# Wei Fei's Note

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ABSTRACT. Note on Wei Fei's lecture.

## 1. NOTATIONS

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz class, i.e.,

$$\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : sup_{t\in\mathbb{R}}|t^k \frac{d^l}{dt^l} f(t)| < \infty \text{ for any } k, l \in \mathbb{N} \cup \{0\}\}.$$

Let $f \in \mathcal{S}(\mathbb{R})$. The Fourier transform $\hat{f}$ is

$$\hat{f}(s) = \int_{-\infty}^{+\infty} f(t)e^{-2\pi ist}dt, \text{ for any } s \in \mathbb{R}.$$

We have

$$f(t) = \int_{-\infty}^{+\infty} \hat{f}(s)e^{2\pi its}ds, \text{ for any } t \in \mathbb{R}.$$

## 2. IMPORTANT FORMULA

### 2.1. Partial Summation Formula.

**Lemma 2.1.** *Let $f(x) \in C^1([a,b])$. Then*

$$\sum_{a<n\leq b} c_n f(n) = C(b)f(b) - \int_a^b C(x)f'(x)dx,$$

*where*

$$C(x) = \sum_{a<n\leq x} c_n, \qquad c_n \in \mathbb{C}.$$

**Theorem 2.1** (Possion Summation Formula)**.** *Let $f$ in $\mathcal{S}(\mathbb{R})$. We have*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n).$$

*If $f(x) = f(-x)$, then*

$$\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} \hat{f}(n) + \frac{1}{2}(\hat{f}(0) - f(0)).$$

*Proof.* Let

$$g(\theta) = \sum_{n\in\mathbb{Z}} f(\theta + n).$$

Since $f \in \mathcal{S}(\mathbb{R})$, $g(\theta)$ is a well-defined function of period 1 such that

$$g(\theta) = \sum_{n\in Z} \hat{f}(n)e^{2\pi i\theta}.$$

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Let $\theta = 0$, we have

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n).$$

$\square$

**Example 2.1.** *Let $f(x) = e^{-\pi x^2 y}$.*

$$\hat{f}(x) = y^{-\frac{1}{2}} e^{-\frac{\pi x^2}{y}}.$$

## 3. Entire Function

**Lemma 3.1.** *Let $\{a_n\}_n \subset \mathbb{C}$ be a sequence,*

$$0 < |a_1| \le |a_2| \le \cdots \le |a_n| \le \cdots, \text{ and } |a_n| \to \infty.$$

*Then there exists a entire function $f$ such that $f(s) = 0$ if and only if $s \in \{a_n\}_n$.*

*Proof.* Let

$$h_n = (1 - \frac{s}{a_n}) e^{\frac{s}{a_n} + \frac{1}{2}(\frac{s}{a_n})^2 + \frac{1}{3}(\frac{s}{a_n})^3 + \cdots + \frac{1}{n}(\frac{s}{a_n})^n}.$$

Then

$$f(s) = \prod_{n=1}^{\infty} h_n(s)$$

satisfies the condition. $\square$

**Remark 3.1.** *Let $\{a_n\}_n \subset \mathbb{C}$ be a sequence,*

$$0 < |a_1| \le |a_2| \le \cdots \le |a_n| \le \cdots, \text{ and } |a_n| \to \infty.$$

*If $\sum_n \frac{1}{|a_n|^{p+1}} < \infty$, then we could use*

$$h_n = (1 - \frac{s}{a_n}) e^{\frac{s}{a_n} + \frac{1}{2}(\frac{s}{a_n})^2 + \frac{1}{3}(\frac{s}{a_n})^3 + \cdots + \frac{1}{p}(\frac{s}{a_n})^p}.$$

*in the above proof.*

**Example 3.1.** *Let $\{n\}_{n\in\mathbb{Z}}$. Then*

$$f(s) = s \prod_{n\in\mathbb{N}} (1 - \frac{s}{n})(1 + \frac{s}{n}).$$

**Lemma 3.2.** *Let $\{a_n\}_n \subset \mathbb{C}$ be a sequence,*

$$0 < |a_1| \le |a_2| \le \cdots \le |a_n| \le \cdots, \text{ and } |a_n| \to \infty.$$

*Suppose that $f$ satisfies $f(s) = 0$ if and only if $s \in \{a_n\}_n$. Then $f(s) = e^{H(s)} \prod_{n=1}^{\infty} h_n(s)$.*

**Definition 3.1.** *Suppose $G(s)$ is a function and $\mu(r) = max_{|s|\le r} |G(s)|$. Let*

$$\alpha_0 = inf\{\alpha : \mu(r) \le e^{a_0 r^{\alpha}}\}.$$

**Theorem 3.1.** *Let $p$ be the smallest integer such that*

$$\sum_n \frac{1}{|a_n|^{p+1}} < \infty.$$

*Then the degree of*

$$f(s) = \prod_n (1 - \frac{s}{a_n}) e^{\frac{s}{a_n} + \frac{1}{2}(\frac{s}{a_n})^2 + \frac{1}{3}(\frac{s}{a_n})^3 + \cdots + \frac{1}{p}(\frac{s}{a_n})^p}$$

*is $p$.*

**Example 3.2.**

$$\sin(\pi s) = \pi s \prod_n (1 - \frac{s^2}{n^2}).$$

*Proof.* Add proof. □

4. $\Gamma(s)$

Let

$$\Gamma(s) = \int_0^{+\infty} x^{s-1} e^{-x} dx, \qquad Re(S) > 0.$$

It is easy to see that $\Gamma(s+1) = s\Gamma(s)$.

$$\frac{1}{\Gamma(s)} = s e^{\gamma s} \prod_{n=1}^{\infty} (1 + \frac{s}{n}) e^{-\frac{s}{n}},$$

where $\gamma$ is the Euler constant, i.e. $\gamma = \lim_{n\to\infty}(1 + \frac{1}{2} + \cdots \frac{1}{n} - \log(n))$.

**Theorem 4.1.**

$$\frac{1}{\Gamma(s)} = s \prod_{n=1}^{\infty} (1 + \frac{1}{n})^{-s} (1 + \frac{s}{n}).$$

*Proof.* add proof. □

**Theorem 4.2.** *Let* $0 < \delta < \pi$. *We have*

$$\log \Gamma(s) = s \log s - \frac{1}{2} \log s - s + \log(\sqrt{2\pi}) + O_\delta(\frac{1}{|s|}).$$

*Proof.* Add proof. □

This following is incomplete.

**Proposition 4.1.** *Let* $s = \sigma + it$. *Assume* $\alpha < \sigma < \beta$. *We have*

$$\Gamma(s) = |t|^{s-\frac{1}{2}} e^{-\frac{\pi}{2}|t| - it +}$$

5. $\zeta$

**Theorem 5.1.** *If* $Re(s) = \sigma > 1$, *then* $\zeta(s) \neq 0$.

*Proof.* Note

$$\frac{1}{|\zeta(s)|} = |\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}| \leq \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \leq 1 + \int_1^\infty \frac{1}{t^\sigma} dt = \frac{\sigma}{\sigma - 1}.$$

This implies the result. □

**Theorem 5.2.** *For* $Re(s) > 0$, *we have*

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^2} + \frac{N^{1-s}}{s-1} - \frac{N^{-s}}{2} + s \int_N^\infty \frac{\rho(u)}{u^{s+1}} du, \quad N \geq 1,$$

*where* $\rho(u) = \frac{1}{2} - \{u\}$. *Specially,*

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + s \int_1^\infty \frac{\rho(u)}{u^{s+1}} du.$$

**Theorem 5.3.**

$$\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s) = \pi^{\frac{s-1}{2}} \Gamma(\frac{1-s}{2}) \zeta(1-s)$$

*Proof.* Add proof. □

Let

$$\xi(s) = \frac{1}{2}s(s-1)\zeta(s)\Gamma(\frac{S}{2})\pi^{-\frac{s}{2}}.$$

**Theorem 5.4.** *For any $\varepsilon > 0$, we have*

$$|\xi(s)| \ll e^{c|s|^{1+\varepsilon}}.$$

**Remark 5.1.** *The theorem above implies that $\xi$ has infinite many zeros. (provide a proof).*

For $Re(s) > 1$, estimate

$$f(s) = (1-2^{1-s})\zeta(s) = \sum_{n}^{\infty} \frac{(-1)^{n-1}}{n^s}.$$

It is not hard to see that $\zeta(s)$ does not have real zeros.

**Lemma 5.1.** *Let $\{\rho_n\}_n$ be the zeros of $\xi(s)$. Then*

$$\sum_{n}^{\infty} \frac{1}{|\rho_n|} = \infty,$$
$$\sum_{n}^{\infty} \frac{1}{|\rho_n|^{1+\varepsilon}} < \infty,$$

*for any $\varepsilon > 0$.*

**Theorem 5.5.**

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{1-s} + \sum_{n=1}^{\infty}(\frac{1}{s-\rho_n} + \frac{1}{\rho_n}) + \sum_{n=1}^{\infty}(\frac{1}{s+2n} - \frac{1}{2n}) + B_0,$$

*where $B_0$ is a constant.*

*Proof.* Consider the two expressions of $\xi(s)$:

$$\xi(s) = e^{as+b}\prod_{n}(1-\frac{s}{\rho_n})e^{\frac{s}{\rho_n}}$$
$$\xi(s) = \frac{1}{2}s(s-1)\zeta(s)\Gamma(\frac{S}{2})\pi^{-\frac{s}{2}}.$$

Compute the derivative of $\log \xi(s)$ by plugging in the two expressions. □

**Theorem 5.6.** *Let $T \geq 0$ and $\rho_n = \beta_n + i\gamma_n$ be the non-trivial zeros of $\zeta(s)$. Then*

$$\sum_{n=1}^{\infty} \frac{1}{1+(T-\gamma_n)^2} \leq C\log(T+2).$$

*Proof.* Let $s = 2 + iT$.

$$Re(\frac{1}{s-\rho_n}) = \frac{2-\beta_n}{(2-\beta_n)^2 + (T-\gamma_n)^2)^2} \geq \frac{1}{4(1+(T-\gamma_n)^2)}.$$

□

## 6. $n \times 2$ CASE

Let

$$S_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in M_n(\mathbb{C})$$

We would like to find the commutant of $(I - \frac{1}{2}S_n) \otimes (I - \frac{1}{3}S_2)$ in $M_n(\mathbb{C}) \otimes M_2(\mathbb{C})$. If

$$\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \begin{pmatrix} S_n & \frac{2}{3}(I - \frac{1}{2}S_n) \\ 0 & S_n \end{pmatrix} = \begin{pmatrix} T_1 S_n & T_1 \frac{2}{3}(I - \frac{1}{2}S_n) + T_2 S_n \\ T_3 S_n & T_3 \frac{2}{3}(I - \frac{1}{2}S_n) + T_4 S_n \end{pmatrix}$$

$$= \begin{pmatrix} S_n T_1 + \frac{2}{3}(I - \frac{1}{2}S_n) T_3 & S_n T_2 + \frac{2}{3}(I - \frac{1}{2}S_n) T_4 \\ S_n T_3 & S_n T_4 \end{pmatrix} = \begin{pmatrix} S_n & \frac{2}{3}(I - \frac{1}{2}S_n) \\ 0 & S_n \end{pmatrix} \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$$

Since $T_3$ commute with $S_n$, $T_3$ must be a polynomial of $S_n$. Also note that $T_1 S_n - S_n T_1 = \frac{2}{3}(I - \frac{1}{2}S_n) T_3$, this implies that the trace of $T_3$ is zero. Therefore $T_3$ must be upper triangular.

Note that

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \ddots & x_{3n} \\ x_{n-1,1} & x_{n-1,2} & x_{n-1,3} & \cdots & x_{n-1,n} \\ x_{n,1} & x_{n,2} & x_{n,3} & \cdots & x_{n,n} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$- \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \ddots & x_{3n} \\ x_{n-1,1} & x_{n-1,2} & x_{n-1,3} & \cdots & x_{n-1,n} \\ x_{n,1} & x_{n,2} & x_{n,3} & \cdots & x_{n,n} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & x_{11} & x_{12} & \cdots & x_{1,n-1} \\ 0 & x_{21} & x_{22} & \cdots & x_{2,n-1} \\ 0 & x_{31} & x_{32} & \ddots & x_{3,n-1} \\ 0 & x_{n-1,1} & x_{n-1,2} & \cdots & x_{n-1,n-1} \\ 0 & x_{n,1} & x_{n,2} & \cdots & x_{n,n-1} \end{pmatrix} - \begin{pmatrix} x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3n} \\ x_{41} & x_{42} & x_{43} & \ddots & x_{4n} \\ x_{n,1} & x_{n,2} & x_{n,3} & \cdots & x_{n,n} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

If the above is strict upper triangular, we must have

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \ddots & x_{3n} \\ x_{n-1,1} & x_{n-1,2} & x_{n-1,3} & \cdots & x_{n-1,n} \\ x_{n,1} & x_{n,2} & x_{n,3} & \cdots & x_{n,n} \end{pmatrix}$$

is upper triangular.

So we have $T_1, T_4$ are upper triangular. And it is easy to see that for a fixed $T_3$ which commute with $S_n$, we have many elements which commute with $(I - \frac{1}{2}S_n) \otimes (I - \frac{1}{3}S_2)$.

## 7. SPECTRUM OF SUMS OF UNITARY

**Definition 7.1.** *Let $T$ be a densely defined, closed operator on a Hilbert space $\mathcal{H}$.*

*(1) The resolvent set $\rho(T)$ of $T$ is the set of all complex numbers $\lambda$ such that $\lambda - T$ is a bijection (between $\mathfrak{D}(T)$ and $\mathcal{H}$) with bounded inverse.*
*(2) The resolvent of $T$ at $\lambda \in \rho(T) \subset \mathbb{C}$ is $R_\lambda(T) = (\lambda - T)^{-1}$.*
*(3) The spectrum of $T$ is $\sigma(T) = \mathbb{C} \setminus \rho(T)$.*
*(4) $\lambda \in \mathbb{C}$ is a point spectrum of $T$ if $\lambda - T$ is not injective.*
*(5) $\lambda \in \mathbb{C}$ is a residual spectrum of $T$ if $\lambda - T$ is injective but the range of $\lambda - T$ is not dense in $\mathcal{H}$.*

**Example 7.1.** *Let $\mathcal{H} = L^2([0,1])$ and*

$$\mathfrak{D} = \{f(x) = C + \int_0^x \varphi(t)dt : C \in \mathbb{C}, \varphi \in L^2([0,1])\}.$$

- *Assume that $\mathfrak{D}(T) = \mathfrak{D}$ such that $Tf = i\frac{df}{dx}$. Then $\sigma(T) = \mathbb{C}$ since $e^{-i\lambda x}$ is an eigenfunction for $T$ with eigenvalue $\lambda$.*
- *If $\mathfrak{D}(T) = \{f \in \mathfrak{D} : f(0) = 0\}$ with $Tf = i\frac{df}{dx}$, then $\sigma(T) = \varnothing$. In fact, for any $\lambda \in \mathbb{C}$, the resolvent of $(\lambda - T)^{-1}$ is*
$$(R_\lambda(T)f)(x) = i\int_0^x e^{-i\lambda(x-t)} f(t)dt.$$
- *Let $\alpha \in \mathbb{C} \setminus \{0\}$. If $\mathfrak{D}(T) = \{f \in \mathfrak{D} : f(0) = \alpha f(1)\}$ with $Tf = i\frac{df}{dx}$, then $\sigma(T) = \{-i\ln\alpha + 2k\pi : k \in \mathbb{Z}\}$. If $\lambda = -i\ln\alpha + 2k\pi$, then $e^{-i\lambda x}$ is an eigenfunction for $T$. For $\lambda$ not of the form $-i\ln\alpha + 2k\pi$, the resolvent operator $(\lambda - T)^{-1}$ is*
$$(R_\lambda(T)f)(x) = \int_0^1 G_\lambda(x,t)f(t)dt$$
*with*
$$G_\lambda(x,t) = \begin{cases} \frac{i\alpha e^{i\lambda(t-x-1)}}{1-\alpha e^{-i\lambda}} & \text{if } x < t \\ \frac{i\alpha e^{i\lambda(t-x)}}{1-\alpha e^{-i\lambda}} & \text{if } x > t \end{cases}$$

Let $U$ be a unitary such that $(Uf)(t) = f(t+1)$. Then
$$\hat{U}f(s) = e^{2\pi i s}\hat{f}(s).$$

Let
$$\mathfrak{D} = \{(a_l)_{l\in\mathbb{Z}} : \exists m \in \mathbb{N}, a_n = 0, \text{ for any } n > m \text{ or } n < m \text{ and } \textstyle\sum_{-m\le k\le m} a_k = 0\}.$$
It is easy to see that $\mathfrak{D}$ is dense in $\mathcal{H}$.

Let $\mathcal{H} = l^2(\mathbb{Z})$ and
$$T : (a_l)_{l\in\mathbb{Z}} \in \mathfrak{D} \to (\sum_{l\le k<+\infty} a_k)_{l\in\mathbb{Z}},$$
$$S : (a_l)_{l\in\mathbb{Z}} \in \mathfrak{D} \to (\sum_{-\infty<k\le l} a_k)_{l\in\mathbb{Z}}.$$
It is easy to see that $T$ and $S$ are well-defined.

Let
$$\xi = (\ldots, 0, a_{-m}, a_{-m+1}\ldots, a_{m-1}, a_m, 0, \ldots) \in \mathfrak{D},$$
$$\beta = (\ldots, 0, b_{-m}, b_{-m+1}\ldots, b_{m-1}, b_m, 0, \ldots) \in \mathfrak{D}.$$
Then we have
$$\begin{aligned}\langle T\xi, \beta\rangle &= \bar{b}_{-m}(a_{-m} + \cdots + a_m) + \bar{b}_{-m+1}(a_{-m+1} + \cdots + a_m) + \cdots + \bar{b}_m a_m \\ &= a_{-m}\bar{b}_{-m} + a_{-m+1}(\bar{b}_{-m} + \bar{b}_{-m+1}) + \cdots + a_m(\bar{b}_{-m} + \cdots + \bar{b}_m) = \langle \xi, S\beta\rangle.\end{aligned}$$
Therefore, $T$ and $S$ are both closable.

**Example 7.2.** *Let $U : e_k \to e_{k-1}$ where $\{e_k\}_{k\in\mathbb{Z}}$ is the canonical orthonormal basis of $l^2(\mathbb{Z})$. It is well-known that the spectrum of $U$ is $S^1$.*

*Formally, we can write $T$ and $S$ as*

$$T = \sum_{0\le k<\infty} U^k, \qquad S = \sum_{0\le k<\infty} U^{-k}.$$

*Let $\mathfrak{A}$ be the von Neumann algebra generated by $U$. Consider the closure of $T$ and $S$ affiliated with $\mathfrak{A}$.*

*We can identifies $l^2(\mathbb{Z})$ with $L^2(S^1)$ and $e_n$ with $z^n \in L^2(S^1)$. Then $U$ is the multiplication of $\frac{1}{z}$. Note that every vector in $\mathfrak{D}$ corresponding to a function*

$$(\ldots, 0, a_{-m}, a_{-m+1} \ldots, a_{m-1}, a_m, 0, \ldots) \Longleftrightarrow f(z) = \sum_{-m\le k\le m} a_k z^k.$$

*By the definition of $\mathfrak{D}$, we have $f(1) = 0$. Therefore $f(z) = (z-1)g(z)$ where the Fourier coefficients of $g(z)$ is finite supported.*

*It is not hard to check that*

$$(Tf)(z) = (M_{\frac{z}{z-1}} f)(z) = \frac{zf(z)}{z-1} \quad \text{and} \quad (Sf)(z) = (M_{\frac{1}{1-z}} f)(z) = \frac{f(z)}{1-z} \qquad f \in \mathfrak{D}.$$

*It is obvious that $M_{\frac{z}{z-1}}$ and $M_{\frac{1}{1-z}}$ are closed operator affiliated with $\mathfrak{A}$ and extend $T$ and $S$ respectively.*

**Remark 7.1.** *Note that for $z \in S^1$, $\bar{z} = \frac{1}{z}$ and*

$$M^*_{\frac{z}{z-1}} = M_{\frac{\frac{1}{z}}{\frac{1}{z}-1}} = M_{\frac{1}{1-z}},$$

$$M_{\frac{z}{z-1}} + M_{\frac{1}{1-z}} = I.$$

*It is not hard to see that the spectrum of $M_{\frac{1}{1-z}}$ as a closed operator affiliated with $\mathfrak{A}$ is*

$$\{\frac{1}{1-s} : |s| = 1\}.$$

*Let $\gamma \in \mathbb{C} \setminus \{\frac{1}{1-s} : |s| \le 1\}$. If $\gamma = 0$, then $M_{1-z}$ is a bounded inverse of $M_{\frac{1}{1-z}}$ in the Banach algebra generated by $U^*$. If $\gamma \ne 0$, then*

$$\frac{1}{\gamma} - 1 \notin \{s : |s| \le 1\}.$$

*This implies that $|\frac{1-\gamma}{\gamma}| > 1$. Thus*

$$M_{\frac{(\gamma-1)-\gamma z}{1-z}} = \frac{1}{(\gamma-1)}(1-M_z)(I + \frac{\gamma}{\gamma-1}M_z + (\frac{\gamma}{\gamma-1}M_z)^2 + (\frac{\gamma}{\gamma-1}M_z)^3 + \cdots).$$

*Thus $\gamma - M_{\frac{1}{1-z}}$ has a bounded inverse in the Banach algebra generated by $U^* = M_z$.*

*Assume that $\gamma = \frac{1}{1-s}$ where $|s| < 1$. Then*

$$\gamma - \frac{1}{1-z} = \frac{s-z}{(1-s)(1-z)}.$$

*Note that*

$$\frac{(1-s)(1-z)}{s-z}$$

*is not analytic in the unit disk, therefore $\gamma - M_{\frac{1}{1-z}}$ does not has a bounded inverse in the Banach algebra generated by $M_z$.*

*In summary, we have the spectrum of $M_{\frac{1}{1-z}}$ restricted to the Banach algebra generated by $M_z$ is $\{\frac{1}{1-s} : |s| \le 1\}$.*

Let $\omega = e^{\frac{2\pi i}{n}}$ and

$$W = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ \omega & \omega^2 & \omega^3 & \cdots & \omega^{n-1} & 1 \\ \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(n-1)} & 1 \\ \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(n-1)} & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \cdots & \omega^{(n-1)(n-1)} & 1 \end{pmatrix}$$

Recall the definition of Hardy space $H^2(\mathbb{D})$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in Hol(\mathbb{D})$, then $f \in H^2(\mathbb{D})$ if

$$sup_{0<r<1} \int_{\mathbb{T}} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

Let $P : l^2(\mathbb{Z}) \to l^2(\mathbb{N})$ be the orthonormal projection onto the subspace spanned by $e_i$, $i \geq 0$. Identify the Hardy space $H^2(\mathbb{D})$ with $Pl^2(\mathbb{Z})$ and

$$T_1 = PTP : (a_0, a_1, a_2, \ldots) \to (\sum_{k=0} a_k, \sum_{k=1} a_k, \sum_{k=2} a_k, \ldots),$$

$$S_1 = PSP : (a_0, a_1, a_2, \ldots) \to (a_0, \sum_{k=0}^{1} a_k, \sum_{k=0}^{2} a_k, \ldots),$$

where $(a_0, a_1, a_2, \ldots) \in \mathfrak{D}|_{H^2(\mathbb{D})}$.

Let $\xi = (a_0, \ldots, a_n, 0, 0, \ldots)$ and $\beta = (b_0, \ldots, b_n, 0, 0, \ldots)$ in $\mathfrak{D}|_{H^2(\mathbb{D})}$.

$$\begin{aligned} \langle T_1\xi, \beta\rangle &= \bar{b}_0(a_0 + \cdots + a_n) + \bar{b}_1(a_1 + \cdots + a_n) + \cdots + \bar{b}_n a_n \\ &= a_0\bar{b}_0 + a_1(\bar{b}_0 + \bar{b}_1) + \cdots + a_n(\bar{b}_0 + \cdots + \bar{b}_n) = \langle \xi, S_1\beta\rangle. \end{aligned}$$

Note that

$$(I - P)SP\xi = 0 \text{ and } (I - P)TP\xi = 0,$$

for any $\xi \in \mathfrak{D}|_{H^2(\mathbb{D})}$.

**Example 7.3.** *By identify $l^2(\mathbb{Z})$ with $L^2(S^1)$, we have*

$$PM_{\frac{(\gamma-1)-\gamma z}{1-z}}P = \frac{1}{(\gamma-1)}(1 - PM_zP)(I + \frac{\gamma}{\gamma-1}PM_zP + (\frac{\gamma}{\gamma-1}PM_zP)^2 + (\frac{\gamma}{\gamma-1}PM_zP)^3 + \cdots)$$

*is the bounded inverse of $\gamma - M_{\frac{1}{1-z}}$ for $\gamma \in \mathbb{C} \setminus \{\frac{1}{1-s} : |s| \leq 1\}$ and $\gamma \neq 0$. If $\gamma = 0$, it is obvious that $PM_{1-z}P$ is the bounded inverse of $M_{\frac{1}{1-z}}$.*

*Note that the closure of $PM_{\frac{1}{1-z}}P|_{\mathfrak{D}|_{H^2(\mathbb{D})}}$ is a extension of $S_1$.*

*Assume that $\gamma = \frac{1}{1-s}$ where $|s| < 1$. Note that*

$$\gamma - \frac{1}{1-z} = \frac{s-z}{(1-s)(1-z)}.$$

*Let*

$$\xi = 1 + sz + (sz)^2 + (sz)^3 + \cdots,$$

*and $f \in \mathfrak{D}|_{H^2(\mathbb{D})}$, i.e., $f(z) = (1-z)g(z)$. Then*

$$\langle \frac{s-z}{(1-s)(1-z)} f, \xi\rangle = \frac{1}{1-s}\langle g, PM_{\bar{s}-\bar{z}}\xi\rangle = 0.$$

*Therefore the range of $M_{\frac{s-z}{(1-s)(1-z)}}$ is not dense in $H^2(\mathbb{D})$.*

Note that

$$\mathfrak{D}|_{H^2(\mathbb{D})} = span\{e_i - e_j : i \neq j, i, j \in \{0,1,2,\ldots\}\}.$$

Let $\mathfrak{D}_1 = \{(1-z)g(z) : g \in H^2(\mathbb{D})\}$. Let

$$S_2 : \mathfrak{D}_1 \to H^2(\mathbb{D}), (1-z)g(z) \to g(z), \qquad \forall g \in H^2(\mathbb{D}).$$

It is clear that $\mathfrak{D}|_{\mathcal{H}_1} \subset \mathfrak{D}_1$ and $S_1|_{\mathfrak{D}|_{\mathcal{H}_1}} \subset S_2|_{\mathfrak{D}_2}$. The range of $S_2$ is $H^2(\mathbb{D})$ and the graph of $S_2$ is

$$Gr(S_2) = \{((1-z)g(z), g(z)) : g(z) \in H^2(\mathbb{D})\}.$$

$Gr(S_2)$ is closed since $Gr(S_2)$ can be viewed as the graph $Gr(M_{1-z})$ of the bounded operator $M_{1-z}$.

Let $\mathfrak{D}_2 = \{P(1-\bar{z})g(z) : g(z) \in H^2(\mathbb{D})\} \subset H^2(\mathbb{D})$ where $P$ is the projection form $L^2(S^1)$ onto $H^2(\mathbb{D})$. Note that

$$Ker(PM_{1-\bar{z}}|_{H^2(\mathbb{D})}) = \{0\}.$$

Thus

$$T_2 : \mathfrak{D}_2 \to H^2(\mathbb{D}), P(1-\bar{z})g(z) \to g(z)$$

is well-defined. It is not hard to check that

$$\langle S_2\xi, \beta\rangle = \langle \xi, T_2\beta\rangle,$$

where $\xi \in \mathfrak{D}_1$ and $\beta \in \mathfrak{D}_2$. This implies that $S_2^* = T_2$.

Let $X_n = \{f_n(z) : z \in S^1\}$ where $f_n(z) = 1 + z + z^2 + \cdots + z^{n-1} = \frac{1-z^n}{1-z}$, $n = 1, 2, \ldots$. Each $X_n$ is a compact subset of $\mathbb{C}$. We would like to know the limit of $X_n$ as $n \to \infty$.

Let $z = e^{i\theta}$, then

$$\begin{aligned}\frac{1-z^n}{1-z} &= \frac{1-\cos\theta - \cos n\theta + \cos(n-1)\theta}{2-2\cos\theta} + \frac{\sin\theta - \sin n\theta + \sin(n-1)\theta}{2-2\cos\theta} i \\ &= \frac{\sin(\frac{n}{2}\theta)}{\sin(\frac{\theta}{2})}\left(\cos(\frac{(n-1)}{2}\theta) + i\sin(\frac{(n-1)}{2}\theta)\right).\end{aligned}$$

If $n = 2m+1$, then

$$\frac{1-z^n}{1-z} = \frac{\sin(m\theta + \frac{1}{2}\theta)}{\sin(\frac{\theta}{2})}\left(\cos(m\theta) + i\sin(m\theta)\right).$$

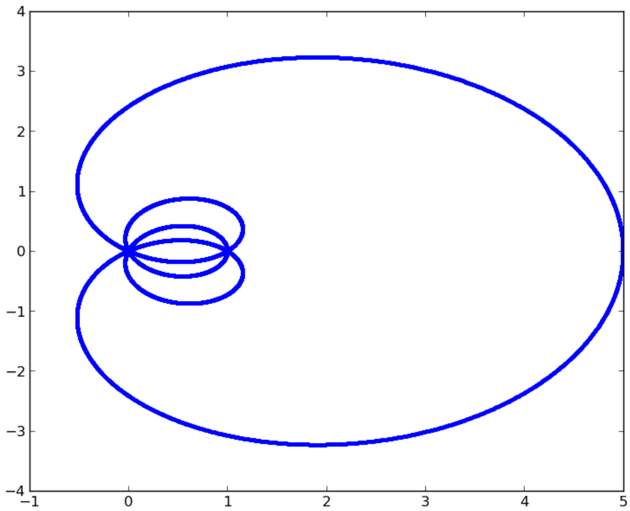


**Figure 1.** $n = 5$

Let $z_1 = e^{i\theta_1}$ and $z_2 = e^{i\theta_2}$ and $\theta_1, \theta_2$ are in $[0, 2\pi)$. Suppose that $f_n(z_1) = f_n(z_2)$ and $\theta_1 < \theta_2$. First assume that $\sin(m\theta_1 + \frac{1}{2}\theta_1) = 0 = \sin(m\theta_2 + \frac{1}{2}\theta_2)$ and $\theta_1, \theta_2$. Note that $\sin(\frac{\theta_1}{2})$ and $\sim (\frac{\theta_2}{2})$ can not equal zero at the same time, since $\frac{\theta_1}{2}$ and $\frac{\theta_1}{2}$ are in $[0, \pi)$. Then

$$\theta_1, \theta_2 \in \{\frac{2k\pi}{2m+1} : k = 1, 2, \ldots, 2m\}.$$

And $f_n(z) = 0$.

Now assume that $\sin(m\theta + \frac{1}{2}\theta) \neq 0$. We have $\theta_2 = \theta_1 + \frac{2k\pi}{m}$ or $\theta_2 = \theta_1 + \frac{(2k+1)\pi}{m}$.

First assume that $\theta_2 = \theta_1 + \frac{2k\pi}{m}$, we have

$$\frac{\sin(m\theta_1 + \frac{1}{2}\theta_1)}{\sin(\frac{\theta_1}{2})} = \frac{\sin(m\theta_2 + \frac{1}{2}\theta_2)}{\sin(\frac{\theta_2}{2})}$$

implies

$$\frac{\sin m\theta_1 \cos(\frac{\theta_1}{2})}{\sin(\frac{\theta_1}{2})} + \cos m\theta_1 = \frac{\sin m\theta_2 \cos(\frac{\theta_2}{2})}{\sin(\frac{\theta_2}{2})} + \cos m\theta_2.$$

If $\sin m\theta_1 \neq 0$ then $\cot(\frac{\theta_1}{2}) = \cot(\frac{\theta_2}{2})$. This implies that $\theta_1 = \theta_2$.

Suppose that $\sin m\theta_1 = 0$, we have $\sin(\frac{\theta_1}{2}) \neq 0$ and $\sin(\frac{\theta_2}{2}) \neq 0$, since $\frac{\theta_1}{2}$ and $\frac{\theta_2}{2}$ are in $[0, \pi)$. This means that $\theta_1$ and $\theta_2$ are in

$$\{\frac{k\pi}{m} : k = 1, 2, \ldots, 2m-1\}.$$

And $f_n(z) = \cos^2 m\theta = 1$.

Now assume that $\theta_2 = \theta_1 + \frac{(2k+1)\pi}{m}$. Then we have

$$\frac{\sin(m\theta_1 + \frac{1}{2}\theta_1)}{\sin(\frac{\theta_1}{2})} = -\frac{\sin(m\theta_2 + \frac{1}{2}\theta_2)}{\sin(\frac{\theta_2}{2})}.$$

This also implies that

$$\frac{\sin m\theta_1 \cos(\frac{\theta_1}{2})}{\sin(\frac{\theta_1}{2})} = \frac{\sin m\theta_1 \cos(\frac{\theta_2}{2})}{\sin(\frac{\theta_2}{2})}.$$

Argue as above, we have $\theta_1$ and $\theta_2$ are in

$$\{\frac{k\pi}{m} : k = 1, 2, \ldots, 2m-1\}.$$

And $f_n(z) = \cos^2 m\theta = 1$.

**Lemma 7.1.** *For any $re^{i\alpha} \in \mathbb{C}$ and any $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ and a $\theta_m \in [0, 2\pi)$ such that*

$$|\frac{\sin(m\theta_m + \frac{1}{2}\theta_m)}{\sin(\frac{\theta_m}{2})}(\cos(m\theta_m) + i\sin(m\theta_m)) - re^{i\alpha}| < \varepsilon$$

*for any $m \geq N$.*

*Proof.* Assume that $\alpha = \frac{2\pi i p}{q}$, where $(p, q) = 1$ and $q > p$. For any $m > 1$, consider the set

$$\{\frac{2\pi i(p + kq)}{qm} : k = 0, 1, \ldots, m-1\}.$$

Let

$$r_k = \frac{\sin(\frac{2\pi i p}{q} + \frac{\pi i(p+kq)}{qm})}{\sin(\frac{\theta_m}{2})} = \cos(\frac{2\pi i p}{q}) + \sin(\frac{2\pi i p}{q})\cot(\frac{\pi i(p+kq)}{qm}).$$

Now it is not hard to see that there exist a $N \in \mathbb{N}$ such that there is a $0 \leq k_m \leq m-1$ such that $|r_{k_m} - r| \leq \varepsilon$ whenever $m > N$. $\square$

AMSS