Wei Fei's Note

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ABSTRACT. Note on Wei Fei's lecture.

1. IMPORTANT FORMULA

Theorem 1.1 (Possion Summation Formula)**.** *Let f in $S(\mathbb{R})$. We have*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n),$$

where $\hat{f}(x) = \int_{-\infty}^{\infty} f(y)e^{2\pi ixy}dy$.

Example 1.1. *Let $f(x) = e^{-\pi x^2 y}$.*

$$\hat{f}(x) = y^{-\frac{1}{2}}e^{-\frac{\pi x^2}{y}}.$$

2. ENTIRE FUNCTION

Lemma 2.1. *Let $\{a_n\}_n \subset \mathbb{C}$ be a sequence,*

$$0 < |a_1| \le |a_2| \le \cdots \le |a_n| \le \cdots, \text{ and } |a_n| \to \infty.$$

Then there exists a entire function f such that $f(s) = 0$ if and only if $s \in \{a_n\}_n$.

Proof. Let

$$h_n = (1 - \frac{s}{a_n})e^{\frac{s}{a_n}+\frac{1}{2}(\frac{s}{a_n})^2+\frac{1}{3}(\frac{s}{a_n})^3+\cdots+\frac{1}{n}(\frac{s}{a_n})^n}.$$

Then

$$f(s) = \prod_{n=1}^{\infty} h_n(s)$$

satisfies the condition. $\square$

Remark 2.1. *Let $\{a_n\}_n \subset \mathbb{C}$ be a sequence,*

$$0 < |a_1| \le |a_2| \le \cdots \le |a_n| \le \cdots, \text{ and } |a_n| \to \infty.$$

If $\sum_n \frac{1}{|a_n|^{p+1}} < \infty$, then we could use

$$h_n = (1 - \frac{s}{a_n})e^{\frac{s}{a_n}+\frac{1}{2}(\frac{s}{a_n})^2+\frac{1}{3}(\frac{s}{a_n})^3+\cdots+\frac{1}{p}(\frac{s}{a_n})^p}.$$

in the above proof.

Example 2.1. *Let $\{n\}_{n\in\mathbb{Z}}$. Then*

$$f(s) = s\prod_{n\in\mathbb{N}}(1 - \frac{s}{n})(1 + \frac{s}{n}).$$

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Lemma 2.2. *Let $\{a_n\}_n \subset \mathbb{C}$ be a sequence,*

$$0 < |a_1| \le |a_2| \le \cdots \le |a_n| \le \cdots, \text{ and } |a_n| \to \infty.$$

Suppose that f satisfies $f(s) = 0$ if and only if $s \in \{a_n\}_n$. Then $f(s) = e^{H(s)} \prod_{n=1}^{\infty} h_n(s)$.

Definition 2.1. *Suppose $G(s)$ is a function and $\mu(r) = max_{|s|\le r}|G(s)|$. Let*

$$\alpha_0 = inf\{\alpha : \mu(r) \le e^{a_0 r^{\alpha}}\}.$$

Theorem 2.1. *Let p be the smallest integer such that*

$$\sum_n \frac{1}{|a_n|^{p+1}} < \infty.$$

Then the degree of

$$f(s) = \prod_n (1 - \frac{s}{a_n}) e^{\frac{s}{a_n} + \frac{1}{2}(\frac{s}{a_n})^2 + \frac{1}{3}(\frac{s}{a_n})^3 + \cdots + \frac{1}{p}(\frac{s}{a_n})^p}$$

is p.

Example 2.2.

$$\sin(\pi s) = \pi s \prod_n (1 - \frac{s^2}{n^2}).$$

Proof. Add proof. □

3. $\Gamma(s)$

Let

$$\Gamma(s) = \int_0^{+\infty} x^{s-1} e^{-x} dx, \qquad Re(S) > 0.$$

It is easy to see that $\Gamma(s+1) = s\Gamma(s)$.

$$\frac{1}{\Gamma(s)} = s e^{\gamma s} \prod_{n=1}^{\infty} (1 + \frac{s}{n}) e^{-\frac{s}{n}},$$

where γ is the Euler constant, i.e. $\gamma = \lim_{n\to\infty}(1 + \frac{1}{2} + \cdots \frac{1}{n} - \log(n))$.

Theorem 3.1.

$$\frac{1}{\Gamma(s)} = s \prod_{n=1}^{\infty} (1 + \frac{1}{n})^{-s} (1 + \frac{s}{n}).$$

Proof. add proof. □

Theorem 3.2. *Let $0 < \delta < \pi$. We have*

$$\log \Gamma(s) = s \log s - \frac{1}{2} \log s - s + \log(\sqrt{2\pi}) + O_{\delta}(\frac{1}{|s|}).$$

Proof. Add proof. □

This following is incomplete.

Proposition 3.1. *Let $s = \sigma + it$. Assume $\alpha < \sigma < \beta$. We have*

$$\Gamma(s) = |t|^{s-\frac{1}{2}} e^{-\frac{\pi}{2}|t| - it +}$$

4. ζ

Theorem 4.1. *If $Re(s) = \sigma > 1$, then $\zeta(s) \neq 0$.*

Proof. Note

$$\frac{1}{|\zeta(s)|} = |\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}| \leq \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \leq 1 + \int_1^\infty \frac{1}{t^\sigma} dt = \frac{\sigma}{\sigma - 1}.$$

This implies the result. □

Theorem 4.2. *For $Re(s) > 0$, we have*

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^2} + \frac{N^{1-s}}{s-1} - \frac{N^{-s}}{2} + s\int_N^\infty \frac{\rho(u)}{u^{s+1}} du, \quad N \geq 1,$$

where $\rho(u) = \frac{1}{2} - \{u\}$. Specially,

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + s\int_1^\infty \frac{\rho(u)}{u^{s+1}} du.$$

Theorem 4.3.

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{\frac{s-1}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s)$$

Proof. Add proof. □

Let

$$\xi(s) = \frac{1}{2}s(s-1)\zeta(s)\Gamma(\frac{S}{2})\pi^{-\frac{s}{2}}.$$

Theorem 4.4. *For any $\varepsilon > 0$, we have*

$$|\xi(s)| \ll e^{c|s|^{1+\varepsilon}}.$$

Remark 4.1. *The theorem above implies that ξ has infinite many zeros. (provide a proof).*

For $Re(s) > 1$, estimate

$$f(s) = (1 - 2^{1-s})\zeta(s) = \sum_{n}^{\infty} \frac{(-1)^{n-1}}{n^s}.$$

It is not hard to see that $\zeta(s)$ does not have real zeros.

Lemma 4.1. *Let $\{\rho_n\}_n$ be the zeros of $\xi(s)$. Then*

$$\sum_{n}^{\infty} \frac{1}{|\rho_n|} = \infty,$$

$$\sum_{n}^{\infty} \frac{1}{|\rho_n|^{1+\varepsilon}} < \infty,$$

for any $\varepsilon > 0$.

Theorem 4.5.

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{1-s} + \sum_{n=1}^{\infty}(\frac{1}{s-\rho_n} + \frac{1}{\rho_n}) + \sum_{n=1}^{\infty}(\frac{1}{s+2n} - \frac{1}{2n}) + B_0,$$

where B_0 is a constant.

Proof. Consider the two expressions of $\xi(s)$:

$$\xi(s) = e^{as+b}\prod_n(1-\frac{s}{\rho_n})e^{\frac{s}{\rho_n}}$$
$$\xi(s) = \frac{1}{2}s(s-1)\zeta(s)\Gamma(\frac{S}{2})\pi^{-\frac{s}{2}}.$$

Compute the derivative of $\log\xi(s)$ by plugging in the two expressions. $\square$

Theorem 4.6. *Let $T \geq 0$ and $\rho_n = \beta_n + i\gamma_n$ be the non-trivial zeros of $\zeta(s)$. Then*

$$\sum_{n=1}^{\infty}\frac{1}{1+(T-\gamma_n)^2} \leq C\log(T+2).$$

Proof. Let $s = 2+iT$.

$$Re(\frac{1}{s-\rho_n}) = \frac{2-\beta_n}{(2-\beta_n)^2+(T-\gamma_n)^2)^2} \geq \frac{1}{4(1+(T-\gamma_n)^2)}.$$

$\square$

5. $n \times 2$ CASE

Let

$$S_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in M_n(\mathbb{C})$$

We would like to find the commutant of $(I-\frac{1}{2}S_n)\otimes(I-\frac{1}{3}S_2)$ in $M_n(\mathbb{C})\otimes M_2(\mathbb{C})$. If

$$\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}\begin{pmatrix} S_n & \frac{2}{3}(I-\frac{1}{2}S_n) \\ 0 & S_n \end{pmatrix} = \begin{pmatrix} T_1S_n & T_1\frac{2}{3}(I-\frac{1}{2}S_n)+T_2S_n \\ T_3S_n & T_3\frac{2}{3}(I-\frac{1}{2}S_n)+T_4S_n \end{pmatrix}$$
$$= \begin{pmatrix} S_nT_1+\frac{2}{3}(I-\frac{1}{2}S_n)T_3 & S_nT_2+\frac{2}{3}(I-\frac{1}{2}S_n)T_4 \\ S_nT_3 & S_nT_4 \end{pmatrix} = \begin{pmatrix} S_n & \frac{2}{3}(I-\frac{1}{2}S_n) \\ 0 & S_n \end{pmatrix}\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$$

Since T_3 commute with S_n, T_3 must be a polynomial of S_n. Also note that $T_1S_n - S_nT_1 = \frac{2}{3}(I-\frac{1}{2}S_n)T_3$, this implies that the trace of T_3 is zero. Therefore T_3 must be upper triangular.

Note that

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \ddots & x_{3n} \\ x_{n-1,1} & x_{n-1,2} & x_{n-1,3} & \cdots & x_{n-1,n} \\ x_{n,1} & x_{n,2} & x_{n,3} & \cdots & x_{n,n} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$- \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \ddots & x_{3n} \\ x_{n-1,1} & x_{n-1,2} & x_{n-1,3} & \cdots & x_{n-1,n} \\ x_{n,1} & x_{n,2} & x_{n,3} & \cdots & x_{n,n} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & x_{11} & x_{12} & \cdots & x_{1,n-1} \\ 0 & x_{21} & x_{22} & \cdots & x_{2,n-1} \\ 0 & x_{31} & x_{32} & \ddots & x_{3,n-1} \\ 0 & x_{n-1,1} & x_{n-1,2} & \cdots & x_{n-1,n-1} \\ 0 & x_{n,1} & x_{n,2} & \cdots & x_{n,n-1} \end{pmatrix} - \begin{pmatrix} x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3n} \\ x_{41} & x_{42} & x_{43} & \ddots & x_{4n} \\ x_{n,1} & x_{n,2} & x_{n,3} & \cdots & x_{n,n} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

If the above is strict upper triangular, we must have

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \ddots & x_{3n} \\ x_{n-1,1} & x_{n-1,2} & x_{n-1,3} & \cdots & x_{n-1,n} \\ x_{n,1} & x_{n,2} & x_{n,3} & \cdots & x_{n,n} \end{pmatrix}$$

is upper triangular.

So we have T_1, T_4 are upper triangular. And it is easy to see that for a fixed T_3 which commute with S_n, we have many elements which commute with $(I - \frac{1}{2}S_n) \otimes (I - \frac{1}{3}S_2)$.

6. Spectrum of sums of unitary

Let $X_n = \{f_n(z) : z \in S^1\}$ where $f_n(z) = 1 + z + z^2 + \cdots + z^{n-1} = \frac{1-z^n}{1-z}$, $n = 1, 2, \ldots$. Each X_n is a compact subset of $\mathbb{C}$. We would like to know the limit of X_n as $n \to \infty$.

Let $z = e^{i\theta}$, then

$$\begin{aligned} \frac{1-z^n}{1-z} &= \frac{1 - \cos\theta - \cos n\theta + \cos(n-1)\theta}{2 - 2\cos\theta} + \frac{\sin\theta - \sin n\theta + \sin(n-1)\theta}{2 - 2\cos\theta} i \\ &= \frac{\sin(\frac{n}{2}\theta)}{\sin(\frac{\theta}{2})} \left(\cos(\frac{(n-1)}{2}\theta) + i \sin(\frac{(n-1)}{2}\theta) \right). \end{aligned}$$

If $n = 2m + 1$, then

$$\frac{1-z^n}{1-z} = \frac{\sin(m\theta + \frac{1}{2}\theta)}{\sin(\frac{\theta}{2})} \left(\cos(m\theta) + i\sin(m\theta)\right).$$

Let $z_1 = e^{i\theta_1}$ and $z_2 = e^{i\theta_2}$ and θ_1, θ_2 are in $[0, 2\pi)$. Suppose that $f_n(z_1) = f_n(z_2)$ and $\theta_1 < \theta_2$. First assume that $\sin(m\theta_1 + \frac{1}{2}\theta_1) = 0 = \sin(m\theta_2 + \frac{1}{2}\theta_2)$ and θ_1, θ_2. Note that $\sin(\frac{\theta_1}{2})$ and $\sim(\frac{\theta_2}{2})$ can not equal zero at the same time, since $\frac{\theta_1}{2}$ and $\frac{\theta_1}{2}$ are in $[0, \pi)$. Then

$$\theta_1, \theta_2 \in \{\frac{2k\pi}{2m+1} : k = 1, 2, \ldots, 2m\}.$$

And $f_n(z) = 0$.

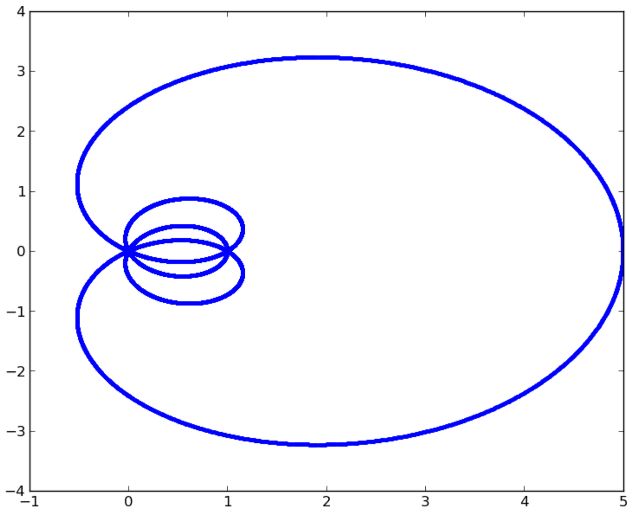

Figure 1. $n = 5$

Now assume that $\sin(m\theta + \frac{1}{2}\theta) \neq 0$. We have $\theta_2 = \theta_1 + \frac{2k\pi}{m}$ or $\theta_2 = \theta_1 + \frac{(2k+1)\pi}{m}$.

First assume that $\theta_2 = \theta_1 + \frac{2k\pi}{m}$, we have

$$\frac{\sin(m\theta_1 + \frac{1}{2}\theta_1)}{\sin(\frac{\theta_1}{2})} = \frac{\sin(m\theta_2 + \frac{1}{2}\theta_2)}{\sin(\frac{\theta_2}{2})}$$

implies

$$\frac{\sin m\theta_1 \cos(\frac{\theta_1}{2})}{\sin(\frac{\theta_1}{2})} + \cos m\theta_1 = \frac{\sin m\theta_2 \cos(\frac{\theta_2}{2})}{\sin(\frac{\theta_2}{2})} + \cos m\theta_2.$$

If $\sin m\theta_1 \neq 0$ then $\cot(\frac{\theta_1}{2}) = \cot(\frac{\theta_2}{2})$. This implies that $\theta_1 = \theta_2$.

Suppose that $\sin m\theta_1 = 0$, we have $\sin(\frac{\theta_1}{2}) \neq 0$ and $\sin(\frac{\theta_2}{2}) \neq 0$, since $\frac{\theta_1}{2}$ and $\frac{\theta_2}{2}$ are in $[0, \pi)$. This means that θ_1 and θ_2 are in

$$\{\frac{k\pi}{m} : k = 1, 2, \ldots, 2m - 1\}.$$

And $f_n(z) = \cos^2 m\theta = 1$.

Now assume that $\theta_2 = \theta_1 + \frac{(2k+1)\pi}{m}$. Then we have

$$\frac{\sin(m\theta_1 + \frac{1}{2}\theta_1)}{\sin(\frac{\theta_1}{2})} = -\frac{\sin(m\theta_2 + \frac{1}{2}\theta_2)}{\sin(\frac{\theta_2}{2})}.$$

This also implies that

$$\frac{\sin m\theta_1 \cos(\frac{\theta_1}{2})}{\sin(\frac{\theta_1}{2})} = \frac{\sin m\theta_1 \cos(\frac{\theta_2}{2})}{\sin(\frac{\theta_2}{2})}.$$

Argue as above, we have θ_1 and θ_2 are in

$$\{\frac{k\pi}{m} : k = 1, 2, \ldots, 2m - 1\}.$$

And $f_n(z) = \cos^2 m\theta = 1$.

Lemma 6.1. *For any $re^{i\alpha} \in \mathbb{C}$ and any $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ and a $\theta_m \in [0, 2\pi)$ such that*

$$|\frac{\sin(m\theta_m + \frac{1}{2}\theta_m)}{\sin(\frac{\theta_m}{2})}(\cos(m\theta_m) + i\sin(m\theta_m)) - re^{i\alpha}| < \varepsilon$$

for any $m \geq N$.

Proof. Assume that $\alpha = \frac{2\pi ip}{q}$, where $(p, q) = 1$ and $q > p$. For any $m > 1$, consider the set

$$\{\frac{2\pi i(p + kq)}{qm} : k = 0, 1, \ldots, m - 1\}.$$

Let

$$r_k = \frac{\sin(\frac{2\pi ip}{q} + \frac{\pi i(p+kq)}{qm})}{\sin(\frac{\theta_m}{2})} = \cos(\frac{2\pi ip}{q}) + \sin(\frac{2\pi ip}{q})\cot(\frac{\pi i(p + kq)}{qm}).$$

Now it is not hard to see that there exist a $N \in \mathbb{N}$ such that there is a $0 \leq k_m \leq m - 1$ such that $|r_{k_m} - r| \leq \varepsilon$ whenever $m > N$. □

Lemma 6.2. *If U is a Haar unitary, then $T = I + U + U^2 + \cdots$ is a densely defined closed operator. The spectrum of T is $\mathbb{C}$.*

AMSS