

ON GENERATORS OF ABELIAN KADISON-SINGER ALGEBRAS IN MATRIX ALGEBRAS

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ABSTRACT. Assume that \mathcal{H} is a Hilbert space of dimension greater than two. We prove that an abelian Kadison-Singer algebra acting on \mathcal{H} can not contain any non-trivial idempotent. Based on this, we show that an abelian KS-algebra in matrix algebra $M_n(\mathbb{C}) (n \geq 3)$ can not be generated by a single element. As a corollary, it is also proved that the lattice of an abelian KS-algebra can not be completely distributive.

1. INTRODUCTION

In 1960, Kadison and Singer introduced and studied a class of non selfadjoint operator algebras which they called “triangular algebras” (subalgebras of a C^* -algebra whose diagonal is maximal abelian selfadjoint) [2]. The most well understood non selfadjoint algebras are the Nest algebras introduced by Ringrose [3]. It is a class of maximal triangular algebras. Reflexive algebras are generalizations of nest algebras. These algebras are completely determined by their lattices of invariant subspace projections. In [14] and [15], Ge and Yuan combine triangularity, reflexivity and von Neumann algebra properties in a single class of algebras and introduce Kadison-Singer algebras (KS-algebras for simplicity).

Before giving the definition of KS-algebras, we will fix some notation and review some definitions. Throughout the paper, \mathcal{H} will be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators acting on \mathcal{H} . If \mathcal{L} is a collection of self-adjoint projections of $\mathcal{B}(\mathcal{H})$, $\text{Alg}(\mathcal{L})$ is used to denote the set of bounded operators that leave the range of every member of \mathcal{L} invariant, i.e. $\text{Alg}(\mathcal{L}) = \{T \in \mathcal{B}(\mathcal{H}) : (I - P)TP = 0, \text{ for any } P \text{ in } \mathcal{L}\}$. Dually, if \mathfrak{A} is a set of operators in $\mathcal{B}(\mathcal{H})$, $\text{Lat}(\mathfrak{A})$ is used to denote the collection of projections whose ranges are left invariant by every element of \mathfrak{A} , i.e. $\text{Lat}(\mathfrak{A}) = \{P \in \mathcal{B}(\mathcal{H}) : P^* = P, P^2 = P \text{ and } (I - P)TP = 0 \text{ for any } T \text{ in } \mathfrak{A}\}$. A subalgebra \mathfrak{A} of $\mathcal{B}(\mathcal{H})$ is called reflexive if $\mathfrak{A} = \text{Alg}(\text{Lat}(\mathfrak{A}))$.

Definition 1.1 (Definition 1[14]). *Let \mathcal{H} be a Hilbert space. A subalgebra \mathfrak{A} of $\mathcal{B}(\mathcal{H})$ is called a Kadison-Singer algebra (or KS-algebra) if \mathfrak{A} is reflexive and maximal with respect to the diagonal subalgebra $\mathfrak{A} \cap \mathfrak{A}^*$ of \mathfrak{A} , in the sense that if there is another reflexive subalgebra \mathfrak{B} of $\mathcal{B}(\mathcal{H})$ such that $\mathfrak{A} \subset \mathfrak{B}$ and $\mathfrak{B} \cap \mathfrak{B}^* = \mathfrak{A} \cap \mathfrak{A}^*$, then $\mathfrak{A} = \mathfrak{B}$.*

It is easy to deduce from the definition that the invariant subspace lattice of a KS-algebra \mathfrak{A} is reflexive and minimal generating the commutant of the diagonal subalgebra of \mathfrak{A} as a von Neumann algebra (for a comprehensive treatment of the theory of von Neumann algebra one may refer to the book [1]). So the theory of KS-algebras is closely related with the theory of von Neumann algebras. Actually,

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one of the main purposes to introduce the KS-algebra, as stated in the introduction of [14], is to recapture the synergy that should exist between the powerful techniques that have developed in selfadjoint operator algebra theory and those of the non-selfadjoint theory. In [14] and [15] many interesting examples of KS-algebras with II_1 factor as their diagonal subalgebras have been constructed. Since these algebras contain II_1 factors, none of them can be commutative. So it is quite natural to ask whether there exist non-trivial abelian KS-algebras. The following example provides an affirmative answer to this question.

Example 1.1 (Example 3.2[16]). *Let $S = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}$ be an idempotent in $M_2(\mathbb{C})$. Then*

$$\mathfrak{A}_a = \text{Alg}(\{P_{\xi_1}, P_{\xi_2}\}) = \{S\}' = \left\{ \begin{pmatrix} x & a(x-y) \\ 0 & y \end{pmatrix} : x, y \in \mathbb{C} \right\},$$

where $\xi_1 = (1, 0)^t$, $\xi_2 = (-a, 1)^t$, and P_{ξ_1} , P_{ξ_2} are the orthogonal projections onto the subspaces spanned by ξ_1 and ξ_2 respectively. It is easy to check that \mathfrak{A}_a is reflexive and similar to the diagonal algebra $\mathfrak{A}_0 = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x, y \in \mathbb{C} \right\}$. We assert that \mathfrak{A}_a is an KS-algebra, provided that $a \neq 0$. If the assertion was false, then there would exist a reflexive algebra \mathfrak{B} contains \mathfrak{A}_a and $\mathfrak{B} \cap \mathfrak{B}^* = \mathfrak{A}_a \cap \mathfrak{A}_a^* = \mathbb{C}I$. It is clear that \mathfrak{B} could only be $M_2(\mathbb{C})$, $\text{Alg}(\{P_{\xi_1}\})$ or $\text{Alg}(\{P_{\xi_2}\})$ if $\mathfrak{B} \neq \mathfrak{A}_a$. However, these algebras either contain P_{ξ_1} or P_{ξ_2} . This contradicts the fact that $\mathfrak{B} \cap \mathfrak{B}^* = \mathbb{C}I$.

So far these are the simplest non-trivial examples of abelian KS-algebras we have. Note that the algebras in Example 1.1 are similar to the algebra of diagonal matrices in $M_2(\mathbb{C})$. So it is interesting to ask whether we can find new abelian KS-algebras by "twisting" the algebra of diagonal matrices in $M_n(\mathbb{C})$ ($n \geq 3$). More generally, we can ask how to find new examples of abelian KS-algebras, besides the algebras given in Example 1.1, if there are any. It is known that there are abelian reflexive subalgebras of matrix algebras generated by a single matrix [10]. Hence, it is also natural to ask whether there exist abelian KS-subalgebras of matrix algebras generated by one element. We will provide a negative answer to the last question if the size of the matrices is greater than 2 and shed some light on other questions.

There are four sections in this paper. In section two, we give some properties of abelian KS-algebras and prove an abelian KS-algebra acting on Hilbert space of dimension greater than 2 can not contain any non-trivial idempotent. Therefore, any abelian von Neumann algebra of dimension greater than 2 can not similar to an abelian KS-algebra. In section 3, by using the results we get in section 2, we show that an abelian KS-subalgebra of $M_n(\mathbb{C})$ ($n \geq 3$) can not be a reflexive algebra determined by a single element. In the last section, we prove that the lattice of an abelian KS-subalgebra of $\mathcal{B}(\mathcal{H})$ ($\dim \mathcal{H} \geq 3$) can not be completely distributive.

2. PROPERTIES OF ABELIAN KS-ALGEBRAS

Note that the algebras in Example 1.1 contain non-trivial idempotents in their centers. However, if an algebra is a KS-algebra (Definition 1.1), its center can not contain any non-trivial orthogonal projections. More precisely, we will show that the center of a KS-algebra contains no non-trivial selfadjoint elements.

The following fact, which will be needed in the proof of Lemma 2.1 and in Section 4, is proved in [3] and [11].

Let ξ and β be two vectors in \mathcal{H} and $\xi \otimes \beta$ the rank one operator defined by $\eta \rightarrow \langle \eta, \xi \rangle \beta$ for all η in \mathcal{H} . If \mathcal{L} is a subspace lattice on \mathcal{H} , then the operator $\xi \otimes \beta$ belongs to $\text{Alg}(\mathcal{L})$ if and only if there is a projection P in \mathcal{L} such that $\beta \in P\mathcal{H}$ and $\xi \in (I - P_-)\mathcal{H}$ where $P_- = \vee \{Q \in \mathcal{L} : P \not\leq Q\}$.

Lemma 2.1. *The center $\mathcal{C}(\mathfrak{A})$ of a KS-algebra $\mathfrak{A}(\subset \mathcal{B}(\mathcal{H}))$ contains no non-trivial selfadjoint elements.*

Proof. Suppose that $\mathcal{C}(\mathfrak{A})$ contains a non-trivial selfadjoint element P . Since \mathfrak{A} is weak operator topology closed, we have the spectrum projections of P are in $\mathcal{C}(\mathfrak{A})$. Thus we may assume that P is a projection. Choose a proper orthonormal basis so that P can be written as $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$. Because P commutes with every element in \mathfrak{A} and $\mathcal{L}at(\mathfrak{A})$, it is easy to see that $\mathfrak{A} \cong \mathfrak{A}_1 \oplus \mathfrak{A}_2$, where $\mathfrak{A}_1 = P\mathfrak{A}P$ and $\mathfrak{A}_2 = (I - P)\mathfrak{A}(I - P)$. Thus

$$\mathcal{L}at(\mathfrak{A}) = \left\{ \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} : Q_1 \in \mathcal{L}at(\mathfrak{A}_1) \text{ and } Q_2 \in \mathcal{L}at(\mathfrak{A}_2) \right\}.$$

Let

$$\begin{aligned} \tilde{\mathcal{L}} &= \{Q \in \mathcal{L}at(\mathfrak{A}) : Q \leq P \text{ or } P \leq Q\} \\ &= \left\{ \begin{pmatrix} Q_1 & 0 \\ 0 & 0 \end{pmatrix} : Q_1 \in \mathcal{L}at(\mathfrak{A}_1) \right\} \cup \left\{ \begin{pmatrix} I & 0 \\ 0 & Q_2 \end{pmatrix} : Q_2 \in \mathcal{L}at(\mathfrak{A}_2) \right\}. \end{aligned}$$

Then $\tilde{\mathcal{L}}$ is a sub-lattice of $\mathcal{L}at(\mathfrak{A})$. Since P is in $\tilde{\mathcal{L}}$, the von Neumann algebras generated by $\tilde{\mathcal{L}}$ and $\mathcal{L}at(\mathfrak{A})$ are the same. Therefore, $\tilde{\mathcal{L}}'' = \mathcal{L}at(\mathfrak{A})''$ and $\tilde{\mathcal{L}}' = \mathcal{L}at(\mathfrak{A})'$. This implies that $\mathcal{A}lg(\tilde{\mathcal{L}})^* \cap \mathcal{A}lg(\tilde{\mathcal{L}}) = \tilde{\mathcal{L}}' = \mathcal{L}at(\mathfrak{A})' = \mathfrak{A}^* \cap \mathfrak{A}$. Let ξ be a vector in $(I - P)\mathcal{H}$ and β a vector in $P\mathcal{H}$. Note that $P_- \subseteq P$, where $P_- = \vee\{Q \in \tilde{\mathcal{L}} : P \not\leq Q\}$. It follows that $\xi \in (I - P_-)\mathcal{H}$ and the rank one operator $\xi \otimes \beta$ is in $\mathcal{A}lg(\tilde{\mathcal{L}})$. However, $P(\xi \otimes \beta)(I - P) = \xi \otimes \beta \neq 0$. Since $I - P$ is in $\mathcal{L}at(\mathfrak{A})$, $\xi \otimes \beta$ is not in \mathfrak{A} . This contradicts the maximality of \mathfrak{A} and the initial assumption must be false. \square

In particular, if the algebra \mathfrak{A} is abelian, then the center of \mathfrak{A} equals to itself and we have the following corollary.

Corollary 2.1. *If $\mathfrak{A}(\subset \mathcal{B}(\mathcal{H}))$ is an abelian KS-algebra, then $\mathfrak{A}^* \cap \mathfrak{A} = \mathbb{C}I$ and $\mathcal{L}at(\mathfrak{A})'' = \mathcal{B}(\mathcal{H})$.*

The following is a technical result that will be used later. This fact might be well known, we sketch a proof here for the reader's convenience.

Lemma 2.2. *Suppose that \mathfrak{A} is an abelian subalgebra of $M_n(\mathbb{C})$ and $\mathcal{L} = \mathcal{L}at(\mathfrak{A})$. If P_1, P_2 are two projections in \mathcal{L} such that $P_1 < P_2$ and $\dim(P_2 - P_1) \geq 2$, then there exists a $P_3 \in \mathcal{L}$ satisfying $P_1 < P_3 < P_2$.*

Proof. Since $\dim(P_2 - P_1) \geq 2$, the abelian algebra $(P_2 - P_1)\mathfrak{A}(P_2 - P_1)$ has a non-trivial invariant subspace Q . Let $P_3 = P_1 + Q$. Then $P_1 < P_3 < P_2$ and it is not hard to check that $P_3 \in \mathcal{L}$. \square

Before proceeding further, let us recall some definitions. A nest \mathcal{N} is a complete totally ordered family of selfadjoint projections on a Hilbert space \mathcal{H} containing 0 and I . A nest is maximal if it is not contained in any larger nest. The algebra $\mathcal{A}lg(\mathcal{N})$ is called the nest algebra with respect to the nest \mathcal{N} . The relevant background of the theory of nest algebras can be found in [9] and [6].

Combine Lemma 2.3 with the fact that nest algebras are non abelian, we proved the following corollary.

Corollary 2.2. *If \mathfrak{A} is an abelian algebra in $M_n(\mathbb{C})$ and $n \geq 2$, then $\mathcal{L}at(\mathfrak{A})$ contains at least two maximum nests.*

Let S be a closed idempotent (may be unbounded), i.e., S is a densely defined closed operator such that $S^2 = S$. The equality is to be understood in the strict sense that S^2 and S have the same domain $\mathcal{D}(S)$ and $S^2\xi = S\xi$ for any $\xi \in \mathcal{D}(S)$. It is not hard to see that $I - S$ is also a closed idempotent, and the range of S and $I - S$ are both closed. Let P, Q be the range and kernel of S respectively. If

$\xi \in \mathcal{D}(S)$, then $\xi = S\xi + (I - S)\xi$. Thus $\mathcal{D}(S) = P + Q = \{\xi + \beta : \xi \in P, \beta \in Q\}$. Note that P equals to the kernel of $I - S$, Q equals to the range of $I - S$. It is easy to check that $P \wedge Q = 0$ and $P \vee Q = I$.

Conversely, if P and Q are two projections such that $P \wedge Q = 0$ and $P \vee Q = I$. Since $\{(\xi + \beta, \xi) : \xi \in P, \beta \in Q\} (\subset \mathcal{H} \oplus \mathcal{H})$ is closed, $S : \xi + \beta \rightarrow \xi$ is a closed operator with dense domain $\mathcal{D}(S) = \{\xi + \beta : \xi \in P, \beta \in Q\}$. It is clear that $S^2 = S$. Therefore, each pair of projections P, Q satisfying $P \wedge Q = 0, P \vee Q = I$ determines a closed idempotent and vice versa.

From the above discussion, the following lemma is immediate, and we omit the proof.

Lemma 2.3. *Let \mathcal{H} be a Hilbert space. If \mathfrak{A} is a subalgebra of $\mathcal{B}(\mathcal{H})$, then there are two projections P, Q in $\mathcal{Lat}(\mathfrak{A})$ satisfying $P \wedge Q = 0, P \vee Q = I$ if and only if there is a closed idempotent S such that $TS \subset ST$ for any $T \in \mathfrak{A}$. In particular, if \mathcal{H} is finite-dimensional, then S is bounded. Therefore, P and Q are in $\mathcal{Lat}(\mathfrak{A})$ iff every elements in \mathfrak{A} commute with S .*

Lemma 2.4. *Let \mathfrak{A} be a subalgebra of $\mathcal{B}(\mathcal{H})$. If the center of \mathfrak{A} contains an idempotent S , then the range projections P, Q of S and $I - S$ are in $\mathcal{Lat}(\mathfrak{A})$, and $E = (E \wedge P) \vee (E \wedge Q)$ for each $E \in \mathcal{Lat}(\mathfrak{A})$.*

Proof. The first part of the lemma is just a restatement of Lemma 2.3. For any $\xi \in E\mathcal{H}$, we have $S\xi \in E \wedge P$ and $(I - S)\xi \in Q \wedge E$. This clearly implies that $E = (E \wedge P) \vee (E \wedge Q)$. \square

The following observation is certainly known, but we could not find a reference. For the sake of completeness, we provide a short proof.

Lemma 2.5. *Let \mathfrak{A} be a reflexive subalgebra of $\mathcal{B}(\mathcal{H})$. If S is an idempotent in the center of \mathfrak{A} , then $P\mathfrak{A}P|_{P\mathcal{H}}$ and $(I - P)\mathfrak{A}(I - P)|_{(I - P)\mathcal{H}}$ are reflexive algebras, where P is the range projections of S .*

Proof. To prove $P\mathfrak{A}P|_{P\mathcal{H}}$ is reflexive, we only need to show that $P\mathfrak{A}P|_{P\mathcal{H}} = \mathcal{Alg}(\mathcal{L})$ where $\mathcal{L} = \{E|_{P\mathcal{H}} : E \leq P, E \in \mathcal{Lat}(\mathfrak{A})\}$. $P\mathfrak{A}P|_{P\mathcal{H}} \subseteq \mathcal{Alg}(\mathcal{L})$ is clear. Suppose now that T is an element in $\mathcal{Alg}(\mathcal{L})$, let $\tilde{T} \in \mathcal{B}(\mathcal{H})$ be the operator such that $\tilde{T}(I - P) = (I - P)\tilde{T} = 0$ and $P\tilde{T}P|_{P\mathcal{H}} = T$. By Lemma 2.4, it is easy to check that $\tilde{T}S$ is in \mathfrak{A} and $P\tilde{T}SP|_{P\mathcal{H}} = T$. Thus $P\mathfrak{A}P|_{P\mathcal{H}} = \mathcal{Alg}(\mathcal{L})$.

Note that S^* is in the center of the reflexive algebra \mathfrak{A}^* and $I - P$ is the range projection of $I - S^*$. Therefore, by the argument above, we have $(I - P)\mathfrak{A}^*(I - P)|_{(I - P)\mathcal{H}}$ is reflexive which implies $(I - P)\mathfrak{A}(I - P)|_{(I - P)\mathcal{H}}$ is reflexive. \square

Remark 2.1. *Note that the projection Q onto the kernel of S is also the range projection of $I - S$, thus by Lemma 2.5, we have $Q\mathfrak{A}Q|_{Q\mathcal{H}}$ and $(I - Q)\mathfrak{A}(I - Q)|_{(I - Q)\mathcal{H}}$ are reflexive algebras.*

Remark 2.2. *Suppose $\mathfrak{A}_1(\subset \mathcal{B}(\mathcal{H}_1))$ and $\mathfrak{A}_2(\subset \mathcal{B}(\mathcal{H}_2))$ are two reflexive algebras. If $S = \begin{pmatrix} I & T \\ 0 & 0 \end{pmatrix}$ is an idempotent in $\mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$, then the following algebra*

$$\mathfrak{A} = \left\{ \begin{pmatrix} A & AT - TB \\ 0 & B \end{pmatrix} : A \in \mathfrak{A}_1, B \in \mathfrak{A}_2 \right\} (\cong \mathfrak{A}_1 \oplus \mathfrak{A}_2)$$

is also reflexive.

Lemma 2.6. *Let \mathfrak{A} be a subalgebra of $\mathcal{B}(\mathcal{H})$. If S is an idempotent in the center of \mathfrak{A} , then $P\mathfrak{A}P|_{P\mathcal{H}} \cong (I - Q)\mathfrak{A}(I - Q)|_{(I - Q)\mathcal{H}}$ and $Q\mathfrak{A}Q|_{Q\mathcal{H}} \cong (I - P)\mathfrak{A}(I - P)|_{(I - P)\mathcal{H}}$, where P and Q are the range projections of S and $I - S$ respectively.*

Proof. We could assume that $S = \begin{pmatrix} I & T \\ 0 & 0 \end{pmatrix}$. It is not difficult to see that $\mathfrak{A} = \left\{ \begin{pmatrix} A & AT-TB \\ 0 & B \end{pmatrix} : A \in P\mathfrak{A}P|_{P\mathcal{H}}, B \in (I-P)\mathfrak{A}(I-P)|_{(I-P)\mathcal{H}} \right\}$ and both $P\mathfrak{A}P|_{P\mathcal{H}}$ and $(I-P)\mathfrak{A}(I-P)|_{(I-P)\mathcal{H}}$ are algebras. Let $S = \sqrt{SS^*}V$ be the polar decomposition of S , then $V = \begin{pmatrix} (I+TT^*)^{-\frac{1}{2}} & (I+TT^*)^{-\frac{1}{2}}T \\ 0 & 0 \end{pmatrix}$ is a partial isometry with initial space $(I-Q)\mathcal{H}$ and final space $P\mathcal{H}$. Since $(I-Q)\mathfrak{A}(I-Q)|_{(I-Q)\mathcal{H}} \cong V\mathfrak{A}V^*|_{P\mathcal{H}} = \left\{ \begin{pmatrix} (I+TT^*)^{-\frac{1}{2}}A(I+TT^*)^{\frac{1}{2}} & 0 \\ 0 & 0 \end{pmatrix} : A \in P\mathfrak{A}P|_{P\mathcal{H}} \right\}$, we have $P\mathfrak{A}P|_{P\mathcal{H}} \cong (I-Q)\mathfrak{A}(I-Q)|_{(I-Q)\mathcal{H}}$. Replacing S by $I-S$, P by Q and Q by P , we have $Q\mathfrak{A}Q|_{Q\mathcal{H}} \cong (I-P)\mathfrak{A}(I-P)|_{(I-P)\mathcal{H}}$. \square

Lemma 2.7. *Let \mathfrak{A} be a KS-subalgebra of $\mathcal{B}(\mathcal{H})$. If the center of \mathfrak{A} contains an idempotent $S = \begin{pmatrix} I & T \\ 0 & 0 \end{pmatrix}$, then $P\mathfrak{A}P$, $(I-P)\mathfrak{A}(I-P)$, $Q\mathfrak{A}Q$ and $(I-Q)\mathfrak{A}(I-Q)$ are all KS-algebras, where P and Q are the range projections of S and $I-S$.*

Proof. If there exist reflexive subalgebras \mathfrak{A}_1 of $\mathcal{B}(P\mathcal{H})$ and \mathfrak{A}_2 of $\mathcal{B}((I-P)\mathcal{H})$ such that $P\mathfrak{A}P \subset \mathfrak{A}_1$, $(I-P)\mathfrak{A}(I-P) \subset \mathfrak{A}_2$ and $\mathfrak{A}_1 \cap \mathfrak{A}_1^* = P\mathfrak{A}P \cap P\mathfrak{A}^*P$, $\mathfrak{A}_2 \cap \mathfrak{A}_2^* = (I-P)\mathfrak{A}(I-P) \cap (I-P)\mathfrak{A}^*(I-P)$. Let $\tilde{\mathfrak{A}} = \left\{ \begin{pmatrix} A & AT-TB \\ 0 & B \end{pmatrix} : A \in \mathfrak{A}_1, B \in \mathfrak{A}_2 \right\}$. By Remark 2.2, the algebra $\tilde{\mathfrak{A}}$ is reflexive. Note that

$$\mathfrak{A} = \left\{ \begin{pmatrix} A & AT-TB \\ 0 & B \end{pmatrix} : A \in P\mathfrak{A}P|_{P\mathcal{H}}, B \in (I-P)\mathfrak{A}(I-P)|_{(I-P)\mathcal{H}} \right\}.$$

Thus $\tilde{\mathfrak{A}} \cap \tilde{\mathfrak{A}}^* = \mathfrak{A} \cap \mathfrak{A}^*$. By the maximality of \mathfrak{A} , we have $\tilde{\mathfrak{A}} = \mathfrak{A}$. Therefore $\mathfrak{A}_1 = P\mathfrak{A}P$ and $\mathfrak{A}_2 = (I-P)\mathfrak{A}(I-P)$.

Replacing S by $I-S$, P by Q and Q by P , we have the same results for $Q\mathfrak{A}Q$ and $(I-Q)\mathfrak{A}(I-Q)$. \square

With the help of the preceding lemmas we can now prove the main theorem of the section.

Theorem 2.1. *Let \mathcal{H} be a Hilbert space such that $\dim \mathcal{H} \geq 3$. Suppose that \mathfrak{A} is a reflexive subalgebra of $\mathcal{B}(\mathcal{H})$ such that $\mathfrak{A} \cap \mathfrak{A}^* = \mathbb{C}I$. If the center of \mathfrak{A} contains an idempotent S such that $\mathcal{L}'_1 = \{E \wedge P|_{P\mathcal{H}} : E \in \mathcal{L}at(\mathfrak{A})\}' = \mathbb{C}I$ and $\mathcal{L}'_2 = \{(E \vee P - P)|_{(I-P)\mathcal{H}} : E \in \mathcal{L}at(\mathfrak{A})\}' = \mathbb{C}I$ where P is the range projection of S , then \mathfrak{A} is not a KS-algebra.*

Proof. Note that $I-S^*$ is in the center of \mathfrak{A}^* and $I-P$ is the range projection of $I-S^*$. Since $\mathcal{L}at(\mathfrak{A}^*) = \{I-E : E \in \mathcal{L}at(\mathfrak{A})\}$, we also have

$$\begin{aligned} \widetilde{\mathcal{L}}_1 &= \{[(I-P) \wedge E]|_{(I-P)\mathcal{H}} : E \in \mathcal{L}at(\mathfrak{A}^*)\} \\ &= \{[(I-P) \wedge (I-E)]|_{(I-P)\mathcal{H}} : E \in \mathcal{L}at(\mathfrak{A})\} \\ &= \{(I-E \vee P)|_{(I-P)\mathcal{H}} : E \in \mathcal{L}at(\mathfrak{A})\} = I - \mathcal{L}_2, \\ \widetilde{\mathcal{L}}_2 &= \{[(I-P) \vee E - (I-P)]|_{P\mathcal{H}} : E \in \mathcal{L}at(\mathfrak{A}^*)\} \\ &= \{[(I-P) \vee (I-E) - (I-P)]|_{P\mathcal{H}} : E \in \mathcal{L}at(\mathfrak{A})\} \\ &= \{(P - P \wedge E)|_{P\mathcal{H}} : E \in \mathcal{L}at(\mathfrak{A})\} = I - \mathcal{L}_1. \end{aligned}$$

Therefore $\widetilde{\mathcal{L}}_1' = \mathbb{C}I$ and $\widetilde{\mathcal{L}}_2' = \mathbb{C}I$. If $\dim(I-P)\mathcal{H} < 2$, we could replace S by $I-S^*$, P by $I-P$, \mathcal{L}_1 by $\widetilde{\mathcal{L}}_1$ and \mathcal{L}_2 by $\widetilde{\mathcal{L}}_2$. Since \mathfrak{A} is a KS-algebra if and only if \mathfrak{A}^* is a KS-algebra, we could assume that $\dim(I-P)\mathcal{H} \geq 2$.

If \mathfrak{A} is a KS-algebra, then S can not be a projection by Lemma 2.1. Therefore we assume that $S = \begin{pmatrix} I & T \\ 0 & 0 \end{pmatrix}$ where $T \neq 0$. Let Q be the range projection of $I-S$. We claim that there is a projection $E \in \mathcal{L}at(\mathfrak{A})$ such that $E < Q$ and $PE(I-P) \neq 0$.

By Lemma 2.4, we have $\mathcal{L}_2 = \{(E \vee P - P)|_{(I-P)\mathcal{H}} : E \in \mathcal{L}at(\mathfrak{A}) \text{ and } E \leq Q\}$. If the claim is false, then $PE = EP = 0$ for any $E < Q$. This implies that $E \leq I-P$ and $\mathcal{L}_2 = \{E|_{(I-P)\mathcal{H}} : E \in \mathcal{L}at(\mathfrak{A}) \text{ and } E < Q \text{ or } E = I\}$. Since $\mathcal{L}_2' = \mathbb{C}I$ and

$\dim Q\mathcal{H} = \dim(I - P)\mathcal{H} \geq 2$, we must have $I - P = \vee\{E : E \in \mathcal{Lat}(\mathfrak{A}), E < Q\} \leq Q$. Since $Q \wedge P = 0$, we have $Q = I - P$. This is in contradiction to $T \neq 0$. Thus the claim holds.

Let \mathcal{L} be the sublattice of $\mathcal{Lat}(\mathfrak{A})$ generated by \mathcal{L}_1 and $P \vee \mathcal{L}_2 = \{P \vee E : E \in \mathcal{Lat}(\mathfrak{A})\}$. By Lemma 2.4, we have $\mathcal{Lat}(\mathfrak{A})$ is generated by $\mathcal{L} \cup \{Q\}$. Let E be a projection in $\mathcal{Lat}(\mathfrak{A})$ such that $E < Q$ and $PE(I - P) \neq 0$. Since $\mathcal{L}'_1 = \mathbb{C}I$ and $\mathcal{L}'_2 = \mathbb{C}I$, we have $(\mathcal{L} \cup \{E\})' = \mathbb{C}I$. Therefore $\mathcal{Alg}(\mathcal{L} \cup \{E\}) \cap \mathcal{Alg}^*(\mathcal{L} \cup \{E\}) = \mathbb{C}I$. Next we will show that $\mathfrak{A} \subsetneq \mathcal{Alg}(\mathcal{L} \cup \{E\})$.

Let $V = \begin{pmatrix} I & T \\ 0 & I \end{pmatrix}$. It is easy to check that the range projection of VQV^{-1} is $I - P$. Thus we may assume that the range projection of VEV^{-1} is $\begin{pmatrix} 0 & 0 \\ 0 & E_0 \end{pmatrix}$ where E_0 is a projection in $\mathcal{B}((I - P)\mathcal{H})$. Note that

$$V\mathfrak{A}V^{-1} = \left\{ \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} : A_1 \in \mathcal{Alg}(\mathcal{L}_1), A_2 \in \mathcal{Alg}(\mathcal{L}_2) \right\} \text{ and}$$

$$V\mathcal{Alg}(\mathcal{L} \cup \{E\})V^{-1} = \left\{ \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} : A_1 \in \mathcal{Alg}(\mathcal{L}_1), A_2 \in \mathcal{Alg}(\mathcal{L}_2), A_{12}E_0 = 0 \right\}.$$

Since $0 < E_0 < I - P_1$, there is an operator $A \in \mathcal{B}((I - P)\mathcal{H}, P\mathcal{H})$ such that $A \neq 0$ and $AE_0 = 0$. Thus we have $\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \in V\mathcal{Alg}(\mathcal{L} \cup \{E\})V^{-1} \setminus V\mathfrak{A}V^{-1}$, and \mathfrak{A} is not a KS-algebra. \square

Corollary 2.3. *Let \mathcal{H} be a Hilbert space such that $\dim \mathcal{H} \geq 3$. If \mathfrak{A} is an abelian KS-algebra, then \mathfrak{A} contains no non-trivial idempotent.*

Proof. If S is a non-trivial idempotent in \mathfrak{A} . Let P be the range projection of S . By Lemma 2.7, we know that $P\mathfrak{A}P$ and $(I - P)\mathfrak{A}(I - P)$ both are abelian KS-algebra. Thus $\mathcal{Lat}(P\mathfrak{A}P)' = \mathbb{C}I$ and $\mathcal{Lat}((I - P)\mathfrak{A}(I - P))' = \mathbb{C}I$ by Corollary 2.1. Therefore Theorem 2.1 implies that \mathfrak{A} is not a KS-algebra. \square

Corollary 2.4. *Let $\mathfrak{A} \subset M_n(\mathbb{C}) (n \geq 3)$ be an abelian KS-algebra, then $\mathfrak{A} = \mathbb{C}I + \mathfrak{A}_0$ where \mathfrak{A}_0 is an abelian algebra contains only nilpotent matrices.*

Proof. If A in \mathfrak{A} has more than one eigenvalue, then \mathfrak{A} must contains a non-trivial idempotent which is impossible by Corollary 2.3. \square

Corollary 2.5. *Let \mathfrak{A} be an abelian von Neumann algebra such that $\dim \mathfrak{A} > 2$, then \mathfrak{A} can not similar to an abelian KS-algebra.*

Proof. Only need to note that every von Neumann algebra contains non-trivial idempotent, and Corollary 2.3 implies the result. \square

3. GENERATOR OF ABELIAN KS-ALGEBRAS

If an subalgebra \mathfrak{A} of $M_n(\mathbb{C}) (n \geq 3)$ is an abelian KS-algebra, then Corollary 2.3 implies that the spectrum of every element in \mathfrak{A} contains only one point. Therefore for any $T \in \mathfrak{A}$, there is a $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is nilpotent. In this section, we will prove that if $T \in M_n(\mathbb{C}) (n \geq 2)$ is a nilpotent, then the $\mathcal{Alg}(\mathcal{Lat}(T))$ can not be an abelian KS-algebra.

Lemma 3.1. *Suppose that \mathfrak{A} is an abelian subalgebra of the matrix algebra $M_n(\mathbb{C})$. Let $\mathcal{S} = \{T : \dim(\text{Ker}(T)) = m\}$ where $m = \sup_{T \in \mathfrak{A} \setminus \{0\}} \dim \text{Ker}(T)$. If $T \in \mathcal{S}$ and A is a nilpotent in \mathfrak{A} , then we have $TA = AT = 0$.*

Proof. Let $\mathcal{K} = \text{Ran}(T)$. Since $AT = TA$, we have $A\mathcal{K} \subset \mathcal{K}$. Note that $A|_{\mathcal{K}}$ is also a nilpotent, therefore there exists a vector $\xi (\neq 0)$ in \mathcal{K} such that $A\xi = 0$. If $AT \neq 0$, we must have $\dim(\text{Ker}(AT)) - \dim(\text{Ker}(T)) \geq 1$ which is in contradiction to the fact that T is in \mathcal{S} . \square

Remark 3.1. With the notations in Lemma 3.1. If T in \mathcal{S} is a nilpotent, then we have $T^2 = 0$. This implies that the range of T is contained in the kernel of T . Let $m = \dim \text{Ker}(T)$. Choosing a right orthonormal basis, T can be written as $\begin{pmatrix} 0 & T_1 \\ 0 & 0 \end{pmatrix}$ where $T_1 \in \mathcal{M}_{m, n-m}(\mathbb{C})$. Note that the columns of the matrix T_1 are linearly independent. Therefore, $m \geq n - m$ and $m \geq \frac{n}{2}$.

Lemma 3.2. Suppose that $A \in M_n(\mathbb{C})$ is a nilpotent matrix with the order k . Then there exists an invertible matrix W such that $W^{-1}AW = \begin{pmatrix} J & A_0 \\ 0 & 0 \end{pmatrix}$, where J is in Jordan normal form such that $J^{k-1} = 0$ and $J^{k-2} \neq 0$, and the rank of A_0 equals to the number of its columns.

Before proving this lemma, we feel that it is worthwhile to examine a special case.

Example 3.1. Let

$$A = \begin{pmatrix} J_3 & \\ & J_3 \end{pmatrix} = \begin{pmatrix} \begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix} & \\ & \begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix} \end{pmatrix}.$$

It is not hard to see that we may put the matrix into the form which is described in the lemma by switching the rows and columns. Indeed, let

$$U = \begin{pmatrix} I_2 & \\ & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \end{pmatrix}.$$

Then $UAU^* = \begin{pmatrix} J' & A_0 \\ 0 & 0 \end{pmatrix}$ where $A_0^t = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, $J' = \text{diag}(J_2, J_2)$ and $J_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is the 2×2 Jordan block of a nilpotent matrix.

The same strategy gives us the proof of the general case as follows.

Proof of Lemma 3.2. We first assume that $M_n(\mathbb{C}) \cong M_k(\mathbb{C}) \otimes M_l(\mathbb{C})$ and $A = J_k \otimes I_l$ where J_k is the $k \times k$ Jordan block. Let $\{e_{jk+i}\}_{i=1, \dots, k; j=0, \dots, l-1}$ be an orthonormal basis of \mathbb{C}^n such that $Ae_{jk+1} = 0$ and $Ae_{jk+i} = e_{jk+i-1}$, $i = 2, \dots, k$, $j = 0, \dots, l-1$.

Let W_0 be a unitary matrix defined by

$$\begin{aligned} W_0 e_{jk+i} &= e_{j(k-1)+i} & 0 \leq j \leq l, \quad 1 \leq i \leq k-1, \\ W_0 e_{jk+k} &= e_{l(k-1)+j+1} & 0 \leq j \leq l-1. \end{aligned}$$

Then an easy computation shows that $W_0 A W_0^* = \begin{pmatrix} J & A_0 \\ 0 & 0 \end{pmatrix}$ such that A_0 is a $l(k-1) \times l$ -matrix with $\text{rank}(A_0) = l$ and $J = J_{k-1} \otimes I_l$ where J_{k-1} is the $k-1 \times k-1$ Jordan block of a single nilpotent matrix.

For the general case, we could assume that $A = \begin{pmatrix} J_0 & 0 \\ 0 & J_1 \end{pmatrix}$ where J_0 is a nilpotent matrix in Jordan normal form such that $J_0^{k-1} = 0$, $J_1 = J_k \otimes I = \text{diag}(J_k, \dots, J_k)$. Then we can construct W_0 for $J_k \otimes I$ as in the first part of the proof, and it is easy to see that $W = \text{diag}(I, W_0)$ satisfies the conditions of the lemma. \square

Theorem 3.1. Let A be a nilpotent in $M_n(\mathbb{C})$ such that $\dim(\text{Ker}(A)) \geq 2$. Then there exists a reflexive sublattice \mathcal{L} of $\mathcal{Lat}(A)$ such that $\mathcal{L}' = \mathbb{C}I$ and $\text{Alg}(\mathcal{L}) \setminus \text{Alg}(\mathcal{Lat}(A))$ contains a non-trivial idempotent. In particular $\mathcal{Lat}(A)' = \mathbb{C}I$ and $\text{Alg}(\mathcal{Lat}(A))$ is not a KS-algebra.

Proof. We will prove this result by induction on the size n of $M_n(\mathbb{C})$. If $n = 2$, the only nilpotent satisfies the condition in the proposition is 0, and the conclusion is trivial (see Example 1.1).

Let $n \geq 3$ be given and suppose the conclusion of the proposition is true for $l \leq n-1$. Let A be a nonzero nilpotent in $M_n(\mathbb{C})$ such that $\dim(\text{Ker}(A)) \geq 2$ and $A^k = 0$ and $A^{k-1} \neq 0$ where $k \geq 2$.

Let $P = \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}$ be the projection onto the kernel of A^{k-1} , where $m = \dim \text{Ker}(A^{k-1})$. By the argument in Remark 3.1, we may write A^{k-1} as $\begin{pmatrix} 0 & \tilde{A} \\ 0 & 0 \end{pmatrix}$ such that $\text{rank}(\tilde{A}) = n - m$. Since $(I - P)AP = 0$, we have $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$ where $A_1 \in M_m(\mathbb{C})$, $A_2 \in M_{m, n-m}(\mathbb{C})$ and $A_3 \in M_{n-m}(\mathbb{C})$. Also note that $A^{k-1} \times A = 0$ implies $A_3 = 0$. Therefore we assume that $A = \begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix}$.

Since $\text{rank}(\tilde{A}) = \text{rank}(A_1^{k-2}A_2) = n - m$, we have $A_1^{k-2}\xi \neq 0$, where $\xi = A_2e_1$ is the 1-st column of A_2 and $e_1 = (1, 0, \dots, 0)^t \in (I - P)\mathbb{C}^{n+1}$. It is not hard to see that

$$\text{span}\{\xi, A_1\xi, \dots, A_1^{k-2}\xi\} \cap \text{Ker}(A_1) = \text{span}\{A_1^{k-2}\xi\}.$$

Since the kernel of A is contained in $P\mathbb{C}^n$, we have $\dim(\text{Ker}(A_1)) \geq 2$ and there exists a nonzero vector $\beta \in \text{Ker}(A_1)$ such that $\beta \perp A_1^{k-2}\xi$. Note that β and $\{\xi, A_1\xi, \dots, A_1^{k-2}\xi\}$ are linearly independent.

Let $\alpha = \beta \oplus e_1 = (\beta, 1, 0, \dots, 0)^t$ and Q be the orthogonal projection onto to the subspace spanned by $\begin{pmatrix} \xi \\ 0 \end{pmatrix}, \begin{pmatrix} A_1\xi \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} A_1^{k-2}\xi \\ 0 \end{pmatrix}, \alpha$. Since $A\alpha = \begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix}\alpha = \begin{pmatrix} \xi \\ 0 \end{pmatrix}$, we have $Q \in \mathcal{Lat}(A)$. Let

$$\mathcal{L}_1 = \left\{ \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} : E \in \mathcal{Lat}(A_1) \right\}$$

$$\mathcal{L}_2 = \left\{ \begin{pmatrix} I & 0 \\ 0 & E \end{pmatrix} : E \text{ is a projection in } M_{n-m}(\mathbb{C}) \right\}$$

and $\mathcal{L} = \mathcal{Lat}(\text{Alg}(\mathcal{L}_1 \cup \mathcal{L}_2 \cup \{Q\}))$. It is clear that $\mathcal{L} \subset \mathcal{Lat}(A)$. By the induction assumption, we know that $\mathcal{L}_1|_{P\mathbb{C}^n} = \mathbb{C}I_m$. This implies that $\mathcal{L}' \subset \left\{ \begin{pmatrix} aI_m & 0 \\ 0 & bI_{n-m} \end{pmatrix} : a, b \in \mathbb{C} \right\}$. If $\mathcal{L}' \neq \mathbb{C}I$, then we have $P \in \mathcal{L}'$ and $P\alpha = \begin{pmatrix} \beta \\ 0 \end{pmatrix} \in Q\mathbb{C}^n$. However, this can not be true since β and $\{\xi, A_1\xi, \dots, A_1^{k-2}\xi\}$ are linearly independent. Therefore \mathcal{L}' must equals to $\mathbb{C}I$.

Let $B \in M_{m, n-m}(\mathbb{C})$ be a matrix such that $Be_1 = \beta$. Then $\begin{pmatrix} 0 & B \\ 0 & I \end{pmatrix}$ is a non-trivial idempotent in $\text{Alg}(\mathcal{L})$. Let \tilde{Q} be the projection onto the subspace spanned by $\begin{pmatrix} \xi \\ 0 \end{pmatrix}, \begin{pmatrix} A_1\xi \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} A_1^{k-2}\xi \\ 0 \end{pmatrix}, 0 \oplus e_1$. Since $A(0 \oplus e_1) = \begin{pmatrix} \xi \\ 0 \end{pmatrix}$, we have $\tilde{Q} \in \mathcal{Lat}(A)$. However, $\begin{pmatrix} 0 & B \\ 0 & I \end{pmatrix}(0 \oplus e_1) = \alpha \notin \tilde{Q}\mathbb{C}^n$ and $\begin{pmatrix} 0 & B \\ 0 & I \end{pmatrix} \notin \text{Alg}(\mathcal{Lat}(A))$. \square

It is well-known that if $T \in M_n(\mathbb{C})$ is a nilpotent and $\dim(\text{Ker}(T)) = 1$, then $\mathcal{Lat}(T)$ is a maximal nest and $\text{Alg}(\mathcal{Lat}(T))$ is non-commutative(it is a KS-algebra by [14]). Therefore we have the following corollary.

Corollary 3.1. *Let A be a nilpotent element in $M_n(\mathbb{C})$, $n \geq 2$, then $\text{Alg}(\mathcal{Lat}(A))$ is not an abelian KS-algebra.*

Corollary 3.2. *Suppose that $A \in M_n(\mathbb{C})$ ($n \geq 3$). Then the algebra generated by A and I is not a KS-algebra.*

4. ABELIAN KS-ALGEBRAS AND COMPLETELY DISTRIBUTIVE LATTICES

In this part, we will show that if \mathfrak{A} is an abelian KS-subalgebra of $\mathcal{B}(\mathcal{H})$, $\dim \mathcal{H} \geq 3$, then $\mathcal{Lat}(\mathfrak{A})$ can not be completely distributive.

It is well known that a subspace lattice \mathcal{L} is completely distributive if and only if $E_{\sharp} = E$ for any $E \in \mathcal{L}$ [11], where $E_{\sharp} = \vee\{F \in \mathcal{L} : E \not\leq F_{-}\}$ and $E_{-} = \vee\{F \in \mathcal{L} : E \not\leq F\}$.

Lemma 4.1. *If $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$ ($\dim \mathcal{H} \geq 3$) is an abelian KS-algebra, then $E \leq E_{-}$ for any $E \in \mathcal{Lat}(\mathfrak{A})$.*

Proof. If $E = 0$ or I , it is trivial. Assume that $E \in \mathcal{Lat}(\mathfrak{A})$ is a projection such that $E \not\leq E_{-}$, then there is a vector $\xi \in E\mathcal{H}$ such that $\eta = (E_{-})^{\perp}\xi \neq 0$. Let $r = \langle \xi, \eta \rangle = \langle \xi, (E_{-})^{\perp}\xi \rangle = \|(E_{-})^{\perp}\xi\|^2 > 0$ and $\eta' = \frac{\eta}{r}$. Then the rank one

operator $\eta' \otimes \xi$ is in \mathfrak{A} . Since $(\eta' \otimes \xi)^2 = \langle \xi, \eta' \rangle (\eta' \otimes \xi) = \eta' \otimes \xi$, $\eta' \otimes \xi$ is a non-trivial idempotent. This is contradict with Corollary 2.3, thus $E \leq E_-$. \square

Remark 4.1. Suppose \mathfrak{A} is an abelian KS-algebra such that $\dim \mathfrak{A} > 1$. If $I_- < I$, then for any $E \in \mathcal{Lat}(\mathfrak{A})$, we have $E I_- = I_- E$. Thus I_- is a non-trivial projection in \mathfrak{A} (note that $\mathcal{Lat}(\mathfrak{A}) \neq \{0, I\}$). This contradicts with Corollary 2.1. Hence $I_- = I$.

Lemma 4.2. Suppose that $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$ is an abelian subalgebra. Let $\mathfrak{J} = \{E \in \mathcal{Lat}(\mathfrak{A}) : E \neq 0, E_- \neq I\}$. Then for any $E_1 \neq E_2 \in \mathfrak{J}$, we have $E_1 \leq (E_2)_-$ and $E_2 \leq (E_1)_-$.

Proof. If $E_1 \not\leq E_2$ and $E_2 \not\leq E_1$, then $E_2 \leq (E_1)_-$ and $E_1 \leq (E_2)_-$.

Without lose generosity, we could assume $E_1 < E_2$, then $E_1 \leq (E_2)_-$. Let $\xi_i \in E_i \mathcal{H}$ and $\eta_i \in ((E_i)_-)^{\perp} \mathcal{H}$ ($i = 1, 2$) be nonzero unit vectors, then we have $\eta_i \otimes \xi_i$ ($i = 1, 2$) $\in \mathfrak{A}$. Since \mathfrak{A} is abelian, we have

$$(\eta_1 \otimes \xi_1)(\eta_2 \otimes \xi_2) = \langle \xi_2, \eta_1 \rangle (\eta_2 \otimes \xi_1) = \langle \xi_1, \eta_2 \rangle (\eta_1 \otimes \xi_2) = (\eta_2 \otimes \xi_2)(\eta_1 \otimes \xi_1).$$

Note that $E_1 \leq (E_2)_-$, $\xi_1 \in E_1 \mathcal{H}$ and $\eta_2 \in ((E_2)_-)^{\perp} \mathcal{H}$. We have $\langle \xi_1, \eta_2 \rangle = 0$. Therefore $\langle \xi_2, \eta_1 \rangle$ must also equals 0. This implies $E_2 \leq (E_1)_-$. \square

Theorem 4.1. If $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$ ($\dim \mathcal{H} \geq 3$) is an abelian KS-algebra, then $\mathcal{Lat}(\mathfrak{A})$ is not completely distributive.

Proof. If $E_- = I$ for every nonzero $E \in \mathcal{Lat}(\mathfrak{A})$, then $I_{\#} = 0 \neq I$. If $E_- < I$, we have $I_{\#} \leq E_- \neq I$ by Lemma 4.1 and Lemma 4.2. Thus $\mathcal{Lat}(\mathfrak{A})$ is not completely distributive. \square

At the end of this paper, we state the following question whose answer we believe is negative.

Question 4.1. Besides the algebras given in Example 1.1, dose there exist any other non-trivial abelian KS-algebras?

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