Minimal Generating Reflexive Lattices of Projections in Finite von Neumann Algebras

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Abstract We show that the reflexive lattice generated by a double triangle lattice of projections in a finite von Neumann algebra is topologically homeomorphic to the two-dimensional sphere S^2 (plus two distinct points corresponding to zero and I). Furthermore, such a reflexive lattice is in general minimally generating for the von Neumann algebra it generates. As an application, we show that if a reflexive lattice $\mathcal{F}$ generates the algebra $M_n(\mathbb{C})$ of all $n \times n$ complex matrices, for some $n \geq 3$, then $\mathcal{F} \setminus \{0, I\}$ is connected if and only if it is homeomorphic to S^2.

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MSC(2010) [1] 46L10, 47L75, 47L60

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1 Introduction

In their seminal article [11], Kadison and Singer initiated the study of non selfadjoint algebras of bounded linear operators on Hilbert spaces. They introduced and studied a class of operator algebras which they called triangular (operator) algebras. An algebra $\mathcal{T}$ is said to be triangular in a factor $\mathcal{M}$ (a von Neumann algebra with a trivial center) when its diagonal subalgebra $\mathcal{T} \cap \mathcal{T}^*$ is a maximal abelian selfadjoint subalgebra of $\mathcal{M}$. Nest algebras were introduced by Ringrose ([16]) as generalizations of certain triangular algebras. Reflexive (operator) algebras are more general than nest algebras ([7]). Let $\mathcal{H}$ be a complex Hilbert space and $B(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. By a nest $\mathcal{N}$ we mean a totally ordered family of (orthogonal) projections on a Hilbert space $\mathcal{H}$ containing 0 and the identity operator I which is closed in the strong operator topology. The nest algebra associated with $\mathcal{N}$ is the set of all operators in $B(\mathcal{H})$ that leave the range of each projection in $\mathcal{N}$ invariant. A subalgebra $\mathfrak{A}$ of $B(\mathcal{H})$ is called reflexive if $\mathfrak{A}$ is equal to the algebra of all operators in $B(\mathcal{H})$ which leave invariant the range of each projection in a family of projections in $B(\mathcal{H})$. Similarly, a lattice of

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projections in $B(\mathcal{H})$ is called reflexive if it is equal to the set of all projections (with their ranges) invariant under each operator in a family of operators in $B(\mathcal{H})$. Hence reflexive algebras are completely determined by their lattices of invariant projections. In general, non selfadjoint operator algebras are closely related to operator theory and invariant subspaces of operators. Parallel to the selfadjoint theory (C^*- and von Neumann algebras), non selfadjoint theory has undergone a vigorous development in the past 50 years. Many important results were obtained by many authors, e.g., Arveson ([1]), Larson ([13]), Davidson ([3]) and Lance ([12]).

Recently, motivated by the work of Kadison and Singer on maximal triangular algebras, Ge and Yuan ([5]) introduced a new class of non selfadjoint algebras which they called Kadison-Singer algebras, or KS-algebras for simplicity. KS-algebras combine triangularity ([11]), reflexivity ([7, 3]) and von Neumann algebra properties into one package. Recall that a subalgebra $\mathfrak{A}$ of $B(\mathcal{H})$ is called a KS-algebra if $\mathfrak{A}$ is reflexive and maximal with respect to the diagonal subalgebra $\mathfrak{A}\cap\mathfrak{A}^*$ of $\mathfrak{A}$, in the sense that if there is another reflexive subalgebra $\mathcal{B}$ of $B(\mathcal{H})$ such that $\mathfrak{A}\subseteq\mathcal{B}$ and $\mathcal{B}\cap\mathcal{B}^*=\mathfrak{A}\cap\mathfrak{A}^*$, then $\mathfrak{A}=\mathcal{B}$. The lattice of all invariant projections of a KS-algebra is called a KS-lattice. Equivalently, a reflexive lattice $\mathcal{L}$ of projections in $B(\mathcal{H})$ is called a KS-lattice if $\mathcal{L}$ is a minimally generating reflexive lattice of the von Neumann algebra it generates, which means that $\mathcal{L}$ and any proper reflexive sublattice of $\mathcal{L}$ do not generate the same von Neumann subalgebra of $B(\mathcal{H})$. Hence the study of KS-lattices is closely related to the generator problem for von Neumann algebras ([4]).

In [5, 6], Ge and Yuan constructed KS-algebras with hyperfinite factors as their diagonals. They also proved that the reflexive lattice determined by three free projections with trace $\frac{1}{2}$ is homeomorphic to the two-dimensional sphere S^2 (plus two distinct points corresponding to zero and I). It is easy to see that the lattice generated by three free projections with trace $\frac{1}{2}$ is a double triangle lattice, which means that it is a five-element lattice containing three nontrivial elements such that the intersection of any two is zero and the union of any two is I. In [6], the authors claimed that if a double triangle lattice of projections in a type II_1 factor satisfies some specified conditions, then the reflexive lattice generated by it is also homeomorphic to S^2 (plus zero and I). Hence, for these cases, Halmos's problem, which asks whether any realization of a double triangle lattice is reflexive, was settled. These results also indicate that many type II_1 factors can be minimally generated by reflexive lattices of projections which are topologically homeomorphic to S^2 (plus zero and I). Some examples of KS-lattices with different topological structures were also given in [8, 19].

In this paper, we study the reflexive lattice generated by a double triangle lattice of projections in a finite von Neumann algebra. Our methods are different from those in [6] where heavy free probability techniques were involved. In the section following the introduction, by using the theory of unbounded operators affiliated with a von Neumann algebra, we show that the reflexive lattice generated by any double triangle lattice of projections in a finite von Neumann algebra is topologically homeomorphic to S^2 (plus zero and I). In particular, we prove that each nontrivial projection in such a reflexive lattice has trace $\frac{1}{2}$, and that such a reflexive lattice is a KS-lattice if the von Neumann algebra generated by the double triangle lattice cannot be generated by any two nontrivial projections. In Section 3, we apply our results in Section 2 to describe the "connected" reflexive lattices in matrix algebras. We show that if a reflexive lattice $\mathcal{F}$ generates the matrix algebra $M_n(\mathbb{C})$ for $n \geq 3$, and $\mathcal{F} \setminus \{0, I\}$ is connected, then $\mathcal{F} \setminus \{0, I\}$ is topologically homeomorphic to S^2. Moreover, $\mathcal{F}$ is a KS-lattice.

2 Three projections in finite von Neumann algebras

We first recall some definitions in the theory of non selfadjoint operator algebras. For basic facts on operator algebras, we refer to [9, 15].

Suppose $\mathcal{H}$ is a separable complex Hilbert space and $B(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. For a set $\mathcal{L}$ of orthogonal projections in $B(\mathcal{H})$, we denote by $Alg\,\mathcal{L}$ the set of all bounded linear operators on $\mathcal{H}$ leaving (the range of) each projection in $\mathcal{L}$ invariant. Then $Alg\,\mathcal{L}$ is a unital, weak-operator closed subalgebra of $B(\mathcal{H})$. Similarly, for a subset $\mathcal{S}$ of $B(\mathcal{H})$, let $Lat\,\mathcal{S}$ be the set of all projections (with their ranges) invariant under each operator in $\mathcal{S}$. Then $Lat\,\mathcal{S}$ is a strong-operator closed lattice of projections. A subalgebra $\mathcal{A}$ of $B(\mathcal{H})$ is said to be reflexive if $\mathcal{A} = Alg\,Lat\,\mathcal{A}$. Similarly, a lattice $\mathcal{L}$ of projections in $B(\mathcal{H})$ is called reflexive if $\mathcal{L} = Lat\,Alg\,\mathcal{L}$.

Throughout this section, $\mathcal{M}$ denotes a finite von Neumann algebra acting on $\mathcal{H}$, and tr is a faithful, normal trace on $\mathcal{M}$. Let $\mathfrak{A} = M_2(\mathbb{C}) \otimes \mathcal{M}$. Then $\mathfrak{A}$ is a finite von Neumann algebra acting on $\mathcal{H} \oplus \mathcal{H}$. Let $\{E_{i,j}\}_{i,j=1}^{2}$ be the standard matrix unit system in $M_2(\mathbb{C})$. Each operator T in $B(\mathcal{H} \oplus \mathcal{H})$ can be expressed as an operator matrix with respect to the units:

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \quad T_{ij} \in B(\mathcal{H}).$$

Then the state τ, defined by $\tau(T) = \frac{1}{2}(tr(T_{11}) + tr(T_{22}))$ for $T \in \mathfrak{A}$, is a faithful, normal trace on $\mathfrak{A}$. We use the same symbol I to denote the identity operator in $\mathcal{M}$ and in $\mathfrak{A}$ when there is no risk of ambiguity. Let $P_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ $(= E_{11} \otimes I)$ in $\mathfrak{A}$. Now we state our first lemma, the proof of which might be well-known.

Lemma 2.1 *With the above notation, let P be a projection in $\mathfrak{A}$ such that $P \vee P_1 = I$ and $P \wedge P_1 = 0$. Then there are a unitary operator V in $\mathcal{M}$ and a positive contractive operator H in $\mathcal{M}$ such that $I - H$ is injective and*

$$P = \begin{pmatrix} H & \sqrt{H(I-H)}V \\ V^*\sqrt{H(I-H)} & V^*(I-H)V \end{pmatrix}.$$

Furthermore, $\tau(P) = \frac{1}{2}$.

Proof By Kaplansky formula ([9, Theorem 6.1.7]), we have $\tau(P_1 \vee P) = \tau(P_1) + \tau(P) - \tau(P_1 \wedge P)$. Thus $\tau(P) = \frac{1}{2}$. Let us write formally

$$P = \begin{pmatrix} H & H_1 V \\ V^* H_1 & H_2 \end{pmatrix},$$

where H, H_1 and H_2 are positive operators in $\mathcal{M}$, and V is a unitary operator in $\mathcal{M}$. By $P^2 = P$, we have

$$H = H^2 + H_1^2, \quad H_2 = V^* H_1^2 V + H_2^2 \quad \text{and} \quad H_1 V = H H_1 V + H_1 V H_2.$$

Hence we have

$$H_1 = \sqrt{H(I-H)}, \tag{1}$$

$$V^* H(I-H)V = H_2(I-H_2), \tag{2}$$

$$(I-H)\sqrt{H(I-H)}V = \sqrt{H(I-H)}VH_2. \tag{3}$$

Now we show that $I - H$ and H_2 are injective. Indeed, for each vector ξ in $\mathcal{H}$ with $(I-H)\xi = 0$, we have $\xi = H\xi$ and $\sqrt{I-H}\xi = 0$, which imply $\xi \oplus 0 \in P_1 \wedge P$. Hence $\xi = 0$, and thus, $I - H$ is injective. Similarly, for each vector η in $\mathcal{H}$ with $H_2\eta = 0$, we have $H_1 V\eta = \sqrt{H(I-H)}V\eta = 0$ by (1) and (3). Thus $0 \oplus \eta \in P_1^\perp \wedge P^\perp$, which implies $\eta = 0$. Hence H_2 is injective.

Next we show that $H_2 = V^*(I-H)V$. Clearly, (3) and the injectivity of $I - H$ imply that $H(I-H)V = HVH_2$. Hence $V^*H(I-H)V = V^*HVH_2$, which, together with (2), gives us that $V^*HVH_2 = H_2(I-H_2)$. Therefore, by the injectivity of H_2, we have $H_2 = V^*(I-H)V$. This completes the proof of our lemma.

Before we proceed to the proof of our main result, we recall some basic facts on unbounded operators affiliated with a finite von Neumann algebra.

For a closed, densely defined operator T, we denote its domain, range, null space, range projection and null projection by $\mathcal{D}(T)$, $Ran(T)$, $Ker(T)$, $R(T)$ and $N(T)$, respectively. A dense subspace $\mathcal{D}_0$ of $\mathcal{D}(T)$ is called a core for T if $\mathcal{G}(T|_{\mathcal{D}_0})^- = \mathcal{G}(T)$, where $T|_{\mathcal{D}_0}$ is the restriction of T to $\mathcal{D}_0$, $\mathcal{G}(T)$ is the graph of T and $\mathcal{G}(T|_{\mathcal{D}_0})^-$ is the closure of $\mathcal{G}(T|_{\mathcal{D}_0})$. We say that the operator T is affiliated with a von Neumann algebra $\mathcal{R}$ when $U^*TU = T$ for each unitary operator U commuting with $\mathcal{R}$. In [14], Murray and von Neumann showed that the family of all operators affiliated with a finite von Neumann algebra $\mathcal{M}$ forms an associative algebra, denoted by $\widetilde{\mathcal{M}}$. For arbitrary operators X and Y in $\widetilde{\mathcal{M}}$, $X+Y$ and XY are densely defined and closable. Their closures, denoted by $X\hat{+}Y$ and $X\hat{\cdot}Y$, respectively, are in $\widetilde{\mathcal{M}}$. If $\xi \in \mathcal{D}(X\hat{+}Y) \cap \mathcal{D}(X)$, then $\xi \in \mathcal{D}((X\hat{+}Y) - X)$. Since $\widetilde{\mathcal{M}}$ is an algebra and $(X\hat{+}Y)\hat{-}X = Y$, we have $\xi \in \mathcal{D}(Y)$ and $(X\hat{+}Y)\xi = X\xi + Y\xi$. Moreover, $Ker(X\hat{+}Y) = 0$ if and only if $Ker(X+Y) = 0$. Indeed, if $Ker(X+Y) = 0$ and $Ker(X\hat{+}Y) \neq 0$, then there exists a projection Q in $\mathcal{M}$ such that $tr(Q) > 0$, and for any ξ in $Q(\mathcal{H})$, $(X\hat{+}Y)\xi = 0$. Hence we can choose two projections E and F in $\mathcal{M}$ such that $E(\mathcal{H}) \subset \mathcal{D}(X)$, $F(\mathcal{H}) \subset \mathcal{D}(Y)$ and $tr(E \wedge F \wedge Q) > 0$. This means that we can find a nonzero vector ξ in $(E \wedge F \wedge Q)\mathcal{H}$, which contradicts the fact that $Ker(X+Y) = 0$. For more details on unbounded operators, we refer to [10, Exercises 6.9.54 and 8.7.60] and [14].

Lemma 2.2 *Let P_1 be as before, P_2 and P_3 be two projections in $\mathfrak{A}$ such that $P_1 \wedge P_i = 0$ and $P_1 \vee P_i = I$ for $i = 2, 3$. We assume*

$$P_2 = \begin{pmatrix} H_1 & \sqrt{H_1(I-H_1)}V_1 \\ V_1^*\sqrt{H_1(I-H_1)} & V_1^*(I-H_1)V_1 \end{pmatrix} \quad \text{and}$$

$$P_3 = \begin{pmatrix} H_2 & \sqrt{H_2(I-H_2)}V_2 \\ V_2^*\sqrt{H_2(I-H_2)} & V_2^*(I-H_2)V_2 \end{pmatrix},$$

where V_i is a unitary operator and H_i is a positive contractive operator in $\mathcal{M}$ such that $Ker(I-H_i) = 0$ for $i = 1, 2$. Then

(i) $P_2 \wedge P_3 = 0$ if and only if $Ker(\sqrt{H_1(I-H_1)^{-1}}V_1 - \sqrt{H_2(I-H_2)^{-1}}V_2) = 0$;

(ii) $P_2 \vee P_3 = I$ if and only if $Ker(V_1^*\sqrt{H_1(I-H_1)^{-1}} - V_2^*\sqrt{H_2(I-H_2)^{-1}}) = 0$.

Proof (i) If $Ker(\sqrt{H_1(I-H_1)^{-1}}V_1 - \sqrt{H_2(I-H_2)^{-1}}V_2) \neq 0$, then there is a nonzero vector ξ in $\mathcal{D}(\sqrt{H_1(I-H_1)^{-1}}V_1) \cap \mathcal{D}(\sqrt{H_2(I-H_2)^{-1}}V_2)$ such that

$$\sqrt{H_1(I-H_1)^{-1}}V_1\xi = \sqrt{H_2(I-H_2)^{-1}}V_2\xi \stackrel{\text{def}}{=} \eta.$$

Then $\sqrt{H_i(I-H_i)}\eta = H_iV_i\xi$ for $i = 1, 2$. So $\eta \oplus \xi \in P_2 \wedge P_3$, which yields $P_2 \wedge P_3 \neq 0$. We can reverse the above argument to obtain the other direction.

Note that $P_i \vee P_j = I$ if and only if $(I - P_i) \wedge (I - P_j) = 0$, for $i, j = 1, 2, 3$ and $i \neq j$. We can obtain (ii) by applying a similar argument to $I - P_1$, $I - P_2$ and $I - P_3$.

Remark 2.1 *In the lemma above, we may assume that $V_1 = I$ by changing P_2 to U^*P_2U, and P_3 to U^*P_3U, where $U = \begin{pmatrix} I & 0 \\ 0 & V_1^* \end{pmatrix}$.*

From now on, let P_1 be as before, P_2 and P_3 be two projections in $\mathfrak{A}$ such that $P_i \wedge P_j = 0$ and $P_i \vee P_j = I$, for $i, j = 1, 2, 3$ and $i \neq j$. By Lemma 2.1, we have $\tau(P_i) = \frac{1}{2}$ for $i = 1, 2, 3$. Let $\mathcal{L} = \{0, P_1, P_2, P_3, I\}$. Then $\mathcal{L}$ is a double triangle lattice of projections in $\mathfrak{A}$. In the following, we study the reflexive lattice determined by $\mathcal{L}$, as well as the corresponding reflexive algebra. By Lemma 2.2 and its remark, we may assume

$$P_2 = \begin{pmatrix} H_1 & \sqrt{H_1(I-H_1)} \\ \sqrt{H_1(I-H_1)} & I-H_1 \end{pmatrix} \quad \text{and}$$

$$P_3 = \begin{pmatrix} H_2 & \sqrt{H_2(I-H_2)}V \\ V^*\sqrt{H_2(I-H_2)} & V^*(I-H_2)V \end{pmatrix},$$

where H_i is a positive contractive operator in $\mathcal{M}$ such that $Ker(I-H_i) = 0$ for $i = 1, 2$, and V is a unitary operator in $\mathcal{M}$. Let

$$S = \sqrt{H_1(I-H_1)^{-1}} \,\widehat{-}\, \sqrt{H_2(I-H_2)^{-1}}V.$$

Then, by Lemma 2.2 and the discussion before it, we know that S is in general an unbounded, invertible operator affiliated with $\mathcal{M}$, and its inverse is also in $\widetilde{\mathcal{M}}$.

Lemma 2.3 *With the above notation, suppose* $T = \begin{pmatrix} T_1 & T_2 \\ T_4 & T_3 \end{pmatrix} \in M_2(\mathbb{C}) \otimes B(\mathcal{H})$. *Then* T *is in* $Alg\,\mathcal{L}$ *if and only if* $T_4 = 0$ *and the following equations hold:*

$$\sqrt{I-H_1}T_2\sqrt{I-H_1} = \sqrt{H_1}T_3\sqrt{I-H_1} - \sqrt{I-H_1}T_1\sqrt{H_1}, \tag{4}$$

$$\sqrt{I-H_2}T_2V^*\sqrt{I-H_2} = \sqrt{H_2}VT_3V^*\sqrt{I-H_2} - \sqrt{I-H_2}T_1\sqrt{H_2}. \tag{5}$$

Proof Note that $P_2 = W_2^*P_1W_2$ and $P_3 = W_3^*P_1W_3$, where

$$W_2 = \begin{pmatrix} \sqrt{H_1} & \sqrt{I-H_1} \\ \sqrt{I-H_1} & -\sqrt{H_1} \end{pmatrix} \quad \text{and} \quad W_3 = \begin{pmatrix} \sqrt{H_2} & \sqrt{I-H_2}V \\ \sqrt{I-H_2} & -\sqrt{H_2}V \end{pmatrix}$$

are unitary operators. Hence T is in $Alg\,\mathcal{L}$ if and only if $(I-P_1)TP_1 = (I-P_1)W_2TW_2^*P_1 = (I-P_1)W_3TW_3^*P_1 = 0$. A straightforward computation shows that (4),(5) and $T_4 = 0$ are equivalent to these equalities.

Lemma 2.4 *With the above notation, if* $T \in Alg\,\mathcal{L}$, *then there is an operator* T_1 *in* $B(\mathcal{H})$ *such that*

$$T = \begin{pmatrix} T_1 & \sqrt{H_1(I-H_1)^{-1}}S^{-1}T_1S - T_1\sqrt{H_1(I-H_1)^{-1}} \\ 0 & S^{-1}T_1S \end{pmatrix}.$$

Conversely, if $T_1 \in B(\mathcal{H})$ *such that* $\sqrt{H_1(I-H_1)^{-1}}S^{-1}T_1S - T_1\sqrt{H_1(I-H_1)^{-1}} = T_2$ *and* $S^{-1}T_1S = T_3$ *for some bounded operators* T_2 *and* T_3, *then* T *belongs to* $Alg\,\mathcal{L}$.

Before the proof of Lemma 2.4, we note that the equalities $\sqrt{H_1(I-H_1)^{-1}}S^{-1}T_1S - T_1\sqrt{H_1(I-H_1)^{-1}} = T_2$ and $S^{-1}T_1S = T_3$ in the lemma hold in the sense that, for each vector ξ in $\mathcal{D}(\sqrt{H_1(I-H_1)^{-1}}) \cap \mathcal{D}(S)$ $(= \mathcal{D}(\sqrt{H_1(I-H_1)^{-1}}) \cap \mathcal{D}(\sqrt{H_2(I-H_2)^{-1}}V))$, $S^{-1}T_1S\xi$ and $\sqrt{H_1(I-H_1)^{-1}}S^{-1}T_1S\xi - T_1\sqrt{H_1(I-H_1)^{-1}}\xi$ are defined and equal to $T_3\xi$ and $T_2\xi$, respectively.

Proof By Lemma 2.3, $T \in Alg\,\mathcal{L}$ if and only if T has the form $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$, where T_1, T_2 and T_3 satisfy (4) and (5). Since $\mathcal{D}(\sqrt{H_1(I-H_1)^{-1}}) \cap \mathcal{D}(\sqrt{H_2(I-H_2)^{-1}}V)$ is a common core for both $\sqrt{H_1(I-H_1)^{-1}}$ and $\sqrt{H_2(I-H_2)^{-1}}V$, we conclude that (4) and (5) are true if and only if, for each vector ξ in $\mathcal{D}(\sqrt{(I-H_1)^{-1}}) \cap \mathcal{D}(\sqrt{(I-H_2)^{-1}}V)$,

$$\begin{aligned} T_2\xi &= \sqrt{H_1(I-H_1)^{-1}}T_3\xi - T_1\sqrt{H_1(I-H_1)^{-1}}\xi \qquad \text{and} \\ T_2\xi &= \sqrt{H_2(I-H_2)^{-1}}VT_3\xi - T_1\sqrt{H_2(I-H_2)^{-1}}V\xi. \end{aligned}$$

These are equivalent to

$$T_2\xi = \sqrt{H_1(I-H_1)^{-1}}T_3\xi - T_1\sqrt{H_1(I-H_1)^{-1}}\xi \quad \text{and} \quad T_3\xi = S^{-1}T_1S\xi.$$

This completes our proof.

Remark 2.2 *Actually, there exists a family* $\{E_\epsilon : \epsilon > 0\}$ *of projections in* $\mathcal{M}$ *such that* $S^{-1}E_\epsilon$, $E_\epsilon S$, $E_\epsilon\sqrt{H_1(I-H_1)^{-1}}$ *and* $\sqrt{H_1(I-H_1)^{-1}}S^{-1}E_\epsilon$ *are bounded for each* ϵ, *and* $\{E_\epsilon\}$ *is convergent to* I *under the strong operator topology (as* $\epsilon \to 0$*). Hence the set of all operators* T_1 *that satisfy the conditions in Lemma 2.4 is dense in* $B(\mathcal{H})$ *under the strong operator topology.*

Corollary 2.1 *For Q in $Lat\,Alg\,\mathcal{L} \setminus \{0, I, P_1\}$, we have $Q \wedge P_1 = 0$, $Q \vee P_1 = I$ and $\tau(Q) = \frac{1}{2}$. Moreover, for any two distinct nontrivial projections Q_1 and Q_2 in $Lat\,Alg\,\mathcal{L}$, we have $Q_1 \wedge Q_2 = 0$ and $Q_1 \vee Q_2 = I$.*

Proof For any given Q as in the corollary, let $Q \wedge P_1 = \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}$, where P is a projection in $\mathcal{M}$. Since $Q \wedge P_1 \in Lat\,Alg\,\mathcal{L}$, we have that P is invariant under $P_1TP_1|_{\mathcal{H}}$ for all T in $Alg\,\mathcal{L}$. By Remark 2.2, $P = 0$ or I. Thus $Q \wedge P_1 = 0$ or P_1. Similarly, we have $Q \vee P_1 = P_1$ or I. It follows that $Q \wedge P_1 = 0$, $Q \vee P_1 = I$ and $\tau(Q) = \frac{1}{2}$.

For any distinct projections Q_1 and Q_2 in $Lat\,Alg\,\mathcal{L}\setminus\{0, I, P_1\}$, we have $Q_i \wedge P_1 = 0$, $Q_i \vee P_1 = I$ and $\tau(Q_i) = \frac{1}{2}$ for $i = 1, 2$. Clearly, $Q_1 \wedge Q_2 \neq P_1$. Suppose that $Q_1 \wedge Q_2 \neq 0$. Since $Q_1 \wedge Q_2 \in Lat\,Alg\,\mathcal{L} \setminus \{0, I\}$, we have $\tau(Q_1 \wedge Q_2) = \frac{1}{2}$. This implies that $Q_1 \wedge Q_2 = Q_1 = Q_2$, which contradicts the fact that $Q_1 \neq Q_2$. Hence $Q_1 \wedge Q_2 = 0$ and $Q_1 \vee Q_2 = I$.

Proposition 2.1 *For any projection Q in $Lat\,Alg\,\mathcal{L} \setminus \{0, I, P_1\}$, there are K_a and U_a in $\mathcal{M}$ such that*

$$Q = \begin{pmatrix} K_a & \sqrt{K_a(I - K_a)}U_a \\ U_a^*\sqrt{K_a(I - K_a)} & U_a^*(I - K_a)U_a \end{pmatrix},$$

where K_a and U_a are determined by the polar decomposition $\sqrt{K_a(I - K_a)^{-1}}U_a$ of $aS\hat{+}\sqrt{H_1(I - H_1)^{-1}}$ for some scalar a in $\mathbb{C}$. Conversely, for any given a in $\mathbb{C}$, the polar decomposition determines U_a and K_a, which give rise to a projection Q (in the above form) in $Lat\,Alg\,\mathcal{L}$.

Proof Suppose Q is in $Lat\,Alg\,\mathcal{L}\setminus\{0, I, P_1\}$. From Corollary 2.1, we know that $Q \wedge P_1 = 0$ and $Q \vee P_1 = I$. It follows from Lemma 2.1 that Q has the form

$$Q = \begin{pmatrix} K & \sqrt{K(I - K)}U \\ U^*\sqrt{K(I - K)} & U^*(I - K)U \end{pmatrix},$$

where K is a positive contractive operator in $\mathcal{M}$ such that $Ker(I - K) = 0$, and U is a unitary operator in $\mathcal{M}$. Let $\{E_\epsilon\}_{\epsilon>0}$ be an increasing net (as $\epsilon \to 0$) of projections in $\mathcal{M}$ with the strong-operator limit I such that, for each ϵ, $E_\epsilon\sqrt{H_1(I - H_1)^{-1}}$, $S^{-1}E_\epsilon$, $\sqrt{H_1(I - H_1)^{-1}}S^{-1}E_\epsilon$, $E_\epsilon S$ and $(SU^*\sqrt{I - K})^{-1}E_\epsilon$ are all bounded operators. Then, for each operator A in $B(\mathcal{H})$ with $E_\epsilon A = AE_\epsilon = A$, we have $S^{-1}AS$, $\sqrt{H_1(I - H_1)^{-1}}S^{-1}AS$ and $A\sqrt{H_1(I - H_1)^{-1}}$ are bounded operators. By Lemma 2.4,

$$T = \begin{pmatrix} A & \sqrt{H_1(I - H_1)^{-1}}S^{-1}AS - A\sqrt{H_1(I - H_1)^{-1}} \\ 0 & S^{-1}AS \end{pmatrix} \in Alg\,\mathcal{L}.$$

Exactly as in the proof of Lemma 2.3, the equation $(I - Q)TQ = 0$ implies that

$$\begin{aligned} &\sqrt{I - K}A\sqrt{K} + \sqrt{I - K}\left[\sqrt{H_1(I - H_1)^{-1}}S^{-1}AS\right. \\ &\qquad \left. - A\sqrt{H_1(I - H_1)^{-1}}\right]U^*\sqrt{I - K} - \sqrt{K}US^{-1}ASU^*\sqrt{I - K} = 0. \end{aligned}$$

This is equivalent to

$$\begin{aligned}&\sqrt{I-K}A\left[\sqrt{K}-E_\epsilon\sqrt{H_1(I-H_1)^{-1}}U^*\sqrt{I-K}\right]\\&\qquad=\left[\sqrt{K}US^{-1}E_\epsilon-\sqrt{I-K}\sqrt{H_1(I-H_1)^{-1}}S^{-1}E_\epsilon\right]ASU^*\sqrt{I-K}.\end{aligned}$$

Multiplying both sides of the equation by $\sqrt{(I-K)^{-1}}$ on the left and $(SU^*\sqrt{I-K})^{-1}E_\epsilon$ on the right, we have

$$\begin{aligned}&A\left[\sqrt{K}-E_\epsilon\sqrt{H_1(I-H_1)^{-1}}U^*\sqrt{I-K}\right](SU^*\sqrt{I-K})^{-1}E_\epsilon\\&\qquad=\sqrt{(I-K)^{-1}}\left[\sqrt{K}US^{-1}E_\epsilon-\sqrt{I-K}\sqrt{H_1(I-H_1)^{-1}}S^{-1}E_\epsilon\right]AE_\epsilon.\end{aligned}$$

Since $\left[E_\epsilon\sqrt{K}-E_\epsilon\sqrt{H_1(I-H_1)^{-1}}U^*\sqrt{I-K}\right](SU^*\sqrt{I-K})^{-1}E_\epsilon$ is bounded, by letting $A=E_\epsilon$, we know that $\sqrt{(I-K)^{-1}}\left[\sqrt{K}US^{-1}E_\epsilon-\sqrt{I-K}\sqrt{H_1(I-H_1)^{-1}}S^{-1}E_\epsilon\right]$ is also bounded. Hence there is a unique a in $\mathbb{C}$ such that

$$\begin{aligned}&[E_\epsilon\sqrt{K}-E_\epsilon\sqrt{H_1(I-H_1)^{-1}}U^*\sqrt{I-K}](SU^*\sqrt{I-K})^{-1}E_\epsilon\\&\qquad=\sqrt{(I-K)^{-1}}\left[\sqrt{K}US^{-1}E_\epsilon-\sqrt{I-K}\sqrt{H_1(I-H_1)^{-1}}S^{-1}E_\epsilon\right]=aE_\epsilon.\end{aligned}$$

Letting $\epsilon\to 0$, we have

$$\sqrt{K(I-K)^{-1}}U=aS\widehat{+}\sqrt{H_1(I-H_1)^{-1}}.$$

Conversely, if K and U are given by the above equation as a polar decomposition of the right hand side, then, by Lemma 2.4, one can check easily that Q given in the proposition is in $Lat\,Alg\,\mathcal{L}$.

By this proposition, we have the following corollary.

Corollary 2.2 *For any two distinct projections Q_1 and Q_2 in $Lat\,Alg\,\mathcal{L}\setminus\{0,I,P_1\}$, we have $Alg\,\mathcal{L}=Alg\,\{P_1,Q_1,Q_2\}$ and $Lat\,Alg\,\{P_1,Q_1,Q_2\}=Lat\,Alg\,\mathcal{L}$.*

Proof Obviously, $Alg\,\mathcal{L}\subseteq Alg\,\{P_1,Q_1,Q_2\}$. On the other hand, following Proposition 2.1, we write

$$Q_i=\begin{pmatrix}K_{a_i} & \sqrt{K_{a_i}(I-K_{a_i})}U_{a_i}\\ U_{a_i}^*\sqrt{K_{a_i}(I-K_{a_i})} & U_{a_i}^*(I-K_{a_i})U_{a_i}\end{pmatrix},$$

where K_{a_i} and U_{a_i} are determined by the polar decomposition $a_iS\widehat{+}\sqrt{H_1(I-H_1)^{-1}}=\sqrt{K_{a_i}(I-K_{a_i})^{-1}}U_{a_i}$, for $a_i\in\mathbb{C}$, $i=1,2$ and $a_1\neq a_2$. Then

$$\sqrt{K_{a_1}(I-K_{a_1})^{-1}}U_{a_1}\widehat{-}\sqrt{K_{a_2}(I-K_{a_2})^{-1}}U_{a_2}=(a_1-a_2)S.$$

A similar argument to the proof of Lemma 2.4 gives us that $T\in Alg\,\{P_1,Q_1,Q_2\}$ if and only if there exists an operator T_1 in $B(\mathcal{H})$ such that

$$T=\begin{pmatrix}T_1 & \sqrt{K_{a_1}(I-K_{a_1})^{-1}}U_{a_1}S^{-1}T_1S-T_1\sqrt{K_{a_1}(I-K_{a_1})^{-1}}U_{a_1}\\ 0 & S^{-1}T_1S\end{pmatrix}.$$

However, for any ξ in $\mathcal{D}(\sqrt{K_{a_1}(I-K_{a_1})^{-1}}U_{a_1})\cap\mathcal{D}(\sqrt{K_{a_2}(I-K_{a_2})^{-1}}U_{a_2})$ which is equal to $\mathcal{D}(\sqrt{H_1(I-H_1)^{-1}})\cap\mathcal{D}(\sqrt{H_2(I-H_2)^{-1}}V)$, we have

$$\begin{aligned}&\sqrt{K_{a_1}(I-K_{a_1})^{-1}}U_{a_1}S^{-1}T_1S\xi - T_1\sqrt{K_{a_1}(I-K_{a_1})^{-1}}U_{a_1}\xi\\ &=(a_1S\hat{+}\sqrt{H_1(I-H_1)^{-1}})S^{-1}T_1S\xi - T_1(a_1S\hat{+}\sqrt{H_1(I-H_1)^{-1}})\xi\\ &=\sqrt{H_1(I-H_1)^{-1}}S^{-1}T_1S\xi - T_1\sqrt{H_1(I-H_1)^{-1}}\xi.\end{aligned}$$

It follows from Lemma 2.4 that T is in $Alg\,\mathcal{L}$. Consequently, $Alg\,\mathcal{L}=Alg\,\{P_1,Q_1,Q_2\}$.

Proposition 2.2 *With the above notation, the mapping, from $\mathbb{C}$ into Lat Alg $\mathcal{L}$, given by*

$$a\to Q_a=\begin{pmatrix}K_a & \sqrt{K_a(I-K_a)}U_a\\ U_a^*\sqrt{K_a(I-K_a)} & U_a^*(I-K_a)U_a\end{pmatrix},$$

is one to one and continuous with respect to the usual topology on $\mathbb{C}$ and the trace norm on Lat Alg $\mathcal{L}$. Moreover, as $a\to\infty$, $\|Q_a-P_1\|_2\to 0$. Hence Lat Alg $\mathcal{L}\setminus\{0,I\}$ is homeomorphic to S^2.

Proof For two scalars a and a_0 in $\mathbb{C}$, let Q_a and Q_{a_0} be defined as in the proposition. By Corollary 2.1, $\tau(Q_a)=\tau(Q_{a_0})=\frac{1}{2}$. We only need to show that $\|Q_a-Q_{a_0}\|_2\to 0$ as a tends to a_0. Since

$$\begin{aligned}\|Q_a-Q_{a_0}\|_2^2&=1-2\tau(Q_aQ_{a_0})\\ &=1-tr(K_aK_{a_0})-tr(\sqrt{K_a(I-K_a)}U_aU_{a_0}^*\sqrt{K_{a_0}(I-K_{a_0})})\\ &\quad-tr(U_a^*\sqrt{K_a(I-K_a)}\sqrt{K_{a_0}(I-K_{a_0})}U_{a_0})\\ &\quad-tr(U_a^*(I-K_a)U_aU_{a_0}^*(I-K_{a_0})U_{a_0}),\end{aligned}$$

it is sufficient to show that the following statements hold as $a\to a_0$:

$$|tr((K_a-K_{a_0})K_{a_0})|\to 0, \tag{6}$$

$$|tr((\sqrt{K_a(I-K_a)}U_a-\sqrt{K_{a_0}(I-K_{a_0})}U_{a_0})U_{a_0}^*\sqrt{K_{a_0}(I-K_{a_0})})|\to 0, \tag{7}$$

$$|tr((U_a^*\sqrt{K_a(I-K_a)}-U_{a_0}^*\sqrt{K_{a_0}(I-K_{a_0})})\sqrt{K_{a_0}(I-K_{a_0})}U_{a_0})|\to 0, \tag{8}$$

$$|tr(K_a-K_{a_0})|\to 0,\quad |tr((U_a^*K_aU_a-U_{a_0}^*K_{a_0}U_{a_0})U_{a_0}^*K_{a_0}U_{a_0})|\to 0. \tag{9}$$

Recall that $\sqrt{K_a(I-K_a)^{-1}}U_a=aS\hat{+}\sqrt{H_1(I-H_1)^{-1}}$. Let

$$\begin{aligned}F(a)&=K_a(I-K_a)^{-1}\\ &=|a|^2SS^*\hat{+}aS\sqrt{H_1(I-H_1)^{-1}}\hat{+}\bar{a}\sqrt{H_1(I-H_1)^{-1}}S^*\hat{+}(\sqrt{H_1(I-H_1)^{-1}})^2.\end{aligned}$$

Then $I-K_a=(I+F(a))^{-1}$.

For each positive number ϵ, there exists a projection F_ϵ in $\mathcal{M}$ such that SS^*E_ϵ, $S\sqrt{H_1(I-H_1)^{-1}}E_\epsilon$, $\sqrt{H_1(I-H_1)^{-1}}S^*E_\epsilon$ and $H_1(I-H_1)^{-1}E_\epsilon$ are bounded operators, and $tr(I-F_\epsilon)<\epsilon^2$, where $E_\epsilon=\mathcal{R}((I+F(a_0))^{-1}F_\epsilon)$. Note that $tr(I-F_\epsilon)=tr(I-E_\epsilon)<$

ϵ^2. So there exists a constant $\delta > 0$ such that, if $|a-a_0| < \delta$, then $\|F(a)E_\epsilon - F(a_0)E_\epsilon\| < \epsilon$ and

$$\begin{aligned} |tr((K_a - K_{a_0})K_{a_0})| &\le |tr(K_{a_0}(K_a - K_{a_0})F_\varepsilon)| + |tr(K_{a_0}(K_a - K_{a_0})(I - F_\varepsilon))| \\ &\le |tr(K_{a_0}(I + F(a))^{-1}(F(a) - F(a_0))(I + F(a_0))^{-1}F_\varepsilon)| + 2\varepsilon \\ &\le 3\varepsilon. \end{aligned}$$

This implies (6). By a similar argument, we have $|tr(K_a - K_{a_0})| \to 0$ as $a \to a_0$.

Next we prove (7). For each positive number ϵ, we choose a projection P_ϵ in $\mathcal{M}$ such that SP_ε and $\sqrt{H_1(I - H_1)^{-1}}P_\varepsilon$ are bounded operators, and $tr(I - P_\varepsilon) < \varepsilon^2$. Thus there is a positive constant δ_1 such that, if $|a - a_0| < \delta_1$, then

$$\|\sqrt{K_a(I - K_a)^{-1}}U_aP_\epsilon - \sqrt{K_{a_0}(I - K_{a_0})^{-1}}U_{a_0}P_\epsilon\| = \|(a - a_0)SP_\epsilon\| \le \epsilon.$$

Note that

$$\begin{aligned} &\sqrt{K_a(I - K_a)}U_aP_\varepsilon - \sqrt{K_{a_0}(I - K_{a_0})}U_{a_0}P_\varepsilon \\ &\quad = (I - K_a)\sqrt{K_a(I - K_a)^{-1}}U_aP_\varepsilon - (I - K_{a_0})\sqrt{K_{a_0}(I - K_{a_0})^{-1}}U_{a_0}P_\varepsilon \\ &\quad = (I - K_a)(\sqrt{K_a(I - K_a)^{-1}}U_aP_\varepsilon - \sqrt{K_{a_0}(I - K_{a_0})^{-1}}U_{a_0}P_\varepsilon) \\ &\quad - (K_a - K_{a_0})\sqrt{K_{a_0}(I - K_{a_0})^{-1}}U_{a_0}P_\varepsilon. \end{aligned}$$

Hence we have

$$\begin{aligned} &|tr((\sqrt{K_a(I - K_a)}U_a - \sqrt{K_{a_0}(I - K_{a_0})}U_{a_0})U_{a_0}^*\sqrt{K_{a_0}(I - K_{a_0})})| \\ &\quad \le |tr(U_{a_0}^*\sqrt{K_{a_0}(I - K_{a_0})}(\sqrt{K_a(I - K_a)}U_a - \sqrt{K_{a_0}(I - K_{a_0})}U_{a_0})P_\varepsilon)| \\ &\quad + |tr(U_{a_0}^*\sqrt{K_{a_0}(I - K_{a_0})}(\sqrt{K_a(I - K_a)}U_a - \sqrt{K_{a_0}(I - K_{a_0})}U_{a_0})(I - P_\varepsilon))| \\ &\quad \le |tr(U_{a_0}^*\sqrt{K_{a_0}(I - K_{a_0})}(I - K_a)[\sqrt{K_a(I - K_a)^{-1}}U_aP_\varepsilon - \sqrt{K_{a_0}(I - K_{a_0})^{-1}}U_{a_0}P_\varepsilon])| \\ &\quad + |tr(\sqrt{K_{a_0}(I - K_{a_0})^{-1}}U_{a_0}P_\varepsilon U_{a_0}^*\sqrt{K_{a_0}(I - K_{a_0})}(K_{a_0} - K_a))| + 2\varepsilon \\ &\quad \le |tr(\sqrt{K_{a_0}(I - K_{a_0})^{-1}}U_{a_0}P_\varepsilon U_{a_0}^*\sqrt{K_{a_0}(I - K_{a_0})}(K_{a_0} - K_a))| + 3\varepsilon. \end{aligned}$$

To show $|tr(\sqrt{K_{a_0}(I - K_{a_0})^{-1}}U_{a_0}P_\varepsilon U_{a_0}^*\sqrt{K_{a_0}(I - K_{a_0})}(K_{a_0} - K_a))| \to 0$, we only need to apply the same argument as in the proof of (6). Therefore, we have (7). The proof of (8) is similar to that of (7).

For the second statement in (9), we choose, for each positive number ϵ, a projection P_ϵ in $\mathcal{M}$ such that SP_ε, $\sqrt{H_1(I - H_1)^{-1}}P_\varepsilon$, $P_\varepsilon S^*$ and $P_\varepsilon\sqrt{H_1(I - H_1)^{-1}}$ are all bounded operators, and $tr(I - P_\varepsilon) < \varepsilon^2$. Hence there is a positive constant δ_1 such that, if $|a - a_0| < \delta_1$, then

$$\begin{aligned} \|\sqrt{K_a(I - K_a)^{-1}}U_aP_\epsilon - \sqrt{K_{a_0}(I - K_{a_0})^{-1}}U_{a_0}P_\epsilon\| &= \|(a - a_0)SP_\epsilon\| \le \epsilon \;\text{ and} \\ \|P_\epsilon U_a^*\sqrt{K_a(I - K_a)^{-1}} - P_\epsilon U_{a_0}^*\sqrt{K_{a_0}(I - K_{a_0})^{-1}}\| &= \|(a - a_0)P_\epsilon S^*\| \le \epsilon. \end{aligned}$$

Note that

$$\begin{aligned} &P_\epsilon U_a^*K_aU_aP_\varepsilon - P_\epsilon U_{a_0}^*K_{a_0}U_{a_0}P_\varepsilon \\ &\quad = P_\epsilon U_a^*\sqrt{K_a(I - K_a)^{-1}}(1 - K_a)\sqrt{K_a(I - K_a)^{-1}}U_aP_\varepsilon \\ &\quad - P_\epsilon U_{a_0}^*\sqrt{K_{a_0}(I - K_{a_0})^{-1}}(1 - K_{a_0})\sqrt{K_{a_0}(I - K_{a_0})^{-1}}U_{a_0}P_\varepsilon. \end{aligned}$$

By the same argument as in the proof of (7), we can obtain (9).

Finally, we show that if $a \to \infty$, then $\|Q_a - P_1\|_2 \to 0$. Since $\|Q_a - P_1\|_2^2 = 1 - tr(K_a)$, we only need to show that, when $a \to \infty$, $tr(I - K_a) \to 0$.

Since SS^* is invertible, for any positive number ϵ, we can choose a positive constant β and a projection E in $\mathcal{M}$ such that $tr(E) > 1 - \epsilon^2$, $ESS^*E \geq \beta E > 0$, and ESS^*E, $ES\sqrt{H_1(I - H_1)^{-1}}E$ and $EH_1(I - H_1)^{-1}E$ are all bounded. Thus there exists a positive constant c such that if $|a| > c$, we have

$$\begin{aligned} & ESS^*E + \frac{1}{a}ES\sqrt{H_1(I - H_1)^{-1}}E \\ & \qquad + \frac{1}{a}E\sqrt{H_1(I - H_1)^{-1}}S^*E + \frac{1}{|a|^2}EH_1(I - H_1)^{-1}E > \frac{\beta}{2}E. \end{aligned}$$

From the definition of $F(a)$, we have $EF(a)E > |a|^2\frac{\beta}{2}E$. By [2, Lemma 3.2], if we let e denote the spectral projection of $F(a)$ on $[|a|^2\frac{\beta}{2}, +\infty)$, then $tr(e) \geq tr(E) > 1 - \epsilon^2$ and $eF(a)e \geq \frac{|a|^2\beta}{2}e$. Choosing a scalar a with $|a|$ large enough, we will have

$$tr(I - K_a) = tr((I + F(a))^{-1}e) + tr((I + F(a))^{-1}(1 - e)) \leq \frac{2}{|a|^2\beta} + \epsilon \leq 2\epsilon.$$

This implies that, when $a \to \infty$, $tr(I - K_a) \to 0$. This finishes the proof of the proposition.

Let us now recall the definitions of Kadison-Singer algebras and Kadison-Singer lattices:

Definition 2.1 *A subalgebra $\mathfrak{A}$ of $B(\mathcal{H})$ is called a Kadison-Singer algebra (or KS-algebra) if $\mathfrak{A}$ is reflexive and maximal with respect to the diagonal subalgebra $\mathfrak{A} \cap \mathfrak{A}^*$ of $\mathfrak{A}$, in the sense that if there is another reflexive subalgebra $\mathcal{B}$ of $B(\mathcal{H})$ such that $\mathfrak{A} \subseteq \mathcal{B}$ and $\mathcal{B} \cap \mathcal{B}^* = \mathfrak{A} \cap \mathfrak{A}^*$, then $\mathfrak{A} = \mathcal{B}$. A lattice $\mathcal{L}$ of projections in $B(\mathcal{H})$ is called a Kadison-Singer lattice (or KS-lattice) if $\mathcal{L}$ is a minimal reflexive lattice that generates the von Neumann algebra $\mathcal{L}''$, or equivalently, if $\mathcal{L}$ is reflexive and $Alg\,\mathcal{L}$ is a Kadison-Singer algebra.*

Next we will show that the lattice given in Proposition 2.1 is a KS-lattice in general.

Lemma 2.5 *For any three distinct projections Q_1, Q_2 and Q_3 in $Lat\,Alg\,\mathcal{L} \setminus \{0, I\}$, we have $P_1 \in Lat\,Alg\,\{Q_1, Q_2, Q_3\}$. Hence $Lat\,Alg\,\{Q_1, Q_2, Q_3\} = Lat\,Alg\,\mathcal{L}$.*

Proof We may assume that $Q_i \in Lat\,Alg\,\mathcal{L} \setminus \{0, I, P_1\}$ for $i = 1, 2, 3$. By Proposition 2.1,

$$Q_i = \begin{pmatrix} K_i & \sqrt{K_i(I - K_i)}U_i \\ U_i^*\sqrt{K_i(I - K_i)} & U_i^*(I - K_i)U_i \end{pmatrix},$$

where K_i and U_i are determined by the polar decomposition $a_i S \widehat{+} \sqrt{H_1(I - H_1)^{-1}} = \sqrt{K_i(I - K_i)^{-1}}U_i$ for distinct scalars a_i in $\mathbb{C}$.

To prove the lemma, we only need to show that if

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in Alg\,\{Q_1, Q_2, Q_3\},$$

then $A_{21} = 0$. Since $(I - Q_i)AQ_i = 0$ for $i = 1, 2, 3$, by a similar argument to the proof of Lemma 2.3, we have

$$\sqrt{I - K_i}(A_{11}\sqrt{K_i} + A_{12}U_i^*\sqrt{I - K_i}) = \sqrt{K_i}U_i(A_{21}\sqrt{K_i} + A_{22}U_i^*\sqrt{I - K_i}).$$

Since the set $(\cap_{i=1}^3 \mathcal{D}(\sqrt{(I - K_i)^{-1}}U_i) \cap \mathcal{D}(S) \cap \mathcal{D}(\sqrt{H_1(I - H_1)^{-1}})$, denoted by $\mathfrak{D}$, is dense in $\mathcal{H}$, the invertibility of $I - K_i$ (and $\sqrt{I - K_i}$) implies that, for each ξ in $\mathfrak{D}$,

$$A_{11}\sqrt{K_i(I - K_i)^{-1}}U_i\xi + A_{12}\xi = \sqrt{K_i(I - K_i)^{-1}}U_i[A_{21}\sqrt{K_i(I - K_i)^{-1}}U_i\xi + A_{22}\xi].$$

Let $\{E_\epsilon\}_{\epsilon>0}$ be an increasing net (as $\epsilon \to 0$) of projections in $\mathcal{M}$ with the strong-operator topology limit I such that $E_\epsilon S$ and $E_\epsilon\sqrt{H_1(I - H_1)^{-1}}$ are bounded operators for each ϵ. Then

$$E_\epsilon\sqrt{K_i(I - K_i)^{-1}}U_i = a_iE_\epsilon S + E_\epsilon\sqrt{H_1(I - H_1)^{-1}}$$

is bounded for $i = 1, 2, 3$. Thus, for i, $j \in \{1, 2, 3\}$, we have

$$\begin{aligned}(a_i - a_j)[E_\varepsilon A_{11}S\xi - E_\varepsilon SA_{22}\xi] = &\\ &E_\epsilon\sqrt{K_i(I - K_i)^{-1}}U_iA_{21}\sqrt{K_i(I - K_i)^{-1}}U_i\xi \\ &- E_\epsilon\sqrt{K_j(I - K_j)^{-1}}U_jA_{21}\sqrt{K_j(I - K_j)^{-1}}U_j\xi.\end{aligned}$$

Replacing $E_\epsilon\sqrt{K_i(I - K_i)^{-1}}U_i$ with $a_iE_\epsilon S + E_\epsilon\sqrt{H_1(I - H_1)^{-1}}$ in the above equation, we show that

$$\begin{aligned}(a_i - a_j)&[E_\varepsilon A_{11}S\xi - E_\varepsilon SA_{22}\xi] \\ &= (a_i^2 - a_j^2)E_\epsilon SA_{21}S\xi + (a_i - a_j)E_\epsilon SA_{21}\sqrt{H_1(I - H_1)^{-1}}\xi \\ &+ (a_i - a_j)E_\epsilon\sqrt{H_1(I - H_1)^{-1}}A_{21}S\xi.\end{aligned}$$

A simple calculation gives us that $(a_1 + a_2)E_\epsilon SA_{21}S\xi = (a_1 + a_3)E_\epsilon SA_{21}S\xi$. Since $a_2 \neq a_3$, $E_\epsilon SA_{21}S\xi = 0$. Note that $Ker(S) = 0$ and $\{S\xi : \xi \in \mathfrak{D}\}$ is dense in $\mathcal{H}$. Letting $\epsilon \to 0$, we have $A_{21} = 0$.

By Corollary 2.2, we have $Lat\,Alg\,\mathcal{L} = Lat\,Alg\,\{P_1, Q_1, Q_2\} \subseteq Lat\,Alg\,\{Q_1, Q_2, Q_3\} \subseteq Lat\,Alg\,\mathcal{L}$. Hence $Lat\,Alg\,\{Q_1, Q_2, Q_3\} = Lat\,Alg\,\mathcal{L}$. This proves our lemma.

Proposition 2.3 *With the above notation, $Lat\,Alg\,\mathcal{L}$ is a Kadison-Singer lattice if $\mathcal{L}''$, the von Neumann algebra generated by $\mathcal{L}$, cannot be generated by two nontrivial projections.*

Proof Suppose that $\mathcal{F}$ is a reflexive lattice in $M_2(\mathbb{C}) \otimes B(\mathcal{H})$ such that $\mathcal{F} \subseteq Lat\,Alg\,\mathcal{L}$ and $\mathcal{F}'' = (Lat\,Alg\,\mathcal{L})''$. From our assumption, there are at least three nontrivial projections Q_1, Q_2 and Q_3 in $\mathcal{F}$. Since $\mathcal{F}$ is reflexive, we have $Lat\,Alg\,\{Q_1, Q_2, Q_3\} \subseteq \mathcal{F} \subseteq Lat\,Alg\,\mathcal{L}$. By Lemma 2.5, we have $\mathcal{F} = Lat\,Alg\,\mathcal{L}$. Hence $Lat\,Alg\,\mathcal{L}$ is a Kadison-Singer lattice.

Remark 2.3 *When $\mathcal{L}$ is a double triangle lattice of projections and $\mathcal{L}''$ is generated by two nontrivial projections, Lat Alg $\mathcal{L}$ may not be a Kadison-Singer lattice. For example, let* $P_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$, $P_2 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$ *and* $P_3 = \begin{pmatrix} \frac{I}{2} & \frac{I}{2} \\ \frac{I}{2} & \frac{I}{2} \end{pmatrix}$. *Then, for $i \neq j$, $P_i \wedge P_j = 0$ and $P_i \vee P_j = I$. In this case, if let $\mathcal{L} = \{0, I, P_1, P_2, P_3\}$, then $\mathcal{L}$ is a double triangle lattice of projections, which generates the von Neumann algebra $M_2(\mathbb{C}) \otimes \mathbb{C}I$. Thus $\mathcal{L}''$ can be generated by two nontrivial projections. Let* $\mathcal{L}_1 = \left\{0, I, \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{I}{2} & \frac{I}{2} \\ \frac{I}{2} & \frac{I}{2} \end{pmatrix}\right\}$. *Then $\mathcal{L}_1$ is a reflexive sublattice of $\mathcal{L}$ that generates the same von Neumann algebra as $\mathcal{L}$. Hence Lat Alg $\mathcal{L}$ is not a Kadison-Singer lattice.*

Proposition 2.4 *Let Q_1, Q_2 and Q_3 be any three projections acting on a separable Hilbert space $\mathcal{K}$ such that $Q_i \wedge Q_j = 0$ and $Q_i \vee Q_j = I$ for $i \neq j$. Assume that the von Neumann algebra $\mathfrak{A}$ generated by these three projections is finite. Then Q_1 is equivalent to $I - Q_1$ in $\mathfrak{A}$ and $\mathfrak{A}$ is *-isomorphic to $Q_1 \mathfrak{A} Q_1 \otimes M_2(\mathbb{C})$.*

Proof Let τ be a faithful, normal, tracial state on $\mathfrak{A}$. By Kaplansky formula, we have $\tau(Q_i) = \frac{1}{2}$ for $i = 1, 2, 3$. Let $\mathcal{W} = \mathfrak{A} * M_2(\mathbb{C})$ be the reduced (von Neumann algebra) free product of $\mathfrak{A}$ with $M_2(\mathbb{C})$. Since there exists a trace half projection which is free with Q_1, it follows that Q_1 is equivalent to $I - Q_1$ in $\mathcal{W}$ ([18, 6]). Therefore, we may choose a system $\{E_{ij}\}_{i,j=1}^2$ of matrix units in $\mathcal{W}$ such that $E_{11} = Q_1$. In this case, $\mathcal{W}$ is $*$-isomorphic to $Q_1 \mathcal{W} Q_1 \otimes M_2(\mathbb{C})$.

By Lemma 2.1, we write

$$Q_2 = \begin{pmatrix} H_1 & \sqrt{H_1(I - H_1)}V_1 \\ V_1^* \sqrt{H_1(I - H_1)} & V_1^*(I - H_1)V_1 \end{pmatrix} \quad \text{and}$$
$$Q_3 = \begin{pmatrix} H_2 & \sqrt{H_2(I - H_2)}V_2 \\ V_2^* \sqrt{H_2(I - H_2)} & V_2^*(I - H_2)V_2 \end{pmatrix},$$

where V_i is a unitary operator in $Q_1 \mathcal{W} Q_1$ and H_i is a positive contractive operator in $Q_1 \mathcal{W} Q_1$ such that $Ker(I - H_i) = 0$ for $i = 1, 2$.

Note that

$$\begin{pmatrix} H_{i-1} & 0 \\ 0 & 0 \end{pmatrix} = Q_1 Q_i Q_1 \text{ and } \begin{pmatrix} 0 & \sqrt{H_{i-1}(I - H_{i-1})}V_{i-1} \\ 0 & 0 \end{pmatrix} = Q_1 Q_i (I - Q_1).$$

Let

$$T = \begin{pmatrix} 0 & \sqrt{H_1(I - H_1)^{-1}}V_1 \widehat{-} \sqrt{H_2(I - H_2)^{-1}}V_2) \\ 0 & 0 \end{pmatrix}.$$

Then T is affiliated with $\mathfrak{A}$. By Lemma 2.2, $Ker(\sqrt{H_1(I - H_1)^{-1}}V_1 - \sqrt{H_2(I - H_2)^{-1}}V_2) = 0$ and $Ker(V_1^* \sqrt{H_1(I - H_1)^{-1}} - V_2^* \sqrt{H_2(I - H_2)^{-1}}) = 0$. Let HU be the polar decomposition of T. Then $U \in \mathfrak{A}$, $U^*U = I - Q_1$ and $UU^* = Q_1$. Thus Q_1 is equivalent to $I - Q_1$ in $\mathfrak{A}$. This proves our proposition.

We summarize what we have proved into the following theorem.

Theorem 2.1 *Let P_1, P_2 and P_3 be three projections acting on a separable complex Hilbert space $\mathcal{H}$ such that $P_i \wedge P_j = 0$ and $P_i \vee P_j = I$ for $i \neq j$. Suppose the von Neumann algebra $\mathfrak{A}$ generated by P_1, P_2 and P_3 is finite. Then Lat Alg $\{P_1, P_2, P_3\} \setminus \{0, I\}$ is homeomorphic to S^2. In addition, each nontrivial projection in Lat Alg $\{P_1, P_2, P_3\}$ has trace $\frac{1}{2}$ and the reflexive lattice can be generated by arbitrary three nontrivial projections in it.*

Furthermore, if the von Neumann algebra $\mathfrak{A}$ cannot be generated by two nontrivial projections, then Lat Alg $\{P_1, P_2, P_3\}$ is a KS-lattice.

With the notation in the above theorem, let $\mathcal{F} = Lat\,Alg\{P_1, P_2, P_3\}$. For any three projections in $\mathcal{F} \setminus \{0, I\}$, they determine a coordinate chart of $\mathcal{F} \setminus \{0, I\}$ (exclude one point corresponding to ∞) by Proposition 2.1. It can be shown that the transition map between two coordinate charts is a Möbius transformation. We will address this issue in a forthcoming paper. As for now, we will determine all "connected" reflexive lattices acting on a finite-dimensional Hilbert space.

3 Connected reflexive lattices in $M_n(\mathbb{C})$

For $n \geq 3$, we know that $M_n(\mathbb{C})$ cannot be generated by two projections. When n is an even number greater than 3, $M_n(\mathbb{C})$ can be generated by three projections of (normalized) trace $\frac{1}{2}$. By Theorem 2.1, we know that every nontrivial projection in the reflexive lattice generated by a double triangle lattice of projections in $M_n(\mathbb{C})$ has trace $\frac{1}{2}$. In this case, such a reflexive lattice (with the $\|\cdot\|_2$-topology) is homeomorphic to S^2 (plus 0 and I). In this section, we will show that if each nontrivial projection in a reflexive lattice $\mathcal{F}$ of projections in $M_n(\mathbb{C})$ has trace $\frac{1}{2}$ and $\mathcal{F}$ has more than three nontrivial projections, then $\mathcal{F} \setminus \{0, I\}$ (with the $\| \, \|_2$-topology) is homeomorphic to S^2. Moreover, if $n \geq 3$ and $\mathcal{F}$ generates $M_n(\mathbb{C})$, then $\mathcal{F}$ is a Kadison-Singer lattice.

Proposition 3.1 *Let $\mathcal{F}$ be a reflexive lattice of projections in $M_n(\mathbb{C})$. If $\mathcal{F} \setminus \{0, 1\}$ has only one connected component under the $\| \, \|_2$-topology, then $\mathcal{F} = \{0, P, I\}$ for some nontrivial projection P, or $\mathcal{F}$ is generated by a double triangle lattice of projections in $M_n(\mathbb{C})$. Furthermore, n must be even in the latter case.*

Proof Suppose $\mathcal{F}$ contains at least two nontrivial projections. Let τ be the normalized trace on $M_n(\mathbb{C})$. Since $\mathcal{F} \setminus \{0, 1\}$ has only one connected component, the range of the restriction of τ to $\mathcal{F} \setminus \{0, 1\}$ contains only one point. This implies that $P \vee Q = I$, $P \wedge Q = 0$ and $\tau(P) = \tau(Q) = \frac{1}{2}$ for any two distinct projections P and Q in $\mathcal{F} \setminus \{0, 1\}$. Thus, n must be even. Let $n = 2k$ and write $M_n(\mathbb{C}) = M_2(\mathbb{C}) \otimes M_k(\mathbb{C})$.

Next we show that $\mathcal{F}$ is generated by a double triangle lattice of projections in $M_n(\mathbb{C})$. Note that if $\mathcal{F}$ contains at least two nontrivial projections, then $\mathcal{F}$ must contain infinitely many projections. Otherwise, $\mathcal{F} \setminus \{0, I\}$ has more than one connected components. Given P_1, P_2 and P_3 in $\mathcal{F} \setminus \{0, I\}$, let $\mathcal{L} = \{0, P_1, P_2, P_3, I\}$. Then $\mathcal{L}$ is a double triangle lattice of projections in $M_n(\mathbb{C})$, and $Lat\,Alg\,\mathcal{L} \subseteq \mathcal{F}$, since $\mathcal{F}$ is reflexive. Now we show that $Lat\,Alg\,\mathcal{L} = \mathcal{F}$.

If there is a projection P_4 in $\mathcal{F} \setminus Lat\, Alg\, \mathcal{L}$, then $P_i \wedge P_j = 0$ and $P_i \vee P_j = I$ for $i, j = 1, 2, 3, 4$, and $i \neq j$. Up to unitary equivalence, we may assume that $P_1 = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$, where I_k is the identity matrix in $M_k(\mathbb{C})$. Then, by Lemma 2.1, we have

$$P_j = \begin{pmatrix} H_{j-1} & \sqrt{H_{j-1}(I-H_{j-1})^{-1}}V_{j-1} \\ V_{j-1}^* \sqrt{H_{j-1}(I-H_{j-1})^{-1}} & V_{j-1}^*(I-H_{j-1})V_{j-1} \end{pmatrix}, \quad \text{for } j = 2, 3, 4,$$

where H_j is a positive matrix and V_j is a unitary matrix in $M_k(\mathbb{C})$ such that $I - H_j$ is positive definite (and hence, invertible). Without loss of generality, we may assume $V_1 = I(= I_k)$.

Let $\mathcal{L}_0 = \{0, P_1, P_2, P_4, I\}$. Then we have $Alg\,\{P_1, P_2, P_3, P_4\} = Alg\,\mathcal{L} \cap Alg\,\mathcal{L}_0$. By Lemma 2.4, we obtain that

$$\begin{aligned} &Alg\,\{P_1, P_2, P_3, P_4\} \\ &\quad = \left\{ \begin{pmatrix} A & \sqrt{H_1(I-H_1)^{-1}}S_1^{-1}AS_1 - A\sqrt{H_1(I-H_1)^{-1}} \\ 0 & S_1^{-1}AS_1 \end{pmatrix} : \begin{array}{l} A \in M_k(\mathbb{C}) \\ A\widetilde{S} = \widetilde{S}A \end{array} \right\}, \end{aligned}$$

here $\widetilde{S} = S_2 S_1^{-1}$, $S_1 = \sqrt{H_1(I-H_1)^{-1}} - \sqrt{H_2(I-H_2)^{-1}}V_2$ and $S_2 = \sqrt{H_1(I-H_1)^{-1}} - \sqrt{H_3(I-H_3)^{-1}}V_3$. When $k = 1$, $\widetilde{S}$, S_1 and S_2 are all scalars. It is easy to see that $Alg\,\mathcal{L} = Alg\,\{P_1, P_2, P_3, P_4\} = Alg\,\mathcal{L}_0 = \mathbb{C}I$. Hence we have a contradiction with the assumption that $P_4 \notin Lat\, Alg\,\mathcal{L}$. The proposition follows.

For $k > 1$, we first remark that $\widetilde{S} \notin \mathbb{C}I$. Otherwise, $S_2 = \lambda S_1$. Then $S_2^{-1}AS_2 = S_1^{-1}AS_1$ for any A in $M_k(\mathbb{C})$. Hence we have $Alg\,\mathcal{L} = Alg\,\mathcal{L}_0$ by Lemma 2.4. Thus P_4 is in $Lat\, Alg\,\mathcal{L}$, which contradicts our assumption.

Let λ be an eigenvalue of $\widetilde{S}$ and q be the orthogonal projection onto the eigenspace of $\widetilde{S}$ at λ. For each A in $M_k(\mathbb{C})$ with $A\widetilde{S} = \widetilde{S}A$, we have $(I-q)Aq = 0$. Thus $(I-Q)TQ = 0$ for any T in $Alg\,\{P_1, P_2, P_3, P_4\}$, where $Q = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$. This implies that Q is in $Lat\, Alg\{P_1, P_2, P_3, P_4\}$. Since $\widetilde{S} \neq \lambda I$, we have $0 < q < I$ and $0 < \tau(Q) < \frac{1}{2}$. Hence Q is a projection with trace less than $\frac{1}{2}$, which contradicts the fact that $\mathcal{F} \setminus \{0, I\}$ has only one connected component. Consequently, $\mathcal{F}$ is generated by $\mathcal{L}$. This completes the proof.

By Proposition 3.1 and Theorem 2.1, we have the following corollary.

Corollary 3.1 *Let $\mathcal{F}$ be a reflexive lattice of projections in $M_n(\mathbb{C})$. Suppose there exist at least two nontrivial projections in $\mathcal{F}$. Then the following statements are equivalent:*

(i) $\mathcal{F} \setminus \{0, 1\}$ has only one connected component under the $\|\cdot\|_2$-topology;

(ii) $\mathcal{F}$ is generated by three projections P_1, P_2 and P_3 such that $P_i \wedge P_j = 0$ and $P_i \vee P_j = I$ for $i \neq j$;

(iii) $\mathcal{F} \setminus \{0, 1\}$ is homeomorphic to S^2.

In addition, if any of the above conditions is satisfied, then n must be even and each nontrivial projection in $\mathcal{F}$ has trace $\frac{1}{2}$. Conversely, if $\mathcal{F}$ contains at least three nontrivial projections, then the above conditions are equivalent to:

For each projection P in $\mathcal{F} \setminus \{0, I\}$, $\tau(P) = \frac{1}{2}$, where τ is the normalized trace on $M_n(\mathbb{C})$.

Proof By Proposition 3.1, we have that (i) implies (ii). By Theorem 2.1, we know that (ii) implies (iii). It is obvious that (iii) implies (i). If any of these equivalent conditions holds, by Theorem 2.1, we have $\tau(P) = \frac{1}{2}$ for each nontrivial projection P in $\mathcal{F}$. Hence n is even.

Suppose $\mathcal{F}$ contains at least three nontrivial projections and each nontrivial projection P in $\mathcal{F}$ has trace $\frac{1}{2}$. Then n is even. For each pair of distinct nontrivial projections P and Q in $\mathcal{F}$, $P \wedge Q = 0$ and $P \vee Q = I$. Let P_1, P_2 and P_3 be three distinct nontrivial projections in $\mathcal{F}$. By a similar argument to the proof of Proposition 3.1, we can show that $Lat\, Alg\, \{P_1, P_2, P_3\} = \mathcal{F}$. This proves that our last statement in the corollary implies (ii). Hence our corollary follows.

Remark 3.1 *When $\mathcal{F}$ contains only two nontrivial projections, the last statement in Corollary 3.1 cannot imply (i). For example, let* $\mathcal{F} = \left\{0, I, \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2}I_k & \frac{1}{2}I_k \\ \frac{1}{2}I_k & \frac{1}{2}I_k \end{pmatrix}\right\}$. *Then $\mathcal{F}$ is a reflexive lattice of projections in $M_2(\mathbb{C}) \otimes M_k(\mathbb{C})$ such that each nontrivial projection in $\mathcal{F}$ has trace $\frac{1}{2}$. Obviously, $\mathcal{F} \setminus \{0, I\}$ has two connected components.*

Corollary 3.2 *Let $\mathcal{F}$ be a reflexive lattice of projections in $M_n(\mathbb{C})$ such that $\mathcal{F}$ generates $M_n(\mathbb{C})$, for $n \geq 3$. Suppose each nontrivial projection in $\mathcal{F}$ has trace $\frac{1}{2}$. Then $\mathcal{F} \setminus \{0, I\}$ is homeomorphic to S^2. Moreover, $\mathcal{F}$ is a Kadison-Singer lattice and the Kadison-Singer algebra $Alg\, \mathcal{F}$ has dimension $\frac{n^2}{4}$.*

Proof When $n \geq 3$, $M_n(\mathbb{C})$ cannot be generated by two nontrivial projections. Thus $\mathcal{F}$ satisfies the conditions in Corollary 3.1. Hence $\mathcal{F} \setminus \{0, I\}$ is homeomorphic to S^2. Let $n = 2k$. It follows from Lemma 2.4 that $Alg\, \mathcal{F}$ is isomorphic to $M_k(\mathbb{C})$, which has dimension $\frac{n^2}{4}$.

Remark 3.2 *Let $n = 2k > 2$. Then there exists a reflexive lattice of projections in $M_n(\mathbb{C})$ satisfying the conditions in Corollary 3.2. For example, choose two positive matrices H_1 and H_2 in $M_k(\mathbb{C})$ such that $I - H_i$ is positive definite for $i = 1, 2$, $\sqrt{H_1(I-H_1)^{-1}} - \sqrt{H_2(I-H_2)^{-1}}$ is invertible, and $\{H_1, H_2\}$ generates $M_k(\mathbb{C})$. Let $\mathcal{L}$ be the lattice generated by* $\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} H_1 & \sqrt{H_1(I-H_1)} \\ \sqrt{H_1(I-H_1)} & I - H_1 \end{pmatrix}$ *and* $\begin{pmatrix} H_2 & \sqrt{H_2(I-H_2)} \\ \sqrt{H_2(I-H_2)} & I - H_2 \end{pmatrix}$. *By Theorem 2.1, $Lat\, Alg\, \mathcal{L}$ is a KS-lattice generating $M_n(\mathbb{C})$ and satisfies the conditions in Corollary 3.2.*

Proposition 3.2 [8, 19] *Let E_{ij}, $i, j = 1, 2, \cdots, n$, be the standard matrix units of $M_n(\mathbb{C})$, where $n \geq 2$. Let $P_i = \sum\limits_{j=1}^{i} E_{ii}$, for $i = 1, 2, \cdots, n$, and $Q = \frac{1}{n} \sum\limits_{i,j=1}^{n} E_{ij}$. Let $\mathcal{F}$ be the lattice generated by $P_1, \cdots, P_n$ and Q. Then $\mathcal{F}$ is a Kadison-Singer lattice of projections in $M_n(\mathbb{C})$ and the corresponding Kadiosn-Singer algebra $Alg\, \mathcal{F}$ has dimension $1 + \frac{n(n-1)}{2}$.*

Remark 3.3 *When $n = 2$ or 3, it was proved that if a Kadison-Singer lattice $\mathcal{L}$ generates $M_n(\mathbb{C})$, then $\mathcal{L}$ or $I - \mathcal{L}$ is similar to the one given in Proposition 3.2 ([17]). Hence each Kadison-Singer algebra in $M_2(\mathbb{C})$ (or $M_3(\mathbb{C})$) with diagonal $\mathbb{C}I$ has dimension 2 (or 4). For a general n, we have the following conjecture.*

Conjecture 3.1. *Let $\mathfrak{A}$ be a Kadison-Singer algebra in $M_n(\mathbb{C})$ with a trivial diagonal. Then $\frac{n^2}{4} \leq dim(\mathfrak{A}) \leq 1 + \frac{n(n-1)}{2}$, where $dim(\mathfrak{A})$ is the dimension of $\mathfrak{A}$.*

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