Kadison-Singer Algebras, I ——Hyperfinite Case

Liming Ge [*] [†], **Wei Yuan** [†]

[*] University of New Hampshire, Durham,USA, and [†]Chinese Academy of Sciences, Beijing, China

Submitted to Proceedings of the National Academy of Sciences of the United States of America

This is abstract

von Neumann algebra | Kadison-Singer algebra | Kadison-Singer lattice | Reflexive

Introduction

In [1], Kadison and Singer initiate the study of non-self-adjoint algebras of bounded operators on Hilbert spaces. They introduce a class of algebras they call *triangular operator algebras*. An algebra $\mathcal{T}$ is triangular (relative to a factor $\mathcal{M}$) when $\mathcal{T} \cap \mathcal{T}^*$ is a maximal abelian (self-adjoint) algebra in the factor $\mathcal{M}$. (See section 2 for definitions.) When the factor is the algebra of all $n \times n$ complex matrices, this condition guarantees that there is a unitary matrix U such that the mapping $A \to UAU^*$ transforms $\mathcal{T}$ onto a subalgebra of the upper triangular matrices.

Beginning with [1], the theory of non-self-adjoint operator algebras has undergone a vigorous development parallel to, but not nearly as explosive as, that of the self-adjoint theory, the C*- and von Neumann algebra theories. Of course, the self-adjoint theory began with the 1929-30 von Neumann article [2] — well before the 1960 [1] article appeared. Surprisingly, to the present authors, and apparently to Kadison and Singer as well (from private conversations), this parallel development has not produced the synergistic interactions we would have expected from subjects that are so closely and naturally related, and so likely to benefit from cross connections with one another.

Considerable effort has gone into the study of triangular operator algebras (see, for example, [3]) and [4]) and another class of non-self-adjoint operator algebras, the *reflexive algebras* (see, for example, [5], [6], [7], and [8]). Many definitive and interesting results are obtained during the course of these investigations. For the most part, these more detailed results rely on relations to compact, or even finite-rank, operators. This direction is taken in the seminal article [1], as well. In Section 3.2 of [1], a detailed and complete classification is given for an important class of (maximal) triangular algebras; but much depends on the analysis of those $\mathcal{T}$ for which (the "diagonal") $\mathcal{T} \cap \mathcal{T}^*$ is generated by one-dimensional projections. On the other hand, the emphasis of C*- and von Neumann algebra theory is on those algebras where compact operators are (almost) absent.

One of our main goals in this article is to recapture the synergy that should exist between the powerful techniques that have developed in selfadjoint-operator-algebra theory and those of the non-selfadjoint theory by conjoining the two theories. We do this by embodying those theories in a single class of algebras. For this, we mimic the defining relation for the triangular algebra removing the commutativity assumption on the diagonal subalgebra $\mathcal{T} \cap \mathcal{T}^*$ of $\mathcal{T}$ and imposing suitable maximality and reflexivity conditions on $\mathcal{T}$ (compare Definition 2.1). Our particular focus is the case where the diagonal algebra is a factor.

The new non self-adjoint operator algebras will combine triangularity, reflexivity and von Neumann algebra properties in their structure. These algebras will be called *Kadison-Singer algebras* or *KS-algebras* for simplicity. They are reflexive and maximal triangular with respect to their "diagonal subalgebras." Kadison-Singer factors (or KS-factors) are those with factors as their diagonal algebras. These are highly noncommutative and non selfadjoint operator algebras. In standard form, where the diagonal and its commutant share a cyclic vector, such "standard" Kadison-Singer algebras have a large selfadjoint part. Many selfadjoint features are preserved in them and concepts can be borrowed directly from the theory of von Neumann algebras. In fact, a more direct connection of Kadison-Singer algebras and von Neumann algebras is through the lattice of invariant projections of a KS-algebra. The lattice is reflexive and "minimally generating" in the sense that it generates the commutant of the diagonal as a von Neumann algebra. Most factors are generated by three projections (see [9]). One of our main results shows that the reflexive algebra which leaves three generating projections of a factor invariant is often a Kadison-Singer algebra, which agrees with the fact that three projections are "minimally generating" for a factor. Moreover Kadison-Singer algebras associated with three projections contain compact operators, and the reflexive lattice generated by the three projections is often homeomorphic to the two-dimensional sphere. We believe that the reflexive algebra given by four or more free projections is a Kadison-Singer algebra and does not contain any nonzero compact operators. Indeed, we shall show that the reflexive algebra associated with infinitely many free projections contains no nonzero compact operators. Through the study of minimal reflexive lattice generators of a von Neumann algebra, we may better understand the generator problem for von Neumann algebras and hence the isomorphism problem for free group factors. These are some of the deepest, most diffcult, and longest standing problems in von Neumann algebra theory. The techniques we use are closely related to those of the theory of selfadjoint operator algebras, especially some of the recently developed theory of free probability [10]. For some of the important results and approaches in non selfadjoint theory, we refer to [11], [12], [13], [14], [3], [15] and many references in [16] and [7].

There are four sections in this paper, the first in a series. In Section 2, we give the definition of Kadison-Singer algebras (as well as corresponding Kadison-Singer lattices) and a basic classification according to their diagonals. In Section 3, we

Reserved for Publication Footnotes

construct Kadison-Singer factors with hyperfinite factors as their diagonals. In Section 4, we describe the corresponding Kadison-Singer lattices in detail. A new lattice invariant is introduced to distinguish these lattices.

We hope that new examples and constructions of non selfadjoint algebras will lead to new insights for some puzzling, old questions in operator theory (see [17] and [7]).

Definitions

For basic theory on operator algebras, we refer to [18]. We recall the definitions of some well known classes of non selfadjoint operator algebras. For details on triangular algebras, we refer to [1]. For others, we refer to [7].

Suppose $\mathcal{H}$ is a separable Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. Let $\mathcal{M}$ be a von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$. A *triangular (operator) algebra* is a subalgebra $\mathcal{T}$ of $\mathcal{M}$ such that $\mathcal{T}\cap\mathcal{T}^* = \mathcal{A}$, a maximal abelian selfadjoint subalgebra (masa) of $\mathcal{M}$. One of the interesting cases is when $\mathcal{M} = \mathcal{B}(\mathcal{H})$.

Let $\mathcal{P}$ be a set of (orthogonal) projections in $\mathcal{B}(\mathcal{H})$. Define $\mathrm{A}lg(\mathcal{P}) = \{T \in \mathcal{B}(\mathcal{H}) : TP = PTP,\ \textit{for all}\ P \in \mathcal{P}\}$. Then $\mathrm{A}lg(\mathcal{P})$ is a weak-operator closed subalgebra of $\mathcal{B}(\mathcal{H})$. Similarly, for a subset $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$, define $\mathrm{L}at(\mathcal{S}) = \{P \in \mathcal{B}(\mathcal{H}) : P\ \textit{a projection},\ TP = PTP,\ \textit{for all}\ T \in \mathcal{S}\}$. Then $\mathrm{L}at(\mathcal{S})$ is a strong-operator closed lattice of projections. A subalgebra $\mathcal{B}$ of $\mathcal{B}(\mathcal{H})$ is called a *reflexive (operator) algebra* if $\mathcal{B} = \mathrm{A}lg(\mathrm{L}at(\mathcal{B}))$. Similarly, a lattice $\mathcal{L}$ of projections in $\mathcal{B}(\mathcal{H})$ is called a *reflexive lattice (of projections)* if $\mathcal{L} = \mathrm{L}at(\mathrm{A}lg(\mathcal{L}))$. A *nest* is a totally ordered reflexive lattice. If $\mathcal{L}$ is a nest, then $\mathrm{A}lg(\mathcal{L})$ is called a *nest algebra*. Nest algebras are generalizations of (hyperreducible) "maximal triangular" algebras introduced by Kadison and Singer in [1]. Kadison and Singer also show that nest algebras are the only maximal triangular reflexive algebras (with a commutative lattice of invariant projections). Motivated by this, we give the following definition:

Definition 1. *A subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ is called a Kadison-Singer (operator) algebra (or KS-algebra) if $\mathcal{A}$ is reflexive and maximal with respect to the diagonal subalgebra $\mathcal{A}\cap\mathcal{A}^*$ of $\mathcal{A}$, in the sense that if there is another reflexive subalgebra $\mathfrak{B}$ of $\mathcal{B}(\mathcal{H})$ such that $\mathcal{A}\subseteq\mathfrak{B}$ and $\mathfrak{B}\cap\mathfrak{B}^* = \mathcal{A}\cap\mathcal{A}^*$, then $\mathcal{A} = \mathfrak{B}$. When the diagonal of a KS-algebra is a factor, we call the KS-algebra a KS-factor or a Kadison-Singer factor. A lattice $\mathcal{L}$ of projections in $\mathcal{B}(\mathcal{H})$ is called a Kadison-Singer lattice (or KS-lattice) if $\mathcal{L}$ is a minimal reflexive lattice that generates the von Neumann algebra $\mathcal{L}''$, or equivalently $\mathcal{L}$ is reflexive and $\mathrm{A}lg(\mathcal{L})$ is a Kadison-Singer algebra.*

Clearly nest algebras are KS-algebras. Since a nest generates an abelian von Neumann algebra, we may view nest algebras as "type I" KS-algebras and general KS-algebras as "quantized" nest algebras. The maximality condition for a KS-algebra requires that the associated lattice is "reflexive and minimal" in the sense that there is no smaller reflexive sublattice that generates the commutant of the diagonal algebra. We believe that the following statement is true:

Conjecture 1. *If $\mathcal{A}$ is a KS-algebra in $\mathcal{B}(\mathcal{H})$ and $P \in \mathrm{Lat}(\mathcal{A})$ $(\neq 0, I)$, then $I - P \notin \mathrm{Lat}(\mathcal{A})$, i.e., a KS-algebra has no nontrivial reducing invariant subspaces.*

The following lemma is an immediate consequence of the above definition.

Lemma 1. *Suppose $\mathcal{A}$ is a Kadison-Singer algebra in $\mathcal{B}(\mathcal{H})$ and $\mathcal{M}$ is the commutant of $\mathcal{A}\cap\mathcal{A}^*$ in $\mathcal{B}(\mathcal{H})$. Then $\mathrm{Lat}(\mathcal{A}) \subseteq \mathcal{M}$ and generates $\mathcal{M}$ as a von Neumann algebra.*

When $\mathcal{A}$ is a KS-algebra and $\mathcal{A}\cap\mathcal{A}^*$ is a factor of type I, II or III, then $\mathcal{A}$ is called a KS-factor of the same type. In the same way, we can further classify KS-factors into type II_1, II_∞, etc., similar to usual factors. A KS-algebra $\mathcal{A}$ is said to be in a *standard form*, or a *standard* KS-algebra, if the diagonal $\mathcal{A}\cap\mathcal{A}^*$ of $\mathcal{A}$ is in a standard form, i.e., $\mathcal{A}\cap\mathcal{A}^*$ has a cyclic and separating vector in $\mathcal{H}$. In this case, the von Neumann algebra generated by $\mathrm{L}at(\mathcal{A})$ (or the core, see [1]) is also in a standard form.

In the present article, one of our main goals is to give some nontrivial examples of KS-algebras, in particular, KS-factors of type II and III. The following theorem shows that all type II and type III KS-algebras are truly non selfadjoint algebras.

Theorem 2. *If $\mathcal{A}$ is a KS-algebra of type II or type III in $\mathcal{B}(\mathcal{H})$, then $\mathcal{A}$ is not selfadjoint.*

Proof Assume on the contrary that $\mathcal{A}$ is selfadjoint. From our assumption we know that $\mathcal{A}'$ contains a 2×2 matrix subalgebra $\mathcal{M}_2$. Let E_{ij}, $i, j = 1, 2$, be a matrix unit system for $\mathcal{M}_2$. Then one can construct a reflexive lattice $\mathcal{L}$ generated by all projections in the relative commutant of $\mathcal{M}_2$ in $\mathcal{A}'$ and two non commuting projections E_{11} and $\frac{1}{2}\sum_{i,j} E_{ij}$ in $\mathcal{M}_2$. It is easy to see that $\mathcal{L}$ generates $\mathcal{A}'$ as a von Neumann algebra. One easily checks that $\mathrm{A}lg(\mathcal{L})$ is non selfadjoint but reflexive. Moreover its diagonal is equal to the commutant of $\mathcal{L}$, which agrees with $\mathcal{A}$. This contradicts to the assumption that $\mathcal{A}$ is a KS-algebra.

Similar argument shows that any nontrivial standard KS-algebra, even in the case of type I, is not selfadjoint. Standard KS-algebras can be viewed as *maximal* upper triangular algebras with a von Neumann algebra as its diagonal.

Definition 2. *Two Kadison-Singer algebras are said to be isomorphic if there is a norm preserving (algebraic) isomorphism between the two algebras. Two KS-algebras are called unitarily equivalent if there is a unitary operator between the underlying Hilbert spaces that induces an isomorphism between the KS-algebras.*

It is easy to see that an isomorphism between two Kadison-Singer algebras induces a * isomorphism between the diagonal subalgebras.

For lattices of projections on a Hilbert space, the definition of an isomorphism is subtle. We consider a simple example where a lattice $\mathcal{L}_0$ contains two free projections of trace $\frac{1}{2}$ and $0, I$ in a type II_1 factor. As a lattice (with respect to union, intersection and ordering), it is isomorphic to the lattice generated by two rank-one projections on a two-dimensional euclidean space. We shall call such an isomorphism (which preserves only the lattice structure) an *algebraic (lattice) isomorphism.* An isomorphism between two lattices, in this paper, is an isomorphism that also induces a * isomorphism between the von Neumann algebras they generate. To avoid confusion, sometimes we call such isomorphisms *spatial isomorphisms* between two lattices of projections.

Hyperfinite Kadison-Singer factors

In this section, we shall construct some hyperfinite Kadison-Singer factors. We begin with a UHF C*-algebra $\mathcal{A}_n$ (see [19]) obtained by taking the completion (with respect to operator norm) of $\otimes_1^\infty M_n(\mathbb{C})$, denoted by $\mathfrak{A}$ (or equivalently, $\mathfrak{A} = \cup_{j=1}^\infty M_{n^j}(\mathbb{C})$). We denote by $M_n^{(k)}(\mathbb{C})$ the kth copy of $M_n(\mathbb{C})$ in $\mathfrak{A}$ (or $\mathcal{A}_n$) and $E_{ij}^{(k)}$, $i, j = 1, \ldots, n$, the standard matrix unit system for $M_n^{(k)}(\mathbb{C})$, for $k = 1, 2, \ldots$. Then we may write $\mathfrak{A} = M_n^{(1)}(\mathbb{C}) \otimes M_n^{(2)}(\mathbb{C}) \otimes \cdots$. Let $\mathcal{N}_m = M_n^{(1)}(\mathbb{C}) \otimes M_n^{(2)}(\mathbb{C}) \otimes \cdots \otimes M_n^{(m)}(\mathbb{C})$ $(\cong M_{n^m}(\mathbb{C}))$. Then

$\mathfrak{A} = \cup_{m=1}^{\infty}\mathcal{N}_m$. Now, we construct inductively a family of projections in $\mathcal{N}_m$.

When $m = 1$, define $P_{1j} = \sum_{i=1}^{j} E_{ii}^{(1)}$, for $j = 1, \dots, n-1$, and $P_{1n} = \frac{1}{n}\sum_{s,t=1}^{n} E_{st}^{(1)}$. Suppose for $k = m-1$, $P_{kj} \in \mathcal{N}_k$ are defined, for $j = 1, \dots, n$. Now we define

$$P_{mj} = P_{m-1,n-1} + (I - P_{m-1,n-1})\sum_{i=1}^{j} E_{ii}^{(m)}, j = 1, \dots, n-1, \qquad [1]$$

$$P_{mn} = P_{m-1,n-1} + (I - P_{m-1,n-1})\left(\frac{1}{n}\sum_{s,t=1}^{n} E_{st}^{(m)}\right). \qquad [2]$$

Denote by $\mathcal{L}_m$ the lattice generated by $\{P_{kj} : 1 \le k \le m, 1 \le j \le n\}$ and $\mathcal{L}_\infty = \cup_m \mathcal{L}_m$, the lattice generated by $\{P_{kj} : k \ge 1, 1 \le j \le n\}$. We can easily show inductively that $\mathcal{N}_m$ is generated by $\mathcal{L}_m$ (as a finite-dimensional von Neumann algebra).

Let ρ_n be a faithful state on $M_n(\mathbb{C})$. We extend ρ_n to a state on $\mathcal{A}_n$, denoted by ρ, i.e., $\rho = \rho_n \otimes \rho_n \otimes \cdots$. Let $\mathcal{H}$ be the Hilbert space obtained by GNS construction on $(\mathcal{A}_n, \rho)$. It is well known (see [20]) that the weak-operator closure of $\mathcal{A}_n$ in $\mathcal{B}(\mathcal{H})$ is a hyperfinite factor $\mathcal{R}$ (when ρ is a trace, the factor $\mathcal{R}$ is type II_1). Then $\mathcal{L}_m$ and $\mathcal{L}_\infty$ become lattices of projections in $\mathcal{R}$.

Theorem 3. *With the above notation, we have that $\mathrm{Alg}(\mathcal{L}_\infty)$ is a Kadison-Singer factor containing the hyperfinite factor $\mathcal{R}'$ as its diagonal.*

Our above defined hyperfinite KS-factor depends on n (≥ 2) appeared in the UHF algebra construction. We shall see in Section 4 that, when ρ is a trace, for different n, the Kadison-Singer algebras constructed above are not unitarily equivalent.

To prove Theorem 3, we need some lemmas.

Lemma 2. *With $\mathcal{L}_1 \subset \mathcal{N}_1$ defined above and $E_{ij}^{(1)}$, $i, j = 1, \dots, n$, the matrix units for $\mathcal{N}_1$, we have*

$$\begin{aligned}\mathrm{Alg}(\mathcal{L}_1) = \{T \in \mathcal{B}(\mathcal{H}) :\quad & E_{ii}^{(1)}TE_{jj}^{(1)} = 0, \qquad 1 \le j < i \le n;\\ & \sum_{j=1}^{n} E_{11}^{(1)}TE_{j1}^{(1)} = \sum_{j=2}^{n} E_{12}^{(1)}TE_{j1}^{(1)} = \cdots = E_{1n}^{(1)}TE_{n1}^{(1)}\}.\end{aligned}$$

Proof Let T be an element in $\mathrm{Alg}(\mathcal{L}_1)$. Since $P_{1j} = \sum_{i=1}^{j} E_{ii}^{(1)} \in \mathcal{L}_1$ for $j = 1, \dots, n-1$, we know that $E_{ii}^{(1)}TE_{jj}^{(1)} = 0, 1 \le j < i \le n$. From $TP_{1n}^{(1)} = P_{1n}^{(1)}TP_{1n}^{(1)}$, we have

$$nT\sum_{i,j=1}^{n} E_{ij}^{(1)} = (\sum_{i,j=1}^{n} E_{ij}^{(1)})T(\sum_{i,j=1}^{n} E_{ij}^{(1)}).$$

Multiplying the above equation by $E_{1l}^{(1)}$ on left and $E_{11}^{(1)}$ on right, we have

$$\begin{aligned} nE_{1l}^{(1)}T\sum_{i=1}^{n} E_{i1}^{(1)} &= n\sum_{i=1}^{n} E_{1l}^{(1)}TE_{i1}^{(1)}\\ &= (\sum_{j=1}^{n} E_{1j}^{(1)})T(\sum_{i=1}^{n} E_{i1}^{(1)}) = \sum_{i,j=1}^{n} E_{1i}^{(1)}TE_{j1}^{(1)}.\end{aligned}$$

The right hand side is independent of l. By letting $l = 1, \dots, n$ and applying $E_{ii}^{(1)}TE_{jj}^{(1)} = 0$ when $1 \le j < i \le n$, we have that $\sum_{i=1}^{n} E_{11}^{(1)}TE_{i1}^{(1)} = \sum_{i=2}^{n} E_{12}^{(1)}TE_{i1}^{(1)} = \cdots = E_{1n}^{(1)}TE_{n1}^{(1)}$. It is easy to check that when T satisfies those identities in the lemma, T must be an element in $\mathrm{Alg}(\mathcal{L}_1)$.

In terms of matrix representations of elements in $\mathrm{Alg}(\mathcal{L}_1)$ with respect to matrix units in $\mathcal{N}_1$, we know from Lemma 2 that such an element T is upper triangular. Moreover, one can arbitrarily choose the strictly upper triangular part of T and use equations

$$\sum_{j=1}^{n} E_{11}^{(1)}TE_{j1}^{(1)} = \sum_{j=2}^{n} E_{12}^{(1)}TE_{j1}^{(1)} = \cdots = E_{1n}^{(1)}TE_{n1}^{(1)}$$

to determine the diagonal entries of T so that $T \in \mathrm{Alg}(\mathcal{L}_1)$.

Lemma 3. *For any T in $\mathrm{Alg}(\mathcal{L}_1)$, there are T_1 in $\mathrm{Alg}(\mathcal{L}_1) \cap \mathcal{L}_1'$ and T_2 in $\mathrm{Alg}(\mathcal{L}_\infty)$ ($\subseteq \mathrm{Alg}(\mathcal{L}_1)$) such that $T = T_1 + T_2$. In particular, when $E_{nn}^{(1)}TE_{nn}^{(1)} = 0$, $T = T_2 \in \mathrm{Alg}(\mathcal{L}_\infty)$.*

Proof Suppose $T \in \mathrm{Alg}(\mathcal{L}_1)$ and let

$$T_1 = \sum_{i=1}^{n} E_{in}^{(1)}TE_{ni}^{(1)}, \qquad T_2 = T - T_1. \qquad [3]$$

It is easy to check that $E_{ii}^{(1)}T_1E_{jj}^{(1)} = 0$ when $i \ne j$ and, by Lemma 2, $T_1 \in \mathrm{Alg}(\mathcal{L}_1)$. Moreover, for all l, k, $E_{lk}^{(1)}T_1 = E_{lk}^{(1)}\sum_{i=1}^{n} E_{in}^{(1)}TE_{ni}^{(1)} = E_{ln}^{(1)}TE_{nk}^{(1)} = T_1E_{lk}^{(1)}$. This implies that $T_1 \in \mathcal{L}_1'$ ($= \mathcal{N}_1'$).

Clearly $T_2 \in \mathrm{Alg}(\mathcal{L}_1)$. Thus $T_2P_{1k} = P_{1k}T_2P_{1k}$, for $k = 1, \dots, n$. We need to show that $T_2P_{jk} = P_{jk}T_2P_{jk}$, for $j \ge 2$ and $k = 1, \dots, n$. By the definition of P_{jk} in (1), we know that $I - P_{jk} \le E_{nn}^{(1)}$ for $j \ge 2$. Now, from $T_2 \in \mathrm{Alg}(\mathcal{L}_1)$, we have

$$\begin{aligned} E_{nn}^{(1)}T_2 &= E_{nn}^{(1)}\sum_{1\le l\le k\le n} E_{ll}^{(1)}T_2E_{kk}^{(1)}\\ &= E_{nn}^{(1)}T_2E_{nn}^{(1)} = E_{nn}^{(1)}(T - T_1)E_{nn}^{(1)} = 0.\end{aligned}$$

This implies that $0 = (I - P_{jk})T_2 = (I - P_{jk})T_2P_{jk}$. Thus we have $T_2 \in \mathrm{Alg}(\mathcal{L}_\infty)$.

Lemma 4. *If $T \in \mathrm{Alg}(\mathcal{L}_m)$ and $(I - P_{m,n-1})T = 0$ for some $m \ge 1$, then $T \in \mathrm{Alg}(\mathcal{L}_\infty)$.*

When $m = 1$, the proof is given above. For a general m, the argument is similar. We omit its details here. From the construction of P_{mk}'s, we know that the differences between elements in $\mathrm{Alg}(\mathcal{L}_m)$ and those in $\mathrm{Alg}(\mathcal{L}_{m+1})$ only occur within $I - P_{m,n-1}$ ($= E_{nn}^{(1)} \otimes \cdots \otimes E_{nn}^{(m)}$). Thus we have the following lemma.

Lemma 5. *If $T \in \mathrm{Alg}(\mathcal{L}_m)$, then $T \in \mathrm{Alg}(\mathcal{L}_{m+1})$ if and only if, for $j = 1, \dots, n$, the projections $(I - P_{m,n-1})P_{m+1,j}(I - P_{m,n-1})$ are invariant under $(I - P_{m,n-1})T(I - P_{m,n-1})$.*

Inductively, we can easily prove the following lemma which generalizes Lemma 3.

Lemma 6. *If $T \in \mathrm{Alg}(\mathcal{L}_m)$, then there are $T_1, \dots, T_{m+1}$ in $\mathrm{Alg}(\mathcal{L}_m)$ such that $T = T_1 + \cdots + T_{m+1}$, where $T_i \in \mathcal{N}_{i-1}' \cap \mathrm{Alg}(\mathcal{L}_\infty)$, $(I - P_{i,n-1})T_i = 0$ for $i = 1, \dots, m$ (here we let $\mathcal{N}_0 = \mathbb{C}I$), and $T_{m+1} \in \mathcal{N}_m' \cap \mathrm{Alg}(\mathcal{L}_m)$.*

Lemma 7. *Suppose T is an element in $\mathcal{B}(\mathcal{H})$ and $\mathfrak{A}$ is the algebra generated by T and $\mathrm{Alg}(\mathcal{L}_\infty)$. If $\mathfrak{A} \cap \mathfrak{A}^* = \mathrm{Alg}(\mathcal{L}_\infty) \cap \mathrm{Alg}(\mathcal{L}_\infty)^* = \mathcal{R}'$, then $T \in \mathrm{Alg}(\mathcal{L}_1)$.*

Proof Suppose $T \in \mathfrak{A}$ is given. From the comments preceding Lemma 3 and by taking a difference from an element in $\mathrm{Alg}(\mathcal{L}_\infty)$, we may assume that, with respect to matrix units $E_{ij}^{(1)}$ in $\mathcal{N}_1$, T is lower triangular, i.e., $E_{ii}^{(1)}TE_{jj}^{(1)} = 0$ for $i < j$. Now we want to show that T is diagonal. If the strictly lower triangular entries of T are not all zero, then let i_0 be the largest integer such that $E_{i_0i_0}^{(1)}TE_{jj}^{(1)} \ne 0$ for some $j < i_0$. Among all such j, let j_0 be the largest. Then we have that $E_{ii}^{(1)}TE_{jj}^{(1)} = 0$ if $i > j$ and $i > i_0$; or

$i = i_0 > j > j_0$. It is easy to check (from Lemma 4) that $E^{(1)}_{j_0,i_0-1} - E^{(1)}_{j_0 i_0} \in \mathrm{A}lg(\mathcal{L}_\infty)$. Then $T(E^{(1)}_{j_0,i_0-1} - E^{(1)}_{j_0 i_0}) \in \mathfrak{A}$. Define $T_1 = T(E^{(1)}_{j_0,i_0-1} - E^{(1)}_{j_0 i_0})$. Then

$$\begin{aligned} T_1 &= \sum_{n\geq k\geq l\geq 1} E^{(1)}_{kk} T E^{(1)}_{ll} (E^{(1)}_{j_0,i_0-1} - E^{(1)}_{j_0 i_0}) \\ &= \sum_{i_0\geq k\geq j_0} E^{(1)}_{kk} T (E^{(1)}_{j_0,i_0-1} - E^{(1)}_{j_0 i_0}). \end{aligned}$$

Let $T_2 = E^{(1)}_{i_0 i_0} T (E^{(1)}_{j_0,i_0-1} - E^{(1)}_{j_0 i_0})$, and

$$T_3 = \sum_{i_0 > k\geq j_0} E^{(1)}_{kk} T (E^{(1)}_{j_0,i_0-1} - E^{(1)}_{j_0 i_0}).$$

Then $T_1 = T_2 + T_3$. From Lemma 4 again, $T_3 \in \mathrm{A}lg(\mathcal{L}_\infty)$. This implies that $T_2 \in \mathfrak{A}$.

Let $E^{(1)}_{i_0 i_0} T E^{(1)}_{j_0 i_0} = HV$ be the polar decomposition (in $\mathcal{B}(\mathcal{H})$), where H is positive and V a partial isometry. From our assumption that $E^{(1)}_{i_0 i_0} T E^{(1)}_{j_0 j_0} \neq 0$, we have $H \neq 0$, $E^{(1)}_{i_0 i_0} H = H E^{(1)}_{i_0 i_0} = H$ and $E^{(1)}_{i_0 i_0} V = V E^{(1)}_{i_0 i_0} = V$. Then $T_2 = HVE^{(1)}_{i_0,i_0-1} - HV$. Define

$$\begin{aligned} T_4 &= -E^{(1)}_{i_0-1,i_0} V^* E^{(1)}_{i_0,i_0-1} + E^{(1)}_{i_0-1,i_0} V^* E^{(1)}_{i_0 i_0}, \\ T_5 &= E^{(1)}_{i_0-1,i_0} H E^{(1)}_{i_0,i_0-1} - E^{(1)}_{i_0-1,i_0} H E^{(1)}_{i_0 i_0}. \end{aligned}$$

It is easy to check, from Lemma 3, that $T_4, T_5 \in \mathrm{A}lg(\mathcal{L}_\infty)$. Let

$$\begin{aligned} T_6 &= T_2 T_4 + T_5 \\ &= (HVE^{(1)}_{i_0,i_0-1} - HV)(-E^{(1)}_{i_0-1,i_0} V^* E^{(1)}_{i_0,i_0-1} \\ &+ E^{(1)}_{i_0-1,i_0} V^* E^{(1)}_{i_0 i_0}) + E^{(1)}_{i_0-1,i_0} H E^{(1)}_{i_0,i_0-1} - E^{(1)}_{i_0-1,i_0} H E^{(1)}_{i_0 i_0} \\ &= -HE^{(1)}_{i_0,i_0-1} + HE^{(1)}_{i_0 i_0} \\ &+ E^{(1)}_{i_0-1,i_0} H E^{(1)}_{i_0,i_0-1} - E^{(1)}_{i_0-1,i_0} H E^{(1)}_{i_0 i_0}. \\ &= -HE^{(1)}_{i_0,i_0-1} + H + E^{(1)}_{i_0-1,i_0} H E^{(1)}_{i_0,i_0-1} - E^{(1)}_{i_0-1,i_0} H. \end{aligned}$$

Clearly $T_6 \in \mathfrak{A}$ and $T_6^* = T_6$. But T_6 is not upper triangular. Thus $T_6 \notin \mathrm{A}lg(\mathcal{L}_\infty)$. This implies that $\mathfrak{A} \cap \mathfrak{A}^* \neq \mathrm{A}lg(\mathcal{L}_\infty) \cap \mathrm{A}lg(\mathcal{L}_\infty)^*$. This contradiction shows that T must be diagonal. Thus we have that $T = \sum_{j=1}^n E^{(1)}_{jj} T E^{(1)}_{jj}$. Now we show that $E^{(1)}_{11} T E^{(1)}_{11} = E^{(1)}_{1j} T E^{(1)}_{j1}$ for $j = 1, \ldots, n$.

Assume that there is an i such that $E^{(1)}_{11} T E^{(1)}_{11} \neq E^{(1)}_{1i} T E^{(1)}_{i1}$. Define

$$T_7 = (E^{(1)}_{11} - E^{(1)}_{1i}) T = E^{(1)}_{11} T E^{(1)}_{11} - E^{(1)}_{1i} T E^{(1)}_{ii}.$$

Then similar to the construction of T_1, we see that $T_7 \in \mathfrak{A}$. Again write $T_8 = -E^{(1)}_{1i} T E^{(1)}_{i1} + E^{(1)}_{1i} T E^{(1)}_{ii}$. One checks (by Lemma 3) that $T_8 \in \mathrm{A}lg(\mathcal{L}_\infty)$. Then

$$0 \neq T_7 + T_8 = E^{(1)}_{11} T E^{(1)}_{11} - E^{(1)}_{1i} T E^{(1)}_{i1} \in \mathcal{A}.$$

Set $T_7 + T_8 = V'H'$, the polar decomposition with V' a partial isometry. One easily checks that $V'^* - V'^* E^{(1)}_{12} \in \mathrm{A}lg(\mathcal{L}_\infty)$. Then $(V'^* - V'^* E^{(1)}_{12})(T_7 + T_8) = H' \in \mathcal{A}$. Since H' is selfadjoint, $H' \in \mathcal{A} \cap \mathcal{A}^*$. But $E^{(1)}_{11} H' E^{(1)}_{11} \neq 0$ (with $E^{(1)}_{22} H' E^{(1)}_{22} = \cdots = E^{(1)}_{nn} H' E^{(1)}_{nn} = 0$). Thus $H' \notin \mathcal{N}_1'$ ($\supseteq \mathrm{A}lg(\mathcal{L}_\infty) \cap \mathrm{A}lg(\mathcal{L}_\infty)^*$). This implies that $\mathfrak{A} \cap \mathfrak{A}^* \neq \mathrm{A}lg(\mathcal{L}_\infty) \cap \mathrm{A}lg(\mathcal{L}_\infty)^*$. This contradiction shows that $E^{(1)}_{11} T E^{(1)}_{11} = \cdots = E^{(1)}_{1n} T E^{(1)}_{n1}$. Therefore $T \in \mathcal{L}_1' \subseteq \mathrm{A}lg(\mathcal{L}_1)$.

Now we restate our Theorem 3 in a slightly stronger form.

Theorem 4. *Suppose $\mathfrak{A}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$ such that $\mathrm{A}lg(\mathcal{L}_\infty) \subseteq \mathfrak{A}$ and $\mathfrak{A} \cap \mathfrak{A}^* = \mathrm{A}lg(\mathcal{L}_\infty) \cap \mathrm{A}lg(\mathcal{L}_\infty)^*$. Then $\mathfrak{A} = \mathrm{A}lg(\mathcal{L}_\infty)$.*

Proof Without the loss of generality, we may assume that $\mathfrak{A}$ is generated by T and $\mathrm{A}lg(\mathcal{L}_\infty)$. From the above lemma, we have that $T \in \mathrm{A}lg(\mathcal{L}_1)$. Suppose $T \in \mathrm{A}lg(\mathcal{L}_m)$ but $T \notin \mathrm{A}lg(\mathcal{L}_{m+1})$. From Lemma 6, we write $T = S + T'$ such that $S \in \mathrm{A}lg(\mathcal{L}_\infty)$ and $T' \in \mathcal{N}_m' \cap \mathrm{A}lg(\mathcal{L}_m)$. When we restrict all operators to the commutant of $\mathcal{N}_m$ and working with matrix units $E^{(m+1)}_{ij}$, similar computation as in the proof of Lemma 7 will show that $T' \in \mathrm{A}lg(\mathcal{L}_{m+1})$. This contradiction shows that $T \in \cap_{m=1}^\infty \mathrm{A}lg(\mathcal{L}_m) = \mathrm{A}lg(\mathcal{L}_\infty)$.

In the above theorem, we did not assume the closedness of $\mathfrak{A}$ under any topology. Thus $\mathrm{A}lg(\mathcal{L}_\infty)$ has an algebraic maximality property. Next section, we will show that $\mathrm{L}at(\mathrm{A}lg(\mathcal{L}_\infty))$ is the strong-operator closure of $\mathcal{L}_\infty$.

Kadison-Singer lattices

It is hard to determine whether a given lattice is a Kadison-Singer lattice. The only known class is the family of nests [1]. Some finite distributive lattices (see [5] and [21]) are Kadison-Singer lattices if they have a minimal generating property. In this section, we will show that the strong-operator closure of $\mathcal{L}_\infty$ defined in Section 3 is a Kadison-Singer lattice. For simplicity of description, we shall assume that the state ρ on $\mathfrak{A}_n$ is a trace, now denoted by τ. Let $\mathcal{R}$ be the hyperfinite II_1 factor generated by $\mathcal{L}_\infty$ (or $\mathfrak{A}_n$). The commutant $\mathcal{R}'$ of $\mathcal{R}$ is the diagonal subalgebra of $\mathrm{A}lg(\mathcal{L}_\infty)$.

Theorem 5. *Let $\mathcal{L}^{(n)}$ be the strong-operator closure of $\mathcal{L}_\infty$, $\mathcal{H}$ the Hilbert space obtained by GNS construction on $(\mathfrak{A}_n, \tau)$. Then we have that $\mathcal{L}^{(n)} = \mathrm{Lat}(\mathrm{Alg}(\mathcal{L}_\infty))$. For any $r \in (0,1)$, if there are $a, l \in \mathbb{N}$ such that $r = \frac{a}{n^l}$, then there are two distinct projections in $\mathcal{L}^{(n)}$ with trace value r; otherwise there is only one projection in $\mathcal{L}^{(n)}$ with trace r.*

To understand the lattice structure of $\mathcal{L}^{(n)}$, we first analyze the lattice properties of $\mathcal{L}_m$. From the definition of P_{1j}, $j = 1, \ldots, n$, the generators of $\mathcal{L}_1$, we know that $\mathcal{L}_1$ consists of a nest $\{0, P_{11}, \ldots, P_{1,n-1}, I\}$ in $\mathcal{N}_1$ ($\cong M_n(\mathbb{C})$) on the diagonal and a minimal projection P_{1n}. It is easy to see that $P_{1n} \wedge P_{1j} = 0$ for $1 \leq j \leq n-1$, and their unions give rise to another nest $\{0, P_{1n}, P_{1n} \vee P_{11}, \ldots, P_{1n} \vee P_{1,n-1} = I\}$ in $\mathcal{N}_1$. The lattice $\mathcal{L}_1$ is the union of these two nests. For any $1 \leq k \leq n-1$, there are two distinct projections in $\mathcal{L}_1$ such that they have the same trace $\frac{k}{n}$. This pattern of double nests appears in $\mathcal{L}_2$ between any two trace values $\frac{k}{n}$ and $\frac{k+1}{n}$, $0 \leq k \leq n-1$. To describe all these projections, we need more notation. For $k = 1, 2, \ldots$, define

$$\begin{aligned} E^{(k)}_i &= \sum_{l=1}^{i} E^{(k)}_{ll} \quad i = 1, \ldots, n; \\ F^{(k)}_i &= \frac{1}{i} \sum_{n\geq l, m > n-i} E^{(k)}_{lm}, \quad i = 1, \ldots, n. \end{aligned}$$

Since $E^{(k)}_{ij}$ and $E^{(k')}_{i'j'}$ are tensorial relations for $k \neq k'$, we have $E^{(k)}_i$ and $F^{(k)}_i$ are projections in $\mathcal{N}_{k-1}' \cap \mathcal{N}_k$ ($\mathcal{N}_0 = \mathbb{C}I$). Also $E^{(k)}_i F^{(k)}_j = 0$ when $j \leq n-i$. We shall use τ to denote the unique trace on both $\mathcal{R}$ and $\mathcal{R}'$, $\tau(E^{(k)}_i) = \frac{i}{n}$, $\tau(F^{(k)}_i) = \frac{1}{n}$.

Lemma 8. *Suppose $P \in \mathrm{Lat}(\mathrm{Alg}(\mathcal{L}_\infty))$ and $P \neq 0, I$. Then $P = E_i^{(1)} + F_{n-i}^{(1)} Q$, for some $i \in \{0, 1, \dots, n-1\}$, $Q \in \mathcal{N}_1'$, and also $Q \in \mathrm{Lat}(\mathcal{N}_1' \cap \mathrm{Alg}(\mathcal{L}_\infty))$. When P is given in this form, $P \in \mathrm{Lat}(\mathrm{Alg}(\mathcal{L}_\infty))$.*

Proof First we show that if $P = E_i^{(1)} + F_{n-i}^{(1)} Q$ with Q described in the lemma, then $P \in \mathrm{Lat}(\mathrm{Alg}(\mathcal{L}_\infty))$.

For any $T \in \mathrm{Alg}(\mathcal{L}_\infty)$, let $T = T_1 + T_2$, given by (3) in the proof of Lemma 3. Then it is easy to check that $T_1 \in \mathrm{Alg}(\mathcal{L}_\infty)$, $(I - P_{1,n-1})T_1 = 0$ and $T_2 \in \mathcal{N}_1' \cap \mathrm{Alg}(\mathcal{L}_\infty)$. Since $E_i^{(1)} = P_{1i} \in \mathcal{L}_1 \subseteq \mathcal{L}_\infty$, we have $(I - E_i^{(1)})T_j E_i^{(1)} = 0$, for $j \in \{1, 2\}$. One can check directly that $F_{n-i}^{(1)}(I - E_i^{(1)}) = F_{n-i}^{(1)} = (I - E_i^{(1)})F_{n-i}^{(1)}$. Thus $F_{n-i}^{(1)} T E_i^{(1)} = 0$. Since Q commutes with $\mathcal{N}_1$, we have

$$\begin{aligned}
(I-P)TP &= (I - E_i^{(1)} - F_{n-i}^{(1)}Q)T(E_i^{(1)} + F_{n-i}^{(1)}Q) \\
&= (I - E_i^{(1)})TF_{n-i}^{(1)}Q - F_{n-i}^{(1)}QTE_i^{(1)} \\
&\quad - F_{n-i}^{(1)}QTF_{n-i}^{(1)}Q \\
&= (I - E_i^{(1)})TF_{n-i}^{(1)}Q - QF_{n-i}^{(1)}(I - E_i^{(1)})TF_{n-i}^{(1)}Q.
\end{aligned}$$

The above equations hold when T is replaced by T_1 or T_2. From our assumptions that $Q \in \mathrm{Lat}(\mathcal{N}_1' \cap \mathrm{Alg}(\mathcal{L}_\infty))$, $E_i^{(1)}, F_{n-i}^{(1)} \in \mathcal{N}_1$ and $T_2 \in \mathcal{N}_1' \cap \mathrm{Alg}(\mathcal{L}_\infty)$, we have

$$\begin{aligned}
(I-P)T_2P &= ((I - E_i^{(1)})F_{n-i}^{(1)} - QF_{n-i}^{(1)}(I - E_i^{(1)}))T_2Q \\
&= F_{n-1}^{(1)}(I - Q)T_2Q = 0.
\end{aligned}$$

Next we show that $(I - E_i^{(1)})T_1F_{n-i}^{(1)} = 0$, which implies that $(I-P)T_1P = 0$. Note that

$$\begin{aligned}
(n-i)(I - E_i^{(1)})T_1F_{n-i}^{(1)} &= (\sum_{j=i+1}^{n} E_{jj}^{(1)})T_1(\sum_{l,m=i+1}^{n} E_{lm}^{(1)}) \\
&= \sum_{j,m=i+1}^{n} \sum_{l=j}^{n} E_{jj}^{(1)} T_1 E_{lm}^{(1)}.
\end{aligned}$$

By Lemma 3 and $(I - P_{n-1}^{(1)})T_1 = E_{nn}^{(1)}T_1 = 0$, we have $\sum_{l=j}^{n} E_{jj}^{(1)}T_1E_{lm}^{(1)} = 0$. So $(n-i)(I - E_i^{(1)})T_1F_{n-i}^{(1)} = 0$. Thus $(I-P)TP = 0$ which implies that $P \in \mathrm{Lat}(\mathrm{Alg}(\mathcal{L}_\infty))$.

Now for any $P \in \mathrm{Lat}(\mathrm{Alg}(\mathcal{L}_\infty))$, let i_0, $1 \le i_0 \le n$, be the smallest integer such that $E_{i_0i_0}^{(1)} P E_{i_0i_0}^{(1)} \neq E_{i_0i_0}^{(1)}$. Then $E_{ii}^{(1)}PE_{ii}^{(1)} = E_{ii}^{(1)}$ for $1 \le i \le i_0 - 1$ and $P = E_{i_0-1}^{(1)} + P_1$, where P_1 is a projection and $E_i^{(1)}P_1 = 0$ for $i \le i_0 - 1$. First we assume that $i_0 \le n-1$. For any $A \in \mathcal{B}(\mathcal{H})$ and $i_1 \ge i_0 + 1$, define $A_{i_1} = E_{i_0i_0}^{(1)}A(E_{i_0i_0}^{(1)} - E_{i_0i_1}^{(1)})$. Then $A_{i_1} \in \mathrm{Alg}(\mathcal{L}_\infty)$. Since $P \in \mathrm{Lat}(\mathrm{Alg}(\mathcal{L}_\infty))$, we have

$$\begin{aligned}
0 &= (I - E_{i_0-1}^{(1)} - P_1)A_{i_1}(E_{i_0-1}^{(1)} + P_1) \\
&= (I - P_1)E_{i_0i_0}^{(1)}A(E_{i_0i_0}^{(1)} - E_{i_0i_1}^{(1)})P_1.
\end{aligned}$$

From $E_{i_0i_0}^{(1)}(I - P_1)E_{i_0i_0}^{(1)} \neq 0$, the above equation implies that $E_{i_0i_0}^{(1)}P_1 = E_{i_0i_1}^{(1)}P_1$, for all $i_1 \ge i_0$. So, multiplying by $P_1E_{i_0i_0}^{(1)} = P_1E_{ji_0}^{(1)}$ (the adjoint of the above equation) on the right hand side, we have $E_{i_0i_0}^{(1)}P_1E_{i_0i_0}^{(1)} = E_{i_0i_1}^{(1)}P_1E_{ji_0}^{(1)}$ for all $i_1, j \ge i_0$. This implies that $P_1 = F_{n-i_0+1}^{(1)}Q$, where Q is a projection in $\mathcal{N}_1'$. If $i_0 = n$, then P_1 can be written as $F_1^{(1)}Q$ for $Q \in \mathcal{N}_1'$. From $P \in \mathrm{Lat}(\mathrm{Alg}(\mathcal{L}_\infty))$, it is easy to see that $Q \in \mathrm{Lat}(\mathcal{N}_1' \cap \mathrm{Alg}(\mathcal{L}_\infty))$.

Lemma 9. *Suppose $P \in \mathrm{Lat}(\mathrm{Alg}(\mathcal{L}_\infty))$. Then there exist $Q \in \mathcal{N}_k' \cap \mathrm{Lat}(\mathcal{N}_k' \cap \mathrm{Alg}(\mathcal{L}_\infty))$ and integers a_k such that*

$$P = E_{a_1}^{(1)} + F_{n-a_1}^{(1)}E_{a_2}^{(2)} + \cdots + (\prod_{i=1}^{k-1} F_{n-a_i}^{(i)})E_{a_k}^{(k)} + (\prod_{i=1}^{k} F_{n-a_i}^{(i)})Q,$$

where $0 \le a_i \le n-1$ and $\tau(P) = \sum_{i=1}^{k} \frac{a_i}{n^i} + \frac{\tau(Q)}{n^k}$ which is a real number lies in the closed interval $\left[\sum_{i=1}^{k} \frac{a_i}{n^i},\ \sum_{i=1}^{k} \frac{a_i}{n^i} + \frac{1}{n^k}\right] \subseteq [0,1]$.

The above lemma follows easily from induction. The details are similar to the proof of Lemma 8.

Proof of Theorem 5 To describe an arbitrary projection P in $\mathrm{Lat}(\mathrm{Alg}(\mathcal{L}_\infty))$ in more details, we need to know the trace value of P. First when $\tau(P) = \sum_{i=1}^{k} \frac{a_i}{n^i}$, where $0 \le a_i \le n-1$ and $a_k \neq 0$, then there are two cases, either

$$\begin{aligned}
P &= \sum_{j=1}^{k} (\prod_{i=1}^{j-1} F_{n-a_i}^{(i)})E_{a_j}^{(j)}, \qquad (\text{let } \prod_{i=1}^{0} F_{n-a_i}^{(i)} = I) \quad \text{or} \\
P &= \sum_{j=1}^{k-1} (\prod_{i=1}^{j-1} F_{n-a_i}^{(i)})E_{a_j}^{(j)} + (\prod_{i=1}^{k-1} F_{n-a_i}^{(i)})E_{a_k-1}^{(k)} \\
&\qquad + (\prod_{i=1}^{k-1} F_{n-a_i}^{(i)})F_{n-a_k+1}^{(k)}.
\end{aligned}$$

Note that the above two projections correspond to the case when $Q = 0$ for the decomposition $\tau(P) = \sum_{i=1}^{k} \frac{a_i}{n^i}$, or respectively $Q = I$ for $\tau(P) = \sum_{i=1}^{k-1} \frac{a_i}{n^i} + \frac{a_k - 1}{n^k} + \frac{1}{n^k}$ in Lemma 9. So $P \in \mathrm{Lat}(\mathrm{Alg}(\mathcal{L}_\infty))$. Thus for any $r = \frac{a}{n^l}$ for some integer $l > 0$ and any integer a such that $0 < a < n^l$, there are exactly two projections in $\mathrm{Lat}(\mathrm{Alg}(\mathcal{L}_\infty))$ with trace r.

Secondly, when $r \in (0,1)$ and $r \neq \frac{a}{n^l}$ for any positive integer l and any integer a with $0 < a < n^l$, we shall show that there is a unique P in $\mathrm{Lat}(\mathrm{Alg}(\mathcal{L}_\infty))$ with trace r. For the given r, there is a unique expansion $r = \sum_{k=1}^{\infty} \frac{a_k}{n^k}$, where a_k is an integer with $0 \le a_k \le n-1$, there are infinitely many non zero a_k's and infinitely many $a_k \neq n-1$. (This is because repeating $n-1$ as coefficients from certain place on will result r being $\frac{a}{n^l}$, e.g., $0.09999\cdots = 0.1$ when $n = 10$.) In fact, Lemma 9 gives the existence and uniqueness of such a projection:

$$P = E_{a_1}^{(1)} + F_{n-a_1}^{(1)}E_{a_2}^{(2)} + F_{n-a_1}^{(1)}F_{n-a_2}^{(2)}E_{a_3}^{(3)} + \cdots.$$

It is not hard to see that P is the strong-operator limit of finite sums. The finite sums

$$Q_k = \sum_{j=1}^{k} (\prod_{i=1}^{j-1} F_{n-a_i}^{(i)})E_{a_j}^{(j)} \in \mathcal{L}_\infty, \quad k = 1, 2, \dots,$$

$Q_1 < Q_2 < \cdots < Q_k < \cdots < P$ and $\lim_{k\to\infty} \tau(Q_k) = r = \tau(P)$.

The following theorem gives us infinitely many non isomorphic Kadison-Singer lattices.

Theorem 6. *For $n \neq k$, $\mathcal{L}^{(n)}$ and $\mathcal{L}^{(k)}$ are not algebraically isomorphic as lattices.*

Proof For any $n \ge 2$, first we observe from Lemma 9 that if $E \in \mathcal{L}^{(n)}$ is a minimal projection, then $E = (\prod_{i=1}^{m} F_n^{(i)})E_1^{(m+1)}$ for $m = 0, 1, 2, \dots$.

Suppose that $P \in \mathcal{L}^{(n)}$ is given as in Lemma 9,

$$\begin{aligned} P =& E^{(1)}_{a_1} + F^{(1)}_{n-a_1}E^{(2)}_{a_2} + \cdots + (\prod_{i=1}^{m} F^{(i)}_{n-a_i})E^{(m+1)}_{a_{m+1}} \\ &+ (\prod_{i=1}^{m} F^{(i)}_{n-a_i})F^{(m+1)}_{n-a_{m+1}}Q, \end{aligned}$$

where $0 \le a_i \le n-1$ and $Q \in \mathcal{N}'_{m+1} \cap \mathrm{L}at(\mathcal{N}'_{m+1} \cap \mathrm{A}lg(\mathcal{L}^{(n)}))$. We shall show that $P \wedge (\prod_{i=1}^{m} F^{(i)}_n)E^{(m+1)}_1 = 0$ for some $m \ge 0$ if and only if $a_{m+1} = 0$.

First if $a_{m+1} > 0$, it is easy to check that the minimal projection $(\prod_{i=1}^{m} F^{(i)}_n)E^{(m+1)}_1 \le P$. Conversely, if $a_{m+1} = 0$, then

$$\begin{aligned} P =& E^{(1)}_{a_1} + F^{(1)}_{n-a_1}E^{(2)}_{a_2} + \cdots + (\prod_{i=1}^{m-1} F^{(i)}_{n-a_i})E^{(m)}_{a_m} \\ &+ (\prod_{i=1}^{m} F^{(i)}_{n-a_i})F^{(m+1)}_{n}Q. \end{aligned}$$

Let $\xi \in P(\mathcal{H}) \wedge (\prod_{i=1}^{m} F^{(i)}_n)E^{(m+1)}_1(\mathcal{H})$, and $E = (I - E^{(1)}_{n-1})(I - E^{(2)}_{n-1}) \cdots (I - E^{(m+1)}_{n-1})$. We have $E\xi = E(\prod_{i=1}^{m} F^{(i)}_n)E^{(m+1)}_1\xi = 0$. But

$$\begin{aligned} 0 = PEP\xi &= (\prod_{i=1}^{m} F^{(i)}_{n-a_i})F^{(m+1)}_{n}E(\prod_{i=1}^{m} F^{(i)}_{n-a_i})F^{(m+1)}_{n}Q\xi \\ &= (\prod_{i=1}^{m} F^{(i)}_{n-a_i}(I - E^{(i)}_{n-1})F^{(i)}_{n-a_i})F^{(m+1)}_{n}(I - E^{m+1}_{n-1})F^{(m+1)}_{n}Q\xi \\ &= \frac{1}{n}(\prod_{i=1}^{m} \frac{1}{n-a_i})(\prod_{i=1}^{m} F^{(i)}_{n-a_i})F^{(m+1)}_{n}Q\xi. \end{aligned}$$

This shows that $\xi \in (\prod_{i=1}^{m} F^{(i)}_n)E^{(m+1)}_1(\mathcal{H}) \wedge P_1(\mathcal{H})$, here $P_1 = P - (\prod_{i=1}^{m} F^{(i)}_{n-a_i})F^{(m+1)}_{n}Q$. Let $\widetilde{E} = (I - E^{(1)}_{n-1})(I - E^{(2)}_{n-1}) \cdots (I - E^{(m)}_{n-1})$. Then $\widetilde{E}\xi = \widetilde{E}P_1\xi = 0$ and

$$\begin{aligned} 0 &= (\prod_{i=1}^{m} F^{(i)}_n)E^{(m+1)}_1\widetilde{E}(\prod_{i=1}^{m} F^{(i)}_n)E^{(m+1)}_1\xi \\ &= (\prod_{i=1}^{m} F^{(i)}_n(I - E^{(i)}_n)F^{(i)}_n)E^{(m+1)}_1\xi \\ &= \frac{1}{n^m}(\prod_{i=1}^{m} F^{(i)}_n)E^{(m+1)}_1\xi = \frac{1}{n^m}\xi. \end{aligned}$$

This shows that $\xi = 0$ and thus $P \wedge (\prod_{i=1}^{m} F^{(i)}_n)E^{(m+1)}_1 = 0$.

For any $\mathcal{S} \subset \mathcal{L}^{(n)}$, we define

$$\mathcal{Z}(\mathcal{S}) = \{P \in \mathcal{L}^{(n)} : \ P \wedge Q = 0, \text{ for all } Q \in \mathcal{S}\}.$$

From the above, we know that, for any minimal projection $(\prod_{i=1}^{m} F^{(i)}_n)E^{(m+1)}_1$,

$$\begin{aligned} &\mathcal{Z}(\{(\prod_{i=1}^{m} F^{(i)}_n)E^{(m+1)}_1\}) \\ &\quad = \{E^{(1)}_{a_1} + F^{(1)}_{n-a_1}E^{(2)}_{a_2} + \cdots + (\prod_{i=1}^{m-1} F^{(i)}_{n-a_i})E^{(m)}_{a_m} \\ &\qquad + (\prod_{i=1}^{m} F^{(i)}_{n-a_i})F^{(m+1)}_{n}Q : \\ &\qquad 0 \le a_i \le n-1, Q \in \mathcal{N}'_{m+1} \cap Lat(\mathcal{N}'_{m+1} \cap \mathrm{Alg}(\mathcal{L}^{(n)}))\}. \end{aligned}$$

Now it is not hard to show that $\mathcal{Z}(\mathcal{Z}(\{(\prod_{i=1}^{m} F^{(i)}_n)E^{(m+1)}_1\})) = \{(\prod_{i=1}^{m} F^{(i)}_n)E^{(m+1)}_k : \ k = 0, 1, \ldots, n-1\}$. The number of elements in this set is an invariant of $\mathcal{L}^{(n)}$.

ACKNOWLEDGMENTS. Research supported in part by Chinese Academy of Sciences and President Fund of Academy of Mathematics and Systems Science, Chinese Academy of Sciences

1. R. Kadison and I. Singer, Triangular operator algebras. Fundamentals and hyperreducible theory, Amer. J. Math., 82 (1960), 227–259.
2. J. von Neumann, Zur Algebra der Functionaloperationen und Theorie der normalen Operatoren, Math. Ann. 102 (1930), 370–427.
3. P. S. Muhly, K. Saito and B. Solel, Coordinates for triangular operator algebras, Ann. of Math., 127 (1988), 245–278.
4. A. Hopenwasser, Completely isometric maps and triangular operator algebras, Proc. London Math. Soc. 25 (1972), 96–114.
5. P. Halmos, Reflexive lattices of subspaces, J. London Math. Soc. 4 (1971), 257–263.
6. D. Hadwin, A general view of reflexivity, Trans. Amer. Math. Soc. 344 (1994), 325–360.
7. H. Radjavi and P. Rosenthal, "Invariant Subspaces", Springer–Verlag, Berlin, 1973.
8. D. Larson, Reflexivity, algebraic reflexivity and linear interpolation, Amer. J. Math. 110 (1998), 283–299.
9. L. Ge and J. Shen, On the generator problem of von Neumann algebras, AMS/IP Studies in Adv. Math. 42 (2008), 257–275.
10. D. Voiculescu, K. Dykema and A. Nica, "Free Random Variables," CRM Monograph Series, vol. 1, 1992.
11. J. Ringrose, On some algebras of operators, Proc. London Math. Soc. (3) 15 (1965), 61–83.
12. W. Arveson, Operator algebras and invariant subspaces, Ann. of Math. 100 (1974), 433–532.
13. E. C. Lance, Cohomology and perturbations of nest algebras, Proc. London Math. Soc. 43 (1981), 334–356.
14. D. Larson, Similarity of nest algebras, Ann. of Math. 121 (1988), 409–427.
15. K. Davidson, E. Katsoulis and D. Pitts, The structure of free semigroup Algebras, J. Reine Ang. Math. (Crelle's Journal), 533 (2001), 99–125.
16. K. Davidson, "Nest Algebras," Research Notes in Math. vol. 191, Pitman, Boston-Londodn Melbourne, 1988.
17. R. Kadison, On the orthogonalization of operator representations, Amer. J. Math. 78 (1955), 600–621.
18. R. Kadison and J. Ringrose, "Fundamentals of the Operator Algebras," vols. I and II, Academic Press, Orlando, 1983 and 1986.
19. J. Glimm, Type I C*-algebras, Ann. Math. 73 (1961), 572–611.
20. R. Powers, Representations of Uniformly Hyperfinite Algebras and Their Associated von Neumann Rings, Ann. Math. 86 (1967), 138–171.
21. K. Harrison, On lattices of invariant subspaces, Doctoral Thesis, Monash University, Melbourne, 1970.