

# Note on Sarnak Conjecture

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ABSTRACT. Enter abstract here

## 1. INTRODUCTION

The Möbius function  $\mu(n)$ ,  $n = 1, 2, 3, \dots$  is defined by

$$\mu(n) = \begin{cases} 1, & \text{if } n=1 \\ 0, & \text{if } n \text{ is not square free} \\ (-1)^t, & \text{if } n \text{ is a product of } t \text{ distinct primes} \end{cases}$$

Peter Sarnak give the disjointness conjecture concerning the Möbius function  $\mu(n)$ .

**Conjecture 1.1** (P. Sarnak). *Let  $(X, T)$  be a deterministic (i.e. the topological entropy  $h(T) = 0$ ) topological dynamical system.*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_n \mu(n) f(T^n x) = 0$$

where  $x \in X$  and  $f \in C(X)$ .

## 2. REVIEW ON ENTROPY

“ You should call it entropy, for two reasons. In the first place, your uncertainty function has been used in statistical mechanics under that name, so it already has a name. In the second place, and more important, no one knows what entropy really is, so in a debate you will always have the advantage. ”

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von Neumann said to Shannon, *Mathematical Theory of Entropy*, 1981

Throughout this section  $(X, \mathcal{A}, \mu)$  denotes a measure space.

**2.1. Some ergodic results.** Let  $T$  be a  $\mathcal{A}$ -measurable transformation from  $X$  to  $X$ .  $\mu$  is  $T$ -invariant iff  $\mu(T^{-1}B) = \mu(B)$  for every  $B \in \mathcal{A}$ .

**Definition 2.1.** Let  $T : X \rightarrow X$  be a transformation. The periodic entropy of  $T$  is

$$p(T) = \lim_{m \rightarrow \infty} \frac{1}{m} \log(\#\{x \in X : T^m(x) = x\}).$$

**Remark 2.1.** The periodic entropy measures the complexity of  $T$  from the point of view of the periodic points. For example, let  $T : z \rightarrow z^m$  from  $S^1$  to  $S^1$ ,  $p(T) = \log|m| > 0$ .

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**Theorem 2.1** (Poincaré's recurrence theorem). *Let  $T : X \rightarrow X$  be a measurable transformation and let  $\mu$  be a finite  $T$ -invariant measure in  $X$ . If  $A \subset X$  is measurable, then the set*

$$B = \{x \in A : T^n(x) \in A, \text{ for infinitely many integers } n \in \mathbb{N}\}$$

*has measure  $\mu(B) = \mu(A)$ .*

**Definition 2.2.** *Given a transformation  $T : X \rightarrow X$ , we say that a function  $\varphi$  is  $T$ -invariant if  $\varphi(T(x)) = \varphi(x)$  for every  $x \in X$ .  $\varphi$  is  $T$ -invariant almost everywhere if there is a  $T$ -invariant subset  $B \subset X$ , i.e.  $T^{-1}B = B$ , with  $\mu(X \setminus B) = 0$  and  $\varphi|_B$  is  $T$ -invariant.*

**Theorem 2.2** (Birkhoff's ergodic theorem). *Let  $T : X \rightarrow X$  be a measurable transformation and let  $\mu$  be a finite  $T$ -invariant measure in  $X$ . If  $\varphi \in L^1(X, d\mu)$ , then the limit*

$$\varphi_T = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(T^k(x))$$

- $\varphi_T$  is  $T$ -invariant almost everywhere.
- $\varphi_T \in L^1(X, d\mu)$  and  $\int_X \varphi_T d\mu = \int_X \varphi d\mu$ .

**2.2. Metric(measure-theoretical) Entropy.** In this subsection we assume that  $\mu(X) = 1$ . Let

$$\psi(x) = \begin{cases} x \log x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

**Definition 2.3.** *Let  $\xi \subset \mathcal{A}$ . If*

- (1)  $\mu(\bigcup_{C \in \xi} C) = 1$ ,
- (2)  $\mu(C \cap D) = 0$  for any distinct  $C, D$  in  $\xi$ ,

*then  $\xi$  is a measurable partition of  $(X, \mathcal{A}, \mu)$ .*

**Definition 2.4** (Entropy of a measurable partition). *Let  $\xi$  be a measurable partition of  $X$  w.r.t  $\mu$ . The entropy of  $\xi$  is given by*

$$H_\mu(\xi) = - \sum_{C \in \xi} \mu(C) \log \mu(C) = - \sum_{C \in \xi} \psi(\mu(C)).$$

**Definition 2.5.** *Given two measurable partition  $\xi, \eta$  of  $X$ , we define a new measurable partition*

$$\xi \vee \eta = \{C \cap D : C \in \xi, D \in \eta\}.$$

**Definition 2.6.** *Let  $T : X \rightarrow X$  be a measurable transformation preserving the probability measure  $\mu$  and  $\xi$  a measurable partition of  $X$ . The metric entropy of  $T$  with respect to  $\mu$  and  $\xi$  is*

$$h_\mu(T, \xi) = \lim_{n \rightarrow \infty} \frac{1}{n} H_\mu(\xi_n) = \inf_{n \in \mathbb{N}} H_\mu(\xi_n),$$

*where  $\xi_n = \bigvee_{k=0}^{n-1} T^{-k} \xi$ . The metric entropy of  $T$  with respect to  $\mu$  is*

$$h_\mu(T) = \sup\{h_\mu(T, \xi) : \xi \text{ is a measurable partition of } X\}.$$

**Lemma 2.1.** *Let  $\xi$  and  $\eta$  be measurable partitions of  $X$ . If  $\eta$  is a refinement of  $\xi$ , i.e., each element of  $\xi$  is a union of elements of  $\eta$ , then  $H_\mu(\xi) \leq H_\mu(\eta)$ .*

**Theorem 2.3.** Let  $T : X \rightarrow X$  be a measurable transformation preserving the probability measure  $\mu$ . If  $(\xi_n)_{n \in \mathbb{N}}$  is a sequence of measurable partitions of  $X$  with  $\bigvee_{n=1}^{\infty} \mathcal{A}(\xi_n)$  ( $\mathcal{A}(\xi)$  is the  $\sigma$ -algebra generated by  $\xi$ ) such that  $\xi_{n+1}$  is a refinement of  $\xi_n$  for each  $n \in \mathbb{N}$ , then

$$h_\mu(T) = \lim_{n \rightarrow \infty} h_\mu(T, \xi_n) = \sup_{n \in \mathbb{N}} h_\mu(T, \xi_n).$$

Specially, if  $\bigvee_{k=0}^{\infty} \mathcal{A}(T^{-k}\xi) = \mathcal{A}$  or  $\bigvee_{k=-\infty}^{\infty} \mathcal{A}(T^{-k}\xi) = \mathcal{A}$ , then we have the following.

**Corollary 2.1.** Let  $T : X \rightarrow X$  be a measurable transformation preserving the probability measure  $\mu$ . If  $\bigvee_{k=0}^{\infty} \mathcal{A}(T^{-k}\xi) = \mathcal{A}$  or  $\bigvee_{k=-\infty}^{\infty} \mathcal{A}(T^{-k}\xi) = \mathcal{A}$ , then  $h_\mu(T) = h_\mu(T, \xi)$ .

Note that if  $\xi$  is a measurable partition of  $X$  w.r.t.  $\mu$ , then for almost every  $x \in X$ , there exists a single element  $\xi_n(x)$  (depending on  $x$  such that  $\xi_n(x) \in \xi_n = \bigvee_{k=0}^{n-1} T^{-k}\xi$ ).

**Theorem 2.4** (Shannon-McMillan-Breiman). If  $T : X \rightarrow X$  is a measurable transformation preserving a probability measure  $\mu$  in  $X$  and  $\xi$  is a measurable partition  $X$ , then the limit

$$h_\mu(T, \xi, x) := \lim_{n \rightarrow \infty} -\frac{1}{n} \log \mu(\xi_n(x))$$

exists for almost every  $x \in X$ . Moreover, the function  $x \rightarrow h_\mu(T, \xi, x)$  is  $T$ -invariant almost everywhere, is integrable and

$$h_\mu(T, \xi) = \int_X h_\mu(T, \xi, x) d\mu(x).$$

### 2.3. Topological Entropy.

**Definition 2.7** (Cover entropy of continuous map). For a cover  $\mathcal{U}$  of a compact topological space  $X$ , define  $N(\mathcal{U})$  to be the smallest cardinality of a subcover of  $\mathcal{U}$ , and define the entropy of  $\mathcal{U}$  to be  $H(\mathcal{U}) = \log N(\mathcal{U})$ . Let  $T : X \rightarrow X$  be a continuous map. The cover entropy of  $T$  with respect to  $\mathcal{U}$  is defined to be

$$h_{\text{cover}}(T, \mathcal{U}) = \lim_{n \rightarrow \infty} \frac{1}{n} \log N\left(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}\right) = \inf_{n \geq 1} \frac{1}{n} \log N\left(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}\right),$$

and the cover entropy of  $T$  is defined to be

$$h_{\text{cover}}(T) = \sup_{\mathcal{U}} h_{\text{cover}}(T, \mathcal{U})$$

where the supremum is taken over all open covers of  $X$ .

Throughout the rest of this subsection, let  $(X, d)$  be a **compact metric space** and  $T$  a continuous transformation from  $X$  to  $X$ .

**Definition 2.8.** Let  $T : (X, d) \rightarrow (X, d)$  be a continuous map on a compact metric space. A one side generator for  $T$  is a finite open cover  $\mathcal{U} = \{U_i\}_{i \in I}$  with the property that for any sequence  $(i_k)_{k \geq 0}$ , the set

$$\bigcap_{k \geq 0} T^{-k}(\overline{U_{i_k}})$$

contains at most a single point.

**Theorem 2.5.** Let  $\mathcal{U}$  be a one-sided generator for a continuous map  $T : (X, d) \rightarrow (X, d)$  on a compact metric space. Then  $h_{\text{cover}}(T, \mathcal{U}) = h_{\text{cover}}(T)$ .

**Theorem 2.6.** *Let  $T : X \rightarrow X$  be a continuous map on a compact metric space  $(X, d)$ . If  $(\mathcal{U}_n)$  is a sequence of open covers of  $X$  with  $\text{diam}(\mathcal{U}_n) \rightarrow 0$  as  $n \rightarrow \infty$ , then*

$$\lim_{n \rightarrow \infty} h_{\text{cover}}(T, \mathcal{U}_n) = h_{\text{cover}}(T).$$

For each  $n \in \mathbb{N}$ , we introduce a new distance in  $X$  by

$$d_n(x, y) = \max\{d(T^k(x), T^k(y)) : 0 \leq k \leq n-1\}.$$

And  $N(d, \varepsilon)$  denotes the maximum number of points in  $X$  at a  $d$ -distance at least  $\varepsilon$ .

**Definition 2.9** (Separated Set Topological Entropy). *The separated set topological entropy of  $T$  is*

$$h(T)_{\text{sep}} = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \log N(d_n, \varepsilon).$$

**Remark 2.2.**  $h_{\text{sep}}(T)$  only depends on the topology induced by the distance  $d$  and  $h_{\text{sep}}(T) = h_{\text{cover}}(T)$ . The common value is called the **topological entropy of  $T$** , denoted  $h_{\text{top}}(T)$ .

Some facts about  $h_{\text{top}}(T)$ :

- If  $T : X \rightarrow X$  is a **homeomorphism** of a compact metric space, then  $h_{\text{top}}(T^{-1}) = h_{\text{top}}(T)$ .
- If  $T : X \rightarrow X$  is a continuous map of a compact metric space, then  $h_{\text{top}}(T^k) = kh_{\text{top}}(T)$  for  $k \geq 1$ .

**Theorem 2.7** (Variational principle for the topological entropy). *If  $T : X \rightarrow X$  is a continuous transformation of a compact metric space, then*

$$h_{\text{top}}(T) = \sup\{h_\mu(T) : \mu \text{ is a } T\text{-invariant probability measure in } X\}.$$

**2.4. For AF algebra.** Let  $\sigma$  be an automorphism of  $\mathfrak{A}$ ,  $\mathfrak{A}$  is an AF-algebra.  $\Sigma = \{A_i\}_{i=1}^n \subset \mathfrak{A}$ , and  $\mathcal{M}$  be a finite dimensional subalgebra of  $\mathfrak{A}$  and  $\Sigma \subset_\delta \mathcal{M}$ . Let  $r(\Sigma, \delta) = \min_{\Sigma \subset_\delta \mathcal{M}} \{\max \dim \text{ of masa of } \mathcal{M}\}$ .

$$\Sigma_n = \cup_{i=0}^n \sigma^i(\Sigma)$$

$$h(\sigma) = \lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \frac{\log(\Sigma_n, \delta)}{n}$$

### 3. FINITE MATRIX CASE

Let  $\mathfrak{A} = M_n(\mathbb{C})$  and  $\rho$  be a pure state of  $\mathfrak{A}$ . We may ask if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_n \mu(n) \rho(\sigma^n(T)) = 0$$

where  $\sigma$  is an automorphism of  $\mathfrak{A}$  and  $T \in \mathfrak{A}$ .

Since automorphism preserve the norm of  $\mathfrak{A}$ , it is not hard to see that, as a continuous map from the unit ball of  $\mathfrak{A}$  onto itself, the topological entropy of  $\sigma$  is 0, by considering the definition of Bowen and Dianburg.

Without lose of generality, we may assume that  $\sigma = Ad(U)$  where  $U$  is a unitary and  $\rho(T) = \langle T\xi, \xi \rangle$ . The question can be restated as if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_n \mu(n) \langle TU^n \xi, U^n \xi \rangle = 0$$

Assume  $\xi = \xi_1 + \xi_2 + \cdots + \xi_n$  such that

$$U\xi_k = e^{i\theta_k} \xi_k \quad k = 1, 2, \dots, n.$$

Then we have

$$\langle TU^m \xi, U^m \xi \rangle = \sum_{k,l=1}^n e^{im(\theta_k - \theta_l)} \langle T \xi_k, \xi_l \rangle.$$

Let  $C_{kl} = \langle T \xi_k, \xi_l \rangle$ . Davenport and Hua showed in [2, 5] that for any fixed  $h > 0$ ,  $m \in \mathbb{N}^+$  and  $0 \leq l < m$ ,

$$(1) \quad \frac{1}{N} \sum_{\substack{n \leq N \\ n \equiv l \pmod{m}}} \mu(n) e^{2\pi i(a_d n^d + a_{d-1} n^{d-1} + \dots + a_1 n + a_0)} = O((\log N)^{-h}),$$

such that the implied constant depends only on  $h$  and  $p$ , but is independent of any of coefficients  $a_d, \dots, a_0$ . In particular,

$$\frac{1}{N} \sum_{n \leq N} \mu(n) e^{2\pi i n \theta} = O((\log N)^{-h})$$

uniformly in  $\theta$ . Therefore we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_n \mu(n) e^{im(\theta_k - \theta_l)} C_{kl} = \lim_{n \rightarrow \infty} \frac{C_{kl}}{n} \sum_n \mu(n) e^{im(\theta_k - \theta_l)} = 0$$

**Theorem 3.1.** *Suppose that  $\mathfrak{A}$  be a finite dimensional  $C^*$ -algebra. Let  $\rho$  be a continuous functional of  $\mathfrak{A}^*$  and  $\sigma$  be an automorphism of  $\mathfrak{A}$ . Then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_n \mu(n) \rho(\sigma^n(T)) = 0.$$

*Proof.* We could assume that  $\mathfrak{A}$  is a direct sum of finite matrix algebras and  $\rho$  is a vector state (otherwise we can always apply GNS construction to make  $\rho$  a vector state). Then the discussion above give the result.  $\square$

#### 4. VON NEUMANN ALGEBRA

Let  $\mathcal{M}$  be a von Neumann algebra acting on the Hilbert space  $\mathcal{H}$  and  $\sigma$  be a  $*$ -automorphism of  $\mathcal{M}$ . For any  $T \in \mathcal{M}$  and any  $\rho \in \mathcal{M}_*^+$ , we will study the following limit.

$$(2) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_n \mu(n) \rho(\sigma^n(T))$$

Without lose of generality, we could assume that  $\mathcal{M}$  is in standard form  $\|T\| = 1$  and  $\|\rho\| = 1$ . By Lemma 2.10 and Theorem 3.2 in [4], we may also assume that  $\rho(T) = \langle T \xi, \xi \rangle$  and  $\sigma(T) = U^* T U$ , where  $\xi$  is a unit vector in  $\mathcal{H}$  and  $U$  is a unitary in  $\mathcal{B}(\mathcal{H})$ . For any fixed  $N \in \mathbb{N}$  and any  $\varepsilon > 0$ , we can find a unitary  $V = \sum_{k=1}^n e^{i\theta_k} P_k$  such that  $\|V^m - U^m\| \leq \varepsilon$ , where  $P_k, k = 1 \dots n$ , are orthogonal projections and  $m \leq N$ . Then we

have

$$\begin{aligned}
\Delta(N) &= \left| \frac{1}{N} \sum_{m=1}^N \mu(m) \langle TU^m \xi, U^m \xi \rangle \right| \\
&\leq \left| \frac{1}{N} \sum_{m=1}^N \mu(m) \langle T(U^m - V^m) \xi, U^m \xi \rangle \right| + \left| \frac{1}{N} \sum_{m=1}^N \mu(m) \langle TV^m \xi, (U^m - V^m) \xi \rangle \right| \\
&\quad + \left| \frac{1}{N} \sum_{m=1}^N \mu(m) \langle TV^m \xi, V^m \xi \rangle \right| \\
&\leq \frac{2}{N} \sum_{m=1}^N \| (U^m - V^m) \xi \| + \left| \sum_{l,k=1}^n \left( \frac{1}{N} \sum_m \mu(m) e^{im(\theta_l - \theta_k)} \right) \langle TP_l \xi, P_k \xi \rangle \right| \\
&\leq 2\varepsilon + C \log(N)^{-h} \sum_{l,k=1}^n \|P_l \xi\|_2 \|P_k \xi\|_2
\end{aligned}$$

If  $\sum_{i=1}^n a_i^2 = 1$ , then we have that

$$\left( \sum_{i=1}^n a_i \right)^2 = \sum_{i=0}^{n-1} \left( \sum_{j=1}^n a_j a_{(j+i) \% n} \right) \leq \sum_{i=0}^{n-1} \left( \sum_{i=1}^n a_i^2 \right) = n$$

This seems lead to a dead end. Actually, the estimation above is quiet trivial.

However, if  $\mathfrak{A}$  is a finite von Neumann algebra and  $\omega$  a trace vector in  $L^2(\mathfrak{A}, \tau)$ , then there exists a  $A \in \mathfrak{A}$  such that  $\|A\omega - \xi\| \leq \varepsilon$ . Furthermore, we assume  $U$  is in  $\mathfrak{A}$ , therefore  $V$  can also be constructed in  $\mathfrak{A}$ .

$$\begin{aligned}
\Delta(N) &= \left| \frac{1}{N} \sum_{m=1}^N \mu(m) \langle TU^m \xi, U^m \xi \rangle \right| \\
&\leq \left| \frac{1}{N} \sum_{m=1}^N \mu(m) \langle TU^m (\xi - A\omega), U^m \xi \rangle \right| + \left| \frac{1}{N} \sum_{m=1}^N \mu(m) \langle TU^m A\omega, U^m (\xi - A\omega) \rangle \right| \\
&\quad + \left| \frac{1}{N} \sum_{m=1}^N \mu(m) \langle T(U^m - V^m) A\omega, U^m A\omega \rangle \right| + \left| \frac{1}{N} \sum_{m=1}^N \mu(m) \langle TV^m A\omega, (U^m - V^m) A\omega \rangle \right| \\
&\quad + \left| \frac{1}{N} \sum_{m=1}^N \mu(m) \langle TV^m A\omega, V^m A\omega \rangle \right| \\
&\leq 4\varepsilon(1 + \varepsilon)^2 + \left| \sum_{l,k=1}^n \left( \frac{1}{N} \sum_m \mu(m) e^{im(\theta_l - \theta_k)} \right) \langle TP_l A\omega, P_k A\omega \rangle \right| \\
&= 4\varepsilon(1 + \varepsilon)^2 + \left| \sum_{l,k=1}^n \left( \frac{1}{N} \sum_m \mu(m) e^{im(\theta_l - \theta_k)} \right) \tau(P_k TP_l AA^*) \right| \\
&\leq 4\varepsilon(1 + \varepsilon)^2 + \left| \frac{1}{N} \sum_m \mu(m) e^{im(\theta_l - \theta_k)} \right| \left( \sum_{l,k=1}^n \|P_k TP_l\|_2 \|P_l AA^* P_k\|_2 \right) \\
&\leq 4\varepsilon(1 + \varepsilon)^2 + \left| \frac{1}{N} \sum_m \mu(m) e^{im(\theta_l - \theta_k)} \right| \left( \sum_{l,k=1}^n \|P_k TP_l\|_2^2 \right)^{\frac{1}{2}} \left( \sum_{l,k=1}^n \|P_l AA^* P_k\|_2^2 \right)^{\frac{1}{2}} \\
&\leq 4\varepsilon(1 + \varepsilon)^2 + \left| \frac{1}{N} \sum_m \mu(m) e^{im(\theta_l - \theta_k)} \right| \|T\|_2 \|AA^*\|_2
\end{aligned}$$

Fix  $A$ , we have that  $\lim_{n \rightarrow \infty} \sup \Delta(n) \leq 4\varepsilon(1 + \varepsilon)^2$  for any  $\varepsilon$ . Then we have that  $\lim_{n \rightarrow \infty} \Delta(n) = 0$ .

## 5. COMPACT SET WITH FINITE ACCUMULATION POINTS

## 5.1. Topological Entropy.

**Lemma 5.1.** *Let  $X$  be a compact space and  $\sigma : X \rightarrow X$  a continuous map. If there exists  $n \in \mathbb{N}$  such that  $\sigma^{m_1}(x) = \sigma^{m_2}(x)$  for each  $x \in X$  where  $0 \leq m_1 < m_2 \leq n$ , then  $h_{top}(\sigma) = 0$ .*

*Proof.* By the hypothesis,  $\sigma^n(x)$  is a periodic point with period less than  $n$  for each  $x \in X$ . Let  $k = n!$ , we have  $\sigma^{n+k} = \sigma^n(x)$ . Since  $h_{top}(\sigma) = 0$  if and only if  $h_{top}(\sigma^n) = 0$ . We may assume that  $\sigma^k(x) = \sigma(x)$  for any  $x \in X$  where  $k > 1$ . Let  $U$  be an open set of  $X$ ,  $1 \leq m \leq k-1$  and  $l \geq 0$ ,

$$\sigma^{-(l(k-1)+m)}U = \{y : \sigma^{l(k-1)+m}(y) \in U\} = \{y : \sigma^m(y) \in U\} = \sigma^{-m}U.$$

Let  $\mathcal{U}$  be an open cover of  $X$ , we have  $\bigvee_{i=0}^m \sigma^{-i}\mathcal{U} = \bigvee_{i=0}^k \sigma^{-i}\mathcal{U}$  if  $m > k$ . Therefore

$$h_{cover}(\sigma, \mathcal{U}) = \lim_{m \rightarrow \infty} \frac{1}{m} \log N\left(\bigvee_{i=0}^{m-1} \sigma^{-i}\mathcal{U}\right) = 0.$$

Thus  $h_{top}(\sigma) = 0$ . □

Let  $X$  be a compact space with only one accumulation point  $c$ . If  $\sigma : X \rightarrow X$  is a continuous map such that  $\sigma(c) = y (\neq c)$ , then there exist a neighborhood  $U$  of  $c$  such that  $\sigma(U) = \{y\}$ . It is not hard to see that there exist  $n_0$  such that  $\sigma^{n_0}(y)$  is a periodic point of  $\sigma$ . Indeed, note that  $X \setminus U$  contains only finite many points ( $U$  is closed since it contains all accumulation points), then either there exists a  $k \in \mathbb{N}$  such that  $\sigma^k(y) \in U$  or there exists  $n_0$  such that  $\sigma^{n_0}(y)$  is a periodic point. The same argument also tells us that  $\sigma^m(x)$  is periodic for some  $m \in \mathbb{N}$  for each  $x \in X \setminus U$ . Since there are only finite many points in  $X \setminus U$ , there exists  $n \in \mathbb{N}$  such that  $\sigma^{m_1}(x) = \sigma^{m_2}(x)$  for each  $x \in X$  where  $0 \leq m_1 < m_2 \leq n$ . Hence  $h_{top}(\sigma) = 0$  by lemma 5.1.

**Lemma 5.2.** *Let  $X$  be a compact space with only one accumulation point  $c$ . If  $\sigma : X \rightarrow X$  is a continuous map such that  $\sigma(c) = y (\neq c)$ , then  $h_{top}(\sigma) = 0$ .*

If a compact Hausdorff space with only finite many accumulation points is metrizable, then it only contains countable many points. If  $X$  is a countable, compact metric space and  $\sigma : X \rightarrow X$  is continuous mapping, then by Proposition 5.1 in [8],  $h_{top}(\sigma) = 0$ . However, a compact Hausdorff space  $X$  may not be metrizable even if it only has one accumulation point. Indeed, let  $X = \mathbb{R} \cup \{\infty\}$  be the one point compactification of  $\mathbb{R}$  with discrete topology, it is clear that  $X$  is not second-countable. Therefore, it can not be metrizable since a compact Hausdorff space  $X$  is metrizable if and only if it is second-countable.

**Theorem 5.1.** *Let  $X$  be a compact Hausdorff space with finite many accumulation points  $\{c_1, \dots, c_l\}$ . If  $\sigma : X \rightarrow X$  is a continuous map, then  $h_{top}(\sigma) = 0$ .*

*Proof.* Suppose that  $\sigma(c_i) = q_i, i = 1, \dots, l$ . Let  $\mathcal{V}$  be a cover of  $X$ . There exists a cover  $\mathcal{U} = \{U_1, \dots, U_l, \{p_1\}, \dots, \{p_m\}\}$  such that  $U_i$  is a neighborhood of  $c_i$  and

- $q_i \notin U_j$  if  $q_i \neq c_j$ ,
- $U_i \cap U_j = \emptyset, i \neq j$ ,
- $\sigma(U_i) = \{q_i\}$  if  $q_i \notin \{c_1, \dots, c_l\}$ ,
- $\sigma(U_i) \cap U_j = \emptyset$  if  $q_i \neq U_j$ ,
- $U_i \subset V_i$  for some  $V_i \in \mathcal{V}$ ,
- $\{p_j\} \subset V_j$  for some  $V_j \in \mathcal{V}$ ,
- $X \setminus (\bigcup_{i=1}^l U_i) = \{p_1, \dots, p_m\}$ .

Thus  $h_{\text{cover}}(\sigma, \mathcal{V}) \leq h_{\text{cover}}(\sigma, \mathcal{U})$ . If we can show that  $h_{\text{cover}}(\sigma, \mathcal{U}) = 0$ , then we have  $h_{\text{top}}(\sigma) = 0$ .

Let

$$\alpha(U_i) = \begin{cases} U_j, & \text{if } \sigma(c_i) = c_j, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Note that the non empty set in  $\bigvee_{i=0}^{n-1} \sigma^{-i} \mathcal{U}$  can only be  $\{p_j\}$ ,

$$U_i \cap \sigma^{-1} \alpha(U_i) \cap \dots \cap \sigma^{-n+1} \alpha^{n-1}(U_i)$$

or

$$U_i \cap \sigma^{-1} \alpha(U_i) \cap \dots \cap \sigma^{-k+1} \alpha^{k-1}(U_i) \cap \sigma^{-k}(\{p_j\}) \cap \sigma^{-k-1} V_{k+1} \dots \cap \sigma^{1-n} V_{n-1},$$

where  $1 \leq k \leq n-1$ ,  $1 \leq i \leq l$ ,  $1 \leq j \leq m$  and  $V_j \in \mathcal{U}$  are uniquely determined by  $p_j$ . Therefore  $N(\bigvee_{i=0}^{n-1} \sigma^{-i} \mathcal{U}) \leq ml(n-1) + l + m$  and

$$h_{\text{cover}}(\sigma, \mathcal{U}) \leq \lim_{n \rightarrow \infty} \frac{1}{n} \log(ml(n-1) + l + m) = 0.$$

□

**5.2. Sarnak's conjecture is true for these spaces.** Throughout this section  $X$  is a compact space with only finite many accumulation points and  $\sigma : X \rightarrow X$  a continuous map.

It is easy to get the following two lemmas by eq. (1).

**Lemma 5.3.** *Let  $X$  be a compact space and  $\sigma$  a continuous map from  $X$  to  $X$ . If  $\{\sigma^n(x)\}$  has only one cluster point  $c \in X$ , then*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} \mu(n) f(\sigma^n(x)) = 0, \forall f \in C(X).$$

*Proof.* Let  $X' = \{\sigma^n(x)\}_{n=1}^\infty = \{\sigma^n(x)\} \cup \{c\}$ . Since  $\sigma(c)$  is a cluster point of  $\sigma(\{\sigma^n(x)\}) \subset \{\sigma^n(x)\}$ , we have  $\sigma(c) = c$ . Therefore  $\sigma(X') \subset X'$  and  $\sigma$  is a continuous map from  $X'$  to  $X'$ . Note that  $X'$  is compact. For any  $\varepsilon > 0$ , there exists an open neighborhood of  $c$  such that  $X' \setminus U$  contains only finite many elements and  $|f(y) - f(c)| < \varepsilon$  if  $y \in U$ . Now it is easy to see  $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} \mu(n) f(\sigma^n(x)) = 0$ . □

**Lemma 5.4.** *Let  $X$  be a compact space,  $f$  a function on  $X$  and  $\sigma$  a map from  $X$  to  $X$ . If  $x \in X$  and there is a  $n_0$  and  $n_1 \in \mathbb{N}$  such that  $\sigma^{n_0+n_1}(x) = \sigma^{n_0}(x)$  then*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} \mu(n) f(\sigma^n(x)) = 0.$$

**Example 5.1.** *Let  $X$  be a compact Hausdorff space with a unique accumulation point  $c_0$  and  $f$  a continuous function on  $X$ . By lemmas 5.3 and 5.4,  $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} \mu(n) f(\sigma^n(x)) = 0$ .*

**Lemma 5.5.** *Let  $X$  be a compact Hausdorff space and  $\sigma$  a continuous map from  $X$  to  $X$ . Let  $x$  be a element in  $X$  and  $c_1$  a cluster point of  $\{\sigma^n(x)\}_{n=1}^\infty$ . If  $\sigma(c_1) = y$  where  $y$  is an isolated point of  $X$ , then there exist  $n_1$  and  $n_2$  such that  $\sigma^{n_1+n_2}(x) = \sigma^{n_1}(x) = y$ , and  $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} \mu(n) f(\sigma^n(x)) = 0$  for any  $f \in C(X)$ .*

*Proof.* By the continuity of  $\sigma$  there is a neighborhood  $U$  of  $c_1$  such that  $\sigma(U) = \{y\}$ . Since  $c_1$  is a cluster point of  $\{\sigma^n(x)\}_{n=1}^\infty$  and  $X$  is Hausdorff, then there exist  $n_1$  and  $n_2$  such that  $\sigma^{n_1+n_2}(x) = \sigma^{n_1}(x) = y$ . By lemma 5.4,  $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} \mu(n) f(\sigma^n(x)) = 0$ . □



**Theorem 5.2.** *If  $X$  is a compact Hausdorff space with finite many limit points, then we have*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} \mu(n) f(\sigma^n(x)) = 0,$$

for any continuous map  $\sigma$  from  $X$  to  $X$  and any continuous function  $f$  on  $X$ .

*Proof.* Assume that  $\{c_1, \dots, c_m\}$  are the only limit points of  $X$ . By lemma 5.4, we could assume that  $\sigma^n(x) \neq \sigma^m(x)$ ,  $n \neq m$  and  $c_1$  is a cluster point of  $\{\sigma^n(x)\}_{n=1}^\infty$ . If  $\sigma(c_1) = y$  and  $y \notin \{c_1, \dots, c_m\}$ , then by lemma 5.5, we have the result. Therefore, we assume that  $\sigma(c_1) \in \{c_1, \dots, c_m\}$  and consider the following cases.

- (1) Suppose that  $\sigma(c_1) = c_1$ . Let  $U_i$  be the open neighborhood of  $c_i$  and such that  $U_i \cap U_j = \emptyset$ ,  $i \neq j$ . Since  $\sigma(c_1) = c_1$  and  $\sigma$  is continuous, there is neighborhood  $V_1$  of  $c_1$  such that  $V_1 \subset U_1$  and  $\sigma(V_1) \subset U_1$ . Note that  $U_1 \setminus V_1$  contains only finite many elements of  $X$ . Since all points in  $\{\sigma^n(x)\}$  are different and  $c_1$  is a cluster point of  $\{\sigma^n(x)\}$ , there exist a  $N$  such that  $\sigma^m(x) \in V_1$  if  $m \geq N$ . Therefore,  $c_1$  is the only cluster point of  $\{\sigma^n(x)\}_{n=1}^\infty$ . By lemma 5.3,  $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} \mu(n) f(\sigma^n(x)) = 0$ .

We restate the above result as the following lemma.

**Lemma 5.6.** *Let  $X$  be a compact Hausdorff space with finite many limit points and  $\sigma$  a continuous map from  $X$  to  $X$ . If  $c$  is a cluster point of  $\{\sigma^n(x)\}_{n=1}^\infty$  and  $\sigma(c) = c$ , then  $c$  is the only cluster point of  $\{\sigma^n(x)\}_{n=1}^\infty$ .*

Now we consider the other case.

- (2) Suppose that  $\sigma(c_1) = c_2$ . Since  $c_1$  is a cluster point of  $\{\sigma^n(x)\}$ ,  $c_2$  must also be a cluster point of  $\{\sigma^n(x)\}$ . Hence  $\sigma(c_2) \neq c_2$ . By an inductive argument as above, we could assume that  $\sigma(c_1) = c_2, \dots, \sigma(c_k) = c_1$ . For any  $f \in C(X)$  and  $\varepsilon > 0$ , there exists neighborhoods  $U_i$  and  $V_i$  of  $c_i$ ,  $i = 1, \dots, m$  such that
- $U_i \cap U_j = \emptyset$ ,  $i \neq j$ .
  - $V_i \subset U_i$ ,
  - $\sigma(V_i) \subset U_{i+1}$ ,  $i = 1, \dots, k-1$ , and  $\sigma(V_k) \subset U_1$ .
  - $|f(x) - f(c_i)| \leq \varepsilon$  if  $x \in U_i$ .

Since  $\cup_{i=1}^k (U_i \setminus V_i)$  contains only finite many points, there exists  $k \in \mathbb{N}$  such that  $\sigma^{n+1}(x) \in V_{i+1}$  whenever  $n \geq k$  and  $\sigma^n(x) \in V_i$  for  $i = 1, \dots, k-1$  and  $\sigma^{n+1}(x) \in V_1$  if  $\sigma^n(x) \in V_k$ . Therefore,  $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} \mu(n) f(\sigma^n(x)) = 0$ .  $\square$

**Corollary 5.1.** *Let  $X$  be a compact Hausdorff space and  $f$  a continuous function on  $X$ . If the sequence  $\{\sigma^n(x)\}$  has only finite many limit points, then*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} \mu(n) f(\sigma^n(x)) = 0.$$

*Proof.* Let  $X' = \{\sigma^n(x)\}^-$ . Then  $X'$  is compact. If  $y \in X'$  is a limit point, then  $\sigma(y)$  is also a cluster point of  $\{\sigma^{n_i}(x)\}$ . Therefore  $\sigma(X') \subset X'$ . Now, apply theorem 5.2, we have the result.  $\square$

## 6. COMPACT SPACE WITH COUNTABLY MANY POINTS

Let  $X$  be a countable, compact Hausdorff space. Since a compact Hausdorff space is metrizable if and only if it is second-countable,  $X$  is metrizable. Hence, throughout this section  $X$  denotes a countable *compactum*, i.e. a compact metric space has countable

cardinality. Therefore, by Proposition 5.1 in [8],  $h_{top}(\sigma) = 0$  for any continuous map from  $X$  to  $X$ .

Argue as in the proof of corollary 5.1, we assume in this section that  $\{\sigma^n(x)\}$  is dense in  $X$ .

Since  $X$  only contains countable many points,  $X$  must contains isolated point by Baire category theorem. Recall that a topological space is totally disconnected if the connected components in the space are the one-point sets. Let  $Y$  be a connected component of  $X$ .  $Y$  is closed since connected component is closed. Thus  $Y$  is also a complete metric space with only countable many points. Hence  $Y$  contain isolated points. Therefore  $Y$  can only contain one point and  $X$  is totally disconnected and zero-dimensional. Since every compact totally disconnected metric space is homeomorphic to a subset of the Cantor set (Corollary 2-99 [1]), we could assume that  $X$  is a closed subset of the Cantor set.

For a metric space  $X$ , let  $\iota(X)$  be the set of all isolated points of  $X$  and  $X^d = X \setminus \iota(X)$  the set of limit points of  $X$ . The derived set of  $X$  of order  $\alpha$  is defined by  $X^{(1)} = X^d$ ,  $X^{(\alpha+1)} = (X^{(\alpha)})^d$  and  $X^{(\lambda)} = \bigcap_{\alpha < \lambda} X^{(\alpha)}$  if  $\lambda$  is a limit ordinal number. If  $X^{(\alpha)} \neq \emptyset$  and  $X^{(\alpha+1)} = \emptyset$ , then we put  $d(X) = \alpha$ . And  $d(X)$  is called the *derived degree* of  $X$ . It is well known that a compactum  $X$  is a countable set if and only if  $d(X)$  exists and it is a countable ordinal number. In this case, if  $d(X) = \alpha$ , then  $X^{(\alpha)}$  is a finite set.

**Proposition 6.1** (Proposition 2.1 of [7]). *Let  $X$  and  $Y$  be countable compacta. If  $d(X) = d(Y) = \alpha$  and  $X^{(\alpha)}$  is homeomorphic to  $Y^{(\alpha)}$ , then  $X$  is homeomorphic to  $Y$ .*

**Lemma 6.1.** *Let  $X$  be a countably infinite, compact metric space and  $x$  a isolated point in  $X$ . If  $\{\sigma^n(x)\}_{n=0}^{\infty}$  is dense in  $X$ , then  $\sigma^n(x)$  is isolated for all  $n$  and  $\sigma^n(x)$  are all the isolated points in  $X$ .*

*Proof.* Since  $X$  is a countable, compact metric space contains infinitely many points,  $X$  contains infinite isolated points. It is easy to see that  $\{\sigma^n(x)\}$  must contains all isolated points. Otherwise,  $\{\sigma^n(x)\}$  can not be dense in  $X$ . Assume that there is a  $n$  such that  $\sigma^n(x)$  is a limit point in  $X$ . Let  $m \in \mathbb{N}$  be a number such that  $\sigma^m(x)$  is a limit point and  $\sigma^{m+1}(x)$  is an isolated point. By lemma 5.5, there is  $n_1$  and  $n_2$  such that  $\sigma^{n_1+n_2}(x) = \sigma^{n_1}(x)$ . Therefore  $\{\sigma^n(x)\}$  contains only finite many different elements. It contradicts the fact that  $\{\sigma^n(x)\}$  contains infinite many isolated point.  $\square$

**Lemma 6.2.** *Let  $X$  be a countably infinite, compact metric space and  $x$  a isolated point in  $X$ . If  $\{\sigma^n(x)\}_{n=0}^{\infty}$  is dense in  $X$  and  $X^d$  contains infinite many points, then  $y$  is not periodic for any isolated point in  $y$  in  $X^d$ .*

*Proof.* Suppose that there exist  $n \geq 1$  such that  $\sigma^n(y) = y$ . Let  $\{U_i\}_{i=0}^{n-1}$  be neighborhoods of  $\sigma^i(y)$  such that  $(\bigcup_{i=0}^{n-1} U_i \cap X^d)^- \neq X^d$ ,  $U_0 \cap X^d = \{y\}$ ,  $\sigma(U_i) \subset U_{i+1}$  for  $i = 1, \dots, n-2$  and  $\sigma(U_{n-1}) \subset U_0$ . We can also find a neighborhood  $V_0$  of  $y$  such that  $V_0 \subset U_0$  and  $\sigma(V_0) \subset U_1$ . Since  $U_0$  do not contains limit point except  $y$ , we have  $U_0 \setminus V_0$  contains finite many points. Therefore, there exist a  $m \in \mathbb{N}$  such that  $\sigma^k \notin U_0 \setminus V_0$  for any  $k \geq m$ , it is easy to see that all limit points of  $\{\sigma^n(x)\}$  must contains in  $(\bigcup_{i=0}^{n-1} U_i \cap X^d)^- \subsetneq X^d$ . It is a contradiction.  $\square$

**Example 6.1.** *Let*

$$X = \bigcup_{n=1}^{\infty} \left( \left\{ \left( \frac{1}{n}, \frac{1}{n+k} \right) : k = 0, 1, 2, \dots \right\} \cup \left\{ \left( \frac{1}{n}, 0 \right) \right\} \right) \cup \{(0, 0)\} \subset \mathbb{R}^2.$$

Then  $X^{(1)} = \{(\frac{1}{n}, 0) : n = 1, 2, \dots\} \cup \{(0, 0)\}$  and  $X^{(2)} = \{(0, 0)\}$ . Let  $\sigma$  be a map  $X \rightarrow X$  defined by

$$\begin{aligned} (1, 0) &\rightarrow (0, 0) \text{ and } (0, 0) \rightarrow (0, 0) \\ (\frac{1}{n}, 0) &\rightarrow (\frac{1}{n-1}, 0) \quad (\text{for } n > 1) \\ (1, \frac{1}{k+1}) &\rightarrow (\frac{1}{k+2}, \frac{1}{k+2}) \\ (\frac{1}{n}, \frac{1}{n+k}) &\rightarrow (\frac{1}{n-1}, \frac{1}{n+k}) \quad (\text{for } n > 1). \end{aligned}$$

It is easy to see that  $\{\sigma^n((1, 1))\} = X \setminus X^{(1)}$ .

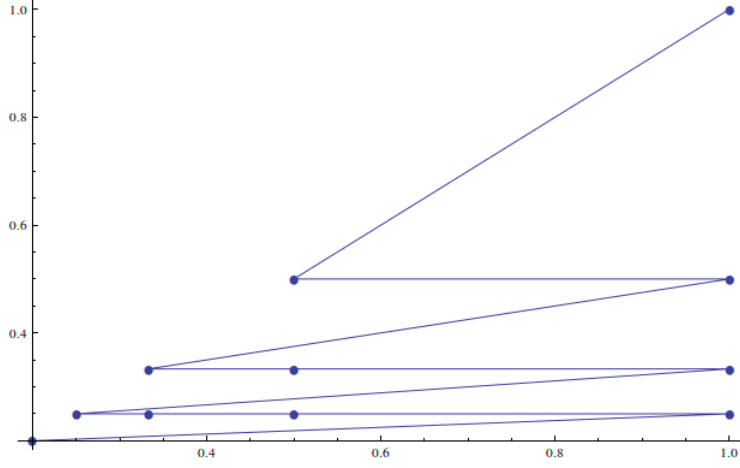


Figure 1. example 6.1

**Lemma 6.3.** Let  $X$  be a countably infinite, compact metric space such that  $d(X) = 2$  and  $X^{(2)} = \{c\}$ . If  $\{\sigma^n(x)\}_{n=0}^\infty$  is dense in  $X$  for a isolated point  $x$  in  $X$ , then  $\sigma(c) = c$ .

*Proof.* Suppose that  $\sigma(c) = y \in X^{(1)}$ . If  $y \neq c$ , then  $y$  is an isolated point in  $X^{(1)}$ . There exist a neighborhood  $U$  of  $c$  such that  $\sigma(U) = \{y\}$  and  $X^{(1)} \setminus U$  contains only finite many points. By lemma 6.2, there exist a  $n$  such that  $\sigma^n(y) \in U$ , since no isolated point in  $X^{(1)}$  is periodic. However we have  $\sigma^{n+1}(y) = y$ . It is a contradiction.  $\square$

Let  $(X, \sigma)$  be a dynamic system,  $A$  a subset of  $X$  and  $x$  a point in  $X$ . Let  $A(x) = \{m : \sigma^m(x) \in A, \}$ . Recall that the upper asymptotic density  $\bar{d}(A(x))$  is  $\limsup_{n \rightarrow \infty} \frac{|A(x, n)|}{n}$  and lower asymptotic density  $\underline{d}(A(x))$  is  $\liminf_{n \rightarrow \infty} \frac{|A(x, n)|}{n}$ , where  $A(x, n) = \{1, 2, \dots, n\} \cap A(x)$ .  $A(x)$  has asymptotic density  $\text{den}(A(x))$  if  $\bar{d}(A(x)) = \underline{d}(A(x))$ , in which case  $\text{den}(A(x))$  is equal to this common value.

**Lemma 6.4.** Let  $X$  be a countably infinite, compact metric space such that  $d(X) \geq 2$ . Suppose that  $\{\sigma^n(x)\}_{n=0}^\infty$  is dense in  $X$  for a isolated point  $x$  in  $X$ . Let  $A = \{\sigma^{n_k}(x)\}$  be a subsequence converge to  $y$ . If  $y$  is not a periodic point, then  $\text{den}(A(x)) = 0$ .

*Proof.* We consider the following two cases.

- (1)  $\sigma^{m_1}(y) \neq \sigma^{m_2}(y)$  if  $m_1 \neq m_2$ . Hence  $\lim_{m \rightarrow \infty} \sigma^m(y) = c$ . Fix a positive integer  $k$ . We choose  $k+1$  neighborhoods  $\{U_i\}_{i=0}^k$  of  $\sigma^i(y)$  such that  $U_i \cap U_j = \emptyset$  and  $\sigma(U_i) \subset U_{i+1}$ . Note that  $A \setminus U_0$  contains finite many points. Therefore there exists a  $l \in \mathbb{N}$  such that  $\{\sigma^n(x) : n \geq l\}$  do not contains any element in  $A \setminus U_0$ . If  $n \geq l$  and  $n \in A(x)$ , we have  $n+1, \dots, n+k$  are not in  $A$ . It is easy to see that  $\bar{d}(A(x)) \leq \frac{1}{k}$ . Hence  $\text{den}(A(x)) = 0$ .
- (2)  $\sigma^m(y) = c_0$  and  $c_0$  is a periodic point such that  $\sigma^j(c_0) = c_j$  and  $\sigma^n(c_0) = c_0$ . Fix a positive integer  $k$ . We choose  $m$  neighborhoods  $U_i$  of  $\sigma^i(y)$  for  $i = 0, \dots, m-1$  and  $k$  neighborhoods  $\{U_{m+nl+j}\}_{l=0}^{k-1}$  of  $c_j$ ,  $j = 0, \dots, n-1$  such that
- $U_{m+nl+j} \subset U_{m+n(l+1)+j}$ ,  $l = 0, \dots, k-1$ ,
  - $U_{i_1} \cap U_{i_2} = \emptyset$  for  $i_1 \neq i_2$  and  $i_1, i_2 \in \{0, \dots, m-1\}$ ,
  - $U_i \cap U_{m+nk+j} = \emptyset$  for  $i = 0, \dots, m-1$  and  $j = 0, \dots, n-1$ ,
  - $U_{m+nk+j_1} \cap U_{m+nk+j_2} = \emptyset$  for  $j_1 \neq j_2$ ,
  - $\sigma(U_i) \subset U_{i+1}$  for  $i = 0, 1, \dots, m+nk-1$ .

Note that  $A \setminus U_0$  contains finite many points. Therefore there exists a  $N_0 \in \mathbb{N}$  such that  $\{\sigma^n(x) : n \geq N_0\}$  do not contains any element in  $A \setminus U_0$ . If  $s \geq N_0$  and  $s \in A(x)$ , we have  $s+1, \dots, s+nk+m-1$  are not in  $A$ . It is easy to see that  $\bar{d}(A(x)) \leq \frac{1}{nk+m-1}$ . Hence  $\text{den}(A(x)) = 0$

□

**Corollary 6.1.** *Let  $X$  be a countably infinite, compact metric space such that  $d(X) = 2$  and  $X^{(2)} = \{c\}$ . Suppose that  $\{\sigma^n(x)\}_{n=0}^\infty$  is dense in  $X$  for a isolated point  $x$  in  $X$ . Let  $B$  be any neighborhood of  $c$  and  $A = X \setminus B$ . Then  $\text{den}(A(x)) = 0$ .*

*Proof.* Note that  $A \cap X^{(1)}$  contains finite elements  $\{c_1, \dots, c_m\}$ . Since  $A$  is closed,  $A = \bigcup_{k=1}^m A_k$ , where  $A_k$  is a subsequence of  $\{\sigma^n(x)\}$  converges to  $c_k$ . Each  $c_k$  is not periodic by lemma 6.2. Then  $\text{den}(A_k(x)) = 0$  by lemma 6.4,. Therefore  $\text{den}(A(x)) = 0$ . □

**Corollary 6.2.** *Let  $(X, \sigma)$  be a dynamic system and  $x \in X$ . Let  $Y$  be the closure of  $\{\sigma^n(x)\}_{n=0}^\infty$ . If  $d(Y) = 2$  and  $Y^{(2)} = \{c\}$ , then  $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} \mu(n) f(\sigma^n(x)) = 0$  for any  $f \in C(X)$ .*

*Proof.* We could assume that  $f(c) = 0$ . For any  $\varepsilon > 0$ , there is neighborhood  $U$  of  $c$  such that  $|f(y)| < \varepsilon$  for any  $y \in U$ . By corollary 6.1,  $\text{den}(A(x)) = 0$  where  $A = \{\sigma^n(x) : \sigma^n(x) \text{ not in } U\}$ . This implies the result. □

**Example 6.2.** *Let*

$$X_1 = \bigcup_{n=1}^{\infty} \left( \left\{ \left( 2 + \frac{1}{n}, \frac{1}{n+k} \right) : k = 0, 1, 2, \dots \right\} \cup \left\{ \left( 2 + \frac{1}{n}, 0 \right) \right\} \right) \cup \{(2, 0)\},$$

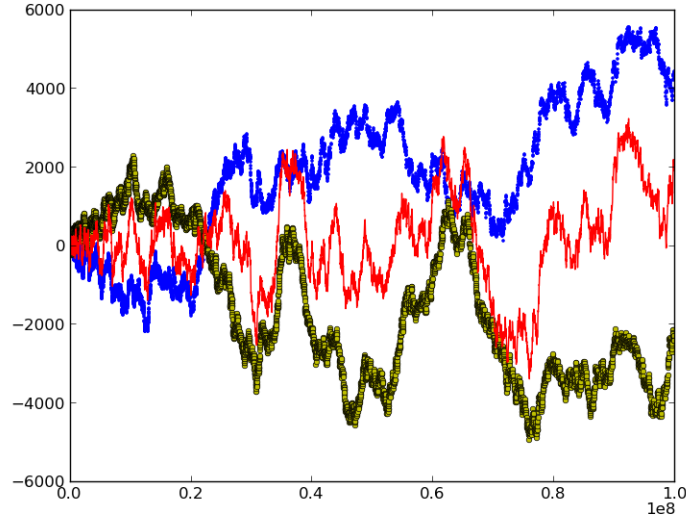
$$X_2 = \bigcup_{n=2}^{\infty} \left( \left\{ \left( \frac{1}{n}, \frac{1}{n+k} \right) : k = 0, 1, 2, \dots \right\} \cup \left\{ \left( \frac{1}{n}, 0 \right) \right\} \right) \cup \left\{ \left( 1, \frac{1}{2+k} \right) : k = 0, 1, 2, \dots \right\} \cup \left\{ \left( \frac{1}{n}, 0 \right) \right\} \cup \{(0, 0)\},$$

$X = X_1 \cup X_2$ . Then  $X^{(1)} = \{(\frac{1}{n}, 0) : n = 1, 2, \dots\} \cup \{(0, 0)\} \cup \{(2 + \frac{1}{n}, 0) : n = 1, 2, \dots\} \cup \{(2, 0)\}$  and  $X^{(2)} = \{(0, 0)\} \cup \{(2, 0)\}$ . Let  $\sigma$  be a map  $X \rightarrow X$  defined by

$$\begin{aligned} (3, 0) &\rightarrow (0, 0) \text{ and } (2, 0) \rightarrow (2, 0) \\ (1, 0) &\rightarrow (2, 0) \text{ and } (0, 0) \rightarrow (0, 0) \\ (2 + \frac{1}{n}, 0) &\rightarrow (2 + \frac{1}{n-1}, 0) \quad (\text{for } n > 1) \\ (\frac{1}{n}, 0) &\rightarrow (\frac{1}{n-1}, 0) \quad (\text{for } n > 1) \\ (3, \frac{1}{k+1}) &\rightarrow (\frac{1}{k+2}, \frac{1}{k+2}) \\ (1, \frac{1}{2+k}) &\rightarrow (2 + \frac{1}{k+2}, \frac{1}{k+2}) \\ (2 + \frac{1}{n}, \frac{1}{n+k}) &\rightarrow (2 + \frac{1}{n-1}, \frac{1}{n+k}) \quad (\text{for } n > 1). \\ (\frac{1}{n}, \frac{1}{n+k}) &\rightarrow (\frac{1}{n-1}, \frac{1}{n+k}) \quad (\text{for } n > 1). \end{aligned}$$

It is easy to see that  $\{\sigma^n((3, 1))\} = X \setminus X^{(1)}$ . Let

$$\mathcal{X}(n) = \begin{cases} 1, & n \in [m^2 - m + 1, m^2] \text{ for } m = 1, 2, \dots \\ 0, & n \in [m^2 + 1, m(m+1)] \text{ for } m = 1, 2, \dots \end{cases}$$

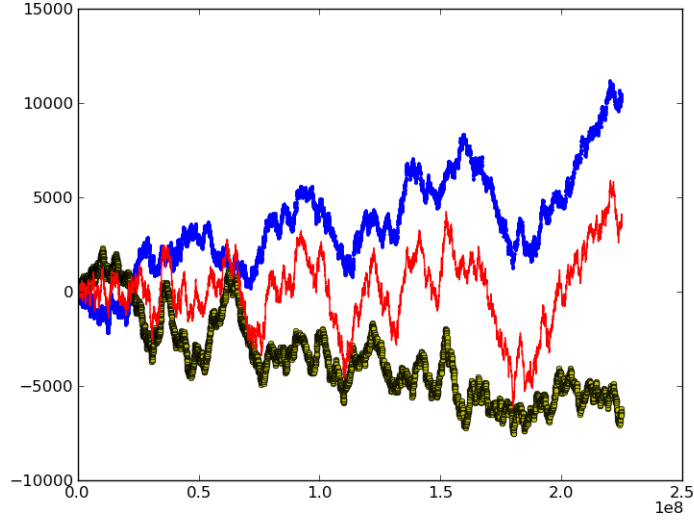


**Figure 2.** example 6.2, up to 100000000

Blue:  $\sum_n^t \mu(n)\mathcal{X}(n)$ , max:5561, min:-2197. Yellow:  $\sum_n^t \mu(n)(1 - \mathcal{X}(n))$ , max:2278, min:-4954. Red:  $\sum_n^t \mu(n)$ , max:3225, min:-3402.

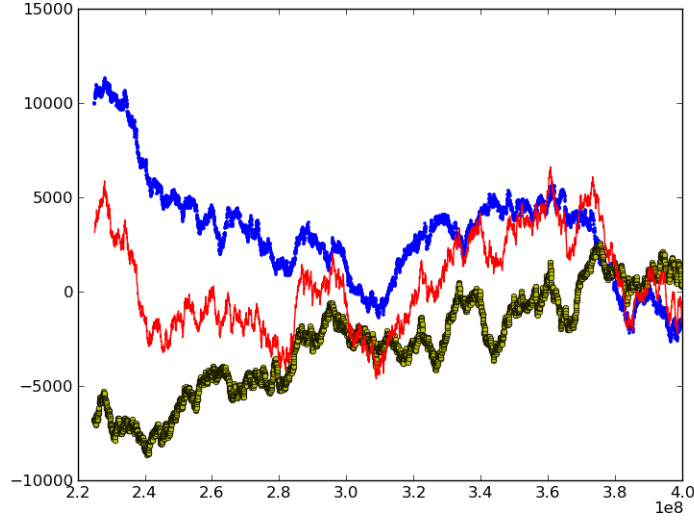
In number theory, the Mertens function  $M(x)$  is defined as

$$M(x) = \sum_{k \leq x} \mu(k).$$



**Figure 3.** example 6.2, up to 225000000

Blue:  $\sum_n^t \mu(n)\mathcal{X}(n)$ , max:11175, min:-2197. Yellow:  $\sum_n^t \mu(n)(1 - \mathcal{X}(n))$ , max:2278, min:-7547. Red:  $\sum_n^t \mu(n)$ , max:5890, min:-6136.



**Figure 4.** example 6.2, up to 400000000

Blue:  $\sum_n^t \mu(n)\mathcal{X}(n)$ , max:11319, min:-2663. Yellow:  $\sum_n^t \mu(n)(1 - \mathcal{X}(n))$ , max:2527, min:-8688. Red:  $\sum_n^t \mu(n)$ , max:6602, min:-4645.

By Theorem 333 in [6], the probabilit that a number should be squarefree is  $\frac{6}{\pi^2}$ , more precisely

$$Q(x) = \sum_{n \leq x} |\mu(n)| = \frac{6x}{\pi^2} + O(\sqrt{x}).$$

**Lemma 6.5.**

$$\sum_{n \leq x} \mathcal{X}(n) \log n = \frac{x}{2} \log x + O(x).$$

*Proof.* Let  $x = m^2 + m + l$  where  $0 \leq l < 2(m+1)$ . Note

$$\begin{aligned} \sum_{n \leq x} (1 - \mathcal{X}(n)) \log n - \mathcal{X}(n) \log n &= \sum_{k=1}^m \sum_{i=1}^k \log\left(\frac{k^2+i}{k^2-k+i}\right) + \sum_{j=m^2+m+1}^{m^2+m+l} \log(j) \\ &= \sum_{k=1}^m \sum_{i=1}^k O\left(\frac{1}{k}\right) + O(\sqrt{x} \log x) = O(\sqrt{x} \log x). \end{aligned}$$

By Theorem 423 in [6],

$$\sum_{n \leq x} \log n = \sum_{n \leq x} (1 - \mathcal{X}(n)) \log n + \mathcal{X}(n) \log n = x \log x + O(x).$$

Therefore

$$\sum_{n \leq x} \mathcal{X}(n) \log n = \frac{x}{2} \log x + O(x).$$

□

**Lemma 6.6.** Let  $S_l = \bigcup_{m=1}^{\infty} \{l \times m^2 + t : t \in [0, lm]\}$ . If  $j/i = k^2 > 1$ , then

$$\lim_{x \rightarrow \infty} \frac{\#\{n \leq x : n \in S_i \cap S_j\}}{x} = \begin{cases} \frac{k+1}{4k} & \text{if } k \text{ is odd} \\ \frac{1}{4} & \text{if } k \text{ is even} \end{cases}.$$

*Proof.* Assume that  $x = ka$  and  $y = a$ . Then  $ix^2 = ik^2a^2 = ja^2 = jy^2$ . Consider the interval

$$s_y = [jy^2, j(y+1)^2 - 1] = [jy^2, jy^2 + jy] \cup [jy^2 + jy + 1, j(y+1)^2 - 1] = s_y^1 + s_y^2.$$

Note that  $[jy^2, j(y+1)^2 - 1]$  contains  $k$  intervals

$$[ix^2, ix^2 + ix], [i(x+1)^2, i(x+1)^2 + i(x+1)], \dots, [i(x+k-1)^2, i(x+k-1)^2 + i(x+k-1)].$$

If  $k$  is even, then

$$[ix^2, ix^2 + ix], [i(x+1)^2, i(x+1)^2 + i(x+1)], \dots, [i(x + \frac{k}{2} - 1)^2, i(x + \frac{k}{2} - 1)^2 + i(x + \frac{k}{2} - 1)]$$

are in  $s_y^1$  for  $x$  large enough, since  $i(x + \frac{k}{2} - \frac{k^2}{4}) > 0$ . And

$$[i(x + \frac{k}{2})^2, i(x + \frac{k}{2})^2 + i(x + \frac{k}{2})], \dots, [i(x+k-1)^2, i(x+k-1)^2 + i(x+k-1)]$$

are in  $s_y^2$ . This implies that  $\lim_{x \rightarrow \infty} \frac{\#\{n \leq x : n \in S_i \cap S_j\}}{x} = \frac{1}{4}$ .

If  $k$  is odd, then

$$i(x + \frac{k-1}{2})^2 + i(x + \frac{k-1}{2}) - jy^2 - jy = \frac{i(k^2-1)}{4}.$$

Therefore  $s_y^1$  contains almost  $\frac{k+1}{2}$  intervals. It is clear implies the result. □

**Lemma 6.7.** Suppose that  $p, q$  are two coprime positive integers, i.e.  $(p, q) = 1$ . Let

$$S_1 = \bigcup_{k=1}^p \left[ \frac{(2k-2)}{2p}, \frac{(2k-1)}{2p} \right] \text{ and } S_2 = \bigcup_{k=1}^q \left[ \frac{(2k-2)}{2q}, \frac{(2k-1)}{2q} \right].$$

$$\text{len}(S_1 \cap S_2) = \begin{cases} \frac{1}{4} & \text{if } p \text{ or } q \text{ is even,} \\ \frac{1}{4} + \frac{4}{15pq} & \text{otherwise.} \end{cases}$$

*Proof.* Let  $\mathcal{X}_1$  and  $\mathcal{X}_2$  be the characteristic function of  $S_1$  and  $S_2$  respectively.

$$\begin{aligned}\langle \mathcal{X}_1, e^{2\pi i n \theta} \rangle &= \sum_{k=1}^p \int_{\frac{2k-2}{2p}}^{\frac{2k-1}{2p}} e^{-2\pi i n \theta} \\ &= \frac{1 - e^{-\frac{\pi i n}{p}}}{2\pi i n} \sum_{k=0}^{p-1} e^{-\frac{2\pi i n k}{p}} = \begin{cases} \frac{1}{2}, & n = 0 \\ \frac{p}{\pi i n}, & \text{if } n = p(2a+1), a \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases}.\end{aligned}$$

Similiarly,

$$\begin{aligned}\langle \mathcal{X}_2, e^{2\pi i n \theta} \rangle &= \sum_{k=1}^q \int_{\frac{2k-2}{2q}}^{\frac{2k-1}{2q}} e^{-2\pi i n \theta} \\ &= \frac{1 - e^{-\frac{\pi i n}{q}}}{2\pi i n} \sum_{k=0}^{q-1} e^{-\frac{2\pi i n k}{q}} = \begin{cases} \frac{1}{2}, & n = 0 \\ \frac{q}{\pi i n}, & \text{if } n = q(2a+1), a \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases}\end{aligned}$$

If  $p$  or  $q$  is even, then  $\langle \mathcal{X}_1, \mathcal{X}_2 \rangle = \frac{1}{4}$ . If both  $p$  and  $q$  are odd, then

$$\begin{aligned}\text{len}(S_1 \cap S_2) &= \langle \mathcal{X}_1, \mathcal{X}_2 \rangle \\ &= \frac{1}{4} + \frac{1}{\pi^2 p q} \sum_{k \in \mathbb{Z}} \frac{1}{(2k+1)^2} \\ &= \frac{1}{4} + \frac{8}{5\pi^2 p q} \zeta(2) = \frac{1}{4} + \frac{4}{15 p q} \quad (\zeta(2) = \frac{\pi^2}{6}).\end{aligned}$$

□

**Theorem 6.1** (Dirichlet, c. 1840). *For any real number  $\theta$  and any integer  $N \geq 1$ , there exist integers  $p$  and  $q$  such that  $1 \leq q \leq N$  and*

$$|q\theta - p| \leq \frac{1}{N+1}.$$

**Lemma 6.8.** *Let  $S_l = \bigcup_{m=1}^{\infty} \{l \times m^2 + t : t \in [0, lm]\}$ . If  $\sqrt{\frac{l}{j}} \in \mathbb{R} \setminus \mathbb{Q}$ , then*

$$\lim_{x \rightarrow \infty} \frac{\#\{n \leq x : n \in S_l \cap S_j\}}{x} = \frac{1}{4}.$$

*Proof.* We assume that  $\sqrt{\frac{l}{j}} < 1$  and  $|\sqrt{\frac{l}{j}}q - p| \leq \frac{1}{q}$ . Since

$$\lim_{m \rightarrow \infty} \frac{iq^2 l^2}{iq^2(l+1)^2} = 1,$$

we only need to consider the case for  $x = iq^2 l^2$ , where  $l \in \mathbb{N}$ . The interval  $[iq^2 l^2, iq^2(l+1)^2]$  contains  $q$  intervals in  $S_1$ :

$$[i(ql)^2, i((ql)^2 + ql)], \dots, [i(ql + q - 1)^2, i(ql + q)(ql + q - 1)].$$

Let

$$\begin{aligned}S_{1l} &= \bigcup_{k=1}^q [i(ql + k - 1)^2, i(ql + k)(ql + k - 1)] \\ S'_{1l} &= \bigcup_{k=1}^q [i(ql)^2 + (2k-2)iq l, i(ql)^2 + (2k-1)iq l].\end{aligned}$$



Note that

$$\lim_{l \rightarrow 0} \frac{\#(S_{1l} \cap S'_{1l})}{iq^2 l} = 1.$$

If  $iq^2 l^2 \leq jy^2 \leq iq^2(l+1)^2$ , then

$$\sqrt{\frac{i}{j}}ql \leq y \leq \sqrt{\frac{i}{j}}q(l+1).$$

Since

$$p - \frac{1}{q} \leq \sqrt{\frac{i}{j}}q \leq p + \frac{1}{q},$$

$[iq^2 l^2, iq^2(l+1)^2]$  contains at least  $p-3$  and at most  $p+1$  intervals in  $S_2$ . Let  $y_0$  be the first  $y$  such that  $iq^2 l^2 \leq jy^2 \leq iq^2(l+1)^2$ . Let

$$S_{2l} = \bigcup_{k=1}^{p-3} [j(y_0 + k - 1)^2, j(y_0 + k - 1)^2 + j(y_0 + k - 1)] \subset S_2 \cap [iq^2 l^2, iq^2(l+1)^2].$$

Note that

$$\liminf_{l \rightarrow \infty} \frac{\#S_{2l}}{\#(S_2 \cap [iq^2 l^2, iq^2(l+1)^2])} \geq 1 - \frac{5}{p-3}.$$

Let

$$S'_{2l} = \bigcup_{k=1}^{p-3} [jy_0^2 + (2k-2)jy_0, jy_0^2 + (2k-1)jy_0].$$

Also we have

$$\lim_{l \rightarrow \infty} \frac{\#(S_{2l} \cap S'_{2l})}{iq^2 l} = 1.$$

We would like to compare  $S'_{1l}$  with  $S'_{2l}$ . Since  $y_0 \leq \sqrt{\frac{i}{j}}ql + 1$ , we have

$$\begin{aligned} \Delta &= i(ql)^2 + 2iq^2 l - jy_0^2 + (2p-7)jy_0 \\ &\geq i(ql)^2 + 2iq^2 l - j(\sqrt{\frac{i}{j}}ql + 1)^2 + (2p-7)j(\sqrt{\frac{i}{j}}ql + 1) \\ &= 2iq^2 l(1 - (\frac{p}{q} - \frac{5}{2q})\sqrt{\frac{j}{i}} - \frac{(2p-6)j}{2iq^2 l}) \\ &\geq 2iq^2 l(1 - (\sqrt{\frac{i}{j}} + \frac{1}{q^2} - \frac{5}{2q})\sqrt{\frac{j}{i}} - \frac{(2p-6)j}{2iq^2 l}) \quad (\frac{p}{q} \leq \sqrt{\frac{i}{j}} + \frac{1}{q^2}) \\ &= 2iq^2 l((\frac{5}{2q} - \frac{1}{q^2})\sqrt{\frac{j}{i}} - \frac{(2p-6)j}{2iq^2 l}) \geq 0, \end{aligned}$$

provide that  $l$  is large enough.

This means that  $S'_{2l}$  is contains in  $[i(ql)^2, i(ql)^2 + 2iq^2 l]$ . A similar caculation as in the proof of lemma 6.8 shows that

$$\lim_{l \rightarrow \infty} \frac{\#(S'_{1l} \cap S'_{2l})}{2iq^2 l} = \frac{1}{4} + O(\frac{1}{pq}).$$

Therefore

$$\lim_{l \rightarrow \infty} \frac{\#(S_{1l} \cap S_{2l})}{2iq^2 l} = \frac{1}{4} + O(\frac{1}{p}).$$

This implies that

$$\lim_{x \rightarrow \infty} \frac{\#\{n \leq x : n \in S_i \cap S_j\}}{x} = \frac{1}{4} + O\left(\frac{1}{p}\right).$$

By increasing  $p$  and  $q$ , the result is proved.  $\square$

**Lemma 6.9.** Let  $S_l = \bigcup_{m=1}^{\infty} \{l \times m^2 + l \times i : i \in [0, m]\}$ . If  $\sqrt{\frac{i}{j}} \in \mathbb{R} \setminus \mathbb{Q}$ , then

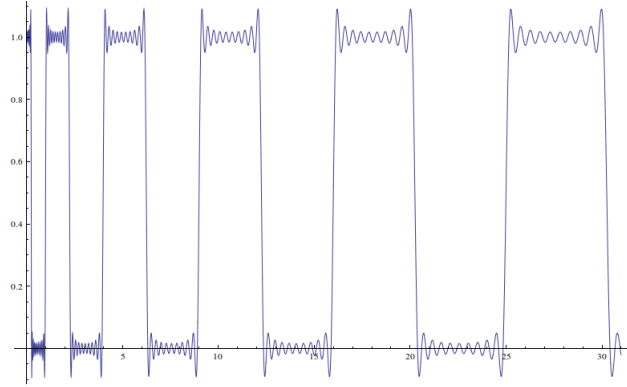
$$\lim_{x \rightarrow \infty} \frac{\#\{n \leq x : n \in S_i \cap S_j\}}{x} = \frac{1}{4ij} \quad (i \neq j).$$

*Proof.*  $\square$

**Theorem 6.2** (Carleson's theorem). Let  $f$  be an  $L^p$  periodic function for some  $p \in (1, \infty)$ , with Fourier coefficient  $\hat{f}(n)$ . Then

$$\lim_{N \rightarrow \infty} \sum_{|n| \leq N} \hat{f}(n) e^{2\pi i n x} = f(x)$$

for almost every  $x$ .



**Figure 5.**  $\frac{1}{2} + \sum_{k=1}^{10} \frac{2}{(2k-1)\pi} \sin(2\pi i(2k-1)\sqrt{x}), (x \in [0, 31])$

**Definition 6.1.** A function  $f(x)$  is piecewise continuous on an interval  $I$  if it is continuous on  $I$  except perhaps for a finite number of points, and if  $a \in I$  is a point of discontinuity for  $f(x)$  then  $f(a_+)$  and  $f(a_-)$  exist: that is

$$f(a_+) = \lim_{x \rightarrow a_+} f(x), \quad f(a_-) = \lim_{x \rightarrow a_-} f(x)$$

are required to exist. We denote the space of piecewise continuous functions on  $I$  by  $E(I)$ .

**Definition 6.2.** The space  $E'$  is defined as the space of all functions  $f(x) \in E([-\pi, \pi])$  such that the right-hand derivative  $D_+ f(x)$  and left-hand derivative  $D_- f(x)$  exists. Recall that

$$D_+ f(x) = \lim_{h \rightarrow 0_+} \frac{f(x+h) - f(x_+)}{h}$$

$$D_- f(x) = \lim_{h \rightarrow 0_-} \frac{f(x+h) - f(x_-)}{h}.$$

**Theorem 6.3.** *If  $f \in E'$ , then the Fourier series of  $f$*

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx),$$

where

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx, \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx,$$

converges pointwise to

$$\frac{f(x_+) + f(x_-)}{2}.$$

**Theorem 6.4.** *Suppose that*

- $f(x)$  is continuous on  $[-\pi, \pi]$
- $f(-\pi) = f(\pi)$
- $f'(x) \in E([-\pi, \pi])$ .

Then

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

converges uniformly to  $f(x)$  on  $[-\pi, \pi]$ .

**Corollary 6.3.** *Let  $\mathcal{X}$  be the characteristic function of  $\bigcup_{m=1}^{\infty} [m^2, m^2 + m + \frac{1}{4}]$ . Then*

$$\mathcal{X} = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{(2k-1)\pi} \sin(2\pi(2k-1)\sqrt{x}) \quad (x \geq 1)$$

at every point where  $\mathcal{X}$  is continuous.

**Lemma 6.10.** [Bourgain-Sarnak-Ziegler] *Let  $F : \mathbb{N} \rightarrow \mathbb{C}$  with  $|F| \leq 1$  and let  $v$  be a multiplicative function with  $|v| \leq 1$ . Let  $\tau > 0$  be a small parameter and assume that for all primes  $p_1, p_2 \leq e^{1/\tau}$ ,  $p_1 \neq p_2$ , we have that for  $M$  large enough*

$$\left| \sum_{m \leq M} F(p_1 m) \overline{F(p_2 m)} \right| \leq \tau M.$$

Then for  $N$  large enough

$$\left| \sum_{n \leq N} v(n) F(n) \right| \leq 2N \sqrt{\tau \log(1/\tau)}.$$

**Lemma 6.11** (Kusmin-Landau). *If  $f$  is continuously differentiable,  $f'$  is monotonic, and  $0 < \lambda \leq f'(x) \leq 1 - \lambda$  on the interval  $(a, b]$ , then*

$$\left| \sum_{a < n \leq b} e(f(n)) \right| \ll \lambda^{-1}.$$

**Lemma 6.12.** *Let  $0 < \gamma < 1$ . For arbitrarily small  $\tau > 0$ , and for  $0 \neq |k| \leq x^{(1-\gamma)/2}$  and multiplicative function  $v$  with  $|v| \leq 1$ , we have*

$$\left| \sum_{n \leq x} v(n) e(kn^\gamma) \right| \ll_{\tau} x$$

*Proof.* With the help of Lemma lemma 6.10, if we prove that for all primes  $p_1 \neq p_2 \ll e^{1/\tau}$ , the estimate

$$\left| \sum_{n \leq x} e(k(p_1 n)^\gamma - k(p_2 n)^\gamma) \right| \ll_\tau x$$

holds for sufficiently large  $x$ , then

$$\left| \sum_{n \leq x} v(n) e(kn^\gamma) \right| \ll_\tau x$$

also holds. Since

$$\left| \sum_{n \leq x(\log x)^{-1}} e(k(p_1 n)^\gamma - k(p_2 n)^\gamma) \right| \ll x(\log x)^{-1} \ll_\tau x$$

for  $x$  sufficiently large, it is sufficient to prove

$$\left| \sum_{x(\log x)^{-1} \leq n \leq x} e(k(p_1 n)^\gamma - k(p_2 n)^\gamma) \right| \ll_\tau x$$

Take  $f(n) := k(p_1^\gamma - p_2^\gamma)n^\gamma$  in Lemma lemma 6.11. Note that

$$|f'(n)| \asymp k(p_1^\gamma - p_2^\gamma)n^{\gamma-1} \gg_\tau |k|x^{\gamma-1}$$

and

$$|f'(n)| \asymp k(p_1^\gamma - p_2^\gamma)n^{\gamma-1} \ll_\tau |k|x^{\gamma-1}(\log x)^{1-\gamma} = o_\tau(1)$$

when  $k \leq x^{(1-\gamma)/2}$ . By Lemma lemma 6.11, we conclude that

$$\left| \sum_{x(\log x)^{-1} \leq n \leq x} e(k(p_1 n)^\gamma - k(p_2 n)^\gamma) \right| \ll_\tau x^{1-\gamma}.$$

□

**Theorem 6.5.** Let  $0 < \gamma < 1$  and write  $\beta = \gamma^{-1}$ . Define  $A = \bigcup_{m=1}^\infty [m^\beta, (m+\gamma)^\beta)$ . Then for any multiplicative function  $v$  with  $|v| \leq 1$ , we have

$$\sum_{n \leq x} v(n) 1_A(n) = \gamma \sum_{n \leq x} v(n) + o(x).$$

In particular, we have

$$\sum_{n \leq x} \mu(n) 1_A(n) = o(x)$$

and

$$\sum_{n \leq x} \mu^2(n) 1_A(n) = \frac{6\gamma}{\pi^2} x + o(x).$$

*Proof.* Let  $\tau > 0$  be an arbitrarily small parameter to be determined later. Define

$$\tilde{A} = \bigcup_{m=1}^\infty [m, m+\gamma]$$

and

$$\eta(y) = \begin{cases} \exp\left(-\frac{1}{1-((y-\tau-m)/\tau)^2} + 1\right), & y \in \bigcup_{m=1}^\infty [m, m+\tau), \\ 1, & y \in \bigcup_{m=1}^\infty [m+\tau, m+\gamma-\tau), \\ \exp\left(-\frac{1}{1-((y-\gamma+\tau-m)/\tau)^2} + 1\right), & y \in \bigcup_{m=1}^\infty [m+\gamma-\tau, m+\gamma), \\ 0, & y \in \bigcup_{m=1}^\infty [m+\gamma, m+1). \end{cases}$$

The function  $\eta$  has the Fourier expansion

$$\eta(y) = \sum_{k \in \mathbb{Z}} \hat{\eta}(k) e(ky), \quad y \geq 1,$$

where the Fourier coefficients  $\hat{\eta}(k) = \int_0^1 \eta(t) e(-kt) dt$  satisfy  $|\hat{\eta}(k)| \ll_{\tau, A} (2 + |k|)^{-A}$ . Write  $\delta = (1 - \gamma)/2$ , we have

$$\begin{aligned} \eta(y) &= \sum_{|k| \leq x^{\delta/2}} \hat{\eta}(k) e(ky) + O_{\tau, A} \left( \sum_{|k| \geq x^{\delta/2}} |k|^{-A} \right) \\ &= \sum_{|k| \leq x^{\delta/2}} \hat{\eta}(k) e(ky) + O_{\tau, A} (x^{-100}). \end{aligned}$$

Now

$$\begin{aligned} \sum_{n \leq x} \nu(n) 1_A(n) &= \sum_{n \leq x} \nu(n) 1_{\tilde{A}}(n^\gamma) \\ &= \sum_{n \leq x} \nu(n) \eta(n^\gamma) + \sum_{n \leq x} \nu(n) (1_{\tilde{A}}(n^\gamma) - \eta(n^\gamma)) \\ &:= W_1 + W_2. \end{aligned}$$

The first summation is

$$\begin{aligned} W_1 &= \sum_{n \leq x} \nu(n) \left( \sum_{|k| \leq x^{\delta/2}} \hat{\eta}(k) e(kn^\gamma) + O_A(x^{-100}) \right) \\ &= \sum_{|k| \leq x^{\delta/2}} \hat{\eta}(k) \left( \sum_{n \leq x} \nu(n) e(kn^\gamma) \right) + O(1) \\ &= \hat{\eta}(0) \sum_{n \leq x} \nu(n) + O \left( 1 + \sum_{|k| \leq x^{\delta/2}} |\hat{\eta}(k)| \left| \sum_{n \leq x} \nu(n) e(kn^\gamma) \right| \right) \end{aligned}$$

In view of Lemma lemma 6.12, the summation in the error term is  $O_\tau(x)$ . The coefficient

$$\hat{\eta}(0) = \int_0^1 \eta(y) dy = \gamma + O(\tau).$$

As a result, it follows that

$$W_1 = \gamma \sum_{n \leq x} \nu(n) + O(\tau x).$$

Now we start to bound  $W_2$ . Write  $h(n) = 1_{\tilde{A}}(n^\gamma) - \eta(n^\gamma)$ , then

$$\text{Supp}(h) \cap [1, x] \subseteq \bigcup_{m=1}^{x^\gamma} [m^\beta, (m+\tau)^\beta] \cup [(m+\gamma-\tau)^\beta, (m+\gamma)^\beta].$$

Hence

$$W_2 \ll \sum_{m \leq x^\gamma} \tau m^{\beta-1} \ll \tau x.$$

Putting every thing together, we see that

$$\sum_{n \leq x} \nu(n) 1_A(n) = \gamma \sum_{n \leq x} \nu(n) + O(\tau x).$$

Since  $\tau > 0$  can be chosen arbitrarily small, it can be conclude that

$$\sum_{n \leq x} \nu(n) 1_A(n) = \gamma \sum_{n \leq x} \nu(n) + o(x).$$

□

**Lemma 6.13.**

$$\sum_{n \leq x} \left[ \frac{x}{n} \right] \mathcal{X}(n) \Lambda(n) = \frac{x}{2} \log x + O(x).$$

*Proof.* Assume that  $x = m^2 + m$ . We claim that

$$\sum_{k=1}^m \sum_{l=1}^k \left[ \frac{m^2 + m}{k^2 + l} \right] \Lambda(k^2 + l) - \sum_{k=1}^m \sum_{l=1}^k \left[ \frac{m^2 + m}{k^2 - k + l} \right] \Lambda(k^2 - k + l) = O(x).$$

First note that

$$\begin{aligned} & \sum_{k=1}^m \sum_{l=1}^k \left[ \frac{m^2 + m}{k^2 + l} \right] \Lambda(k^2 + l) - \sum_{k=1}^m \sum_{l=1}^k \left[ \frac{m^2 + m}{k^2 - k + l} \right] \Lambda(k^2 - k + l) \\ &= \sum_{k=1}^m \sum_{l=1}^k \frac{m^2 + m}{k^2 + l} \Lambda(k^2 + l) - \sum_{k=1}^m \sum_{l=1}^k \frac{m^2 + m}{k^2 - k + l} \Lambda(k^2 - k + l) + O(x). \end{aligned}$$

However

$$\begin{aligned} & \sum_{k=1}^m \sum_{l=1}^k \left( \frac{m^2 + m}{k^2 + l} - \frac{m^2 + m}{k^2} \right) \Lambda(k^2 + l) \\ &= x \sum_{k=1}^m \sum_{l=1}^k \left( \frac{-l}{k^2(k^2 + l)} \right) \Lambda(k^2 + l) = xO\left(\sum_{k=1}^m \frac{\log 2k}{k^3}\right) = O(x). \end{aligned}$$

Similarly

$$\begin{aligned} & \sum_{k=1}^m \sum_{l=1}^k \left( \frac{m^2 + m}{k^2 - k + l} - \frac{m^2 + m}{k^2} \right) \Lambda(k^2 - k + l) \\ &= x \sum_{k=1}^m \sum_{l=1}^k \left( \frac{k-l}{k^2(k^2 - k + l)} \right) \Lambda(k^2 - k + l) = xO\left(\sum_{k=1}^m \frac{\log 2k}{k^3}\right) = O(x). \end{aligned}$$

Therefore, we only need to show

$$\sum_{k=1}^m \frac{m^2 + m}{k^2} \sum_{l=1}^k \left( \left[ \frac{k^2 + k}{k^2 + l} \right] \Lambda(k^2 + l) - \left[ \frac{k^2}{k^2 - k + l} \right] \Lambda(k^2 - k + l) \right)$$

□

Let  $\Psi_{\mathcal{X}}(k, x) = \sum_{d \leq x} \mathcal{X}(dk) \Lambda(d)$  and  $\Psi_{1-\mathcal{X}}(k, x) = \sum_{d \leq x} (1 - \mathcal{X}(dk)) \Lambda(d)$ . Recall that

$$\Psi(x) = \sum_{d \leq x} \Lambda(d) = O(x).$$

Therefore  $\Psi_{\mathcal{X}}(k, x) = O(x)$  and  $\Psi_{1-\mathcal{X}}(k, x) = O(x)$ .

**Lemma 6.14.** *Let*

$$\Psi(k, x) = \sum_{d \leq x} \mathcal{X}(dk) \Lambda(d).$$

*Then*  $\Psi(k, x) - \frac{1}{2}[x] = o(x)$ .

*Proof.* Let  $\Psi(x) = \sum_{d \leq x} \Lambda(d) = \Psi(k, x)$

□

**Lemma 6.15.** *Let*  $M_{\mathcal{X}}(x) = \sum_{n \leq x} \mathcal{X}(n) \mu(n)$ . *Then*  $M_{\mathcal{X}}(x) = o(x)$ .

*Proof.* By Theorem 434 in [6], we have

$$\sum_{n \leq x} \mathcal{X}(n) \mu(n) \log\left(\frac{x}{n}\right) = O(x).$$

Hence

$$M_{\mathcal{X}}(x) \log(x) = \sum_{n \leq x} \mathcal{X}(n) \mu(n) \log n + O(x).$$

By Theorem 297 in [6],

$$\begin{aligned} - \sum_{n \leq x} \mathcal{X}(n) \mu(n) \log n &= \sum_{n \leq x} \mathcal{X}(n) \sum_{d|n} \mu\left(\frac{n}{d}\right) \Lambda(d) = \sum_{dk \leq x} \mathcal{X}(dk) \mu(k) \Lambda(d) \\ &= \sum_{k \leq x} \mu(k) \sum_{d \leq \frac{x}{k}} \mathcal{X}(dk) \Lambda(d) \end{aligned}$$

□

**Lemma 6.16 (Conjecture).** *Let  $(X, \sigma)$  be a dynamic system where  $X$  is a compact Hausdorff metric space and  $\iota(X)$  is the set of all isolated points of  $X$ . Let  $X' = X \setminus \iota(X)$ . If  $\sigma(X') \subset X'$  and  $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} \mu(n) f(\sigma^n(x)) = 0$  for any  $f \in C(X')$  and any  $x \in X'$ , then  $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} \mu(n) f(\sigma^n(x)) = 0$  for any  $f$  in  $C(X)$  and  $x \in X$ .*

## 7. WEIGHTED SUM OF $\mu(n)$

Let  $f : \mathbb{N} \rightarrow \mathbb{N}$  be a function defined on  $\mathbb{N}$ . Consider the weighted sum

$$\sum_{n \leq N} f(n) \mu(n)$$

Let  $\rho : \mathbb{N} \rightarrow \mathbb{N}$  be a map defined by

$$\rho : p_1^{\gamma(1)} p_2^{\gamma(2)} \cdots p_n^{\gamma(n)} \rightarrow 1^{\gamma(1)} 2^{\gamma(2)} \cdots n^{\gamma(n)},$$

where  $p_i$  is the  $i$ -th prime number.

**Remark 7.1.** *If  $k < \sqrt{N}$ , then  $\lfloor \frac{N}{k} \rfloor > \lfloor \frac{N}{k+1} \rfloor$ . Assume that  $N = ak + b$  where  $0 \leq b < k$  and  $a \geq k$ . If  $\lfloor \frac{N}{k} \rfloor = \lfloor \frac{N}{k+1} \rfloor$ , then  $ak + b = a(k+1) + c$ . It is impossible.*

**Lemma 7.1.**

$$\sum_{\gamma \in (\mathbb{Z}_2)^N \text{ and } 1^{\gamma(1)} 2^{\gamma(2)} \cdots N^{\gamma(N)} \leq M} (-1)^{\sum_i \gamma(i)} = 0.$$

*Proof.*

$$\begin{aligned} &\sum_{\gamma \in (\mathbb{Z}_2)^N \text{ and } 1^{\gamma(1)} 2^{\gamma(2)} \cdots N^{\gamma(N)} \leq M} (-1)^{\sum_i \gamma(i)} \\ &= \sum_{\gamma \in (\mathbb{Z}_2)^N \text{ and } 2^{\gamma(2)} \cdots N^{\gamma(N)} \leq M} (-1)^{\sum_{i=2}^N \gamma(i)} - \sum_{\gamma \in (\mathbb{Z}_2)^N \text{ and } 2^{\gamma(2)} \cdots N^{\gamma(N)} \leq M} (-1)^{\sum_{i=2}^N \gamma(i)} \\ &= 0. \end{aligned}$$

□

Estimate

$$\sum_{\gamma \in (\mathbb{Z}_2)^N \text{ and } 2^{\gamma(2)} \cdots N^{\gamma(N)} \leq N} (-1)^{\sum_i \gamma(i)}.$$

Note that if  $j > \lfloor \frac{N}{2} \rfloor$  and  $\gamma(j) = 1$ , then  $\gamma(i) = 0$  for any  $i \neq j$ . Therefore

$$\begin{aligned} & \sum_{\gamma \in (\mathbb{Z}_2)^N \text{ and } 2^{\gamma(2)} \dots N^{\gamma(N)} \leq N} (-1)^{\sum_i \gamma(i)} \\ &= \sum_{\gamma \in (\mathbb{Z}_2)^{\lfloor \frac{N}{2} \rfloor} \text{ and } 2^{\gamma(2)} \dots (\lfloor \frac{N}{2} \rfloor)^{\gamma(\lfloor \frac{N}{2} \rfloor)} \leq N} (-1)^{\sum_i \gamma(i)} - (N - \lfloor \frac{N}{2} \rfloor). \end{aligned}$$

Similar argument implies that

$$\sum_{\gamma \in (\mathbb{Z}_2)^{\frac{N}{2}} \text{ and } 1^{\gamma(1)} 2^{\gamma(2)} \dots (\lfloor \frac{N}{2} \rfloor)^{\gamma(\lfloor \frac{N}{2} \rfloor)} \leq N} (-1)^{\sum_i \gamma(i)} = \sum_{\gamma \in (\mathbb{Z}_2)^{\frac{N}{3}} \text{ and } 1^{\gamma(1)} 2^{\gamma(2)} \dots (\lfloor \frac{N}{3} \rfloor)^{\gamma(\lfloor \frac{N}{3} \rfloor)} \leq N} (-1)^{\sum_i \gamma(i)}.$$

Suppose that  $\gamma(j) = 1$  and  $\lfloor \frac{N}{4} \rfloor < j \leq \lfloor \frac{N}{3} \rfloor$ . The possibilities are

$$\begin{aligned} \gamma(1) = \gamma(2) = \gamma(3) = 0 & \quad \gamma(1) = 1, \gamma(2) = \gamma(3) = 0 \\ \gamma(1) = 0, \gamma(2) = 1, \gamma(3) = 0 & \quad \gamma(1) = 0, \gamma(2) = 0, \gamma(3) = 1 \\ \gamma(1) = 0, \gamma(2) = 0, \gamma(3) = 1 & \quad \gamma(1) = 1, \gamma(2) = 1, \gamma(3) = 0 \\ \gamma(1) = 1, \gamma(2) = 0, \gamma(3) = 1. & \end{aligned}$$

This implies that

$$\sum_{\gamma \in (\mathbb{Z}_2)^{\frac{N}{3}} \text{ and } 1^{\gamma(1)} 2^{\gamma(2)} \dots (\lfloor \frac{N}{3} \rfloor)^{\gamma(\lfloor \frac{N}{3} \rfloor)} \leq N} (-1)^{\sum_i \gamma(i)} = \sum_{\gamma \in (\mathbb{Z}_2)^{\frac{N}{4}} \text{ and } 1^{\gamma(1)} 2^{\gamma(2)} \dots (\lfloor \frac{N}{4} \rfloor)^{\gamma(\lfloor \frac{N}{4} \rfloor)} \leq N} (-1)^{\sum_i \gamma(i)}$$

Similarly, we have

$$\sum_{\gamma \in (\mathbb{Z}_2)^{\frac{N}{4}} \text{ and } 1^{\gamma(1)} 2^{\gamma(2)} \dots (\lfloor \frac{N}{4} \rfloor)^{\gamma(\lfloor \frac{N}{4} \rfloor)} \leq N} (-1)^{\sum_i \gamma(i)} = \sum_{\gamma \in (\mathbb{Z}_2)^{\frac{N}{6}} \text{ and } 1^{\gamma(1)} 2^{\gamma(2)} \dots (\lfloor \frac{N}{6} \rfloor)^{\gamma(\lfloor \frac{N}{6} \rfloor)} \leq N} (-1)^{\sum_i \gamma(i)}$$

Assume that  $k + 1 \leq \sqrt{N}$  and

$$\sum_{\gamma \in (\mathbb{Z}_2)^N \text{ and } 1^{\gamma(1)} 2^{\gamma(2)} \dots N^{\gamma(N)} \leq N} (-1)^{\sum_i \gamma(i)} = \sum_{\gamma \in (\mathbb{Z}_2)^{\lfloor \frac{N}{k} \rfloor} \text{ and } 1^{\gamma(1)} 2^{\gamma(2)} \dots (\lfloor \frac{N}{k} \rfloor)^{\gamma(\lfloor \frac{N}{k} \rfloor)} \leq N} (-1)^{\sum_i \gamma(i)}$$

**Lemma 7.2.** *Let  $k \leq \sqrt{N}$ . Then*

$$\sum_{\gamma \in (\mathbb{Z}_2)^N \text{ and } 1^{\gamma(1)} 2^{\gamma(2)} \dots N^{\gamma(N)} \leq N} (-1)^{\sum_i \gamma(i)} = \sum_{\gamma \in (\mathbb{Z}_2)^{\lfloor \frac{N}{k} \rfloor} \text{ and } 1^{\gamma(1)} 2^{\gamma(2)} \dots (\lfloor \frac{N}{k} \rfloor)^{\gamma(\lfloor \frac{N}{k} \rfloor)} \leq N} (-1)^{\sum_i \gamma(i)}$$

*Proof.* For  $k = 1$ , the statement is trivial. Let  $k = 2$ . Assume that

$$\sum_{\gamma \in (\mathbb{Z}_2)^N \text{ and } 1^{\gamma(1)} 2^{\gamma(2)} \dots N^{\gamma(N)} \leq N} (-1)^{\sum_i \gamma(i)} = \sum_{\gamma \in (\mathbb{Z}_2)^{\lfloor \frac{N}{k-1} \rfloor} \text{ and } 1^{\gamma(1)} 2^{\gamma(2)} \dots (\lfloor \frac{N}{k-1} \rfloor)^{\gamma(\lfloor \frac{N}{k-1} \rfloor)} \leq N} (-1)^{\sum_i \gamma(i)}.$$

Suppose that  $\gamma(j) = 1$  and  $\lfloor \frac{N}{k} \rfloor < j \leq \lfloor \frac{N}{k-1} \rfloor$ . Note that

$$\sum_{\gamma \in (\mathbb{Z}_2)^{k-1} \text{ and } 1^{\gamma(1)} 2^{\gamma(2)} \dots (k-1)^{\gamma(k-1)} \leq N} -(-1)^{\sum_i \gamma(i)}$$

□



## 8. Stone – Čech COMPACTIFICATION

**Definition 8.1.** A filter on a set  $X$  is a collection  $\mathcal{F}$  of subsets of  $X$  satisfying:

- (1)  $X \in \mathcal{F}$ , but  $\emptyset \notin \mathcal{F}$ .
- (2) If  $A \in \mathcal{F}$  and  $A \subset B \subset X$ , then  $B \in \mathcal{F}$ .
- (3) A finite intersection of sets in  $\mathcal{F}$  is in  $\mathcal{F}$ .

A filter is a ultrafilter if

- (4) For every set  $A \subset X$  either  $A \in \mathcal{F}$  or  $A^c = X \setminus A \in \mathcal{F}$ .

or

- (4)' For every finite cover  $\{A_i\}_{i=1}^n$  of a set  $A \in \mathcal{F}$ ,  $A_i \in \mathcal{F}$  for some  $i$ .

It is well-known that the Stone – Čech Compactification  $\beta\mathbb{N}$  of  $\mathbb{N}$  can be identified with the set of all ultrafilters on  $\mathbb{N}$ . The topology of  $\beta\mathbb{N}$  is given by the basis  $\mathcal{B} = \{U_A : A \subset \mathbb{N}\}$ , where for any set  $A \subset \mathbb{N}$ ,

$$U_A = \{\mathcal{F} \in \beta\mathbb{N} : A \in \mathcal{F}\}.$$

**Example 8.1.** Let  $\sigma$  be a continuous map from  $\mathbb{N}$  to  $\mathbb{N}$  defined by  $\sigma(n) = n + 1$ . By the universal property of  $\beta\mathbb{N}$ ,  $\sigma$  lifts uniquely to a continuous map  $\sigma : \beta\mathbb{N} \rightarrow \beta\mathbb{N}$ . Then the möbius function  $\mu$  can be viewed as a continuous function on  $\beta\mathbb{N}$  and  $\sum_{k=1}^n \mu(k)^2$  is the number of square-free numbers below  $n$ . It is well-known

$$\frac{1}{n} \sum_{k=0}^{n-1} \mu(k)^2 = \frac{6}{\pi^2} + o(1).$$

Therefore  $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(k)^2 = \frac{6}{\pi^2} \neq 0$ . However, this is not a counter-example of sarnak's conjecture, since  $h_{\text{top}}(\sigma) = \infty$ . Indeed, for any  $N \in \mathbb{N}$ , let

$$A_N = \bigcup_{k=1}^{\infty} \left\{ kn + 1 + \frac{k(k-1)}{2}, \dots, kn + 1 + \frac{k(k-1)}{2} + k \right\}.$$

It is easy to see that for any  $n_1, \dots, n_m$ , we have  $\sigma^{n_1}(A_N) \cap \dots \cap \sigma^{n_m}(A_N) \neq \emptyset$ , and  $\bigcup_{i=0}^{N-1} \sigma^i(A_N) = \mathbb{N}$ . Therefore,  $\mathcal{U} = \{U_{\sigma^i(A_N)}\}_{i=0}^{N-1}$  is an open cover of  $\beta\mathbb{N}$  and  $h_{\text{cover}}(\sigma, \mathcal{U}) = \log N$ .

**Remark 8.1.** Recall that a thick set is a set of integers that contains arbitrarily long intervals. And a syndetic set is a subset of  $\mathbb{N}$ , having the property of "bounded gaps", i.e. the sizes of the gaps in the sequence of natural numbers is bounded. It is easy to see that  $A_N$  in the example 8.1 is thick and syndetic.

**Definition 8.2.** A dynamical system  $(X, \sigma)$  is call minimal if  $X$  does not contain any non-empty, proper, closed  $\sigma$ -invariant subset, i.e. every orbit is dense in  $X$ .

**Remark 8.2.** If  $(X, \sigma)$  is a minimal dynamical system and  $X$  is a compact Hausdorff space, then  $\sigma$  must be surjective.

**Lemma 8.1.**  $\beta\mathbb{N} \setminus \mathbb{N}$  is not separable.

*Proof.* Let  $\{\mathcal{F}_n\}_{n=1}^{\infty}$  be a sequence of ultrafilters. To prove the result, we will construct a subset  $A$  of  $\mathbb{N}$  recursively such that  $A \notin \mathcal{F}_n$  for any  $n$ . Let  $A_1$  be a infinite set that  $A_1 \notin \mathcal{F}_1$  and  $a_1 = \min A_1$ . Assume that we have  $A_n \notin \mathcal{F}_n$ . Let  $a_n = \min A_n$ . Choose a infinite subset  $A_{n+1}$  of  $A_n$  such that  $a_{n+1} = \min A_{n+1} > a_n$  and  $A_{n+1} \notin \mathcal{F}_{n+1}$ . If  $A \in \mathcal{F}_n$ , then  $A \cap A_n = A \cap \{1, \dots, a_n - 1\}^c \in \mathcal{F}_n$ . Since  $A_n^c \in \mathcal{F}_n$ , we have  $A \cap A_n \cap A_n^c = \emptyset \in \mathcal{F}_n$ . It is a contradiction.  $\square$

**Corollary 8.1.** Let  $(\beta\mathbb{N}, \sigma)$  be the dynamical system where  $\sigma$  is the map induced by the shift on  $\mathbb{N}$ , i.e.  $\sigma(n) = n + 1$ . Then  $(\beta\mathbb{N}, \sigma)$  is not minimal.

## 9. WHAT WE TALK ABOUT WHEN WE TALK ABOUT SMOOTHNESS

Let  $C(X)$  and  $C(Y)$  be two abelian  $C^*$ -algebras, where  $X$  and  $Y$  are compact Hausdorff spaces. Suppose  $\Phi$  is a  $*$ -homomorphism from  $C(X)$  to  $C(Y)$ . It is clear that  $\Phi$  induces a map  $\Phi^*$  from  $Y$  to  $X$ ,

$$\Phi^* : \rho \rightarrow \rho \circ \Phi, \quad \rho \in Y.$$

Similar, if  $\varphi : X \rightarrow Y$  be a continuous map, then  $\varphi$  induces a map from  $C(Y) \rightarrow C(X)$ ,

$$\varphi^* : f \rightarrow f \circ \varphi, \quad f \in C(Y).$$

It is clear that  $\Phi^{**} = \Phi$  and  $\varphi^{**} = \varphi$ .

Consider the  $C^*$ -algebra  $L^\infty(\mathbb{N})$  acting on  $l^2(\mathbb{N})$ . It is well-known that  $L^\infty(\mathbb{N}) \simeq C(\beta\mathbb{N})$ . Let  $\sigma$  be the continuous map from  $\beta\mathbb{N}$  to  $\beta\mathbb{N}$  induced by the shift on  $\mathbb{N}$ :  $\sigma(n) = n + 1$ . Throughout this section, we will use  $\sigma$  to denote this continuous map.

Let  $V$  be the unilateral shift, i.e.,  $Ve_i = e_{\sigma(i)}$ . Then

$$\Sigma : M_f \rightarrow V^* M_f V = M_{f \circ \sigma}, \text{ for all } f \in L^\infty(\mathbb{N}),$$

is a homomorphism from  $L^\infty(\mathbb{N})$  onto  $L^\infty(\mathbb{N})$ .

Note that

$$V^* M_{f_1} V V^* M_{f_2} V = V^* M_{f_1 f_2} V,$$

where  $f_1$  and  $f_2$  are in  $L^\infty(\mathbb{N})$ . It is easy to see that  $\sigma^* = \Sigma$  and  $\Sigma^* = \sigma$ .

Let  $f \in L^\infty(\mathbb{N})$  and  $C(f)$  be the  $C^*$ -algebra generated by  $M_{f \circ \sigma^n}$ ,  $n = 0, 1, \dots$  and  $I$ . This is an abelian  $C^*$ -algebra. By Gelfand transform,  $C(f)$  can be identified as  $C(\Omega)$ , where  $\Omega$  is a compact Hausdorff space.  $\Omega$  is metrizable since  $C(f)$  is separable.

**Remark 9.1.** Let  $C(f) \simeq C(\Omega)$  where  $\Omega$  is the spectrum of  $C(f)$ .  $\{M_{f \circ \sigma^n} : n = 0, 1, 2, \dots\}$  separates points in  $\Omega$ .

**Theorem 9.1** (Tietze extension theorem). *If  $X$  is a normal topological space (e.g., every compact Hausdorff space is normal) and  $f : A \rightarrow \mathbb{R}$  is a continuous map from a closed subset  $A$  of  $X$  into the real numbers carrying the standard topology, then there exists a continuous map  $F : X \rightarrow \mathbb{R}$  with  $F(x) = f(x)$  for all  $x \in A$ . Moreover,  $F$  may be chosen such that  $\sup\{|f(a)| : a \in A\} = \sup\{|F(x)| : x \in X\}$ , i.e., if  $f$  is bounded,  $F$  may be chosen to be bounded (with the same bound as  $f$ ).  $F$  is called a continuous extension of  $f$ .*

**Definition 9.1.** Let  $T : X \rightarrow X$  and  $S : Y \rightarrow Y$  be continuous maps of compact (metric) spaces (that is, topological dynamical systems). Then a homeomorphism  $\theta : X \rightarrow Y$  with  $\theta \circ T = S \circ \theta$  is called a topological conjugacy, and if there such a conjugacy then  $T$  and  $S$  are topologically conjugate. A continuous surjective map  $\varphi : X \rightarrow Y$  with  $\varphi \circ T = S \circ \varphi$  is called a topological factor map, and in this case  $S$  is said to be a factor of  $T$ .

**Lemma 9.1.** Let  $\mathfrak{A} \simeq C(Y)$  be a  $C^*$ -subalgebra of  $C(X)$ , and  $\varphi$  be the continuous surjective from  $X$  onto  $Y$  induced by the embedding of  $C(Y)$  into  $C(X)$ . Suppose  $\Phi$  is a homomorphism from  $C(X)$  to  $C(X)$  such that  $\Phi$  fixes  $\mathfrak{A}$ , i.e.,  $\Phi(\mathfrak{A}) \subset \mathfrak{A}$ . Let  $\Psi = \Phi|_{\mathfrak{A}}$ . Then  $\Psi^*$  is a factor of  $\Phi^*$ .

*Proof.* We only need to show that  $\varphi \circ \Phi^* = \Psi^* \circ \varphi$ . Let  $\rho$  in  $X$  and  $A \in \mathfrak{A}$ . Then

$$\varphi \circ \Phi^*(\rho)(A) = \rho(\Phi(A)) = \Psi^* \circ \varphi(\rho)(A).$$

□

**Lemma 9.2.** Let  $T : X \rightarrow X$  and  $S : Y \rightarrow Y$  be continuous maps of compact (metric) spaces. Let  $\varphi : X \rightarrow Y$  be a topological factor map. Then  $\varphi^*$  is injective. And  $\{f \circ \varphi : f \in C(Y)\}$  is invariant under the map  $g \rightarrow g \circ T$ ,  $g \in C(X)$ .

**Lemma 9.3.** *Let  $T : X \rightarrow X$  and  $S : Y \rightarrow Y$  be continuous maps of compact (metric) spaces. If  $S$  is a factor of  $T$ , then  $h_{top}(T) \geq h_{top}(S)$ .*

*Proof.* Let  $\varphi$  be a factor map. Suppose that  $\{U_i\}_{i \in I}$  be an open cover of  $Y$ . Let  $V_i = \varphi^{-1}(U_i)$ . Then  $\{V_i\}_{i \in I}$  is an open cover of  $X$ . Let  $U_i^n = S^{-n}(U_i)$  and  $V_i^n = T^{-n}(V_i)$ . We claim that  $V_i^n = \varphi^{-1}(U_i^n)$ . Indeed

$$\begin{aligned} V_i^n &= \{x \in X : T^n(x) \in V_i\} \\ &= \{x \in X : \varphi \circ T^n(x) \in U_i\} \\ &= \{x \in X : S^n \circ \varphi(x) \in U_i\} \\ &= \{x \in X : \varphi(x) \in U_i^n\} = \varphi^{-1}(U_i^n). \end{aligned}$$

This implies that

$$\begin{aligned} V_{i_1} \cap V_{i_2}^2 \cap \cdots \cap V_{i_n}^n &= \varphi^{-1}(U_{i_1}) \cap \varphi^{-1}(U_{i_2}^2) \cap \cdots \cap \varphi^{-1}(U_{i_n}^n) \\ &= \varphi^{-1}(U_{i_1} \cap U_{i_2}^2 \cap \cdots \cap U_{i_n}^n). \end{aligned}$$

Thus

$$N\left(\bigvee_{i=0}^m T^{-i}(\{V_i\}_{i \in I})\right) = N\left(\bigvee_{i=0}^m S^{-i}(\{U_i\}_{i \in I})\right).$$

This clearly implies that  $h_{top}(T) \geq h_{top}(S)$ .  $\square$

**Lemma 9.4.** *Let  $f \in L^\infty(\mathbb{N})$  and  $(Y, \varphi)$  be a topological dynamical system. Assume that  $C(f) \simeq C(X)$ . If there exists  $h \in C(Y)$  such that  $f(n) = h(\varphi^{(n)}(x))$ , and  $\{h \circ \varphi^{(n)} : n = 0, 1, \dots\} \cup \{I\}$  generate  $C(Y)$ , then there is a continuous injective map from  $X$  into  $Y$ . Furthermore, the range of the continuous map is the closure of  $\{x, \varphi(x), \varphi^{(2)}(x), \dots\}$  in  $Y$ .*

*Proof.* Let  $\Omega$  be the closure of  $\{x, \varphi(x), \varphi^{(2)}(x), \dots\}$  in  $Y$ . By the Tietze extension theorem, we have

$$g \rightarrow g|_\Omega, \quad \text{for any } g \in C(Y),$$

is a surjective map from  $C(Y)$  to  $C(\Omega)$ . Therefore  $\{h \circ \varphi^{(n)}|_\Omega : n = 0, 1, \dots\} \cup \{I\}$  also generate  $C(\Omega)$ . It is not hard to see that

$$\Phi : h \circ \varphi^n|_\Omega \rightarrow f \circ \sigma^{(n)}, \quad n = 0, 1, \dots$$

induce a \*-isomorphism from  $C(\Omega)$  to  $C(f) \simeq C(X)$ . Then

$$\Phi^* : \rho \rightarrow \rho \circ \Phi,$$

where  $\rho \in X$ , is a homeomorphism from  $X$  to  $\Omega$ .

Note that

$$\Phi(g \circ \varphi) = \Phi(g) \circ \sigma, \quad g \in C(Y).$$

Therefore, for  $g \in C(Y)$  and  $\rho \in X$ , we have

$$g(\varphi(\Phi^*(\rho))) = g \circ \varphi(\Phi^*(\rho)) = \Phi(g \circ \varphi)(\rho) = (\Phi(g) \circ \sigma)(\rho) = \Phi(g)(\sigma(\rho)) = g(\Phi^*(\sigma(\rho))).$$

This means that  $\varphi \circ \Phi^* = \Phi^* \circ \sigma$ .  $\square$

**Corollary 9.1.** *Let  $f \in L^\infty(\mathbb{N})$  and  $(Y, \varphi)$  be a topological dynamical system. Assume that  $C(f) \simeq C(X)$ . If there exists  $h \in C(Y)$  such that  $f(n) = h(\varphi^{(n)}(x))$ ,  $\{h \circ \varphi^{(n)} : n = 0, 1, \dots\} \cup \{I\}$  generate  $C(Y)$  and  $\{x, \sigma(x), \sigma^{(2)}(x), \dots\}$  is dense in  $Y$ . Then  $\sigma$  and  $\varphi$  are topologically conjugate.*

**Lemma 9.5.** *Let  $f \in L^\infty(\mathbb{N})$  and  $(Y, \varphi)$  be a topological dynamical system. Assume that  $C(f) \simeq C(X)$ . If there exists  $h \in C(Y)$  such that  $f(n) = h(\varphi^{(n)}(x))$ , then  $h_{\text{top}}(\varphi) \geq h_{\text{top}}(\sigma)$ .*

*Proof.* By Variational principle for the topological entropy.  $\square$

**Example 9.1** (Theorem 8 in [3]). *Let  $\Omega^{(2)} = \{0, 1\}^{\mathbb{N}}$ . The square-free flow  $S$  is the subflow of the full shift on  $\Omega^{(2)}$  by the point  $\theta = (\mu^2(1), \mu^2(2), \dots)$ , that is  $S = (X_S, T)$  with  $X_S$  the closure of the  $T$ -orbit of  $\theta$  in  $\Omega^{(2)}$ .*

- $T : X_S \rightarrow X_S$  is surjective, it is topologically ergodic and  $h_{\text{top}}(S) = \frac{6}{\pi^2} \log 2$ .
- $S$  is proximal (i.e. for any  $x, y \in X_S$ ,  $\inf_{n \geq 1} d(T^{(n)}(x), T^{(n)}(y)) = 0$ ) and  $\{(0, 0, 0, \dots)\}$  is the unique  $T$ -minimal subset of  $X_S$ .

**Remark 9.2.** *Let  $h$  be the characteristic function on of the subset  $\{(a_1, a_2, \dots) : a_1 = 1\}$  of  $\Omega^{(2)}$ . It is clear that  $\{h, h \circ T, h \circ T^{(2)}, \dots\}$  separates points. By Stone–Weierstrass theorem, we have  $\{h, h \circ T, h \circ T^{(2)}, \dots\}$  generates the  $C^*$ -algebra  $C(X_S)$ . By corollary 9.1,  $C(\mu^2) \simeq C(X_S)$ . Let  $\rho \in X$  where  $X$  is the spectrum of  $C(\mu^2)$ . Then*

$$\rho \rightarrow (\rho(\mu^2), \rho(\mu^2 \circ \sigma), \rho(\mu^2 \circ \sigma^{(2)}), \dots) \in \Omega^{(2)}$$

*is the homeomorphism from  $X$  to  $X_S$  ( $X$  endowed with weak\* topology and  $X_S$  endowed with the product topology).*

In general, we have the following fact.

**Proposition 9.1.** *Suppose that  $f$  is a function in  $L^\infty(\mathbb{N})$ . Let  $\Omega$  be the spectrum of  $M_f$ , and  $S$  be the left shift map :  $\Omega^{\mathbb{N}} \rightarrow \Omega^{\mathbb{N}}$  defined by*

$$T(a_1, a_2, \dots) = (a_2, a_3, \dots).$$

*Note that  $\Omega^{\mathbb{N}}$  is a compact Hausdorff space endowed with the product topology. Then the spectrum of  $C(f)$  is homeomorphic to the closure of  $\{\omega, T(\omega), T^{(2)}(\omega), \dots\}$  in  $\Omega^{\mathbb{N}}$ , where  $\omega = (f(1), f(2), \dots)$ . And the homeomorphism is given by*

$$\rho \rightarrow (\rho(f), \rho(f \circ \sigma), \rho(f \circ \sigma^{(2)}), \dots),$$

*where  $\rho$  is a point in the spectrum of  $C(f)$ .*

*Proof.* By corollary 9.1, we only need to show that there exists a continuous function  $h \in C(\Omega^{\mathbb{N}})$  such that  $h(T^{(n)}(\omega)) = f(n)$  and  $h$  separates points. It is easy to check that the function given by the projection to the first coordinate, i.e.  $h((a_1, a_2, \dots)) = a_1$ , satisfies all conditions.  $\square$

**Example 9.2.** *Let  $f(n) = e^{2\pi i \sqrt{n}}$ . It is not hard to see that  $\{e^{2\pi i \sqrt{n}}\}_{n \in \mathbb{N}}$  is dense in the unit circle  $S^1$ . Indeed, we only need to show that  $\{\frac{k}{\sqrt{n^2+k+n}} : k = 0, 1, \dots, 2n, n \in \mathbb{N}\}$  is dense in  $[0, 1]$ . If  $\frac{k}{2n} \rightarrow \theta$ , then*

$$\frac{\frac{k}{n}}{\sqrt{1 + \frac{k}{n^2} + 1}} \rightarrow \theta.$$

*Let  $T$  be the left shift of  $(S^1)^{\mathbb{N}}$  and  $X$  be the closure of  $\{\omega, T(\omega), T^{(2)}(\omega), \dots\}$  in  $(S^1)^{\mathbb{N}}$ , where  $\omega = (e^{2\pi i}, e^{2\pi i \sqrt{2}}, e^{2\pi i \sqrt{3}}, \dots)$ . We claim that The accumulation points of  $\{\omega, T(\omega), T^{(2)}(\omega), \dots\}$  in  $(S^1)^{\mathbb{N}}$  is*

$$\{(e^{2\pi i \theta}, e^{2\pi i \theta}, e^{2\pi i \theta}, \dots) : \theta \in [0, 1)\}.$$

And the closure of  $\{\omega, T(\omega), T^{(2)}(\omega), \dots\}$  is

$$\{\omega, T(\omega), T^{(2)}(\omega), \dots\} \cup \{(e^{2\pi i\theta}, e^{2\pi i\theta}, e^{2\pi i\theta}, \dots) : \theta \in [0, 1)\}.$$

*Proof of the claim.* Let  $x = \{(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}, e^{2\pi i\theta_3}, \dots) : \theta \in [0, 1)\}$  be an accumulation point of  $\{\omega, T(\omega), T^{(2)}(\omega), \dots\}$ . Since  $(S^1)^{\mathbb{N}}$  is a compact metric space, there exists a sequence of nature number  $n_k, k = 1, 2, \dots$  such that  $T^{(n_k)}(\omega) \rightarrow x$ . Assume that  $e^{2\pi i\sqrt{n_k}} \rightarrow e^{2\pi i\theta}$ , then

$$e^{2\pi i\sqrt{n_k+l}} \rightarrow e^{2\pi i\theta}, \quad l \in \mathbb{N}.$$

This implies that  $T^{(n_k)} \rightarrow (e^{2\pi i\theta}, e^{2\pi i\theta}, e^{2\pi i\theta}, \dots)$ .  $\square$

We quote the following fact in [8].

**Lemma 9.6** (Lemma 4.2 in [8]). *Let  $Y$  be a compact metric space and  $S : Y \rightarrow Y$  a continuous mapping. If  $C \subset Y$  is a closed  $S$ -invariant subset of  $Y$  and  $Y \setminus C$  is countable and  $S$ -invariant, then  $h_{\text{top}}(S) = h_{\text{top}}(S|_C)$ .*

**Proposition 9.2.** *The topological entropy of the left shift  $T$  of the spectrum of  $C(f)$  is 0, where  $f(n) = e^{2\pi i\sqrt{n}}$ .*

*Proof.* Let  $C = \{(e^{2\pi i\theta}, e^{2\pi i\theta}, e^{2\pi i\theta}, \dots) : \theta \in [0, 1)\}$ . By lemma 9.6, we have  $h_{\text{top}}(T) = h_{\text{top}}(T|_C)$ . It is clear that  $h_{\text{top}}(T|_C) = 0$  since  $T|_C$  is identity.  $\square$

**Definition 9.2.** *A sequence  $x_n = (x_n^1, \dots, x_n^k) \in \mathbb{R}^k$  is said to be uniformly distributed mod 1 if, for each choice of  $k$  intervals  $[a_1, b_1], \dots, [a_k, b_k] \subset [0, 1)$ , we have that*

$$\frac{1}{n} \sum_{j=1}^n \prod_{i=1}^k \chi_{[a_i, b_i]}(\{x_j^i\}) \rightarrow \prod_{i=1}^k (b_i - a_i), \quad \text{as } n \rightarrow \infty.$$

**Theorem 9.2** (Weyl's Criterion). *The sequence  $x_n \in \mathbb{R}^k$  is uniformly distributed mod 1 if and only if*

$$\frac{1}{n} \sum_{j=1}^n e^{2\pi i(l_1 x_j^1 + l_2 x_j^2 + \dots + l_k x_j^k)} \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

for all  $l = (l_1, \dots, l_k) \in \mathbb{Z}^k \setminus \{0\}$ .

**Theorem 9.3** (Weyl). *Let  $p(n) = a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n + a_0$ . If any  $a_1, a_2, \dots, a_k$  is irrational, then*

$$\frac{1}{n} \sum_{j=1}^n e^{2\pi i p(j)} \rightarrow 0$$

as  $n \rightarrow \infty$ .

**Corollary 9.2.** *Let  $p(n) = \theta n^k$  such that  $\theta$  is irrational. Then  $\{(p(n), p(n+1), \dots, p(n+k-1))\}_{n=1}^{\infty}$  is uniformly distributed mod 1.*

*Proof.* Assumet that

$$\frac{1}{n} \sum_{j=1}^n e^{2\pi i \theta (l_0 n^k + \dots + l_{k-1} (n+k-1)^k)} \rightarrow 0$$

where  $(l_0, \dots, l_{k-1}) \in \mathbb{Z}^k$ . By theorem 9.3, we have

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & \cdots & k-1 \\ 0 & 1 & 2^2 & \cdots & (k-1)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 2^{k-1} & \cdots & (k-1)^{k-1} \end{pmatrix} \begin{pmatrix} l_0 \\ l_1 \\ l_2 \\ \vdots \\ l_{k-1} \end{pmatrix} = 0.$$

This implies that  $(l_0, \dots, l_{k-1}) = (0, \dots, 0)$ .  $\square$

**Example 9.3.** Let  $f(n) = e^{2\pi i n^2 \theta}$ . Let  $T$  be the left shift of  $(S^1)^\mathbb{N}$  and  $X$  be the closure of  $\{\omega, T(\omega), T^2(\omega), \dots\}$ , where  $\omega = (f(1), f(2), \dots)$ . By corollary 9.2,

$$\{(\theta n^2, \theta(n+1)^2) : n \in \mathbb{N}\}$$

is uniformly distributed mod 1. This implies that  $X \simeq \mathbb{T}^2$ , the two dimensional torus  $\mathbb{R}^2/\mathbb{Z}^2$ . Indeed, for any  $(e^{2\pi i \alpha_1}, e^{2\pi i \alpha_2}) \in \mathbb{T}^2$ , there is a subsequence  $n_k$  such that

$$\theta n_k^2 \rightarrow \alpha_1 \pmod{1} \text{ and } \theta(n_k+1)^2 \rightarrow \alpha_2 \pmod{1}$$

as  $k \rightarrow \infty$ . This implies that

$$2\theta n_k \rightarrow \alpha_2 - \alpha_1 - \theta \pmod{1}.$$

Therefore

$$(n_k + m)^2 \rightarrow \alpha_1 + m(\alpha_2 - \alpha_1 - \theta) + m^2 \theta \pmod{1}, \text{ for any } m \in \mathbb{N}.$$

And the homomorphism from  $X$  onto  $\mathbb{T}^2$  is the projection onto the first two coordinates. Now, consider the map induced by the left shift on  $(S^1)^\mathbb{N}$ . Note that

$$(e^{2\pi i \theta n^2}, e^{2\pi i \theta(n+1)^2}) \rightarrow (e^{2\pi i \theta(n+1)^2}, e^{2\pi i \theta(n+2)^2}).$$

Therefore the map is

$$\varphi : (e^{2\pi i \alpha_1}, e^{2\pi i \alpha_2}) \rightarrow (e^{2\pi i \alpha_2}, e^{2\pi i (2\alpha_2 - \alpha_1 + 2\theta)}).$$

If we identifies  $\mathbb{T}^2$  with  $[0, 1) \times [0, 1)$ , then

$$\varphi((\alpha_1, \alpha_2)) = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2\theta \end{pmatrix}.$$

Or, we can represent the transformation by affine transformation matrix as following:

$$\varphi((\alpha_1, \alpha_2)) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & 2\theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 1 \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}.$$

**Remark 9.3.** A continuous automorphism  $\varphi$  of the  $n$ -dimensional torus  $\mathbb{T}^n$  is associated with a unimodular matrix  $\Phi = (\psi_{ij})$ , i.e.,  $\psi_{ij}$  are integers and  $\det \Phi = \pm 1$ . This transformation preserves Haar measure; and it is ergodic if and only if no characteristic value of  $\Phi$  is a root of unity.

Note that  $\varphi$  is a continuous automorphism of  $\mathbb{T}^2$ . By induction, we have

$$A^n = \begin{pmatrix} 1-n & n \\ -n & 1+n \end{pmatrix}.$$

From this, it is easy too see that  $\varphi$  is distal. By [9], we have  $h_{top}(\varphi) = 0$ .

**Example 9.4.** *Let*

$$f(n) = \begin{cases} \frac{1}{\log p}, & \text{if } n = p, p \text{ is prime,} \\ 0, & \text{otherwise.} \end{cases}$$

*It is clear that  $C(f) = C_0(\mathbb{N}) \cup \{CI\}$ . Therefore the spectrum of  $C(f)$  is just the one-point compactification of  $\mathbb{N}$ . And  $h_{\text{top}}(\sigma) = 0$ , where  $\sigma$  is the left shift on  $\mathbb{N} \cup \{\infty\}$ .*

**Example 9.5.** *Let  $X = \{\log p : p \text{ is prime}\} \cup \{\infty\}$  and  $\omega = \{f(1), f(2), \dots\}$  where*

$$f(n) = \begin{cases} \log p, & \text{if } n = p, p \text{ is prime,} \\ 0, & \text{otherwise.} \end{cases}$$

*Consider the dynamic system  $(X, T)$  where  $T$  is the left shift of  $X$ .*

*It will be interesting to consider the dynamic system determined by the von Mangoldt function. Recall that von Mangoldt is defined as*

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^k, p \text{ is prime and } k \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

**Example 9.6.** *Let*

$$f(n) = \begin{cases} \frac{1}{\log p}, & \text{if } n = p^m, p \text{ is prime,} \\ 0, & \text{otherwise.} \end{cases}$$

## 10. AF ALGEBRAS

**Definition 10.1.** *Let  $\mathfrak{A}$  be a unital  $C^*$ -algebra and  $\text{Inn}(\mathfrak{A}) = \{AdU : U \in \mathcal{U}(\mathfrak{A})\}$ . A automorphism  $\alpha$  on  $\mathfrak{A}$  is called approximately inner if, for every finite subset  $F$  of  $\mathfrak{A}$  and for every  $\varepsilon > 0$ , there is a unitary  $U$  such that  $\|\alpha(A) - U^*AU\| < \varepsilon$  for all  $A \in F$ .*

**Remark 10.1.** *One can check that if  $\mathfrak{A}$  is separable and if  $\alpha$  is an automorphism on  $\mathfrak{A}$ , then  $\alpha$  is approximately inner if and only if there exists a sequence  $\{U_n\}$  in  $\mathcal{U}(\mathfrak{A})$  such that*

$$\lim_{n \rightarrow \infty} U_n^*AU_n = \alpha(A), \forall A \in \mathfrak{A}.$$

**Definition 10.2.** *Let  $\mathfrak{A}$  be a unital separable  $C^*$ -algebra. Denote by  $A\text{Inn}(A)$  the group of all asymptotically inner automorphisms. An automorphism  $\alpha$  is said to be strongly asymptotically inner if there is a continuous path of unitaries  $\{U(t) : t \in [0, \infty)\}$  of  $\mathfrak{A}$  such that*

$$U(0) = I \text{ and } \lim_{t \rightarrow \infty} U(t)^*AU(t) = \alpha(A) \text{ for all } A \in \mathfrak{A}.$$

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