Central sequence algebras of the tensor product of II_1 facotrs

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Abstract: Let $\mathcal{M}$ and $\mathcal{N}$ be two type II_1 factors and ω a free ultrafilter for the natural number $\mathbb{N}$. It is shown that, if the central sequence algebra $\mathcal{N}_\omega$ is abelian and there is a non-atomic abelian subalgebra $\mathcal{A}$ in $\mathcal{M}$ such that $(\mathcal{M}\otimes\mathcal{N})_\omega \subseteq (\mathcal{A}\otimes\mathcal{N})^\omega$, then $(\mathcal{M}\otimes\mathcal{N})_\omega$ is abelian. It is also given two specify factors of type II_1 which satisfy the above-mentioned conditions.

Keywords: Type II_1 factor; Central sequence algebras; Tensor product; Crossed product

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1 Introduction and Preliminaries

J.von Neumann introduced one kind of rings, which now are called von Neumann algebras, consisting of some bounded linear operators acting on some Hilbert space. These algebras are strong-operator closed self-adjoint subalgebras of the algebra of all bounded linear operators acting on a Hilbert space. Factors are von Neumann algebras whose center consists of the scalar multiples of the identity operator. Every von Neumann algebras is the direct sum or direct integral of factors. Thus factors are the building blocks for all von Neumann algebras. Murray and von Neumann classified factors by means of a relative dimension function. Finite factors are those for which this dimension function has a bounded range. Finite factors whose dimension function has an infinite range are referred as the factors of type II_1. For II_1 factors, this dimension function gives rise to a unique normalized faithful tracial state.

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By using of the ideas of central sequences, Murray and von Neumann showed that there are two non-isomorphic factors of type II_1. Central sequences in a finite factor form an algebra. The central sequence algebra for a factor of type II_1 can be viewed as the relative commutant of the algebra in the ultrapower of the algebra constructed from a given free ultrafilter on the natural number $\mathbb{N}$. Dixmier [2] showed that the central sequence algebra of a factor of type II_1 is either trivial (the scale multiples of the identity) or non-atomic. Indeedly the central sequence algebra for a factor of type II_1 is non-trivial if and only if the origin factor has the property Γ. Furthermore McDuff [5] showed that if the central sequence algebra of a II_1 factor is non-commutative, then it is a von Neumann algebra of type II_1. She also showed that the central sequence algebra of a II_1 factor is noncommutative if and only if the origin factor is isomorphic to the tensor product of the hyperfinite II_1 factor with itself.

The factor with trivial central sequence algebra, which is equivalent to that it has not the property Γ, is referred to as the full factor. A.Connes [1] shown that the tensor product von Neumann algebra of two II_1 factors is full if and only if both the two factors are full. Thus the central sequence algebra of the tensor product of two II_1 factors with trivial sequence algebra is still full. Of course, if one of the central sequence algebras of two II_1 factors is non-commutative (i.e.one of two factors is McDuff factor), then the central sequence algebra of the tensor product of them is non-commutative. However, given two type II_1 factors with abelian and nontrivial central sequence algebra, it is not known whether the central sequence algebra of the tensor product of these factors is still abelian and nontrivial. Recently, Fang, Ge and Li [3] have gotten that the central sequence algebra of the injective II_1 factor is prime, then the central sequence algebra of the tensor product of two factors may not be the tensor product of the central sequence algebras of the factors. Nowadays the above-mentioned question is still open.

In this paper, we partly answer the previous problem in positive. We in fact restrict our attention to the factors of type II_1 whose central sequence algebras satisfy some particular conditions. Let $\mathcal{M}$ and $\mathcal{N}$ be two II_1 factors with separable predual. If the central sequence algebra of one of the two factors, say $\mathcal{N}$, is abelian and the central sequence algebra of their tensor product is contained in the ultrapower of the tensor product of $\mathcal{N}$ and a non-atomic abelian von Neumann subalgebra of $\mathcal{M}$, then the central sequence algebra of the two factors is abelian. As a corollary, under the previous conditions, we can see that the tensor product of two II_1 factors, which have the property Γ and are not McDuff factors, also has the property Γ and are not McDuff factor.

This paper is organized as follows. In the rest part of this section, we firstly recall the basic concepts and properties of the central sequence algebras. In the second section, we get the main result of this paper. In the last section, we give two specify II_1 factors which satisfy the above-mentioned conditions.

Now let's recall the basic facts of the central sequence in II_1 factors. Let $\mathcal{M}$ be a II_1 factor with separable predual and τ the unique faithful normal tracial state on $\mathcal{M}$. For any $X \in \mathcal{M}$, $\|X\|_2 = \tau(X^*X)^{1/2}$ is the trace norm induced by τ. Suppose that (T_n) is a bounded sequence of elements of $\mathcal{M}$. Then (T_n) is said to be a *central sequence* in $\mathcal{M}$ if

$$\|[T_n, A]\|_2 = \|T_nA - AT_n\|_2 \to 0$$

as $n \to \infty$ for any $A \in \mathcal{M}$. Note that any subsequence of a central sequence is still a central sequence. The central sequence (T_n) is called *non-trivial* if $\|T_n - \tau(T_n)I\|_2 \nrightarrow 0$ as $n \to \infty$.

Suppose that ω is a free ultrafilter of the natural number $\mathbb{N}$. Let $l^\infty(\mathcal{M})$ be the set of all bounded sequence in $\mathcal{M}$, and let

$$I_\omega = \{(T_n) \in l^\infty(\mathcal{M}) : \lim_\omega \|T_n\|_2 = 0\}.$$

Then I_ω is a maximal ideal in $l^\infty(\mathcal{M})$. Thus $l^\infty(\mathcal{M})/I_\omega$ forms a II_1 factor [8] which is said to be the ultrapoqer of $\mathcal{M}$ and denoted by $\mathcal{M}^\omega$. There is a canonical imbedding of $\mathcal{M}$ into $\mathcal{M}^\omega$. Then under the meaning of imbedding, for any element (T_n) in the relative commutant $\mathcal{M}' \cap \mathcal{M}^\omega$, there is a subsequence of (T_n) which is a central sequence. Thus we say that the relative commutant $\mathcal{M}' \cap \mathcal{M}^\omega$ is the *central sequence algebra* of $\mathcal{M}$ and denoted by $\mathcal{M}_\omega$.

2 Main result

Suppose that $\mathcal{M}$ and $\mathcal{N}$ are factors of type II_1 acting on the separable Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively. Let τ be the faithful normal tracial states on $\mathcal{M}$ and $\mathcal{N}$ for the convenience. Let ω be a free ultrafilter on $\mathbb{N}$ and $\mathcal{M}_\omega$ and $\mathcal{N}_\omega$ be the central sequence algebras of $\mathcal{M}$ and $\mathcal{N}$ respectively.

Theorem 2.1. With notations as above. If $\mathcal{N}_\omega$ is abelian and there is a maxiaml abelian von Neumann subalgebra $\mathcal{A}$ of $\mathcal{M}$ such that

$$(\mathcal{M} \otimes \mathcal{N})_\omega \subset (\mathcal{A} \otimes \mathcal{N})^\omega,$$

then $(\mathcal{M} \otimes \mathcal{N})_\omega$ is abelian.

As McDuff showed that the central sequence algebra of a factor of type II_1 is abelian if and only if any two central sequences in it are commutating [5], thus we just need to show that any two central sequences in $\mathcal{M} \otimes \mathcal{N}$ are commutating.

Now we firstly discuss the tensor product $\mathcal{A} \otimes \mathcal{N}$. As $\mathcal{A}$ is a maximal abelian von Neumann subalgebra of $\mathcal{M}$, then $\mathcal{A}$ is non-atomic. Thus $\mathcal{A}$ is unitary equivalent to $\mathcal{A}_0 \otimes \mathbb{C}I$ where $\mathcal{H} \cong \mathcal{H}_1 \otimes \mathcal{H}_2$ and $\mathcal{A}_0$ is a maximal non-atomic abelian von Neumann subalgebra of $\mathcal{B}(\mathcal{H}_1)$ [4]. Then there is a unitary operator W from $\mathcal{H}_1$ onto the Hilbert space $L^2([0,1],\mu)$ such that $\varphi(\mathcal{A}_0)$ is identified with the multiplication algebra $L^\infty([0,1],\mu)$, where φ is the mapping defined as $A \to WAW^*$ for any $A \in \mathcal{A}_0$ and μ is the Lebesgue measure on the interval $[0,1]$. Furthermore, we have $\tau(A) = \int_{[0,1]} \varphi(A)(x) d\mu(x)$ for any $A \in \mathcal{A}_0$. For the convenience, we identify $\mathcal{H}_1$ with $L^2([0,1],\mu)$, $\mathcal{A}$ with $L^\infty([0,1],\mu)$ and A with $\varphi(A)$. Thus $\mathcal{H} \otimes \mathcal{K} = \mathcal{H}_2 \otimes L^2([0,1],\mu) \otimes \mathcal{K}$.

Note that the Hilbert space $L^2([0,1],\mu) \otimes \mathcal{K}$ is identified with $L^2([0,1],\mathcal{K})$ where the inner product is given by [9]

$$\langle \xi, \eta \rangle = \int_{[0,1]} \langle \xi(x), \eta(x) \rangle d\mu(x), \xi, \eta \in L^2([0,1],\mathcal{K}).$$

Furthermore, let $\eta \in \mathcal{K}$ and E be a measurable subset of $[0,1]$ with respect to the Lebesgue measure μ. Then $\chi_E \eta$ is a vector of $L^2([0,1],\mu) \otimes \mathcal{K}$ where χ_E is the characteristic function of E. Each element in the linear span of all $\chi_E \eta$ is referred to as the *simple functions* with values in $\mathcal{K}$ on the interval $[0,1]$.

Similar to the elementary real analysis, we have the following technique results.

Lemma 2.2. For any $\xi \in L^2([0,1], \mathcal{K})$, there is a sequence of simple functions $\{\xi_n(= \sum_k \chi_{A_{n,k}} \eta_{n,k})\}$ such that $\xi_n(x) \to \xi(x)$ for almost every $x \in [0,1]$ and $\xi_n \to \xi$ under the norm of $L^2([0,1], \mathcal{K})$ as $n \to \infty$.

Proof. For any $\xi \in L^2([0,1], \mathcal{K})$, there is a sequence $(\sum_k f_{n,k} \otimes \eta_{n,k})_n$ of the linear combinations of simple tensors, where $f_{n,k} \in L^2([0,1], \mu)$ and $\eta_{n,k} \in \mathcal{K}$, such that

$$\sum_k f_{n,k} \otimes \eta_{n,k} \to \xi$$

in $L^2([0,1], \mathcal{K})$ as $n \to \infty$. It is equivalent to

$$\int_{[0,1]} ||\xi(x) - (\sum_k f_{n,k} \otimes \eta_{n,k})(x)||^2 d\mu(x) \to 0$$

as $n \to \infty$. Thus there is a subsequence of $(\sum_k f_{n,k} \otimes \eta_{n,k})_n$ which is converges to ξ for almost every $x \in [0,1]$. So we just need to show that the result holds for any simple tensor in $L^2([0,1], \mathcal{K})$.

Suppose that $f \otimes \eta \in L^2([0,1], \mu) \otimes \mathcal{K}(= L^2([0,1], \mathcal{K}))$, then there is a sequence of simple functions $(f_n) \in L^2([0,1], \mu)$ such that $f_n \to f$ in $L^2([0,1], \mu)$ and $f_n(x) \to f(x)$ for almost every $x \in [0,1]$. Thus the sequence $(f_n \otimes \eta)(= f_n \eta)$ is convergent to $f \otimes \eta$ and $f_n(x)\eta \to f(x)\eta$ for almost every $x \in [0,1]$. ■

Now we discuss the action of the von Neumann algebra $L^\infty([0,1]) \bar{\otimes} \mathcal{B}(\mathcal{K})$ on $L^2([0,1], \mathcal{K})$.

Lemma 2.3. Let $T \in \mathcal{B}(\mathcal{K})$ and $f \in L^\infty([0,1])$. For any $\xi \in L^2([0,1], \mu) \otimes \mathcal{K}$, we have

$$((I \otimes T)\xi)(x) = T\xi(x), \quad ((f \otimes I)\xi)(x) = f(x)\xi(x)$$

for almost every $x \in [0,1]$.

Proof. Let $(\xi_n(= \sum_k \chi_{E_{n,k}} \eta_{n,k}))$ be a sequence of simple functions in $L^2([0,1], \mathcal{K})$ such that $\xi_n(x) \to \xi(x)$ for almost every $x \in [0,1]$ and $\xi_n \to \xi$ in $L^2([0,1], \mathcal{K})$.

Then we have $(I \otimes T)\xi_n \to (I \otimes T)\xi$. Thus there is a subsequence (ξ_{n_j}) of (ξ_n) such that $((I \otimes T)\xi_{n_j})(x) \to ((I \otimes T)\xi)(x)$ for almost every $x \in [0,1]$. For the convenience, we use the sequence (ξ_n) to denote the subsequence. Hence, we have

$$((I \otimes T)\xi_n)(x) = \sum_k \chi_{E_{n,k}}(x) T\eta_{n,k} \to ((I \otimes T)\xi)(x)$$

for almost every $x \in [0,1]$.

At the same time, as $\xi_n(x) \to \xi(x)$ for almost every $x \in [0,1]$, thus we have

$$T\xi_n(x) = T(\sum_k \chi_{E_{n,k}} \otimes \eta_{n,k})(x) = \sum_k \chi_{E_{n,k}}(x) T\eta_{n,k} \to T\xi(x)$$

as $n \to \infty$ almost everywhere. Therefore, we get

$$((I \otimes T)\xi)(x) = T\xi(x)$$

almost everywhere.

Similar to the above argument, we also have $((f \otimes I)\xi)(x) = f(x)\xi(x)$ almost everywhere. ■

According to the above results, it is easy to know that $f \otimes T$ is a decomposable operator for any $f \in L^\infty([0,1],\mu)$ and any $T \in \mathcal{B}(\mathcal{K})$. As $L^\infty([0,1],\mu) \otimes \mathcal{N}$ commutes with the diagonal algebra $L^\infty([0,1]) \otimes \mathbb{C}I$, thus $L^\infty([0,1],\mu) \otimes \mathcal{N}$ is a von Neumann algebra consisting of decomposable operators. Furthermore, we have the following conclusion.

Lemma 2.4. For any $T \in L^\infty([0,1],\mu) \otimes \mathcal{N}$, we have $T(x) \in \mathcal{N}$ almost everywhere.

Proof. As the Hilbert space $\mathcal{K}$ is separable, then there exists a strong operator topology dense sequence $\{A'_n\}$ of operators in the unit ball $(\mathcal{N}')_1$ of the commutant of $\mathcal{N}$. Then for any $\xi \in L^2([0,1],\mu) \otimes \mathcal{K}$, as T is decomposable, we have

$$||T(I \otimes A'_n)\xi - (I \otimes A'_n)T\xi||^2 = \int_{[0,1]} ||T(x)A'_n\xi(x) - A'_nT(x)\xi(x)||^2 d\mu(x) = 0.$$

Thus $\|T(x)A'_n\xi(x) - A'_nT(x)\xi(x)\| = 0$ almost everywhere for any $n = 1, 2, \cdots$ and $\xi \in L^2([0,1],\mu) \otimes \mathcal{K}$. In particular, pick a dense sequence $\{\eta_k : k \in \mathbb{N}\}$ in $\mathcal{K}$, then we have

$$\|T(x)A'_n\eta_k - A'_nT(x)\eta_k|| = 0$$

almost everywhere for any n and k. Thus we get $\|T(x)A'_n - A'_nT(x)\| = 0$ almost everywhere. Hence we have $T(x) \in \mathcal{N}$ almost everywhere. ■

Remark 2.5. Note that $L^\infty([0,1],\mu) \otimes \mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M} \otimes \mathcal{N}$ up to some isomorphism. For any $T \in L^\infty([0,1],\mu) \otimes \mathcal{N}$, we have

$$\begin{aligned}\|T\|_2 = \tau(T^*T)^{\frac{1}{2}} &= (\int_{[0,1]} \tau(T(x)^*T(x))d\mu(x))^{\frac{1}{2}} \\ &= (\int_{[0,1]} \|T(x)\|_2^2 d\mu(x))^{\frac{1}{2}}.\end{aligned}$$

Now according to the result of McDuff, to get theorem 1, we just need to show that any two central sequences of $\mathcal{M} \otimes \mathcal{N}$ are commutating. In the lemma that follows, we prove a technique result about the central sequences of $\mathcal{M} \otimes \mathcal{N}$ under the condition $(\mathcal{M} \otimes \mathcal{N}) \subset (\mathcal{A} \otimes \mathcal{N})^\omega$.

Lemma 2.6. Suppose that $(T_n) \in (L^\infty([0,1],\mu) \otimes \mathcal{N})^\omega$ is a central sequence of $\mathcal{M} \otimes \mathcal{N}$. There is a subsequence (T_{n_k}) of (T_n) such that $(T_{n_k}(x)) \in \mathcal{N}_\omega$ almost everywhere.

Proof. Without loss of generality, we can suppose that $\|T_n\| = 1$. As (T_n) is a central sequence, then for any $A \in \mathcal{N}$, we have

$$\|T_n(I \otimes A) - (I \otimes A)T_n\|_2^2 = \int_{[0,1]} \|T_n(x)A - AT_n(x)\|_2^2 d\mu(x) \to 0 \tag{1}$$

as $n \to \infty$. Since $\mathcal{N}$ has separable predual, we can pick a $\| \cdot \|_2-$dense sequence $\{A_k\}$ in the unit ball $\mathcal{N}_1$ of $\mathcal{N}$.

For the operator A_1, by the equation (1), there is a subsequence $(T_n^{(1)})$ of (T_n) such that

$$\|T_n^{(1)}(x)A_1 - A_1 T_n^{(1)}(x)\|_2^2 \to 0$$

as $n \to \infty$ almost everywhere. Note that (1) also holds for the subsequence (T_n^1). Then for the operator A_2, there exists a subsequence $(T_n^{(2)})$ of $(T_n^{(1)})$ such that

$$\|T_n^{(2)}(x)A_i - A_i T_n^{(2)}(x)\|_2^2 \to 0$$

as $n \to \infty$ almost everywhere for $i = 1, 2$.

By induction on the positive integer k, we can find a subsequence $(T_n^{(k+1)})$ of $(T_n^{(k)})$ such that

$$\|T_n^{(k+1)}(x)A_i - A_i T_n^{(k+1)}(x)\|_2^2 \to 0$$

as $n \to \infty$ almost everywhere for $i = 1, 2, \cdots, k+1$. Thus we can construct a sequence $\{\{T_n^{(k)}\} : k = 1, 2, \cdots\}$ of subsequences of $\{T_n\}$ such that

$$\|T_n^{(k)}(x)A_i - A_i T_n^{(k)}(x)\|_2^2 \to 0$$

as $n \to \infty$ almost everywhere for $1 \leq i \leq k$.

Now we define $R_n = T_n^{(n)}$ for any $n \in \mathbb{N}$. Then $\{R_n\}$ is a subsequence of $\{T_n\}$ such that for any $A_k, k = 1, 2, \cdots$, we have

$$\|R_n(x)A_k - A_k R_n(x)\|_2^2 \to 0$$

as $n \to \infty$ almost everywhere.

Note that the sequence $(R_n(x)) \in \mathcal{N}^\omega$ almost everywhere since $\{R_n\}$ is a subsequence of $\{T_n\}$ and all $T_n, n = 1, 2, \cdots$ are decomposable operators. Thus $(R_n(x)) \in \mathcal{N}_\omega$ almost everywhere. ■

Note that any subsequence of a central sequence of $\mathcal{M} \otimes \mathcal{N}$ is still a central sequence. Now we can show that the theorem 2.1. holds.

Proof of theorem 1. Let (T_n) and (S_n) be two central sequences of $\mathcal{M} \otimes \mathcal{N}$. Then we have

$$(T_n), \quad (S_n) \in (L^\infty([0,1], \mu) \otimes \mathcal{N})^\omega.$$

Let $a_n = \|T_n S_n - S_n T_n\|_2$. Then $0 \leq a_n \leq 2\|(T_n)\|\|(S_n)\|$, thus (a_n) is a bounded real number sequence.

Claim. For any subsequence $(a_n^{(1)})$ of (a_n), there is a subsequence $(a_n^{(2)})$ of $(a_n^{(1)})$ such that $a_n^{(2)} \to 0$ as $n \to \infty$.

In fact, for the subsequence $(a_n^{(1)})$, let $(T_n^{(1)})$ and $(S_n^{(1)})$ be the corresponding subsequences of (T_n) and (S_n) such that

$$a_n^{(1)} = \|T_n^{(1)} S_n^{(1)} - S_n^{(1)} T_n^{(1)}\|_2.$$

Then according to the above lemmas, there is a subsequence $(T_n^{(2)})$ of $(T_n^{(1)})$ such that $(T_n^{(2)}(x)) \in \mathcal{N}_\omega$ almost everywhere. As the corresponding subsequence $(S_n^{(2)})$ of (S_n^1) still is a central sequence and $(S_n^{(2)}) \in (L^\infty([0,1],\mu)\otimes\mathcal{N})^\omega$, thus there is a subsequence $(S_n^{(3)})$ of $(S_n^{(2)})$ such that $(S_n^{(3)}(x)) \in \mathcal{N}_\omega$ almost everywhere.

Up to a measurable subset of $[0,1]$ with measure zero, we define a sequence of essential bounded measurable functions (f_n) as following

$$f_n(x) = \|T_n^{(3)}(x)S_n^{(3)}(x) - S_n^{(3)}(x)T_n^{(3)}(x)\|_2, \quad x \in [0,1].$$

Then $\|f_n(x)\| \leq 2\|(T_n)\|\|(S_n)\|$ almost everywhere. Since $(S_n^{(3)}(x)) \in \mathcal{N}_\omega$ and $(T_n^{(3)}(x)) \in \mathcal{N}_\omega$ almost everywhere and $\mathcal{N}_\omega$ is abelian, then $f_n(x) \to 0 (n\to\infty)$ almost everywhere. Hence by the Lesbegue dominated converge theorem, we have

$$\begin{aligned}\lim_{n\to\infty} a_n^{(3)} &= \lim_{n\to\infty}(\int_{[0,1]} |f_n(x)|^2 d\mu(x))^{\frac{1}{2}} \\ &= (\int_{[0,1]} \lim_{n\to\infty} f_n(x)^2 d\mu(x))^{\frac{1}{2}} = 0.\end{aligned}$$

Thus the claim holds.

Now as for any subsequence $(a_n^{(1)})$ of (a_n), there is a subsequence $(a_n^{(2)})$ of $(a_n^{(1)})$ such that $a_n^{(2)} \to 0$, thus we have $a_n \to 0$. Hence $\|T_nS_n - S_nT_n\|_2 \to 0$ as $n \to \infty$. Thus we have shown that $(\mathcal{M}\otimes\mathcal{N})_\omega$ is abelian. ∎

Remark 2.7. According to the above argument, the result also holds under the further assumption which $\mathcal{A}$ is a non-atomic abelian von Neumann subalgebra of $\mathcal{M}$. Furthermore, under the condition of the theorem 2.1., if the II_1 factors $\mathcal{M}$ and $\mathcal{N}$ has the property Γ and is not McDuff factor, then the tensor product $\mathcal{M}\otimes\mathcal{N}$ also has the property Γ and is not McDuff factor.

3 Specified examples

In the previous section, we have shown that, under some conditions, the central sequence algebra $(\mathcal{M}\otimes\mathcal{N})_\omega$ is abelian. However, there is a natural question if there are two factors of type II_1 satisfying the conditions in the theorem 2.1. In this section, by using of the properties of the free group and the crossed product of von Neumann algebras, we will give specified examples of II_1 factors which satisfy those conditions.

Example 3.1. Let F_2 be the free group on two generators a and b. Suppose that $\mathcal{L}_{F_2}$ is the group factor generated by the operators L_a and L_b where $g \to L_g$ is the left regular representation of F_2 on the Hilbert space $l^2(F_2)$. The shift operator U acting on the Hilbert space $l^2(\mathbb{Z})$ is defined as following

$$Ue_n = e_{n+1}, \quad n \in \mathbb{Z}$$

where e_n is the function on $\mathbb{Z}$ such that $e_n(m) = \delta_{n,m}$ for any $m \in \mathbb{Z}$. Let E_n be the orthogonal projection from $l^2(\mathbb{Z})$ onto the subspace $\mathbb{C}e_n$. It is well known that U is a Haar unitary with respect to the vector state ω_{e_0}. Let $\mathcal{L}_\mathbb{Z}$ be the von Neumann algebra generated by the operator U.

Now we define a unitary representation of F_2 on $l^2(\mathbb{Z})$ as following

$$V_a(e_n) = e_n, \quad V_b(e_n) = e^{-in\theta}e_n, \quad n \in \mathbb{Z},$$

where $\frac{\theta}{2\pi} \in [0, 1)$ is an irrational number. It is easy to check that $V_b E_n = e^{-in\theta}E_n$ for any $n \in \mathbb{Z}$ and $V_b U V_b^{-1} = e^{-i\theta}U$.

By using of the above unitary representation of F_2, we define an implemented action α of F_2 on $\mathcal{L}_{\mathbb{Z}}$ as following

$$\alpha_g(A) = V_g A V_g^*, \quad g \in F_2, \quad A \in \mathcal{L}_{\mathbb{Z}}.$$

The *crossed product* $\mathcal{L}_{\mathbb{Z}} \rtimes_\alpha F_2$ of $\mathcal{L}_{\mathbb{Z}}$ under the action α of F_2 is the von Neumann subalgebra of $\mathcal{B}(l^2(\mathbb{Z}) \otimes l^2(F_2))$ generated by the operators

$$A \otimes I, \quad V_g \otimes L_g, \quad A \in \mathcal{L}_{\mathbb{Z}}, \quad g \in F_2.$$

Note that $\mathcal{L}_{\mathbb{Z}} \rtimes_\alpha F_2 = \{U \otimes I, I \otimes L_a, \sum_n e^{-in\theta}E_n \otimes L_b\}''$. Furthermore, any element in $\mathcal{L}_{\mathbb{Z}} \rtimes_\alpha F_2$ has the form $\sum_{g \in F_2} x_g V_g \otimes L_g$ with $x_g \in \{U \otimes I\}''$.

Recalling the following definitions. A (finite) von Neumann algebra $\mathcal{M}$ is *solid* if for every diffuse von Neumann subalgebra $\mathcal{A}$, the relative commutant $\mathcal{A}' \cap \mathcal{M}$ is injective. A (finite) von Neumann algebra is *semi-solid* if the relative commutant of any type II_1 von Neumann subalgebra is injective. A maximal abelian von Neumann subalgebra $\mathcal{A}$ of $\mathcal{M}$ is *singular* if the normalizer $\mathcal{N}(\mathcal{A}) = \{W : W \text{ is a unitary in } \mathcal{M} \text{ such that } W\mathcal{A}W^* = \mathcal{A}\}$ is contained in $\mathcal{A}$.

Now we discuss the crossed product $\mathcal{L}_{\mathbb{Z}} \rtimes_\alpha F_2$.

Proposition 3.2. The crossed product $\mathcal{L}_{\mathbb{Z}} \rtimes_\alpha F_2$ have the following properties:

(1) $\mathcal{L}_{\mathbb{Z}} \rtimes_\alpha F_2$ is a factor of type II_1,

(2) The crossed product $\mathcal{L}_{\mathbb{Z}} \rtimes_\alpha F_2$ is semisolid but not solid,

(3) Let $\mathcal{A} = \{U \otimes I, V_a \otimes L_a\}''$, then $\mathcal{A}$ is a singular maximal abelian von Neumann subalgebra of $\mathcal{L}_{\mathbb{Z}} \rtimes_\alpha F_2$.

Proof. Let $\tau_0(\sum_{g \in F_2} x_g V_g \otimes L_g) \triangleq \omega_{e_0}(x_e)$. Then it is easy to shown that τ_0 is a tracial state of $\mathcal{L}_{\mathbb{Z}} \rtimes_\alpha F_2$. Thus the crossed product $\mathcal{L}_{\mathbb{Z}} \rtimes_\alpha F_2$ is a finite von Neumann algebra.

Let $\sum_{g \in F_2} x_g V_g \otimes L_g$ be an element in the center of $\mathcal{L}_{\mathbb{Z}} \rtimes_\alpha F_2$. Then according to the following equation

$$(V_a \otimes L_a)(\sum_{g \in F_2} x_g V_g \otimes L_g)(V_a \otimes L_a)^* = \sum_{g \in F_2} x_{a^{-1}ga} V_g \otimes L_g = \sum_{g \in F_2} x_g V_g \otimes L_g,$$

we have $x_g = x_{a^{-1}ga}$ for any $g \in F_2$. Hence if there is $b^{\pm 1}$ in the reduced form of g, then $x_g = 0$. So we can assume that the central element has the form $\sum_{n \in \mathbb{Z}} x_{a^n} V_{a^n} \otimes L_{a^n}$.

Now by the equation

$$(V_b \otimes L_b)(\sum_{n \in \mathbb{Z}} x_{a^n} V_{a^n} \otimes L_{a^n})(V_b \otimes L_b)^* = \sum_{n \in \mathbb{Z}} x_{a^n} V_{a^n} \otimes L_{a^n},$$

we furthermore have $x_g = 0$ for all $g \in F_2 \setminus \{e\}$. However, as $(V_b \otimes L_b)' \cap (\mathcal{L}_{\mathbb{Z}} \otimes I)$ is trivial, thus the center of $\mathcal{L}_{\mathbb{Z}} \rtimes_\alpha F_2$ is also trivial. Hence the claim (1) holds.

As the free group F_2 is a word-hyperbolic group and its action α on $\mathcal{L}_{\mathbb{Z}}$ is trace-preserving, thus the crossed product $\mathcal{L}_{\mathbb{Z}} \rtimes_\alpha F_2$ is semisolid [6]. However, since the operators $V_a \otimes L_a$ and $V_{bab^{-1}} \otimes L_{bab^{-1}}$ generate a type II_1 factor which is isomorphic to the free group factor $\mathcal{L}_{F_2}$ and

$$V_a \otimes L_a, V_{bab^{-1}} \otimes L_{bab^{-1}} \in (\mathcal{L}_{\mathbb{Z}} \otimes I)' \cap (\mathcal{L}_{\mathbb{Z}} \rtimes_\alpha F_2),$$

thus the crossed product $\mathcal{L}_{\mathbb{Z}} \rtimes_\alpha F_2$ is not solid. Hence the claim (2) holds.

It is easy to check that the operator V_a commutates with the operator U, so $\mathcal{A}$ is an abelian von Neumann subalgebra of $\mathcal{L}_{\mathbb{Z}} \rtimes_\alpha F_2$. Note that in the proof of the claim (1), we have shown that $\mathcal{A}' \cap (\mathcal{L}_{\mathbb{Z}} \rtimes_\alpha F_2) \subset \mathcal{A}$. Thus $\mathcal{A}$ is maximal abelian.

Let $\mathcal{A}_0$ be the abelian von Neumann subalgebra of $\mathcal{A}$ which is generated by the operator $V_a \otimes L_a$. As $V_a \otimes L_a$ is a Haar unitary, then $\mathcal{A}_0$ is diffuse. For any element $g \in F_2$ whose reduced form contains the letter b, it is an exercise to check that $(V_g \otimes L_g)\mathcal{A}_0(V_g \otimes L_g)^* \perp \mathcal{A}$. Thus $V_g \otimes L_g$ is orthogonal to the algebra $\mathcal{N}(\mathcal{A})''$ [7]. Hence $\mathcal{N}(\mathcal{A})'' \subset \mathcal{A}$. Thus the claim (3) holds. ■

Remark 3.3. Recalling that a type II_1 factor $\mathcal{M}$ is *prime* if $\mathcal{M} \neq \mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$ for any type II_1 factors $\mathcal{M}_1$ and $\mathcal{M}_2$. As the crossed product $\mathcal{L}_{\mathbb{Z}} \rtimes_\alpha F_2$ is semisolid, thus it is a prime II_1 factor.

Let ω be a free ultrafilter on $\mathbb{N}$. Then we have the following conclusion.

Proposition 3.4. The central sequence algebra $(\mathcal{L}_{\mathbb{Z}} \rtimes_\alpha F_2)_\omega$ is abelian and nontrivial.

Proof. Suppose that $(t_n) \in (\mathcal{L}_{\mathbb{Z}} \rtimes_\alpha F_2)^\omega$ is a nontrivial central sequence. Without loss of generality, we can assume that

$$\|t_n\| = 1, \quad \tau(t_n) = 0, \quad t_n = \sum_{g \in F_2} x_g^{(n)} V_g \otimes L_g$$

with $x_g^{(n)} \in \mathcal{L}_{\mathbb{Z}} \otimes I$.

Let $S = \{g \in F_2 : \text{the reduced form of } g \text{ beginning with } b^{\pm 1}\}$. If $\|t_n|_S\|_2 \nrightarrow 0$, then there is a subsequence (t_{n_k}) of (t_n) such that

$$\lim_{k \to \infty} \sum_{g \in S} \|x_g^{(n_k)}\|_2^2 = c$$

where $c \in \mathbb{R}$ is a positive constant. Since any subsequence of a central sequence is also a central sequence, thus we can assume that $\sum_{g \in S} \|x_g^{(n)}\|_2^2 \to c$. Then there exists an positive integer n_0 such that

$$\|t_n|_S\|_2^2 = \sum_{g \in S} \|x_g^{(n)}\|_2^2 > \frac{c}{2}$$

for any $n > n_0$.

Let $\delta = \frac{c}{8}$ and N be a positive integer such that $\frac{cN}{4} > 1$. As (t_n) is a central sequence, then for $a, a^2, \cdots, a^N$, there is a $n_1 \in \mathbb{N}(\geqslant n_0)$ such that we have

$$\|(V_a \otimes L_a)^i t_n (V_a \otimes L_a)^{-i} - t_n\|_2 < \delta, \quad i = 1, 2, \cdots, N,$$

for any $n > n_1$.

Then by the following equations

$$\|((V_a \otimes L_a)^i t_n (V_a \otimes L_a)^{-i})|_S\|_2^2 = \sum_{g \in a^{-i}Sa^i} \|x_g^{(n)}\|_2^2 = \|t_n|_{a^{-i}Sa^i}\|_2^2,$$

we have

$$|\|t_n|_{a^{-i}Sa^i}\|_2^2 - \|t_n|_S\|_2^2| \leqslant 2\|(a^i t_n a^{-i})|_S - t_n|_S\|_2 < 2\delta.$$

Therefore we get

$$\|t_n|_{a^{-i}Sa^i}\|_2^2 > \|t_n|_S\|_2^2 - 2\delta, \quad i = 1, 2, \cdots, N.$$

As the sets $a^{-i}Sa^i (i = 1, 2, \cdots, N)$ are pairwise disjoint, then we have

$$1 \geqslant \|t_n\|_2^2 \geqslant \sum_{i=1}^{N} \|t_n|_{a^{-i}Sa^i}\|_2^2 \geqslant N(\|t_n|_S\|_2^2 - 2\delta) \geqslant \frac{cN}{4} > 1$$

which is impossible. Hence $\|t_n|_S\|_2 \to 0$. Substituting the (t_n) by $(t_n - (t_n|_S))$, then we can further assume that all t_n vanish at the set S.

Now let $S_0 = \{g \in F_2$: the reduced form of g beginning with $a^{\pm 1}\}$. Then for any $0 \neq i \in \mathbb{Z}$, we have $b^i S_0 b^{-i} \cap S_0 = \emptyset$. As (t_n) vanishes at the set S, thus $t_n|_{b^i S_0 b^{-i}} = 0$ and then we have

$$\|(V_b \otimes L_b)^i t_n (V_b \otimes L_b)^{-i} - t_n\|_2^2 \geqslant \|t_n|_{S_0}\|_2^2.$$

As (t_n) is a central sequence, thus we have $\|t_n|_{S_0}\|_2 \to 0$ as $n \to \infty$.

According to the above arguments, now we can assume that the support $\{g \in F_2 : x^n(g) \neq 0\}$ of t_n is concentrating on the unit element e of F_2. Then the central sequence (t_n) satisfy the condition $t_n \in \mathcal{L}_{\mathbb{Z}} \otimes I$. Thus $(\mathcal{L}_{\mathbb{Z}} \rtimes_\alpha F_2)_\omega$ is abelian as $\mathcal{L}_{\mathbb{Z}}$ is abelian.

Finally, since $\frac{\theta}{2\pi} \in [0, 1)$ is an irrational number, then there is a sequence (n_k) of natural numbers such that $\frac{n_k\theta}{2\pi} - [\frac{n_k\theta}{2\pi}] \to 0$ as $k \to \infty$. For the given sequence (n_k), let $t_k = U^{n_k} \otimes I$. Then $\|t_k\| = 1, \tau(t_k) = 0$ and $\|t_k\|_2 = 1$. It is obvious that (t_k) commutates with U and a. Furthermore, as

$$\|(V_b \otimes L_b) t_k (V_b \otimes L_b)^{-1} - t_k\|_2 = \|(e^{-in_k\theta} - 1)U^{n_k}\|_2 \to 0, \quad k \to \infty,$$

thus (t_k) is a nontrivial central sequence of $\mathcal{L}_{\mathbb{Z}} \rtimes_\alpha F_2$. ∎

Note that in the proof of proposition 3.4., we have shown that the central sequence algebra $(\mathcal{L}_{\mathbb{Z}} \rtimes_\alpha F_2)_\omega$ is contained in the ultrapower of $\mathcal{L}_{\mathbb{Z}} \otimes I$. Thus we get the following consequence.

Corollary 3.5. Let $\mathcal{M} = \mathcal{L}_{\mathbb{Z}} \rtimes_\alpha F_2$ and $\mathcal{A} = \mathcal{L}_{\mathbb{Z}} \otimes I$. Then we have

$$(\mathcal{M} \otimes \mathcal{M})_\omega \subset (\mathcal{A} \otimes \mathcal{M})^\omega.$$

In fact, similar to the arguments in the proof of the proposition 3.4, we can show that the central sequence algebra $(\mathcal{M} \otimes \mathcal{M})_\omega$ is contained in $(\mathcal{A} \otimes \mathcal{A})^\omega$. Hence $(\mathcal{M} \otimes \mathcal{M})_\omega$ is abelian. Of course, $(\mathcal{M} \otimes \mathcal{M})_\omega$ is also nontrivial. Hence we have given a specified example of II_1 factor $\mathcal{M}$, which has the property Γ and is not McDuff factor, such that the tensor product $\mathcal{M} \otimes \mathcal{M}$ also have the property Γ and is not McDuff factor.

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