Wei Fei's Note

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ABSTRACT. Note on Wei Fei's lecture.

1. NOTATIONS

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz class, i.e.,

$$\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : sup_{t\in\mathbb{R}}|t^k \frac{d^l}{dt^l} f(t)| < \infty \text{ for any } k,l \in \mathbb{N} \cup \{0\}\}.$$

Let $f \in \mathcal{S}(\mathbb{R})$. The Fourier transform $\hat{f}$ is

$$\hat{f}(s) = \int_{-\infty}^{+\infty} f(t) e^{-2\pi i s t} dt, \text{ for any } s \in \mathbb{R}.$$

We have

$$f(t) = \int_{-\infty}^{+\infty} \hat{f}(s) e^{2\pi i t s} ds, \text{ for any } t \in \mathbb{R}.$$

2. IMPORTANT FORMULA

2.1. Partial Summation Formula.

Lemma 2.1. *Let $f(x) \in C^1([a,b])$. Then*

$$\sum_{a<n\le b} c_n f(n) = C(b) f(b) - \int_a^b C(x) f'(x) dx,$$

where

$$C(x) = \sum_{a<n\le x} c_n, \qquad c_n \in \mathbb{C}.$$

Theorem 2.1 (Possion Summation Formula)**.** *Let f in $\mathcal{S}(\mathbb{R})$. We have*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n).$$

If $f(x) = f(-x)$, then

$$\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} \hat{f}(n) + \frac{1}{2}(\hat{f}(0) - f(0)).$$

Proof. Let

$$g(\theta) = \sum_{n\in\mathbb{Z}} f(\theta + n).$$

Since $f \in \mathcal{S}(\mathbb{R})$, $g(\theta)$ is a well-defined function of period 1 such that

$$g(\theta) = \sum_{n\in Z} \hat{f}(n) e^{2\pi i \theta}.$$

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Let $\theta = 0$, we have

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n).$$

$\square$

Example 2.1. *Let* $f(x) = e^{-\pi x^2 y}$.

$$\hat{f}(x) = y^{-\frac{1}{2}} e^{-\frac{\pi x^2}{y}}.$$

3. Entire Function

Lemma 3.1. *Let* $\{a_n\}_n \subset \mathbb{C}$ *be a sequence,*

$$0 < |a_1| \le |a_2| \le \cdots \le |a_n| \le \cdots, \text{ and } |a_n| \to \infty.$$

Then there exists a entire function f *such that* $f(s) = 0$ *if and only if* $s \in \{a_n\}_n$.

Proof. Let

$$h_n = (1 - \frac{s}{a_n}) e^{\frac{s}{a_n} + \frac{1}{2}(\frac{s}{a_n})^2 + \frac{1}{3}(\frac{s}{a_n})^3 + \cdots + \frac{1}{n}(\frac{s}{a_n})^n}.$$

Then

$$f(s) = \prod_{n=1}^{\infty} h_n(s)$$

satisfies the condition. $\square$

Remark 3.1. *Let* $\{a_n\}_n \subset \mathbb{C}$ *be a sequence,*

$$0 < |a_1| \le |a_2| \le \cdots \le |a_n| \le \cdots, \text{ and } |a_n| \to \infty.$$

If $\sum_n \frac{1}{|a_n|^{p+1}} < \infty$, *then we could use*

$$h_n = (1 - \frac{s}{a_n}) e^{\frac{s}{a_n} + \frac{1}{2}(\frac{s}{a_n})^2 + \frac{1}{3}(\frac{s}{a_n})^3 + \cdots + \frac{1}{p}(\frac{s}{a_n})^p}.$$

in the above proof.

Example 3.1. *Let* $\{n\}_{n\in\mathbb{Z}}$. *Then*

$$f(s) = s \prod_{n\in\mathbb{N}} (1 - \frac{s}{n})(1 + \frac{s}{n}).$$

Lemma 3.2. *Let* $\{a_n\}_n \subset \mathbb{C}$ *be a sequence,*

$$0 < |a_1| \le |a_2| \le \cdots \le |a_n| \le \cdots, \text{ and } |a_n| \to \infty.$$

Suppose that f *satisfies* $f(s) = 0$ *if and only if* $s \in \{a_n\}_n$. *Then* $f(s) = e^{H(s)} \prod_{n=1}^{\infty} h_n(s)$.

Definition 3.1. *Suppose* $G(s)$ *is a function and* $\mu(r) = max_{|s|\le r} |G(s)|$. *Let*

$$\alpha_0 = inf\{\alpha : \mu(r) \le e^{a_0 r^{\alpha}}\}.$$

Theorem 3.1. *Let* p *be the smallest integer such that*

$$\sum_n \frac{1}{|a_n|^{p+1}} < \infty.$$

Then the degree of

$$f(s) = \prod_n (1 - \frac{s}{a_n}) e^{\frac{s}{a_n} + \frac{1}{2}(\frac{s}{a_n})^2 + \frac{1}{3}(\frac{s}{a_n})^3 + \cdots + \frac{1}{p}(\frac{s}{a_n})^p}$$

is p.

Example 3.2.

$$\sin(\pi s) = \pi s \prod_n (1 - \frac{s^2}{n^2}).$$

Proof. Add proof. $\square$

4. $\Gamma(s)$

Let

$$\Gamma(s) = \int_0^{+\infty} x^{s-1} e^{-x} dx, \qquad Re(S) > 0.$$

It is easy to see that $\Gamma(s+1) = s\Gamma(s)$.

$$\frac{1}{\Gamma(s)} = se^{\gamma s} \prod_{n=1}^{\infty} (1 + \frac{s}{n}) e^{-\frac{s}{n}},$$

where γ is the Euler constant, i.e. $\gamma = \lim_{n\to\infty}(1 + \frac{1}{2} + \cdots \frac{1}{n} - \log(n))$.

Theorem 4.1.

$$\frac{1}{\Gamma(s)} = s \prod_{n=1}^{\infty} (1 + \frac{1}{n})^{-s} (1 + \frac{s}{n}).$$

Proof. add proof. $\square$

Theorem 4.2. *Let* $0 < \delta < \pi$. *We have*

$$\log \Gamma(s) = s \log s - \frac{1}{2} \log s - s + \log(\sqrt{2\pi}) + O_\delta(\frac{1}{|s|}).$$

Proof. Add proof. $\square$

This following is incomplete.

Proposition 4.1. *Let* $s = \sigma + it$. *Assume* $\alpha < \sigma < \beta$. *We have*

$$\Gamma(s) = |t|^{s-\frac{1}{2}} e^{-\frac{\pi}{2}|t| - it +}$$

5. ζ

Theorem 5.1. *If* $Re(s) = \sigma > 1$, *then* $\zeta(s) \neq 0$.

Proof. Note

$$\frac{1}{|\zeta(s)|} = |\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}| \leq \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \leq 1 + \int_1^{\infty} \frac{1}{t^\sigma} dt = \frac{\sigma}{\sigma - 1}.$$

This implies the result. $\square$

Theorem 5.2. *For* $Re(s) > 0$, *we have*

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^2} + \frac{N^{1-s}}{s-1} - \frac{N^{-s}}{2} + s \int_N^{\infty} \frac{\rho(u)}{u^{s+1}} du, \quad N \geq 1,$$

where $\rho(u) = \frac{1}{2} - \{u\}$. *Specially,*

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + s \int_1^{\infty} \frac{\rho(u)}{u^{s+1}} du.$$

Theorem 5.3.

$$\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s) = \pi^{\frac{s-1}{2}} \Gamma(\frac{1-s}{2}) \zeta(1-s)$$

Proof. Add proof. □

Let

$$\xi(s) = \frac{1}{2}s(s-1)\zeta(s)\Gamma(\frac{S}{2})\pi^{-\frac{s}{2}}.$$

Theorem 5.4. *For any $\varepsilon > 0$, we have*

$$|\xi(s)| \ll e^{c|s|^{1+\varepsilon}}.$$

Remark 5.1. *The theorem above implies that ξ has infinite many zeros. (provide a proof).*

For $Re(s) > 1$, estimate

$$f(s) = (1-2^{1-s})\zeta(s) = \sum_{n}^{\infty} \frac{(-1)^{n-1}}{n^s}.$$

It is not hard to see that $\zeta(s)$ does not have real zeros.

Lemma 5.1. *Let $\{\rho_n\}_n$ be the zeros of $\xi(s)$. Then*

$$\sum_{n}^{\infty} \frac{1}{|\rho_n|} = \infty,$$
$$\sum_{n}^{\infty} \frac{1}{|\rho_n|^{1+\varepsilon}} < \infty,$$

for any $\varepsilon > 0$.

Theorem 5.5.

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{1-s} + \sum_{n=1}^{\infty}(\frac{1}{s-\rho_n} + \frac{1}{\rho_n}) + \sum_{n=1}^{\infty}(\frac{1}{s+2n} - \frac{1}{2n}) + B_0,$$

where B_0 is a constant.

Proof. Consider the two expressions of $\xi(s)$:

$$\xi(s) = e^{as+b}\prod_{n}(1-\frac{s}{\rho_n})e^{\frac{s}{\rho_n}}$$
$$\xi(s) = \frac{1}{2}s(s-1)\zeta(s)\Gamma(\frac{S}{2})\pi^{-\frac{s}{2}}.$$

Compute the derivative of $\log \xi(s)$ by plugging in the two expressions. □

Theorem 5.6. *Let $T \geq 0$ and $\rho_n = \beta_n + i\gamma_n$ be the non-trivial zeros of $\zeta(s)$. Then*

$$\sum_{n=1}^{\infty} \frac{1}{1+(T-\gamma_n)^2} \leq C\log(T+2).$$

Proof. Let $s = 2 + iT$.

$$Re(\frac{1}{s-\rho_n}) = \frac{2-\beta_n}{(2-\beta_n)^2 + (T-\gamma_n)^2)^2} \geq \frac{1}{4(1+(T-\gamma_n)^2)}.$$

□

6. Prime number theorem

Lemma 6.1. *Let $s = \sigma + it$. Then*

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{1-s} + \sum_{|\gamma_n - t| \leq 1} \frac{1}{s-\rho_n} + O(\log(|t|+2))$$

where $\rho_n = \beta_n + i\gamma_n$ is the n-th non-trivial zero of $\zeta(s)$.

Proof. Add proof. □

Theorem 6.1 (Peron's Formula)**.** *Let $x \geq 4$, $\|x\| = min_{N\in\mathbb{N}}|x-N| \leq \frac{1}{2}$. For any $b > \sigma_0$ and $T \geq 2$, $0 < \theta < 1$ and $2 \leq H \leq (1-\theta)x$, we have*

$$\sum_{n\leq x} a_n = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} f(s)\frac{x^s}{s}ds + O(\frac{xA_H(x)\log(\frac{x}{H})}{T}) + O(\frac{x^b B(b) H}{T}) + O(|a_N| min(1, \frac{x}{T\|x\|}),$$

where $A_H(x) \geq max_{x-\frac{x}{H}\leq n\leq x+\frac{x}{H}}|a_n|$, and $f(s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s}$ and $B(b) = \sum_{n=1}^{\infty}\frac{|a_n|}{n^b}$, $b > \sigma_0$ converges absolutely.

Theorem 6.2. $\varphi(x) = \sum_{n\leq X}\Lambda(n) = X + O(xe^{-c\sqrt{\log X}})$.

7. $n \times 2$ case

Let

$$S_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in M_n(\mathbb{C})$$

We would like to find the commutant of $(I - \frac{1}{2}S_n) \otimes (I - \frac{1}{3}S_2)$ in $M_n(\mathbb{C}) \otimes M_2(\mathbb{C})$. If

$$\begin{aligned}\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}\begin{pmatrix} S_n & \frac{2}{3}(I-\frac{1}{2}S_n) \\ 0 & S_n \end{pmatrix} &= \begin{pmatrix} T_1S_n & T_1\frac{2}{3}(I-\frac{1}{2}S_n)+T_2S_n \\ T_3S_n & T_3\frac{2}{3}(I-\frac{1}{2}S_n)+T_4S_n \end{pmatrix} \\ &= \begin{pmatrix} S_nT_1+\frac{2}{3}(I-\frac{1}{2}S_n)T_3 & S_nT_2+\frac{2}{3}(I-\frac{1}{2}S_n)T_4 \\ S_nT_3 & S_nT_4 \end{pmatrix} = \begin{pmatrix} S_n & \frac{2}{3}(I-\frac{1}{2}S_n) \\ 0 & S_n \end{pmatrix}\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}\end{aligned}$$

Since T_3 commute with S_n, T_3 must be a polynomial of S_n. Also note that $T_1S_n - S_nT_1 = \frac{2}{3}(I-\frac{1}{2}S_n)T_3$, this implies that the trace of T_3 is zero. Therefore T_3 must be upper triangular.

Note that

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \ddots & x_{3n} \\ x_{n-1,1} & x_{n-1,2} & x_{n-1,3} & \cdots & x_{n-1,n} \\ x_{n,1} & x_{n,2} & x_{n,3} & \cdots & x_{n,n} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$- \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \ddots & x_{3n} \\ x_{n-1,1} & x_{n-1,2} & x_{n-1,3} & \cdots & x_{n-1,n} \\ x_{n,1} & x_{n,2} & x_{n,3} & \cdots & x_{n,n} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & x_{11} & x_{12} & \cdots & x_{1,n-1} \\ 0 & x_{21} & x_{22} & \cdots & x_{2,n-1} \\ 0 & x_{31} & x_{32} & \ddots & x_{3,n-1} \\ 0 & x_{n-1,1} & x_{n-1,2} & \cdots & x_{n-1,n-1} \\ 0 & x_{n,1} & x_{n,2} & \cdots & x_{n,n-1} \end{pmatrix} - \begin{pmatrix} x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3n} \\ x_{41} & x_{42} & x_{43} & \ddots & x_{4n} \\ x_{n,1} & x_{n,2} & x_{n,3} & \cdots & x_{n,n} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

If the above is strict upper triangular, we must have

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \ddots & x_{3n} \\ x_{n-1,1} & x_{n-1,2} & x_{n-1,3} & \cdots & x_{n-1,n} \\ x_{n,1} & x_{n,2} & x_{n,3} & \cdots & x_{n,n} \end{pmatrix}$$

is upper triangular.

So we have T_1, T_4 are upper triangular. And it is easy to see that for a fixed T_3 which commute with S_n, we have many elements which commute with $(I - \frac{1}{2}S_n) \otimes (I - \frac{1}{3}S_2)$.

8. Spectrum of sums of unitary

Definition 8.1. *Let T be a densely defined, closed operator on a Hilbert space $\mathcal{H}$.*

(1) The resolvent set $\rho(T)$ of T is the set of all complex numbers λ such that $\lambda - T$ is a bijection (between $\mathfrak{D}(T)$ and $\mathcal{H}$) with bounded inverse.
(2) The resolvent of T at $\lambda \in \rho(T) \subset \mathbb{C}$ is $R_\lambda(T) = (\lambda - T)^{-1}$.
(3) The spectrum of T is $\sigma(T) = \mathbb{C} \setminus \rho(T)$.
(4) $\lambda \in \mathbb{C}$ is a point spectrum of T if $\lambda - T$ is not injective.
(5) $\lambda \in \mathbb{C}$ is a residual spectrum of T if $\lambda - T$ is injective but the range of $\lambda - T$ is not dense in $\mathcal{H}$.

Example 8.1. *Let $\mathcal{H} = L^2([0,1])$ and*

$$\mathfrak{D} = \{f(x) = C + \int_0^x \varphi(t)dt : C \in \mathbb{C}, \varphi \in L^2([0,1])\}.$$

- *Assume that $\mathfrak{D}(T) = \mathfrak{D}$ such that $Tf = i\frac{df}{dx}$. Then $\sigma(T) = \mathbb{C}$ since $e^{-i\lambda x}$ is an eigenfunction for T with eigenvalue λ.*
- *If $\mathfrak{D}(T) = \{f \in \mathfrak{D} : f(0) = 0\}$ with $Tf = i\frac{df}{dx}$, then $\sigma(T) = \varnothing$. In fact, for any $\lambda \in \mathbb{C}$, the resolvent of $(\lambda - T)^{-1}$ is*

$$(R_\lambda(T)f)(x) = i\int_0^x e^{-i\lambda(x-t)} f(t)dt.$$

- *Let $\alpha \in \mathbb{C} \setminus \{0\}$. If $\mathfrak{D}(T) = \{f \in \mathfrak{D} : f(0) = \alpha f(1)\}$ with $Tf = i\frac{df}{dx}$, then $\sigma(T) = \{-i \ln \alpha + 2k\pi : k \in \mathbb{Z}\}$. If $\lambda = -i \ln \alpha + 2k\pi$, then $e^{-i\lambda x}$ is an eigenfunction for T. For λ not of the form $-i \ln \alpha + 2k\pi$, the resolvent operator $(\lambda - T)^{-1}$ is*
$$(R_\lambda(T)f)(x) = \int_0^1 G_\lambda(x,t)f(t)dt$$
with
$$G_\lambda(x,t) = \begin{cases} \frac{i\alpha e^{i\lambda(t-x-1)}}{1-\alpha e^{-i\lambda}} & \text{if } x < t \\ \frac{i\alpha e^{i\lambda(t-x)}}{1-\alpha e^{-i\lambda}} & \text{if } x > t \end{cases}$$

Let U be a unitary such that $(Uf)(t) = f(t+1)$. Then
$$\hat{Uf}(s) = e^{2\pi i s}\hat{f}(s).$$

Let
$$\mathfrak{D} = \{(a_l)_{l\in\mathbb{Z}} : \exists m \in \mathbb{N}, a_n = 0, \text{ for any } n > m \text{ or } n < m \text{ and } \textstyle\sum_{-m\le k\le m} a_k = 0\}.$$
It is easy to see that $\mathfrak{D}$ is dense in $\mathcal{H}$.

Let $\mathcal{H} = l^2(\mathbb{Z})$ and
$$T : (a_l)_{l\in\mathbb{Z}} \in \mathfrak{D} \to (\sum_{l\le k<+\infty} a_k)_{l\in\mathbb{Z}},$$
$$S : (a_l)_{l\in\mathbb{Z}} \in \mathfrak{D} \to (\sum_{-\infty<k\le l} a_k)_{l\in\mathbb{Z}}.$$
It is easy to see that T and S are well-defined.

Let
$$\xi = (\dots, 0, a_{-m}, a_{-m+1} \dots, a_{m-1}, a_m, 0, \dots) \in \mathfrak{D},$$
$$\beta = (\dots, 0, b_{-m}, b_{-m+1} \dots, b_{m-1}, b_m, 0, \dots) \in \mathfrak{D}.$$
Then we have
$$\begin{aligned}\langle T\xi, \beta\rangle &= \bar{b}_{-m}(a_{-m} + \cdots + a_m) + \bar{b}_{-m+1}(a_{-m+1} + \cdots + a_m) + \cdots + \bar{b}_m a_m \\ &= a_{-m}\bar{b}_{-m} + a_{-m+1}(\bar{b}_{-m} + \bar{b}_{-m+1}) + \cdots + a_m(\bar{b}_{-m} + \cdots + \bar{b}_m) = \langle \xi, S\beta\rangle.\end{aligned}$$
Therefore, T and S are both closable.

Example 8.2. *Let $U : e_k \to e_{k-1}$ where $\{e_k\}_{k\in\mathbb{Z}}$ is the canonical orthonormal basis of $l^2(\mathbb{Z})$. It is well-known that the spectrum of U is S^1.*

Formally, we can write T and S as
$$T = \sum_{0\le k<\infty} U^k, \qquad S = \sum_{0\le k<\infty} U^{-k}.$$

Let $\mathfrak{A}$ be the von Neumann algebra generated by U. Consider the closure of T and S affiliated with $\mathfrak{A}$.

We can identifies $l^2(\mathbb{Z})$ with $L^2(S^1)$ and e_n with $z^n \in L^2(S^1)$. Then U is the multiplication of $\frac{1}{z}$. Note that every vector in $\mathfrak{D}$ corresponding to a function
$$(\dots, 0, a_{-m}, a_{-m+1} \dots, a_{m-1}, a_m, 0, \dots) \Longleftrightarrow f(z) = \sum_{-m\le k\le m} a_k z^k.$$
By the definition of $\mathfrak{D}$, we have $f(1) = 0$. Therefore $f(z) = (z-1)g(z)$ where the Fourier coefficients of $g(z)$ is finite supported.

It is not hard to check that
$$(Tf)(z) = (M_{\frac{z}{z-1}}f)(z) = \frac{zf(z)}{z-1} \quad \text{and} \quad (Sf)(z) = (M_{\frac{1}{1-z}}f)(z) = \frac{f(z)}{1-z} \qquad f \in \mathfrak{D}.$$

It is obvious that $M_{\frac{z}{z-1}}$ *and* $M_{\frac{1}{1-z}}$ *are closed operator affiliated with* $\mathfrak{A}$ *and extend* T *and* S *respectively.*

Remark 8.1. *Note that for* $z \in S^1$, $\bar{z} = \frac{1}{z}$ *and*

$$M^*_{\frac{z}{z-1}} = M_{\frac{\frac{1}{z}}{\frac{1}{z}-1}} = M_{\frac{1}{1-z}},$$
$$M_{\frac{z}{z-1}} + M_{\frac{1}{1-z}} = I.$$

It is not hard to see that the spectrum of $M_{\frac{1}{1-z}}$ *as a closed operator affiliated with* $\mathfrak{A}$ *is*

$$\{\frac{1}{1-s} : |s| = 1\}.$$

Let $\gamma \in \mathbb{C} \setminus \{\frac{1}{1-s} : |s| \leq 1\}$. *If* $\gamma = 0$, *then* M_{1-z} *is a bounded inverse of* $M_{\frac{1}{1-z}}$ *in the Banach algebra generated by* U^*. *If* $\gamma \neq 0$, *then*

$$\frac{1}{\gamma} - 1 \notin \{s : |s| \leq 1\}.$$

This implies that $|\frac{1-\gamma}{\gamma}| > 1$. *Thus*

$$M_{\frac{(\gamma-1)-\gamma z}{1-z}} = \frac{1}{(\gamma-1)}(1 - M_z)(I + \frac{\gamma}{\gamma-1}M_z + (\frac{\gamma}{\gamma-1}M_z)^2 + (\frac{\gamma}{\gamma-1}M_z)^3 + \cdots).$$

Thus $\gamma - M_{\frac{1}{1-z}}$ *has a bounded inverse in the Banach algebra generated by* $U^* = M_z$.

Assume that $\gamma = \frac{1}{1-s}$ *where* $|s| < 1$. *Then*

$$\gamma - \frac{1}{1-z} = \frac{s-z}{(1-s)(1-z)}.$$

Note that

$$\frac{(1-s)(1-z)}{s-z}$$

is not analytic in the unit disk, therefore $\gamma - M_{\frac{1}{1-z}}$ *does not has a bounded inverse in the Banach algebra generated by* M_z.

In summary, we have the spectrum of $M_{\frac{1}{1-z}}$ *restricted to the Banach algebra generated by* M_z *is* $\{\frac{1}{1-s} : |s| \leq 1\}$.

Let $\omega = e^{\frac{2\pi i}{n}}$ and

$$W = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ \omega & \omega^2 & \omega^3 & \cdots & \omega^{n-1} & 1 \\ \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(n-1)} & 1 \\ \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(n-1)} & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \cdots & \omega^{(n-1)(n-1)} & 1 \end{pmatrix}$$

Recall the definition of Hardy space $H^2(\mathbb{D})$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in Hol(\mathbb{D})$, then $f \in H^2(\mathbb{D})$ if

$$sup_{0<r<1} \int_{\mathbb{T}} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

Let $P : l^2(\mathbb{Z}) \to l^2(\mathbb{N})$ be the orthonormal projection onto the subspace spanned by e_i, $i \geq 0$. Identify the Hardy space $H^2(\mathbb{D})$ with $Pl^2(\mathbb{Z})$ and

$$T_1 = PTP : (a_0, a_1, a_2, \dots) \to (\sum_{k=0} a_k, \sum_{k=1} a_k, \sum_{k=2} a_k, \dots),$$

$$S_1 = PSP : (a_0, a_1, a_2, \dots) \to (a_0, \sum_{k=0}^{1} a_k, \sum_{k=0}^{2} a_k, \dots),$$

where $(a_0, a_1, a_2, \dots) \in \mathfrak{D}|_{H^2(\mathbb{D})}$.

Let $\xi = (a_0, \dots, a_n, 0, 0, \dots)$ and $\beta = (b_0, \dots, b_n, 0, 0, \dots)$ in $\mathfrak{D}|_{H^2(\mathbb{D})}$.

$$\begin{aligned}\langle T_1\xi, \beta\rangle &= \bar{b}_0(a_0 + \cdots + a_n) + \bar{b}_1(a_1 + \cdots + a_n) + \cdots + \bar{b}_n a_n \\ &= a_0\bar{b}_0 + a_1(\bar{b}_0 + \bar{b}_1) + \cdots + a_n(\bar{b}_0 + \cdots + \bar{b}_n) = \langle \xi, S_1\beta\rangle.\end{aligned}$$

Note that

$$(I-P)SP\xi = 0 \text{ and } (I-P)TP\xi = 0,$$

for any $\xi \in \mathfrak{D}|_{H^2(\mathbb{D})}$.

Example 8.3. *By identify $l^2(\mathbb{Z})$ with $L^2(S^1)$, we have*

$$PM_{\frac{(\gamma-1)-\gamma z}{1-z}}P = \frac{1}{(\gamma-1)}(1 - PM_zP)(I + \frac{\gamma}{\gamma-1}PM_zP + (\frac{\gamma}{\gamma-1}PM_zP)^2 + (\frac{\gamma}{\gamma-1}PM_zP)^3 + \cdots)$$

is the bounded inverse of $\gamma - M_{\frac{1}{1-z}}$ for $\gamma \in \mathbb{C} \setminus \{\frac{1}{1-s} : |s| \leq 1\}$ and $\gamma \neq 0$. If $\gamma = 0$, it is obvious that $PM_{1-z}P$ is the bounded inverse of $M_{\frac{1}{1-z}}$.

Note that the closure of $PM_{\frac{1}{1-z}}P|_{\mathfrak{D}|_{H^2(\mathbb{D})}}$ is a extension of S_1.

Assume that $\gamma = \frac{1}{1-s}$ where $|s| < 1$. Note that

$$\gamma - \frac{1}{1-z} = \frac{s-z}{(1-s)(1-z)}.$$

Let

$$\xi = 1 + sz + (sz)^2 + (sz)^3 + \cdots,$$

and $f \in \mathfrak{D}|_{H^2(\mathbb{D})}$, i.e., $f(z) = (1-z)g(z)$. Then

$$\langle \frac{s-z}{(1-s)(1-z)} f, \xi\rangle = \frac{1}{1-s}\langle g, PM_{\bar{s}-\bar{z}}\xi\rangle = 0.$$

Therefore the range of $M_{\frac{s-z}{(1-s)(1-z)}}$ is not dense in $H^2(\mathbb{D})$.

Note that

$$\mathfrak{D}|_{H^2(\mathbb{D})} = span\{e_i - e_j : i \neq j, i, j \in \{0, 1, 2, \dots\}\}.$$

Let $\mathfrak{D}_1 = \{(1-z)g(z) : g \in H^2(\mathbb{D})\}$. Let

$$S_2 : \mathfrak{D}_1 \to H^2(\mathbb{D}), (1-z)g(z) \to g(z), \qquad \forall g \in H^2(\mathbb{D}).$$

It is clear that $\mathfrak{D}|_{\mathcal{H}_1} \subset \mathfrak{D}_1$ and $S_1|_{\mathfrak{D}|_{\mathcal{H}_1}} \subset S_2|_{\mathfrak{D}_2}$. The range of S_2 is $H^2(\mathbb{D})$ and the graph of S_2 is

$$Gr(S_2) = \{((1-z)g(z), g(z)) : g(z) \in H^2(\mathbb{D})\}.$$

$Gr(S_2)$ is closed since $Gr(S_2)$ can be viewed as the graph $Gr(M_{1-z})$ of the bounded operator M_{1-z}.

Let $\mathfrak{D}_2 = \{P(1-\bar{z})g(z) : g(z) \in H^2(\mathbb{D})\} \subset H^2(\mathbb{D})$ where P is the projection form $L^2(S^1)$ onto $H^2(\mathbb{D})$. Note that

$$Ker(PM_{1-\bar{z}}|_{H^2(\mathbb{D})}) = \{0\}.$$

Thus

$$T_2 : \mathfrak{D}_2 \to H^2(\mathbb{D}), P(1-\bar{z})g(z) \to g(z)$$

is well-defined. It is not hard to check that

$$\langle S_2\xi, \beta\rangle = \langle \xi, T_2\beta\rangle,$$

where $\xi \in \mathfrak{D}_1$ and $\beta \in \mathfrak{D}_2$. This implies that $S_2^* = T_2$.

Let $X_n = \{f_n(z) : z \in S^1\}$ where $f_n(z) = 1 + z + z^2 + \cdots + z^{n-1} = \frac{1-z^n}{1-z}$, $n = 1, 2, \ldots$. Each X_n is a compact subset of $\mathbb{C}$. We would like to know the limit of X_n as $n \to \infty$.

Let $z = e^{i\theta}$, then

$$\begin{aligned} \frac{1-z^n}{1-z} &= \frac{1-\cos\theta - \cos n\theta + \cos(n-1)\theta}{2-2\cos\theta} + \frac{\sin\theta - \sin n\theta + \sin(n-1)\theta}{2-2\cos\theta}i \\ &= \frac{\sin(\frac{n}{2}\theta)}{\sin(\frac{\theta}{2})}\left(\cos(\frac{(n-1)}{2}\theta) + i\sin(\frac{(n-1)}{2}\theta)\right). \end{aligned}$$

If $n = 2m+1$, then

$$\frac{1-z^n}{1-z} = \frac{\sin(m\theta + \frac{1}{2}\theta)}{\sin(\frac{\theta}{2})}\left(\cos(m\theta) + i\sin(m\theta)\right).$$

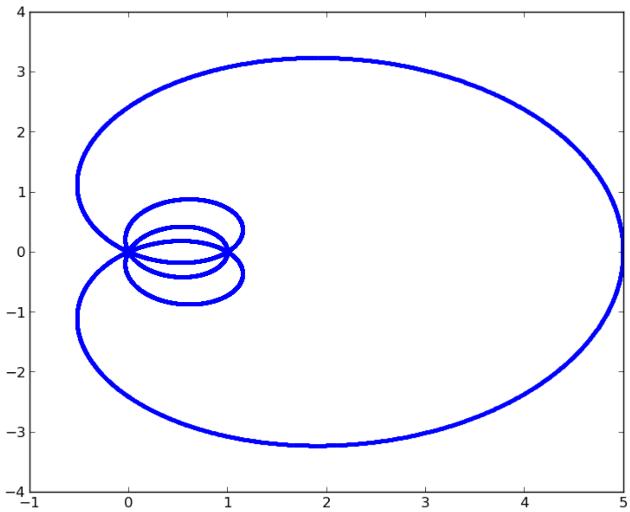

Figure 1. $n = 5$

Let $z_1 = e^{i\theta_1}$ and $z_2 = e^{i\theta_2}$ and θ_1, θ_2 are in $[0, 2\pi)$. Suppose that $f_n(z_1) = f_n(z_2)$ and $\theta_1 < \theta_2$. First assume that $\sin(m\theta_1 + \frac{1}{2}\theta_1) = 0 = \sin(m\theta_2 + \frac{1}{2}\theta_2)$ and θ_1, θ_2. Note that $\sin(\frac{\theta_1}{2})$ and $\sim(\frac{\theta_2}{2})$ can not equal zero at the same time, since $\frac{\theta_1}{2}$ and $\frac{\theta_1}{2}$ are in $[0, \pi)$. Then

$$\theta_1, \theta_2 \in \{\frac{2k\pi}{2m+1} : k = 1, 2, \ldots, 2m\}.$$

And $f_n(z) = 0$.

Now assume that $\sin(m\theta + \frac{1}{2}\theta) \neq 0$. We have $\theta_2 = \theta_1 + \frac{2k\pi}{m}$ or $\theta_2 = \theta_1 + \frac{(2k+1)\pi}{m}$.

First assume that $\theta_2 = \theta_1 + \frac{2k\pi}{m}$, we have

$$\frac{\sin(m\theta_1 + \frac{1}{2}\theta_1)}{\sin(\frac{\theta_1}{2})} = \frac{\sin(m\theta_2 + \frac{1}{2}\theta_2)}{\sin(\frac{\theta_2}{2})}$$

implies

$$\frac{\sin m\theta_1 \cos(\frac{\theta_1}{2})}{\sin(\frac{\theta_1}{2})} + \cos m\theta_1 = \frac{\sin m\theta_2 \cos(\frac{\theta_2}{2})}{\sin(\frac{\theta_2}{2})} + \cos m\theta_2.$$

If $\sin m\theta_1 \neq 0$ then $\cot(\frac{\theta_1}{2}) = \cot(\frac{\theta_2}{2})$. This implies that $\theta_1 = \theta_2$.

Suppose that $\sin m\theta_1 = 0$, we have $\sin(\frac{\theta_1}{2}) \neq 0$ and $\sin(\frac{\theta_2}{2}) \neq 0$, since $\frac{\theta_1}{2}$ and $\frac{\theta_2}{2}$ are in $[0, \pi)$. This means that θ_1 and θ_2 are in

$$\{\frac{k\pi}{m} : k = 1, 2, \ldots, 2m - 1\}.$$

And $f_n(z) = \cos^2 m\theta = 1$.

Now assume that $\theta_2 = \theta_1 + \frac{(2k+1)\pi}{m}$. Then we have

$$\frac{\sin(m\theta_1 + \frac{1}{2}\theta_1)}{\sin(\frac{\theta_1}{2})} = -\frac{\sin(m\theta_2 + \frac{1}{2}\theta_2)}{\sin(\frac{\theta_2}{2})}.$$

This also implies that

$$\frac{\sin m\theta_1 \cos(\frac{\theta_1}{2})}{\sin(\frac{\theta_1}{2})} = \frac{\sin m\theta_1 \cos(\frac{\theta_2}{2})}{\sin(\frac{\theta_2}{2})}.$$

Argue as above, we have θ_1 and θ_2 are in

$$\{\frac{k\pi}{m} : k = 1, 2, \ldots, 2m - 1\}.$$

And $f_n(z) = \cos^2 m\theta = 1$.

Lemma 8.1. *For any $re^{i\alpha} \in \mathbb{C}$ and any $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ and a $\theta_m \in [0, 2\pi)$ such that*

$$\left|\frac{\sin(m\theta_m + \frac{1}{2}\theta_m)}{\sin(\frac{\theta_m}{2})} (\cos(m\theta_m) + i\sin(m\theta_m)) - re^{i\alpha}\right| < \varepsilon$$

for any $m \geq N$.

Proof. Assume that $\alpha = \frac{2\pi i p}{q}$, where $(p, q) = 1$ and $q > p$. For any $m > 1$, consider the set

$$\{\frac{2\pi i(p + kq)}{qm} : k = 0, 1, \ldots, m - 1\}.$$

Let

$$r_k = \frac{\sin(\frac{2\pi i p}{q} + \frac{\pi i(p+kq)}{qm})}{\sin(\frac{\theta_m}{2})} = \cos(\frac{2\pi i p}{q}) + \sin(\frac{2\pi i p}{q})\cot(\frac{\pi i(p + kq)}{qm}).$$

Now it is not hard to see that there exist a $N \in \mathbb{N}$ such that there is a $0 \leq k_m \leq m - 1$ such that $|r_{k_m} - r| \leq \varepsilon$ whenever $m > N$. $\square$

In the sequal, we will use inifinte tensor product of Hilbert. So let us recall some basic facts first.

Let $\{\mathcal{H}_k\}_{k\in I}$ be a family of Hilbert spaces and I is a index set. A incomplete tensor product of $\{\mathcal{H}_k\}$ is constructed as following ([1]).

Let $\{e^k_j\}$ be a orthnormal basis of $\mathcal{H}_k$ and $\xi = \otimes_{k\in I} e^k_0$. A orthnormal basis of the incomplete tensor product $\otimes^{\xi}_{k\in I}\mathcal{H}_k$ is

$$\{\otimes_{k\in I} e^k_{j_k} : j_k \neq 0 \text{ occurs for a finite k's only}\}$$

(Lemma 4.1.4 in [1]).

$\otimes^{\xi}_{k\in I}\mathcal{H}_k$ contains all sequences $(\xi_k)_{k\in I}$ where $\xi_k \in \mathcal{H}_k$ such that

$$\sum_{k\in I} |\langle \xi_k, e^k_0\rangle - 1| \text{ converges.}$$

Let $\otimes^{\infty}_{k=1} H^2(\mathbb{D})$ be the infinte tensor product of $H^2(\mathbb{D})$. More precisely, let $\{e_j\}^{\infty}_{j=0}$ be the canonical basis of $H^2(\mathbb{D})$, a basis of $\otimes^{\infty}_{k=1} H^2(\mathbb{D})$ is $\{\otimes e_{j_k}\}_{(j_1,j_2,\ldots)\in\mathbb{N}\cup\{0\}}$ where only finitely many of $j_k \neq 0$.

Note that $\otimes^n_{j=1} H^2(\mathbb{D}) \to \otimes^{\infty}_{k=1} H^2(\mathbb{D})$ given by

$$\otimes^n_{k=1}\xi_k \to \otimes^n_{k=1}\xi_k \otimes^{\infty}_{l=n+1} e_0,$$

is an embedding.

Let $U : e_n \to e_{n-1}(Ue_0 = 0)$ be the backward shift on $H^2(\mathbb{D})$ and T be a unbounded operator with domain

$$\mathfrak{D} = \cup^{\infty}_{n=1} \otimes^n_{j=1} H^2(\mathbb{D}),$$

given by

$$T : \otimes^n_{k=1}\xi_k \to \otimes^n_{k=1}(I-U)\xi_k.$$

Let $\xi_n = (\frac{\sqrt{n^2-1}}{n} - \frac{1}{n})e_0 - \frac{1}{n}e_1$. Then $(I-U)\xi = \frac{\sqrt{n^2-1}}{n}e_0 - \frac{1}{n}e_1 = \beta$ is a unit vector. Note that

$$T : \underbrace{\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n}_{\text{n}} \to \underbrace{\beta_1 \otimes \beta_2 \otimes \cdots \otimes \beta_n}_{\text{n}}.$$

Note that

$$\lim_{n\to\infty} \|\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n\|^2_2 = \lim_{n\to\infty} \prod^n_{k=1}(1 - \frac{2\sqrt{n^2-1}-1}{n^2}) = 0.$$

We also have

$$\|\beta_n - e_0\|^2_2 = \frac{2(n-\sqrt{n^2-1})}{n} = \frac{2}{n(n+\sqrt{n^2-1})}.$$

Note that

$$\begin{aligned}
\|\beta_1 \otimes \cdots \otimes \beta_n \otimes e_0 \cdots - \otimes^{\infty}_{k=1}\beta_k\|^2_2 &= 2(1 - \prod^{\infty}_{j=1} \frac{\sqrt{(n+j)^2-1}}{n+j}) \\
&= 2(1 - \prod^{\infty}_{j=1}(1 - \frac{1}{(n+j)[(n+j)+\sqrt{(n+j)^2-1}]})) \\
&\leq 2(e^{\sum^{\infty}_{j=1} \frac{1}{(n+j)[(n+j)+\sqrt{(n+j)^2-1}]}} - 1).
\end{aligned}$$

Therefore, $\beta_1 \otimes \cdots \otimes \beta_n$ converges to $\otimes^{\infty}_{k=1}\beta_k$ in $\otimes^{\infty}_{k} H^2(\mathbb{D})$.

Thus, T is not preclosed unbounded operator.

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