ON GENERATORS OF ABELIAN KADISON-SINGER ALGEBRAS IN MATRIX ALGEBRAS

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Abstract. Assume that $\mathcal{H}$ is a Hilbert space of dimension greater than two. We prove that an abelian Kadison-Singer algebra acting on $\mathcal{H}$ can not contain any non-trivial idempotent. Based on this, we show that an abelian KS-algebra in matrix algebra $M_n(\mathbb{C})(n \geq 3)$ can not be generated by a single element. As a corollary, it is also proved that the lattice of an abelian KS-algebra can not be completely distributive.

1. Introduction

In 1960, Kadison and Singer introduced and studied a class of non selfadjoint operator algebras which they called "triangular algebras"(subalgebras of a C*-algebra whose diagonal is maximal abelian selfadjoint)[2]. The most well understood non selfadjoint algebras are the Nest algebras introduced by Ringrose [3]. It is a class of maximal triangular algebras. Reflexive algebras are generalizations of nest algebras. These algebras are completely determined by their lattices of invariant subspace projections. In [14] and [15], Ge and Yuan combine triangularity, reflexivity and von Neumann algebra properties in a single class of algebras and introduce Kadison-Singer algebras(KS-algebras for simplicity).

Before giving the definition of KS-algebras, we will fix some notation and review some definitions. Throughout the paper, $\mathcal{H}$ will be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators acting on $\mathcal{H}$. If $\mathcal{L}$ is a collection of self-adjoint projections of $\mathcal{B}(\mathcal{H})$, $\mathcal{A}lg(\mathcal{L})$ is used to denote the set of bounded operators that leave the range of every member of $\mathcal{L}$ invariant, i.e. $\mathcal{A}lg(\mathcal{L}) = \{T \in \mathcal{B}(\mathcal{H}) : (I-P)TP = 0$, for any P in $\mathcal{L}\}$. Dually, if $\mathfrak{A}$ is a set of operators in $\mathcal{B}(\mathcal{H})$, $\mathcal{L}at(\mathfrak{A})$ is used to denote the collection of projections whose ranges are left invariant by every element of $\mathfrak{A}$, i.e. $\mathcal{L}at(\mathfrak{A}) = \{P \in \mathcal{B}(\mathcal{H}) : P^* = P,\ P^2 = P$ and $(I-P)TP = 0$ for any T in $\mathfrak{A}\}$. A subalgebra $\mathfrak{A}$ of $\mathcal{B}(\mathcal{H})$ is called reflexive if $\mathfrak{A} = \mathcal{A}lg(\mathcal{L}at(\mathfrak{A}))$.

Definition 1.1 (Definition 1[14]). *Let $\mathcal{H}$ be a Hilbert space. A subalgebra $\mathfrak{A}$ of $\mathcal{B}(\mathcal{H})$ is called a Kadison-Singer algebra (or KS-algebra) if $\mathfrak{A}$ is reflexive and maximal with respect to the diagonal subalgebra $\mathfrak{A} \cap \mathfrak{A}^*$ of $\mathfrak{A}$, in the sense that if there is another reflexive subalgebra $\mathfrak{B}$ of $\mathcal{B}(\mathcal{H})$ such that $\mathfrak{A} \subset \mathfrak{B}$ and $\mathfrak{B} \cap \mathfrak{B}^* = \mathfrak{A} \cap \mathfrak{A}^*$, then $\mathfrak{A} = \mathfrak{B}$.*

It is easy to deduce from the definition that the invariant subspace lattice of a KS-algebra $\mathfrak{A}$ is reflexive and minimal generating the commutant of the diagonal subalgebra of $\mathfrak{A}$ as a von Neumann algebra (for a comprehensive treatment of the theory of von Neumann algebra one may refer to the book [1]). So the theory of KS-algebras is closely related with the theory of von Neumann algebras. Actually,

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one of the main purposes to introduce the KS-algebra, as stated in the introduction of [14], is to recapture the synergy that should exist between the powerful techniques that have developed in selfadjoint operator algebra theory and those of the non-selfadjoint theory. In [14] and [15] many interesting examples of KS-algebras with II_1 factor as their diagonal subalgebras have been constructed. Since these algebras contain II_1 factors, none of them can be commutative. So it is quiet natural to ask whether there exist non-trivial abelian KS-algebras. The following example provides an affirmative answer to this question.

Example 1.1 (Example 3.2[16]). *Let $S = \left(\begin{smallmatrix} 1 & a \\ 0 & 0 \end{smallmatrix}\right)$ be an idempotent in $M_2(\mathbb{C})$. Then*

$$\mathfrak{A}_a = \mathcal{A}lg(\{P_{\xi_1}, P_{\xi_2}\}) = \{S\}' = \{\begin{pmatrix} x & a(x-y) \\ 0 & y \end{pmatrix} : x, y \in \mathbb{C}\},$$

where $\xi_1 = (1,0)^t$, $\xi_2 = (-a,1)^t$, and P_{ξ_1}, P_{ξ_2} are the orthogonal projections onto the subspaces spanned by ξ_1 and ξ_2 respectively. It is easy to check that $\mathfrak{A}_a$ is reflexive and similar to the diagonal algebra $\mathfrak{A}_0 = \{\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right) : x, y \in \mathbb{C}\}$. We assert that $\mathfrak{A}_a$ is an KS-algebra, provided that $a \neq 0$. If the assertion was false, then there would exist a reflexive algebra $\mathfrak{B}$ contains $\mathfrak{A}_a$ and $\mathfrak{B} \cap \mathfrak{B}^* = \mathfrak{A}_\mathfrak{a} \cap \mathfrak{A}_\mathfrak{a}{}^* = \mathbb{C}I$. It is clear that $\mathfrak{B}$ could only be $M_2(\mathbb{C})$, $\mathcal{A}lg(\{P_{\xi_1}\})$ or $\mathcal{A}lg(\{P_{\xi_2}\})$ if $\mathfrak{B} \neq \mathfrak{A}_a$. However, these algebras either contain P_{ξ_1} or P_{ξ_2}. This contradicts the fact that $\mathfrak{B} \cap \mathfrak{B}^* = \mathbb{C}I$.

So far these are the simplest non-trivial examples of abelian KS-algebras we have. Note that the algebras in Example 1.1 are similar to the algebra of diagonal matrices in $M_2(\mathbb{C})$. So it is interesting to ask whether we can find new abelian KS-algebras by "twisting" the algebra of diagonal matrices in $M_n(\mathbb{C})$ $(n \geq 3)$. More generally, we can ask how to find new examples of abelian KS-algebras, besides the algebras given in Example 1.1, if there are any. It is known that there are abelian reflexive subalgebras of matrix algebras generated by a single matrix [10]. Hence, it is also natural to ask whether there exist abelian KS-subalgebras of matrix algebras generated by one element. We will provide a negative answer to the last question if the size of the matrices is greater than 2 and shed some light on other questions.

There are four sections in this paper. In section two, we give some properties of abelian KS-algebras and prove an abelian KS-algebra acting on Hilbert space of dimension greater than 2 can not contain any non-trivial idempotent. Therefore, any abelian von Neumann algebra of dimension greater than 2 can not similar to an abelian KS-algebra. In section 3, by using the results we get in section 2, we show that an abelian KS-subalgebra of $M_n(\mathbb{C})$ $(n \geq 3)$ can not be a reflexive algebra determined by a single element. In the last section, we prove that the lattice of an abelian KS-subalgebra of $\mathcal{B}(\mathcal{H})$ $(dim\mathcal{H} \geq 3)$ can not be completely distributive.

2. Properties of Abelian KS-algebras

Note that the algebras in Example 1.1 contain non-trivial idempotents in their centers. However, if an algebra is a KS-algebra(Definition 1.1), its center can not contain any non-trivial orthogonal projections. More precisely, we will show that the center of a KS-algebra contains no non-trivial selfadjoint elements.

The following fact, which will be needed in the proof of Lemma 2.1 and in Section 4, is proved in [3] and [11].

Let ξ and β be two vectors in $\mathcal{H}$ and $\xi \otimes \beta$ the rank one operator defined by $\eta \to \langle \eta, \xi \rangle \beta$ for all η in $\mathcal{H}$. If $\mathcal{L}$ is a subspace lattice on $\mathcal{H}$, then the operator $\xi \otimes \beta$ belongs to $\mathcal{A}lg(\mathcal{L})$ if and only if there is a projection P in $\mathcal{L}$ such that $\beta \in P\mathcal{H}$ and $\xi \in (I - P_-)\mathcal{H}$ where $P_- = \vee\{Q \in \mathcal{L} : P \nleq Q\}$.

Lemma 2.1. *The center $\mathcal{C}(\mathfrak{A})$ of a KS-algebra $\mathfrak{A}(\subset \mathcal{B}(\mathcal{H}))$ contains no non-trivial selfadjoint elements.*

Proof. Suppose that $\mathcal{C}(\mathfrak{A})$ contains a non-trivial selfadjoint element P. Since $\mathfrak{A}$ is weak operator topology closed, we have the spectrum projections of P are in $\mathcal{C}(\mathfrak{A})$. Thus we may assume that P is a projection. Choose a proper orthonormal basis so that P can be written as $\left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Because P commutes with every element in $\mathfrak{A}$ and $\mathcal{L}at(\mathfrak{A})$, it is easy to see that $\mathfrak{A} \cong \mathfrak{A}_1 \oplus \mathfrak{A}_2$, where $\mathfrak{A}_1 = P\mathfrak{A}P$ and $\mathfrak{A}_2 = (I-P)\mathfrak{A}(I-P)$. Thus

$$\mathcal{L}at(\mathfrak{A}) = \{\begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} : Q_1 \in \mathcal{L}at(\mathfrak{A}_1) \text{ and } Q_2 \in \mathcal{L}at(\mathfrak{A}_2)\}.$$

Let

$$\begin{aligned}\widetilde{\mathcal{L}} &= \{Q \in \mathcal{L}at(\mathfrak{A}) : Q \leq P \text{ or } P \leq Q\} \\ &= \{\begin{pmatrix} Q_1 & 0 \\ 0 & 0 \end{pmatrix} : Q_1 \in \mathcal{L}at(\mathfrak{A}_1)\} \cup \{\begin{pmatrix} I & 0 \\ 0 & Q_2 \end{pmatrix} : Q_2 \in \mathcal{L}at(\mathfrak{A}_2)\}.\end{aligned}$$

Then $\widetilde{\mathcal{L}}$ is a sub-lattice of $\mathcal{L}at(\mathfrak{A})$. Since P is in $\widetilde{\mathcal{L}}$, the von Neumann algebras generated by $\widetilde{\mathcal{L}}$ and $\mathcal{L}at(\mathfrak{A})$ are the same. Therefore, $\widetilde{\mathcal{L}}'' = \mathcal{L}at(\mathfrak{A})''$ and $\widetilde{\mathcal{L}}' = \mathcal{L}at(\mathfrak{A})'$. This implies that $\mathcal{A}lg(\widetilde{\mathcal{L}})^* \cap \mathcal{A}lg(\widetilde{\mathcal{L}}) = \widetilde{\mathcal{L}}' = \mathcal{L}at(\mathfrak{A})' = \mathfrak{A}^* \cap \mathfrak{A}$. Let ξ be a vector in $(I-P)\mathcal{H}$ and β a vector in $P\mathcal{H}$. Note that $P_- \subseteq P$, where $P_- = \vee\{Q \in \widetilde{\mathcal{L}} | P \nsubseteq Q\}$. It follows that $\xi \in (I-P_-)\mathcal{H}$ and the rank one operator $\xi \otimes \beta$ is in $\mathcal{A}lg(\widetilde{\mathcal{L}})$. However, $P(\xi \otimes \beta)(I-P) = \xi \otimes \beta \neq 0$. Since $I-P$ is in $\mathcal{L}at(\mathfrak{A})$, $\xi \otimes \beta$ is not in $\mathfrak{A}$. This contradicts the maximality of $\mathfrak{A}$ and the initial assumption must be false. □

In particular, if the algebra $\mathfrak{A}$ is abelian, then the center of $\mathfrak{A}$ equals to itself and we have the following corollary.

Corollary 2.1. *If $\mathfrak{A}(\subset \mathcal{B}(\mathcal{H}))$ is an abelian KS-algebra, then $\mathfrak{A}^* \cap \mathfrak{A} = \mathbb{C}I$ and $\mathcal{L}at(\mathfrak{A})'' = \mathcal{B}(\mathcal{H})$.*

The following is a technical result that will be used later. This fact might be well known, we sketch a proof here for the reader's convenience.

Lemma 2.2. *Suppose that $\mathfrak{A}$ is an abelian subalgebra of $M_n(\mathbb{C})$ and $\mathcal{L} = \mathcal{L}at(\mathfrak{A})$. If P_1, P_2 are two projections in $\mathcal{L}$ such that $P_1 < P_2$ and $dim(P_2 - P_1) \geq 2$, then there exists a $P_3 \in \mathcal{L}$ satisfying $P_1 < P_3 < P_2$.*

Proof. Since $dim(P_2 - P_1) \geq 2$, the abelian algebra $(P_2 - P_1)\mathfrak{A}(P_2 - P_1)$ has a non-trivial invariant subspace Q. Let $P_3 = P_1 + Q$. Then $P_1 < P_3 < P_2$ and it is not hard to check that $P_3 \in \mathcal{L}$. □

Before proceeding further, let us recall some definitions. A nest $\mathcal{N}$ is a complete totally ordered family of selfadjoint projections on a Hilbert space $\mathcal{H}$ containing 0 and I. A nest is maximal if it is not contained in any larger nest. The algebra $\mathcal{A}lg(\mathcal{N})$ is called the nest algebra with respect to the nest $\mathcal{N}$. The relevant background of the theory of nest algebras can be found in [9] and [6].

Combine Lemma 2.3 with the fact that nest algebras are non abelian, we proved the following corollary.

Corollary 2.2. *If $\mathfrak{A}$ is an abelian algebra in $M_n(\mathbb{C})$ and $n \geq 2$, then $\mathcal{L}at(\mathfrak{A})$ contains at least two maximum nests.*

Let S be a closed idempotent(may be unbounded), i.e., S is a densely defined closed operator such that $S^2 = S$. The equality is to be understood in the strict sense that S^2 and S have the same domain $\mathcal{D}(S)$ and $S^2\xi = S\xi$ for any $\xi \in \mathcal{D}(S)$. It is not hard to see that $I - S$ is also a closed idempotent, and the range of S and $I - S$ are both closed. Let P, Q be the range and kernel of S respectively. If

$\xi \in \mathcal{D}(S)$, then $\xi = S\xi + (I - S)\xi$. Thus $\mathcal{D}(S) = P + Q = \{\xi + \beta : \xi \in P, \beta \in Q\}$. Note that P equals to the kernel of $I - S$, Q equals to the range of $I - S$. It is easy to check that $P \wedge Q = 0$ and $P \vee Q = I$.

Conversely, if P and Q are two projections such that $P \wedge Q = 0$ and $P \vee Q = I$. Since $\{(\xi + \beta, \xi) : \xi \in P, \beta \in Q\}$ $(\subset \mathcal{H} \oplus \mathcal{H})$ is closed, $S : \xi + \beta \to \xi$ is a closed operator with dense domain $\mathcal{D}(S) = \{\xi+\beta : \xi \in P, \beta \in Q\}$. It is clear that $S^2 = S$. Therefore, each pair of projections P, Q satisfying $P \wedge Q = 0$, $P \vee Q = I$ determines a closed idempotent and vice versa.

From the above discussion, the following lemma is immediate, and we omit the proof.

Lemma 2.3. *Let $\mathcal{H}$ be a Hilbert space. If $\mathfrak{A}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$, then there are two projections P, Q in $\mathcal{L}at(\mathfrak{A})$ satisfying $P \wedge Q = 0$, $P \vee Q = I$ if and only if there is a closed idempotent S such that $TS \subset ST$ for any $T \in \mathfrak{A}$. In particular, if $\mathcal{H}$ is finite-dimensional, then S is bounded. Therefore, P and Q are in $\mathcal{L}at(\mathfrak{A})$ iff every elements in $\mathfrak{A}$ commute with S.*

Lemma 2.4. *Let $\mathfrak{A}$ be a subalgebra of $\mathcal{B}(\mathcal{H})$. If the center of $\mathfrak{A}$ contains an idempotent S, then the range projections P, Q of S and $I - S$ are in $\mathcal{L}at(\mathfrak{A})$, and $E = (E \wedge P) \vee (E \wedge Q)$ for each $E \in \mathcal{L}at(\mathfrak{A})$.*

Proof. The first part of the lemma is just a restatement of Lemma 2.3. For any $\xi \in E\mathcal{H}$, we have $S\xi \in E \wedge P$ and $(I - S)\xi \in Q \wedge E$. This clearly implies that $E = (E \wedge P) \vee (E \wedge Q)$. □

The following observation is certainly known, but we could not find a reference. For the sake of completeness, we provide a short proof.

Lemma 2.5. *Let $\mathfrak{A}$ be a reflexive subalgebra of $B(\mathcal{H})$. If S is an idempotent in the center of $\mathfrak{A}$, then $P\mathfrak{A}P|_{P\mathcal{H}}$ and $(I - P)\mathfrak{A}(I - P)|_{(I-P)\mathcal{H}}$ are reflexive algebras, where P is the range projections of S.*

Proof. To prove $P\mathfrak{A}P|_{P\mathcal{H}}$ is reflexive, we only need to show that $P\mathfrak{A}P|_{P\mathcal{H}} = \mathcal{A}lg(\mathcal{L})$ where $\mathcal{L} = \{E|_{P\mathcal{H}} : E \le P, E \in \mathcal{L}at(\mathfrak{A})\}$. $P\mathfrak{A}P|_{P\mathcal{H}} \subseteq \mathcal{A}lg(\mathcal{L})$ is clear. Suppose now that T is an element in $\mathcal{A}lg(\mathcal{L})$, let $\widetilde{T} \in \mathcal{B}(\mathcal{H})$ be the operator such that $\widetilde{T}(I - P) = (I - P)\widetilde{T} = 0$ and $P\widetilde{T}P|_{P\mathcal{H}} = T$. By Lemma 2.4, it is easy to check that $\widetilde{T}S$ is in $\mathfrak{A}$ and $P\widetilde{T}SP|_{P\mathcal{H}} = T$. Thus $P\mathfrak{A}P|_{P\mathcal{H}} = \mathcal{A}lg(\mathcal{L})$.

Note that S^* is in the center of the reflexive algebra $\mathfrak{A}^*$ and $I - P$ is the range projection of $I - S^*$. Therefore, by the argument above, we have $(I - P)\mathfrak{A}^*(I - P)|_{(I-P)\mathcal{H}}$ is reflexive which implies $(I - P)\mathfrak{A}(I - P)|_{(I-P)\mathcal{H}}$ is reflexive. □

Remark 2.1. *Note that the projection Q onto the kernel of S is also the range projection of $I - S$, thus by Lemma 2.5, we have $Q\mathfrak{A}Q|_{Q\mathcal{H}}$ and $(I - Q)\mathfrak{A}(I - Q)|_{(I-Q)\mathcal{H}}$ are reflexive algebras.*

Remark 2.2. *Suppose $\mathfrak{A}_1(\subset \mathcal{B}(\mathcal{H}_1))$ and $\mathfrak{A}_2(\subset \mathcal{B}(\mathcal{H}_2))$ are two reflexive algebras. If $S = \left(\begin{smallmatrix} I & T \\ 0 & 0 \end{smallmatrix}\right)$ is an idempotent in $\mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$, then the following algebra*

$$\mathfrak{A} = \left\{ \begin{pmatrix} A & AT - TB \\ 0 & B \end{pmatrix} : A \in \mathfrak{A}_1, B \in \mathfrak{A}_2 \right\} (\cong \mathfrak{A}_1 \oplus \mathfrak{A}_2)$$

is also reflexive.

Lemma 2.6. *Let $\mathfrak{A}$ be a subalgebra of $B(\mathcal{H})$. If S is an idempotent in the center of $\mathfrak{A}$, then $P\mathfrak{A}P|_{P\mathcal{H}} \cong (I-Q)\mathfrak{A}(I-Q)|_{(I-Q)\mathcal{H}}$ and $Q\mathfrak{A}Q|_{Q\mathcal{H}} \cong (I-P)\mathfrak{A}(I-P)|_{(I-P)\mathcal{H}}$, where P and Q are the range projections of S and $I - S$ respectively.*

Proof. We could assume that $S = \left(\begin{smallmatrix} I & T \\ 0 & 0 \end{smallmatrix}\right)$. It is not difficult to see that $\mathfrak{A} = \left\{\left(\begin{smallmatrix} A & AT-TB \\ 0 & B \end{smallmatrix}\right) : A \in P\mathfrak{A}P|_{P\mathcal{H}}, B \in (I-P)\mathfrak{A}(I-P)|_{(I-P)\mathcal{H}}\right\}$ and both $P\mathfrak{A}P|_{P\mathcal{H}}$ and $(I-P)\mathfrak{A}(I-P)|_{(I-P)\mathcal{H}}$ are algebras. Let $S = \sqrt{SS^*}V$ be the polar decomposition of S, then $V = \left(\begin{smallmatrix} (I+TT^*)^{-\frac{1}{2}} & (I+TT^*)^{-\frac{1}{2}}T \\ 0 & 0 \end{smallmatrix}\right)$ is a partial isometry with initial space $(I-Q)\mathcal{H}$ and final space $P\mathcal{H}$. Since $(I-Q)\mathfrak{A}(I-Q)|_{(I-Q)\mathcal{H}} \cong V\mathfrak{A}V^*|_{P\mathcal{H}} = \left\{\left(\begin{smallmatrix} (I+TT^*)^{-\frac{1}{2}}A(I+TT^*)^{\frac{1}{2}} & 0 \\ 0 & 0 \end{smallmatrix}\right) : A \in P\mathfrak{A}P|_{P\mathcal{H}}\right\}$, we have $P\mathfrak{A}P|_{P\mathcal{H}} \cong (I-Q)\mathfrak{A}(I-Q)|_{(I-Q)\mathcal{H}}$. Replacing S by $I-S$, P by Q and Q by P, we have $Q\mathfrak{A}Q|_{Q\mathcal{H}} \cong (I-P)\mathfrak{A}(I-P)|_{(I-P)\mathcal{H}}$. □

Lemma 2.7. *Let $\mathfrak{A}$ be a KS-subalgebra of $B(\mathcal{H})$. If the center of $\mathfrak{A}$ contains an idempotent $S = \left(\begin{smallmatrix} I & T \\ 0 & 0 \end{smallmatrix}\right)$, then $P\mathfrak{A}P$, $(I-P)\mathfrak{A}(I-P)$, $Q\mathfrak{A}Q$ and $(I-Q)\mathfrak{A}(I-Q)$ are all KS-algebras, where P and Q are the range projections of S and $I-S$.*

Proof. If there exist reflexive subalgebras $\mathfrak{A}_1$ of $\mathcal{B}(P\mathcal{H})$ and $\mathfrak{A}_2$ of $\mathcal{B}((I-P)\mathcal{H})$ such that $P\mathfrak{A}P \subset \mathfrak{A}_1$, $(I-P)\mathfrak{A}(I-P) \subset \mathfrak{A}_2$ and $\mathfrak{A}_1 \cap \mathfrak{A}_1^* = P\mathfrak{A}P \cap P\mathfrak{A}^*P$, $\mathfrak{A}_2 \cap \mathfrak{A}_2^* = (I-P)\mathfrak{A}(I-P) \cap (I-P)\mathfrak{A}^*(I-P)$. Let $\widetilde{\mathfrak{A}} = \left\{\left(\begin{smallmatrix} A & AT-TB \\ 0 & B \end{smallmatrix}\right) : A \in \mathfrak{A}_1, B \in \mathfrak{A}_2\right\}$. By Remark 2.2, the algebra $\widetilde{\mathfrak{A}}$ is reflexive. Note that

$$\mathfrak{A} = \left\{\begin{pmatrix} A & AT-TB \\ 0 & B \end{pmatrix} : A \in P\mathfrak{A}P|_{P\mathcal{H}}, B \in (I-P)\mathfrak{A}(I-P)|_{(I-P)\mathcal{H}}\right\}.$$

Thus $\widetilde{\mathfrak{A}} \cap \widetilde{\mathfrak{A}}^* = \mathfrak{A} \cap \mathfrak{A}^*$. By the maximality of $\mathfrak{A}$, we have $\widetilde{\mathfrak{A}} = \mathfrak{A}$. Therefore $\mathfrak{A}_1 = P\mathfrak{A}P$ and $\mathfrak{A}_2 = (I-P)\mathfrak{A}(I-P)$.

Replacing S by $I-S$, P by Q and Q by P, we have the same results for $Q\mathfrak{A}Q$ and $(I-Q)\mathfrak{A}(I-Q)$. □

With the help of the preceding lemmas we can now prove the main theorem of the section.

Theorem 2.1. *Let $\mathcal{H}$ be a Hilbert space such that $dim\mathcal{H} \geq 3$. Suppose that $\mathfrak{A}$ is a reflexive subalgebra of $\mathcal{B}(\mathcal{H})$ such that $\mathfrak{A} \cap \mathfrak{A}^* = \mathbb{C}I$. If the center of $\mathfrak{A}$ contains an idempotent S such that $\mathcal{L}_1' = \{E \wedge P|_{P\mathcal{H}} : E \in \mathcal{L}at(\mathfrak{A})\}' = \mathbb{C}I$ and $\mathcal{L}_2' = \{(E \vee P - P)|_{(I-P)\mathcal{H}} : E \in \mathcal{L}at(\mathfrak{A})\}' = \mathbb{C}I$ where P is the range projection of S, then $\mathfrak{A}$ is not a KS-algebra.*

Proof. Note that $I-S^*$ is in the center of $\mathfrak{A}^*$ and $I-P$ is the range projection of $I-S^*$. Since $\mathcal{L}at(\mathfrak{A}^*) = \{I-E : E \in \mathcal{L}at(\mathfrak{A})\}$, we also have

$$\begin{aligned}
\widetilde{\mathcal{L}_1} &= \{[(I-P) \wedge E]|_{(I-P)\mathcal{H}} : E \in \mathcal{L}at(\mathfrak{A}^*)\} \\
&= \{[(I-P) \wedge (I-E)]|_{(I-P)\mathcal{H}} : E \in \mathcal{L}at(\mathfrak{A})\} \\
&= \{(I-E \vee P)|_{(I-P)\mathcal{H}} : E \in \mathcal{L}at(\mathfrak{A})\} = I - \mathcal{L}_2, \\
\widetilde{\mathcal{L}_2} &= \{[(I-P) \vee E - (I-P)]|_{P\mathcal{H}} : E \in \mathcal{L}at(\mathfrak{A}^*)\} \\
&= \{[(I-P) \vee (I-E) - (I-P)]|_{P\mathcal{H}} : E \in \mathcal{L}at(\mathfrak{A})\} \\
&= \{(P - P \wedge E)|_{P\mathcal{H}} : E \in \mathcal{L}at(\mathfrak{A})\} = I - \mathcal{L}_1.
\end{aligned}$$

Therefore $\widetilde{\mathcal{L}_1}' = \mathbb{C}I$ and $\widetilde{\mathcal{L}_2}' = \mathbb{C}I$. If $dim(I-P)\mathcal{H} < 2$, we could replace S by $I-S^*$, P by $I-P$, $\mathcal{L}_1$ by $\widetilde{\mathcal{L}_1}$ and $\mathcal{L}_2$ by $\widetilde{\mathcal{L}_2}$. Since $\mathfrak{A}$ is a KS-algebra if and only if $\mathfrak{A}^*$ is a KS-algebra, we could assume that $dim(I-P)\mathcal{H} \geq 2$.

If $\mathfrak{A}$ is a KS-algebra, then S can not be a projection by Lemma 2.1. Therefore we assume that $S = \left(\begin{smallmatrix} I & T \\ 0 & 0 \end{smallmatrix}\right)$ where $T \neq 0$. Let Q be the range projection of $I-S$. We claim that there is a projection $E \in \mathcal{L}at(\mathfrak{A})$ such that $E < Q$ and $PE(I-P) \neq 0$.

By Lemma 2.4, we have $\mathcal{L}_2 = \{(E \vee P - P)|_{(I-P)\mathcal{H}} : E \in \mathcal{L}at(\mathfrak{A}) \text{ and } E \leq Q\}$. If the claim is false, then $PE = EP = 0$ for any $E < Q$. This implies that $E \leq I-P$ and $\mathcal{L}_2 = \{E|_{(I-P)\mathcal{H}} : E \in \mathcal{L}at(\mathfrak{A}) \text{ and } E < Q \text{ or } E = I\}$. Since $\mathcal{L}_2' = \mathbb{C}I$ and

$dimQ\mathcal{H} = dim(I-P)\mathcal{H} \geq 2$, we must have $I - P = \vee\{E : E \in \mathcal{L}at(\mathfrak{A}), E < Q\} \leq Q$. Since $Q \wedge P = 0$, we have $Q = I - P$. This is in contradiction to $T \neq 0$. Thus the claim holds.

Let $\mathcal{L}$ be the sublattice of $\mathcal{L}at(\mathfrak{A})$ generated by $\mathcal{L}_1$ and $P \vee \mathcal{L}_2 = \{P \vee E : E \in \mathcal{L}at(\mathfrak{A})\}$. By Lemma 2.4, we have $\mathcal{L}at(\mathfrak{A})$ is generated by $\mathcal{L} \cup \{Q\}$. Let E be a projection in $\mathcal{L}at(\mathfrak{A})$ such that $E < Q$ and $PE(I-P) \neq 0$. Since $\mathcal{L}_1' = \mathbb{C}I$ and $\mathcal{L}_2' = \mathbb{C}I$, we have $(\mathcal{L} \cup \{E\})' = \mathbb{C}I$. Therefore $\mathcal{A}lg(\mathcal{L} \cup \{E\}) \cap \mathcal{A}lg^*(\mathcal{L} \cup \{E\}) = \mathbb{C}I$. Next we will show that $\mathfrak{A} \subsetneq \mathcal{A}lg(\mathcal{L} \cup \{E\})$.

Let $V = \left(\begin{smallmatrix} I & T \\ 0 & I \end{smallmatrix}\right)$. It is easy to check that the range projection of VQV^{-1} is $I - P$. Thus we may assume that the range projection of VEV^{-1} is $\left(\begin{smallmatrix} 0 & 0 \\ 0 & E_0 \end{smallmatrix}\right)$ where E_0 is a projection in $\mathcal{B}((I-P)\mathcal{H})$. Note that

$$V\mathfrak{A}V^{-1} = \{\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} : A_1 \in \mathcal{A}lg(\mathcal{L}_1), A_2 \in \mathcal{A}lg(\mathcal{L}_2)\} \text{ and}$$

$$V\mathcal{A}lg(\mathcal{L} \cup \{E\})V^{-1} = \{\begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} : A_1 \in \mathcal{A}lg(\mathcal{L}_1), A_2 \in \mathcal{A}lg(\mathcal{L}_2), A_{12}E_0 = 0\}.$$

Since $0 < E_0 < I - P_1$, there is an operator $A \in \mathcal{B}((I-P)\mathcal{H}, P\mathcal{H})$ such that $A \neq 0$ and $AE_0 = 0$. Thus we have $\left(\begin{smallmatrix} 0 & A \\ 0 & 0 \end{smallmatrix}\right) \in V\mathcal{A}lg(\mathcal{L} \cup \{E\})V^{-1} \setminus V\mathfrak{A}V^{-1}$, and $\mathfrak{A}$ is not a KS-algebra. □

Corollary 2.3. *Let $\mathcal{H}$ be a Hilbert space such that $dim\mathcal{H} \geq 3$. If $\mathfrak{A}$ is an abelian KS-algebra, then $\mathfrak{A}$ contains no non-trivial idempotent.*

Proof. If S is a non-trivial idempotent in $\mathfrak{A}$. Let P be the range projection of S. By Lemma 2.7, we know that $P\mathfrak{A}P$ and $(I-P)\mathfrak{A}(I-P)$ both are abelian KS-algebra. Thus $\mathcal{L}at(P\mathfrak{A}P)' = \mathbb{C}I$ and $\mathcal{L}at((I-P)\mathfrak{A}(I-P))' = \mathbb{C}I$ by Corollary 2.1. Therefore Theorem 2.1 implies that $\mathfrak{A}$ is not a KS-algebra. □

Corollary 2.4. *Let $\mathfrak{A} \subset M_n(\mathbb{C})(n \geq 3)$ be an abelian KS-algebra, then $\mathfrak{A} = \mathbb{C}I + \mathfrak{A}_0$ where $\mathfrak{A}_0$ is an abelian algebra contains only nilpotent matrices.*

Proof. If A in $\mathfrak{A}$ has more than one eigenvalue, then $\mathfrak{A}$ must contains a non-trivial idempotent which is impossible by Corollary 2.3. □

Corollary 2.5. *Let $\mathfrak{A}$ be an abelian von Neumann algebra such that $dim\mathfrak{A} > 2$, then $\mathfrak{A}$ can not similar to an abelian KS-algebra.*

Proof. Only need to note that every von Neumann algebra contains non-trivial idempotent, and Corollary 2.3 implies the result. □

3. Generator of abelian KS-algebras

If an subalgebra $\mathfrak{A}$ of $M_n(\mathbb{C})(n \geq 3)$ is an abelian KS-algebra, then Corollary 2.3 implies that the spectrum of every element in $\mathfrak{A}$ contains only one point. Therefore for any $T \in \mathfrak{A}$, there is a $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is nilpotent. In this section, we will prove that if $T \in M_n(\mathbb{C})(n \geq 2)$ is a nilpotent, then the $\mathcal{A}lg(\mathcal{L}at(T))$ can not be an abelian KS-algebra.

Lemma 3.1. *Suppose that $\mathfrak{A}$ is an abelian subalgebra of the matrix algebra $M_n(\mathbb{C})$. Let $\mathcal{S} = \{T : dim(Ker(T)) = m\}$ where $m = sup_{T \in \mathfrak{A} \setminus \{0\}} dimKer(T)$. If $T \in \mathcal{S}$ and A is a nilpotent in $\mathfrak{A}$, then we have $TA = AT = 0$.*

Proof. Let $\mathcal{K} = Ran(T)$. Since $AT = TA$, we have $A\mathcal{K} \subset \mathcal{K}$. Note that $A|_{\mathcal{K}}$ is also a nilpotent, therefore there exists a vector $\xi (\neq 0)$ in $\mathcal{K}$ such that $A\xi = 0$. If $AT \neq 0$, we must have $dim(Ker(AT)) - dim(Ker(T)) \geq 1$ which is in contradiction to the fact that T is in $\mathcal{S}$. □

Remark 3.1. *With the notations in Lemma 3.1. If T in $\mathcal{S}$ is a nilpotent, then we have $T^2 = 0$. This implies that the range of T is contained in the kernel of T. Let $m = dimKer(T)$. Choosing a right orthonormal basis, T can be written as $\left(\begin{smallmatrix} 0 & T_1 \\ 0 & 0 \end{smallmatrix}\right)$ where $T_1 \in \mathcal{M}_{m,n-m}(\mathbb{C})$. Note that the columns of the matrix T_1 are linearly independent. Therefore, $m \geq n - m$ and $m \geq \frac{n}{2}$.*

Lemma 3.2. *Suppose that $A \in M_n(\mathbb{C})$ is a nilpotent matrix with the order k. Then there exists an invertible matrix W such that $W^{-1}AW = \left(\begin{smallmatrix} J & A_0 \\ 0 & 0 \end{smallmatrix}\right)$, where J is in Jordan normal form such that $J^{k-1} = 0$ and $J^{k-2} \neq 0$, and the rank of A_0 equals to the number of its columns.*

Before proving this lemma, we feel that it is worthwhile to examine a special case.

Example 3.1. *Let*

$$A = \begin{pmatrix} J_3 & \\ & J_3 \end{pmatrix} = \begin{pmatrix} \begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix} & \\ & \begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix} \end{pmatrix}.$$

It is not hard to see that we may put the matrix into the form which is described in the lemma by switching the rows and columns. Indeed, let

$$U = \begin{pmatrix} I_2 & & \\ & \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{smallmatrix}\right) & \\ & & 1 \end{pmatrix}.$$

Then $UAU^* = \left(\begin{smallmatrix} J' & A_0 \\ 0 & 0 \end{smallmatrix}\right)$ where $A_0^t = \left(\begin{smallmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right)$, $J' = daig(J_2, J_2)$ and $J_2 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ is the 2×2 Jordan block of a nilpotent matrix.

The same strategy gives us the proof of the general case as follows.

Proof of Lemma 3.2. We first assume that $M_n(\mathbb{C}) \cong M_k(\mathbb{C}) \otimes M_l(\mathbb{C})$ and $A = J_k \otimes I_l$ where J_k is the $k \times k$ Jordan block. Let $\{e_{jk+i}\}_{i=1,\ldots,k;j=0,\ldots,l-1}$ be an orthonormal basis of $\mathbb{C}^n$ such that $Ae_{jk+1} = 0$ and $Ae_{jk+i} = e_{jk+i-1}$, $i = 2, \ldots, k$, $j = 0, \ldots l-1$.

Let W_0 be a unitary matrix defined by

$$\begin{aligned} W_0 e_{jk+i} &= e_{j(k-1)+i} \qquad 0 \leq j \leq l, \quad 1 \leq i \leq k-1, \\ W_0 e_{jk+k} &= e_{l(k-1)+j+1} \qquad 0 \leq j \leq l-1. \end{aligned}$$

Then an easy computation shows that $W_0 A W_0^* = \left(\begin{smallmatrix} J & A_0 \\ 0 & 0 \end{smallmatrix}\right)$ such that A_0 is a $l(k-1) \times l$-matrix with $rank(A_0) = l$ and $J = J_{k-1} \otimes I_l$ where J_{k-1} is the $k-1 \times k-1$ Jordan black of a single nilpotent matrix.

For the general case, we could assume that $A = \left(\begin{smallmatrix} J_0 & 0 \\ 0 & J_1 \end{smallmatrix}\right)$ where J_0 is a nilpotent matrix in Jordan normal form such that $J_0^{k-1} = 0$, $J_1 = J_k \otimes I = diag(J_k, \ldots, J_k)$. Then we can construct W_0 for $J_k \otimes I$ as in the first part of the proof, and it is easy to see that $W = diag(I, W_0)$ satisfies the conditions of the lemma. $\square$

Theorem 3.1. *Let A be a nilpotent in $M_n(\mathbb{C})$ such that $dim(Ker(A)) \geq 2$. Then there exists a reflexive sublattice $\mathcal{L}$ of $\mathcal{L}at(A)$ such that $\mathcal{L}' = \mathbb{C}I$ and $\mathcal{A}lg(\mathcal{L}) \setminus \mathcal{A}lg(\mathcal{L}at(A))$ contains a non-trivial idempotent. In particular $\mathcal{L}at(A)' = \mathbb{C}I$ and $\mathcal{A}lg(\mathcal{L}at(A))$ is not a KS-algebra.*

Proof. We will prove this result by induction on the size n of $M_n(\mathbb{C})$. If $n = 2$, the only nilpotent satisfies the condition in the proposition is 0, and the conclusion is trivial(see Example 1.1).

Let $n \geq 3$ be given and suppose the conclusion of the proposition is true for $l \leq n-1$. Let A be a nonzero nilpotent in $M_n(\mathbb{C})$ such that $dim(Ker(A)) \geq 2$ and $A^k = 0$ and $A^{k-1} \neq 0$ where $k \geq 2$.

Let $P = \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}$ be the projection onto the kernel of A^{k-1}, where $m = dimKer(A^{k-1})$. By the argument in Remark 3.1, we may write A^{k-1} as $\begin{pmatrix} 0 & \tilde{A} \\ 0 & 0 \end{pmatrix}$ such that $rank(\tilde{A}) = n - m$. Since $(I - P)AP = 0$, we have $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$ where $A_1 \in M_m(\mathbb{C})$, $A_2 \in M_{m,n-m}(\mathbb{C})$ and $A_3 \in M_{n-m}(\mathbb{C})$. Also note that $A^{k-1} \times A = 0$ implies $A_3 = 0$. Therefore we assume that $A = \begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix}$.

Since $rank(\tilde{A}) = rank(A_1^{k-2}A_2) = n - m$, we have $A_1^{k-2}\xi \neq 0$, where $\xi = A_2 e_1$ is the 1-st column of A_2 and $e_1 = (1, 0, \ldots, 0)^t \in (I - P)\mathbb{C}^{n+1}$. It is not hard to see that

$$span\{\xi, A_1\xi, \cdots, A_1^{k-2}\xi\} \cap Ker(A_1) = span\{A_1^{k-2}\xi\}.$$

Since the kernel of A is contained in $P\mathbb{C}^n$, we have $dim(Ker(A_1)) \geq 2$ and there exists a nonzero vector $\beta \in Ker(A_1)$ such that $\beta \perp A_1^{k-2}\xi$. Note that β and $\{\xi, A_1\xi, \cdots, A_1^{k-2}\xi\}$ are linearly independent.

Let $\alpha = \beta \oplus e_1 = (\beta, 1, 0, \cdots, 0)^t$ and Q be the orthogonal projection onto to the subspace spanned by $\begin{pmatrix} \xi \\ 0 \end{pmatrix}, \begin{pmatrix} A_1\xi \\ 0 \end{pmatrix}, \cdots, \begin{pmatrix} A_1^{k-2}\xi \\ 0 \end{pmatrix}, \alpha$. Since $A\alpha = \begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix}\alpha = \begin{pmatrix} \xi \\ 0 \end{pmatrix}$, we have $Q \in \mathcal{L}at(A)$. Let

$$\mathcal{L}_1 = \{\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} : E \in \mathcal{L}at(A_1)\}$$
$$\mathcal{L}_2 = \{\begin{pmatrix} I & 0 \\ 0 & E \end{pmatrix} : E \text{ is a projection in } M_{n-m}(\mathbb{C})\}$$

and $\mathcal{L} = \mathcal{L}at(\mathcal{A}lg(\mathcal{L}_1 \cup \mathcal{L}_2 \cup \{Q\}))$. It is clear that $\mathcal{L} \subset \mathcal{L}at(A)$. By the induction assumption, we know that $\mathcal{L}_1|'_{P\mathbb{C}^n} = \mathbb{C}I_m$. This implies that $\mathcal{L}' \subset \{\begin{pmatrix} aI_m & 0 \\ 0 & bI_{n-m} \end{pmatrix} : a, b \in \mathbb{C}\}$. If $\mathcal{L}' \neq \mathbb{C}I$, then we have $P \in \mathcal{L}'$ and $P\alpha = \begin{pmatrix} \beta \\ 0 \end{pmatrix} \in Q\mathbb{C}^n$. However, this can not be true since β and $\{\xi, A_1\xi, \cdots, A_1^{k-2}\xi\}$ are linearly independent. Therefore $\mathcal{L}'$ must equals to $\mathbb{C}I$.

Let $B \in M_{m,n-m}(\mathbb{C})$ be a matrix such that $Be_1 = \beta$. Then $\begin{pmatrix} 0 & B \\ 0 & I \end{pmatrix}$ is a non-trivial idempotent in $Alg(\mathcal{L})$. Let $\widetilde{Q}$ be the projection onto the subspace spanned by $\begin{pmatrix} \xi \\ 0 \end{pmatrix}, \begin{pmatrix} A_1\xi \\ 0 \end{pmatrix}, \cdots, \begin{pmatrix} A_1^{k-2}\xi \\ 0 \end{pmatrix}, 0 \oplus e_1$. Since $A(0 \oplus e_1) = \begin{pmatrix} \xi \\ 0 \end{pmatrix}$, we have $\widetilde{Q} \in \mathcal{L}at(A)$. However, $\begin{pmatrix} 0 & B \\ 0 & I \end{pmatrix}(0 \oplus e_1) = \alpha \notin \widetilde{Q}\mathbb{C}^n$ and $\begin{pmatrix} 0 & B \\ 0 & I \end{pmatrix} \notin \mathcal{A}lg(\mathcal{L}at(A))$. □

It is well-known that if $T \in M_n(\mathbb{C})$ is a nilpotent and $dim(Ker(T)) = 1$, then $\mathcal{L}at(T)$ is a maximal nest and $\mathcal{A}lg(\mathcal{L}at(T))$ is non-commutative(it is a KS-algebra by [14]). Therefore we have the following corollary.

Corollary 3.1. *Let A be a nilpotent element in $M_n(\mathbb{C})$, $n \geq 2$, then $\mathcal{A}lg(\mathcal{L}at(A))$ is not an abelian KS-algebra.*

Corollary 3.2. *Suppose that $A \in M_n(\mathbb{C})(n \geq 3)$. Then the algebra generated by A and I is not a KS-algebra.*

4. Abelian KS-algebras and Completely Distributive Lattices

In this part, we will show that if $\mathfrak{A}$ is an abelian KS-subalgebra of $\mathcal{B}(\mathcal{H})$, $dim\mathcal{H} \geq 3$, then $\mathcal{L}at(\mathfrak{A})$ can not be completely distributive.

It is well known that a subspace lattice $\mathcal{L}$ is completely distributive if and only if $E_\sharp = E$ for any $E \in \mathcal{L}$ [11], where $E_\sharp = \vee\{F \in \mathcal{L} : E \nleq F_-\}$ and $E_- = \vee\{F \in \mathcal{L} : E \nleq F\}$.

Lemma 4.1. *If $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})(dim\mathcal{H} \geq 3)$ is an abelian KS-algebra, then $E \leq E_-$ for any $E \in \mathcal{L}at(\mathfrak{A})$.*

Proof. If $E = 0$ or I, it is trivial. Assume that $E \in \mathcal{L}at(\mathfrak{A})$ is a projection such that $E \nleq E_-$, then there is a vector $\xi \in E\mathcal{H}$ such that $\eta = (E_-)^\perp\xi \neq 0$. Let $r = \langle\xi, \eta\rangle = \langle\xi, (E_-)^\perp\xi\rangle = ||(E_-)^\perp\xi||^2 > 0$ and $\eta' = \frac{\eta}{r}$. Then the rank one operator $\eta' \otimes \xi$ is in $\mathfrak{A}$. Since $(\eta' \otimes \xi)^2 = \langle \xi, \eta' \rangle (\eta' \otimes \xi) = \eta' \otimes \xi$, $\eta' \otimes \xi$ is a non-trivial idempotent. This is contradict with Corollary 2.3, thus $E \leq E_-$. □

Remark 4.1. *Suppose $\mathfrak{A}$ is an abelian KS-algebra such that $dim\mathfrak{A} > 1$. If $I_- < I$, then for any $E \in \mathcal{L}at(\mathfrak{A})$, we have $EI_- = I_-E$. Thus I_- is a non-trivial projection in $\mathfrak{A}$(note that $\mathcal{L}at(\mathfrak{A}) \neq \{0, I\}$). This contradicts with Corollary 2.1. Hence $I_- = I$.*

Lemma 4.2. *Suppose that $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$ is an abelian subalgebra. Let $\mathfrak{I} = \{E \in \mathcal{L}at(\mathfrak{A}) : E \neq 0, E_- \neq I\}$. Then for any $E_1 \neq E_2 \in \mathfrak{I}$, we have $E_1 \leq (E_2)_-$ and $E_2 \leq (E_1)_-$.*

Proof. If $E_1 \nleq E_2$ and $E_2 \nleq E_1$, then $E_2 \leq (E_1)_-$ and $E_1 \leq (E_2)_-$.

Without lose generosity, we could assume $E_1 < E_2$, then $E_1 \leq (E_2)_-$. Let $\xi_i \in E_i\mathcal{H}$ and $\eta_i \in ((E_i)_-)^\perp\mathcal{H}(i = 1, 2)$ be nonzero unit vectors, then we have $\eta_i \otimes \xi_i (i = 1, 2) \in \mathfrak{A}$. Since $\mathfrak{A}$ is abelian, we have

$$(\eta_1 \otimes \xi_1)(\eta_2 \otimes \xi_2) = \langle \xi_2, \eta_1 \rangle (\eta_2 \otimes \xi_1) = \langle \xi_1, \eta_2 \rangle (\eta_1 \otimes \xi_2) = (\eta_2 \otimes \xi_2)(\eta_1 \otimes \xi_1).$$

Note that $E_1 \leq (E_2)_-$, $\xi_1 \in E_1\mathcal{H}$ and $\eta_2 \in ((E_2)_-)^\perp\mathcal{H}$. We have $\langle \xi_1, \eta_2 \rangle = 0$. Therefore $\langle \xi_2, \eta_1 \rangle$ must also equals 0. This implies $E_2 \leq (E_1)_-$. □

Theorem 4.1. *If $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})(dim\mathcal{H} \geq 3)$ is an abelian KS-algebra, then $\mathcal{L}at(\mathfrak{A})$ is not completely distributive.*

Proof. If $E_- = I$ for every nonzero $E \in \mathcal{L}at(\mathfrak{A})$, then $I_\sharp = 0 \neq I$. If $E_- < I$, we have $I_\sharp \leq E_- \neq I$ by Lemma 4.1 and Lemma 4.2. Thus $\mathcal{L}at(\mathfrak{A})$ is not completely distributive. □

At the end of this paper, we state the following question whose answer we believe is negative.

Question 4.1. *Besides the algebras given in Example 1.1, dose there exist any other non-trivial abelian KS-algebras?*

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