On Some Examples of Kadison-Singer Algebras

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Abstract In this paper, we first show that the algebra corresponding to the one point extension of a maximal nest on a Hilbert space is a Kadison-Singer algebra with diagonal equal to $\mathbb{C}I$. Three examples are also given. Let $\mathscr{N}$ be a von Neumann algebra in $\mathcal{B}(\mathscr{H})$ with a normal faithful tracial state τ. Suppose $\{P_1, P_2, \cdots, P_n\}$ is a nest with $P_1 \leq P_2 \leq \cdots \leq P_n = I$ and $\tau(P_k) = \frac{k}{n}$, Q is a projection with $\tau(Q) = \frac{1}{n}$. If Q is free from $\{P_1, P_2, \cdots, P_n\}$ with respect to τ, then the lattice generated by $\{P_1, P_2, \cdots, P_n\}$ and Q is also a Kadison-Singer lattice and $\mathcal{A}lg(\mathscr{L})$ is a Kadison-Singer algebra.

Keywords Kadison-Singer algebra; Kadison-singer Lattice; Nest; Free.

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1 Introduction

Bounded linear operators on an infinite-dimensional separable Hilbert space are generalizations of finite matrices on Euclidean spaces. Non selfadjoint operator algebras were introduced to study questions concerning the structural properties of operators. For example, whether every bounded linear operator has a (nontrivial) invariant closed subspace is a longstanding open question in operator theory (see [1] and [2]). Considerable effort has been made to the study of two classes of non selfadjoint operator algebras: triangular

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algebras (introduced in [3]) and reflexive algebras (see [12]). Nest algebras belong to the intersection of these two (see [5]). Many people have tried to extend the theory of selfadjoint operator algebras (C*-algebras and von Neumann algebras) as well as its techniques to non selfadjoint operator algebras. One of such attempts is to use an arbitrary factor to replace the algebra of all bounded linear operators (a factor of type I_∞). But it has not been a fruitful attempt. Many of the definitive results on non selfadjoint algebras (or single operators) rely on some relations to compact operators (or finite-rank operators).

Suppose $\mathscr{H}$ is a separable Hilbert space and $\mathcal{B}(\mathscr{H})$ is the algebra of all bounded linear operators on $\mathscr{H}$. Let $\mathscr{P}$ be a set of (orthogonal) projections in $\mathcal{B}(\mathscr{H})$. Define

$$\mathcal{A}lg(\mathscr{P}) = \{T \in \mathcal{B}(\mathscr{H}) : TP = PTP, \text{ for all } P \in \mathscr{P}\}.$$

Then $\mathcal{A}lg(\mathscr{P})$ is a weak-operator closed subalgebra of $\mathcal{B}(\mathscr{H})$. Similarly, for a subset $\mathscr{S}$ of $\mathcal{B}(\mathscr{H})$, define

$$\mathcal{L}at(\mathscr{S}) = \{P \in \mathcal{B}(\mathscr{H}) : P \text{ is a projection, } TP = PTP, \text{ for all } T \in \mathscr{S}\}.$$

Then $\mathcal{L}at(\mathscr{S})$ is a strong-operator closed lattice of projections. A subalgebra $\mathscr{B}$ of $\mathcal{B}(\mathscr{H})$ is said to be a reflexive (operator) algebra if $\mathscr{B} = \mathcal{A}lg(\mathcal{L}at(\mathscr{B}))$. Similarly, a lattice $\mathscr{L}$ of projections in $\mathcal{B}(\mathscr{H})$ is called a reflexive lattice (of projections) if $\mathscr{L} = \mathcal{L}at(\mathcal{A}lg(\mathscr{L}))$. A nest is a totally ordered reflexive lattice. If $\mathscr{L}$ is a nest, then $\mathcal{A}lg(\mathscr{L})$ is called a nest algebra. Nest algebras are generalizations of (hyperreducible) "maximal triangular" algebras introduced by Kadison and Singer in [3]. Kadison and Singer also showed that nest algebras are the only maximal triangular reflexive algebras (with a commutative lattice of invariant projections). Motivated by this, Liming Ge and Wei Yuan introduced the following definition in [6].

Definition A subalgebra $\mathscr{A}$ of $\mathcal{B}(\mathscr{H})$ is called a Kadison-Singer (operator) algebra (or KS-algebra) if $\mathscr{A}$ is reflexive and maximal with respect to the diagonal subalgebra $\mathscr{A} \cap \mathscr{A}^*$ of $\mathscr{A}$, in the sense that if there is another reflexive subalgebra $\mathscr{B}$ of $\mathcal{B}(\mathscr{H})$ such that $\mathscr{A} \subseteq \mathscr{B}$ and $\mathscr{B} \cap \mathscr{B}^* = \mathscr{A} \cap \mathscr{A}^*$, then $\mathscr{A} = \mathscr{B}$. When the diagonal of a KS-algebra $\mathscr{A}$ is a factor, we say $\mathscr{A}$ is a Kadison-Singer factor (or KS-factor). A lattice $\mathscr{L}$ of projections in $\mathcal{B}(\mathscr{H})$ is called a Kadison-Singer lattice (or KS-lattice) if $\mathscr{L}$ is a minimal reflexive lattice that generates the von Neumann algebra $\mathscr{L}''$, or equivalently $\mathscr{L}$ is reflexive and $\mathcal{A}lg(\mathscr{L})$ is a Kadison-Singer algebra.

In [6], Liming Ge and Wei Yuan constructed examples of Kadison-Singer algebras with hyperfinite factors as diagonal. Let G_n be the free product of $\mathbb{Z}_2$ with itself n times, for $n \geq 2$ or $n = \infty$. Let $\mathscr{L}_{G_n}$ be the group von Neumann algebra associated to the group G_n ([7]). If $U_1, \ldots, U_n$ are canonical generators for $\mathscr{L}_{G_n}$ corresponding to the generators of G_n with $U_j^2 = I$. Then $\frac{I-U_j}{2}$ $(j = 1, \cdots, n)$ are projections of trace $\frac{1}{2}$. Let $\mathscr{F}_n$ be the lattice consisting of these n free projections with trace $\frac{1}{2}$ and $0, I$. Then $\mathscr{F}_n$ is a minimal lattice which generates $\mathscr{L}_{G_n}$ as a von Neumann algebra. In [8], Ge and Yuan showed that

$\mathcal{A}lg(\mathscr{F}_n)$ ($n \leq 3$) is a Kadison-Singer algebra and $\mathcal{L}at(\mathcal{A}lg(\mathscr{F}_3)) \setminus \{0, I\}$ is homeomorphic to the two-dimensional sphere $\mathbb{S}^2$.

In this paper we will give some examples of Kadison-Singer algebras with trivial diagonal. This paper contains three sections. In Section 2, we give our main results. We show that the one point extension of a maximal nest on a separable infinite dimensional Hilbert space is a Kadison-singer lattice and the corresponding algebra is a Kadison-Singer algebra. We also give three examples coming from the one point extension of an $\mathbb{N}$-ordered maximal nest, a $\mathbb{Z}$-ordered maximal nest and a continuous maximal nest respectively. Let $\mathscr{N}$ be a von Neumann algebra in $\mathcal{B}(\mathscr{H})$ with a normal faithful tracial state τ. Suppose $\{P_1, P_2, \cdots, P_n\}$ is a nest with $P_1 \leq P_2 \leq \cdots \leq P_n = I$ and $\tau(P_k) = \frac{k}{n}$, Q is a projection with $\tau(Q) = \frac{1}{n}$. In Section 3, we show that if Q is free from $\{P_1, P_2, \cdots, P_n\}$ with respect to τ, then the lattice generated by $\{P_1, P_2, \cdots, P_n\}$ and Q is also a Kadison-Singer lattice and $\mathcal{A}lg(\mathscr{L})$ is a Kadison-Singer algebra.

2 One point extension of a maximal nest

Let $\mathscr{H}$ be a separable Hilbert space with $dim\mathscr{H} = \infty$. Let $\mathcal{N}$ be a nest in $\mathcal{B}(\mathscr{H})$ such that $\mathfrak{A} = \mathcal{N}'$ is a maximal abelian self-adjoint subalgebra of $\mathcal{B}(\mathscr{H})$. Suppose $\xi \in \mathscr{H}$ is a separating vector for $\mathfrak{A}$. Let P_ξ be the orthogonal projection from $\mathscr{H}$ onto the one dimensional subspace of $\mathscr{H}$ generated by ξ.

Lemma 1 *The von Neumann algebra generated by $\mathcal{N}$ and P_ξ is $\mathcal{B}(\mathscr{H})$, i.e., $\{\mathcal{N}, P_\xi\}'' = \mathcal{B}(\mathscr{H})$.*

Proof. Let Q be a projection in $\{\mathcal{N}, P_\xi\}' \subset \mathcal{N}' = \mathfrak{A}$. Then $QP_\xi = P_\xi Q$ and therefore

$$Q\xi = P_\xi Q\xi = \langle Q\xi, \xi\rangle \xi.$$

Since ξ is a separating vector for $\mathfrak{A}$, we have $Q = \langle Q\xi, \xi\rangle I$. Since Q is a projection, $Q = 0$ or $Q = I$. Hence $Q\xi = 0$ or $Q\xi = \xi$. Thus $\{\mathcal{N}, P_\xi\}' = \mathbb{C}I$ and the result follows. ∎

Since ξ is a separating vector for $\mathfrak{A}$, the map $\Phi(Q) = \langle Q\xi, \xi\rangle$ is an order preserving homeomorphism of $(\mathcal{N}, <)$ onto a compact subset $\mathcal{S}$ of $[0, 1]$. From now on, for any $t \in \mathcal{S}$, we denote $Q_t = \Phi^{-1}(t)$ the inverse image of t under Φ. In other words, $Q_t \in \mathcal{N}$ and $\langle Q_t\xi, \xi\rangle = t$. By the definition, we have $Q_{t_1} < Q_{t_2}$ for $t_1 < t_2 \in \mathcal{S}$ and $Q_0 = 0$, $Q_1 = I$. Also we have $Q_{t_1} = Q_{t_2}$ if and only if $t_1 = t_2$.

Lemma 2 *Let $\mathscr{L} = \{Q_t, Q_t \vee P_\xi : t \in \mathcal{S}\}$. Then $\mathscr{L}$ is a lattice.*

Proof. It suffices to show

$$Q_{t_1} \wedge (Q_{t_2} \vee P_\xi) = Q_{min(t_1,t_2)},$$

for t_1 and t_2 in $\mathcal{S}\backslash\{1\}$. It is clear that the above equality holds when $t_1 \leq t_2$. It remains to consider the case when $t_1 > t_2$. Note that

$$Q_{t_2} \leq Q_{t_1} \wedge (Q_{t_2} \vee P_\xi) = P.$$

If $P \neq Q_{t_2}$, then $Q_{t_2} \vee P_\xi = P \leq Q_{t_1}$. It follows that $\xi \in Q_{t_1}(\mathscr{H})$ and $Q_{t_1}\xi = \xi$. Since ξ is separating for $\mathfrak{A}$, we have $Q_{t_1} = I$. This contradicts with the assumption that $t_1 \neq 1$ and ends the proof. ■

Next we will show that $\mathscr{L}$ is a reflexive lattice, i.e., $\mathscr{L} = \mathcal{L}at(\mathcal{A}lg(\mathscr{L}))$.

Lemma 3 *Suppose $E \in \mathcal{L}at(\mathcal{A}lg(\mathscr{L}))$. If there exist a unit vector ζ in $E(\mathscr{H})$ and a $t \in \mathcal{S}\backslash\{1\}$ such that $(I - Q_t)\zeta \notin span\{(I - Q_t)\xi\}$, then $Q_t \leq E$.*

Proof. Let

$$\eta = (I - Q_t)\zeta - \left\langle \zeta, \frac{(I - Q_t)\xi}{\|(I - Q_t)\xi\|} \right\rangle \frac{(I - Q_t)\xi}{\|(I - Q_t)\xi\|}.$$

Then $\eta \neq 0$. For any $\beta \in Q_t(\mathscr{H})$, we consider the map T_β from $\mathscr{H}$ into $\mathscr{H}$ defined by

$$T_\beta(\alpha) = \langle \alpha, \eta \rangle \beta, \quad \alpha \in \mathscr{H}.$$

Then T_β belongs to $\mathcal{A}lg(\mathscr{L})$. Indeed, for any $s \in \mathcal{S}$, $s \leq t$, if $\alpha \in Q_s(\mathscr{H})$, then

$$T_\beta(\alpha) = \langle \alpha, \eta \rangle \beta = \langle (I - Q_t)\alpha, \eta \rangle \beta = 0.$$

This shows that T_β leaves Q_s ($s \in \mathcal{S}$, $s \leq t$) invariant. And obviously T_β leaves $P_\xi(\mathscr{H})$ and $Q_s(\mathscr{H})(s > t)$ invariant. Thus

$$T_\beta \zeta = \left(\|(I - Q_t)\zeta\|^2 - |\langle \zeta, \frac{(I - Q_t)\xi}{\|(I - Q_t)\xi\|} \rangle|^2 \right) \beta \in E(\mathscr{H}).$$

It follows that $\beta \in E(\mathscr{H})$. Since β is arbitrary in $Q_t(\mathscr{H})$, we have $Q_t \leq E$. ■

Lemma 4 *Let $\zeta \in \mathscr{H}$. If there exist t_1 and t_2 in $\mathcal{S}\backslash\{0, 1\}$ such that*

$$(I - Q_{t_i})\zeta = a_{t_i}(I - Q_{t_i})\xi, \ i = 1, 2,$$

then $a_{t_1} = a_{t_2}$.

Proof. Without loss of generality, we may assume that $a_{t_1} = 0$ (from replacing ζ by $\zeta - a_{t_1}\xi$). Then we have

$$(I - Q_{t_1})\zeta = 0 \ \text{ and } \ (I - Q_{t_2})\zeta = a_{t_2}(I - Q_{t_2})\xi.$$

This is equivalent to

$$Q_{t_1}\zeta = \zeta \ \text{ and } \ Q_{t_2}(\zeta - a_{t_2}\xi) = \zeta - a_{t_2}\xi.$$

It follows from Lemma 2 that $\zeta \in Q_{t_1} \wedge (Q_{t_2} \vee P_\xi) = Q_{min(t_1,t_2)}$. This implies that $a_{t_2}Q_{t_2}\xi = a_{t_2}\xi$. If $a_{t_2} \neq 0$, then $\xi \in Q_{t_2}(\mathscr{H})$. Since ξ is a separating vector of $\mathfrak{A}$, we have $Q_{t_2} = I$. This contradicts with the fact that $t_2 \neq 1$. This completes the proof. ■

The following Corollary is an immediate consequence of the above lemma.

Corollary 5 *Let $\zeta \in \mathscr{H}$ and $\{t_i\}_{i=1}^{\infty} \subset \mathcal{S} \setminus \{0\}$ be a sequence of numbers with* $\lim_i t_i = 0$. *If, for any t_i, there is a complex number a_{t_i} such that $(I - Q_{t_i})\zeta = a_{t_i}(I - Q_{t_i})\xi$, then there is a complex number a such that $\zeta = a\xi$.*

Lemma 6 *If $E \in \mathcal{L}at(\mathcal{A}lg(\mathscr{L})) \setminus \{0\}$ and $E \neq P_\xi$, then there exists a $t \in \mathcal{S}\setminus\{0\}$ such that $Q_t \leq E$.*

Proof. Since $E \neq P_\xi$, there exists a $\zeta \in E(\mathscr{H})$ such that $\zeta \notin span\{\xi\}$. If there is a $t \in \mathcal{S} \setminus \{0, 1\}$ such that

$$(I - Q_t)\zeta \notin span\{(I - Q_t)\xi\},$$

then we have $Q_t \leq E$ by Lemma 3. Otherwise we know that for any $t \in \mathcal{S} \setminus \{0, 1\}$, $(I - Q_t)\zeta \in span\{(I - Q_t)\xi\}$. Because $\zeta \notin span\{\xi\}$, there is an $\epsilon > 0$ such that $(0, \epsilon) \cap \mathcal{S} = \emptyset$ by Corollary 5. Let

$$t_0 = inf\{t \mid (0, t) \cap \mathcal{S} \neq \emptyset, t \in \mathcal{S}\}.$$

Since $\mathcal{S}$ is compact, we have $t_0 \in \mathcal{S}$. Thus Q_{t_0} is a minimal projection. Denote by e the unit vector which spans $Q_{t_0}(\mathscr{H})$. Let

$$\beta = \zeta - \langle \zeta, \xi \rangle \xi.$$

Then the linear operator T_e on $\mathscr{H}$ defined by

$$T_e(\alpha) = \langle \alpha, \beta \rangle e, \quad \alpha \in \mathscr{H},$$

is in $\mathcal{A}lg(\mathscr{L})$. This implies that $e \in E(\mathscr{H})$ and $Q_{t_0} \leq E$. ■

Theorem 7 *$\mathscr{L}$ is a reflexive lattice.*

Proof. Suppose $E \in \mathcal{L}at(\mathcal{A}lg(\mathscr{L})) \setminus \{0, I\}$. When $E \neq P_\xi$, let

$$t_0 = sup\{t \in \mathcal{S} | Q_t \leq E\}.$$

It follows from Lemma 6 that $t_0 > 0$. If $E = Q_{t_0}$, then the result follows. Now assume $E \neq Q_{t_0}$. Given $\zeta \in E(\mathscr{H})$ and $(I - Q_{t_0})\zeta \neq 0$. By Lemma 3, we have for any $t \in \mathcal{S}$, $t > t_0$,

$$(I - Q_t)\zeta \in span\{(I - Q_t)\xi\}.$$

If $(I - Q_{t_0})\zeta \notin span\{(I - Q_{t_0})\xi\}$, then we have $(t_0, 1) \cap \mathcal{S} \neq \emptyset$. Let

$$t_1 = inf\{t \in \mathcal{S} | t > t_0\}.$$

It is not hard to show that $1 > t_1 > t_0$, $t_1 \in \mathcal{S}$ and $(t_0, t_1) \cap \mathcal{S} = \emptyset$. Let e be the unit vector that spans $(Q_{t_1} - Q_{t_0})\mathscr{H}$ and

$$\beta = (I - Q_{t_0})\zeta - \left\langle \zeta, \frac{(I - Q_{t_0})\xi}{\|(I - Q_{t_0})\xi\|} \right\rangle \frac{(I - Q_{t_0})\xi}{\|(I - Q_{t_0})\xi\|}.$$

Then the operator T_e on $\mathscr{H}$ defined by

$$T_e(\alpha) = \langle \alpha, \beta \rangle e, \quad \alpha \in \mathscr{H},$$

is in $\mathcal{A}lg(\mathscr{L})$ and $e \in E(\mathscr{H})$. This means that $Q_{t_1} \leq E$ and we get a contradiction. So we have for any $\zeta \in E(\mathscr{H})$,

$$(I - Q_{t_0})\zeta = a(I - Q_{t_0})\xi$$

for some $a \in \mathbb{C}$. Note that there is a vector ζ which makes $a \neq 0$ (otherwise $E = Q_{t_0}$). Hence $(I - Q_{t_0})\xi \in E(\mathscr{H})$. Since $Q_{t_0}\xi \in E(\mathscr{H})$, we have $\xi \in E(\mathscr{H})$ and $\zeta - a\xi \in Q_{t_0}(\mathscr{H})$. Therefore we have $E = Q_{t_0} \vee P_\xi$. ∎

Next we will show that $\mathscr{L}$ is a KS-lattice. Let $\mathscr{L}_0$ be a reflexive sub-lattice of $\mathscr{L}$.

Lemma 8 *If $\{\mathscr{L}_0\}'' = \mathcal{B}(\mathscr{H})$, then $P_\xi \in \mathscr{L}_0$.*

Proof. Suppose $P_\xi \notin \mathscr{L}_0$. Let

$$t_0 = \inf\{t \in \mathcal{S} | Q_t \vee P_\xi \in \mathscr{L}_0\}.$$

Since $P_\xi \notin \mathscr{L}_0$, $t_0 > 0$. Also since $\{\mathscr{L}_0\}'' = \mathcal{B}(\mathscr{H})$, there must be a $t \in \mathcal{S} \setminus \{1\}$ such that $Q_t \vee P_\xi \in \mathscr{L}_0$. Hence $t_0 < 1$. Since $\mathcal{S}$ is compact, we have $t_0 \in \mathcal{S}$. Finally it is easy to see that $Q_{t_0} \in \mathscr{L}_0'$. This is a contradiction and the result follows. ∎

If there is no non-trival projection $Q_t (\neq 0, I)$ in $\mathscr{L}_0$, then $\{\mathscr{L}_0\}''$ will be abelian. Hence

$$\mathcal{S}_0 = \{t \in \mathcal{S} | Q_t \in \mathscr{L}_0\} \neq \{0, 1\}.$$

By the completeness of $\mathscr{L}_0$, we have that $\mathcal{S}_0$ is a closed (and hence compact) subset of $\mathcal{S}$.

Lemma 9 *Suppose $\{\mathscr{L}_0\}'' = \mathcal{B}(\mathscr{H})$. If $t \in \mathcal{S} \setminus \mathcal{S}_0$, then $(t, 1) \cap \mathcal{S}_0 \neq \emptyset$.*

Proof. Assume that $(t, 1) \cap \mathcal{S}_0 = \emptyset$. Then $dim(I - Q_t)(\mathscr{H}) \geq 1$. Let

$$t_0 = \sup\{s \in \mathcal{S}_0 | s < t\} < t.$$

Then $dim(I - Q_{t_0})\mathscr{H} \geq 2$ and $(t_0, 1) \cap \mathcal{S}_0 = \emptyset$. Now it is easy to see that $I \neq Q_{t_0} \vee P_\xi \in \mathscr{L}_0'$. This is a contradiction and the result follows. ∎

Theorem 10 *If $\mathscr{L}_0$ generates $\mathcal{B}(\mathscr{H})$, then $\mathscr{L}_0 = \mathscr{L}$.*

Proof. Suppose there is a t in $\mathcal{S} \setminus \mathcal{S}_0$. By the lemma above there is a t_0 in $(t, 1) \cap \mathcal{S}_0$. Let

$$t_1 = \sup\{s \in \mathcal{S}_0 | s < t\} < t,$$
$$t_2 = \inf\{s \in \mathcal{S}_0 | s > t\} \in (t, 1).$$

Note that for any $s \in (t_1, t_2) \cap \mathcal{S}$, $Q_s \vee P_\xi \notin \mathscr{L}_0$ (otherwise, $Q_s = Q_{t_0} \wedge (Q_s \vee P_\xi) \in \mathscr{L}_0$ which implies that $s \in \mathcal{S}_0$, but this contradicts with the fact that $t_1 < s < t_2$.) Since $dim(Q_{t_2} - Q_{t_1})\mathscr{H} \geq 2$, we can choose a subprojection F of $(Q_{t_2} - Q_{t_1})$ such that $F(Q_{t_2} - Q_{t_1})\xi = 0$. Then $F \in \mathscr{L}_0'$ and we get a contradiction. Thus $\mathcal{S}_0 = \mathcal{S}$. Since $P_\xi \in \mathscr{L}_0$, we have $\mathscr{L}_0 = \mathscr{L}$. ■

The following theorem is our main result which follows from above results.

Theorem 11 *$\mathscr{L}$ is a Kadison-Singer lattice and $\mathcal{A}lg(\mathscr{L})$ is a Kadison-Singer algebra.*

In the following, we will give some examples of Kadison-Singer lattices and Kadison-Singer algebras coming from one point extension of a maximal nest. Our first examples comes from the one point extension of a $\mathbb{Z}$-ordered maximal nest.

Example 12 *Suppose $\mathscr{H}$ is a separable infinite dimensional Hilbert space with an orthogonal bases $\{e_i : i \in \mathbb{N}\}$. For each $i \in \mathbb{N}$, let P_n be the orthogonal projection of $\mathscr{H}$ onto the linear subspace of $\mathscr{H}$ generated by $\{e_1, e_2, \cdots, e_n\}$. Let $\xi = \sum_{n=1}^{\infty} a_n e_n \in \mathscr{H}$ be a vector with all $a_n \neq 0$. Without loss of generality, we can assume that $a_1 = 1$. Let P_ξ be the orthogonal projection of $\mathscr{H}$ onto the one dimensional subspace generated by $\mathbb{C}\xi$. Let 0 and I be the zero operator and identity operator on $\mathscr{H}$ respectively. Since $P_n \wedge P_\xi = 0$ for any $n \in \mathbb{N}$,*

$$\mathscr{L} = \{0, I, P_n, P_\xi, P_n \vee P_\xi : n \in \mathbb{N}\}$$

is the lattice generated by $\{P_n : n \in \mathbb{N}\}$ and P_ξ. It follows from section 2 that $\mathscr{L}$ is a Kadison-Singer lattice and $Alg(\mathscr{L})$ is a Kadison-Singer algebra with diagonal equal to $\mathbb{C}I$..

The second example concerns the one point extension of a $\mathbb{Z}$-ordered type nest.

Example 13 *Suppose $\mathscr{H}$ is an infinite dimensional Hilbert space with orthogonal basis $\{e_n : n \in \mathbb{Z}\}$. For each $n \in \mathbb{Z}$, let P_n be the orthogonal projection of $\mathscr{H}$ onto the closed subspace spanned by $\{e_k : k \in \mathbb{Z}, k \leq n\}$. Note that $lim_{n \to -\infty} P_n = 0$ and $lim_{n\to\infty} P_n = I$ in the strong operator topology. Given a vector $\xi = \sum_{-\infty}^{\infty} a_k e_k \in \mathscr{H}$ with $a_k \neq 0$ for all $k \in \mathbb{Z}$. Let P_ξ be the orthogonal projection of $\mathscr{H}$ onto the closed subspace of $\mathscr{H}$ spanned by ξ. Let $\mathscr{L}$ be the lattice generated by P_n ($n \in \mathbb{Z}$) and P_ξ. Then since $P_n \wedge P_\xi = 0$ for all $n \in \mathbb{Z}$,*

$$\mathscr{L} = \{P_n, P_\xi, P_n \vee P_\xi, 0, I : n \in \mathbb{Z}\}.$$

From results obtained in section 2, $\mathscr{L}$ is a Kadison-Singer lattice and $Alg(\mathscr{L})$ is a Kadison-Singer algebra with diagonal equal to $\mathbb{C}I$..

Our last examples is about the continuous nest which is of order $[0, 1]$.

Example 14 *Let $\mathscr{H} = L^2[0,1]$ equipped with the inner product:*

$$\langle f, g \rangle = \int_{[0,1]} \overline{g(x)} f(x) dx.$$

For every $f \in L^\infty[0,1]$, define M_f on $\mathscr{H}$ by

$$(M_f(g))(x) = f(x)g(x), \forall g \in L^2[0,1], x \in [0,1].$$

Then M_f is a bounded linear operator on $\mathscr{H}$. Let

$$\mathscr{D} = \{M_f : f \in L^\infty[0,1]\}.$$

Then $\mathscr{D}$ is a maximal abelian subalgebra of $B(\mathscr{H})$ (for the proof of these facts, see [7]).

For any $t \in [0,1]$, let $\chi_{[0,t]}$ be the characteristic function of $[0,t]$ and $P_t = M_{\chi_{[0,t]}}$ be the orthogonal projection on $L^2[0,1]$ defined by $P_t(g) = \chi_{[0,t]} g$. Suppose that $\xi \in \mathscr{H}$ is a measurable function that is nonzero almost everywhere (we may suppose that $\xi(t) \neq 0$ for all $t \in [0,1]$). Let P_ξ be the orthogonal projection of $\mathscr{H}$ onto the one dimensional closed subspace $\mathbb{C}\xi$. Let $\mathscr{L}$ be the lattice generated by $\{P_t : t \in [0,1]\}$ and P_ξ. Then by results in section 2, we see that $\mathscr{L}$ is a Kadison-Singer lattice and $Alg(\mathscr{L})$ is a Kadison-Singer algebra with diagonal equal to $\mathbb{C}I$.

3 The lattice generated by a finite nest and a projection free with it

First we recall some definitions from free probability theory ([9]). Suppose $\mathscr{N}$ is finite von Neumann algebra $\mathscr{N}$ and a faithful normal tracial state τ. Elements in $\mathscr{N}$ are called random variables. Let $(\mathscr{N}, \tau)$ be a W^*-probability space and $\mathcal{I}$ be a fixed index set. A family of unital subalgebras $\{\mathscr{N}_i : i \in \mathcal{I}\}$ of $\mathscr{N}$ are free with respect to the trace τ if $\tau(A_1 A_2 \cdots A_n) = 0$ whenever $A_j \in \mathscr{N}_{i_j}$ $(i_j \in \mathcal{I})$, $i_1 \neq i_2 \neq \cdots \neq i_n$ and $\tau(A_j) = 0$ for $1 \leq j \leq n$ and every n in $\mathbb{N}$. The random variables $X_1, X_2, \cdots, X_n$ in $\mathscr{N}$ are free with respect to τ if the von Neumann subalgebras $\mathscr{A}_i$ of $\mathscr{N}$ generated by X_i, respectively, are free.

Let n be a positive integer such that $n \geq 2$. Suppose $\{P_1, P_2, \cdots, P_n = I\}$ is a family of projections in a finite von Neumann algebra $\mathscr{N} (\subset B(\mathscr{H})$, where $\mathscr{H}$ is a Hilbert space) with faithful tracial state τ such that

$$P_1 \leq P_2 \leq \cdots \leq P_n$$

and $\tau(P_k) = \frac{k}{n}$ for $k = 1, 2, \cdots, n$. Let Q be a projection in $\mathscr{N}$ free with $\{P_1, P_2, \cdots, P_n\}$ and $\tau(Q) = \frac{1}{n}$.

K. Dykema [11] and F. Radulescu [12] introduced, independently, the interpolated free group factors $\mathscr{L}(F_t)$, for $t > 1$. These factors can be obtained from the free group factors by suitable compression with projections. The following lemma is an easy consequence of Theorem 2.3 in [10].

Lemma 15 *The von Neumann algebra $\mathscr{M}$ generated by $\{P_1, P_2, \cdots, P_n\}$ and Q is*

$$\{P_1, P_2, \cdots, P_n\}'' * \{Q\}'' \cong \mathscr{L}(F_{1+\frac{1}{n}-\frac{2}{n^2}}).$$

Lemma 16 *For any $i \in \{1, 2, \cdots, n-1\}$, we have $P_i \wedge Q = 0$.*

Proof. It suffice to show that $P_1 \wedge Q = 0$. It is not difficult to that the $\tau((PQP)^n) \to 0$ as $n \to \infty$. Since $(PQP)^n \to P \wedge Q$ in weak operator topology, we have $\tau(P \wedge Q) = 0$ which shows that $P \wedge Q = 0$. This completes the proof. ∎

Lemma 17 *Let*

$$\mathscr{L} = \{0, P_i, Q, P_i \vee Q : i = 1, 2, \cdots, n\}.$$

Then $\mathscr{L}$ is the reflexive lattice generated by $\{P_1, P_2, \cdots, P_n\}$ and Q.

Proof. Note that $P_i \wedge Q = 0$ for $i = 1, 2, \cdots, n-1$. It is easy to see that $\mathscr{L}$ is a lattice generated by $\{P_1, P_2, \cdots, P_n\}$ and Q. To show $\mathscr{L}$ is distributive, it suffice to show that $\mathscr{L}$ is distributive. Note we have the following equalities.

(1) $(P_i \vee Q) \wedge P_j = P_{min\{i,j\}}$. We may assume $i \leq j$. Then $P_i \leq P_j$ and $P_i \leq P_i \vee Q$, thus $(P_i \vee Q) \wedge P_j \geq P_{min\{i,j\}}$. But they have same trace (use $\tau(E) + \tau(F) = \tau(E \vee F) + \tau(E \wedge F)$). Hence $(P_i \vee Q) \wedge P_j = P_{min\{i,j\}}$.

(2) For $i, j \in \{1, 2, \cdots, n-1\}$, we have $(P_i \vee P_j) \wedge Q = 0$.

Use this two formulas it is easy to show that $\mathscr{L}$ is distributive. It follows from [13] that $\mathscr{L}$ is a reflexive lattice. ∎

Lemma 18 *$\mathscr{L}$ is the minimal lattice that generate the von Neumann algebra $\mathscr{M}$.*

Proof. Let $\mathscr{L}_0$ be a reflexive sublattice of $\mathscr{L}$ such that $\mathscr{L}_0{}'' = \mathscr{M} = \mathscr{L}''$. We want to show that $\mathscr{L}_0 = \mathscr{L}$. First we show that $Q \in \mathscr{L}_0$. Indeed if $Q \notin \mathscr{L}_0$, then $P_1 \in \mathscr{L}_0{}' = \mathscr{L}'$ which contradicts with the fact that $PQ \neq QP$. Thus $Q \in \mathscr{L}_0$. Now we show $\forall i \in \{1, 2, \cdots, n-1\}$, we have $P_i \in \mathscr{L}_0$. Let k be the minimal number in $\{1, 2, \cdots, n-1\}$ such that $P_1, \cdots, P_{k-1} \in \mathscr{L}_0$, $P_k \notin \mathscr{L}_0$. Then we must have $P_k \wedge Q \notin \mathscr{L}_0$ (otherwise suppose $P_k \wedge Q \in \mathscr{L}_0$. If $P_j \notin \mathscr{L}_0$ for all $j \neq k+1$, $j \neq n$, then $P_k \wedge Q \in \mathscr{L}_0{}' = \mathscr{L}'$ which is a contradiction). Hence $\mathscr{L}_0''$ is a von Neumann algebra containing in $\mathscr{L}(F_t)$ with $t = 1 + \frac{n-4}{n^2}$ and $\mathscr{L}_0'' \neq \mathscr{L}''$. This is again a contradiction and shows that $\{P_1, P_2, \cdots, P_n - 1\} \subset \mathscr{L}_0$. This shows that $\mathscr{L}_0 = \mathscr{L}$ and completes the proof. ∎

It follows from above Lemmas that we have the following result.

Theorem 19 *$\mathscr{L}$ is a Kadison-Singer lattice and $\mathcal{A}lg\mathscr{L}$ is a Kadison-Singer algebra.*

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