$$L_2(S) = \frac{4}{\pi}\int_0^{\infty} \frac{log(t)}{t^2 + 4}dt = log(2).$$

It is clear that $L_2(S^*) = L_2(S)$ and $L_2(|z_1 - z_2|) = log(|z_1 - z_2|)$, so we have

$$\begin{aligned} L_1(I - Q_\infty WQ_{z_2}WQ_\infty) &= log(|z_1 - z_2|) + log(2) - log(\Delta_{z_1} + 1) - log(\Delta_{z_2} + 1) \\ &= log(\frac{2|z_2 - z|}{(1 + |z| + |z + 1|)(1 + |z_2| + |z_2 + 1|)}). \end{aligned}$$

□

Corollary 3.4. $G[\mathrm{Lat}(\mathrm{Alg}(\{Q_\infty, Q_0, Q_{-1}\}))] \approx S_3$.

Proof. Suppose φ is a automorphism in $G[\mathrm{Lat}(\mathrm{Alg}(\{Q_\infty, Q_0, Q_{-1}\}))]$, and ρ is the Möbinus transformation such that $\varphi(Q_z) = Q_{\rho_z}$, $\forall z \in \widehat{\mathbb{C}}$. Let $\rho(\infty) = z_1$, $\rho(0) = z_2$ and $\rho(-1) = z_3$. First assume that z_1, z_2 and z_3 are in $\mathbb{C}$. From Theorem 3.1 and the freeness of $\{Q_{z_1}, Q_{z_2}, Q_{z_3}\}$, we have

$$2log(\frac{1}{\sqrt{2}}) = L(I - Q_{z_i}Q_{z_j}Q_{z_i}) = 2log(dist(z_i, z_j)), \quad i \neq j, i, j \in \{1, 2, 3\}.$$

This is impossible by Corollary 3.3. Thus one of z_1, z_2 and z_3 is ∞. Without loss of generality, we could assume that $z_1 = \infty$. Since $\|Q_\infty - Q_{z_k}\| = dist(\infty, z_k) = \frac{1}{\sqrt{2}}$, $k = 2, 3$. z_2, z_3 must be in $[-1, 0]$. Apply Corollary 3.3 again, we have $\{z_2, z_3\} = \{0, -1\}$, since $dist(z_2, z_3) = \frac{1}{\sqrt{2}}$. Therefore φ must be in the group generated by the two automorphisms in Lemma 3.1. □

Appendix A. Some technique results on reflexive lattices

The purpose of this appendix is to prove that the reflexivity of a lattice of projections in a von Neumann algebra is independent of any particular faithful representation. Precisely, if $\mathcal{L}$ is a reflexive subspace lattice in a von Neumann algebra $\mathfrak{A}$, φ is a *-isomorphism of $\mathfrak{A}$, and $\varphi(\mathfrak{A})$ is in $\mathcal{B}(\mathcal{K})$, then $\varphi(\mathcal{L})$is also reflexive. This result was mentioned by Morhan in [4], but his proof was incomplete. In the following we will provide a detailed proof of this fact.

Lemma A.1. *Let $\mathfrak{A}$ $(\subset \mathcal{B}(\mathcal{H}))$ be a von Neumann algebra, $\mathcal{L}$ is a subspace lattice in $\mathfrak{A}$. Suppose P', Q' are two projections in $\mathfrak{A}'$ such that $P' \precsim Q'$, and $C_{P'} = C_{Q'}$, where $C_{P'}$ and $C_{Q'}$ are the central carriers of P' and Q' respectively. Let $P'\mathcal{L} = \{P'E | E \in \mathcal{L}\}$. If $P'\mathcal{L}$ is reflexive as a subspace lattice in $\mathcal{B}(P'\mathcal{H})$, then we have $Q'\mathcal{L}(= \{Q'E | E \in \mathcal{L}\})$ is also reflexive as a subspace lattice in $\mathcal{B}(Q'\mathcal{H})$.*

Proof. Without loss of generality, we may assume that $C_{P'} = I$ and $Q' = I$. So we only need to show that if $P'\mathcal{L}$ is reflexive and $C_{P'} = I$, then $\mathcal{L}$ is also reflexive.

For any T_1 in $\mathrm{Alg}(P'\mathcal{L})$, let T be the operator in $\mathcal{B}(\mathcal{H})$ such that $T(I - P') = (I - P')T = 0$, and $TP'|_{P'\mathcal{H}} = T_1$. If E is in $\mathcal{L}$, then $(I - E)TE = (P' - P'E)TP'E = 0$, since T_1 is in $\mathrm{Alg}(P'\mathcal{L})$. Because $\mathrm{Lat}(\mathrm{Alg}(\mathcal{L}))$ is a subset of $\mathfrak{A}$, we have $(P' - P'F)P'TP'F = P'(I - F)TFP' = 0$ for any F belongs to $\mathrm{Lat}(\mathrm{Alg}(\mathcal{L}))$. This implies $FP'|_{P'\mathcal{H}} \in \mathrm{Lat}(\mathrm{Alg}(P'\mathcal{L})) = P'\mathcal{L}$. The map

$$\mathfrak{A} \to P'\mathfrak{A} : A \to AP'|_{P'\mathcal{H}}$$

is $*-$isomorphism since $C_{P'} = I$, therefore F must be in $\mathcal{L}$. □

Example A.1. *The condition $C_{P'} = I$ in the above lemma cannot be removed. Let*

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, P' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and $\mathcal{L} = \{0, E_1, E_2, E_3, I\}$. Then $C_{P'} = P' \neq I$, and $P'\mathcal{L}$ is reflexive. However, it is not hard to check that $\mathcal{L}$ is not reflexive.

In following lemma is easy, and we omit the proof.

Lemma A.2. *Suppose $\{\mathcal{H}_i\}_{i\in\mathcal{I}}$ is a family of Hilbert spaces, where $\mathcal{I}$ is an index set. Let $\mathcal{H} = \oplus_i \mathcal{H}_i$, and E_i is the orthonormal projection form $\mathcal{H}$ onto $\mathcal{H}_i$. If $\mathcal{L}$ is a subset of projections in ${\{E_i\}'}_{i\in\mathcal{I}}$. For each projection $P \in \mathcal{L}$, let $P_i = E_iP|_{\mathcal{H}_i}$. Then*

$$\mathrm{Alg}(\mathcal{L}) = \{A \in \mathcal{B}(\mathcal{H}) | (E_i - P_i)E_iAE_jP_j = 0, \text{ for all } i, j \in \mathcal{I} \text{ and } P \in \mathcal{L}\}.$$

With the same notations in the above lemma, we give the following definition.

Definition A.1. *An extension index ex of $\mathcal{L}$ is a map assigning a cardinal number $ex(i)$ to each i in $\mathcal{I}$. Let $\mathcal{I}^{ex} = \{(i,j) | i \in \mathcal{I}, \text{ and } j \in \mathcal{I}(ex(i))\}$, and $\mathcal{H}^{ex} = \oplus_{i\in\mathcal{I}} \mathcal{H}_i \otimes l^2(ex(i)) = \oplus_{(i,j)\in\mathcal{I}^{ex}} \mathcal{H}_i$, where $\mathcal{I}(ex(i))$ is a set with cardinal $ex(i)$, and $l^2(ex(i))$ is the Hilbert space with dimension $ex(i)$. For each $P \in \mathcal{L}$, let P^{ex} be a projection in $\mathcal{B}(\mathcal{H}^{ex})$ such that $E_i \otimes F_j^i P^{ex} = P^{ex} E_i \otimes F_j^i = P_i|_{\mathcal{H}_i}$, where $E_i \otimes F_j^i$ is the projection form $\mathcal{H}^{ex}$ onto the jth copy of $\mathcal{H}_i$ in $\mathcal{H}^{ex}$. And we will use $\mathcal{L}^{ex}$ to denote the set $\{P^{ex} | \text{for all } P \in \mathcal{L}\}$.*

Lemma A.3. *With the notations in Lemma A.2 and DefinitionA.1, let ex be an extension index, then $\mathcal{L}$ is reflexive if and only if $\mathcal{L}^{ex}$ is reflexive.*

Proof. First note $\bigoplus_i I \otimes \mathcal{B}(l^2(ex(i))$ is in $\{\mathcal{L}^{ex}\}'$, this implies that $\mathrm{Lat}(\mathrm{Alg}(\mathcal{L}^{ex})) \subset \bigoplus_i \mathcal{B}(\mathcal{H}_i) \otimes I_{ex(i)}$. Thus for any $Q^{ex} \in \mathrm{Lat}(\mathrm{Alg}(\mathcal{L}^{ex}))$, there exist $Q_i \in \mathcal{B}(\mathcal{H}_i)$, $i \in \mathcal{I}$, and $Q^{ex} = \bigoplus_{i\in\mathcal{I}} Q_i \otimes I_{ex(i)} = \oplus_{(i,j)\in\mathcal{I}^{ex}} Q_i$.

We need to show

$$Q^{ex} = \oplus_{(i,j)\in\mathcal{I}^{ex}} Q_i \in \mathrm{Lat}(\mathrm{Alg}(\mathcal{L}^{ex})) \Longleftrightarrow Q = \oplus_{i\in\mathcal{I}} Q_i \in \mathrm{Lat}(\mathrm{Alg}(\mathcal{L})).$$

By LemmaA.2, $Q = \oplus_i Q_i$ is in $\mathrm{Lat}(\mathrm{Alg}(\mathcal{L}))$ if and only if

$$(I - Q_i)A_{ij}Q_j = 0, \forall i, j \in \mathcal{I},$$

where A_{ij} is any operator in $\mathcal{B}(\mathcal{H}_j, \mathcal{H}_i)$ such that $(I - P_i)A_{ij}P_j = 0$ for any $P \in \mathcal{L}$. Similarly, $\oplus_{(i,j)\in\mathcal{I}^{ex}} Q_i \in \mathrm{Lat}(\mathrm{Alg}(\mathcal{L}^{ex})$ if and only if

$$(I - Q_i)A_{(i,k),(j,l)}Q_j = 0, \forall (i,k), (j,l) \in \mathcal{I}^{ex},$$

where $A_{(i,k),(j,l)}$ is any operator in $\mathcal{B}(E_j \otimes F_l^j \mathcal{H}^{ex}, E_i \otimes F_k^i \mathcal{H}^{ex})$ such that $(I - P_i)A_{(i,k),(j,l)}P_j = 0$ for any $P \in \mathcal{L}$. This end the proof, since $E_j \otimes F_l^j \mathcal{H}^{ex}$ is just $\mathcal{H}_j$. $\square$

Let $\mathfrak{A}$ be a von Neumann algebra, we say that a projection E with central carrier C_E in $\mathfrak{A}$ is monic if $E \neq 0$ and there exist a positive integer k and projections $E_1, \ldots, E_k$ in $\mathfrak{A}$ such that

$$E_1 \sim E_2 \sim \cdots \sim E_k \sim E, \; E_1 + E_2 + \cdots + E_k = C_E.$$

The following lemma is Proposition 8.2.1 in [1].

Lemma A.4. *Each non-zero projection E in a finite von Neumann algebra $\mathfrak{A}$ is the sum of an orthogonal family of monic projections in $\mathfrak{A}$.*

The following corollary is then immediate consequence of this lemma.

Corollary A.1. *Suppose $\mathfrak{A}$ is a finite von Neumann algebra, then for any non-zero projection P in $\mathfrak{A}$, there exists a non-zero projection Q in $\mathfrak{A}$ such that $Q \leq P$ and $C_Q = Q + \sum_i Q_i$, where $\{Q_i\}$ is a mutually orthogonal family of projections in $\mathfrak{A}$ such that $Q_i \sim Q$ for each i.*

In the following, we will prove that the statement in Corollary A.1 is true for any von Neumann algebra. Since any von Neumann algebra $\mathfrak{A}$ is a direct sum of algebras of types $\mathrm{I}_n(n = 1, 2, 3 \ldots)$, II_1, II_∞ and III, it is suffices to consider only the three cases in which $\mathfrak{A}$ is type I_∞, II_∞ or III. For basic comparison theory of projections, we refer to [1].

Lemma A.5. *Suppose $\mathfrak{A}$ is a type III von Neumann algebra, then for any non-zero projection P in $\mathfrak{A}$, there exists a non-zero projection Q in $\mathfrak{A}$ such that $Q \le P$, and $C_Q = Q + \sum_i Q_i$, where $\{Q_i\}$ is a mutually orthogonal family of projections in $\mathfrak{A}$ such that $Q_i \sim Q$ for each i.*

Proof. By considering $C_P\mathfrak{A}$, we could assume that $C_P = I$. Since any properly infinite projection can be "halved," we can find two projection P_1 and P_2 such that $P = P_1 + P_2$ and $P \sim P_1 \sim P_2$. Let $\{P_\alpha\}$ be a maximal family of equivalent mutually orthogonal projections that contains $\{P_1, P_2\}$. If $\sum_\alpha P_\alpha = I$, let $Q = P_1$. Otherwise, by maximality of $\{P_\alpha\}$, there exists a non-zero central projection E such that

$$0 \ne F = (I - \sum_\alpha P_\alpha)E \prec P_1E.$$

Thus we have

$$EP_1 \precsim EP_2 + F \precsim E(P_1 + P_2) \sim EP_1,$$

which implies $EP_1 \sim EP_2+F$. So we could let $Q = EP_1$ (Note that $C_{EP_1} = E$). $\square$

Lemma A.6. *If a von Neumann algebra $\mathfrak{A}$ is type I_α(α is a infinite cardinal number) or type II_∞, and P is a non-zero projection in $\mathfrak{A}$, then there exists a non-zero subprojection Q of P such that $C_Q = Q + \sum_i Q_i$, where $\{Q_i\}$ is a mutually orthogonal family of projections in $\mathfrak{A}$ such that $Q_i \sim Q$ for each i.*

Proof. By choosing a proper finite projection with central carrier I, we may assume $\mathfrak{A} = \mathfrak{A}_1\overline{\otimes}\mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a infinite dimensional Hilbert space, and $\mathfrak{A}_1$ is a finite von Neumann algebra. Let $\{E_{\alpha,\beta}\}$ be a system of matrix units for $\mathcal{B}(\mathcal{H})$. If there is a central projection F such that $0 \ne FE_{\alpha,\alpha} \sim Q \le FP$ for some $E_{\alpha,\alpha}$, then Q satisfies all the requests in the lemma. Indeed, since $FE_{\alpha,\alpha} \sim Q$ and $FE_{\alpha,\alpha}$ is a finite projection, there exists a unitary U in $\mathfrak{A}$ such that $Q = U^*FE_{\alpha,\alpha}U$. It clear that $Q \sim U^*FE_{\beta,\beta}U$ and $\sum_\beta FE_{\beta,\beta} = F$, thus we have

$$C_Q = F = U^*FE_{\alpha,\alpha}U + \sum_{\beta\ne\alpha} U^*FE_{\beta,\beta}U = Q + \sum_{\beta\ne\alpha} U^*FE_{\beta,\beta}U.$$

We may, thus, assume that $P \prec E_\alpha$. Let U be a unitary in $\mathfrak{A}$ such that $P \le U^*E_\alpha U$. Replacing $\{E_{\alpha,\beta}\}$ by $\{U^*E_{\alpha,\beta}U\}$, we may now assume that $P \le E_{\alpha,\alpha}$. Remember $\mathfrak{A}_1 \cong E_{\alpha,\alpha}\mathfrak{A}E_{\alpha,\alpha}$ is a finite von Neumann algebra. From Corollary A.1, there is a monic projection Q in $E_{\alpha,\alpha}\mathfrak{A}E_{\alpha,\alpha}$ such that $Q \le P$ and $\widetilde{C_Q} = Q + \sum_i Q_i$, where $\widetilde{C_Q}$ is the central carrier of Q in $E_{\alpha,\alpha}\mathfrak{A}E_{\alpha,\alpha}$. Since the central carrier of Q in $\mathfrak{A}$ is $C_Q = \sum_\beta E_{\beta,\alpha}\widetilde{C_Q}E_{\alpha,\beta}$, we have

$$C_Q = Q + \sum_i Q_i + \sum_{\beta\ne\alpha} E_{\beta,\alpha}(Q + \sum_i Q_i)E_{\alpha,\beta}.$$

$\square$

By combining the above lemmas, we have the following theorem.

Theorem A.1. *Suppose P is a non-zero projection in a von Neumann algebra $\mathfrak{A}$, then there exists a non-zero subprojection Q of P and a mutually orthogonal family of projections $\{Q_i\}$ in $\mathfrak{A}$ such that $C_Q = Q + \sum_i Q_i$ and $Q_i \sim Q$ for each i.*

The following corollary is immediate by a maximality argument, and we omit the proof.

Corollary A.2. *Let $\mathfrak{A}$ be a von Neumann algebra and P be a projection in $\mathfrak{A}$, then there is an orthogonal family $\{Q_\alpha\}$ of subprojections of P in $\mathfrak{A}$ such that $C_P = \sum_\alpha C_{Q_\alpha}(C_{\alpha_1} \perp C_{\alpha_2}, \alpha_1 \neq \alpha_2)$. Moreover, $C_{Q_\alpha} = Q_\alpha + \sum_i Q_\alpha^i$ where $\{Q_\alpha^i\}$ is a orthogonal family of projections in $\mathfrak{A}$ such that $Q_\alpha^i \sim Q$ for each i.*

Theorem A.2. *Suppose $\mathcal{L}$ is a reflexive subspace lattice in a von Neumann algebra $\mathfrak{A}_1(\subset \mathcal{B}(\mathcal{H}_1))$. If φ is a *-isomorphism of $\mathfrak{A}_1$ onto $\mathfrak{A}_2(\subset \mathcal{B}(\mathcal{H}_2))$, then $\varphi(\mathcal{L})$ is also reflexive as a subspace lattice in $\mathcal{B}(\mathcal{H}_2)$.*

Proof. By [3, Theorem 5.5], there exists a Hilbert space $\mathcal{K}$, a projection $P' \in \mathfrak{A}'\overline{\otimes}\mathcal{B}(\mathcal{K})$ with central carrier I, and an unitary U of $P'(\mathcal{H}_1 \otimes \mathcal{K})$ onto to $\mathcal{H}_2$ such that

$$\varphi(A) = U(A \otimes I_\mathcal{K})P'|_{P'(\mathcal{H}_1\otimes\mathcal{K})}U^*, A \in \mathfrak{A}.$$

In other words, φ can be decomposed into the composition of an amplification, an induction, and a spatial isomorphism. Since amplification and spatial isomorphism preserve reflexivity, we could assume that φ is an induction. And we only need to show $P'\mathcal{L} \subset P'\mathfrak{A}|_{P'\mathcal{H}}$ is reflexive if $\mathcal{L} \subset \mathfrak{A}$ is reflexive , where P' is a projection in $\mathfrak{A}'$ such that $C_{P'} = I$.

By Theorem A.2, there is a family $\{Q'_\alpha\}$ of subprojections of P' in $\mathfrak{A}'$ such that $\sum_\alpha C_{Q'_\alpha} = I = C_P$, and $C_{Q'_\alpha} = Q'_\alpha + \sum_i {Q'}^i_\alpha$, where ${Q'}^i_\alpha \sim Q'_\alpha$. Let $Q' = \sum_\alpha Q'_\alpha \leq P'$. It is obvious $C_{Q'} = I$, so if $Q'\mathcal{L} \subset Q'\mathfrak{A}|_{Q'\mathcal{H}}$ is reflexive, then $P'\mathcal{L}$ is reflexive by Lemma A.1.

Let W_α^i be a partial isometry in $\mathfrak{A}'$ with initial projection ${Q'}^i_\alpha$ and final projection Q'_α, i.e. ${W_\alpha^i}^* W_\alpha^i = {Q'}^i_\alpha$ and $W_\alpha^i {W_\alpha^i}^* = Q'_\alpha$. Then the equation

$$V = \oplus_\alpha(Q'_\alpha \oplus (\oplus_i W_\alpha^i))$$

defines a unitary operator from $\mathcal{H}$ onto $\bigoplus_\alpha(Q'_\alpha\mathcal{H} \oplus (\oplus_i Q'_\alpha\mathcal{H}))$.

Replacing $\mathcal{H}$ by $\bigoplus_\alpha(Q'_\alpha\mathcal{H} \oplus (\oplus_i Q'_\alpha\mathcal{H}))$, $\mathfrak{A}$ by $V\mathfrak{A}V^*$($\mathfrak{A}'$ by $V\mathfrak{A}'V^*$) and Q' by $VQ'V^* = \oplus_\alpha(I_\alpha \oplus_i (0))$, we may now assume that $\mathfrak{A}' = \bigoplus_\alpha Q'_\alpha \mathfrak{A}' Q'_\alpha \otimes \mathcal{B}(l^2(ex(\alpha))$, where $ex(\alpha)$ equals the cardinal of set $\{Q'_\alpha\} \cup \{{Q'}^i_\alpha\}$, and $l^2(ex(\alpha))$ is a Hilbert space with dimension $ex(\alpha)$. Under this assumption, $\mathcal{L}$ is just $(Q'\mathcal{L})^{ex}$. Thus by Lemma A.3, $\mathcal{L}$ is reflexive if and only if $Q'\mathcal{L}$ is reflexive, and the theorem follows. □

Corollary A.3. *Suppose $\mathcal{L}$ is a KS-lattice for a von Neumann algebra $\mathfrak{A}$ acting on some Hilbert space $\mathcal{H}$), if φ is a *-isomorphism of $\mathfrak{A}$ and $\varphi(\mathfrak{A}) \subset \mathcal{B}(\mathcal{K})$, then $\varphi(\mathcal{L})$ is also KS-lattice for $\varphi(\mathfrak{A})$.*

Proof. If $\varphi(\mathcal{L})$ is not a KS-lattice, then there is a reflexive sublattice $\mathcal{L}_0$ of $\varphi(\mathcal{L})$ such that $\mathcal{L}_0$ generates $\varphi(\mathfrak{A})$ as von Neumann algebra. So $\varphi^{-1}(\mathcal{L}_0)$ is a reflexive lattice that generates $\mathfrak{A}$ by Theorem A.2, which contradicts the minimality of $\mathcal{L}$. □

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L. K. Hua Key Laboratory of Mathematics, Chinese Academy of Sciences, Beijing, China and Department of Mathematics, University of New Hampshire, Durham, NH 03824

E-mail address: `liming@math.ac.cn`

L. K. Hua Key Laboratory of Mathematics, Chinese Academy of Sciences, Beijing, China

E-mail address: `wyuan@math.ac.com`