Holomorphic Fields of Hilbert Spaces in Finite von Neumann algebra

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ABSTRACT. Enter abstract here

1. INTRODUCTION

Let $\mathcal{H}$ be a Hilbert space, $Proj(\mathcal{H})$ be the set of self-adjoint projections and $Uint(\mathcal{H})$ be the set of unitary operators. We will routinely identify a closed subspace with its associated orthogonal projection in $B(\mathcal{H})$. For a set $\mathcal{L}$ of orthogonal projections in $Proj(\mathcal{H})$, we denote by $Alg\mathcal{L}$ the set of all bounded linear operators on $\mathcal{H}$ leaving each element in $\mathcal{L}$ invariant. Then $Alg\mathcal{L}$ is an unital weak-operator closed subalgebra of $B(\mathcal{H})$. Similarly, for a subset $\mathcal{S}$ of $B(\mathcal{H})$, let $Lat\mathcal{S}$ be the set of invariant projections for every operators in $\mathcal{S}$. Then $Lat\mathcal{S}$ is a strong-operator closed lattice of projections. A subalgebra $\mathcal{A}$ of $B(\mathcal{H})$ is said to be reflexive if $\mathcal{A} = Alg\mathcal{L}at\mathcal{A}$, similarly a lattice $\mathcal{L}$ of projections is reflexive if $\mathcal{L} = LatAlg\mathcal{L}$.

2. HILBERT FIELDS OVER SPHERE

Let

$$\mathfrak{I}_1 = \begin{pmatrix} 0 & T_1 \\ 0 & I \end{pmatrix} \text{ and } \mathfrak{I}_2 = \begin{pmatrix} 0 & T_2 \\ 0 & I \end{pmatrix}$$

be two (unbounded) idempotents acting on $\mathcal{H} \oplus \mathcal{H}$, where T_1 and T_2 are two closed operators affiliated with a finite von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$. The ranges of $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are closed subspaces of $\mathcal{H} \oplus \mathcal{H}$.

Recall that the set of closed, densely defined operators affiliated with a finite von Neumann algebra $\mathfrak{A}$ admits algebraic operations of addition and multiplication. In another word, the set is an unital * algebra (cf. [23]), and the elements in it can be manipulated as if they were bounded operators. In the rest of the paper, we will repeatedly make use this fact without mentioning it explicitly. For a elegant treatment of this subject, we refer readers to [24].

Let $Q(\infty) = Ker(\mathfrak{I}_1)$, $Q(0) = Ran(\mathfrak{I}_1)$ and $Q(-1) = Ran(\mathfrak{I}_2)$ be there trace half projections in $\mathfrak{A} \otimes M_2(\mathbb{C})$. Note that

$$\mathcal{A}lg(Q(\infty), Q(0), Q(-1)) = \{A | A\mathfrak{I}_i \subset \mathfrak{I}_i A, i = 1, 2\}.$$

By conjugating a unitary, we could assume that

$$T_1 - T_2 = \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}$$

where $K \geq 0$ and K has a closed inverse.

2010 *Mathematics Subject Classification.* Primary ; Secondary .

We also write T_1 as $\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$. It is not hard to check that $\mathcal{A}lg(Q(\infty), Q(0), Q(-1))$ contains the following elements

$$\begin{pmatrix} A_1 & 0 & T_{11}K^{-1}A_1K - A_1T_{11} & -A_1T_{12} \\ 0 & 0 & T_{21}K^{-1}A_1K & 0 \\ 0 & 0 & K^{-1}A_1K & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_2 & -A_2T_{21} & -A_2T_{22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & A_3 & -A_3T_{21} & -A_3T_{22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & T_{12}D_1 & 0 \\ 0 & 0 & T_{22}D_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & D_1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & T_{12}D_2 \\ 0 & 0 & 0 & T_{22}D_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_2 \end{pmatrix}$$

If $Q \in \mathcal{L}at(\mathcal{A}lg(Q(\infty), Q(0), Q(-1)))$ such that $Q \wedge Q(\infty) = 0$ and $Q \vee Q(\infty) = I$, then there is an idempotent

$$\mathfrak{I} = \begin{pmatrix} 0 & T_1 + S_1 \\ 0 & I \end{pmatrix}$$

such that $\mathfrak{I}A = A\mathfrak{I}$ for any $A \in \mathcal{A}lg(Q(\infty), Q(0), Q(-1))$. This implies that

$$\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = 0 \text{ and } \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} = 0.$$

Therefore S_{12}, S_{21} and S_{22} are all equals 0. Since

$$\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} S_{11} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} S_{11} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} K^{-1}A_1K & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{for all } A_1 \in Ran(K)\mathfrak{A}Ran(K).$$

This implies that $S = zK$ for some $z \in \mathbb{C}$. We will denote this idempotent by $\mathfrak{I}(z)$ and its range projection by $Q(z)$. Note that $\mathfrak{I}_1 = \mathfrak{I}(0)$ and $\mathfrak{I}_2 = \mathfrak{I}(-1)$.

If $Q \wedge Q(\infty) = 0$ and $Q \vee Q(\infty) \neq I$, then

$$E \equiv Q \vee Q(\infty) = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}.$$

Therefore, there must exist a $\beta = (\xi_1, \xi_2, \xi_3, \xi_4)^T \in Q\mathcal{H}$ such that $\xi_4 \neq 0$. This implies that

$$\{(T_{12}\xi, T_{22}\xi, 0, \xi)^t | \xi\} \subset Q\mathcal{H}.$$

Consider the trace of Q, we know that

$$\{(T_{12}\xi, T_{22}\xi, 0, \xi)^t | \xi\} = Q\mathcal{H}, \text{ and } Q = Ran(\begin{pmatrix} 0 & T_1(I - Ran(T_1 - T_2)) \\ 0 & (I - Ran(T_1 - T_2)) \end{pmatrix}.$$

It is not hard to check that

$$Q(z_1) \wedge Q(z_2) = \{(T_{12}\xi, T_{22}\xi, 0, \xi)^t | \xi\}$$

for any $z_1 \neq z_2 (\in \mathbb{C}))$.

If $Q \wedge Q(\infty) \neq 0$, then it is easy to see that

$$F \equiv Q \wedge Q(\infty) = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If $Q \neq F$, we have $E \leq Q \vee Q(\infty)$ and $Q(z_1) \wedge Q(z_2) \leq Q$. If $Q \vee Q(\infty) = E$, then $\tau(Q) = \frac{1}{2}$. Hence

$$Q\mathcal{H} = \{(\xi_1, T_{22}\xi, 0, \xi)^t | \xi_1, \xi\} = F \vee (Q(z_1) \wedge Q(z_2)).$$

The last possibility is $Q \vee Q(\infty) = I$. Then there must exists a vector $(\xi_1, \xi_2, \xi_3, \xi_4)^T \in Q$ such that $\xi_3 \neq 0$. Note that $\tau(Q) = \frac{1}{2} + \tau(F)$ and

$$\{(0, T_{21}\xi, \xi, 0)^t | \xi\} \subset Q\mathcal{H}.$$

Then it is not hard to see that

$$Q = \{(\xi_1, T_{21}\xi_2 + T_{22}\xi_3, \xi_2, \xi_3)^t | \xi_1, \xi_2, \xi_3\} = F \vee Q(z) = Q(z_1) \vee Q(z_2)$$

for any z_1 and $z_2 \in C$.

Remark 2.1. *$E = Q(\infty) \vee (Q(0) \wedge Q(-1))$ and $F = (Q(0) \vee Q(-1)) \wedge Q(\infty)$.*

Example 2.1 (Tautological line bundle over $\mathbb{CP}^1$)**.** *Let*

$$Q(\infty) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, Q(0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, Q(-1) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

then

$$Q(z) = \frac{1}{1+|z|^2} \begin{pmatrix} |z|^2 & z \\ \overline{z} & 1 \end{pmatrix}.$$

Note that the map $\phi : \mathbb{CP}^1 \to S^2$ given by $\phi([z_1, z_2]) = z_1/z_2$ is a homeomorphism. We have

$$\phi^*(Q)([z_1, z_2]) = \frac{1}{|z_1|^2 + |z_2|^2} \begin{pmatrix} |z_1|^2 & z_1\overline{z_2} \\ \overline{z_1}z_2 & |z_2|^2 \end{pmatrix}$$

is a line bundle over $\mathbb{CP}^1$. It is the tautological line bundle (or Hopf bundle) over $\mathbb{CP}^1$.

If T_1 and T_2 are not bouneded, then the connected component of $\mathcal{L}at(\mathcal{A}lg(Q(\infty), Q(0), Q(-1))$ is not a Hilbert bundle.

Definition 2.1 (Definition 2.2.1 and Definition 2.2.2 in [27])**.** *Let X be a smooth manifold. A field of Hilbert spaces in a map $p : \mathcal{H} \to X$, with each fiber $\mathcal{H}_x = p^{-1}(x)$ endowed with the structure of a Hilbert space. A smooth structure on a field $\mathcal{H} \to X$ of Hilbert spaces in given by spcifiying a set Γ^∞ of sections of $\mathcal{H}$, closed under addition and under multiplicatin by elements of $C^\infty(X)$, and linear operators $\nabla_\xi : \Gamma^\infty \to \Gamma^\infty$ for each $\xi \in Vect(X)$, such that for ξ, $\eta \in Vect(X)$, $f \in C^\infty(X)$ and φ, $\psi \in \Gamma^\infty$*

(1) $\nabla_{\xi+\eta} = \nabla_\xi + \nabla\eta, \nabla_{f\xi} = f\nabla_\xi, \nabla_\xi(f\varphi) = (\xi f)\varphi + f\nabla_\xi\varphi;$

(2) $\langle\varphi, \psi\rangle \in C^\infty(X)$ *and* $\xi\langle\varphi, \psi\rangle = \langle\nabla_\xi\varphi, \pi\rangle + \langle\varphi, \nabla_\xi\pi\rangle;$

(3) $\{\varphi(x) : \varphi \in \Gamma^\infty\} \subset \mathcal{H}_x$ *is dense, for all* $x \in X$.

Let $Proj(1, \mathcal{H})$ denote the set of projections in $\mathcal{B}(\mathcal{H})$ such that $dimP\mathcal{H} = 1$ for any $P \in Proj(1, \mathcal{H})$. Altougth $Proj(1, \mathcal{H})$ is only a real analytic manifold, there exists a canonical complex structure on $Proj(1, \mathcal{H})$.

In general, an almost complex structure on a Banach manifold X is a smooth tensor field J on X which associates to each $x \in X$ a linear operator J on the tangent space $T_x(X)$ to X at x which satisfies $J_x(J\xi) = -\xi$ for all $\xi \in T_x(X)$. An almost complex structure J on X is said to be a complex structure on X if and only if, given any point $x \in X$, there exists an open neighbourhood Ω of x in X and a smooth map $\Phi : \Omega \to B$, where B is some complex Banach space which satisfies the following conditions:

(1) Φ maps Ω diffeomorphically onto some open set in B;
(2) for each $x \in \Omega$ the derivative $(d\Phi)_x : T_m(\Omega) \to B$ of the map Ω at x satisfies $(d\Phi)_m(J_m\xi) = i(d\Phi)_m\xi$ for all $\xi \in T_m(X)$.

We will dsecibe the complex structure of $Proj(1, \mathcal{H})$. Fixed a unit vector β in $P\mathcal{H}$, it easy to see that the tangent space $T_P Proj(1, \mathcal{H})$ at P is

$$\begin{aligned} T_P Proj(1, \mathcal{H}) &= \{H | H^* = H, PHP = 0, (I-P)H(I-P) = 0, H \in \mathcal{B}(\mathcal{H})\} \\ &= \{|\xi\rangle \langle\beta| + (|\xi\rangle \langle\beta|)^* | \xi \in (I-P)\mathcal{H}\} \cong (I-P)\mathcal{H}. \end{aligned}$$

Let

$$J_P : \{|\xi\rangle \langle\beta| + (|\xi\rangle \langle\beta|)^* \to |i\xi\rangle \langle\beta| + (|i\xi\rangle \langle\beta|)^*$$

It is easy to see that $J_P^2 = -I$. Actually the operator is just the mutiplication of i on the Hilbert space $(I-P)\mathcal{H}$. Let $\Omega = \{Q | \langle Q\beta, \beta\rangle > 0\}$ and

$$\Phi : Q \to \frac{(I-P)Q\beta}{\langle Q\beta, \beta\rangle}.$$

Then

$$d\Phi : H \to H\beta, \qquad \forall H \in T_P Proj(1, \mathcal{H}).$$

Now it obvious that $(d\Phi)_P(J_P H) = iH\beta = i(d\Phi)_P H$ (Much general result is proved for the Flag manifolds in a C*-algebra).

Remark 2.2. *If we identify the projection with its range, then the complex structure descirbed above is the same as the complex structure of the 1-dimensional Grassmann manifold $Gr(1, \mathcal{H})$, the set of all 1-dimensional subspaces of $\mathcal{H}$, as the complex projective space.*

Definition 2.2. *Suppose that $\Phi : X \to Proj(1, \mathcal{H})$ is a holomorphic embedding, where X is a complex manifold. We say the embedding is locally constant with respect to a von Neumann algebra $\mathfrak{A}$ if for any $x \in X$*

Definition 2.3. *Let $\mathfrak{A}$ be a von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$. Suppose that $\Phi : X \to Proj(1, \mathcal{H})$ is a holomorphic embeding, where X is a complex manifold. Then the field of Hilbert spaces induce by the holomorphic embeding and $\mathfrak{A}$ is*

$$\mathfrak{F}(\Phi, \mathfrak{A}) \equiv \amalg_{x \in X} \bigvee_{U \in U(\mathfrak{A})} U^*\Phi(x)U.$$

The following fact is immediate from the Definition 2.3.

Proposition 2.1. *Let $\mathfrak{A}$ be a von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$ and $\Phi : X \to Proj(1, \mathcal{H})$ is a holomorphic embeding of a complex manifold. Then for any $x \in X$, $\mathfrak{F}(\Phi, \mathfrak{A})(x) \in \mathfrak{A}'$.*

3. Conclusion

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