Note On Quantum Field Theory

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ABSTRACT. Enter abstract here

1. FIELD OPERATORS

1.1. Distributions.

Definition 1.1. *Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of rapidly decreasing smooth functions, that is the complex vector space of all functions $f : \mathbb{R}^n \to \mathbb{C}$ with continuous partial derivatives of any order for which*

$$|f|_{p,k} = \sup_{|\alpha|\leq q} \sup_{x\in\mathbb{R}^n} |\partial^\alpha f(x)|(1+|x|^2)^k < \infty, \tag{1}$$

for all $p, k \in \mathbb{N}$ and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n$.

Definition 1.2. *A tempered distribution T is a linear functional $T : \mathcal{S} \to \mathbb{C}$ which is continuous with respect to all the seminorms $|.|_{p,k}$ defined in eq. (1), $p, k \in \mathbb{N}$.*

Example 1.1. (1) *Let $g \in L^\infty(\mathbb{R}^n)$.*

$$T_g(f) = \int_{\mathbb{R}^n} g(x)f(x)dx, \qquad f \in \mathcal{S}.$$

(2) *The delta distribution given by*

$$\delta_y : f \to f(y), \qquad f \in \mathcal{S}.$$

If T is a tempered distribution, then

$$\langle \partial^\alpha T, f\rangle = \langle T, (-1)^{|\alpha|}\partial^\alpha f\rangle,$$

i.e.,

$$\partial^\alpha T(f) = (-1)^{|\alpha|}T(\partial^\alpha f), \qquad f \in \mathcal{S}.$$

Example 1.2. *Let $\varkappa$ be the characteristic function of $[0,\infty)$. Then*

$$\frac{d}{dt}T_\varkappa(f) = -\int_0^\infty f'(x)dx = f(0) = \delta_0(f).$$

Proposition 1.1. *Every tempered distribution T has a representation as a finite sum of derivatives of continuous functions of polynomial growth, that is there exist $g_{alpha} : \mathbb{R}^n \to \mathbb{C}$ such that*

$$T = \sum_{0\leq|\alpha|\leq k} \partial^\alpha T_{g_\alpha}.$$

For a polynomial $P(x) = c_\alpha X^\alpha \in \mathbb{C}[x_1, \dots, x_n]$ in n variables with complex coefficients $c_\alpha \in \mathbb{C}$ on obtains the partial differential operator

$$P(i\partial) = c_\alpha(i\partial)^\alpha = \sum (i)^{|\alpha|} c_{\alpha_1,\dots,\alpha_2}\partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}.$$

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Example 1.3. (1) *$P(x) = -(x_1^2 + \cdots + x_n^2)$ gives the Laplace operator*

$$\Delta = \partial_1^2 + \cdots + \partial_n^2.$$

(2) *$P(x) = -x_0^2 + x_1^2 + \cdots + x_n^2$ gives the Laplace-Beltrami operator*

$$\Box = \partial_0^2 - (\partial_1^2 + \cdots + \partial_n^2).$$

For any $f \in \mathcal{S}$, let

$$(\mathfrak{F} f)(x) = \hat{f}(p) = \int_{\mathbb{R}^n} f(x) e^{-i\langle x,p\rangle} dx$$

be the Fourier transform of f. The inverse Fourier transform of f is

$$(\mathfrak{F}^{-1}\hat{f})(x) = f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(p) e^{i\langle p,x\rangle} dp.$$

Note that

$$\mathfrak{F}(\partial_k f)(p) = \int_{\mathbb{R}^n} \partial_k f(x) e^{-i\langle x,p\rangle} dx = -\int_{\mathbb{R}^n} f(x) \partial_k e^{-i\langle x,p\rangle} dx = ip_k \mathfrak{F}(f)(p).$$

Similarly, we have

$$\mathfrak{F}^{-1}(\partial_k \hat{f})(x) = -ix_k \mathfrak{F}^{-1}(\hat{f})(x).$$

Definition 1.3. *For any tampered distribution T, let*

$$\mathfrak{F}(T)(f) = (T \circ \mathfrak{F})(f) = T(\mathfrak{F}(f)).$$

If $P(p)$ is a polynomial, then

$$PT(f) = T(Pf).$$

Example 1.4. (1) *$\mathfrak{F}(\delta_0)(f) = \mathfrak{F}(f)(0) = \int_{\mathbb{R}^n} f(x)dx$, therefore*

$$\mathfrak{F}(\delta_0) = 1.$$

(2)

$$\mathfrak{F}^{-1}(e^{i\langle p,y\rangle}) = \delta_{x-y}.$$

Remark 1.1. *Let T be a tempered distribution and $f \in \mathcal{S}$.*

$$\partial_k \circ \mathfrak{F}(T)(f) = T(\mathfrak{F}(\partial_k f)) = T(ip_k \mathfrak{F}(f)) = \mathfrak{F}(ip_k T)(f).$$

And

$$\mathfrak{F}(i\partial_k T)(f) = T(i\partial_k \mathfrak{F}(f)) = T(\mathfrak{F}(x_k f)) = (x_k \mathfrak{F}(T))(f)$$

1.2. **Klein-Gordon Equation.** Consider the Klein-Gordon Equation with mass $m > 0$:

$$(\Box + m^2)T_1 = T_2, \tag{2}$$

where T_i, $i = 1, 2$ are tempered distribution.

If $T_2 = \delta_0$, then by applying the Fourier transform on both side of eq. (2), we only need to solve the division problem

$$(-\mathbf{p}^2 + m^2)T = 1,$$

where $\mathbf{p}^2 = p_0^2 - (p_1^2 + \cdots + p_{n-1}^2)$.

Consider the homogeneous equation

$$(\Box - m^2)f = 0 \Longleftrightarrow (\mathbf{p}^2 - m^2)\hat{f} = 0.$$

1.3. **Fields Operators.**

Definition 1.4. *Let A be a closed operator. The spectrum of A is*

$$\sigma(A) = \{z \in \mathbb{C} : (zI - A)^{-1} \text{ is not a bounded operator.}\}$$

Definition 1.5. *Let $\mathfrak{O}(\mathcal{H})$ be the set of all densely defined operators in $\mathcal{H}$. A field operator or quantum field is an operator-valued distribution (on $\mathbb{R}^n$), this is a map*

$$\Phi : \mathcal{S}(\mathbb{R}^n) \to \mathfrak{O}(\mathcal{H})$$

such that there exists a dense subspace $\mathfrak{D} \subset \mathcal{H}$ satisfying

(1) *For each $f \in \mathcal{S}$ the domain of $\Phi(f)$ contains $\mathfrak{D}$.*
(2) *The induced map $f \to \Phi(f)|_{\mathfrak{D}}$ is linear.*
(3) *For each $\xi \in \mathfrak{D}$ and $\beta \in \mathcal{H}$ the assignment $f \to \langle \Phi(f)\xi, \beta\rangle$ is a tempered distribution.*

1.4. **Wightman Axioms.** Let $M = \mathbb{R}^{1,D-1}$ be the D-dimensional Minkowski space with the (Lorentz) metric

$$\langle x, x\rangle = (x^0)^2 - \sum_{j=1}^{D-1} (x^j)^2, \qquad x = (x^0, \dots, x^{D-1}) \in M.$$

Two subsets $X, Y \subset M$ are called to be space-like separated if for any $x \in X$ and $y \in Y$ we have

$$(x^0 - y^0) - \sum_{j=1}^{D-1} (x^j - y^j)^2 < 0.$$

The forward cone is

$$C_+ = \{x \in M : \langle x, x\rangle \geq 0, x^0 \geq 0\}$$

and the **causal order** is given by $x \geq y$ iff $x - y \in C_+$.

Let $P = P(1, D-1)$ be the *Poincaré* group and $L = SO_0(1, D-1) \subset GL(D, \mathbb{R})$ be the identity component of the orthogonal group $O(1, D-1)$ preserving the metric. It is know that $P = \mathbb{R}^D \rtimes L$, where $\mathbb{R}^D$ represents the translation group.

The *Poincaré* group acts on $\mathcal{S}$ by

$$g \cdot f(x) = f(g^{-1}x), \qquad g \in P, \quad f \in \mathcal{S}.$$

If $g = (q, \Lambda) \in \mathbb{R}^D \rtimes L$, then

$$g \cdot f(x) = (q, \Lambda) f(x) = f(\Lambda^{-1}(x - q)).$$

Let $\widetilde{P} = \mathbb{R}^D \rtimes Spin(1, D-1)$ for $D > 2$ where $Spin(1, D-1) = \widetilde{L}$ is the spin group, the universal covering group of the Lorentz group $L = SO(1, D-1)$.

Remark 1.2. *The Lorentz group is six-dimensional. The following gives a basis of the Lie algebra:*

$$J_{01} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, J_{02} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, J_{03} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$J_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, J_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, J_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The subgroup generated by J_{01} is

$$\begin{pmatrix} \cosh\theta & \sinh\theta & 0 & 0 \\ \sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The subgroup generated by J_{12} is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We can identify $\mathbb{R}^4$ with the space of 2 by 2 complex self-adjoint matrices by

$$(p_0, p_1, p_2, p_3) \leftrightarrow \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix}$$

and observe that

$$\det \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix} = p_0^2 - p_1^2 - p_2^2 - p_3^2.$$

The for any $\Lambda \in SL_2(\mathbb{C})$, it is obviously that the following map preserves the determinant and self-adjointness:

$$\begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix} \to \Lambda \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix} \Lambda^*$$

Note that both Λ and $-\Lambda$ give the same linear transformation. This implies that $SL_2(\mathbb{C})$ is a double covering of $SO(1,3)$. Thus $spin(1,3) \simeq SL_2(\mathbb{C})$.

Now assume that we have a unitary representation of $\widetilde{P}$ which will be denoted by

$$U : \widetilde{P} \to U(\mathcal{H}), \qquad (q, \Lambda) \to U(q, \Lambda), \qquad (q, \Lambda) \in \mathbb{R}^D \rtimes \widetilde{L}.$$

By Stone's Theorem, there exist D self-adjoint closed operator $P_0, P_1, \dots, P_{D-1}$ such that

$$U(q, 1) = e^{i(q^0 P_0 - q^1 P_1 - \cdots - q^{D-1} P_{D-1})},$$

where $q_i \in \mathbb{R}$. P_0 is interpreted as the energy operators and P_j, $j > 0$ as the components of th momentum.

Wightmen Axioms. A Wightman quantum field theory in dimension D consists of the following data:

- the space of states, which is the projective space $P(\mathcal{H})$ of a separable complex Hilbert space $\mathcal{H}$;
- the vacuum vector $\Omega \in \mathcal{H}$ of norm 1;
- a unitary representation $U : \widetilde{P} \to U(\mathcal{H})$ of $\widetilde{P}$;
- a collection of field operator Φ_a, $a \in I$,

  $$\Phi_a : \mathcal{S} \to \mathfrak{D}(\mathcal{H}),$$

  with a dense subspace $D \subset \mathcal{H}$ as their common domain (that is the domain $\mathfrak{D}(\Phi_a(f))$ contains $\mathfrak{D}$ for all $a \in I$ and $f \in S$) such that $\Omega \in \mathfrak{D}$.

This data satisfy the following three axioms:

Axiom 1(Covariance)

(1) Ω is $\widetilde{P}$-invariant, i.e.,

$$U(q,\Lambda)\Omega = \Omega, \qquad \forall (q,\Lambda) \in \widetilde{P},$$

and $\mathfrak{D}$ is $\widetilde{P}$-invariant, i.e.,

$$U(q,\Lambda)\mathfrak{D} \subset \mathfrak{D}, \qquad \forall (q,\Lambda) \in \widetilde{P};$$

(2)

$$\Phi_a(f)\mathfrak{D} \subset \mathfrak{D}, \qquad \forall f \in \mathcal{S}, \text{and } a \in I;$$

(3)

$$U(q,\Lambda)\Phi_a(f)U(q,\Lambda)^* = \Phi_a((q,\Lambda)f), \qquad \forall f \in \mathcal{S} \text{ and } (q,\Lambda) \in \widetilde{P}.$$

Axiom 2(Locality) $\Phi_a(f)$ and $\Phi_b(g)$ commute on $\mathfrak{D}$ if the supports of $f, g \in \mathcal{S}$ are space-like separated, that is on $\mathfrak{D}$

$$[\Phi_a(f), \Phi_b(f)] = 0.$$

Axiom 3(Spectrum Condition) The joint spectrum of the operators P_j is contained in the forward cone C_+, i.e., the eigenvalues rector $(p_0, p_1, \dots, p_{D-1}) \in C_+$.

Remark 1.3. *It is customary to require that the vacuum is cyclic in the sense that the subspace $\mathfrak{D}_0 \subset \mathfrak{D}$ spanned by all vectors*

$$\Phi_{a_1}(f_1)\Phi_{a_2}(f_2)\cdots\Phi_{a_m}(f_m)\Omega$$

is dense in $\mathfrak{D}$.

As an additional axiom, one can require the vacuum Ω to be unique:

Axiom 4(Uniqueness of the Vacuum) The only vectors in $\mathcal{H}$ left invariant by the translations $U(q,1)$, $q \in \mathbb{R}^D$, are the scalar multiples of the vacuum Ω.

Example 1.5 (Free Bosonic QFT). *Let*

$$\Gamma_m = \{(p_0,p_1,p_2,p_3) \in \mathbb{R}^{(1,3)} : \langle p, p\rangle = p_0^2 - p_1^2 - p_2^2 - p_3^2 = m^2, p_0 > 0\}.$$

Then

$$\rho : \mathbb{R}^3 \to \Gamma_m, (p_1,p_2,p_3) \to (\sqrt{p_1^2+p_2^2+p_3^2+m^2}, p_1, p_2, p_3),$$

is an isomorphism. In the sequel, denote $\sqrt{p_1^2+p_2^2+p_3^2+m^2}$ with $\omega(p)$ for any $p = (p_1,p_2,p_3) \in \mathbb{R}^3$.

Let λ_m be the invariant measure on Γ_m given by

$$\int_{\Gamma_m} f(\rho(\mathbf{p}))d\lambda_m(\rho(\mathbf{p})) = \int_{\mathbb{R}^3} \frac{h(\mathbf{p})}{2\sqrt{\mathbf{p}^2+m^2}}d\mathbf{p},$$

where $\mathbf{p}^2 = p_1^2 + p_2^2 + p_3^2$.

Let

$$\Lambda = \begin{pmatrix} \cosh(x) & \sinh(x) & 0 & 0 \\ \sinh(x) & \cosh(x) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The isometry

$$\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} \to \begin{pmatrix} \cosh(x) & \sinh(x) & 0 & 0 \\ \sinh(x) & \cosh(x) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix},$$

induced the map form $\mathbb{R}^3$ *onto* $\mathbb{R}^3$

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \to \begin{pmatrix} \sinh(x)\sqrt{m^2+\mathbf{p}^2}+\cosh(x)p_1 \\ p_2 \\ p_3 \end{pmatrix}.$$

The Jacobi of this transformation is

$$\frac{\sinh(x)p_1+\cosh(x)p_0}{\sqrt{m^2+\mathbf{p}^2}}.$$

Therefore it is easy to see that the measure $d\lambda_m$ *is invariant.*

Let $\mathcal{H}_1 = L^2(\Gamma_m, d\lambda_m)$ *and* H_n *be the space of rapidly decreasing functions on the n-fold product of the upper hyperboloid* Γ_m *which are symmetric in the variables* $(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n) \in \Gamma_m^n$ *(*$H = \mathcal{S}(\Gamma_m)$*).* H_n *has the inner product*

$$\langle u, v\rangle = \int_{\Gamma_m^n} u(\mathbf{p}_1, \ldots, \mathbf{p}_n)\bar{v}(\mathbf{p}_1, \ldots, \mathbf{p}_n)d\lambda_m(\mathbf{p}_1)\ldots d\lambda_m(\mathbf{p}_n).$$

The Hilbert space completion of H_n *well be denoted by* $\mathcal{H}_n$*. Let*

$$\mathfrak{D} = \bigoplus_{n=0}^{\infty} H_n$$

($H_0 = \mathbb{C}$ *with the vacuum* $\Omega = 1 \in H_0$*) has a natural inner product given by*

$$\langle f, g\rangle = f_0\bar{g}_0 + \sum_{n\geq 1}\frac{1}{n!}\langle f_n, g_n\rangle,$$

where $f = (f_0, f_1, \ldots)$, $(g_0, g_1, \ldots) \in \mathfrak{D}$*. The completion of* $\mathfrak{D}$ *w.r.t this inner product is the Fock space* $\mathcal{H}$*.*

Let $f \in \mathcal{S}(\mathbb{R}^4)$ *and* $g = (g_0, g_1, \ldots) \in \mathfrak{D}$*. Define* $\Phi(f)$ *by*

$$\begin{aligned}(\Phi(f)g)_n(\mathbf{p_1}, \ldots, \mathbf{p_n}) = &\int_{\Gamma_m} \hat{f}(\mathbf{p})g_{N+1}(\mathbf{p}, \mathbf{p}_1, \ldots, \mathbf{p}_n)d\lambda_n(\mathbf{p}) \\ &+ \sum_{j=1}^{n}\hat{f}(-\mathbf{p}_j)g_{n-1}(\mathbf{p}_1, \ldots \hat{\mathbf{p}}_j, \ldots, \mathbf{p}_n).\end{aligned}$$

Note that for any $f \in \mathcal{S}(\mathbb{R}^4)$*, we have* $\Phi(\Box f - m^2 f) = 0$ *since*

$$\mathfrak{F}(\Box f - m^2 f) = (-\mathbf{p}^2 + m^2)\hat{f}$$

vanishes on Γ_m*.*

References

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