

Note on Sarnak Conjecture

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ABSTRACT. Enter abstract here

1. INTRODUCTION

The Möbius function $\mu(n)$, $n = 1, 2, 3, \dots$ is defined by

$$\mu(n) = \begin{cases} 1, & \text{if } n=1 \\ 0, & \text{if } n \text{ is not square free} \\ (-1)^t, & \text{if } n \text{ is a product of } t \text{ distinct primes} \end{cases}$$

Peter Sarnak give the disjointness conjecture concerning the Möbius function $\mu(n)$.

Conjecture 1.1 (P. Sarnak). *Let (X, T) be a deterministic (i.e. the topological entropy $h(T) = 0$) topological dynamical system.*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_n \mu(n) f(T^n x) = 0$$

where $x \in X$ and $f \in C(X)$.

2. REVIEW ON ENTROPY

“ You should call it entropy, for two reasons. In the first place, your uncertainty function has been used in statistical mechanics under that name, so it already has a name. In the second place, and more important, no one knows what entropy really is, so in a debate you will always have the advantage. ”

von Neumann said to Shannon, *Mathematical Theory of Entropy*, 1981

Throughout this section (X, \mathcal{A}, μ) denotes a measure space.

2.1. Some ergodic results. Let T be a \mathcal{A} -measurable transformation from X to X . μ is T -invariant iff $\mu(T^{-1}B) = \mu(B)$ for every $B \in \mathcal{A}$.

Definition 2.1. Let $T : X \rightarrow X$ be a transformation. The periodic entropy of T is

$$p(T) = \lim_{m \rightarrow \infty} \frac{1}{m} \log(\#\{x \in X : T^m(x) = x\}).$$

Remark 2.1. The periodic entropy measures the complexity of T from the point of view of the periodic points. For example, let $T : z \rightarrow z^m$ from S^1 to S^1 , $p(T) = \log|m| > 0$.

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Theorem 2.1 (Poincaré's recurrence theorem). *Let $T : X \rightarrow X$ be a measurable transformation and let μ be a finite T -invariant measure in X . If $A \subset X$ is measurable, then the set*

$$B = \{x \in A : T^n(x) \in A, \text{ for infinitely many integers } n \in \mathbb{N}\}$$

has measure $\mu(B) = \mu(A)$.

Definition 2.2. *Given a transformation $T : X \rightarrow X$, we say that a function φ is T -invariant if $\varphi(T(x)) = \varphi(x)$ for every $x \in X$. φ is T -invariant almost everywhere if there is a T -invariant subset $B \subset X$, i.e. $T^{-1}B = B$, with $\mu(X \setminus B) = 0$ and $\varphi|_B$ is T -invariant.*

Theorem 2.2 (Birkhoff's ergodic theorem). *Let $T : X \rightarrow X$ be a measurable transformation and let μ be a finite T -invariant measure in X . If $\varphi \in L^1(X, d\mu)$, then the limit*

$$\varphi_T = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(T^k(x))$$

- φ_T is T -invariant almost everywhere.
- $\varphi_T \in L^1(X, d\mu)$ and $\int_X \varphi_T d\mu = \int_X \varphi d\mu$.

2.2. Metric(measure-theoretical) Entropy. In this subsection we assume that $\mu(X) = 1$. Let

$$\psi(x) = \begin{cases} x \log x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Definition 2.3. *Let $\xi \subset \mathcal{A}$. If*

- (1) $\mu(\cup_{C \in \xi} C) = 1$,
- (2) $\mu(C \cap D) = 0$ for any distinct C, D in ξ ,

then ξ is a measurable partition of (X, \mathcal{A}, μ) .

Definition 2.4 (Entropy of a measurable partition). *Let ξ be a measurable partition of X w.r.t μ . The entropy of ξ is given by*

$$H_\mu(\xi) = - \sum_{C \in \xi} \mu(C) \log \mu(C) = - \sum_{C \in \xi} \psi(\mu(C)).$$

Definition 2.5. *Given two measurable partition ξ, η of X , we define a new measurable partition*

$$\xi \vee \eta = \{C \cap D : C \in \xi, D \in \eta\}.$$

Definition 2.6. *Let $T : X \rightarrow X$ be a measurable transformation preserving the probability measure μ and ξ a measurable partition of X . The metric entropy of T with respect to μ and ξ is*

$$h_\mu(T, \xi) = \lim_{n \rightarrow \infty} \frac{1}{n} H_\mu(\xi_n) = \inf_{n \in \mathbb{N}} H_\mu(\xi_n),$$

where $\xi_n = \bigvee_{k=0}^{n-1} T^{-k} \xi$. The metric entropy of T with respect to μ is

$$h_\mu(T) = \sup\{h_\mu(T, \xi) : \xi \text{ is a measurable partition of } X\}.$$

Lemma 2.1. *Let ξ and η be measurable partitions of X . If η is a refinement of ξ , i.e., each element of ξ is a union of elements of η , then $H_\mu(\xi) \leq H_\mu(\eta)$.*

Theorem 2.3. Let $T : X \rightarrow X$ be a measurable transformation preserving the probability measure μ . If $(\xi_n)_{n \in \mathbb{N}}$ is a sequence of measurable partitions of X with $\bigvee_{n=1}^{\infty} \mathcal{A}(\xi_n)$ ($\mathcal{A}(\xi)$ is the σ -algebra generated by ξ) such that ξ_{n+1} is a refinement of ξ_n for each $n \in \mathbb{N}$, then

$$h_\mu(T) = \lim_{n \rightarrow \infty} h_\mu(T, \xi_n) = \sup_{n \in \mathbb{N}} h_\mu(T, \xi_n).$$

Specially, if $\bigvee_{k=0}^{\infty} \mathcal{A}(T^{-k}\xi) = \mathcal{A}$ or $\bigvee_{k=-\infty}^{\infty} \mathcal{A}(T^{-k}\xi) = \mathcal{A}$, then we have the following.

Corollary 2.1. Let $T : X \rightarrow X$ be a measurable transformation preserving the probability measure μ . If $\bigvee_{k=0}^{\infty} \mathcal{A}(T^{-k}\xi) = \mathcal{A}$ or $\bigvee_{k=-\infty}^{\infty} \mathcal{A}(T^{-k}\xi) = \mathcal{A}$, then $h_\mu(T) = h_\mu(T, \xi)$.

Note that if ξ is a measurable partition of X w.r.t. μ , then for almost every $x \in X$, there exists a single element $\xi_n(x)$ (depending on x such that $\xi_n(x) \in \xi_n = \bigvee_{k=0}^{n-1} T^{-k}\xi$).

Theorem 2.4 (Shannon-McMillan-Breiman). If $T : X \rightarrow X$ is a measurable transformation preserving a probability measure μ in X and ξ is a measurable partition X , then the limit

$$h_\mu(T, \xi, x) := \lim_{n \rightarrow \infty} -\frac{1}{n} \log \mu(\xi_n(x))$$

exists for almost every $x \in X$. Moreover, the function $x \rightarrow h_\mu(T, \xi, x)$ is T -invariant almost everywhere, is integrable and

$$h_\mu(T, \xi) = \int_X h_\mu(T, \xi, x) d\mu(x).$$

2.3. Topological Entropy.

Definition 2.7 (Cover entropy of continuous map). For a cover \mathcal{U} of a compact topological space X , define $N(\mathcal{U})$ to be the smallest cardinality of a subcover of \mathcal{U} , and define the entropy of \mathcal{U} to be $H(\mathcal{U}) = \log N(\mathcal{U})$. Let $T : X \rightarrow X$ be a continuous map. The cover entropy of T with respect to \mathcal{U} is defined to be

$$h_{\text{cover}}(T, \mathcal{U}) = \lim_{n \rightarrow \infty} \frac{1}{n} \log N\left(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}\right) = \inf_{n \geq 1} \frac{1}{n} \log N\left(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}\right),$$

and the cover entropy of T is defined to be

$$h_{\text{cover}}(T) = \sup_{\mathcal{U}} h_{\text{cover}}(T, \mathcal{U})$$

where the supremum is taken over all open covers of X .

Throughout the rest of this subsection, let (X, d) be a **compact metric space** and T a continuous transformation from X to X .

Definition 2.8. Let $T : (X, d) \rightarrow (X, d)$ be a continuous map on a compact metric space. A one side generator for T is a finite open cover $\mathcal{U} = \{U_i\}_{i \in I}$ with the property that for any sequence $(i_k)_{k \geq 0}$, the set

$$\bigcap_{k \geq 0} T^{-k}(\overline{U_{i_k}})$$

contains at most a single point.

Theorem 2.5. Let \mathcal{U} be a one-sided generator for a continuous map $T : (X, d) \rightarrow (X, d)$ on a compact metric space. Then $h_{\text{cover}}(T, \mathcal{U}) = h_{\text{cover}}(T)$.

Theorem 2.6. Let $T : X \rightarrow X$ be a continuous map on a compact metric space (X, d) . If (\mathcal{U}_n) is a sequence of open covers of X with $\text{diam}(\mathcal{U}_n) \rightarrow 0$ as $n \rightarrow \infty$, then

$$\lim_{n \rightarrow \infty} h_{\text{cover}}(T, \mathcal{U}_n) = h_{\text{cover}}(T).$$

For each $n \in \mathbb{N}$, we introduce a new distance in X by

$$d_n(x, y) = \max\{d(T^k(x), T^k(y)) : 0 \leq k \leq n-1\}.$$

And $N(d, \varepsilon)$ denotes the maximum number of points in X at a d -distance at least ε .

Definition 2.9 (Separated Set Topological Entropy). The separated set topological entropy of T is

$$h(T)_{\text{sep}} = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \log N(d_n, \varepsilon).$$

Remark 2.2. $h_{\text{sep}}(T)$ only depends on the topology induced by the distance d and $h_{\text{sep}}(T) = h_{\text{cover}}(T)$. The common value is called the **topological entropy of T** , denoted $h_{\text{top}}(T)$.

Some facts about $h_{\text{top}}(T)$:

- If $T : X \rightarrow X$ is a **homeomorphism** of a compact metric space, then $h_{\text{top}}(T^{-1}) = h_{\text{top}}(T)$.
- If $T : X \rightarrow X$ is a continuous map of a compact metric space, then $h_{\text{top}}(T^k) = kh_{\text{top}}(T)$ for $k \geq 1$.

Theorem 2.7 (Variational principle for the topological entropy). If $T : X \rightarrow X$ is a continuous transformation of a compact metric space, then

$$h_{\text{top}}(T) = \sup\{h_\mu(T) : \mu \text{ is a } T\text{-invariant probability measure in } X\}.$$

2.4. For AF algebra. Let σ be an automorphism of \mathfrak{A} , \mathfrak{A} is an AF-algebra. $\Sigma = \{A_i\}_{i=1}^n \subset \mathfrak{A}$, and \mathcal{M} be a finite dimensional subalgebra of \mathfrak{A} and $\Sigma \subset_\delta \mathcal{M}$. Let $r(\Sigma, \delta) = \min_{\Sigma \subset_\delta \mathcal{M}} \{\max \dim \text{ of masa of } \mathcal{M}\}$.

$$\Sigma_n = \cup_{i=0}^n \sigma^i(\Sigma)$$

$$h(\sigma) = \lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \frac{\log(\Sigma_n, \delta)}{n}$$

3. FINITE MATRIX CASE

Let $\mathfrak{A} = M_n(\mathbb{C})$ and ρ be a pure state of \mathfrak{A} . We may ask if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_n \mu(n) \rho(\sigma^n(T)) = 0$$

where σ is an automorphism of \mathfrak{A} and $T \in \mathfrak{A}$.

Since automorphism preserve the norm of \mathfrak{A} , it is not hard to see that, as a continuous map from the unit ball of \mathfrak{A} onto itself, the topological entropy of σ is 0, by considering the definition of Bowen and Dianburg.

Without lose of generality, we may assume that $\sigma = Ad(U)$ where U is a unitary and $\rho(T) = \langle T\xi, \xi \rangle$. The question can be restated as if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_n \mu(n) \langle TU^n \xi, U^n \xi \rangle = 0$$

Assume $\xi = \xi_1 + \xi_2 + \cdots + \xi_n$ such that

$$U\xi_k = e^{i\theta_k} \xi_k \quad k = 1, 2, \dots, n.$$

Then we have

$$\langle TU^m \xi, U^m \xi \rangle = \sum_{k,l=1}^n e^{im(\theta_k - \theta_l)} \langle T \xi_k, \xi_l \rangle.$$

Let $C_{kl} = \langle T \xi_k, \xi_l \rangle$. Davenport and Hua showed in [2, 5] that for any fixed $h > 0$, $m \in \mathbb{N}^+$ and $0 \leq l < m$,

$$(1) \quad \frac{1}{N} \sum_{\substack{n \leq N \\ n \equiv l \pmod{m}}} \mu(n) e^{2\pi i(a_d n^d + a_{d-1} n^{d-1} + \dots + a_1 n + a_0)} = O((\log N)^{-h}),$$

such that the implied constant depends only on h and p , but is independent of any of coefficients a_d, \dots, a_0 . In particular,

$$\frac{1}{N} \sum_{n \leq N} \mu(n) e^{2\pi i n \theta} = O((\log N)^{-h})$$

uniformly in θ . Therefore we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_n \mu(n) e^{im(\theta_k - \theta_l)} C_{kl} = \lim_{n \rightarrow \infty} \frac{C_{kl}}{n} \sum_n \mu(n) e^{im(\theta_k - \theta_l)} = 0$$

Theorem 3.1. *Suppose that \mathfrak{A} be a finite dimensional C^* -algebra. Let ρ be a continuous functional of \mathfrak{A}^* and σ be an automorphism of \mathfrak{A} . Then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_n \mu(n) \rho(\sigma^n(T)) = 0.$$

Proof. We could assume that \mathfrak{A} is a direct sum of finite matrix algebras and ρ is a vector state (otherwise we can always apply GNS construction to make ρ a vector state). Then the discussion above give the result. \square

4. VON NEUMANN ALGEBRA

Let \mathcal{M} be a von Neumann algebra acting on the Hilbert space \mathcal{H} and σ be a $*$ -automorphism of \mathcal{M} . For any $T \in \mathcal{M}$ and any $\rho \in \mathcal{M}_*^+$, we will study the following limit.

$$(2) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_n \mu(n) \rho(\sigma^n(T))$$

Without lose of generality, we could assume that \mathcal{M} is in standard form $\|T\| = 1$ and $\|\rho\| = 1$. By Lemma 2.10 and Theorem 3.2 in [4], we may also assume that $\rho(T) = \langle T \xi, \xi \rangle$ and $\sigma(T) = U^* T U$, where ξ is a unit vector in \mathcal{H} and U is a unitary in $\mathcal{B}(\mathcal{H})$. For any fixed $N \in \mathbb{N}$ and any $\varepsilon > 0$, we can find a unitary $V = \sum_{k=1}^n e^{i\theta_k} P_k$ such that $\|V^m - U^m\| \leq \varepsilon$, where $P_k, k = 1 \dots n$, are orthogonal projections and $m \leq N$. Then we

have

$$\begin{aligned}
\Delta(N) &= \left| \frac{1}{N} \sum_{m=1}^N \mu(m) \langle TU^m \xi, U^m \xi \rangle \right| \\
&\leq \left| \frac{1}{N} \sum_{m=1}^N \mu(m) \langle T(U^m - V^m) \xi, U^m \xi \rangle \right| + \left| \frac{1}{N} \sum_{m=1}^N \mu(m) \langle TV^m \xi, (U^m - V^m) \xi \rangle \right| \\
&\quad + \left| \frac{1}{N} \sum_{m=1}^N \mu(m) \langle TV^m \xi, V^m \xi \rangle \right| \\
&\leq \frac{2}{N} \sum_{m=1}^N \| (U^m - V^m) \xi \| + \left| \sum_{l,k=1}^n \left(\frac{1}{N} \sum_m \mu(m) e^{im(\theta_l - \theta_k)} \right) \langle TP_l \xi, P_k \xi \rangle \right| \\
&\leq 2\varepsilon + C \log(N)^{-h} \sum_{l,k=1}^n \|P_l \xi\|_2 \|P_k \xi\|_2
\end{aligned}$$

If $\sum_{i=1}^n a_i^2 = 1$, then we have that

$$\left(\sum_{i=1}^n a_i \right)^2 = \sum_{i=0}^{n-1} \left(\sum_{j=1}^n a_j a_{(j+i) \% n} \right) \leq \sum_{i=0}^{n-1} \left(\sum_{i=1}^n a_i^2 \right) = n$$

This seems lead to a dead end. Actually, the estimation above is quiet trivial.

However, if \mathfrak{A} is a finite von Neumann algebra and ω a trace vector in $L^2(\mathfrak{A}, \tau)$, then there exists a $A \in \mathfrak{A}$ such that $\|A\omega - \xi\| \leq \varepsilon$. Furthermore, we assume U is in \mathfrak{A} , therefore V can also be constructed in \mathfrak{A} .

$$\begin{aligned}
\Delta(N) &= \left| \frac{1}{N} \sum_{m=1}^N \mu(m) \langle TU^m \xi, U^m \xi \rangle \right| \\
&\leq \left| \frac{1}{N} \sum_{m=1}^N \mu(m) \langle TU^m (\xi - A\omega), U^m \xi \rangle \right| + \left| \frac{1}{N} \sum_{m=1}^N \mu(m) \langle TU^m A\omega, U^m (\xi - A\omega) \rangle \right| \\
&\quad + \left| \frac{1}{N} \sum_{m=1}^N \mu(m) \langle T(U^m - V^m) A\omega, U^m A\omega \rangle \right| + \left| \frac{1}{N} \sum_{m=1}^N \mu(m) \langle TV^m A\omega, (U^m - V^m) A\omega \rangle \right| \\
&\quad + \left| \frac{1}{N} \sum_{m=1}^N \mu(m) \langle TV^m A\omega, V^m A\omega \rangle \right| \\
&\leq 4\varepsilon(1 + \varepsilon)^2 + \left| \sum_{l,k=1}^n \left(\frac{1}{N} \sum_m \mu(m) e^{im(\theta_l - \theta_k)} \right) \langle TP_l A\omega, P_k A\omega \rangle \right| \\
&= 4\varepsilon(1 + \varepsilon)^2 + \left| \sum_{l,k=1}^n \left(\frac{1}{N} \sum_m \mu(m) e^{im(\theta_l - \theta_k)} \right) \tau(P_k TP_l AA^*) \right| \\
&\leq 4\varepsilon(1 + \varepsilon)^2 + \left| \frac{1}{N} \sum_m \mu(m) e^{im(\theta_l - \theta_k)} \right| \left(\sum_{l,k=1}^n \|P_k TP_l\|_2 \|P_l AA^* P_k\|_2 \right) \\
&\leq 4\varepsilon(1 + \varepsilon)^2 + \left| \frac{1}{N} \sum_m \mu(m) e^{im(\theta_l - \theta_k)} \right| \left(\sum_{l,k=1}^n \|P_k TP_l\|_2^2 \right)^{\frac{1}{2}} \left(\sum_{l,k=1}^n \|P_l AA^* P_k\|_2^2 \right)^{\frac{1}{2}} \\
&\leq 4\varepsilon(1 + \varepsilon)^2 + \left| \frac{1}{N} \sum_m \mu(m) e^{im(\theta_l - \theta_k)} \right| \|T\|_2 \|AA^*\|_2
\end{aligned}$$

Fix A , we have that $\lim_{n \rightarrow \infty} \sup \Delta(n) \leq 4\varepsilon(1 + \varepsilon)^2$ for any ε . Then we have that $\lim_{n \rightarrow \infty} \Delta(n) = 0$.

5. COMPACT SET WITH FINITE ACCUMULATION POINTS

5.1. Topological Entropy.

Lemma 5.1. *Let X be a compact space and $\sigma : X \rightarrow X$ a continuous map. If there exists $n \in \mathbb{N}$ such that $\sigma^{m_1}(x) = \sigma^{m_2}(x)$ for each $x \in X$ where $0 \leq m_1 < m_2 \leq n$, then $h_{top}(\sigma) = 0$.*

Proof. By the hypothesis, $\sigma^n(x)$ is a periodic point with period less than n for each $x \in X$. Let $k = n!$, we have $\sigma^{n+k} = \sigma^n(x)$. Since $h_{top}(\sigma) = 0$ if and only if $h_{top}(\sigma^n) = 0$. We may assume that $\sigma^k(x) = \sigma(x)$ for any $x \in X$ where $k > 1$. Let U be an open set of X , $1 \leq m \leq k-1$ and $l \geq 0$,

$$\sigma^{-(l(k-1)+m)}U = \{y : \sigma^{l(k-1)+m}(y) \in U\} = \{y : \sigma^m(y) \in U\} = \sigma^{-m}U.$$

Let \mathcal{U} be an open cover of X , we have $\bigvee_{i=0}^m \sigma^{-i}\mathcal{U} = \bigvee_{i=0}^k \sigma^{-i}\mathcal{U}$ if $m > k$. Therefore

$$h_{cover}(\sigma, \mathcal{U}) = \lim_{m \rightarrow \infty} \frac{1}{m} \log N\left(\bigvee_{i=0}^{m-1} \sigma^{-i}\mathcal{U}\right) = 0.$$

Thus $h_{top}(\sigma) = 0$. □

Let X be a compact space with only one accumulation point c . If $\sigma : X \rightarrow X$ is a continuous map such that $\sigma(c) = y (\neq c)$, then there exist a neighborhood U of c such that $\sigma(U) = \{y\}$. It is not hard to see that there exist n_0 such that $\sigma^{n_0}(y)$ is a periodic point of σ . Indeed, note that $X \setminus U$ contains only finite many points (U is closed since it contains all accumulation points), then either there exists a $k \in \mathbb{N}$ such that $\sigma^k(y) \in U$ or there exists n_0 such that $\sigma^{n_0}(y)$ is a periodic point. The same argument also tells us that $\sigma^m(x)$ is periodic for some $m \in \mathbb{N}$ for each $x \in X \setminus U$. Since there are only finite many points in $X \setminus U$, there exists $n \in \mathbb{N}$ such that $\sigma^{m_1}(x) = \sigma^{m_2}(x)$ for each $x \in X$ where $0 \leq m_1 < m_2 \leq n$. Hence $h_{top}(\sigma) = 0$ by lemma 5.1.

Lemma 5.2. *Let X be a compact space with only one accumulation point c . If $\sigma : X \rightarrow X$ is a continuous map such that $\sigma(c) = y (\neq c)$, then $h_{top}(\sigma) = 0$.*

If a compact Hausdorff space with only finite many accumulation points is metrizable, then it only contains countable many points. If X is a countable, compact metric space and $\sigma : X \rightarrow X$ is continuous mapping, then by Proposition 5.1 in [8], $h_{top}(\sigma) = 0$. However, a compact Hausdorff space X may not be metrizable even if it only has one accumulation point. Indeed, let $X = \mathbb{R} \cup \{\infty\}$ be the one point compactification of \mathbb{R} with discrete topology, it is clear that X is not second-countable. Therefore, it can not be metrizable since a compact Hausdorff space X is metrizable if and only if it is second-countable.

Theorem 5.1. *Let X be a compact Hausdorff space with finite many accumulation points $\{c_1, \dots, c_l\}$. If $\sigma : X \rightarrow X$ is a continuous map, then $h_{top}(\sigma) = 0$.*

Proof. Suppose that $\sigma(c_i) = q_i, i = 1, \dots, l$. Let \mathcal{V} be a cover of X . There exists a cover $\mathcal{U} = \{U_1, \dots, U_l, \{p_1\}, \dots, \{p_m\}\}$ such that U_i is a neighborhood of c_i and

- $q_i \notin U_j$ if $q_i \neq c_j$,
- $U_i \cap U_j = \emptyset, i \neq j$,
- $\sigma(U_i) = \{q_i\}$ if $q_i \notin \{c_1, \dots, c_l\}$,
- $\sigma(U_i) \cap U_j = \emptyset$ if $q_i \neq U_j$,
- $U_i \subset V_i$ for some $V_i \in \mathcal{V}$,
- $\{p_j\} \subset V_j$ for some $V_j \in \mathcal{V}$,
- $X \setminus (\bigcup_{i=1}^l U_i) = \{p_1, \dots, p_m\}$.

Thus $h_{\text{cover}}(\sigma, \mathcal{V}) \leq h_{\text{cover}}(\sigma, \mathcal{U})$. If we can show that $h_{\text{cover}}(\sigma, \mathcal{U}) = 0$, then we have $h_{\text{top}}(\sigma) = 0$.

Let

$$\alpha(U_i) = \begin{cases} U_j, & \text{if } \sigma(c_i) = c_j, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Note that the non empty set in $\bigvee_{i=0}^{n-1} \sigma^{-i} \mathcal{U}$ can only be $\{p_j\}$,

$$U_i \cap \sigma^{-1} \alpha(U_i) \cap \dots \cap \sigma^{-n+1} \alpha^{n-1}(U_i)$$

or

$$U_i \cap \sigma^{-1} \alpha(U_i) \cap \dots \cap \sigma^{-k+1} \alpha^{k-1}(U_i) \cap \sigma^{-k}(\{p_j\}) \cap \sigma^{-k-1} V_{k+1} \dots \cap \sigma^{1-n} V_{n-1},$$

where $1 \leq k \leq n-1$, $1 \leq i \leq l$, $1 \leq j \leq m$ and $V_j (\in \mathcal{U})$ are uniquely determined by p_j . Therefore $N(\bigvee_{i=0}^{n-1} \sigma^{-i} \mathcal{U}) \leq ml(n-1) + l + m$ and

$$h_{\text{cover}}(\sigma, \mathcal{U}) \leq \lim_{n \rightarrow \infty} \frac{1}{n} \log(ml(n-1) + l + m) = 0.$$

□

5.2. Sarnak's conjecture is true for these spaces. Throughout this section X is a compact space with only finite many accumulation points and $\sigma : X \rightarrow X$ a continuous map.

It is easy to get the following two lemmas by eq. (1).

Lemma 5.3. *Let X be a compact space and σ a continuous map from X to X . If $\{\sigma^n(x)\}$ has only one cluster point $c \in X$, then*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} \mu(n) f(\sigma^n(x)) = 0, \forall f \in C(X).$$

Proof. Let $X' = \{\sigma^n(x)\}_{n=1}^\infty = \{\sigma^n(x)\} \cup \{c\}$. Since $\sigma(c)$ is a cluster point of $\sigma(\{\sigma^n(x)\}) \subset \{\sigma^n(x)\}$, we have $\sigma(c) = c$. Therefore $\sigma(X') \subset X'$ and σ is a continuous map from X' to X' . Note that X' is compact. For any $\varepsilon > 0$, there exists an open neighborhood of c such that $X' \setminus U$ contains only finite many elements and $|f(y) - f(c)| < \varepsilon$ if $y \in U$. Now it is easy to see $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} \mu(n) f(\sigma^n(x)) = 0$. □

Lemma 5.4. *Let X be a compact space, f a function on X and σ a map from X to X . If $x \in X$ and there is a n_0 and $n_1 \in \mathbb{N}$ such that $\sigma^{n_0+n_1}(x) = \sigma^{n_0}(x)$ then*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} \mu(n) f(\sigma^n(x)) = 0.$$

Example 5.1. *Let X be a compact Hausdorff space with a unique accumulation point c_0 and f a continuous function on X . By lemmas 5.3 and 5.4, $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} \mu(n) f(\sigma^n(x)) = 0$.*

Lemma 5.5. *Let X be a compact Hausdorff space and σ a continuous map from X to X . Let x be a element in X and c_1 a cluster point of $\{\sigma^n(x)\}_{n=1}^\infty$. If $\sigma(c_1) = y$ where y is an isolated point of X , then there exist n_1 and n_2 such that $\sigma^{n_1+n_2}(x) = \sigma^{n_1}(x) = y$, and $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} \mu(n) f(\sigma^n(x)) = 0$ for any $f \in C(X)$.*

Proof. By the continuity of σ there is a neighborhood U of c_1 such that $\sigma(U) = \{y\}$. Since c_1 is a cluster point of $\{\sigma^n(x)\}_{n=1}^\infty$ and X is Hausdorff, then there exist n_1 and n_2 such that $\sigma^{n_1+n_2}(x) = \sigma^{n_1}(x) = y$. By lemma 5.4, $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} \mu(n) f(\sigma^n(x)) = 0$. □

Theorem 5.2. *If X is a compact Hausdorff space with finite many limit points, then we have*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} \mu(n) f(\sigma^n(x)) = 0,$$

for any continuous map σ from X to X and any continuous function f on X .

Proof. Assume that $\{c_1, \dots, c_m\}$ are the only limit points of X . By lemma 5.4, we could assume that $\sigma^n(x) \neq \sigma^m(x)$, $n \neq m$ and c_1 is a cluster point of $\{\sigma^n(x)\}_{n=1}^\infty$. If $\sigma(c_1) = y$ and $y \notin \{c_1, \dots, c_m\}$, then by lemma 5.5, we have the result. Therefore, we assume that $\sigma(c_1) \in \{c_1, \dots, c_m\}$ and consider the following cases.

- (1) Suppose that $\sigma(c_1) = c_1$. Let U_i be the open neighborhood of c_i and such that $U_i \cap U_j = \emptyset$, $i \neq j$. Since $\sigma(c_1) = c_1$ and σ is continuous, there is neighborhood V_1 of c_1 such that $V_1 \subset U_1$ and $\sigma(V_1) \subset U_1$. Note that $U_1 \setminus V_1$ contains only finite many elements of X . Since all points in $\{\sigma^n(x)\}$ are different and c_1 is a cluster point of $\{\sigma^n(x)\}$, there exist a N such that $\sigma^m(x) \in V_1$ if $m \geq N$. Therefore, c_1 is the only cluster point of $\{\sigma^n(x)\}_{n=1}^\infty$. By lemma 5.3, $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} \mu(n) f(\sigma^n(x)) = 0$.

We restate the above result as the following lemma.

Lemma 5.6. *Let X be a compact Hausdorff space with finite many limit points and σ a continuous map from X to X . If c is a cluster point of $\{\sigma^n(x)\}_{n=1}^\infty$ and $\sigma(c) = c$, then c is the only cluster point of $\{\sigma^n(x)\}_{n=1}^\infty$.*

Now we consider the other case.

- (2) Suppose that $\sigma(c_1) = c_2$. Since c_1 is a cluster point of $\{\sigma^n(x)\}$, c_2 must also be a cluster point of $\{\sigma^n(x)\}$. Hence $\sigma(c_2) \neq c_2$. By an inductive argument as above, we could assume that $\sigma(c_1) = c_2, \dots, \sigma(c_k) = c_1$. For any $f \in C(X)$ and $\varepsilon > 0$, there exists neighborhoods U_i and V_i of c_i , $i = 1, \dots, m$ such that
- $U_i \cap U_j = \emptyset$, $i \neq j$.
 - $V_i \subset U_i$,
 - $\sigma(V_i) \subset U_{i+1}$, $i = 1, \dots, k-1$, and $\sigma(V_k) \subset U_1$.
 - $|f(x) - f(c_i)| \leq \varepsilon$ if $x \in U_i$.

Since $\cup_{i=1}^k (U_i \setminus V_i)$ contains only finite many points, there exists $k \in \mathbb{N}$ such that $\sigma^{n+1}(x) \in V_{i+1}$ whenever $n \geq k$ and $\sigma^n(x) \in V_i$ for $i = 1, \dots, k-1$ and $\sigma^{n+1}(x) \in V_1$ if $\sigma^n(x) \in V_k$. Therefore, $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} \mu(n) f(\sigma^n(x)) = 0$. \square

Corollary 5.1. *Let X be a compact Hausdorff space and f a continuous function on X . If the sequence $\{\sigma^n(x)\}$ has only finite many limit points, then*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} \mu(n) f(\sigma^n(x)) = 0.$$

Proof. Let $X' = \{\sigma^n(x)\}^-$. Then X' is compact. If $y \in X'$ is a limit point, then $\sigma(y)$ is also a cluster point of $\{\sigma^{n_i}(x)\}$. Therefore $\sigma(X') \subset X'$. Now, apply theorem 5.2, we have the result. \square

6. COMPACT SPACE WITH COUNTABLY MANY POINTS

Let X be a countable, compact Hausdorff space. Since a compact Hausdorff space is metrizable if and only if it is second-countable, X is metrizable. Hence, throughout this section X denotes a countable *compactum*, i.e. a compact metric space has countable

cardinality. Therefore, by Proposition 5.1 in [8], $h_{top}(\sigma) = 0$ for any continuous map from X to X .

Argue as in the proof of corollary 5.1, we assume in this section that $\{\sigma^n(x)\}$ is dense in X .

Since X only contains countable many points, X must contains isolated point by Baire category theorem. Recall that a topological space is totally disconnected if the connected components in the space are the one-point sets. Let Y be a connected component of X . Y is closed since connected component is closed. Thus Y is also a complete metric space with only countable many points. Hence Y contain isolated points. Therefore Y can only contain one point and X is totally disconnected and zero-dimensional. Since every compact totally disconnected metric space is homeomorphic to a subset of the Cantor set (Corollary 2-99 [1]), we could assume that X is a closed subset of the Cantor set.

For a metric space X , let $\iota(X)$ be the set of all isolated points of X and $X^d = X \setminus \iota(X)$ the set of limit points of X . The derived set of X of order α is defined by $X^{(1)} = X^d$, $X^{(\alpha+1)} = (X^{(\alpha)})^d$ and $X^{(\lambda)} = \bigcap_{\alpha < \lambda} X^{(\alpha)}$ if λ is a limit ordinal number. If $X^{(\alpha)} \neq \emptyset$ and $X^{(\alpha+1)} = \emptyset$, then we put $d(X) = \alpha$. And $d(X)$ is called the *derived degree* of X . It is well known that a compactum X is a countable set if and only if $d(X)$ exists and it is a countable ordinal number. In this case, if $d(X) = \alpha$, then $X^{(\alpha)}$ is a finite set.

Proposition 6.1 (Proposition 2.1 of [7]). *Let X and Y be countable compacta. If $d(X) = d(Y) = \alpha$ and $X^{(\alpha)}$ is homeomorphic to $Y^{(\alpha)}$, then X is homeomorphic to Y .*

Lemma 6.1. *Let X be a countably infinite, compact metric space and x a isolated point in X . If $\{\sigma^n(x)\}_{n=0}^{\infty}$ is dense in X , then $\sigma^n(x)$ is isolated for all n and $\sigma^n(x)$ are all the isolated points in X .*

Proof. Since X is a countable, compact metric space contains infinitely many points, X contains infinite isolated points. It is easy to see that $\{\sigma^n(x)\}$ must contains all isolated points. Otherwise, $\{\sigma^n(x)\}$ can not be dense in X . Assume that there is a n such that $\sigma^n(x)$ is a limit point in X . Let $m \in \mathbb{N}$ be a number such that $\sigma^m(x)$ is a limit point and $\sigma^{m+1}(x)$ is an isolated point. By lemma 5.5, there is n_1 and n_2 such that $\sigma^{n_1+n_2}(x) = \sigma^{n_1}(x)$. Therefore $\{\sigma^n(x)\}$ contains only finite many different elements. It contradicts the fact that $\{\sigma^n(x)\}$ contains infinite many isolated point. \square

Lemma 6.2. *Let X be a countably infinite, compact metric space and x a isolated point in X . If $\{\sigma^n(x)\}_{n=0}^{\infty}$ is dense in X and X^d contains infinite many points, then y is not periodic for any isolated point in y in X^d .*

Proof. Suppose that there exist $n \geq 1$ such that $\sigma^n(y) = y$. Let $\{U_i\}_{i=0}^{n-1}$ be neighborhoods of $\sigma^i(y)$ such that $(\bigcup_{i=0}^{n-1} U_i \cap X^d)^- \neq X^d$, $U_0 \cap X^d = \{y\}$, $\sigma(U_i) \subset U_{i+1}$ for $i = 1, \dots, n-2$ and $\sigma(U_{n-1}) \subset U_0$. We can also find a neighborhood V_0 of y such that $V_0 \subset U_0$ and $\sigma(V_0) \subset U_1$. Since U_0 do not contains limit point except y , we have $U_0 \setminus V_0$ contains finite many points. Therefore, there exist a $m \in \mathbb{N}$ such that $\sigma^k \notin U_0 \setminus V_0$ for any $k \geq m$, it is easy to see that all limit points of $\{\sigma^n(x)\}$ must contains in $(\bigcup_{i=0}^{n-1} U_i \cap X^d)^- \subsetneq X^d$. It is a contradiction. \square

Example 6.1. *Let*

$$X = \bigcup_{n=1}^{\infty} \left(\left\{ \left(\frac{1}{n}, \frac{1}{n+k} \right) : k = 0, 1, 2, \dots \right\} \cup \left\{ \left(\frac{1}{n}, 0 \right) \right\} \right) \cup \{(0, 0)\} \subset \mathbb{R}^2.$$

Then $X^{(1)} = \{(\frac{1}{n}, 0) : n = 1, 2, \dots\} \cup \{(0, 0)\}$ and $X^{(2)} = \{(0, 0)\}$. Let σ be a map $X \rightarrow X$ defined by

$$\begin{aligned} (1, 0) &\rightarrow (0, 0) \text{ and } (0, 0) \rightarrow (0, 0) \\ (\frac{1}{n}, 0) &\rightarrow (\frac{1}{n-1}, 0) \quad (\text{for } n > 1) \\ (1, \frac{1}{k+1}) &\rightarrow (\frac{1}{k+2}, \frac{1}{k+2}) \\ (\frac{1}{n}, \frac{1}{n+k}) &\rightarrow (\frac{1}{n-1}, \frac{1}{n+k}) \quad (\text{for } n > 1). \end{aligned}$$

It is easy to see that $\{\sigma^n((1, 1))\} = X \setminus X^{(1)}$.

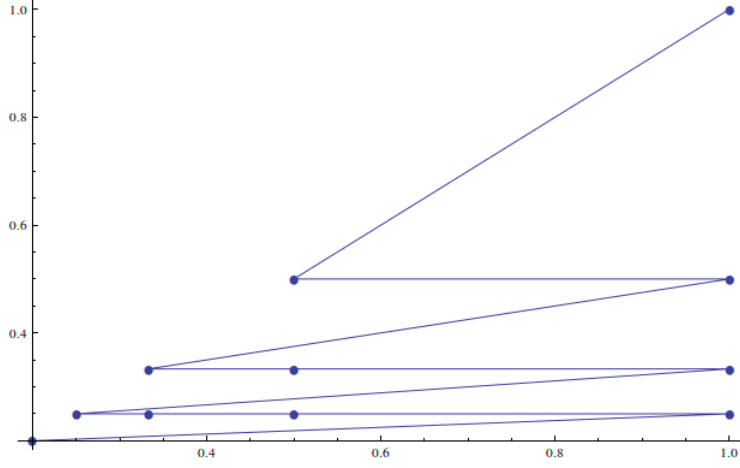


Figure 1. example 6.1

Lemma 6.3. Let X be a countably infinite, compact metric space such that $d(X) = 2$ and $X^{(2)} = \{c\}$. If $\{\sigma^n(x)\}_{n=0}^\infty$ is dense in X for a isolated point x in X , then $\sigma(c) = c$.

Proof. Suppose that $\sigma(c) = y \in X^{(1)}$. If $y \neq c$, then y is an isolated point in $X^{(1)}$. There exist a neighborhood U of c such that $\sigma(U) = \{y\}$ and $X^{(1)} \setminus U$ contains only finite many points. By lemma 6.2, there exist a n such that $\sigma^n(y) \in U$, since no isolated point in $X^{(1)}$ is periodic. However we have $\sigma^{n+1}(y) = y$. It is a contradiction. \square

Let (X, σ) be a dynamic system, A a subset of X and x a point in X . Let $A(x) = \{m : \sigma^m(x) \in A, \}$. Recall that the upper asymptotic density $\bar{d}(A(x))$ is $\limsup_{n \rightarrow \infty} \frac{|A(x, n)|}{n}$ and lower asymptotic density $\underline{d}(A(x))$ is $\liminf_{n \rightarrow \infty} \frac{|A(x, n)|}{n}$, where $A(x, n) = \{1, 2, \dots, n\} \cap A(x)$. $A(x)$ has asymptotic density $\text{den}(A(x))$ if $\bar{d}(A(x)) = \underline{d}(A(x))$, in which case $\text{den}(A(x))$ is equal to this common value.

Lemma 6.4. Let X be a countably infinite, compact metric space such that $d(X) \geq 2$. Suppose that $\{\sigma^n(x)\}_{n=0}^\infty$ is dense in X for a isolated point x in X . Let $A = \{\sigma^{n_k}(x)\}$ be a subsequence converge to y . If y is not a periodic point, then $\text{den}(A(x)) = 0$.

Proof. We consider the following two cases.

- (1) $\sigma^{m_1}(y) \neq \sigma^{m_2}(y)$ if $m_1 \neq m_2$. Hence $\lim_{m \rightarrow \infty} \sigma^m(y) = c$. Fix a positive integer k . We choose $k+1$ neighborhoods $\{U_i\}_{i=0}^k$ of $\sigma^i(y)$ such that $U_i \cap U_j = \emptyset$ and $\sigma(U_i) \subset U_{i+1}$. Note that $A \setminus U_0$ contains finite many points. Therefore there exists a $l \in \mathbb{N}$ such that $\{\sigma^n(x) : n \geq l\}$ do not contains any element in $A \setminus U_0$. If $n \geq l$ and $n \in A(x)$, we have $n+1, \dots, n+k$ are not in A . It is easy to see that $\bar{d}(A(x)) \leq \frac{1}{k}$. Hence $\text{den}(A(x)) = 0$.
- (2) $\sigma^m(y) = c_0$ and c_0 is a periodic point such that $\sigma^j(c_0) = c_j$ and $\sigma^n(c_0) = c_0$. Fix a positive integer k . We choose m neighborhoods U_i of $\sigma^i(y)$ for $i = 0, \dots, m-1$ and k neighborhoods $\{U_{m+nl+j}\}_{l=0}^{k-1}$ of c_j , $j = 0, \dots, n-1$ such that
- $U_{m+nl+j} \subset U_{m+n(l+1)+j}$, $l = 0, \dots, k-1$,
 - $U_{i_1} \cap U_{i_2} = \emptyset$ for $i_1 \neq i_2$ and $i_1, i_2 \in \{0, \dots, m-1\}$,
 - $U_i \cap U_{m+nk+j} = \emptyset$ for $i = 0, \dots, m-1$ and $j = 0, \dots, n-1$,
 - $U_{m+nk+j_1} \cap U_{m+nk+j_2} = \emptyset$ for $j_1 \neq j_2$,
 - $\sigma(U_i) \subset U_{i+1}$ for $i = 0, 1, \dots, m+nk-1$.
- Note that $A \setminus U_0$ contains finite many points. Therefore there exists a $N_0 \in \mathbb{N}$ such that $\{\sigma^n(x) : n \geq N_0\}$ do not contains any element in $A \setminus U_0$. If $s \geq N_0$ and $s \in A(x)$, we have $s+1, \dots, s+nk+m-1$ are not in A . It is easy to see that $\bar{d}(A(x)) \leq \frac{1}{nk+m-1}$. Hence $\text{den}(A(x)) = 0$

□

Corollary 6.1. *Let X be a countably infinite, compact metric space such that $d(X) = 2$ and $X^{(2)} = \{c\}$. Suppose that $\{\sigma^n(x)\}_{n=0}^\infty$ is dense in X for a isolated point x in X . Let B be any neighborhood of c and $A = X \setminus B$. Then $\text{den}(A(x)) = 0$.*

Proof. Note that $A \cap X^{(1)}$ contains finite elements $\{c_1, \dots, c_m\}$. Since A is closed, $A = \bigcup_{k=1}^m A_k$, where A_k is a subsequence of $\{\sigma^n(x)\}$ converges to c_k . Each c_k is not periodic by lemma 6.2. Then $\text{den}(A_k(x)) = 0$ by lemma 6.4,. Therefore $\text{den}(A(x)) = 0$. □

Corollary 6.2. *Let (X, σ) be a dynamic system and $x \in X$. Let Y be the closure of $\{\sigma^n(x)\}_{n=0}^\infty$. If $d(Y) = 2$ and $Y^{(2)} = \{c\}$, then $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} \mu(n) f(\sigma^n(x)) = 0$ for any $f \in C(X)$.*

Proof. We could assume that $f(c) = 0$. For any $\varepsilon > 0$, there is neighborhood U of c such that $|f(y)| < \varepsilon$ for any $y \in U$. By corollary 6.1, $\text{den}(A(x)) = 0$ where $A = \{\sigma^n(x) : \sigma^n(x) \text{ not in } U\}$. This implies the result. □

Example 6.2. *Let*

$$X_1 = \bigcup_{n=1}^{\infty} (\{(2 + \frac{1}{n}, \frac{1}{n+k}) : k = 0, 1, 2, \dots\} \cup \{(2 + \frac{1}{n}, 0)\}) \cup \{(2, 0)\},$$

$$X_2 = \bigcup_{n=2}^{\infty} (\{(\frac{1}{n}, \frac{1}{n+k}) : k = 0, 1, 2, \dots\} \cup \{(\frac{1}{n}, 0)\}) \cup \{(1, \frac{1}{2+k}) : k = 0, 1, 2, \dots\} \cup \{(\frac{1}{n}, 0)\} \cup \{(0, 0)\},$$

$X = X_1 \cup X_2$. Then $X^{(1)} = \{(\frac{1}{n}, 0) : n = 1, 2, \dots\} \cup \{(0, 0)\} \cup \{(2 + \frac{1}{n}, 0) : n = 1, 2, \dots\} \cup \{(2, 0)\}$ and $X^{(2)} = \{(0, 0)\} \cup \{(2, 0)\}$. Let σ be a map $X \rightarrow X$ defined by

$$\begin{aligned} (3, 0) &\rightarrow (0, 0) \text{ and } (2, 0) \rightarrow (2, 0) \\ (1, 0) &\rightarrow (2, 0) \text{ and } (0, 0) \rightarrow (0, 0) \\ (2 + \frac{1}{n}, 0) &\rightarrow (2 + \frac{1}{n-1}, 0) \quad (\text{for } n > 1) \\ (\frac{1}{n}, 0) &\rightarrow (\frac{1}{n-1}, 0) \quad (\text{for } n > 1) \\ (3, \frac{1}{k+1}) &\rightarrow (\frac{1}{k+2}, \frac{1}{k+2}) \\ (1, \frac{1}{2+k}) &\rightarrow (2 + \frac{1}{k+2}, \frac{1}{k+2}) \\ (2 + \frac{1}{n}, \frac{1}{n+k}) &\rightarrow (2 + \frac{1}{n-1}, \frac{1}{n+k}) \quad (\text{for } n > 1). \\ (\frac{1}{n}, \frac{1}{n+k}) &\rightarrow (\frac{1}{n-1}, \frac{1}{n+k}) \quad (\text{for } n > 1). \end{aligned}$$

It is easy to see that $\{\sigma^n((3, 1))\} = X \setminus X^{(1)}$. Let

$$\mathcal{X}(n) = \begin{cases} 1, & n \in [m^2 - m + 1, m^2] \text{ for } m = 1, 2, \dots \\ 0, & n \in [m^2 + 1, m(m+1)] \text{ for } m = 1, 2, \dots \end{cases}$$

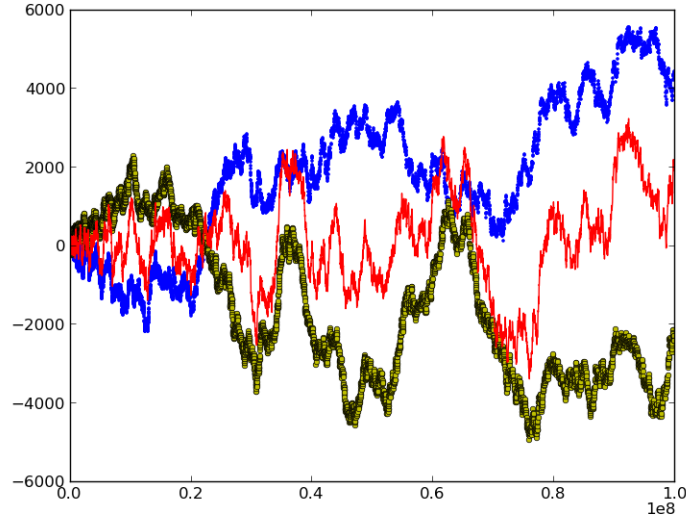


Figure 2. example 6.2, up to 100000000

Blue: $\sum_n^t \mu(n)\mathcal{X}(n)$, max:5561, min:-2197. Yellow: $\sum_n^t \mu(n)(1 - \mathcal{X}(n))$, max:2278, min:-4954. Red: $\sum_n^t \mu(n)$, max:3225, min:-3402.

In number theory, the Mertens function $M(x)$ is defined as

$$M(x) = \sum_{k \leq x} \mu(k).$$

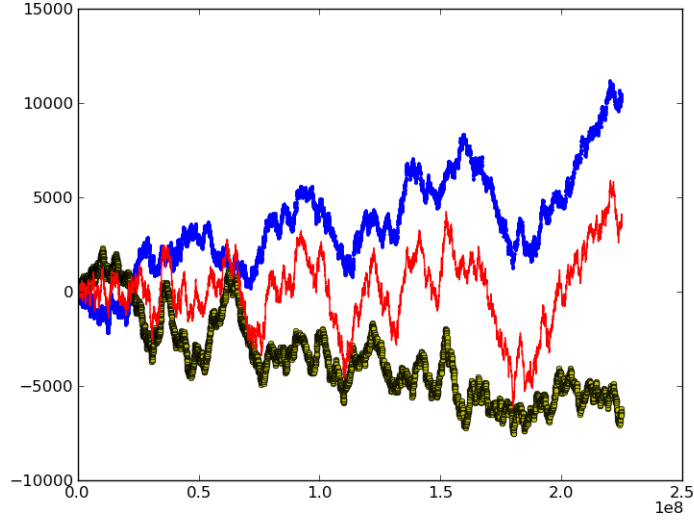


Figure 3. example 6.2, up to 225000000

Blue: $\sum_n^t \mu(n)\mathcal{X}(n)$, max:11175, min:-2197. Yellow: $\sum_n^t \mu(n)(1 - \mathcal{X}(n))$, max:2278, min:-7547. Red: $\sum_n^t \mu(n)$, max:5890, min:-6136.

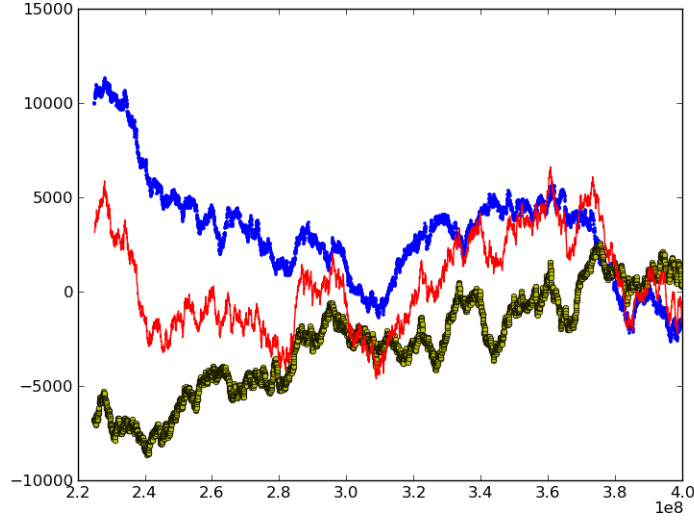


Figure 4. example 6.2, up to 400000000

Blue: $\sum_n^t \mu(n)\mathcal{X}(n)$, max:11319, min:-2663. Yellow: $\sum_n^t \mu(n)(1 - \mathcal{X}(n))$, max:2527, min:-8688. Red: $\sum_n^t \mu(n)$, max:6602, min:-4645.

By Theorem 333 in [6], the probabilit that a number should be squarefree is $\frac{6}{\pi^2}$, more precisely

$$Q(x) = \sum_{n \leq x} |\mu(n)| = \frac{6x}{\pi^2} + O(\sqrt{x}).$$

Lemma 6.5.

$$\sum_{n \leq x} \mathcal{X}(n) \log n = \frac{x}{2} \log x + O(x).$$

Proof. Let $x = m^2 + m + l$ where $0 \leq l < 2(m+1)$. Note

$$\begin{aligned} \sum_{n \leq x} (1 - \mathcal{X}(n)) \log n - \mathcal{X}(n) \log n &= \sum_{k=1}^m \sum_{i=1}^k \log\left(\frac{k^2+i}{k^2-k+i}\right) + \sum_{j=m^2+m+1}^{m^2+m+l} \log(j) \\ &= \sum_{k=1}^m \sum_{i=1}^k O\left(\frac{1}{k}\right) + O(\sqrt{x} \log x) = O(\sqrt{x} \log x). \end{aligned}$$

By Theorem 423 in [6],

$$\sum_{n \leq x} \log n = \sum_{n \leq x} (1 - \mathcal{X}(n)) \log n + \mathcal{X}(n) \log n = x \log x + O(x).$$

Therefore

$$\sum_{n \leq x} \mathcal{X}(n) \log n = \frac{x}{2} \log x + O(x).$$

□

Lemma 6.6. Let $S_l = \bigcup_{m=1}^{\infty} \{l \times m^2 + t : t \in [0, lm]\}$. If $j/i = k^2 > 1$, then

$$\lim_{x \rightarrow \infty} \frac{\#\{n \leq x : n \in S_i \cap S_j\}}{x} = \begin{cases} \frac{k+1}{4k} & \text{if } k \text{ is odd} \\ \frac{1}{4} & \text{if } k \text{ is even} \end{cases}.$$

Proof. Assume that $x = ka$ and $y = a$. Then $ix^2 = ik^2a^2 = ja^2 = jy^2$. Consider the interval

$$s_y = [jy^2, j(y+1)^2 - 1] = [jy^2, jy^2 + jy] \cup [jy^2 + jy + 1, j(y+1)^2 - 1] = s_y^1 + s_y^2.$$

Note that $[jy^2, j(y+1)^2 - 1]$ contains k intervals

$$[ix^2, ix^2 + ix], [i(x+1)^2, i(x+1)^2 + i(x+1)], \dots, [i(x+k-1)^2, i(x+k-1)^2 + i(x+k-1)].$$

If k is even, then

$$[ix^2, ix^2 + ix], [i(x+1)^2, i(x+1)^2 + i(x+1)], \dots, [i(x + \frac{k}{2} - 1)^2, i(x + \frac{k}{2} - 1)^2 + i(x + \frac{k}{2} - 1)]$$

are in s_y^1 for x large enough, since $i(x + \frac{k}{2} - \frac{k^2}{4}) > 0$. And

$$[i(x + \frac{k}{2})^2, i(x + \frac{k}{2})^2 + i(x + \frac{k}{2})], \dots, [i(x+k-1)^2, i(x+k-1)^2 + i(x+k-1)]$$

are in s_y^2 . This implies that $\lim_{x \rightarrow \infty} \frac{\#\{n \leq x : n \in S_i \cap S_j\}}{x} = \frac{1}{4}$.

If k is odd, then

$$i(x + \frac{k-1}{2})^2 + i(x + \frac{k-1}{2}) - jy^2 - jy = \frac{i(k^2-1)}{4}.$$

Therefore s_y^1 contains almost $\frac{k+1}{2}$ intervals. It is clear implies the result. □

Lemma 6.7. Suppose that p, q are two coprime positive integers, i.e. $(p, q) = 1$. Let

$$S_1 = \bigcup_{k=1}^p \left[\frac{(2k-2)}{2p}, \frac{(2k-1)}{2p} \right] \text{ and } S_2 = \bigcup_{k=1}^q \left[\frac{(2k-2)}{2q}, \frac{(2k-1)}{2q} \right].$$

$$\text{len}(S_1 \cap S_2) = \begin{cases} \frac{1}{4} & \text{if } p \text{ or } q \text{ is even,} \\ \frac{1}{4} + \frac{4}{15pq} & \text{otherwise.} \end{cases}$$

Proof. Let \mathcal{X}_1 and \mathcal{X}_2 be the characteristic function of S_1 and S_2 respectively.

$$\begin{aligned}\langle \mathcal{X}_1, e^{2\pi i n \theta} \rangle &= \sum_{k=1}^p \int_{\frac{2k-2}{2p}}^{\frac{2k-1}{2p}} e^{-2\pi i n \theta} \\ &= \frac{1 - e^{-\frac{\pi i n}{p}}}{2\pi i n} \sum_{k=0}^{p-1} e^{-\frac{2\pi i n k}{p}} = \begin{cases} \frac{1}{2}, & n = 0 \\ \frac{p}{\pi i n}, & \text{if } n = p(2a+1), a \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases}.\end{aligned}$$

Similiarly,

$$\begin{aligned}\langle \mathcal{X}_2, e^{2\pi i n \theta} \rangle &= \sum_{k=1}^q \int_{\frac{2k-2}{2q}}^{\frac{2k-1}{2q}} e^{-2\pi i n \theta} \\ &= \frac{1 - e^{-\frac{\pi i n}{q}}}{2\pi i n} \sum_{k=0}^{q-1} e^{-\frac{2\pi i n k}{q}} = \begin{cases} \frac{1}{2}, & n = 0 \\ \frac{q}{\pi i n}, & \text{if } n = q(2a+1), a \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases}\end{aligned}$$

If p or q is even, then $\langle \mathcal{X}_1, \mathcal{X}_2 \rangle = \frac{1}{4}$. If both p and q are odd, then

$$\begin{aligned}\text{len}(S_1 \cap S_2) &= \langle \mathcal{X}_1, \mathcal{X}_2 \rangle \\ &= \frac{1}{4} + \frac{1}{\pi^2 p q} \sum_{k \in \mathbb{Z}} \frac{1}{(2k+1)^2} \\ &= \frac{1}{4} + \frac{8}{5\pi^2 p q} \zeta(2) = \frac{1}{4} + \frac{4}{15 p q} \quad (\zeta(2) = \frac{\pi^2}{6}).\end{aligned}$$

□

Theorem 6.1 (Dirichlet, c. 1840). *For any real number θ and any integer $N \geq 1$, there exist integers p and q such that $1 \leq q \leq N$ and*

$$|q\theta - p| \leq \frac{1}{N+1}.$$

Lemma 6.8. *Let $S_l = \bigcup_{m=1}^{\infty} \{l \times m^2 + t : t \in [0, lm]\}$. If $\sqrt{\frac{l}{j}} \in \mathbb{R} \setminus \mathbb{Q}$, then*

$$\lim_{x \rightarrow \infty} \frac{\#\{n \leq x : n \in S_l \cap S_j\}}{x} = \frac{1}{4}.$$

Proof. We assume that $\sqrt{\frac{l}{j}} < 1$ and $|\sqrt{\frac{l}{j}}q - p| \leq \frac{1}{q}$. Since

$$\lim_{m \rightarrow \infty} \frac{iq^2 l^2}{iq^2(l+1)^2} = 1,$$

we only need to consider the case for $x = iq^2 l^2$, where $l \in \mathbb{N}$. The interval $[iq^2 l^2, iq^2(l+1)^2]$ contains q intervals in S_1 :

$$[i(ql)^2, i((ql)^2 + ql)], \dots, [i(ql + q - 1)^2, i(ql + q)(ql + q - 1)].$$

Let

$$\begin{aligned}S_{1l} &= \bigcup_{k=1}^q [i(ql + k - 1)^2, i(ql + k)(ql + k - 1)] \\ S'_{1l} &= \bigcup_{k=1}^q [i(ql)^2 + (2k-2)iq l, i(ql)^2 + (2k-1)iq l].\end{aligned}$$

Note that

$$\lim_{l \rightarrow 0} \frac{\#(S_{1l} \cap S'_{1l})}{iq^2 l} = 1.$$

If $iq^2 l^2 \leq jy^2 \leq iq^2(l+1)^2$, then

$$\sqrt{\frac{i}{j}}ql \leq y \leq \sqrt{\frac{i}{j}}q(l+1).$$

Since

$$p - \frac{1}{q} \leq \sqrt{\frac{i}{j}}q \leq p + \frac{1}{q},$$

$[iq^2 l^2, iq^2(l+1)^2]$ contains at least $p-3$ and at most $p+1$ intervals in S_2 . Let y_0 be the first y such that $iq^2 l^2 \leq jy^2 \leq iq^2(l+1)^2$. Let

$$S_{2l} = \bigcup_{k=1}^{p-3} [j(y_0 + k - 1)^2, j(y_0 + k - 1)^2 + j(y_0 + k - 1)] \subset S_2 \cap [iq^2 l^2, iq^2(l+1)^2].$$

Note that

$$\liminf_{l \rightarrow \infty} \frac{\#S_{2l}}{\#(S_2 \cap [iq^2 l^2, iq^2(l+1)^2])} \geq 1 - \frac{5}{p-3}.$$

Let

$$S'_{2l} = \bigcup_{k=1}^{p-3} [jy_0^2 + (2k-2)jy_0, jy_0^2 + (2k-1)jy_0].$$

Also we have

$$\lim_{l \rightarrow \infty} \frac{\#(S_{2l} \cap S'_{2l})}{iq^2 l} = 1.$$

We would like to compare S'_{1l} with S'_{2l} . Since $y_0 \leq \sqrt{\frac{i}{j}}ql + 1$, we have

$$\begin{aligned} \Delta &= i(ql)^2 + 2iq^2 l - jy_0^2 + (2p-7)jy_0 \\ &\geq i(ql)^2 + 2iq^2 l - j(\sqrt{\frac{i}{j}}ql + 1)^2 + (2p-7)j(\sqrt{\frac{i}{j}}ql + 1) \\ &= 2iq^2 l(1 - (\frac{p}{q} - \frac{5}{2q})\sqrt{\frac{j}{i}} - \frac{(2p-6)j}{2iq^2 l}) \\ &\geq 2iq^2 l(1 - (\sqrt{\frac{i}{j}} + \frac{1}{q^2} - \frac{5}{2q})\sqrt{\frac{j}{i}} - \frac{(2p-6)j}{2iq^2 l}) \quad (\frac{p}{q} \leq \sqrt{\frac{i}{j}} + \frac{1}{q^2}) \\ &= 2iq^2 l((\frac{5}{2q} - \frac{1}{q^2})\sqrt{\frac{j}{i}} - \frac{(2p-6)j}{2iq^2 l}) \geq 0, \end{aligned}$$

provide that l is large enough.

This means that S'_{2l} is contains in $[i(ql)^2, i(ql)^2 + 2iq^2 l]$. A similar caculation as in the proof of lemma 6.8 shows that

$$\lim_{l \rightarrow \infty} \frac{\#(S'_{1l} \cap S'_{2l})}{2iq^2 l} = \frac{1}{4} + O(\frac{1}{pq}).$$

Therefore

$$\lim_{l \rightarrow \infty} \frac{\#(S_{1l} \cap S_{2l})}{2iq^2 l} = \frac{1}{4} + O(\frac{1}{p}).$$

This implies that

$$\lim_{x \rightarrow \infty} \frac{\#\{n \leq x : n \in S_i \cap S_j\}}{x} = \frac{1}{4} + O\left(\frac{1}{p}\right).$$

By increasing p and q , the result is proved. \square

Lemma 6.9. Let $S_l = \bigcup_{m=1}^{\infty} \{l \times m^2 + l \times i : i \in [0, m]\}$. If $\sqrt{\frac{i}{j}} \in \mathbb{R} \setminus \mathbb{Q}$, then

$$\lim_{x \rightarrow \infty} \frac{\#\{n \leq x : n \in S_i \cap S_j\}}{x} = \frac{1}{4ij} \quad (i \neq j).$$

Proof. \square

Theorem 6.2 (Carleson's theorem). Let f be an L^p periodic function for some $p \in (1, \infty)$, with Fourier coefficient $\hat{f}(n)$. Then

$$\lim_{N \rightarrow \infty} \sum_{|n| \leq N} \hat{f}(n) e^{2\pi i n x} = f(x)$$

for almost every x .

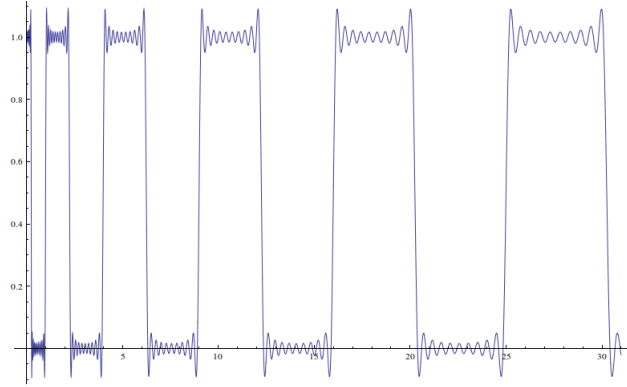


Figure 5. $\frac{1}{2} + \sum_{k=1}^{10} \frac{2}{(2k-1)\pi} \sin(2\pi i(2k-1)\sqrt{x}), (x \in [0, 31])$

Definition 6.1. A function $f(x)$ is piecewise continuous on an interval I if it is continuous on I except perhaps for a finite number of points, and if $a \in I$ is a point of discontinuity for $f(x)$ then $f(a_+)$ and $f(a_-)$ exist: that is

$$f(a_+) = \lim_{x \rightarrow a_+} f(x), \quad f(a_-) = \lim_{x \rightarrow a_-} f(x)$$

are required to exist. We denote the space of piecewise continuous functions on I by $E(I)$.

Definition 6.2. The space E' is defined as the space of all functions $f(x) \in E([-\pi, \pi])$ such that the right-hand derivative $D_+ f(x)$ and left-hand derivative $D_- f(x)$ exists. Recall that

$$D_+ f(x) = \lim_{h \rightarrow 0_+} \frac{f(x+h) - f(x_+)}{h}$$

$$D_- f(x) = \lim_{h \rightarrow 0_-} \frac{f(x+h) - f(x_-)}{h}.$$

Theorem 6.3. *If $f \in E'$, then the Fourier series of f*

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx),$$

where

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx, \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx,$$

converges pointwise to

$$\frac{f(x_+) + f(x_-)}{2}.$$

Theorem 6.4. *Suppose that*

- $f(x)$ is continuous on $[-\pi, \pi]$
- $f(-\pi) = f(\pi)$
- $f'(x) \in E([-\pi, \pi])$.

Then

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

converges uniformly to $f(x)$ on $[-\pi, \pi]$.

Corollary 6.3. *Let \mathcal{X} be the characteristic function of $\bigcup_{m=1}^{\infty} [m^2, m^2 + m + \frac{1}{4}]$. Then*

$$\mathcal{X} = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{(2k-1)\pi} \sin(2\pi(2k-1)\sqrt{x}) \quad (x \geq 1)$$

at every point where \mathcal{X} is continuous.

Lemma 6.10. [Bourgain-Sarnak-Ziegler] *Let $F : \mathbb{N} \rightarrow \mathbb{C}$ with $|F| \leq 1$ and let v be a multiplicative function with $|v| \leq 1$. Let $\tau > 0$ be a small parameter and assume that for all primes $p_1, p_2 \leq e^{1/\tau}$, $p_1 \neq p_2$, we have that for M large enough*

$$\left| \sum_{m \leq M} F(p_1 m) \overline{F(p_2 m)} \right| \leq \tau M.$$

Then for N large enough

$$\left| \sum_{n \leq N} v(n) F(n) \right| \leq 2N \sqrt{\tau \log(1/\tau)}.$$

Lemma 6.11 (Kusmin-Landau). *If f is continuously differentiable, f' is monotonic, and $0 < \lambda \leq f'(x) \leq 1 - \lambda$ on the interval $(a, b]$, then*

$$\left| \sum_{a < n \leq b} e(f(n)) \right| \ll \lambda^{-1}.$$

Lemma 6.12. *Let $0 < \gamma < 1$. For arbitrarily small $\tau > 0$, and for $0 \neq |k| \leq x^{(1-\gamma)/2}$ and multiplicative function v with $|v| \leq 1$, we have*

$$\left| \sum_{n \leq x} v(n) e(kn^\gamma) \right| \ll_{\tau} x$$

Proof. With the help of Lemma lemma 6.10, if we prove that for all primes $p_1 \neq p_2 \ll e^{1/\tau}$, the estimate

$$\left| \sum_{n \leq x} e(k(p_1 n)^\gamma - k(p_2 n)^\gamma) \right| \ll_\tau x$$

holds for sufficiently large x , then

$$\left| \sum_{n \leq x} v(n) e(kn^\gamma) \right| \ll_\tau x$$

also holds. Since

$$\left| \sum_{n \leq x(\log x)^{-1}} e(k(p_1 n)^\gamma - k(p_2 n)^\gamma) \right| \ll x(\log x)^{-1} \ll_\tau x$$

for x sufficiently large, it is sufficient to prove

$$\left| \sum_{x(\log x)^{-1} \leq n \leq x} e(k(p_1 n)^\gamma - k(p_2 n)^\gamma) \right| \ll_\tau x$$

Take $f(n) := k(p_1^\gamma - p_2^\gamma)n^\gamma$ in Lemma lemma 6.11. Note that

$$|f'(n)| \asymp k(p_1^\gamma - p_2^\gamma)n^{\gamma-1} \gg_\tau |k|x^{\gamma-1}$$

and

$$|f'(n)| \asymp k(p_1^\gamma - p_2^\gamma)n^{\gamma-1} \ll_\tau |k|x^{\gamma-1}(\log x)^{1-\gamma} = o_\tau(1)$$

when $k \leq x^{(1-\gamma)/2}$. By Lemma lemma 6.11, we conclude that

$$\left| \sum_{x(\log x)^{-1} \leq n \leq x} e(k(p_1 n)^\gamma - k(p_2 n)^\gamma) \right| \ll_\tau x^{1-\gamma}.$$

□

Theorem 6.5. Let $0 < \gamma < 1$ and write $\beta = \gamma^{-1}$. Define $A = \bigcup_{m=1}^\infty [m^\beta, (m+\gamma)^\beta)$. Then for any multiplicative function v with $|v| \leq 1$, we have

$$\sum_{n \leq x} v(n) 1_A(n) = \gamma \sum_{n \leq x} v(n) + o(x).$$

In particular, we have

$$\sum_{n \leq x} \mu(n) 1_A(n) = o(x)$$

and

$$\sum_{n \leq x} \mu^2(n) 1_A(n) = \frac{6\gamma}{\pi^2} x + o(x).$$

Proof. Let $\tau > 0$ be an arbitrarily small parameter to be determined later. Define

$$\tilde{A} = \bigcup_{m=1}^\infty [m, m+\gamma]$$

and

$$\eta(y) = \begin{cases} \exp\left(-\frac{1}{1-((y-\tau-m)/\tau)^2} + 1\right), & y \in \bigcup_{m=1}^\infty [m, m+\tau), \\ 1, & y \in \bigcup_{m=1}^\infty [m+\tau, m+\gamma-\tau), \\ \exp\left(-\frac{1}{1-((y-\gamma+\tau-m)/\tau)^2} + 1\right), & y \in \bigcup_{m=1}^\infty [m+\gamma-\tau, m+\gamma), \\ 0, & y \in \bigcup_{m=1}^\infty [m+\gamma, m+1). \end{cases}$$

The function η has the Fourier expansion

$$\eta(y) = \sum_{k \in \mathbb{Z}} \hat{\eta}(k) e(ky), \quad y \geq 1,$$

where the Fourier coefficients $\hat{\eta}(k) = \int_0^1 \eta(t) e(-kt) dt$ satisfy $|\hat{\eta}(k)| \ll_{\tau, A} (2 + |k|)^{-A}$. Write $\delta = (1 - \gamma)/2$, we have

$$\begin{aligned} \eta(y) &= \sum_{|k| \leq x^{\delta/2}} \hat{\eta}(k) e(ky) + O_{\tau, A} \left(\sum_{|k| \geq x^{\delta/2}} |k|^{-A} \right) \\ &= \sum_{|k| \leq x^{\delta/2}} \hat{\eta}(k) e(ky) + O_{\tau, A} (x^{-100}). \end{aligned}$$

Now

$$\begin{aligned} \sum_{n \leq x} \nu(n) 1_A(n) &= \sum_{n \leq x} \nu(n) 1_{\tilde{A}}(n^\gamma) \\ &= \sum_{n \leq x} \nu(n) \eta(n^\gamma) + \sum_{n \leq x} \nu(n) (1_{\tilde{A}}(n^\gamma) - \eta(n^\gamma)) \\ &:= W_1 + W_2. \end{aligned}$$

The first summation is

$$\begin{aligned} W_1 &= \sum_{n \leq x} \nu(n) \left(\sum_{|k| \leq x^{\delta/2}} \hat{\eta}(k) e(kn^\gamma) + O_A(x^{-100}) \right) \\ &= \sum_{|k| \leq x^{\delta/2}} \hat{\eta}(k) \left(\sum_{n \leq x} \nu(n) e(kn^\gamma) \right) + O(1) \\ &= \hat{\eta}(0) \sum_{n \leq x} \nu(n) + O \left(1 + \sum_{|k| \leq x^{\delta/2}} |\hat{\eta}(k)| \left| \sum_{n \leq x} \nu(n) e(kn^\gamma) \right| \right) \end{aligned}$$

In view of Lemma lemma 6.12, the summation in the error term is $O_\tau(x)$. The coefficient

$$\hat{\eta}(0) = \int_0^1 \eta(y) dy = \gamma + O(\tau).$$

As a result, it follows that

$$W_1 = \gamma \sum_{n \leq x} \nu(n) + O(\tau x).$$

Now we start to bound W_2 . Write $h(n) = 1_{\tilde{A}}(n^\gamma) - \eta(n^\gamma)$, then

$$\text{Supp}(h) \cap [1, x] \subseteq \bigcup_{m=1}^{x^\gamma} [m^\beta, (m+\tau)^\beta] \cup [(m+\gamma-\tau)^\beta, (m+\gamma)^\beta].$$

Hence

$$W_2 \ll \sum_{m \leq x^\gamma} \tau m^{\beta-1} \ll \tau x.$$

Putting every thing together, we see that

$$\sum_{n \leq x} \nu(n) 1_A(n) = \gamma \sum_{n \leq x} \nu(n) + O(\tau x).$$

Since $\tau > 0$ can be chosen arbitrarily small, it can be conclude that

$$\sum_{n \leq x} \nu(n) 1_A(n) = \gamma \sum_{n \leq x} \nu(n) + o(x).$$

□

Lemma 6.13.

$$\sum_{n \leq x} \left[\frac{x}{n} \right] \mathcal{X}(n) \Lambda(n) = \frac{x}{2} \log x + O(x).$$

Proof. Assume that $x = m^2 + m$. We claim that

$$\sum_{k=1}^m \sum_{l=1}^k \left[\frac{m^2 + m}{k^2 + l} \right] \Lambda(k^2 + l) - \sum_{k=1}^m \sum_{l=1}^k \left[\frac{m^2 + m}{k^2 - k + l} \right] \Lambda(k^2 - k + l) = O(x).$$

First note that

$$\begin{aligned} & \sum_{k=1}^m \sum_{l=1}^k \left[\frac{m^2 + m}{k^2 + l} \right] \Lambda(k^2 + l) - \sum_{k=1}^m \sum_{l=1}^k \left[\frac{m^2 + m}{k^2 - k + l} \right] \Lambda(k^2 - k + l) \\ &= \sum_{k=1}^m \sum_{l=1}^k \frac{m^2 + m}{k^2 + l} \Lambda(k^2 + l) - \sum_{k=1}^m \sum_{l=1}^k \frac{m^2 + m}{k^2 - k + l} \Lambda(k^2 - k + l) + O(x). \end{aligned}$$

However

$$\begin{aligned} & \sum_{k=1}^m \sum_{l=1}^k \left(\frac{m^2 + m}{k^2 + l} - \frac{m^2 + m}{k^2} \right) \Lambda(k^2 + l) \\ &= x \sum_{k=1}^m \sum_{l=1}^k \left(\frac{-l}{k^2(k^2 + l)} \right) \Lambda(k^2 + l) = xO\left(\sum_{k=1}^m \frac{\log 2k}{k^3}\right) = O(x). \end{aligned}$$

Similarly

$$\begin{aligned} & \sum_{k=1}^m \sum_{l=1}^k \left(\frac{m^2 + m}{k^2 - k + l} - \frac{m^2 + m}{k^2} \right) \Lambda(k^2 - k + l) \\ &= x \sum_{k=1}^m \sum_{l=1}^k \left(\frac{k - l}{k^2(k^2 - k + l)} \right) \Lambda(k^2 - k + l) = xO\left(\sum_{k=1}^m \frac{\log 2k}{k^3}\right) = O(x). \end{aligned}$$

Therefore, we only need to show

$$\sum_{k=1}^m \frac{m^2 + m}{k^2} \sum_{l=1}^k \left(\left[\frac{k^2 + k}{k^2 + l} \right] \Lambda(k^2 + l) - \left[\frac{k^2}{k^2 - k + l} \right] \Lambda(k^2 - k + l) \right)$$

□

Let $\Psi_{\mathcal{X}}(k, x) = \sum_{d \leq x} \mathcal{X}(dk) \Lambda(d)$ and $\Psi_{1-\mathcal{X}}(k, x) = \sum_{d \leq x} (1 - \mathcal{X}(dk)) \Lambda(d)$. Recall that

$$\Psi(x) = \sum_{d \leq x} \Lambda(d) = O(x).$$

Therefore $\Psi_{\mathcal{X}}(k, x) = O(x)$ and $\Psi_{1-\mathcal{X}}(k, x) = O(x)$.

Lemma 6.14. *Let*

$$\Psi(k, x) = \sum_{d \leq x} \mathcal{X}(dk) \Lambda(d).$$

Then $\Psi(k, x) - \frac{1}{2}[x] = o(x)$.

Proof. Let $\Psi(x) = \sum_{d \leq x} \Lambda(d) = \Psi(k, x)$

□

Lemma 6.15. *Let* $M_{\mathcal{X}}(x) = \sum_{n \leq x} \mathcal{X}(n) \mu(n)$. *Then* $M_{\mathcal{X}}(x) = o(x)$.

Proof. By Theorem 434 in [6], we have

$$\sum_{n \leq x} \mathcal{X}(n) \mu(n) \log\left(\frac{x}{n}\right) = O(x).$$

Hence

$$M_{\mathcal{X}}(x) \log(x) = \sum_{n \leq x} \mathcal{X}(n) \mu(n) \log n + O(x).$$

By Theorem 297 in [6],

$$\begin{aligned} - \sum_{n \leq x} \mathcal{X}(n) \mu(n) \log n &= \sum_{n \leq x} \mathcal{X}(n) \sum_{d|n} \mu\left(\frac{n}{d}\right) \Lambda(d) = \sum_{dk \leq x} \mathcal{X}(dk) \mu(k) \Lambda(d) \\ &= \sum_{k \leq x} \mu(k) \sum_{d \leq \frac{x}{k}} \mathcal{X}(dk) \Lambda(d) \end{aligned}$$

□

Lemma 6.16 (Conjecture). *Let (X, σ) be a dynamic system where X is a compact Hausdorff metric space and $\iota(X)$ is the set of all isolated points of X . Let $X' = X \setminus \iota(X)$. If $\sigma(X') \subset X'$ and $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} \mu(n) f(\sigma^n(x)) = 0$ for any $f \in C(X')$ and any $x \in X'$, then $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} \mu(n) f(\sigma^n(x)) = 0$ for any f in $C(X)$ and $x \in X$.*

7. WEIGHTED SUM OF $\mu(n)$

Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a function defined on \mathbb{N} . Consider the weighted sum

$$\sum_{n \leq N} f(n) \mu(n)$$

Let $\rho : \mathbb{N} \rightarrow \mathbb{N}$ be a map defined by

$$\rho : p_1^{\gamma(1)} p_2^{\gamma(2)} \cdots p_n^{\gamma(n)} \rightarrow 1^{\gamma(1)} 2^{\gamma(2)} \cdots n^{\gamma(n)},$$

where p_i is the i -th prime number.

Remark 7.1. *If $k < \sqrt{N}$, then $\lfloor \frac{N}{k} \rfloor > \lfloor \frac{N}{k+1} \rfloor$. Assume that $N = ak + b$ where $0 \leq b < k$ and $a \geq k$. If $\lfloor \frac{N}{k} \rfloor = \lfloor \frac{N}{k+1} \rfloor$, then $ak + b = a(k+1) + c$. It is impossible.*

Lemma 7.1.

$$\sum_{\gamma \in (\mathbb{Z}_2)^N \text{ and } 1^{\gamma(1)} 2^{\gamma(2)} \cdots N^{\gamma(N)} \leq M} (-1)^{\sum_i \gamma(i)} = 0.$$

Proof.

$$\begin{aligned} &\sum_{\gamma \in (\mathbb{Z}_2)^N \text{ and } 1^{\gamma(1)} 2^{\gamma(2)} \cdots N^{\gamma(N)} \leq M} (-1)^{\sum_i \gamma(i)} \\ &= \sum_{\gamma \in (\mathbb{Z}_2)^N \text{ and } 2^{\gamma(2)} \cdots N^{\gamma(N)} \leq M} (-1)^{\sum_{i=2}^N \gamma(i)} - \sum_{\gamma \in (\mathbb{Z}_2)^N \text{ and } 2^{\gamma(2)} \cdots N^{\gamma(N)} \leq M} (-1)^{\sum_{i=2}^N \gamma(i)} \\ &= 0. \end{aligned}$$

□

Estimate

$$\sum_{\gamma \in (\mathbb{Z}_2)^N \text{ and } 2^{\gamma(2)} \cdots N^{\gamma(N)} \leq N} (-1)^{\sum_i \gamma(i)}.$$

Note that if $j > \lfloor \frac{N}{2} \rfloor$ and $\gamma(j) = 1$, then $\gamma(i) = 0$ for any $i \neq j$. Therefore

$$\begin{aligned} & \sum_{\gamma \in (\mathbb{Z}_2)^N \text{ and } 2^{\gamma(2)} \dots N^{\gamma(N)} \leq N} (-1)^{\sum_i \gamma(i)} \\ &= \sum_{\gamma \in (\mathbb{Z}_2)^{\lfloor \frac{N}{2} \rfloor} \text{ and } 2^{\gamma(2)} \dots (\lfloor \frac{N}{2} \rfloor)^{\gamma(\lfloor \frac{N}{2} \rfloor)} \leq N} (-1)^{\sum_i \gamma(i)} - (N - \lfloor \frac{N}{2} \rfloor). \end{aligned}$$

Similar argument implies that

$$\sum_{\gamma \in (\mathbb{Z}_2)^{\frac{N}{2}} \text{ and } 1^{\gamma(1)} 2^{\gamma(2)} \dots (\lfloor \frac{N}{2} \rfloor)^{\gamma(\lfloor \frac{N}{2} \rfloor)} \leq N} (-1)^{\sum_i \gamma(i)} = \sum_{\gamma \in (\mathbb{Z}_2)^{\frac{N}{3}} \text{ and } 1^{\gamma(1)} 2^{\gamma(2)} \dots (\lfloor \frac{N}{3} \rfloor)^{\gamma(\lfloor \frac{N}{3} \rfloor)} \leq N} (-1)^{\sum_i \gamma(i)}.$$

Suppose that $\gamma(j) = 1$ and $\lfloor \frac{N}{4} \rfloor < j \leq \lfloor \frac{N}{3} \rfloor$. The possibilities are

$$\begin{aligned} \gamma(1) = \gamma(2) = \gamma(3) = 0 & \quad \gamma(1) = 1, \gamma(2) = \gamma(3) = 0 \\ \gamma(1) = 0, \gamma(2) = 1, \gamma(3) = 0 & \quad \gamma(1) = 0, \gamma(2) = 0, \gamma(3) = 1 \\ \gamma(1) = 0, \gamma(2) = 0, \gamma(3) = 1 & \quad \gamma(1) = 1, \gamma(2) = 1, \gamma(3) = 0 \\ \gamma(1) = 1, \gamma(2) = 0, \gamma(3) = 1. & \end{aligned}$$

This implies that

$$\sum_{\gamma \in (\mathbb{Z}_2)^{\frac{N}{3}} \text{ and } 1^{\gamma(1)} 2^{\gamma(2)} \dots (\lfloor \frac{N}{3} \rfloor)^{\gamma(\lfloor \frac{N}{3} \rfloor)} \leq N} (-1)^{\sum_i \gamma(i)} = \sum_{\gamma \in (\mathbb{Z}_2)^{\frac{N}{4}} \text{ and } 1^{\gamma(1)} 2^{\gamma(2)} \dots (\lfloor \frac{N}{4} \rfloor)^{\gamma(\lfloor \frac{N}{4} \rfloor)} \leq N} (-1)^{\sum_i \gamma(i)}$$

Similarly, we have

$$\sum_{\gamma \in (\mathbb{Z}_2)^{\frac{N}{4}} \text{ and } 1^{\gamma(1)} 2^{\gamma(2)} \dots (\lfloor \frac{N}{4} \rfloor)^{\gamma(\lfloor \frac{N}{4} \rfloor)} \leq N} (-1)^{\sum_i \gamma(i)} = \sum_{\gamma \in (\mathbb{Z}_2)^{\frac{N}{6}} \text{ and } 1^{\gamma(1)} 2^{\gamma(2)} \dots (\lfloor \frac{N}{6} \rfloor)^{\gamma(\lfloor \frac{N}{6} \rfloor)} \leq N} (-1)^{\sum_i \gamma(i)}$$

Assume that $k + 1 \leq \sqrt{N}$ and

$$\sum_{\gamma \in (\mathbb{Z}_2)^N \text{ and } 1^{\gamma(1)} 2^{\gamma(2)} \dots N^{\gamma(N)} \leq N} (-1)^{\sum_i \gamma(i)} = \sum_{\gamma \in (\mathbb{Z}_2)^{\lfloor \frac{N}{k} \rfloor} \text{ and } 1^{\gamma(1)} 2^{\gamma(2)} \dots (\lfloor \frac{N}{k} \rfloor)^{\gamma(\lfloor \frac{N}{k} \rfloor)} \leq N} (-1)^{\sum_i \gamma(i)}$$

Lemma 7.2. *Let $k \leq \sqrt{N}$. Then*

$$\sum_{\gamma \in (\mathbb{Z}_2)^N \text{ and } 1^{\gamma(1)} 2^{\gamma(2)} \dots N^{\gamma(N)} \leq N} (-1)^{\sum_i \gamma(i)} = \sum_{\gamma \in (\mathbb{Z}_2)^{\lfloor \frac{N}{k} \rfloor} \text{ and } 1^{\gamma(1)} 2^{\gamma(2)} \dots (\lfloor \frac{N}{k} \rfloor)^{\gamma(\lfloor \frac{N}{k} \rfloor)} \leq N} (-1)^{\sum_i \gamma(i)}$$

Proof. For $k = 1$, the statement is trivial. Let $k = 2$. Assume that

$$\sum_{\gamma \in (\mathbb{Z}_2)^N \text{ and } 1^{\gamma(1)} 2^{\gamma(2)} \dots N^{\gamma(N)} \leq N} (-1)^{\sum_i \gamma(i)} = \sum_{\gamma \in (\mathbb{Z}_2)^{\lfloor \frac{N}{k-1} \rfloor} \text{ and } 1^{\gamma(1)} 2^{\gamma(2)} \dots (\lfloor \frac{N}{k-1} \rfloor)^{\gamma(\lfloor \frac{N}{k-1} \rfloor)} \leq N} (-1)^{\sum_i \gamma(i)}.$$

Suppose that $\gamma(j) = 1$ and $\lfloor \frac{N}{k} \rfloor < j \leq \lfloor \frac{N}{k-1} \rfloor$. Note that

$$\sum_{\gamma \in (\mathbb{Z}_2)^{k-1} \text{ and } 1^{\gamma(1)} 2^{\gamma(2)} \dots (k-1)^{\gamma(k-1)} \leq N} -(-1)^{\sum_i \gamma(i)}$$

□

8. Stone – Čech COMPACTIFICATION

Definition 8.1. A filter on a set X is a collection \mathcal{F} of subsets of X satisfying:

- (1) $X \in \mathcal{F}$, but $\emptyset \notin \mathcal{F}$.
- (2) If $A \in \mathcal{F}$ and $A \subset B \subset X$, then $B \in \mathcal{F}$.
- (3) A finite intersection of sets in \mathcal{F} is in \mathcal{F} .

A filter is a ultrafilter if

- (4) For every set $A \subset X$ either $A \in \mathcal{F}$ or $A^c = X \setminus A \in \mathcal{F}$.

or

- (4)' For every finite cover $\{A_i\}_{i=1}^n$ of a set $A \in \mathcal{F}$, $A_i \in \mathcal{F}$ for some i .

It is well-known that the Stone – Čech Compactification $\beta\mathbb{N}$ of \mathbb{N} can be identified with the set of all ultrafilters on \mathbb{N} . The topology of $\beta\mathbb{N}$ is given by the basis $\mathcal{B} = \{U_A : A \subset \mathbb{N}\}$, where for any set $A \subset \mathbb{N}$,

$$U_A = \{\mathcal{F} \in \beta\mathbb{N} : A \in \mathcal{F}\}.$$

Example 8.1. Let σ be a continuous map from \mathbb{N} to \mathbb{N} defined by $\sigma(n) = n + 1$. By the universal property of $\beta\mathbb{N}$, σ lifts uniquely to a continuous map $\sigma : \beta\mathbb{N} \rightarrow \beta\mathbb{N}$. Then the möbius function μ can be viewed as a continuous function on $\beta\mathbb{N}$ and $\sum_{k=1}^n \mu(k)^2$ is the number of square-free numbers below n . It is well-known

$$\frac{1}{n} \sum_{k=0}^{n-1} \mu(k)^2 = \frac{6}{\pi^2} + o(1).$$

Therefore $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(k)^2 = \frac{6}{\pi^2} \neq 0$. However, this is not a counter-example of sarnak's conjecture, since $h_{\text{top}}(\sigma) = \infty$. Indeed, for any $N \in \mathbb{N}$, let

$$A_N = \bigcup_{k=1}^{\infty} \left\{ kn + 1 + \frac{k(k-1)}{2}, \dots, kn + 1 + \frac{k(k-1)}{2} + k \right\}.$$

It is easy to see that for any n_1, \dots, n_m , we have $\sigma^{n_1}(A_N) \cap \dots \cap \sigma^{n_m}(A_N) \neq \emptyset$, and $\bigcup_{i=0}^{N-1} \sigma^i(A_N) = \mathbb{N}$. Therefore, $\mathcal{U} = \{U_{\sigma^i(A_N)}\}_{i=0}^{N-1}$ is an open cover of $\beta\mathbb{N}$ and $h_{\text{cover}}(\sigma, \mathcal{U}) = \log N$.

Remark 8.1. Recall that a thick set is a set of integers that contains arbitrarily long intervals. And a syndetic set is a subset of \mathbb{N} , having the property of "bounded gaps", i.e. the sizes of the gaps in the sequence of natural numbers is bounded. It is easy to see that A_N in the example 8.1 is thick and syndetic.

Definition 8.2. A dynamical system (X, σ) is call minimal if X does not contain any non-empty, proper, closed σ -invariant subset, i.e. every orbit is dense in X .

Remark 8.2. If (X, σ) is a minimal dynamical system and X is a compact Hausdorff space, then σ must be surjective.

Lemma 8.1. $\beta\mathbb{N} \setminus \mathbb{N}$ is not separable.

Proof. Let $\{\mathcal{F}_n\}_{n=1}^{\infty}$ be a sequence of ultrafilters. To prove the result, we will construct a subset A of \mathbb{N} recursively such that $A \notin \mathcal{F}_n$ for any n . Let A_1 be a infinite set that $A_1 \notin \mathcal{F}_1$ and $a_1 = \min A_1$. Assume that we have $A_n \notin \mathcal{F}_n$. Let $a_n = \min A_n$. Choose a infinite subset A_{n+1} of A_n such that $a_{n+1} = \min A_{n+1} > a_n$ and $A_{n+1} \notin \mathcal{F}_{n+1}$. If $A \in \mathcal{F}_n$, then $A \cap A_n = A \cap \{1, \dots, a_n - 1\}^c \in \mathcal{F}_n$. Since $A_n^c \in \mathcal{F}_n$, we have $A \cap A_n \cap A_n^c = \emptyset \in \mathcal{F}_n$. It is a contradiction. \square

Corollary 8.1. Let $(\beta\mathbb{N}, \sigma)$ be the dynamical system where σ is the map induced by the shift on \mathbb{N} , i.e. $\sigma(n) = n + 1$. Then $(\beta\mathbb{N}, \sigma)$ is not minimal.

9. WHAT WE TALK ABOUT WHEN WE TALK ABOUT SMOOTHNESS

Let $C(X)$ and $C(Y)$ be two abelian C^* -algebras, where X and Y are compact Hausdorff spaces. Suppose Φ is a $*$ -homomorphism from $C(X)$ to $C(Y)$. It is clear that Φ induces a map Φ^* from Y to X ,

$$\Phi^* : \rho \rightarrow \rho \circ \Phi, \quad \rho \in Y.$$

Similar, if $\varphi : X \rightarrow Y$ be a continuous map, then φ induces a map from $C(Y) \rightarrow C(X)$,

$$\varphi^* : f \rightarrow f \circ \varphi, \quad f \in C(Y).$$

It is clear that $\Phi^{**} = \Phi$ and $\varphi^{**} = \varphi$.

Consider the C^* -algebra $L^\infty(\mathbb{N})$ acting on $l^2(\mathbb{N})$. It is well-known that $L^\infty(\mathbb{N}) \simeq C(\beta\mathbb{N})$. Let σ be the continuous map from $\beta\mathbb{N}$ to $\beta\mathbb{N}$ induced by the shift on \mathbb{N} : $\sigma(n) = n + 1$. Throughout this section, we will use σ to denote this continuous map.

Let V be the unilateral shift, i.e., $Ve_i = e_{\sigma(i)}$. Then

$$\Sigma : M_f \rightarrow V^* M_f V = M_{f \circ \sigma}, \text{ for all } f \in L^\infty(\mathbb{N}),$$

is a homomorphism from $L^\infty(\mathbb{N})$ onto $L^\infty(\mathbb{N})$.

Note that

$$V^* M_{f_1} V V^* M_{f_2} V = V^* M_{f_1 f_2} V,$$

where f_1 and f_2 are in $L^\infty(\mathbb{N})$. It is easy to see that $\sigma^* = \Sigma$ and $\Sigma^* = \sigma$.

Let $f \in L^\infty(\mathbb{N})$ and $C(f)$ be the C^* -algebra generated by $M_{f \circ \sigma^n}$, $n = 0, 1, \dots$ and I . This is an abelian C^* -algebra. By Gelfand transform, $C(f)$ can be identified as $C(\Omega)$, where Ω is a compact Hausdorff space. Ω is metrizable since $C(f)$ is separable.

Remark 9.1. Let $C(f) \simeq C(\Omega)$ where Ω is the spectrum of $C(f)$. $\{M_{f \circ \sigma^n} : n = 0, 1, 2, \dots\}$ separates points in Ω .

Theorem 9.1 (Tietze extension theorem). *If X is a normal topological space (e.g., every compact Hausdorff space is normal) and $f : A \rightarrow \mathbb{R}$ is a continuous map from a closed subset A of X into the real numbers carrying the standard topology, then there exists a continuous map $F : X \rightarrow \mathbb{R}$ with $F(x) = f(x)$ for all $x \in A$. Moreover, F may be chosen such that $\sup\{|f(a)| : a \in A\} = \sup\{|F(x)| : x \in X\}$, i.e., if f is bounded, F may be chosen to be bounded (with the same bound as f). F is called a continuous extension of f .*

Definition 9.1. Let $T : X \rightarrow X$ and $S : Y \rightarrow Y$ be continuous maps of compact (metric) spaces (that is, topological dynamical systems). Then a homeomorphism $\theta : X \rightarrow Y$ with $\theta \circ T = S \circ \theta$ is called a topological conjugacy, and if there such a conjugacy then T and S are topologically conjugate. A continuous surjective map $\varphi : X \rightarrow Y$ with $\varphi \circ T = S \circ \varphi$ is called a topological factor map, and in this case S is said to be a factor of T .

Lemma 9.1. Let $\mathfrak{A} \simeq C(Y)$ be a C^* -subalgebra of $C(X)$, and φ be the continuous surjective from X onto Y induced by the embedding of $C(Y)$ into $C(X)$. Suppose Φ is a homomorphism from $C(X)$ to $C(X)$ such that Φ fixes \mathfrak{A} , i.e., $\Phi(\mathfrak{A}) \subset \mathfrak{A}$. Let $\Psi = \Phi|_{\mathfrak{A}}$. Then Ψ^* is a factor of Φ^* .

Proof. We only need to show that $\varphi \circ \Phi^* = \Psi^* \circ \varphi$. Let ρ in X and $A \in \mathfrak{A}$. Then

$$\varphi \circ \Phi^*(\rho)(A) = \rho(\Phi(A)) = \Psi^* \circ \varphi(\rho)(A).$$

□

Lemma 9.2. Let $T : X \rightarrow X$ and $S : Y \rightarrow Y$ be continuous maps of compact (metric) spaces. Let $\varphi : X \rightarrow Y$ be a topological factor map. Then φ^* is injective. And $\{f \circ \varphi : f \in C(Y)\}$ is invariant under the map $g \rightarrow g \circ T$, $g \in C(X)$.

Lemma 9.3. *Let $T : X \rightarrow X$ and $S : Y \rightarrow Y$ be continuous maps of compact (metric) spaces. If S is a factor of T , then $h_{top}(T) \geq h_{top}(S)$.*

Proof. Let φ be a factor map. Suppose that $\{U_i\}_{i \in I}$ be an open cover of Y . Let $V_i = \varphi^{-1}(U_i)$. Then $\{V_i\}_{i \in I}$ is an open cover of X . Let $U_i^n = S^{-n}(U_i)$ and $V_i^n = T^{-n}(V_i)$. We claim that $V_i^n = \varphi^{-1}(U_i^n)$. Indeed

$$\begin{aligned} V_i^n &= \{x \in X : T^n(x) \in V_i\} \\ &= \{x \in X : \varphi \circ T^n(x) \in U_i\} \\ &= \{x \in X : S^n \circ \varphi(x) \in U_i\} \\ &= \{x \in X : \varphi(x) \in U_i^n\} = \varphi^{-1}(U_i^n). \end{aligned}$$

This implies that

$$\begin{aligned} V_{i_1} \cap V_{i_2}^2 \cap \cdots \cap V_{i_n}^n &= \varphi^{-1}(U_{i_1}) \cap \varphi^{-1}(U_{i_2}^2) \cap \cdots \cap \varphi^{-1}(U_{i_n}^n) \\ &= \varphi^{-1}(U_{i_1} \cap U_{i_2}^2 \cap \cdots \cap U_{i_n}^n). \end{aligned}$$

Thus

$$N\left(\bigvee_{i=0}^m T^{-i}(\{V_i\}_{i \in I})\right) = N\left(\bigvee_{i=0}^m S^{-i}(\{U_i\}_{i \in I})\right).$$

This clearly implies that $h_{top}(T) \geq h_{top}(S)$. \square

Lemma 9.4. *Let $f \in L^\infty(\mathbb{N})$ and (Y, φ) be a topological dynamical system. Assume that $C(f) \simeq C(X)$. If there exists $h \in C(Y)$ such that $f(n) = h(\varphi^{(n)}(x))$, and $\{h \circ \varphi^{(n)} : n = 0, 1, \dots\} \cup \{I\}$ generate $C(Y)$, then there is a continuous injective map from X into Y . Furthermore, the range of the continuous map is the closure of $\{x, \varphi(x), \varphi^{(2)}(x), \dots\}$ in Y .*

Proof. Let Ω be the closure of $\{x, \varphi(x), \varphi^{(2)}(x), \dots\}$ in Y . By the Tietze extension theorem, we have

$$g \rightarrow g|_\Omega, \quad \text{for any } g \in C(Y),$$

is a surjective map from $C(Y)$ to $C(\Omega)$. Therefore $\{h \circ \varphi^{(n)}|_\Omega : n = 0, 1, \dots\} \cup \{I\}$ also generate $C(\Omega)$. It is not hard to see that

$$\Phi : h \circ \varphi^n|_\Omega \rightarrow f \circ \sigma^{(n)}, \quad n = 0, 1, \dots$$

induce a *-isomorphism from $C(\Omega)$ to $C(f) \simeq C(X)$. Then

$$\Phi^* : \rho \rightarrow \rho \circ \Phi,$$

where $\rho \in X$, is a homeomorphism from X to Ω .

Note that

$$\Phi(g \circ \varphi) = \Phi(g) \circ \sigma, \quad g \in C(Y).$$

Therefore, for $g \in C(Y)$ and $\rho \in X$, we have

$$g(\varphi(\Phi^*(\rho))) = g \circ \varphi(\Phi^*(\rho)) = \Phi(g \circ \varphi)(\rho) = (\Phi(g) \circ \sigma)(\rho) = \Phi(g)(\sigma(\rho)) = g(\Phi^*(\sigma(\rho))).$$

This means that $\varphi \circ \Phi^* = \Phi^* \circ \sigma$. \square

Corollary 9.1. *Let $f \in L^\infty(\mathbb{N})$ and (Y, φ) be a topological dynamical system. Assume that $C(f) \simeq C(X)$. If there exists $h \in C(Y)$ such that $f(n) = h(\varphi^{(n)}(x))$, $\{h \circ \varphi^{(n)} : n = 0, 1, \dots\} \cup \{I\}$ generate $C(Y)$ and $\{x, \sigma(x), \sigma^{(2)}(x), \dots\}$ is dense in Y . Then σ and φ are topologically conjugate.*

Lemma 9.5. *Let $f \in L^\infty(\mathbb{N})$ and (Y, φ) be a topological dynamical system. Assume that $C(f) \simeq C(X)$. If there exists $h \in C(Y)$ such that $f(n) = h(\varphi^{(n)}(x))$, then $h_{\text{top}}(\varphi) \geq h_{\text{top}}(\sigma)$.*

Proof. By Variational principle for the topological entropy. \square

Example 9.1 (Theorem 8 in [3]). *Let $\Omega^{(2)} = \{0, 1\}^{\mathbb{N}}$. The square-free flow S is the subflow of the full shift on $\Omega^{(2)}$ by the point $\theta = (\mu^2(1), \mu^2(2), \dots)$, that is $S = (X_S, T)$ with X_S the closure of the T -orbit of θ in $\Omega^{(2)}$.*

- $T : X_S \rightarrow X_S$ is surjective, it is topologically ergodic and $h_{\text{top}}(S) = \frac{6}{\pi^2} \log 2$.
- S is proximal (i.e. for any $x, y \in X_S$, $\inf_{n \geq 1} d(T^{(n)}(x), T^{(n)}(y)) = 0$) and $\{(0, 0, 0, \dots)\}$ is the unique T -minimal subset of X_S .

Remark 9.2. *Let h be the characteristic function on of the subset $\{(a_1, a_2, \dots) : a_1 = 1\}$ of $\Omega^{(2)}$. It is clear that $\{h, h \circ T, h \circ T^{(2)}, \dots\}$ separates points. By Stone–Weierstrass theorem, we have $\{h, h \circ T, h \circ T^{(2)}, \dots\}$ generates the C^* -algebra $C(X_S)$. By corollary 9.1, $C(\mu^2) \simeq C(X_S)$. Let $\rho \in X$ where X is the spectrum of $C(\mu^2)$. Then*

$$\rho \rightarrow (\rho(\mu^2), \rho(\mu^2 \circ \sigma), \rho(\mu^2 \circ \sigma^{(2)}), \dots) \in \Omega^{(2)}$$

is the homeomorphism from X to X_S (X endowed with weak topology and X_S endowed with the product topology).*

In general, we have the following fact.

Proposition 9.1. *Suppose that f is a function in $L^\infty(\mathbb{N})$. Let Ω be the spectrum of M_f , and S be the left shift map : $\Omega^{\mathbb{N}} \rightarrow \Omega^{\mathbb{N}}$ defined by*

$$T(a_1, a_2, \dots) = (a_2, a_3, \dots).$$

Note that $\Omega^{\mathbb{N}}$ is a compact Hausdorff space endowed with the product topology. Then the spectrum of $C(f)$ is homeomorphic to the closure of $\{\omega, T(\omega), T^{(2)}(\omega), \dots\}$ in $\Omega^{\mathbb{N}}$, where $\omega = (f(1), f(2), \dots)$. And the homeomorphism is given by

$$\rho \rightarrow (\rho(f), \rho(f \circ \sigma), \rho(f \circ \sigma^{(2)}), \dots),$$

where ρ is a point in the spectrum of $C(f)$.

Proof. By corollary 9.1, we only need to show that there exists a continuous function $h \in C(\Omega^{\mathbb{N}})$ such that $h(T^{(n)}(\omega)) = f(n)$ and h separates points. It is easy to check that the function given by the projection to the first coordinate, i.e. $h((a_1, a_2, \dots)) = a_1$, satisfies all conditions. \square

Example 9.2. *Let $f(n) = e^{2\pi i \sqrt{n}}$. It is not hard to see that $\{e^{2\pi i \sqrt{n}}\}_{n \in \mathbb{N}}$ is dense in the unit circle S^1 . Indeed, we only need to show that $\{\frac{k}{\sqrt{n^2+k+n}} : k = 0, 1, \dots, 2n, n \in \mathbb{N}\}$ is dense in $[0, 1]$. If $\frac{k}{2n} \rightarrow \theta$, then*

$$\frac{\frac{k}{n}}{\sqrt{1 + \frac{k}{n^2} + 1}} \rightarrow \theta.$$

Let T be the left shift of $(S^1)^{\mathbb{N}}$ and X be the closure of $\{\omega, T(\omega), T^{(2)}(\omega), \dots\}$ in $(S^1)^{\mathbb{N}}$, where $\omega = (e^{2\pi i}, e^{2\pi i \sqrt{2}}, e^{2\pi i \sqrt{3}}, \dots)$. We claim that The accumulation points of $\{\omega, T(\omega), T^{(2)}(\omega), \dots\}$ in $(S^1)^{\mathbb{N}}$ is

$$\{(e^{2\pi i \theta}, e^{2\pi i \theta}, e^{2\pi i \theta}, \dots) : \theta \in [0, 1)\}.$$

And the closure of $\{\omega, T(\omega), T^{(2)}(\omega), \dots\}$ is

$$\{\omega, T(\omega), T^{(2)}(\omega), \dots\} \cup \{(e^{2\pi i\theta}, e^{2\pi i\theta}, e^{2\pi i\theta}, \dots) : \theta \in [0, 1)\}.$$

Proof of the claim. Let $x = \{(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}, e^{2\pi i\theta_3}, \dots) : \theta \in [0, 1)\}$ be an accumulation point of $\{\omega, T(\omega), T^{(2)}(\omega), \dots\}$. Since $(S^1)^\mathbb{N}$ is a compact metric space, there exists a sequence of nature number $n_k, k = 1, 2, \dots$ such that $T^{(n_k)}(\omega) \rightarrow x$. Assume that $e^{2\pi i\sqrt{n_k}} \rightarrow e^{2\pi i\theta}$, then

$$e^{2\pi i\sqrt{n_k+l}} \rightarrow e^{2\pi i\theta}, \quad l \in \mathbb{N}.$$

This implies that $T^{(n_k)} \rightarrow (e^{2\pi i\theta}, e^{2\pi i\theta}, e^{2\pi i\theta}, \dots)$. \square

We quote the following fact in [8].

Lemma 9.6 (Lemma 4.2 in [8]). *Let Y be a compact metric space and $S : Y \rightarrow Y$ a continuous mapping. If $C \subset Y$ is a closed S -invariant subset of Y and $Y \setminus C$ is countable and S -invariant, then $h_{\text{top}}(S) = h_{\text{top}}(S|_C)$.*

Proposition 9.2. *The topological entropy of the left shift T of the spectrum of $C(f)$ is 0, where $f(n) = e^{2\pi i\sqrt{n}}$.*

Proof. Let $C = \{(e^{2\pi i\theta}, e^{2\pi i\theta}, e^{2\pi i\theta}, \dots) : \theta \in [0, 1)\}$. By lemma 9.6, we have $h_{\text{top}}(T) = h_{\text{top}}(T|_C)$. It is clear that $h_{\text{top}}(T|_C) = 0$ since $T|_C$ is identity. \square

Definition 9.2. *A sequence $x_n = (x_n^1, \dots, x_n^k) \in \mathbb{R}^k$ is said to be uniformly distributed mod 1 if, for each choice of k intervals $[a_1, b_1], \dots, [a_k, b_k] \subset [0, 1)$, we have that*

$$\frac{1}{n} \sum_{j=1}^n \prod_{i=1}^k \chi_{[a_i, b_i]}(\{x_j^i\}) \rightarrow \prod_{i=1}^k (b_i - a_i), \quad \text{as } n \rightarrow \infty.$$

Theorem 9.2 (Weyl's Criterion). *The sequence $x_n \in \mathbb{R}^k$ is uniformly distributed mod 1 if and only if*

$$\frac{1}{n} \sum_{j=1}^n e^{2\pi i(l_1 x_j^1 + l_2 x_j^2 + \dots + l_k x_j^k)} \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

for all $l = (l_1, \dots, l_k) \in \mathbb{Z}^k \setminus \{0\}$.

Theorem 9.3 (Weyl). *Let $p(n) = a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n + a_0$. If any a_1, a_2, \dots, a_k is irrational, then*

$$\frac{1}{n} \sum_{j=1}^n e^{2\pi i p(j)} \rightarrow 0$$

as $n \rightarrow \infty$.

Corollary 9.2. *Let $p(n) = \theta n^k$ such that θ is irrational. Then $\{(p(n), p(n+1), \dots, p(n+k-1))\}_{n=1}^\infty$ is uniformly distributed mod 1.*

Proof. Assumet that

$$\frac{1}{n} \sum_{j=1}^n e^{2\pi i\theta(l_0 n^k + \dots + l_{k-1} (n+k-1)^k)} \rightarrow 0$$

where $(l_0, \dots, l_{k-1}) \in \mathbb{Z}^k$. By theorem 9.3, we have

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & \cdots & k-1 \\ 0 & 1 & 2^2 & \cdots & (k-1)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 2^{k-1} & \cdots & (k-1)^{k-1} \end{pmatrix} \begin{pmatrix} l_0 \\ l_1 \\ l_2 \\ \vdots \\ l_{k-1} \end{pmatrix} = 0.$$

This implies that $(l_0, \dots, l_{k-1}) = (0, \dots, 0)$. \square

Example 9.3. Let $f(n) = e^{2\pi i n^2 \theta}$. Let T be the left shift of $(S^1)^\mathbb{N}$ and X be the closure of $\{\omega, T(\omega), T^2(\omega), \dots\}$, where $\omega = (f(1), f(2), \dots)$. By corollary 9.2,

$$\{(\theta n^2, \theta(n+1)^2) : n \in \mathbb{N}\}$$

is uniformly distributed mod 1. This implies that $X \simeq \mathbb{T}^2$, the two dimensional torus $\mathbb{R}^2 / \mathbb{Z}^2$. Indeed, for any $(e^{2\pi i \alpha_1}, e^{2\pi i \alpha_2}) \in \mathbb{T}^2$, there is a subsequence n_k such that

$$\theta n_k^2 \rightarrow \alpha_1 \pmod{1} \text{ and } \theta(n_k + 1)^2 \rightarrow \alpha_2 \pmod{1}$$

as $k \rightarrow \infty$. This implies that

$$2\theta n_k \rightarrow \alpha_2 - \alpha_1 - \theta \pmod{1}.$$

Therefore

$$(n_k + m)^2 \rightarrow \alpha_1 + m(\alpha_2 - \alpha_1 - \theta) + m^2 \theta \pmod{1}, \text{ for any } m \in \mathbb{N}.$$

And the homomorphism from X onto \mathbb{T}^2 is the projection onto the first two coordinates. Now, consider the map induced by the left shift on $(S^1)^\mathbb{N}$. Note that

$$(e^{2\pi i \theta n^2}, e^{2\pi i \theta(n+1)^2}) \rightarrow (e^{2\pi i \theta(n+1)^2}, e^{2\pi i \theta(n+2)^2}).$$

Therefore the map is

$$\varphi : (e^{2\pi i \alpha_1}, e^{2\pi i \alpha_2}) \rightarrow (e^{2\pi i \alpha_2}, e^{2\pi i (2\alpha_2 - \alpha_1 + 2\theta)}).$$

If we identifies \mathbb{T}^2 with $[0, 1) \times [0, 1)$, then

$$\varphi((\alpha_1, \alpha_2)) = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2\theta \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}.$$

By induction, we have

$$A^n = \begin{pmatrix} 1-n & n \\ -n & 1+n \end{pmatrix}.$$

From this, it is easy to see that φ is distal.

10. AF ALGEBRAS

Definition 10.1. Let \mathfrak{A} be a unital C^* -algebra and $\text{Inn}(\mathfrak{A}) = \{\text{Ad}U : U \in \mathcal{U}(\mathfrak{A})\}$. A automorphism α on \mathfrak{A} is called approximately inner if, for every finite subset F of \mathfrak{A} and for every $\varepsilon > 0$, there is a unitary U such that $\|\alpha(A) - U^*AU\| < \varepsilon$ for all $A \in F$.

Remark 10.1. One can check that if \mathfrak{A} is separable and if α is an automorphism on \mathfrak{A} , then α is approximately inner if and only if there exists a sequence $\{U_n\}$ in $\mathcal{U}(\mathfrak{A})$ such that

$$\lim_{n \rightarrow \infty} U_n^* A U_n = \alpha(A), \forall A \in \mathfrak{A}.$$

Definition 10.2. Let \mathfrak{A} be a unital separable C^* -algebra. Denote by $\text{AInn}(A)$ the group of all asymptotically inner automorphisms. An automorphism α is said to be strongly asymptotically inner if there is a continuous path of unitaries $\{U(t) : t \in [0, \infty)\}$ of \mathfrak{A} such that

$$U(0) = I \text{ and } \lim_{t \rightarrow \infty} U(t)^* A U(t) = \alpha(A) \text{ for all } A \in \mathfrak{A}.$$

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