A note on operators commuted with a unbounded operator afflicated with II_1 factors

WENMING WU AND WEI YUAN

ABSTRACT. Enter abstract here

INTRODUCTION

Introduction here!

1. UNBOUNDED FUGLEDE-PUTNAM THEOREM

The celebrated Fuglede-Putnam theorem in its classical form is as follows:

Theorem 1.1 (Fuglede-Putnam Theorem)**.** *If T is a bounded operator and if M and N are normal operators, then*

$$TN \subset MT \Rightarrow TN^* \subset M^*T.$$

Fuglede [4] proved the foregoing theorem in the case $N = M$ in 1950 and Putnam extended it to the form given above one year later [5]. In 1958, Rosenblum published a very elegant proof of the theorem in the case that M and N are bounded. We will prove the following version of Fuglede-Putnam theorem.

Theorem 1.2. *Let $\mathfrak{A}$ be a finite von Neumann algebra, and T a closed operator affiliated with $\mathfrak{A}$. If N is a normal operators in $\mathfrak{A}$ and $NT = TN$, then $N^*T = TN^*$.*

In order to prove this theorem, we will first prove the following lemma.

Lemma 1.1. *Suppose that $\mathfrak{A}$ is a finite von Neumann algebra. Let N be a normal operator in $\mathfrak{A}$ and P a projection in $\mathfrak{A}$. If $(I-P)NP = 0$, then $PN = NP$.*

Proof. Let τ be a faithful trace on $\mathfrak{A}$ and

$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \qquad N = \begin{pmatrix} N_1 & N_2 \\ 0 & N_3 \end{pmatrix}.$$

Since $\tau(PN^*NP) = \tau(PNN^*P)$, $\tau|_{P\mathfrak{A}P}(N_1^*N_1) = \tau|_{P\mathfrak{A}P}(N_1N_1^* + N_2N_2^*)$. Therefore, $\tau(N_2N_2^*) = 0$ and $N_2 = 0$. $\square$

Lemma 1.2. *Let $\mathfrak{A}$ be a finite von Neumann algebra, and H a closed positive operator affiliated with $\mathfrak{A}$. If M and N are normal operators in $\mathfrak{A}$ and $NH = HM$, then $N^*H = HM^*$, $NH = HN$ and $MH = HM$. Furthermore, if $Ker(H) = \{0\}$, then $M = N$.*

Proof. If $ker(H) \neq \{0\}$, then it is not hard to see that $(I - E_0)ME_0 = 0$, where E_0 is the orthonormal projection onto $ker(H)$. By Lemma 1.1, $E_0M = ME_0$. Since $M^*H = HN^*$, similar argument shows that $E_0N^* = N^*E_0$.

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Therefore we assume $ker(H) = \{0\}$. Let τ be a faithful trace on $\mathfrak{A}$. Let $\{E_\lambda\}$ be the resolution of identity in $\mathfrak{A}$ such that

$$H = \int_0^\infty \lambda dE_\lambda.$$

Fix a $\lambda > 0$, let

$$P = E_\lambda = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}, \quad N = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}, \quad M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

where $H_1 = HE_\lambda$ and $H_2 = H(I - E_\lambda)$. $NH = HM$ implies that

$$\begin{pmatrix} H_1^{-1}N_{11}H_1 & H_1^{-1}N_{12}H_2 \\ H_2^{-1}N_{21}H_1 & H_2^{-1}N_{22}H_2 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

Since M is normal, we have $\tau(PM^*MP) = \tau(PMM^*P)$,

$$\begin{aligned} &\tau\left(\begin{pmatrix} H_1N_{11}^*H_1^{-2}N_{11}H_1 + H_1N_{21}^*H_2^{-2}N_{21}H_1 & 0 \\ 0 & 0 \end{pmatrix}\right) \\ =&\tau\left(\begin{pmatrix} H_1^{-1}N_{11}H_1^2N_{11}^*H_1^{-1} + H_1^{-1}N_{12}H_2^2N_{12}^*H_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}\right). \end{aligned}$$

Note that

$$\tau|_{P\mathfrak{A}P}(H_1N_{11}^*H_1^{-2}N_{11}H_1) = \tau|_{P\mathfrak{A}P}(H_1^{-1}N_{11}H_1^2N_{11}^*H_1^{-1}).$$

We have

$$\tau|_{P\mathfrak{A}P}(H_1N_{21}^*H_2^{-2}N_{21}H_1) = \tau|_{P\mathfrak{A}P}(H_1^{-1}N_{12}H_2^2N_{12}^*H_1^{-1}).$$

Since $\|H_1\| \le \lambda$ and $\|H_2^{-1}\| \le \frac{1}{\lambda}$, we have

$$\begin{aligned} \tau\left(\begin{pmatrix} H_1N_{21}^*H_2^{-2}N_{21}H_1 & 0 \\ 0 & 0 \end{pmatrix}\right) &\le \frac{1}{\lambda^2}\tau\left(\begin{pmatrix} H_1N_{21}^*N_{21}H_1 & 0 \\ 0 & 0 \end{pmatrix}\right) \\ &= \frac{1}{\lambda^2}\tau\left(\begin{pmatrix} 0 & H_1N_{21}^* \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ N_{21}H_1 & 0 \end{pmatrix}\right) \\ &= \frac{1}{\lambda^2}\tau\left(\begin{pmatrix} 0 & 0 \\ 0 & N_{21}H_1^2N_{21}^* \end{pmatrix}\right) \\ &\le \tau\left(\begin{pmatrix} 0 & 0 \\ 0 & N_{21}N_{21}^* \end{pmatrix}\right) = \tau\left(\begin{pmatrix} N_{21}^*N_{21} & 0 \\ 0 & 0 \end{pmatrix}\right) \end{aligned}$$

Let $Q = E_\beta - E_\lambda$ where $\beta > \lambda$.

$$\begin{aligned} &\tau\left(\begin{pmatrix} H_1^{-1}N_{12}H_2^2N_{12}^*H_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}\right) \\ &\ge \beta^2\tau\left(\begin{pmatrix} H_1^{-1}N_{12}(I-Q)N_{12}^*H_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}\right) + \lambda^2\tau\left(\begin{pmatrix} H_1^{-1}N_{12}QN_{12}^*H_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}\right) \\ &= \beta^2\tau\left(\begin{pmatrix} 0 & 0 \\ 0 & (I-Q)N_{12}^*H_1^{-2}N_{12}(I-Q) \end{pmatrix}\right) + \lambda^2\tau\left(\begin{pmatrix} 0 & 0 \\ 0 & QN_{12}^*H_1^{-2}N_{12}Q \end{pmatrix}\right) \\ &\ge \frac{\beta^2}{\lambda^2}\tau\left(\begin{pmatrix} 0 & 0 \\ 0 & (I-Q)N_{12}^*N_{12}(I-Q) \end{pmatrix}\right) + \tau\left(\begin{pmatrix} 0 & 0 \\ 0 & QN_{12}^*N_{12}Q \end{pmatrix}\right) \\ &= \frac{\beta^2}{\lambda^2}\tau\left(\begin{pmatrix} N_{12}(I-Q)N_{12}^* & 0 \\ 0 & 0 \end{pmatrix}\right) + \tau\left(\begin{pmatrix} N_{12}QN_{12}^* & 0 \\ 0 & \end{pmatrix}\right) \end{aligned}$$

Since $N^*N = NN^*$, we have

$$\tau\left(\begin{pmatrix} N_{12}N_{12}^* & 0 \\ 0 & 0 \end{pmatrix}\right) = \tau\left(\begin{pmatrix} N_{21}^*N_{21} & 0 \\ 0 & 0 \end{pmatrix}\right).$$

Therefore

$$\frac{\beta^2}{\lambda^2}\tau\left(\begin{pmatrix} N_{12}(I-Q)N_{12}^* & 0 \\ 0 & 0 \end{pmatrix}\right) + \tau\left(\begin{pmatrix} N_{12}QN_{12}^* & 0 \\ 0 & \end{pmatrix}\right) \leq \tau\left(\begin{pmatrix} N_{12}N_{12}^* & 0 \\ 0 & 0 \end{pmatrix}\right).$$

Thus

$$\frac{\beta^2}{\lambda^2}\tau\left(\begin{pmatrix} N_{12}(I-Q)N_{12}^* & 0 \\ 0 & 0 \end{pmatrix}\right) \leq \tau\left(\begin{pmatrix} N_{12}(I-Q)N_{12}^* & 0 \\ 0 & 0 \end{pmatrix}\right).$$

This implies that $N_{12}(I-Q)N_{12}^* = 0$. Since $E_\lambda = \wedge_{\alpha>\lambda}E_\alpha$, we have $N_{12}N_{12}^* = 0$. By Lemma 1.1, we have $E_\lambda N = NE_\lambda$. Since λ is arbitrary, $NH = HN$. By the hypothesis, $(N-M)H = 0$. Thus, $N = M$ since $ker(H) = \{0\}$. □

Now we can prove Theorem 1.2.

Proof of Theorem 1.2. Let $T = UH$ be the polar decomposition of T (Since $\mathfrak{A}$ is a finite von Neumann algebra, we could also assume that U is a unitary). $NT = TN$ is equivalent to $U^*NUH = HN$. Let $M = U^*NU$. We have $MH = HN$. By Lemma 1.2, $M^*H = HN^*$, and $N^*T = TN^*$. □

Corollary 1.1. *Let $\mathfrak{A}$ be a finite von Neumann algebra, and T a closed operator affiliated with $\mathfrak{A}$. If N is a normal operators in $\mathfrak{A}$ and $NT = TN$, then $AT = TA$ for each A in the von Neumann algebra generated by N.*

Proof. By Theorem 1.2 and Lemma 5.6.13 in [2], we have the result. □

Corollary 1.2. *Let $\mathfrak{A}$ be a finite von Neumann algebra, and T a closed operator affiliated with $\mathfrak{A}$. If N is a normal operator affiliated with $\mathfrak{A}$ and $NT = TN$, then $N^*T = TN^*$.*

Proof. Let E_n be the spectral projections for N corresponding to the set $\{z : |z| \leq n\}$ for each positive integer n. Thus $\{E_n\}$ is a increasing sequence of projections with strong-operator limit I. By $NT = TN$, we have

$$(E_nNE_n)(E_nTE_n) = (E_nTE_n)(E_nNE_n).$$

By Theorem 1.2,

$$E_nN^*TE_n = (E_nN^*E_n)(E_nTE_n) = (E_nTE_n)(E_nN^*E_n) = E_nTN^*E_n.$$

Note that $E_n \leq E_m$ if $n \leq m$. Multiply E_n from right on both side of the equation $E_mN^*TE_m = E_mTN^*E_m$, we have $E_mN^*TE_n = E_mTN^*E_n$. Let m tend to ∞, we get $N^*TE_n = TN^*E_n$. Thus, $E_nT^*N = E_nNT^*$. Let n tend to ∞ we have $T^*N = NT^*$ and $N^*T = TN^*$. □

Corollary 1.3. *Let $\mathfrak{A}$ be a finite von Neumann algebra, and T a closed operator affiliated with $\mathfrak{A}$. If N and M are normal operators affiliated with $\mathfrak{A}$ and $MT = TN$, then $M^*T = TN^*$.*

Proof. Note that

$$\mathfrak{A} \otimes M_s(\mathbb{C}) = \{\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} : A_{ij} \in \mathfrak{A}\}$$

is also a finite von Neumann algebra. Consider

$$N_1 = \begin{pmatrix} N & 0 \\ 0 & M \end{pmatrix} \text{ and } T_1 = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix}.$$

Note that N_1 is normal, $N_1T_1 = T_1N_1$. By Corollary 1.2, we have $N_1^*T_1 = T_1N_1^*$. Comparing the $(2,1)$ entry then gives $M^*T = TN^*$. □

Corollary 1.4. *Let $\mathfrak{A}$ be a finite von Neumann algebra with a faithful trace τ, and T is a closed operator affiliated with $\mathfrak{A}$. If N and M are normal operators in $\mathfrak{A}$ and $MT = TN$, then $f(M)T = Tf(N)$ for any measurable function f.*

Let T be a closed operator. The numerical range of T, denoted by $W(T)$, is defined as

$$W(T) = \{\langle T\xi, \xi\rangle : \xi \in \mathfrak{D}(T), \|\xi\|_2 = 1\}.$$

In [7], Embry proved the following theorem.

Theorem 1.3. *Let N and M be two commuting bounded normal operators and T a bounded operator such that $0 \notin W(T)$. If $MT = TN$, then $N = M$.*

By Corollary 1.4 and an argument parallel to that used in [7], we have the following fact.

Corollary 1.5. *Let $\mathfrak{A}$ be a finite von Neumann algebra, and T a closed operator affiliated with $\mathfrak{A}$. If N and M are two commuting normal operators in $\mathfrak{A}$, $MT = TN$ and $0 \notin W(T)$, then $N = M$.*

The following result is well known.

Lemma 1.3. *Let $\mathfrak{A}$ be a separable II_1 factor. There exist two maximal abelian subalgebras $\mathfrak{M}_1$, $\mathfrak{M}_2$ such that $\mathfrak{M}_1 \cap \mathfrak{M}_2 = \mathbb{C}I$.*

Proof. By Corollary 4.1 in [9], there is a hyperfinite subfactor $\mathcal{R}$ such that $\mathcal{R}' \cap \mathfrak{A} = \mathbb{C}I$. Let $\widetilde{\mathfrak{M}_1}$ and $\widetilde{\mathfrak{M}_2}$ be two orthognal maximal abelian subalgebras generate $\mathcal{R}$. There exist two maximal abelian subalgebras $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of $\mathfrak{A}$ contains $\widetilde{\mathfrak{M}_1}$ and $\widetilde{\mathfrak{M}_2}$ respectively. If $T \in \mathfrak{M}_1 \cap \mathfrak{M}_2$, then T in $\mathfrak{A}$ commute with all elements in $\widetilde{\mathfrak{M}_1}$ and $\widetilde{\mathfrak{M}_2}$. Hence T commutes with $\mathcal{R}$ and T is a scalar. □

Corollary 1.6. *If $\mathfrak{A}$ be a separable II_1 factor, then there exists a closed operator T affiliated with $\mathfrak{A}$ such that $NT \neq TN$ for any nontrivial normal operator affiliated with $\mathfrak{A}$.*

Proof. By Lemma 1.3, there exist two maximal abelian subalgebras $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of $\mathfrak{A}$ such that $\mathfrak{M}_1 \cap \mathfrak{M}_2 = \mathbb{C}I$. Let $T = H_1 + iH_2$ where H_1 and H_2 are two positive invertible (the inverse is a bouned positive operator in $\mathfrak{A}$) operators generate $\mathfrak{M}_1$ and $\mathfrak{M}_2$ respective. Suppose $N\eta\mathfrak{A}$ is a nontrivial normal operator and $NT = TN$. By Corollary 1.2, $N^*T = TN^*$. Hence $NT^* = T^*N$. This implise that $TH_1 = H_1T$ and $TH_2 = TH_2$. If T is a unitary, then T is in $\mathfrak{M}_1 \cap \mathfrak{M}_2 = \mathbb{C}I$. If T is not a unitary, then $(I + T^*T)^{-1}$ is in $\mathfrak{M}_1 \cap \mathfrak{M}_2 = \mathbb{C}I$. Thus T must be a scalar. □

2. Examples

Throughout this section $(X, \mathcal{B}, \mu)$ will denote a probability space. For simplicity of notation, we will use h to denote the multiplication operator associated with the measurable function h whenever there is no danger of ambiguity. Let G be a group acts on X ergodically, i.e., if $X \in \mathcal{B}$ and $\mu(g(X) \setminus X) = 0$ for each g in G, then either $m(X) = 0$ or $m(S \setminus X) = 0$, and preserves the measure. This induces an automorphic representation of G on $L^\infty(X)$: $\alpha_g(h)(x) = h(g^{-1}(x))$. The cross product $L^\infty(X) \rtimes_\alpha G$ is the von Neumann algebra, acting on $L^2(X, d\mu) \otimes l^2(G)$, generated by the operators

$$\psi(h) = \sum_{g\in G} \alpha_g^{-1}(h) \otimes E_g, \qquad L_g = I \otimes l_g \qquad (h \in L^\infty(X), g \in G),$$

where E_g is the projection onto the one dimensional subspace spanned by e_g in $l^2(G)$.

Since G preserves the measure, $L^\infty(X) \rtimes_\alpha G$ is a II_1 factor. Let $s_1, s_2, \ldots, s_n$ be n elements in G and $\{h_{s_i}\}_{i=1}^n$ be a measurable functions on X such that $\mu(\{x : h_{s_i}(x) = 0 \text{ or } \infty\}) = 0$. It is easy to see that the operator

$$T = \sum_{i=1}^n \psi(h_{s_i}) L_{s_i} = \sum_{i=1}^n \sum_{g \in G} \alpha_g^{-1}(h_{s_i}) \otimes E_g l_{s_i}$$

is affiliated with $L^\infty(X) \rtimes_\alpha G$. Let

$$A = \sum_s \sum_g \alpha_g^{-1}(f_s) \otimes E_g l_s$$

be any operator in $L^\infty(X) \rtimes_\alpha G$. We have

$$\begin{aligned}
\langle AT\xi \otimes e_l, \beta \otimes e_g \rangle &= \left\langle (\sum_{i=1}^n \sum_{g_2} \alpha_{g_2}^{-1}(h_{s_i}) \otimes E_{g_2} l_{s_i})\xi \otimes e_l, (\sum_s \sum_{g_1} \alpha_{g_1}^{-1}(\bar{f}_s) \otimes l_{s^{-1}} E_{g_1})\beta \otimes e_g \right\rangle \\
&= \left\langle \sum_{i=1}^n \alpha_{s_i l}^{-1}(h_{s_i})\xi \otimes e_{s_i l}, \sum_s \alpha_g^{-1}(\bar{f}_s)\beta \otimes e_{s^{-1}g} \right\rangle \\
&= \sum_{i=1}^n \left\langle \alpha_{s_i l}^{-1}(h_{s_i})\xi, \alpha_g^{-1}(\overline{f_{gl^{-1}s_i^{-1}}})\beta \right\rangle \\
&= \left\langle \sum_{i=1}^n \alpha_g^{-1}(f_{gl^{-1}s_i^{-1}})\alpha_{s_i l}^{-1}(h_{s_i})\xi, \beta \right\rangle,
\end{aligned}$$

and

$$\begin{aligned}
\langle TA\xi \otimes e_l, \beta \otimes e_g \rangle &= \left\langle (\sum_s \sum_{g_1} \alpha_{g_1}^{-1}(f_s) \otimes E_{g_1} L_s)\xi \otimes e_l, (\sum_{i=1}^n \sum_{g_2} \alpha_{g_2}^{-1}(\overline{h_{s_i}}) \otimes l_{s_i^{-1}} E_{g_2})\beta \otimes e_g \right\rangle \\
&= \left\langle \sum_s \alpha_{sl}^{-1}(f_s)\xi \otimes e_{sl}, \sum_{i=1}^n \alpha_g^{-1}(\overline{h_{s_i}})\beta \otimes e_{s_i^{-1}g} \right\rangle \\
&= \left\langle \sum_{i=1}^n \alpha_g^{-1}(h_{s_i})\alpha_{s_i^{-1}g}^{-1}(f_{s_i^{-1}gl^{-1}})\xi, \beta \right\rangle
\end{aligned}$$

If $AT = TA$, then (let $gl^{-1} = s$)

$$\sum_{i=1}^n \alpha_g^{-1}(f_{ss_i^{-1}})\alpha_{s_i s^{-1}g}^{-1}(h_{s_i}) = \sum_{i=1}^n \alpha_g^{-1}(h_{s_i})\alpha_{s_i^{-1}g}^{-1}(f_{s_i^{-1}s}). \tag{1}$$

2.1. **Hyperfinite case.** Let $G = \mathbb{Z}$, $s_1 = 1$, $g = n$, $s = m + 1$. By eq. (1) we have

$$\alpha_{-n}(f_m)\alpha_{m-n}(h_1) = \alpha_{-n}(h_1)\alpha_{1-n}(f_m).$$

Apply α_n to both side of the equation above, we have

$$f_m \alpha_m(h_1) = h_1 \alpha(f_m).$$

Recall that h_i is a measurable functions on X such that $\mu(\{x : h_{s_i}(x) = 0 \text{ or } \infty\}) = 0$. Therefore we have

$$\frac{\alpha(f_m)}{f_m} = \frac{\alpha_m(h_1)}{h_1}.$$

Let

$$k_m = \begin{cases} h_1\alpha_1(h_1)\cdots\alpha_{m-1}(h_1) & \text{if } m > 0, \\ 1 & \text{if } m = 0, \\ \alpha_{-1}(\frac{1}{h_1})\alpha_{-2}(\frac{1}{h_1})\cdots\alpha_m(\frac{1}{h_1}) & \text{if } m < 0. \end{cases}$$

We have

$$\frac{\alpha(f_m)}{f_m} = \frac{\alpha(k_m)}{k_m} \text{ and } \alpha(\frac{f_m}{k_m}) = \frac{f_m}{k_m}.$$

By lemma 8.6.6 in [2], there exist c_k, $k = 0, \pm 1, \pm 2, \ldots$, such that $f_m = c_m k_m$, a.e.. It is not hard to choose h_1, e.g. let $X = S^1$ and $h_1 = \frac{e^{2\pi i\theta}+1}{e^{2\pi i\theta}-1}$, such that each $k_m (m \neq 0)$ is an unbounded measurable function.

Recall that a Cartan subalgebra $\mathcal{M}$ in a II_1 factor $\mathfrak{A}$ is a maximal abelian *-subalgebra with normalizer $\mathcal{N}_{\mathfrak{A}}(\mathcal{M}) = \{U \in \mathcal{U}(\mathfrak{A}) : U^*\mathcal{M}U = \mathcal{M}\}$ generating $\mathfrak{A}$.

Lemma 2.1. *Let $\mathcal{R}$ be the hyperfinite von Neumann algebra. There exists a closed operator $T = UH\eta\mathcal{R}$ such that $T' \cap \mathcal{R} = \mathbb{C}I$, T generates $\mathcal{R}$ i.e. U and H generate $\mathcal{R}$ and H generates a Cartan subalgebra of $\mathcal{R}$.*

Since any two Cartan subalgebra of the hyperfinite II_1 factor are conjugate by an automorphism of $\mathcal{R}$, we have the following result.

Corollary 2.1. *Let $\mathfrak{A}$ be a separable II_1 factor. There exists a hyperfinite II_1 subfactor $\mathcal{M}$ of $\mathfrak{A}$ and a $T\eta\mathcal{M}$ such that $NT \neq TN$ for any nontrivial normal operator affiliated with $\mathfrak{A}$.*

Proof. By Corollary 4.2 in [9], there is a hyperfinite II_1 subfactor $\mathcal{M}$ such that $\mathcal{M}' \cap \mathfrak{A} = \mathbb{C}I$. Let $T = UH$ be the closed operator in Lemma 2.1. If N is a normal operator in $\mathfrak{A}$ and $NT = TN$, then $N \in \mathcal{M}' \cap \mathfrak{A}$. □

2.2. **Non Γ case.** Consider the example in [10]. Let $G = F_2$ be the free group generated by a, b. Assume that the subgroup generated by a acts on X ergodically and the subgroup generated by b acts on X trivially.

Let $T = \sum_g \alpha_g^{-1}(h_a) \otimes E_g L_a$. If $A = \sum_s \sum_g \alpha_g(f_s) \otimes E_g L_s$ is in $L^\infty(X) \rtimes_\alpha G$ and commute with T, then by eq. (1)

$$\alpha_g^{-1}(f_{sa^{-1}})\alpha_{as^{-1}g}^{-1}(h_a) = \alpha_g^{-1}(h_a)\alpha_{a^{-1}g}^{-1}(f_{a^{-1}s}), \qquad \forall g, s \in G.$$

Let $s = sa$ and $g = a$. For simplicity of notation, we will use α^n to denote α_{a^n}. Let ρ be the group homomorphism from F_2 to $\mathbb{Z}$ such that $\rho(a) = 1$, $\rho(b) = 0$. Let $\rho(s) = m$, we have

$$\alpha^{-1}(f_s)\alpha^{m-1}(h_a) = \alpha^{-1}(h_a)f_{a^{-1}sa}, \qquad \forall s \in G \text{ and } m \in \mathbb{Z}.$$

Therefore, $\frac{\alpha(f_{a^{-1}sa})}{f_s} = \frac{\alpha^m(h_a)}{h_a}$.

Let

$$k_m = \begin{cases} h_a\alpha^1(h_a)\cdots\alpha^{m-1}(h_a) & \text{if } m > 0, \\ 1 & \text{if } m = 0, \\ \alpha^{-1}(\frac{1}{h_a})\alpha^{-2}(\frac{1}{h_a})\cdots\alpha^m(\frac{1}{h_a}) & \text{if } m < 0. \end{cases}$$

For any $s \in G$, if $\rho(s) = m$, then $\frac{f_{a^{-1}sa}}{k_m} = \alpha^{-1}(\frac{f_s}{k_m})$. Therefore $f_{a^{-n}sa^n} = \alpha^{-n}(\frac{f_s}{k_m})k_m$.

Let $m = 0$. If s contains $b^{\pm 1}$ in the reduced form, then $f_s = 0$ since $\sum_{n \in Z} \|f_{a^{-n}sa^n}\|_2^2 = \sum_{n \in Z} \|\alpha_n^{-1}(f_s)\|_2^2 = \sum_{n \in Z} \|f_s\|_2^2 \leq \infty$.

If s contains $b^{\pm 1}$ in the reduced form and $m = \rho(s) \neq 0$, then $\frac{f_s}{k_m} = 0$. Indeed, if $\frac{f_s}{k_m} = h \neq 0$, then there exists a measurable subset A in $\mathcal{B}$, $\mu(A) > 0$, $h(x) > c$ for almost every $x \in A$ and $|c\chi_A k_m| \geq \delta > 0$, $\delta > 0$. By Furstenberg's multiple recurrence theorem [3][Theorem 7.4], we have

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap \alpha^{-n}(A)) = \varepsilon > 0.$$

This implies that there exists a subsequence n_i such that $\mu(A \cap \alpha^{-n_i}(A)) >= \varepsilon$. Therefore, we have

$$\infty = \sum_i \delta^2 \varepsilon \leq \sum_i \int \|f_{a^{-n_i} s a^{n_i}}\|^2 d\mu < \infty. \tag{2}$$

It is a contradiction. Hence, $\frac{f_s}{k_m} = 0$ if s contains $b^{\pm 1}$ in the reduced form.

Remark 2.1. *Note that in the above proof, we do not need* 0 *and* ∞ *to be the cluster point of the range of h_a. In another word, this is just a tedious way to show that A must be in the von Neumann subalgebra generated by L_a and $L^\infty(X)$.*

Suppose that $T = KW$ is a unbounded operator affiliated with a II_1 factor such that $T' \cap \mathfrak{A} = \mathbb{C}I$. Here $K = \sqrt{TT^*}$ and W is a unitary.

Let

$$P_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \qquad P_2 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \qquad P_3 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$P_4 = \begin{pmatrix} H & \sqrt{H(I-H)}V \\ V^*\sqrt{H(I-H)} & V^*(I-H)V. \end{pmatrix}$$

We have

$$Alg(\{P_1, P_2, P_3, P_4\}) = \{\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} = \begin{pmatrix} T & 0 \\ 0 & S^{-1}TS \end{pmatrix} | T \in P_1\mathfrak{A}P_1\},$$

where $S = -\sqrt{H(I-H)^{-1}}V$. Just make $V = -W$ and $H = \frac{K^2}{I+K^2}$, we get a transitive lattice contains 4 elements.

3. General

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