

Wei Fei's Note

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ABSTRACT. Note on Wei Fei's lecture.

1. NOTATIONS

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz class, i.e.,

$$\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : \sup_{t \in \mathbb{R}} |t^k \frac{d^l}{dt^l} f(t)| < \infty \text{ for any } k, l \in \mathbb{N} \cup \{0\}\}.$$

Let $f \in \mathcal{S}(\mathbb{R})$. The Fourier transform \hat{f} is

$$\hat{f}(s) = \int_{-\infty}^{+\infty} f(t) e^{-2\pi i s t} dt, \text{ for any } s \in \mathbb{R}.$$

We have

$$f(t) = \int_{-\infty}^{+\infty} \hat{f}(s) e^{2\pi i s t} ds, \text{ for any } t \in \mathbb{R}.$$

2. IMPORTANT FORMULA

2.1. Partial Summation Formula.

Lemma 2.1. Let $f(x) \in C^1([a, b])$. Then

$$\sum_{a < n \leq b} c_n f(n) = C(b) f(b) - \int_a^b C(x) f'(x) dx,$$

where

$$C(x) = \sum_{a < n \leq x} c_n, \quad c_n \in \mathbb{C}.$$

Theorem 2.1 (Poisson Summation Formula). Let f in $\mathcal{S}(\mathbb{R})$. We have

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n).$$

If $f(x) = f(-x)$, then

$$\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} \hat{f}(n) + \frac{1}{2}(\hat{f}(0) - f(0)).$$

Proof. Let

$$g(\theta) = \sum_{n \in \mathbb{Z}} f(\theta + n).$$

Since $f \in \mathcal{S}(\mathbb{R})$, $g(\theta)$ is a well-defined function of period 1 such that

$$g(\theta) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi i n \theta}.$$

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Let $\theta = 0$, we have

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n).$$

□

Example 2.1. Let $f(x) = e^{-\pi x^2 y}$.

$$\hat{f}(x) = y^{-\frac{1}{2}} e^{-\frac{\pi x^2}{y}}.$$

3. ENTIRE FUNCTION

Lemma 3.1. Let $\{a_n\}_n \subset \mathbb{C}$ be a sequence,

$$0 < |a_1| \leq |a_2| \leq \cdots \leq |a_n| \leq \cdots, \text{ and } |a_n| \rightarrow \infty.$$

Then there exists an entire function f such that $f(s) = 0$ if and only if $s \in \{a_n\}_n$.

Proof. Let

$$h_n = \left(1 - \frac{s}{a_n}\right) e^{\frac{s}{a_n} + \frac{1}{2}\left(\frac{s}{a_n}\right)^2 + \frac{1}{3}\left(\frac{s}{a_n}\right)^3 + \cdots + \frac{1}{n}\left(\frac{s}{a_n}\right)^n}.$$

Then

$$f(s) = \prod_{n=1}^{\infty} h_n(s)$$

satisfies the condition. □

Remark 3.1. Let $\{a_n\}_n \subset \mathbb{C}$ be a sequence,

$$0 < |a_1| \leq |a_2| \leq \cdots \leq |a_n| \leq \cdots, \text{ and } |a_n| \rightarrow \infty.$$

If $\sum_n \frac{1}{|a_n|^{p+1}} < \infty$, then we could use

$$h_n = \left(1 - \frac{s}{a_n}\right) e^{\frac{s}{a_n} + \frac{1}{2}\left(\frac{s}{a_n}\right)^2 + \frac{1}{3}\left(\frac{s}{a_n}\right)^3 + \cdots + \frac{1}{p}\left(\frac{s}{a_n}\right)^p}.$$

in the above proof.

Example 3.1. Let $\{n\}_{n \in \mathbb{Z}}$. Then

$$f(s) = s \prod_{n \in \mathbb{N}} \left(1 - \frac{s}{n}\right) \left(1 + \frac{s}{n}\right).$$

Lemma 3.2. Let $\{a_n\}_n \subset \mathbb{C}$ be a sequence,

$$0 < |a_1| \leq |a_2| \leq \cdots \leq |a_n| \leq \cdots, \text{ and } |a_n| \rightarrow \infty.$$

Suppose that f satisfies $f(s) = 0$ if and only if $s \in \{a_n\}_n$. Then $f(s) = e^{H(s)} \prod_{n=1}^{\infty} h_n(s)$.

Definition 3.1. Suppose $G(s)$ is a function and $\mu(r) = \max_{|s| \leq r} |G(s)|$. Let

$$\alpha_0 = \inf \{ \alpha : \mu(r) \leq e^{a_0 r^\alpha} \}.$$

Theorem 3.1. Let p be the smallest integer such that

$$\sum_n \frac{1}{|a_n|^{p+1}} < \infty.$$

Then the degree of

$$f(s) = \prod_n \left(1 - \frac{s}{a_n}\right) e^{\frac{s}{a_n} + \frac{1}{2}\left(\frac{s}{a_n}\right)^2 + \frac{1}{3}\left(\frac{s}{a_n}\right)^3 + \cdots + \frac{1}{p}\left(\frac{s}{a_n}\right)^p}$$

is p .

Example 3.2.

$$\sin(\pi s) = \pi s \prod_n \left(1 - \frac{s^2}{n^2}\right).$$

Proof. Add proof. □

4. $\Gamma(s)$

Let

$$\Gamma(s) = \int_0^{+\infty} x^{s-1} e^{-x} dx, \quad \operatorname{Re}(s) > 0.$$

It is easy to see that $\Gamma(s+1) = s\Gamma(s)$.

$$\frac{1}{\Gamma(s)} = se^{\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-\frac{s}{n}},$$

where γ is the Euler constant, i.e. $\gamma = \lim_{n \rightarrow \infty} (1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log(n))$.

Theorem 4.1.

$$\frac{1}{\Gamma(s)} = s \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{-s} \left(1 + \frac{s}{n}\right).$$

Proof. add proof. □

Theorem 4.2. Let $0 < \delta < \pi$. We have

$$\log \Gamma(s) = s \log s - \frac{1}{2} \log s - s + \log(\sqrt{2\pi}) + O_{\delta}\left(\frac{1}{|s|}\right).$$

Proof. Add proof. □

This following is incomplete.

Proposition 4.1. Let $s = \sigma + it$. Assume $\alpha < \sigma < \beta$. We have

$$\Gamma(s) = |t|^{s-\frac{1}{2}} e^{-\frac{\pi}{2}|t|-it} +$$

5. ζ

Theorem 5.1. If $\operatorname{Re}(s) = \sigma > 1$, then $\zeta(s) \neq 0$.

Proof. Note

$$\frac{1}{|\zeta(s)|} = \left| \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}} \leq 1 + \int_1^{\infty} \frac{1}{t^{\sigma}} dt = \frac{\sigma}{\sigma-1}.$$

This implies the result. □

Theorem 5.2. For $\operatorname{Re}(s) > 0$, we have

$$\zeta(s) = \sum_{n=1}^N \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \frac{N^{-s}}{2} + s \int_N^{\infty} \frac{\rho(u)}{u^{s+1}} du, \quad N \geq 1,$$

where $\rho(u) = \frac{1}{2} - \{u\}$. Specially,

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + s \int_1^{\infty} \frac{\rho(u)}{u^{s+1}} du.$$

Theorem 5.3.

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{\frac{s-1}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$$

Proof. Add proof. □

Let

$$\tilde{\zeta}(s) = \frac{1}{2}s(s-1)\zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}.$$

Theorem 5.4. For any $\varepsilon > 0$, we have

$$|\tilde{\zeta}(s)| \ll e^{c|s|^{1+\varepsilon}}.$$

Remark 5.1. The theorem above implies that $\tilde{\zeta}$ has infinite many zeros. (provide a proof).

For $\operatorname{Re}(s) > 1$, estimate

$$f(s) = (1 - 2^{1-s})\zeta(s) = \sum_n \frac{(-1)^{n-1}}{n^s}.$$

It is not hard to see that $\zeta(s)$ does not have real zeros.

Lemma 5.1. Let $\{\rho_n\}_n$ be the zeros of $\tilde{\zeta}(s)$. Then

$$\sum_n \frac{1}{|\rho_n|} = \infty,$$

$$\sum_n \frac{1}{|\rho_n|^{1+\varepsilon}} < \infty,$$

for any $\varepsilon > 0$.

Theorem 5.5.

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{1-s} + \sum_{n=1}^{\infty} \left(\frac{1}{s-\rho_n} + \frac{1}{\rho_n} \right) + \sum_{n=1}^{\infty} \left(\frac{1}{s+2n} - \frac{1}{2n} \right) + B_0,$$

where B_0 is a constant.

Proof. Consider the two expressions of $\tilde{\zeta}(s)$:

$$\tilde{\zeta}(s) = e^{as+b} \prod_n \left(1 - \frac{s}{\rho_n}\right) e^{\frac{s}{\rho_n}}$$

$$\tilde{\zeta}(s) = \frac{1}{2}s(s-1)\zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}.$$

Compute the derivative of $\log \tilde{\zeta}(s)$ by plugging in the two expressions. □

Theorem 5.6. Let $T \geq 0$ and $\rho_n = \beta_n + i\gamma_n$ be the non-trivial zeros of $\zeta(s)$. Then

$$\sum_{n=1}^{\infty} \frac{1}{1 + (T - \gamma_n)^2} \leq C \log(T+2).$$

Proof. Let $s = 2 + iT$.

$$\operatorname{Re}\left(\frac{1}{s-\rho_n}\right) = \frac{2-\beta_n}{(2-\beta_n)^2 + (T-\gamma_n)^2} \geq \frac{1}{4(1+(T-\gamma_n)^2)}.$$

□

6. $n \times 2$ CASE

Let

$$S_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in M_n(\mathbb{C})$$

We would like to find the commutant of $(I - \frac{1}{2}S_n) \otimes (I - \frac{1}{3}S_2)$ in $M_n(\mathbb{C}) \otimes M_2(\mathbb{C})$. If

$$\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \begin{pmatrix} S_n & \frac{2}{3}(I - \frac{1}{2}S_n) \\ 0 & S_n \end{pmatrix} = \begin{pmatrix} T_1 S_n & T_1 \frac{2}{3}(I - \frac{1}{2}S_n) + T_2 S_n \\ T_3 S_n & T_3 \frac{2}{3}(I - \frac{1}{2}S_n) + T_4 S_n \end{pmatrix} \\ = \begin{pmatrix} S_n T_1 + \frac{2}{3}(I - \frac{1}{2}S_n) T_3 & S_n T_2 + \frac{2}{3}(I - \frac{1}{2}S_n) T_4 \\ S_n T_3 & S_n T_4 \end{pmatrix} = \begin{pmatrix} S_n & \frac{2}{3}(I - \frac{1}{2}S_n) \\ 0 & S_n \end{pmatrix} \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$$

Since T_3 commute with S_n , T_3 must be a polynomial of S_n . Also note that $T_1 S_n - S_n T_1 = \frac{2}{3}(I - \frac{1}{2}S_n) T_3$, this implies that the trace of T_3 is zero. Therefore T_3 must be upper triangular.

Note that

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \ddots & x_{3n} \\ x_{n-1,1} & x_{n-1,2} & x_{n-1,3} & \cdots & x_{n-1,n} \\ x_{n,1} & x_{n,2} & x_{n,3} & \cdots & x_{n,n} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ - \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \ddots & x_{3n} \\ x_{n-1,1} & x_{n-1,2} & x_{n-1,3} & \cdots & x_{n-1,n} \\ x_{n,1} & x_{n,2} & x_{n,3} & \cdots & x_{n,n} \end{pmatrix} \\ = \begin{pmatrix} 0 & x_{11} & x_{12} & \cdots & x_{1,n-1} \\ 0 & x_{21} & x_{22} & \cdots & x_{2,n-1} \\ 0 & x_{31} & x_{32} & \ddots & x_{3,n-1} \\ 0 & x_{n-1,1} & x_{n-1,2} & \cdots & x_{n-1,n-1} \\ 0 & x_{n,1} & x_{n,2} & \cdots & x_{n,n-1} \end{pmatrix} - \begin{pmatrix} x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3n} \\ x_{41} & x_{42} & x_{43} & \ddots & x_{4n} \\ x_{n,1} & x_{n,2} & x_{n,3} & \cdots & x_{n,n} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

If the above is strict upper triangular, we must have

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \ddots & x_{3n} \\ x_{n-1,1} & x_{n-1,2} & x_{n-1,3} & \cdots & x_{n-1,n} \\ x_{n,1} & x_{n,2} & x_{n,3} & \cdots & x_{n,n} \end{pmatrix}$$

is upper triangular.

So we have T_1, T_4 are upper triangular. And it is easy to see that for a fixed T_3 which commute with S_n , we have many elements which commute with $(I - \frac{1}{2}S_n) \otimes (I - \frac{1}{3}S_2)$.

7. SPECTRUM OF SUMS OF UNITARY

Let U be a unitary such that $(Uf)(t) = f(t+1)$. Then

$$\hat{U}f(s) = e^{2\pi i s} \hat{f}(s).$$

Example 7.1. Let

$$\mathfrak{D} = \{(a_l)_{l \in \mathbb{Z}} : \exists m \in \mathbb{N}, a_n = 0, \text{ for any } n > m \text{ or } n < -m \text{ and } \sum_{-m \leq k \leq m} a_k = 0\}.$$

It is easy to see that \mathfrak{D} is dense in \mathcal{H} .

Let $\mathcal{H} = l^2(\mathbb{Z})$ and

$$\begin{aligned} T : (a_l)_{l \in \mathbb{Z}} \in \mathfrak{D} &\rightarrow \left(\sum_{l \leq k < +\infty} a_k \right)_{l \in \mathbb{Z}}, \\ S : (a_l)_{l \in \mathbb{Z}} \in \mathfrak{D} &\rightarrow \left(\sum_{-\infty < k \leq l} a_k \right)_{l \in \mathbb{Z}}. \end{aligned}$$

It is easy to see that T and S are well-defined.

Let

$$\begin{aligned} \xi &= (\dots, 0, a_{-m}, a_{-m+1}, \dots, a_{m-1}, a_m, 0, \dots) \in \mathfrak{D}, \\ \beta &= (\dots, 0, b_{-m}, b_{-m+1}, \dots, b_{m-1}, b_m, 0, \dots) \in \mathfrak{D}. \end{aligned}$$

Then we have

$$\begin{aligned} \langle T\xi, \beta \rangle &= \bar{b}_{-m}(a_{-m} + \dots + a_m) + \bar{b}_{-m+1}(a_{-m+1} + \dots + a_m) + \dots + \bar{b}_m a_m \\ &= a_{-m} \bar{b}_{-m} + a_{-m+1}(\bar{b}_{-m} + \bar{b}_{-m+1}) + \dots + a_m(\bar{b}_{-m} + \dots + \bar{b}_m) = \langle \xi, S\beta \rangle. \end{aligned}$$

Therefore, T and S are both closable.

Let $U : e_k \rightarrow e_{k-1}$ where $\{e_k\}_{k \in \mathbb{Z}}$ is the canonical orthonormal basis of $l^2(\mathbb{Z})$. It is well-known that the spectrum of U is S^1 .

Then $T = \sum_{0 \leq k < \infty} U^k$ and $S = \sum_{-\infty < k \leq 0} U^k$.

Let \mathfrak{A} be the von Neumann algebra generated by U .

We can identify $l^2(\mathbb{Z})$ with $L^2(S^1)$ and e_n with $z^n \in L^2(S^1)$. Then U is the multiplication of $\frac{1}{z}$. Note that every vector in \mathfrak{D} corresponding to a function

$$(\dots, 0, a_{-m}, a_{-m+1}, \dots, a_{m-1}, a_m, 0, \dots) \iff f(z) = \sum_{-m \leq k \leq m} a_k z^k.$$

By the definition of \mathfrak{D} , we have $f(1) = 0$. Therefore $f(z) = (z-1)g(z)$ where $g(z) \in \mathfrak{D}$. It is not hard to check that

$$(Tf)(z) = \frac{zf(z)}{z-1} \quad \text{and} \quad (Sf)(z) = \frac{f(z)}{1-z} \quad f \in \mathfrak{D}.$$

Therefore, as the closed unbounded operator affiliated with the abelian von Neumann algebra \mathfrak{A} , $T = M_{\frac{z}{z-1}}$ and $S = M_{\frac{1}{1-z}}$. It is obvious that $T^* = S$.

Let $\gamma \in \mathbb{C} \setminus \{\frac{1}{1-s} : |s| \leq 1\}$. If $\gamma = 0$, then M_{1-z} is a bounded inverse of S in the Banach algebra generated by U^* . If $\gamma \neq 0$, then

$$\frac{1}{\gamma} - 1 \notin \{s : |s| \leq 1\}.$$

This implies that $|\frac{1-\gamma}{\gamma}| > 1$. Thus

$$M_{\frac{(\gamma-1)-\gamma z}{1-z}} = \frac{1}{(\gamma-1)}(1 - M_z)(I + \frac{\gamma}{\gamma-1}M_z + (\frac{\gamma}{\gamma-1}M_z)^2 + (\frac{\gamma}{\gamma-1}M_z)^3 + \dots).$$

Thus $\gamma - S$ has a bounded inverse in the Banach algebra generated by U^* .

Assume that $\gamma = \frac{1}{1-s}$ where $|s| < 1$. Then

$$\gamma - \frac{1}{1-z} = \frac{s-z}{(1-s)(1-z)}.$$

Note that

$$\frac{(1-s)(1-z)}{s-z}$$

is not analytic in the unit disk, there for $\gamma - \frac{1}{1-z}$ does not has a bounded inverse in the Banach algebra generated by M_z .

In summary, we have the spectrum of S is $\{\frac{1}{1-s} : |s| \leq 1\}$.

Let $\omega = e^{\frac{2\pi i}{n}}$ and

$$W = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ \omega & \omega^2 & \omega^3 & \cdots & \omega^{n-1} & 1 \\ \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(n-1)} & 1 \\ \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(n-1)} & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \cdots & \omega^{(n-1)(n-1)} & 1 \end{pmatrix}$$

Example 7.2. Let $P : l^2(\mathbb{Z}) \rightarrow l^2(\mathbb{N})$ be the orthonormal projection onto the subspace spanned by $e_i, i \geq 0$. Let $\mathcal{H}_1 = Pl^2(\mathbb{Z})$ and

$$T_1 = PTP : (a_1, a_2, a_3, \dots) \rightarrow (\sum_{k=1} a_k, \sum_{k=2} a_k, \sum_{k=3} a_k, \dots),$$

where $(a_1, a_2, \dots) \in \mathcal{D}|_{\mathcal{H}_1}$.

Similarly, let

$$S_1 = PSP : (a_1, a_2, a_3, \dots) \rightarrow (a_1, \sum_{k=1}^2 a_k, \sum_{k=1}^3 a_k, \dots),$$

where $(a_1, a_2, \dots) \in \mathcal{D}_{\mathcal{H}_1}$.

Let $\xi = (a_1, \dots, a_n, 0, 0, \dots)$ and $\beta = (b_1, \dots, b_n, 0, 0, \dots)$ in $\mathcal{D}_{\mathcal{H}_1}$.

$$\begin{aligned} \langle T_1 \xi, \beta \rangle &= \bar{b}_1(a_1 + \cdots + a_n) + \bar{b}_2(a_2 + \cdots + a_n) + \cdots + \bar{b}_n a_n \\ &= a_1 \bar{b}_n + a_2(\bar{b}_1 + \bar{b}_2) + \cdots + a_n(\bar{b}_n + \cdots + \bar{b}_n) = \langle \xi, S_1 \beta \rangle. \end{aligned}$$

Note that

$$(I - P)SP\xi = 0 \text{ and } (I - P)TP\xi = 0,$$

for any $\xi \in \mathcal{D}_{\mathcal{H}_1}$.

By identify $l^2(\mathbb{Z})$ with $L^2(S^1)$, we have

$$PM_{\frac{(\gamma-1)-\gamma z}{1-z}}P = \frac{1}{(\gamma-1)}(1 - PM_z P)(I + \frac{\gamma}{\gamma-1}PM_z P + (\frac{\gamma}{\gamma-1}PM_z P)^2 + (\frac{\gamma}{\gamma-1}PM_z P)^3 + \cdots)$$

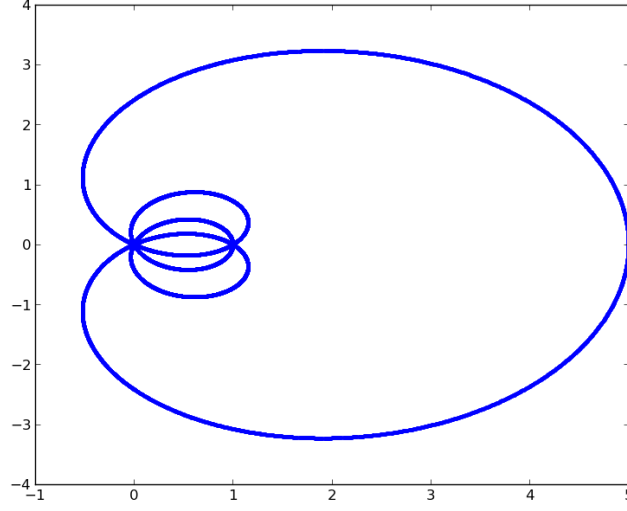
is the bounded inverse of $\gamma - S_1$ for $\gamma \in \mathbb{C} \setminus \{\frac{1}{1-s} : |s| \leq 1\}$ and $\gamma \neq 0$. If $\gamma = 0$, it is obvious that $PM_{1-z}P$ is the bounded inverse of S_1 .

Assume that $\gamma = \frac{1}{1-s}$ where $|s| < 1$. Note that

$$\gamma - \frac{1}{1-z} = \frac{s-z}{(1-s)(1-z)}.$$

Let

$$\xi = 1 + sz + (sz)^2 + (sz)^3 + \cdots.$$

Figure 1. $n = 5$

Let $f \in \mathfrak{D}|\mathcal{H}_1$, i.e., $f(z) = (1-z)g(z)$. Then

$$\left\langle \frac{s-z}{(1-s)(1-z)} f, \xi \right\rangle = \frac{1}{1-s} \langle g, PM_{\bar{s}-z} \xi \rangle = 0.$$

Therefore the range of S_1 is not dense in \mathcal{H}_1 .

Let $X_n = \{f_n(z) : z \in S^1\}$ where $f_n(z) = 1 + z + z^2 + \dots + z^{n-1} = \frac{1-z^n}{1-z}$, $n = 1, 2, \dots$. Each X_n is a compact subset of \mathbb{C} . We would like to know the limit of X_n as $n \rightarrow \infty$.

Let $z = e^{i\theta}$, then

$$\begin{aligned} \frac{1-z^n}{1-z} &= \frac{1 - \cos \theta - \cos n\theta + \cos(n-1)\theta}{2 - 2\cos \theta} + \frac{\sin \theta - \sin n\theta + \sin(n-1)\theta}{2 - 2\cos \theta} i \\ &= \frac{\sin(\frac{n}{2}\theta)}{\sin(\frac{\theta}{2})} \left(\cos(\frac{(n-1)}{2}\theta) + i \sin(\frac{(n-1)}{2}\theta) \right). \end{aligned}$$

If $n = 2m + 1$, then

$$\frac{1-z^n}{1-z} = \frac{\sin(m\theta + \frac{1}{2}\theta)}{\sin(\frac{\theta}{2})} (\cos(m\theta) + i \sin(m\theta)).$$

Let $z_1 = e^{i\theta_1}$ and $z_2 = e^{i\theta_2}$ and θ_1, θ_2 are in $[0, 2\pi)$. Suppose that $f_n(z_1) = f_n(z_2)$ and $\theta_1 < \theta_2$. First assume that $\sin(m\theta_1 + \frac{1}{2}\theta_1) = 0 = \sin(m\theta_2 + \frac{1}{2}\theta_2)$ and θ_1, θ_2 . Note that $\sin(\frac{\theta_1}{2})$ and $\sim(\frac{\theta_2}{2})$ can not equal zero at the same time, since $\frac{\theta_1}{2}$ and $\frac{\theta_2}{2}$ are in $[0, \pi)$. Then

$$\theta_1, \theta_2 \in \left\{ \frac{2k\pi}{2m+1} : k = 1, 2, \dots, 2m \right\}.$$

And $f_n(z) = 0$.

Now assume that $\sin(m\theta + \frac{1}{2}\theta) \neq 0$. We have $\theta_2 = \theta_1 + \frac{2k\pi}{m}$ or $\theta_2 = \theta_1 + \frac{(2k+1)\pi}{m}$.

First assume that $\theta_2 = \theta_1 + \frac{2k\pi}{m}$, we have

$$\frac{\sin(m\theta_1 + \frac{1}{2}\theta_1)}{\sin(\frac{\theta_1}{2})} = \frac{\sin(m\theta_2 + \frac{1}{2}\theta_2)}{\sin(\frac{\theta_2}{2})}$$

implies

$$\frac{\sin m\theta_1 \cos(\frac{\theta_1}{2})}{\sin(\frac{\theta_1}{2})} + \cos m\theta_1 = \frac{\sin m\theta_2 \cos(\frac{\theta_2}{2})}{\sin(\frac{\theta_2}{2})} + \cos m\theta_2.$$

If $\sin m\theta_1 \neq 0$ then $\cot(\frac{\theta_1}{2}) = \cot(\frac{\theta_2}{2})$. This implies that $\theta_1 = \theta_2$.

Suppose that $\sin m\theta_1 = 0$, we have $\sin(\frac{\theta_1}{2}) \neq 0$ and $\sin(\frac{\theta_2}{2}) \neq 0$, since $\frac{\theta_1}{2}$ and $\frac{\theta_2}{2}$ are in $[0, \pi)$. This means that θ_1 and θ_2 are in

$$\{\frac{k\pi}{m} : k = 1, 2, \dots, 2m-1\}.$$

And $f_n(z) = \cos^2 m\theta = 1$.

Now assume that $\theta_2 = \theta_1 + \frac{(2k+1)\pi}{m}$. Then we have

$$\frac{\sin(m\theta_1 + \frac{1}{2}\theta_1)}{\sin(\frac{\theta_1}{2})} = -\frac{\sin(m\theta_2 + \frac{1}{2}\theta_2)}{\sin(\frac{\theta_2}{2})}.$$

This also implies that

$$\frac{\sin m\theta_1 \cos(\frac{\theta_1}{2})}{\sin(\frac{\theta_1}{2})} = \frac{\sin m\theta_1 \cos(\frac{\theta_2}{2})}{\sin(\frac{\theta_2}{2})}.$$

Argue as above, we have θ_1 and θ_2 are in

$$\{\frac{k\pi}{m} : k = 1, 2, \dots, 2m-1\}.$$

And $f_n(z) = \cos^2 m\theta = 1$.

Lemma 7.1. For any $re^{i\alpha} \in \mathbb{C}$ and any $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ and a $\theta_m \in [0, 2\pi)$ such that

$$|\frac{\sin(m\theta_m + \frac{1}{2}\theta_m)}{\sin(\frac{\theta_m}{2})} (\cos(m\theta_m) + i \sin(m\theta_m)) - re^{i\alpha}| < \varepsilon$$

for any $m \geq N$.

Proof. Assume that $\alpha = \frac{2\pi ip}{q}$, where $(p, q) = 1$ and $q > p$. For any $m > 1$, consider the set

$$\{\frac{2\pi i(p+kq)}{qm} : k = 0, 1, \dots, m-1\}.$$

Let

$$r_k = \frac{\sin(\frac{2\pi ip}{q} + \frac{\pi i(p+kq)}{qm})}{\sin(\frac{\theta_m}{2})} = \cos(\frac{2\pi ip}{q}) + \sin(\frac{2\pi ip}{q}) \cot(\frac{\pi i(p+kq)}{qm}).$$

Now it is not hard to see that there exist a $N \in \mathbb{N}$ such that there is a $0 \leq k_m \leq m-1$ such that $|r_{k_m} - r| \leq \varepsilon$ whenever $m > N$. \square

AMSS