# Entropy

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ABSTRACT. Enter abstract here

## 1. INTRODUCTION

Introduction here!

## 2. ENTROPY

By $\mathscr{H}$ we shall denote a complex separable Hilbert space of infinite dimension, by $\mathscr{P}f(\mathscr{H})$ the finite subsets of $\mathscr{H}$ and by $\mathscr{F}(\mathscr{H})$ the finite-dimensional subspaces of $\mathscr{H}$. If $\omega \in \mathscr{P}f(\mathscr{H})$ and $A \subset \mathscr{H}$ we shall write $\omega \subset_\delta A$ if for every $h \in \omega$ we can find $h' \in A$ such that $\|h - h'\| < \delta$.

**Definition 2.1.** *If $\omega \in \mathscr{P}f(\mathscr{H})$ and $\delta > 0$, we define*

$$d(\omega;\delta) = \inf\{dim\chi : \chi \in \mathscr{F}(\mathscr{H}), \omega \subset_\delta \chi\}.$$

Suppose that $G$ be a group with a finite set of generators $\Sigma$. Let $c_n = c_n(\Sigma)$ be number of elements of $G$ whose shortest representative in $\Sigma \cup \Sigma^{-1}$ has exactly length $n$. The growth function $C(z)$ of $(G, \Sigma)$ is the formal power series $C(z) = \sum c_n(\Sigma) z^n$. For $F_m$ the free group on $m$ generators, the growth function with respect to a free basis is

$$C(z) = \frac{1+z}{1-(2m-1)z}.$$

Thus $c_0 = 1$ and $c_n = 2m(2m-1)^{n-1}$ for $n > 1$.

Let $b_n = b_n(G, \Sigma) = \sum_{i=0}^{n} c_i$ be the number of elements of $G$ that can be expressed in terms of words of length at most $n$ in the generating set $\Sigma \cup \Sigma^{-1}$. The growth rate of $G$ with respect to $\sigma$ is defined to be

$$r(G, \Sigma) = \lim_{n \to \infty} \sqrt[n]{b_n(G, \Sigma)}.$$

If $r(G, \Sigma) = 1$, i.e., $G$ has subexponential growth rate, then $G$ is amenable. For $F_m$, we have

$$b_n(F_m, \Sigma) = \begin{cases} 2m & \text{if } m = 1, \\ 1 + \frac{m[(2m-1)^n - 1]}{m-1} & \text{if } m > 1. \end{cases} \qquad m > 1.$$

Thus, $r(F_m, \Sigma) = 2m - 1$. In general, there's no particular connection between rate of growth and amenability between these two extremes. In [2] is showed that for each $m > 1$, there is a sequence of nonamenable groups on $m$ generators whose growth rates approach 1. On the other hand, in [3] is exhibited for each $m > 1$ a sequence of amenable groups on $m$ generators whose growth rates approach $2m - 1$.

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In the rest of this note, we will use $\kappa_m(n)$ to denote $b_n(F_m, \Sigma)$.

On the one hand, Let $\mathfrak{A}$ be a von Neumann algebra and $\mathscr{F}U(\mathfrak{A})$ be the finite subsets of unitaries of $\mathfrak{A}$. Suppose that $\Sigma = \{U_1, \ldots, U_m\} \in \mathscr{F}U(\mathfrak{A})$, let

$$(\Sigma \cup \Sigma^{-1})^n = \{V_{i_1} V_{i_2} \cdots V_{i_n} : V_{i_k} \in \Sigma \cup \Sigma^{-1} \cup \{I\}\}.$$

**Definition 2.2.** *If $\delta > 0$, $\omega \in \mathscr{P}f(\mathscr{H})$ and $\Sigma = \{U_1, \ldots, U_m\} \in \mathscr{F}U(\mathfrak{A})$, we define*

$$\begin{aligned} fh(\Sigma, \omega; \delta) &= \limsup_{n\to\infty} \frac{1}{\kappa_m(n)} d\left((\Sigma \cup \Sigma^{-1})^n \omega; \delta\right) \\ fh(\Sigma, \omega) &= \sup_{\delta>0} h(\Sigma, \omega; \delta), \\ fh(\Sigma, \mathscr{H}) &= \sup\{h(\Sigma, \omega) : \omega \in \mathscr{P}f(\mathscr{H})\}. \end{aligned}$$

**Definition 2.3.** *If $\delta > 0$, $\omega \in \mathscr{P}f(\mathscr{H})$ and $\Sigma = \{U_1, \ldots, U_m\} \in \mathscr{F}U(\mathfrak{A})$. Let $G$ be the group generated by $\Sigma$. We define*

$$\begin{aligned} h(\Sigma, \omega; \delta) &= \limsup_{n\to\infty} \frac{1}{b_n(G, \Sigma)} d\left((\Sigma \cup \Sigma^{-1})^n \omega; \delta\right) \\ h(\Sigma, \omega) &= \sup_{\delta>0} h(\Sigma, \omega; \delta), \\ h(\Sigma, \mathscr{H}) &= \sup\{h(\Sigma, \omega) : \omega \in \mathscr{P}f(\mathscr{H})\}. \end{aligned}$$

Since $\kappa_m(n) \geq b_n(G, \Sigma)$, we have the following lemma.

**Lemma 2.1.** *$fh(\Sigma, \mathscr{H}) \leq h(\Sigma, \mathscr{H})$.*

**Lemma 2.2.** *Let $\Sigma \in \mathscr{F}U(\mathfrak{A})$ and $\omega_j \in \mathscr{P}f(\mathscr{H})$, $j \in \mathbb{N}$, $\omega_1 \subset \omega_2 \subset \ldots$, such that $\bigcup_{j\in\mathbb{N}} \omega_j$ is a dense subset of $\mathscr{H}$. Then*

$$h(\Sigma, \mathscr{H}) = \sup_{j\in\mathbb{N}} h(\Sigma, \omega_j).$$

**Lemma 2.3.** *Let $\Sigma \in \mathscr{F}U(\mathfrak{A})$ and $\omega_j \in \mathscr{P}f(\mathscr{H}) \cap (\mathscr{H})_1$, $j \in \mathbb{N}$, $\omega_1 \subset \omega_2 \subset \ldots$, such that $\bigcup_{j\in\mathbb{N}} \omega_j$ is a dense subset of $(\mathscr{H})_1$. Then*

$$h(\Sigma, \mathscr{H}) = \sup_{j\in\mathbb{N}} h(\Sigma, \omega_j).$$

*Proof.* Let $C = \max\{\|\xi\| : \xi \in \omega\}$. By the assumptions, there is $j$ such that

$$\{\frac{\xi}{\|\xi\|} : \xi \in \omega\} \subset_{\frac{\delta}{2C}} \omega_j$$

It is easy to see that

$$(\Sigma \cup \Sigma^{-1})^n \omega_j \subset_{\frac{\delta}{2C}} B$$

implies

$$(\Sigma \cup \Sigma^{-1})^n \omega \subset_\delta B.$$

Hence

$$\begin{aligned} h(\Sigma, \omega; \delta) &= \limsup_{n\to\infty} \frac{1}{b_n(G, \Sigma)} d\left((\Sigma \cup \Sigma^{-1})^n \omega; \delta\right) \\ &\leq \limsup_{n\to\infty} \frac{1}{b_n(G, \Sigma)} d\left((\Sigma \cup \Sigma^{-1})^n \omega_j; \frac{\delta}{2C}\right) = h(\Sigma, \omega_j; \frac{\delta}{2C}). \end{aligned}$$

□

**Remark 2.1.** *Lemma 2.2 and Lemma 2.3 are also true for $fh(\Sigma)$.*

**Lemma 2.4.** *Let $\Sigma \in \mathscr{F}U(\mathfrak{A})$ and $\omega_j \in \mathscr{P}f(\mathscr{H})$, $j \in \mathbb{N}$, $\omega_1 \subset \omega_2 \subset \ldots$, be such that $\bigcup_{j\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} ((\Sigma \cup \Sigma^{-1})^n(\omega_j)$ spans a dense subspace of $\mathscr{H}$. If the group generated by $\Sigma$ is amenable, then*

$$h(\Sigma) = \sup_{j\in\mathbb{N}} h(\Sigma, \omega_j).$$

*Proof.* It suffices to show that given $\omega \in \mathscr{P}f(\mathscr{H})$ and $\delta > 0$ there is $\delta_1 > 0$ and $\omega_j$ such that

$$h(\Sigma, \omega; \delta) \leq h(\Sigma, \omega_j; \delta_1).$$

By the assumptions, there is $N \in \mathbb{N}$ so that

$$\omega \subset_{\frac{\delta}{2}} Nco\left(\mathbb{T}\left((\Sigma \cup \Sigma^{-1})^N \omega_j)\right)\right),$$

where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $co$ denotes the convex hull.

Let $\delta_1 = \frac{\delta}{2Nn_j^{\kappa m(N)}}$ where $n_j = \#\omega_j$. Then for $B \in \mathscr{F}(\mathscr{H})$ and

$$(\Sigma \cup \Sigma^{-1})^{N+n} \omega_j \subset_{\delta_1} B$$

implies

$$(\Sigma \cup \Sigma^{-1})^n Nco\left(\mathbb{T}\left((\Sigma \cup \Sigma^{-1})^N \omega_j)\right)\right) \subset_{\frac{\delta}{2}} B.$$

Therefore

$$(\Sigma \cup \Sigma^{-1})^n \omega \subset_{\delta} B.$$

Hence

$$\begin{aligned} h(\Sigma, \omega; \delta) &= \limsup_{n\to\infty} \frac{1}{b_n(G, \Sigma)} d\left((\Sigma \cup \Sigma^{-1})^n \omega; \delta\right) \\ &\leq \limsup_{n\to\infty} \frac{b_{n+N}(G, \Sigma)}{b_n(G, \Sigma)} \frac{1}{b_{n+N}(G, \Sigma)} d\left((\Sigma \cup \Sigma^{-1})^{N+n} \omega_j; \delta\right) = h(\Sigma, \omega_j; \delta_1) \end{aligned}$$

□

**Lemma 2.5** (Proposition 8.6 in [1]). *Suppose $\mathfrak{A}$ is a von Neumann algebra acting on $\mathscr{H}$ and $\Sigma = \{U_1, \ldots, U_m\} \in \mathscr{F}U(\mathfrak{A})$, let $\Sigma \otimes I_n = \{U_i \otimes I_n : U_i \in \Sigma\} \in \mathscr{F}(\mathfrak{A} \otimes I_n)$, where $\mathfrak{A} \otimes I_n$ acting on $\mathscr{H} \otimes l^2(\mathbb{Z}_n)$. We have*

$$nh(\Sigma, \mathscr{H}) = h(\Sigma \otimes I, \mathscr{H} \otimes l^2(\mathbb{Z}_n)).$$

**Lemma 2.6** (Proposition 8.4 in [1]). *Let $\mathfrak{A}$ be a von Neumann algebra acting on $\mathscr{H}$. Suppose that $\mathscr{H}_1 \subset \mathscr{H}_2 \subset \ldots \subset \mathscr{H}$ are invariant subspaces of $\mathfrak{A}$ and $\bigcup_j \mathscr{H}_j = \mathscr{H}$. Then*

$$h(\Sigma, \mathscr{H}) = \sup_{j\in\mathbb{N}} h(\Sigma, \mathscr{H}_j).$$

**Proposition 2.1** (Proposition 8.8 in [1]). *Let $\mathfrak{A}$ be a finite factor acting on $\mathscr{H}$. Then*

$$h(\Sigma, \mathscr{H}) = dim_{l^2(\mathfrak{A},\mu)} \mathscr{H} \times h(\Sigma, l^2(\mathfrak{A}, \mu)),$$

*where $dim_{l^2(\mathfrak{A},\mu)}\mathscr{H}$ is the von Neumann dimension of $\mathscr{H}$.*

## 3. Folner Entropy

**Definition 3.1.** *Let $O\mathscr{P}f(\mathscr{H})$ be the set contains the finite orthonormal subsets of $\mathscr{H}$.*

**Remark 3.1.** *By Lemma 7.8 in* [1], *we have*

$$d(\omega;\delta) \geq n(1-\delta^2),$$

*where $\omega = \{e_1, \ldots, e_n\} \in O\mathscr{P}(\mathscr{H})$. Therefore we will use the following definition.*

**Definition 3.2.** *Let $\mathfrak{A}$ be a von Neumann algebra. If $\delta > 0$, $\omega \in O\mathscr{P}f(\mathscr{H})$ and $\Sigma = \{U_1, \ldots, U_m\} \in \mathscr{F}U(\mathfrak{A})$, we define*

$$\begin{aligned} Foh(\Sigma;\delta) &= \inf_{\omega} \frac{1}{dim(\omega)} d\left((\Sigma \cup \Sigma^{-1})\omega;\delta\right) \\ Foh(\Sigma,\mathscr{H}) &= \limsup_{\delta\to 0} Foh(\Sigma;\delta), \\ Foh(\mathfrak{A},\mathscr{H}) &= \inf_{\Sigma}\{Foh(\Sigma,\mathscr{H}) : \omega \in \mathscr{P}f(\mathscr{H}), \text{ and } \Sigma \text{ generates } \mathfrak{A}\ \}. \end{aligned}$$

**Remark 3.2.** *It is easy to see that $Foh(\mathfrak{A},\mathscr{H}) \geq 1$.*

**Lemma 3.1.** *If $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ and $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2$, then*

$$Foh(\mathfrak{A},\mathscr{H}) \leq \min(Foh(\mathfrak{A}_1,\mathscr{H}_1), Foinh(\mathfrak{A}_2,\mathscr{H}_2)).$$

**Lemma 3.2.** *Let $\mathcal{R}$ be the hyperfinite $II_1$ factor. $\mathscr{H} = L^2(\mathcal{R},\tau)$ where $\tau$ is the faithful normal trace on $\mathcal{R}$. For any subset of unitaries $\Sigma = \{U_1, \ldots, U_n\}$ of $\mathcal{R}$, we have*

$$Foh(\Sigma,\mathscr{H}) = 1.$$

*Proof.* Let $\delta > 0$. Since $\mathcal{R}$ is hyperfinite, there exist a type $\mathrm{I}_n$ subfactor $\mathcal{N}$ of $\mathcal{R}$ such that there are $n$ unitaries $V_1 \ldots V_n$ in $\mathcal{N}$ satisfying

$$\|V_i - U_i\|_2 \leq \delta.$$

Let

$$W = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & \gamma & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \gamma^{n-2} & 0 \\ 0 & 0 & \ldots & 0 & \gamma^{n-1} \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}$$

be two unitaries that generates $\mathcal{N}$, where $\gamma = e^{\frac{2\pi i}{n}}$. Then $\omega = \{W^i S^j \Omega : i,j \in \{0, \ldots, n-1\}\}$ is an orthonormal subsets in $F = \{A\Omega : A \in \mathcal{N}\}$, where $\Omega$ is a trace vector. It is not hard to see that

$$\frac{1}{dim(\omega)} d\left((\Sigma \cup \Sigma^{-1})\omega;\delta\right) = 1.$$

This clearly implies that $Foh(\Sigma,\mathscr{H}) = 1$. □

**Lemma 3.3.** *Let $\mathfrak{A}$ be a $II_1$ factor. If there is a subset of unitaries $\Sigma = \{U_1, \ldots, U_n\}$ in $\mathfrak{A}$ such that $\Sigma$ generates $\mathfrak{A}$ and $Foh(\mathfrak{A}, L^2(\mathfrak{A},\tau)) = 0$, then $\mathfrak{A}$ is hyperfinite.*

## REFERENCES

[1] Dan Voiculescu, *Dynamical approximation entropies and topological entropy in operator algebras*, Comm. Math. Phys. Volume 170, Number 2 (1995)

[2] G.N. Arzhantseva, V.S. Guba and L. Guyot. *Growth rates of amenable groups*, Journal of Group Theory, 8 (2005), no.3, 389-394.

[3] R. Grigorchuk and P. de la Harpe, *Limit behaviour of exponential growth rates for finitely generated groups*, Monographie de L'Enseignement Mathematique 38 (2001), 351-370.

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