Note on Sarnak Conjecture

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ABSTRACT. Enter abstract here

1. INTRODUCTION

The Möbius function $\mu(n)$, $n = 1, 2, 3, \ldots$ is defined by

$$\mu(n) = \begin{cases} 1, \text{ if n=1} \\ 0, \text{ if n is not square free} \\ (-1)^t, \text{ if n is a product of t distinct primes} \end{cases}$$

Peter Sarnak give the disjointness conjecture concerning the Möbius function $\mu(n)$.

Conjecture 1.1 (P. Sarnak)**.** *Let (X, T) be a deterministic (i.e. the topological entropy h(T) = 0) topological dynamical system.*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{n} \mu(n) f(T^n x) = 0$$

where $x \in X$ and $f \in C(X)$.

2. REVIEW ON ENTROPY

> *You should call it entropy, for two reasons. In the first place, your uncertainty function has been used in statistical mechanics under that name, so it already has a name. In the second place, and more important, no one knows what entropy really is, so in a debate you will always have the advantage.*
>
> von Neumann said to Shannon, *Mathematical Theory of Entropy*, 1981

Throughout this section $(X, \mathcal{A}, \mu)$ denotes a measure space.

2.1. **Some ergodic results.**
Let T be a $\mathcal{A}$-measurable transformation from X to X. μ is T-invariant iff $\mu(T^{-1}B) = \mu(B)$ for every $B \in \mathcal{A}$.

Definition 2.1. *Let $T : X \to X$ be a transformation. The periodic entropy of T is*

$$p(T) = \lim_{m\to\infty} sum \frac{1}{m} log(\#\{x \in X : T^m(x) = x\}).$$

Remark 2.1. *The periodic entropy measures the complexity of T from the point of view of the periodic points. For example, let $T : z \to z^m$ from S^1 to S^1, $p(T) = log|m| > 0$.*

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Theorem 2.1 (*Poincaŕe*'s recurrence theorem)**.** *Let $T : X \to X$ be a measurable transformation and let μ be a finite T-invariant measure in X. If $A \subset X$ is measurable, then the set*

$$B = \{x \in A : T^n(x) \in A, \textit{for infinitely many integers } n \in \mathbb{N}\}$$

has measure $\mu(B) = \mu(A)$.

Definition 2.2. *Given a transformation $T : X \to X$, we say that a function φ is T-invariant if $\varphi(T(x)) = T(x)$ for every $x \in X$. φ is T-invariant almost everywhere if there is a T-invariant subset $B \subset X$, i.e. $T^{-1}B = B$, with $\mu(X \setminus B) = 0$ and $\varphi|_B$ is T-invariant.*

Theorem 2.2 (Birkhoff's ergodic theorem)**.** *Let $T : X \to X$ be a measurable transformation and let μ be a finite T-invariant measure in X. If $\varphi \in L^1(X, d\mu)$, then the limit*

$$\varphi_T = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \varphi(T^k(x))$$

- *φ_T is T-invariant almost everywhere.*
- *$\varphi_T \in L^1(X, d\mu)$ and $\int_X \varphi_T d\mu = \int_X \varphi d\mu$.*

2.2. **Metric(measure-theoretical) Entropy.** In this subsection we assume that $\mu(X) = 1$. Let

$$\psi(x) = \begin{cases} xlogx, \text{ if x} \neq 0 \\ 0, \text{ if x=0.} \end{cases}$$

Definition 2.3. *Let $\xi \subset \mathcal{A}$. If*

(1) $\mu(\cup_{C\in\xi} C) = 1$,
(2) $\mu(C \cap D) = 0$ for any distinct C, D in ξ,

then ξ is a measurable partition of $(X, \mathcal{A}, \mu)$.

Definition 2.4 (Entropy of a measurable partition)**.** *Let ξ be a measurable partition of X w.r.t μ. The entropy of ξ is given by*

$$H_\mu(\xi) = -\sum_{C\in\xi} \mu(C)log\mu(C) = -\sum_{C\in\xi} \psi(\mu(C)).$$

Definition 2.5. *Given two measurable partition ξ, η of X, we define a new measurable partition*

$$\xi \bigvee \eta = \{C \cap D : C \in \xi, D \in \eta\}.$$

Definition 2.6. *Let $T : X \to X$ be a measurable transformation preserving the probability measure μ and ξ a measurable partition of X. The metric entropy of T with respect to μ and ξ is*

$$h_\mu(T, \xi) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\xi_n) = \inf_{n\in\mathbb{N}} H_\mu(\xi_n),$$

where $\xi_n = \bigvee_{k=0}^{n-1} T^{-k}\xi$. The metric entropy of T with respect to μ is

$$h_\mu(T) = sup\{h_\mu(T, \xi) : \xi \textit{is a measurable partition of} X\}.$$

Lemma 2.1. *Let ξ and η be measurable partitions of X. If η is a refinement of ξ, i.e., each element of ξ is a union of elements of η, then $H_\mu(\xi) \leq H_\mu(\eta)$.*

Theorem 2.3. *Let $T : X \to X$ be a measurable transformation preserving the probability measure μ. If $(\xi_n)_{n\in\mathbb{N}}$ is a sequence of measurable partitions of X with $\bigvee_{n=1}^{\infty} \mathcal{A}(\xi_n)$ ($\mathcal{A}(\xi)$ is the σ-algebra generated by ξ) such that ξ_{n+1} is a refinement of ξ_n for each $n \in N$, then*

$$h_\mu(T) = \lim_{n\to\infty} h_\mu(T, \xi_n) = sup_{n\in\mathbb{N}} h_\mu(T, \xi_n).$$

Specially, if $\bigvee_{k=0}^{\infty} \mathcal{A}(T^{-k}\xi) = \mathcal{A}$ or $\bigvee_{k=-\infty}^{\infty} \mathcal{A}(T^{-k}\xi) = \mathcal{A}$, then we have the following.

Corollary 2.1. *Let $T : X \to X$ be a measurable transformation preserving the probability measure μ. If $\bigvee_{k=0}^{\infty} \mathcal{A}(T^{-k}\xi) = \mathcal{A}$ or $\bigvee_{k=-\infty}^{\infty} \mathcal{A}(T^{-k}\xi) = \mathcal{A}$, then $h_\mu(T) = h_\mu(T, \xi)$.*

Note that if ξ is a measurable partition of X w.r.t. μ, then for almost every $x \in X$, there exists a single element $\xi_n(x)$ (depending on x such that $\xi_n(x) \in \xi_n = \bigvee_{k=0}^{n-1} T^{-k}\xi$.

Theorem 2.4 (Shannon-McMillan-Breiman)**.** *If $T : X \to X$ is a measurable transformation preserving a probability measure μ in X and ξ is a measurable partition X, then the limit*

$$h_\mu(T, \xi, x) := \lim_{n\to\infty} -\frac{1}{n} log\mu(\xi_n(x))$$

exists for almost every $x \in X$. Moreover, the function $x \to h_\mu(T, \xi, x)$ is T-invariant almost everywhere, is integrable and

$$h_\mu(T, \xi) = \int_X h_\mu(T, \xi, x) d\mu(x).$$

2.3. **Topological Entropy.**

Definition 2.7 (Cover entropy of continuous map)**.** *For a cover $\mathcal{U}$ of a compact topological space X, define $N(\mathcal{U})$ to be the smallest cardinality of a subcover of $\mathcal{U}$, and define the entropy of $\mathcal{U}$ to be $H(\mathcal{U}) = \log N(\mathcal{U})$. Let $T : X \to X$ be a continuous map. The cover entropy of T with respect to $\mathcal{U}$ is defined to be*

$$h_{cover}(T, \mathcal{U}) = \lim_{n\to\infty} \frac{1}{n} \log N(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}) = \inf_{n\geq 1} \frac{1}{n} \log N(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}),$$

and the cover entropy of T is defined to be

$$h_{cover}(T) = \sup_{\mathcal{U}} h_{cover}(T, \mathcal{U})$$

where the supremum is taken over all open covers of X.

Throughout the rest of this subsection, let (X, d) be a **compact metric space** and T a continuous transformation form X to X.

Definition 2.8. *Let $T : (X, d) \to (X, d)$ be a continuous map on a compact metric space. A one side generator for T is a finite open cover $\mathcal{U} = \{U_i\}_{i\in I}$ with the property that for any sequence $(i_k)_{k\geq 0}$, the set*

$$\bigcap_{k\geq 0} T^{-k}(\overline{U_{i_k}})$$

contains at most a single point.

Theorem 2.5. *Let $\mathcal{U}$ be a one-sided generator for a continuous map $T : (X, d) \to (X, d)$ on a compact metric space. Then $h_{cover}(T, \mathcal{U}) = h_{cover}(T)$.*

Theorem 2.6. *Let $T : X \to X$ be a continuous map on a compact metric space (X, d). If $(\mathcal{U}_n)$ is a sequence of open covers of X with diam($\mathcal{U}_n$)$\to 0$ as $n \to \infty$, then*

$$\lim_{n\to\infty} h_{cover}(T, \mathcal{U}_n) = h_{cover}(T).$$

For each $n \in \mathbb{N}$, we introduce a new distance in X by

$$d_n(x, y) = max\{d(T^k(x), T^k(y)) : 0 \leq k \leq n-1\}.$$

And $N(d, \varepsilon)$ denotes the maximum number of points in X at a d-distance at least ε.

Definition 2.9 (Separated Set Topological Entropy)**.** *The separated set topological entropy of T is*

$$h(T)_{sep} = \lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{1}{n} logN(d_n, \varepsilon).$$

Remark 2.2. *$h_{sep}(T)$ only depends on the topology induced by the distance d and $h_{sep}(T) = h_{cover}(T)$. The common value is called the **topological entropy of T, denoted** $h_{top}(T)$.*

Some facts about $h_{top}(T)$:

- If $T : X \to X$ is a **homeomorphism** of a compact metric space, then $h_{top}(T^{-1}) = h_{top}(T)$.
- If $T : X \to X$ is a continuous map of a compact metric space, then $h_{top}(T^k) = kh_{top}(T)$ for $k \geq 1$.

Theorem 2.7 (Variational principle for the topological entropy)**.** *If $T : X \to X$ is a continuous transformation of a compact metric space, then*

$$h_{top}(T) = sup\{h_\mu(T) : \mu \text{ is a T-invariant probability measure in } X\}.$$

2.4. **For AF algebra.** Let σ be a automorphism of $\mathfrak{A}$, $\mathfrak{A}$ is an AF-algebra. $\Sigma = \{A_i\}_{i=1}^n \subset \mathfrak{A}$, and $\mathcal{M}$ be a finite dimensional subalgebra of $\mathfrak{A}$ and $\Sigma \subset_\delta \mathcal{M}$. Let $r(\Sigma, \delta) = \min_{\Sigma\subset_\delta \mathcal{M}}\{$ max dim of masa of $\mathcal{M}\}$.

$$\Sigma_n = \cup_{i=0}^n \sigma^i(\Sigma)$$

$$h(\sigma) = \lim_{\delta\to 0}\lim_{n\to 0} \frac{\log(\Sigma_n, \delta)}{n}$$

3. Finite Matrix case

Let $\mathfrak{A} = M_n(\mathbb{C})$ and ρ be a pure state of $\mathfrak{A}$. We may ask if

$$\lim_{n\to\infty} \frac{1}{n} \sum_n \mu(n)\rho(\sigma^n(T)) = 0$$

where σ is an automorphism of $\mathfrak{A}$ and $T \in \mathfrak{A}$.

Since automorphism preserve the norm of $\mathfrak{A}$, it is not hard to see that , as a continuous map from the unit ball of $\mathfrak{A}$ onto itself, the topological entropy of σ is 0, by considering the definition of Bowen and Dianburg.

Without lose of generality, we may assume that $\sigma = Ad(U)$ where U is a unitary and $\rho(T) = \langle T\xi, \xi\rangle$. The question can be restated as if

$$\lim_{n\to\infty} \frac{1}{n} \sum_n \mu(n) \langle TU^n\xi, U^n\xi\rangle = 0$$

Assume $\xi = \xi_1 + \xi_2 + \cdots + \xi_n$ such that

$$U\xi_k = e^{i\theta_k}\xi_k \qquad k = 1, 2, \ldots, n.$$

Then we have

$$\langle TU^m\xi, U^m\xi\rangle = \sum_{k,l=1}^{n} e^{im(\theta_k-\theta_l)} \langle T\xi_k, \xi_l\rangle .$$

Let $C_{kl} = \langle T\xi_k, \xi_l\rangle$. Davenport and Hua showed in [2, 5] that for any fixed $h > 0$, $m \in \mathbb{N}^+$ and $0 \le l < m$,

$$\frac{1}{N} \sum_{\substack{n\le N\\ n\equiv l \mod m}} \mu(n) e^{2\pi i(a_d n^d + a_{d-1}n^{d-1}+\cdots+a_1 n+a_0)} = O((logN)^{-h}), \tag{1}$$

such that the implied constant depends only on h and p, but is independent of any of coefficients $a_d, \ldots, a_0$. In particular,

$$\frac{1}{N}\sum_{n\le N} \mu(n)e^{2\pi ni\theta} = O((logN)^{-h})$$

uniformly in θ. Therefore we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_n \mu(n)e^{im(\theta_k-\theta_l)}C_{kl} = \lim_{n\to\infty} \frac{C_{kl}}{n}\sum_n \mu(n)e^{im(\theta_k-\theta_l)} = 0$$

Theorem 3.1. *Suppose that $\mathfrak{A}$ be a finite dimensional C^*-algebra. Let ρ be a continuous funcitonal of $\mathfrak{A}^*$ and σ be an automorphism of $\mathfrak{A}$. Then*

$$\lim_{n\to\infty} \frac{1}{n}\sum_n \mu(n)\rho(\sigma^n(T)) = 0.$$

Proof. We could assume that $\mathfrak{A}$ is a direct sum of finite matrix algebras and ρ is a vector state (otherwise we can always apply GNS construction to make ρ a vector state). Then the discussion above give the result. □

4. VON NEUMANN ALGEBRA

Let $\mathcal{M}$ be a von Neumann algebra acting on the Hilbert space $\mathcal{H}$ and σ be a *-automorphism of $\mathcal{M}$. For any $T \in \mathcal{M}$ and any $\rho \in \mathcal{M}_*^+$, we will study the following limit.

$$\lim_{n\to\infty} \frac{1}{n}\sum_n \mu(n)\rho(\sigma^n(T)) \tag{2}$$

Without lose of generality, we could assume that $\mathcal{M}$ is in standard form $\|T\| = 1$ and $\|\rho\| = 1$. By Lemma 2.10 and Theorem 3.2 in [4], we may also assume that $\rho(T) = \langle T\xi, \xi\rangle$ and $\sigma(T) = U^*TU$, where ξ is an unit vector in $\mathcal{H}$ and U is a unitary in $\mathcal{B}(\mathcal{H})$. For any fixed $N \in \mathbb{N}$ and any $\varepsilon > 0$, we can find a unitary $V = \sum_{k=1}^{n} e^{i\theta_k}P_k$ such that $\|V^m - U^m\| \le \varepsilon$, where $P_k, k = 1\ldots n$, are orthogonal projections and $m \le N$. Then we have

$$\begin{aligned}
\Delta(N) =& |\frac{1}{N}\sum_{m=1}^{N}\mu(m)\langle TU^m\xi, U^m\xi\rangle| \\
&\leq |\frac{1}{N}\sum_{m=1}^{N}\mu(m)\langle T(U^m - V^m)\xi, U^m\xi\rangle| + |\frac{1}{N}\sum_{m=1}^{N}\mu(m)\langle TV^m\xi, (U^m - V^m)\xi\rangle| \\
&\quad + |\frac{1}{N}\sum_{m=1}^{N}\mu(m)\langle TV^m\xi, V^m\xi\rangle| \\
&\leq \frac{2}{N}\sum_{m=1}^{N}\|(U^m - V^m)\xi\| + |\sum_{l,k=1}^{n}(\frac{1}{N}\sum_{m}\mu(m)e^{im(\theta_l-\theta_k)})\langle TP_l\xi, P_k\xi\rangle| \\
&\leq 2\varepsilon + C\log(N)^{-h}\sum_{l,k=1}^{n}\|P_l\xi\|_2\|P_k\xi\|_2
\end{aligned}$$

If $\sum_{i=1}^{n} a_i^2 = 1$, then we have that

$$\left(\sum_{i=1}^{n} a_i\right)^2 = \sum_{i=0}^{n-1}\left(\sum_{j=1}^{n} a_j a_{(j+i)\%n}\right) \leq \sum_{i=0}^{n-1}\left(\sum_{i=1}^{n} a_i^2\right) = n$$

This seems lead to a dead end. Actually, the estimation above is quiet trivial.

However, if $\mathfrak{A}$ is a finite von Neumann algebra and ω a trace vector in $L^2(\mathfrak{A}, \tau)$, then there exists a $A \in \mathfrak{A}$ such that $\|A\omega - \xi\| \leq \varepsilon$. Furthermore, we assume U is in $\mathfrak{A}$, therefore V can also be constructed in $\mathfrak{A}$.

$$\begin{aligned}
\Delta(N) =& |\frac{1}{N}\sum_{m=1}^{N}\mu(m)\langle TU^m\xi, U^m\xi\rangle| \\
&\leq |\frac{1}{N}\sum_{m=1}^{N}\mu(m)\langle TU^m(\xi - A\omega), U^m\xi\rangle| + |\frac{1}{N}\sum_{m=1}^{N}\mu(m)\langle TU^m A\omega, U^m(\xi - A\omega)\rangle| \\
&\quad |\frac{1}{N}\sum_{m=1}^{N}\mu(m)\langle T(U^m - V^m)A\omega, U^m A\omega\rangle| + |\frac{1}{N}\sum_{m=1}^{N}\mu(m)\langle TV^m A\omega, (U^m - V^m)A\omega\rangle| \\
&\quad + |\frac{1}{N}\sum_{m=1}^{N}\mu(m)\langle TV^m A\omega, V^m A\omega\rangle| \\
&\leq 4\varepsilon(1+\varepsilon)^2 + |\sum_{l,k=1}^{n}(\frac{1}{N}\sum_{m}\mu(m)e^{im(\theta_l-\theta_k)})\langle TP_l A\omega, P_k A\omega\rangle| \\
&= 4\varepsilon(1+\varepsilon)^2 + |\sum_{l,k=1}^{n}(\frac{1}{N}\sum_{m}\mu(m)e^{im(\theta_l-\theta_k)})\tau(P_k TP_l AA^*)| \\
&\leq 4\varepsilon(1+\varepsilon)^2 + |\frac{1}{N}\sum_{m}\mu(m)e^{im(\theta_l-\theta_k)}|(\sum_{l,k=1}^{n}\|P_k TP_l\|_2\|P_l AA^* P_k\|_2) \\
&\leq 4\varepsilon(1+\varepsilon)^2 + |\frac{1}{N}\sum_{m}\mu(m)e^{im(\theta_l-\theta_k)}|(\sum_{l,k=1}^{n}\|P_k TP_l\|_2^2)^{\frac{1}{2}}(\sum_{l,k=1}^{n}\|P_l AA^* P_k\|_2^2)^{\frac{1}{2}} \\
&\leq 4\varepsilon(1+\varepsilon)^2 + |\frac{1}{N}\sum_{m}\mu(m)e^{im(\theta_l-\theta_k)}|\|T\|_2\|AA^*\|_2
\end{aligned}$$

Fix A, we have that $lim_{n\to\infty} sup\Delta(n) \leq 4\varepsilon(1+\varepsilon)^2$ for any ε. Then we have that $lim_{n\to\infty}\Delta(n) = 0$.

5. Compact set with finite accumulation points

5.1. **Topological Entropy.**

Lemma 5.1. *Let X be a compact space and $\sigma : X \to X$ a continuous map. If there exists $n \in \mathbb{N}$ such that $\sigma^{m_1}(x) = \sigma^{m_2}(x)$ for each $x \in X$ where $0 \leq m_1 < m_2 \leq n$, then $h_{top}(\sigma) = 0$.*

Proof. By the hypothesis, $\sigma^n(x)$ is a periodic point with period less than n for each $x \in X$. Let $k = n!$, we have $\sigma^{n+k} = \sigma^n(x)$. Since $h_{top}(\sigma) = 0$ if and only if $h_{top}(\sigma^n) = 0$. We may assume that $\sigma^k(x) = \sigma(x)$ for any $x \in X$ where $k > 1$. Let U be an open set of X, $1 \leq m \leq k-1$ and $l \geq 0$,

$$\sigma^{-(l(k-1)+m)}U = \{y : \sigma^{l(k-1)+m}(y) \in U\} = \{y : \sigma^m(y) \in U\} = \sigma^{-m}U.$$

Let $\mathcal{U}$ be an open cover of X, we have $\bigvee_{i=0}^{m} \sigma^{-i}\mathcal{U} = \bigvee_{i=0}^{k} \sigma^{-i}\mathcal{U}$ if $m > k$. Therefore

$$h_{cover}(\sigma,\mathcal{U}) = \lim_{m\to\infty} \frac{1}{m} \log N(\bigvee_{i=0}^{m-1} \sigma^{-i}\mathcal{U}) = 0.$$

Thus $h_{top}(\sigma) = 0$. □

Let X be a compact space with only one accumulation point c. If $\sigma : X \to X$ is a continuous map such that $\sigma(c) = y(\neq c)$, then there exist a neighborhood U of c such that $\sigma(U) = \{y\}$. It is not hard to see that there exist n_0 such that $\sigma^{n_0}(y)$ is a periodic point of σ. Indeed, note that $X \setminus U$ contains only finite many points(U is closed since it contains all accumulation points), then either there exists a $k \in \mathbb{N}$ such that $\sigma^k(y) \in U$ or there exists n_0 such that $\sigma^{n_0}(y)$ is a periodic point. The same argument also tells us that $\sigma^m(x)$ is periodic for some $m \in \mathbb{N}$ for each $x \in X \setminus U$. Since there are only finite many points in $X \setminus U$, there exists $n \in \mathbb{N}$ such that $\sigma^{m_1}(x) = \sigma^{m_2}(x)$ for each $x \in X$ where $0 \leq m_1 < m_2 \leq n$. Hence $h_{top}(\sigma) = 0$ by lemma 5.1.

Lemma 5.2. *Let X be a compact space with only one accumulation point c. If $\sigma : X \to X$ is a continuous map such that $\sigma(c) = y(\neq c)$, then $h_{top}(\sigma) = 0$.*

If a compact Hausdorff space with only finite many accumulation points is metrizable, then it only contains countable many points. If X is a countable, compact metric space and $\sigma : X \to X$ is continuous mapping, then by Proposition 5.1 in [8], $h_{top}(\sigma) = 0$. However, a compact Hausdorff space X may not be metrizable even if it only has one accumulation point. Indeed, let $X = \mathbb{R} \cup \{\infty\}$ be the on point compactification of $\mathbb{R}$ with discrete topology, it is clear that X is not second-countable. Therefore, it can not be metrizable since a compact Hausdorff space X is metrizable if and only if it is second-countable.

Theorem 5.1. *Let X be a compact Hausdorff space with finite many accumulation points $\{c_1, \ldots, c_l\}$. If $\sigma : X \to X$ is a continuous map, then $h_{top}(\sigma) = 0$.*

Proof. Suppose that $\sigma(c_i) = q_i$, $i = 1, \ldots, l$. Let $\mathcal{V}$ be a cover of X. There exists a cover $\mathcal{U} = \{U_1, \ldots, U_l, \{p_1\}, \ldots, \{p_m\}\}$ such that U_i is a neighborhood of c_i and

- $q_i \notin U_j$ if $q_i \neq c_j$,
- $U_i \cap U_j = \varnothing$, $i \neq j$,
- $\sigma(U_i) = \{q_i\}$ if $q_i \notin \{c_1, \ldots, c_l\}$,
- $\sigma(U_i) \cap U_j = \varnothing$ if $q_i \neq U_j$,
- $U_i \subset V_i$ for some $V_i \in \mathcal{V}$,
- $\{p_j\} \subset V_j$ for some $V_j \in \mathcal{V}$,
- $X \setminus (\cup_{i=1}^{l} U_i) = \{p_1, \ldots, p_m\}$.

Thus $h_{cover}(\sigma,\mathcal{V}) \leq h_{cover}(\sigma,\mathcal{U})$. If we can show that $h_{cover}(\sigma,\mathcal{U}) = 0$, then we have $h_{top}(\sigma) = 0$.

Let

$$\alpha(U_i) = \begin{cases} U_j, \text{ if } \sigma(c_i) = c_j, \\ \varnothing, \text{ otherwise.} \end{cases}$$

Note that the non empty set in $\bigvee_{i=0}^{n-1} \sigma^{-i}\mathcal{U}$ can only be $\{p_j\}$,

$$U_i \cap \sigma^{-1}\alpha(U_i) \cap \cdots \cap \sigma^{-n+1}\alpha^{n-1}(U_i)$$

or

$$U_i \cap \sigma^{-1}\alpha(U_i) \cap \cdots \cap \sigma^{-k+1}\alpha^{k-1}(U_i) \cap \sigma^{-k}(\{p_j\}) \cap \sigma^{-k-1}V_{k+1} \cdots \cap \sigma^{1-n}V_{n-1},$$

where $1 \leq k \leq n-1$, $1 \leq i \leq l$ $1 \leq j \leq m$ and $V_j(\in \mathcal{U})$ are uniquely determined by p_j. Therefore $N(\bigvee_{i=0}^{n-1} \sigma^{-i}\mathcal{U}) \leq ml(n-1) + l + m$ and

$$h_{cover}(\sigma,\mathcal{U}) \leq \lim_{n\to\infty} \frac{1}{n} log(ml(n-1) + l + m) = 0.$$

□

5.2. **Sarnak's conjecture is true for these spaces.** Throughout this section X is a compact space with only finite many accumulation points and $\sigma : X \to X$ a continuous map.

It is easy to get the following two lemmas by eq. (1).

Lemma 5.3. *Let X be a compact space and σ a continuous map from X to X. If $\{\sigma^n(x)\}$ has only one cluster point $c \in X$, then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} \mu(n) f(\sigma^n(x)) = 0, \forall f \in C(X).$$

Proof. Let $X' = \{\sigma^n(x)\}_{n=1}^{-} = \{\sigma^n(x)\} \cup \{c\}$. Since $\sigma(c)$ is a cluster point of $\sigma(\{\sigma^n(x)\}) \subset \{\sigma^n(x)\}$, we have $\sigma(c) = c$. Therefore $\sigma(X') \subset X'$ and σ is a continuous map form X' to X'. Note that X' is compact. For any $\varepsilon > 0$, there exists an open neighborhood of c such that $X' \setminus U$ contains only finite many elements and $|f(y) - f(c)| < \varepsilon$ if $y \in U$. Now it is easy to see $\lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} \mu(n) f(\sigma^n(x)) = 0$. □

Lemma 5.4. *Let X be a compact space, f a function on X and σ a map from X to X. If $x \in X$ and there is a n_0 and $n_1 \in N$ such that $\sigma^{n_0+n_1}(x) = \sigma^{n_0}(x)$ then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} \mu(n) f(\sigma^n(x)) = 0.$$

Example 5.1. *Let X be a compact Hausdorff space with a unique accumulation point c_0 and f a continuous function on X. By lemmas 5.3 and 5.4, $\lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} \mu(n) f(\sigma^n(x)) =$ 0.*

Lemma 5.5. *Let X be a compact Hausdorff space and σ a continuous map form X to X. Let x be a element in X and c_1 a cluster point of $\{\sigma^n(x)\}_{n=1}^{\infty}$. If $\sigma(c_1) = y$ where y is an isolated point of X, then there exist n_1 and n_2 such that $\sigma^{n_1+n_2}(x) = \sigma^{n_1}(x) = y$, and $\lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} \mu(n) f(\sigma^n(x)) = 0$ for any $f \in C(X)$.*

Proof. By the continuity of σ there is a neighborhood U of c_1 such that $\sigma(U) = \{y\}$. Since c_1 is a cluster point of $\{\sigma^n(x)\}_{n=1}^{\infty}$ and X is Hausdorff, then there exist n_1 and n_2 such that $\sigma^{n_1+n_2}(x) = \sigma^{n_1}(x) = y$. By lemma 5.4, $\lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} \mu(n) f(\sigma^n(x)) = 0$. □

Theorem 5.2. *If X is a compact Hausdorff space with finite many limit points, then we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\le N} \mu(n) f(\sigma^n(x)) = 0,$$

for any continuous map σ form X to X and any continuous function f on X.

Proof. Assume that $\{c_1, \dots, c_m\}$ are the only limit points of X. By lemma 5.4, we could assume that $\sigma^n(x) \neq \sigma^m(x)$, $n \neq m$ and c_1 is a cluster point of $\{\sigma^n(x)\}_{n=1}^\infty$. If $\sigma(c_1) = y$ and $y \notin \{c_1, \dots, c_m\}$, then by lemma 5.5, we have the result. Therefore, we assume that $\sigma(c_1) \in \{c_1, \dots, c_m\}$ and consider the following cases.

(1) Suppose that $\sigma(c_1) = c_1$. Let U_i be the open neighborhood of c_i and such that $U_i \cap U_j = \varnothing$, $i \neq j$. Since $\sigma(c_1) = c_1$ and σ is continuous, there is neighborhood V_1 of c_1 such that $V_1 \subset U_1$ and $\sigma(V_1) \subset U_1$. Note that $U_1 \setminus V_1$ contains only finite many elements of X. Since all points in $\{\sigma^n(x)\}$ are different and c_1 is a cluster point of $\{\sigma^n(x)\}$, there exist a N such that $\sigma^m(x) \in V_1$ if $m \geq N$. Therefore, c_1 is the only cluster point of $\{\sigma^n(x)\}_{n=1}^\infty$. By lemma 5.3, $\lim_{N\to\infty} \frac{1}{N} \sum_{n\le N} \mu(n) f(\sigma^n(x)) = 0$.

We restate the above result as the following lemma.

Lemma 5.6. *Let X be a compact Hausdorff space with finite many limit points and σ a continuous map from X to X. If c is a cluster point of $\{\sigma^n(x)\}_{n=1}^\infty$ and $\sigma(c) = c$, then c is the only cluster point of $\{\sigma^n(x)\}_{n=1}^\infty$.*

Now we consider the other case.

(2) Suppose that $\sigma(c_1) = c_2$. Since c_1 is a cluster point of $\{\sigma^n(x)\}$, c_2 must also be a cluster point of $\{\sigma^n(x)\}$. Hence $\sigma(c_2) \neq c_2$. By an inductive argument as above, we could assume that $\sigma(c_1) = c_2, \dots, \sigma(c_k) = c_1$. For any $f \in C(X)$ and $\varepsilon > 0$, there exits neighborhoods U_i and V_i of c_i, $i = 1, \dots, m$ such that
   - $U_i \cap U_j = \varnothing$, $i \neq j$.
   - $V_i \subset U_i$,
   - $\sigma(V_i) \subset U_{i+1}$, $i = 1, \dots, k-1$, and $\sigma(V_k) \subset U_1$.
   - $|f(x) - f(c_i)| \le \varepsilon$ if $x \in U_i$.

   Since $\cup_{i=1}^k (U_i \setminus V_i)$ contains only finite many points, there exists $k \in \mathbb{N}$ such that $\sigma^{n+1}(x) \in V_{i+1}$ whenever $n \geq k$ and $\sigma^n(x) \in V_i$ for $i = 1, \dots, k-1$ and $\sigma^{n+1}(x) \subset V_1$ if $\sigma^n(x) \in V_k$. Therefore, $\lim_{N\to\infty} \frac{1}{N} \sum_{n\le N} \mu(n) f(\sigma^n(x)) = 0$. □

Corollary 5.1. *Let X be a compact Hausdorff space and f a continuous function on X. If the sequence $\{\sigma^n(x)\}$ has only finite many limit points, then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\le N} \mu(n) f(\sigma^n(x)) = 0.$$

Proof. Let $X' = \{\sigma^n(x)\}^-$. Then X' is compact. If $y \in X'$ is a limit point, then $\sigma(y)$ is also a cluster point of $\{\sigma^{n_i}(x)\}$. Therefore $\sigma(X') \subset X'$. Now, apply theorem 5.2, we have the result. □

6. Compact space with countably many points

Let X be a countable, compact Hausdorff space. Since a compact Hausdorff space is metrizable if and only if it is second-countable, X is metrizable. Hence, throughout this section X denotes a countable *compactum*, i.e. a compact metric space has countable cardinality. Therefore, by Proposition 5.1 in [8], $h_{top}(\sigma) = 0$ for any continuous map form X to X.

Argue as in the proof of corollary 5.1, we assume in this section that $\{\sigma^n(x)\}$ is dense in X.

Since X only contains countable many points, X must contains isolated point by Baire category theorem. Recall that a topological space is totally disconnected if the connected components in the space are the one-point sets. Let Y be a connected component of X. Y is closed since connected component is closed. Thus Y is also a complete metric space with only countable many points. Hence Y contain isolated points. Therefore Y can only contain one point and X is totally disconnected and zero-dimensional. Since every compact totally disconnected metric space is homeomorphic to a subset of the Cantor set(Corollary 2-99 [1]), we could assume that X is a closed subset of the Cantor set.

For a metric space X, let $\iota(X)$ be the set of all isolated points of X and $X^d = X \setminus \iota(X)$ the set of limit points of X. The derived set of X of order α is defined by $X^{(1)} = X^d$, $X^{(\alpha+1)} = (X^{(\alpha)})^d$ and $X^{(\lambda)} = \cap_{\alpha<\lambda} X^{(\alpha)}$ if λ is a limit ordinal number. If $X^{(\alpha)} \neq \varnothing$ and $X^{(\alpha+1)} = \varnothing$, then we put $d(X) = \alpha$. And $d(X)$ is called the *derived degree* of X. It is well known that a compactum X is a countable set if and only if $d(X)$ exists and it is a countable ordinal number. In this case, if $d(X) = \alpha$, then $X^{(\alpha)}$ is a finite set.

Proposition 6.1 (Proposition 2.1 of [7]). *Let X and Y be countable compacta. If $d(X) = d(Y) = \alpha$ and $X^{(\alpha)}$ is homeomorphic to $Y^{(\alpha)}$, then X is homeomorphic to Y.*

Lemma 6.1. *Let X be a countably infinite, compact metric space and x a isolated point in X. If $\{\sigma^n(x)\}_{n=0}^{\infty}$ is dense in X, then $\sigma^n(x)$ is isolated for all n and $\sigma^n(x)$ are all the isolated points in X.*

Proof. Since X is a countable, compact metric space contains infinitely many points, X contains infinite isolated points. It is easy to see that $\{\sigma^n(x)\}$ must contains all isolated points. Otherwise, $\{\sigma^n(x)\}$ can not be dense in X. Assume that there is a n such that $\sigma^n(x)$ is a limit point in X. Let $m \in \mathbb{N}$ be a number such that $\sigma^m(x)$ is a limit point and $\sigma^{m+1}(x)$ is an isolated point. By lemma 5.5, there is n_1 and n_2 such that $\sigma^{n_1+n_2}(x) = \sigma^{n_1}(x)$. Therefore $\{\sigma^n(x)\}$ contains only finite many different elements. It contradicts the fact that $\{\sigma^n(x)\}$ contains infinite many isolated point. □

Lemma 6.2. *Let X be a countably infinite, compact metric space and x a isolated point in X. If $\{\sigma^n(x)\}_{n=0}^{\infty}$ is dense in X and X^d contains infinite many points, then y is not periodic for any isolated point in y in X^d.*

Proof. Suppose that there exist $n \geq 1$ such that $\sigma^n(y) = y$. Let $\{U_i\}_{i=0}^{n-1}$ be neighborhoods of $\sigma^i(y)$ such that $(\cup_{i=0}^{n-1} U_i \cap X^d)^- \neq X^d$, $U_0 \cap X^d = \{y\}$, $\sigma(U_i) \subset U_{i+1}$ for $i = 1, \ldots, n-2$ and $\sigma(U_{n-1}) \subset U_0$. We can also find a neighborhood V_0 of y such that $V_0 \subset U_0$ and $\sigma(V_0) \subset U_1$. Since U_0 do not contains limit point except y, we have $U_0 \setminus V_0$ contains finite many points. Therefore, there exist a $m \in \mathbb{N}$ such that $\sigma^k \notin U_0 \setminus V_0$ for any $k \geq m$, it is easy to see that all limit points of $\{\sigma^n(x)\}$ must contains in $(\cup_{i=0}^{n-1} U_i \cap X^d)^- \subsetneq X^d$. It is a contradiction. □

Example 6.1. *Let*

$$X = \bigcup_{n=1}^{\infty} (\{(\frac{1}{n}, \frac{1}{n+k}) : k = 0, 1, 2, \ldots\} \bigcup \{(\frac{1}{n}, 0)\}) \bigcup \{(0,0)\} \subset \mathbb{R}^2.$$

Then $X^{(1)} = \{(\frac{1}{n}, 0) : n = 1, 2, \ldots\} \cup \{(0,0)\}$ *and* $X^{(2)} = \{(0,0)\}$. *Let* σ *be a map* $X \to X$ *defined by*

$$\begin{aligned}
&(1,0) \to (0,0) \text{ and } (0,0) \to (0,0)\\
&(\frac{1}{n}, 0) \to (\frac{1}{n-1}, 0) \qquad (\text{for } n > 1)\\
&(1, \frac{1}{k+1}) \to (\frac{1}{k+2}, \frac{1}{k+2})\\
&(\frac{1}{n}, \frac{1}{n+k}) \to (\frac{1}{n-1}, \frac{1}{n+k}) \qquad (\text{for } n > 1).
\end{aligned}$$

It is easy to see that $\{\sigma^n((1,1))\} = X \setminus X^{(1)}$.

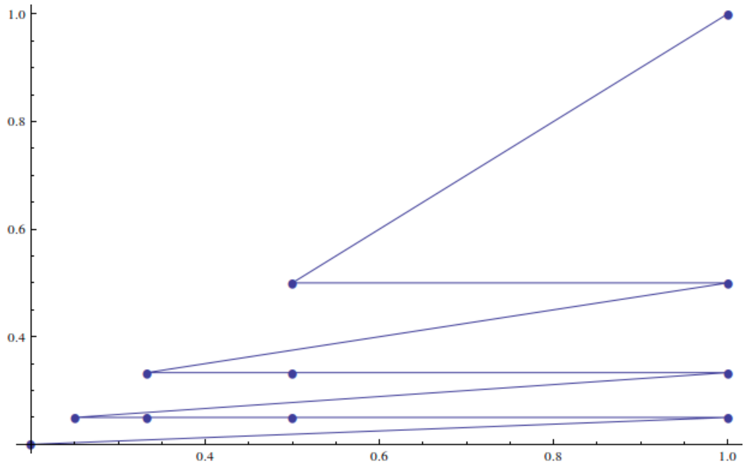

Figure 1. example 6.1

Lemma 6.3. *Let* X *be a countably infinite, compact metric space such that* $d(X) = 2$ *and* $X^{(2)} = \{c\}$. *If* $\{\sigma^n(x)\}_{n=0}^{\infty}$ *is dense in* X *for a isolated point* x *in* X, *then* $\sigma(c) = c$.

Proof. Suppose that $\sigma(c) = y \in X^{(1)}$. If $y \neq c$, then y is an isolated point in $X^{(1)}$. There exist a neighborhood U of c such that $\sigma(U) = \{y\}$ and $X^{(1)} \setminus U$ contains only finite many points. By lemma 6.2, there exist a n such that $\sigma^n(y) \in U$, since no isolated point in $X^{(1)}$ is periodic. However we have $\sigma^{n+1}(y) = y$. It is a contradiction. $\square$

Let (X, σ) be a dynamic system, A a subset of X and x a point in X. Let $A(x) = \{m : \sigma^m(x) \in A, \}$. Recall that the upper asymptotic density $\overline{d}(A(x))$ is $\limsup\limits_{n\to\infty} \frac{|A(x,n)|}{n}$ and lower asymptotic density $\underline{d}(A(x))$ is $\liminf\limits_{n\to\infty} \frac{|A(x,n)|}{n}$, where $A(x,n) = \{1, 2, \ldots, n\} \cap A(x)$. $A(x)$ has asymptotic density $den(A(x))$ if $\overline{d}(A(x)) = \underline{d}(A(x))$, in which case $den(A(x))$ is equal to this common value.

Lemma 6.4. *Let* X *be a countably infinite, compact metric space such that* $d(X) \geq 2$. *Suppose that* $\{\sigma^n(x)\}_{n=0}^{\infty}$ *is dense in* X *for a isolated point* x *in* X. *Let* $A = \{\sigma^{n_k}(x)\}$ *be a subsequence converge to* y. *If* y *is not a periodic point, then* $den(A(x)) = 0$.

Proof. We consider the following two cases.

(1) $\sigma^{m_1}(y) \neq \sigma^{m_2}(y)$ if $m_1 \neq m_2$. Hence $\lim_{m\to\infty} \sigma^m(y) = c$. Fix a positive integer k. We choose $k+1$ neighborhoods $\{U_i\}_{i=0}^{k}$ of $\sigma^i(y)$ such that $U_i \cap U_j = \varnothing$ and $\sigma(U_i) \subset U_{i+1}$. Note that $A \setminus U_0$ contains finite many points. Therefore there exists a $l \in \mathbb{N}$ such that $\{\sigma^n(x) : n \geq l\}$ do not contains any element in $A \setminus U_0$. If $n \geq l$ and $n \in A(x)$, we have $n+1, \ldots n+k$ are not in A. It is easy to see that $\overline{d}(A(x)) \leq \frac{1}{k}$. Hence $den(A(x)) = 0$.

(2) $\sigma^m(y) = c_0$ and c_0 is a periodic point such that $\sigma^j(c_0) = c_j$ and $\sigma^n(c_0) = c_0$. Fix a positive integer k. We choose m neighborhoods U_i of $\sigma^i(y)$ for $i = 0, \ldots, m-1$ and k neighborhoods $\{U_{m+nl+j}\}_{l=0}^{k-1}$ of c_j, $j = 0, \ldots, n-1$ such that
   - $U_{m+nl+j} \subset U_{m+n(l+1)+j}$, $l = 0, \ldots, k-1$,
   - $U_{i_1} \cap U_{i_2} = \varnothing$ for $i_1 \neq i_2$ and $i_1, i_2 \in \{0, \ldots, m-1\}$,
   - $U_i \cap U_{m+nk+j} = \varnothing$ for $i = 0, \ldots, m-1$ and $j = 0 \ldots n-1$,
   - $U_{m+nk+j_1} \cap U_{m+nk+j_2} = \varnothing$ for $j_1 \neq j_2$,
   - $\sigma(U_i) \subset U_{i+1}$ for $i = 0, 1, \ldots, m+nk-1$.

   Note that $A \setminus U_0$ contains finite many points. Therefore there exists a $N_0 \in \mathbb{N}$ such that $\{\sigma^n(x) : n \geq N_0\}$ do not contains any element in $A \setminus U_0$. If $s \geq N_0$ and $s \in A(x)$, we have $s+1, \ldots s+nk+m-1$ are not in A. It is easy to see that $\overline{d}(A(x)) \leq \frac{1}{nk+m-1}$. Hence $den(A(x)) = 0$

□

Corollary 6.1. *Let X be a countably infinite, compact metric space such that $d(X) = 2$ and $X^{(2)} = \{c\}$. Suppose that $\{\sigma^n(x)\}_{n=0}^{\infty}$ is dense in X for a isolated point x in X. Let B be any neighborhood of c and $A = X \setminus B$. Then $den(A(x)) = 0$.*

Proof. Note that $A \cap X^{(1)}$ contains finite elements $\{c_1, \ldots, c_m\}$. Since A is closed, $A = \cup_{k=1}^{m} A_k$, where A_k is a subsequence of $\{\sigma^n(x)\}$ converges to c_k. Each c_k is not periodic by lemma 6.2. Then $den(A_k(x)) = 0$ by lemma 6.4,. Therefore $den(A(x)) = 0$. □

Corollary 6.2. *Let (X, σ) be a dynamic system and $x \in X$. Let Y be the closure of $\{\sigma^n(x)\}_{n=0}^{\infty}$. If $d(Y) = 2$ and $Y^{(2)} = \{c\}$, then $\lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} \mu(n) f(\sigma^n(x)) = 0$ for any $f \in C(X)$.*

Proof. We could assume that $f(c) = 0$. For any $\varepsilon > 0$, there is neighborhood U of c such that $|f(y)| < \varepsilon$ for any $y \in U$. By corollary 6.1, $den(A(x)) = 0$ where $A = \{\sigma^n(x) : \sigma^n(x)$ not in $U\}$. This implies the result. □

Example 6.2. *Let*

$$X_1 = \bigcup_{n=1}^{\infty} (\{(2+\frac{1}{n}, \frac{1}{n+k}) : k = 0,1,2,\ldots\} \bigcup \{(2+\frac{1}{n}, 0)\}) \bigcup \{(2,0)\},$$

$$X_2 = \bigcup_{n=2}^{\infty} (\{(\frac{1}{n}, \frac{1}{n+k}) : k = 0,1,2,\ldots\} \bigcup \{(\frac{1}{n}, 0)\}) \bigcup \{(1, \frac{1}{2+k}) : k = 0,1,2,\ldots\} \bigcup \{(\frac{1}{n}, 0)\} \bigcup \{(0,0)\},$$

$X = X_1 \cup X_2$. Then $X^{(1)} = \{(\frac{1}{n}, 0) : n = 1, 2, \ldots\} \cup \{(0,0)\} \cup \{(2 + \frac{1}{n}, 0) : n = 1, 2, \ldots\} \cup \{(2,0)\}$ and $X^{(2)} = \{(0,0)\} \cup \{(2,0)\}$. Let σ be a map $X \to X$ defined by

$$\begin{aligned}
&(3,0) \to (0,0) \text{ and } (2,0) \to (2,0)\\
&(1,0) \to (2,0) \text{ and } (0,0) \to (0,0)\\
&(2+\frac{1}{n}, 0) \to (2+\frac{1}{n-1}, 0) \qquad (\text{for } n > 1)\\
&(\frac{1}{n}, 0) \to (\frac{1}{n-1}, 0) \qquad (\text{for } n > 1)\\
&(3, \frac{1}{k+1}) \to (\frac{1}{k+2}, \frac{1}{k+2})\\
&(1, \frac{1}{2+k}) \to (2+\frac{1}{k+2}, \frac{1}{k+2})\\
&(2+\frac{1}{n}, \frac{1}{n+k}) \to (2+\frac{1}{n-1}, \frac{1}{n+k}) \qquad (\text{for } n > 1).\\
&(\frac{1}{n}, \frac{1}{n+k}) \to (\frac{1}{n-1}, \frac{1}{n+k}) \qquad (\text{for } n > 1).
\end{aligned}$$

It is easy to see that $\{\sigma^n((3,1))\} = X \setminus X^{(1)}$. Let

$$\mathcal{X}(n) = \begin{cases} 1, n \in [m^2 - m + 1, m^2] \text{ for } m = 1, 2, \ldots \\ 0, n \in [m^2 + 1, m(m+1)] \text{ for } m = 1, 2, \ldots \end{cases}$$

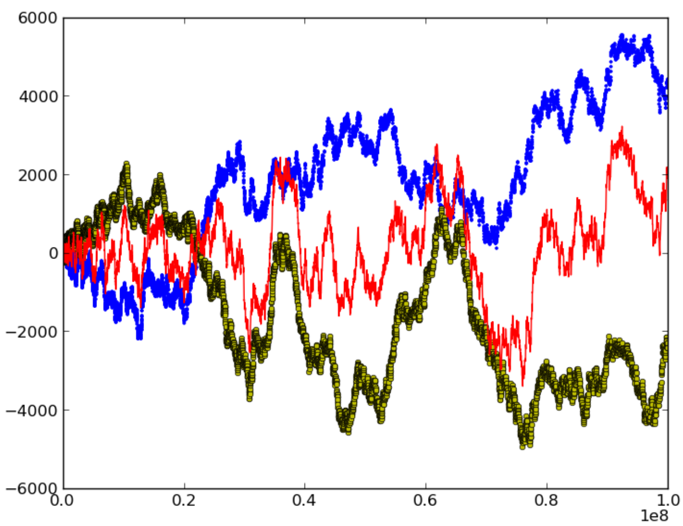

Figure 2. example 6.2, up to 100000000

Blue: $\sum_n^t \mu(n)\mathcal{X}(n)$, max:5561, min:−2197. Yellow: $\sum_n^t \mu(n)(1 - \mathcal{X}(n))$, max:2278, min:−4954. Red: $\sum_n^t \mu(n)$, max:3225, min:−3402.

In number theory, the Mertens function $M(x)$ is defined as

$$M(x) = \sum_{k \leq x} \mu(k).$$

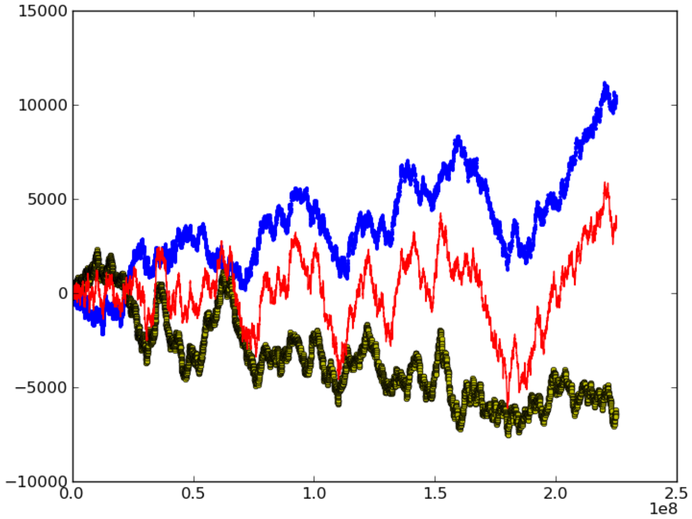

Figure 3. example 6.2, up to 225000000

Blue: $\sum_n^t \mu(n)\mathcal{X}(n)$, max:11175, min:$-2197$. Yellow: $\sum_n^t \mu(n)(1-\mathcal{X}(n))$, max:2278, min:$-7547$. Red: $\sum_n^t \mu(n)$, max:5890, min:-6136.

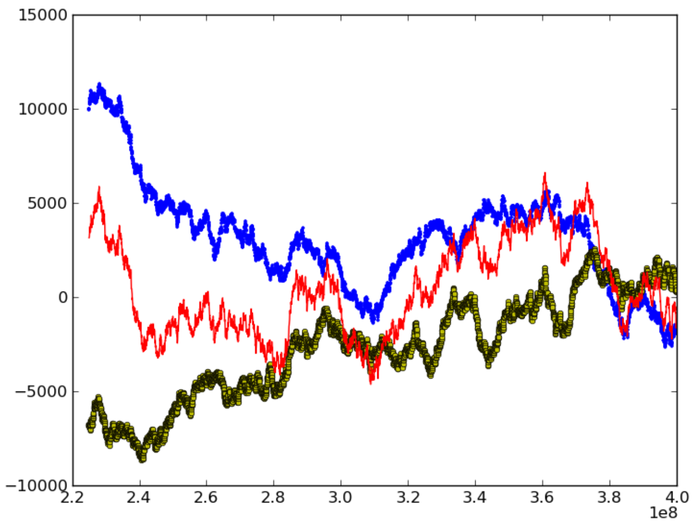

Figure 4. example 6.2, up to 400000000

Blue: $\sum_n^t \mu(n)\mathcal{X}(n)$, max:11319, min:$-2663$. Yellow: $\sum_n^t \mu(n)(1-\mathcal{X}(n))$, max:2527, min:$-8688$. Red: $\sum_n^t \mu(n)$, max:6602, min:-4645.

By Theorem 333 in [6], the probabilit that a number should be squarefree is $\frac{6}{\pi^2}$, more precisely

$$Q(x) = \sum_{n \leq x} |\mu(n)| = \frac{6x}{\pi^2} + O(\sqrt{x}).$$

Lemma 6.5.
$$\sum_{n\le x}\mathcal{X}(n)\log n=\frac{x}{2}logx+O(x).$$

Proof. Let $x=m^2+m+l$ where $0\le l<2(m+1)$. Note
$$\sum_{n\le x}(1-\mathcal{X}(n))\log n-\mathcal{X}(n)\log n=\sum_{k=1}^{m}\sum_{i=1}^{k}\log\left(\frac{k^2+i}{k^2-k+i}\right)+\sum_{j=m^2+m+1}^{m^2+m+l}\log(j)$$
$$=\sum_{k=1}^{m}\sum_{i=1}^{k}O\left(\frac{1}{k}\right)+O(\sqrt{x}\log x)=O(\sqrt{x}\log x).$$
By Theorem 423 in [6],
$$\sum_{n\le x}\log n=\sum_{n\le x}(1-\mathcal{X}(n))\log n+\mathcal{X}(n)\log n=x\log x+O(x).$$
Therefore
$$\sum_{n\le x}\mathcal{X}(n)\log n=\frac{x}{2}logx+O(x).$$
□

Lemma 6.6. *Let $S_l=\bigcup_{m=1}^{\infty}\{l\times m^2+t:t\in[0,lm]\}$. If $j/i=k^2>1$, then*
$$\lim_{x\to\infty}\frac{\#\{n\le x:n\in S_i\cap S_j\}}{x}=\begin{cases}\frac{k+1}{4k} & \text{if } k \text{ is odd}\\ \frac{1}{4} & \text{if } k \text{ is even}\end{cases}.$$

Proof. Assume that $x=ka$ and $y=a$. Then $ix^2=ik^2a^2=ja^2=jy^2$. Consider the interval
$$s_y=[jy^2,j(y+1)^2-1]=[jy^2,jy^2+jy]\cup[jy^2+jy+1,j(y+1)^2-1]=s_y^1+s_y^2.$$
Note that $[jy^2,j(y+1)^2-1]$ contains k intervals
$$[ix^2,ix^2+ix],[i(x+1)^2,i(x+1)^2+i(x+1)],\dots,[i(x+k-1)^2,i(x+k-1)^2+i(x+k-1)].$$
If k is even, then
$$[ix^2,ix^2+ix],[i(x+1)^2,i(x+1)^2+i(x+1)],\dots,[i(x+\frac{k}{2}-1)^2,i(x+\frac{k}{2}-1)^2+i(x+\frac{k}{2}-1)]$$
are in s_y^1 for x large enough, since $i(x+\frac{k}{2}-\frac{k^2}{4})>0$. And
$$[i(x+\frac{k}{2})^2,i(x+\frac{k}{2})^2+i(x+\frac{k}{2})],\dots,[i(x+k-1)^2,i(x+k-1)^2+i(x+k-1)]$$
are in s_y^2. This implies that $\lim_{x\to\infty}\frac{\#\{n\le x:n\in S_i\cap S_j\}}{x}=\frac{1}{4}$.

If k is odd, then
$$i(x+\frac{k-1}{2})^2+i(x+\frac{k-1}{2})-jy^2-jy=\frac{i(k^2-1)}{4}.$$
Therefore s_y^1 contains almost $\frac{k+1}{2}$ intervals. It is clear imples the result. □

Lemma 6.7. *Suppose that p, q are two coprime positive integers, i.e. $(p,q)=1$. Let $S_1=\bigcup_{k=1}^{p}[\frac{(2k-2)}{2p},\frac{(2k-1)}{2p}]$ and $S_2=\bigcup_{k=1}^{q}[\frac{(2k-2)}{2q},\frac{(2k-1)}{2q}]$.*
$$len(S_1\cap S_2)=\begin{cases}\frac{1}{4} & \text{if } p \text{ or } q \text{ is even,}\\ \frac{1}{4}+\frac{4}{15pq} & \text{otherwise.}\end{cases}$$

Proof. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be the charachteristic function of S_1 and S_2 respectively.

$$\begin{aligned}\langle \mathcal{X}_1, e^{2\pi i n\theta}\rangle &= \sum_{k=1}^{p}\int_{\frac{2k-2}{2p}}^{\frac{2k-1}{2p}} e^{-2\pi i n\theta} \\ &= \frac{1-e^{\frac{-\pi i n}{p}}}{2\pi i n}\sum_{k=0}^{p-1} e^{\frac{-2\pi i n k}{p}} = \begin{cases}\frac{1}{2} & n=0\\ \frac{p}{\pi i n}, & \text{if } n=p(2a+1), a\in\mathbb{Z}\\ 0, & \text{otherwise}\end{cases}.\end{aligned}$$

Similiarly,

$$\begin{aligned}\langle \mathcal{X}_2, e^{2\pi i n\theta}\rangle &= \sum_{k=1}^{q}\int_{\frac{2k-2}{2q}}^{\frac{2k-1}{2q}} e^{-2\pi i n\theta} \\ &= \frac{1-e^{\frac{-\pi i n}{q}}}{2\pi i n}\sum_{k=0}^{q-1} e^{\frac{-2\pi i n k}{q}} = \begin{cases}\frac{1}{2} & n=0\\ \frac{q}{\pi i n}, & \text{if } n=q(2a+1), a\in\mathbb{Z}\\ 0, & \text{otherwise}\end{cases}\end{aligned}$$

If p or q is even, then $\langle \mathcal{X}_1, \mathcal{X}_2\rangle = \frac{1}{4}$. If both p and q are odd, then

$$\begin{aligned} len(S_1\cap S_2) &= \langle \mathcal{X}_1, \mathcal{X}_2\rangle \\ &= \frac{1}{4} + \frac{1}{\pi^2 pq}\sum_{k\in\mathbb{Z}}^{\infty}\frac{1}{(2k+1)^2} \\ &= \frac{1}{4} + \frac{8}{5\pi^2 pq}\zeta(2) = \frac{1}{4} + \frac{4}{15pq} \qquad (\zeta(2) = \frac{\pi^2}{6}).\end{aligned}$$

□

Theorem 6.1 (Dirichlet, c. 1840)**.** *For any real number k and any integer $N\geq 1$, there exist integers p and q such that $1\leq q\leq N$ and*

$$|qk-p| \leq \frac{1}{N+1}.$$

Lemma 6.8. *Let $S_l = \bigcup_{m=1}^{\infty}\{l\times m^2 + t : t\in[0, lm]\}$. If $\sqrt{\frac{i}{j}}\in\mathbb{R}\setminus\mathbb{Q}$, then*

$$\lim_{x\to\infty}\frac{\#\{n\leq x : n\in S_i\cap S_j\}}{x} = \frac{1}{4}.$$

Proof. We assume that $\sqrt{\frac{i}{j}} < 1$ and $|\sqrt{\frac{i}{j}}q - p| \leq \frac{1}{q}$. Since

$$\lim_{m\to\infty}\frac{iq^2l^2}{iq^2(l+1)^2} = 1,$$

we only need to consider the case for $x = iq^2l^2$, where $l\in\mathbb{N}$. The interval $[iq^2l^2, iq^2(l+1)^2]$ contains q invervals in S_1:

$$[i(ql)^2, i((ql)^2+ql)], \dots, [i(ql+q-1)^2, i(ql+q)(ql+q-1)].$$

Let

$$\begin{aligned} S_{1l} &= \bigcup_{k=1}^{q}[i(ql+k-1)^2, i(ql+k)(ql+k-1)] \\ S_{1l}' &= \bigcup_{k=1}^{q}[i(ql)^2 + (2k-2)iql, i(ql)^2 + (2k-1)iql].\end{aligned}$$

Note that

$$\lim_{l\to 0}\frac{\#(S_{1l}\cap S'_{1l})}{iq^2l}=1.$$

If $iq^2l^2\le jy^2\le iq^2(l+1)^2$, then

$$\sqrt{\frac{i}{j}}ql\le y\le\sqrt{\frac{i}{j}}q(l+1).$$

Since

$$p-\frac{1}{q}\le\sqrt{\frac{i}{j}}q\le p+\frac{1}{q},$$

$[iq^2l^2,iq^2(l+1)^2]$ contains at least $p-3$ and at most $p+1$ intervals in S_2. Let y_0 be the first y such that $iq^2l^2\le jy^2\le iq^2(l+1)^2$. Let

$$S_{2l}=\bigcup_{k=1}^{p-3}[j(y_0+k-1)^2,j(y_0+k-1)^2+j(y_0+k-1)]\subset S_2\cap[iq^2l^2,iq^2(l+1)^2].$$

Note that

$$\liminf_{l\to\infty}\frac{\#S_{2l}}{\#(S_2\cap[iq^2l^2,iq^2(l+1)^2])}\ge 1-\frac{5}{p-3}.$$

Let

$$S'_{2l}=\bigcup_{k=1}^{p-3}[jy_0^2+(2k-2)jy_0,jy_0^2+(2k-1)jy_0].$$

Also we have

$$\lim_{l\to\infty}\frac{\#(S_{2l}\cap S'_{2l})}{iq^2l}=1.$$

We would like to compare S'_{1l} with S'_{2l}. Since $y_0\le\sqrt{\frac{i}{j}}ql+1$, we have

$$\begin{aligned}
\Delta &= i(ql)^2+2iq^2l-jy_0^2+(2p-7)jy_0\\
&\ge i(ql)^2+2iq^2l-j(\sqrt{\frac{i}{j}}ql+1)^2+(2p-7)j(\sqrt{\frac{i}{j}}ql+1)\\
&=2iq^2l(1-(\frac{p}{q}-\frac{5}{2q})\sqrt{\frac{j}{i}}-\frac{(2p-6)j}{2iq^2l})\\
&\ge 2iq^2l(1-(\sqrt{\frac{i}{j}}+\frac{1}{q^2}-\frac{5}{2q})\sqrt{\frac{j}{i}}-\frac{(2p-6)j}{2iq^2l}) \qquad (\frac{p}{q}\le\sqrt{\frac{i}{j}}+\frac{1}{q^2})\\
&=2iq^2l((\frac{5}{2q}-\frac{1}{q^2})\sqrt{\frac{j}{i}}-\frac{(2p-6)j}{2iq^2l})\ge 0,
\end{aligned}$$

provide that l is large enough.

This means that S'_{2l} is contains in $[i(ql)^2,i(ql)^2+2iq^2l]$. A similar caculation as in the proof of lemma 6.8 shows that

$$\lim_{l\to\infty}\frac{\#(S'_{1l}\cap S'_{2l})}{2iq^2l}=\frac{1}{4}+O(\frac{1}{pq}).$$

Therefore

$$\lim_{l\to\infty}\frac{\#(S_{1l}\cap S_{2l})}{2iq^2l}=\frac{1}{4}+O(\frac{1}{p}).$$

This implies that

$$\lim_{x\to\infty} \frac{\#\{n \le x : n \in S_i \cap S_j\}}{x} = \frac{1}{4} + O(\frac{1}{p}).$$

By increasing p and q, the result is proved. □

Lemma 6.9. *Let $S_l = \bigcup_{m=1}^{\infty}\{l \times m^2 + l \times i : i \in [0, m]\}$. If $\sqrt{\frac{i}{j}} \in \mathbb{R} \setminus \mathbb{Q}$, then*

$$\lim_{x\to\infty} \frac{\#\{n \le x : n \in S_i \cap S_j\}}{x} = \frac{1}{4ij} \qquad (i \ne j).$$

Proof. □

Theorem 6.2 (Carleson's theorem)**.** *Let f be an L^p periodic function for some $p \in (1, \infty)$, with Foruier coefficient $\hat{f}(n)$. Then*

$$\lim_{N\to\infty} \sum_{|n|\le N} \hat{f}(n) e^{2\pi i n x} = f(x)$$

for almost every x.

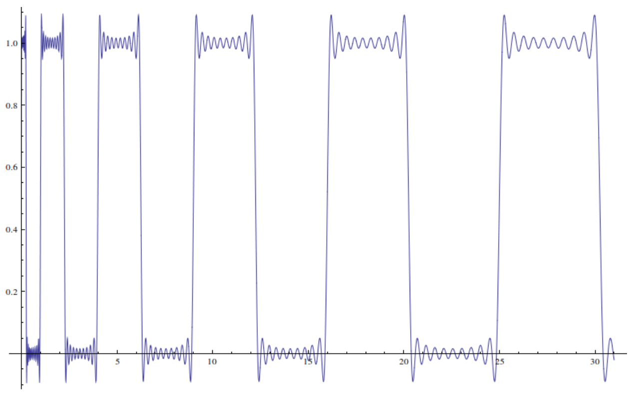

Figure 5. $\frac{1}{2} + \sum_{k=1}^{10} \frac{2}{(2k-1)\pi} \sin(2\pi i(2k-1)\sqrt{x})$, $(x \in [0, 31])$

Definition 6.1. *A function $f(x)$ is piecewise continuous on an interval I if it is continuous on I except perhaps for a finite number of points, and if $a \in I$ is a point of discontinuity for $f(x)$ then $f(a_+)$ and $f(a_-)$ exist: that is*

$$f(a_+) = \lim_{x\to a_+} f(x), \qquad f(a_-) = \lim_{x\to a_-} f(x)$$

are required to exist. We denote the space of piecewise continuous functions on I by $E(I)$.

Definition 6.2. *The space E' is defined as the space of all functions $f(x) \in E([-\pi, \pi]$ such that the right-hand derivative $D_+ f(x)$ and left-hand derivative $D_- f(x)$ exists. Recall that*

$$D_+ f(x) = \lim_{h\to 0_+} \frac{f(x+h) - f(x_+)}{h}$$
$$D_- f(x) = \lim_{h\to 0_-} \frac{f(x+h) - f(x_-)}{h}.$$

Theorem 6.3. *If $f \in E'$, then the Fourier series of f*

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx),$$

where

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx, \qquad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx,$$

converges pointwise to

$$\frac{f(x_+) + f(x_-)}{2}.$$

Theorem 6.4. *Suppose that*

- *$f(x)$ is continuous on $[-\pi, \pi]$*
- *$f(-\pi) = f(\pi)$*
- *$f'(x) \in E([-\pi, \pi])$.*

Then

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

converges uniformly to $f(x)$ on $[-\pi, \pi]$.

Corollary 6.3. *Let $\mathcal{X}$ be the characteristic function of $\bigcup_{m=1}^{\infty} [m^2, m^2 + m + \frac{1}{4}]$. Then*

$$\mathcal{X} = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{(2k-1)\pi} \sin(2\pi(2k-1)\sqrt{x}) \qquad (x \geq 1)$$

at every point where $\mathcal{X}$ is continuous.

Lemma 6.10.

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(n) e^{2\pi i k \sqrt{n}} \to 0.$$

Lemma 6.11.

$$\sum_{n \leq x} \left[\frac{x}{n}\right] \mathcal{X}(n) \Lambda(n) = \frac{x}{2} \log x + O(x).$$

Proof. Assume that $x = m^2 + m$. We claim that

$$\sum_{k=1}^{m} \sum_{l=1}^{k} \left[\frac{m^2+m}{k^2+l}\right] \Lambda(k^2+l) - \sum_{k=1}^{m} \sum_{l=1}^{k} \left[\frac{m^2+m}{k^2-k+l}\right] \Lambda(k^2-k+l) = O(x).$$

First note that

$$\begin{aligned} &\sum_{k=1}^{m} \sum_{l=1}^{k} \left[\frac{m^2+m}{k^2+l}\right] \Lambda(k^2+l) - \sum_{k=1}^{m} \sum_{l=1}^{k} \left[\frac{m^2+m}{k^2-k+l}\right] \Lambda(k^2-k+l) \\ = &\sum_{k=1}^{m} \sum_{l=1}^{k} \frac{m^2+m}{k^2+l} \Lambda(k^2+l) - \sum_{k=1}^{m} \sum_{l=1}^{k} \frac{m^2+m}{k^2-k+l} \Lambda(k^2-k+l) + O(x). \end{aligned}$$

However

$$\begin{aligned} &\sum_{k=1}^{m} \sum_{l=1}^{k} \left(\frac{m^2+m}{k^2+l} - \frac{m^2+m}{k^2}\right) \Lambda(k^2+l) \\ = x &\sum_{k=1}^{m} \sum_{l=1}^{k} \left(\frac{-l}{k^2(k^2+l)}\right) \Lambda(k^2+l) = xO\left(\sum_{k=1}^{m} \frac{\log 2k}{k^3}\right) = O(x). \end{aligned}$$

Similarly

$$\sum_{k=1}^{m}\sum_{l=1}^{k}(\frac{m^2+m}{k^2-k+l}-\frac{m^2+m}{k^2})\Lambda(k^2-k+l)$$
$$=x\sum_{k=1}^{m}\sum_{l=1}^{k}(\frac{k-l}{k^2(k^2-k+l)})\Lambda(k^2-k+l)=xO(\sum_{k=1}^{m}\frac{\log 2k}{k^3})=O(x).$$

Therefore, we only need to show

$$\sum_{k=1}^{m}\frac{m^2+m}{k^2}\sum_{l=1}^{k}\left(\left[\frac{k^2+k}{k^2+l}\right]\Lambda(k^2+l)-\left[\frac{k^2}{k^2-k+l}\right]\Lambda(k^2-k+l)\right)$$

□

Let $\Psi_{\mathcal{X}}(k,x)=\sum_{d\le x}\mathcal{X}(dk)\Lambda(d)$ and $\Psi_{1-\mathcal{X}}(k,x)=\sum_{d\le x}(1-\mathcal{X}(dk))\Lambda(d)$. Recall that

$$\Psi(x)=\sum_{d\le x}\Lambda(d)=O(x).$$

Therefore $\Psi_{\mathcal{X}}(k,x)=O(x)$ and $\Psi_{1-\mathcal{X}}(k,x)=O(x)$.

Lemma 6.12. *Let*

$$\Psi(k,x)=\sum_{d\le x}\mathcal{X}(dk)\Lambda(d).$$

Then $\Psi(k,x)-\frac{1}{2}[x]=o(x)$.

Proof. Let $\Psi(x)=\sum_{d\le x}\Lambda(d)=\Psi(k,x)$ □

Lemma 6.13. *Let* $M_{\mathcal{X}}(x)=\sum_{n\le x}\mathcal{X}(n)\mu(n)$. *Then* $M_{\mathcal{X}}(x)=o(x)$.

Proof. By Theorem 434 in [6], we have

$$\sum_{n\le x}\mathcal{X}(n)\mu(n)log(\frac{x}{n})=O(x).$$

Hence

$$M_{\mathcal{X}}(x)log(x)=\sum_{n\le x}\mathcal{X}(n)\mu(n)\log n+O(x).$$

By Theorem 297 in [6],

$$-\sum_{n\le x}\mathcal{X}(n)\mu(n)\log n=\sum_{n\le x}\mathcal{X}(n)\sum_{d|n}\mu(\frac{n}{d})\Lambda(d)=\sum_{dk\le x}\mathcal{X}(dk)\mu(k)\Lambda(d)$$
$$=\sum_{k\le x}\mu(k)\sum_{d\le\frac{x}{k}}\mathcal{X}(dk)\Lambda(d)$$

□

Lemma 6.14 (Conjecture)**.** *Let (X,σ) be a dynamic system where X is a compact Hausdorff metric space and $\iota(X)$ is the set of all isolated points of X. Let $X'=X\setminus\iota(X)$. If $\sigma(X')\subset X'$ and $\lim_{N\to\infty}\frac{1}{N}\sum_{n\le N}\mu(n)f(\sigma^n(x))=0$ for any $f\in C(X')$ and any $x\in X'$, then $\lim_{N\to\infty}\frac{1}{N}\sum_{n\le N}\mu(n)f(\sigma^n(x))=0$ for any f in $C(X)$ and $x\in X$.*

7. Weighted Sum of $\mu(n)$

Let $f:\mathbb{N}\to\mathbb{N}$ be a function defined on $\mathbb{N}$. Consider the weighted sum

$$\sum_{n\leq N} f(n)\mu(n)$$

Let $\rho:\mathbb{N}\to\mathbb{N}$ be a map defined by

$$\rho: p_1^{\gamma(1)}p_2^{\gamma(2)}\cdots p_n^{\gamma(n)} \to 1^{\gamma(1)}2^{\gamma(2)}\cdots n^{\gamma(n)},$$

where p_i is the i-th prime number.

Remark 7.1. *If $k<\sqrt{N}$, then $\lfloor\frac{N}{k}\rfloor > \lfloor\frac{N}{k+1}\rfloor$. Assume that $N=ak+b$ where $0\leq b<k$ and $a\geq k$. If $\lfloor\frac{N}{k}\rfloor = \lfloor\frac{N}{k+1}\rfloor$, then $ak+b=a(k+1)+c$. It is impossible.*

Lemma 7.1.

$$\sum_{\gamma\in(\mathbb{Z}_2)^N \text{ and } 1^{\gamma(1)}2^{\gamma(2)}\cdots N^{\gamma(N)}\leq M} (-1)^{\sum_i\gamma(i)} = 0.$$

Proof.

$$\begin{aligned}
&\sum_{\gamma\in(\mathbb{Z}_2)^N \text{ and } 1^{\gamma(1)}2^{\gamma(2)}\cdots N^{\gamma(N)}\leq M} (-1)^{\sum_i\gamma(i)}\\
&= \sum_{\gamma\in(\mathbb{Z}_2)^N \text{ and } 2^{\gamma(2)}\cdots N^{\gamma(N)}\leq M} (-1)^{\sum_{i=2}^N\gamma(i)} - \sum_{\gamma\in(\mathbb{Z}_2)^N \text{ and } 2^{\gamma(2)}\cdots N^{\gamma(N)}\leq M} (-1)^{\sum_{i=2}^N\gamma(i)}\\
&= 0.
\end{aligned}$$

□

Estimate

$$\sum_{\gamma\in(\mathbb{Z}_2)^N \text{ and } 2^{\gamma(2)}\cdots N^{\gamma(N)}\leq N} (-1)^{\sum_i\gamma(i)}.$$

Note that if $j>\lfloor\frac{N}{2}\rfloor$ and $\gamma(j)=1$, then $\gamma(i)=0$ for any $i\neq j$. Therefore

$$\begin{aligned}
&\sum_{\gamma\in(\mathbb{Z}_2)^N \text{ and } 2^{\gamma(2)}\cdots N^{\gamma(N)}\leq N} (-1)^{\sum_i\gamma(i)}\\
&= \sum_{\gamma\in(\mathbb{Z}_2)^{\lfloor\frac{N}{2}\rfloor} \text{ and } 2^{\gamma(2)}\cdots(\lfloor\frac{N}{2}\rfloor)^{\gamma(\lfloor\frac{N}{2}\rfloor)}\leq N} (-1)^{\sum_i\gamma(i)} - (N-\lfloor\frac{N}{2}\rfloor).
\end{aligned}$$

Similar argument implies that

$$\sum_{\gamma\in(\mathbb{Z}_2)^{\frac{N}{2}} \text{ and } 1^{\gamma(1)}2^{\gamma(2)}\cdots(\lfloor\frac{N}{2}\rfloor)^{\gamma(\lfloor\frac{N}{2}\rfloor)}\leq N} (-1)^{\sum_i\gamma(i)} = \sum_{\gamma\in(\mathbb{Z}_2)^{\frac{N}{3}} \text{ and } 1^{\gamma(1)}2^{\gamma(2)}\cdots(\lfloor\frac{N}{3}\rfloor)^{\gamma(\lfloor\frac{N}{3}\rfloor)}\leq N} (-1)^{\sum_i\gamma(i)}.$$

Suppose that $\gamma(j)=1$ and $\lfloor\frac{N}{4}\rfloor < j \leq \lfloor\frac{N}{3}\rfloor$. The possibilities are

$$\begin{aligned}
&\gamma(1)=\gamma(2)=\gamma(3)=0 && \gamma(1)=1,\gamma(2)=\gamma(3)=0\\
&\gamma(1)=0,\gamma(2)=1,\gamma(3)=0 && \gamma(1)=0,\gamma(2)=0,\gamma(3)=1\\
&\gamma(1)=0,\gamma(2)=0,\gamma(3)=1 && \gamma(1)=1,\gamma(2)=1,\gamma(3)=0\\
&\gamma(1)=1,\gamma(2)=0,\gamma(3)=1.
\end{aligned}$$

This implies that

$$\sum_{\gamma\in(\mathbb{Z}_2)^{\frac{N}{3}} \text{ and } 1^{\gamma(1)}2^{\gamma(2)}\cdots(\lfloor\frac{N}{3}\rfloor)^{\gamma(\lfloor\frac{N}{3}\rfloor)}\leq N} (-1)^{\sum_i\gamma(i)} = \sum_{\gamma\in(\mathbb{Z}_2)^{\frac{N}{4}} \text{ and } 1^{\gamma(1)}2^{\gamma(2)}\cdots(\lfloor\frac{N}{4}\rfloor)^{\gamma(\lfloor\frac{N}{4}\rfloor)}\leq N} (-1)^{\sum_i\gamma(i)}$$

Similarly, we have

$$\sum_{\gamma\in(\mathbb{Z}_2)^{\frac{N}{4}} \text{ and } 1^{\gamma(1)}2^{\gamma(2)}\cdots(\lfloor\frac{N}{4}\rfloor)^{\gamma(\lfloor\frac{N}{4}\rfloor)}\le N} (-1)^{\sum_i\gamma(i)} = \sum_{\gamma\in(\mathbb{Z}_2)^{\frac{N}{6}} \text{ and } 1^{\gamma(1)}2^{\gamma(2)}\cdots(\lfloor\frac{N}{6}\rfloor)^{\gamma(\lfloor\frac{N}{6}\rfloor)}\le N} (-1)^{\sum_i\gamma(i)}$$

Assume that $k+1\le\sqrt{N}$ and

$$\sum_{\gamma\in(\mathbb{Z}_2)^{N} \text{ and } 1^{\gamma(1)}2^{\gamma(2)}\cdots N^{\gamma(N)}\le N} (-1)^{\sum_i\gamma(i)} = \sum_{\gamma\in(\mathbb{Z}_2)^{\lfloor\frac{N}{k}\rfloor} \text{ and } 1^{\gamma(1)}2^{\gamma(2)}\cdots(\lfloor\frac{N}{k}\rfloor)^{\gamma(\lfloor\frac{N}{k}\rfloor)}\le N} (-1)^{\sum_i\gamma(i)}$$

Lemma 7.2. *Let $k\le\sqrt{N}$. Then*

$$\sum_{\gamma\in(\mathbb{Z}_2)^{N} \textit{ and } 1^{\gamma(1)}2^{\gamma(2)}\cdots N^{\gamma(N)}\le N} (-1)^{\sum_i\gamma(i)} = \sum_{\gamma\in(\mathbb{Z}_2)^{\lfloor\frac{N}{k}\rfloor} \textit{ and } 1^{\gamma(1)}2^{\gamma(2)}\cdots(\lfloor\frac{N}{k}\rfloor)^{\gamma(\lfloor\frac{N}{k}\rfloor)}\le N} (-1)^{\sum_i\gamma(i)}$$

Proof. For $k=1$, the statement is trivial. Let $k=2$. Assume that

$$\sum_{\gamma\in(\mathbb{Z}_2)^{N} \text{ and } 1^{\gamma(1)}2^{\gamma(2)}\cdots N^{\gamma(N)}\le N} (-1)^{\sum_i\gamma(i)} = \sum_{\gamma\in(\mathbb{Z}_2)^{\lfloor\frac{N}{k-1}\rfloor} \text{ and } 1^{\gamma(1)}2^{\gamma(2)}\cdots(\lfloor\frac{N}{k-1}\rfloor)^{\gamma(\lfloor\frac{N}{k-1}\rfloor)}\le N} (-1)^{\sum_i\gamma(i)}.$$

Suppose that $\gamma(j)=1$ and $\lfloor\frac{N}{k}\rfloor<j\le\lfloor\frac{N}{k-1}\rfloor$. Note that

$$\sum_{\gamma\in(\mathbb{Z}_2)^{k-1} \text{ and } 1^{\gamma(1)}2^{\gamma(2)}\cdots(k-1)^{\gamma(k-1)}\le N} -(-1)^{\sum_i\gamma(i)}$$

□

8. What We Talk About When We Talk About Smoothness

Let $C(X)$ and $C(Y)$ be two abelian C*-algebras, where X and Y are compact Hausdorff spaces. Suppose Φ is a *-homomorphism form $C(X)$ to $C(Y)$. It is clear that Φ induces a map Φ^* from Y to X,

$$\Phi^*:\rho\to\rho\circ\Phi,\qquad \rho\in Y.$$

Similar, if $\varphi: X\to Y$ be a continuous map, then φ induces a map from $C(Y)\to C(X)$,

$$\varphi^*: f\to f\circ\varphi,\qquad f\in C(Y).$$

It is cleat that $\Phi^{**}=\Phi$ and $\varphi^{**}=\varphi$.

Consider the C^*-algebra $L^\infty(\mathbb{N})$ acting on $l^2(\mathbb{N})$. It is well-known that $L^\infty(\mathbb{N})\simeq C(\beta\mathbb{N})$. Let σ be the continuous map form $\beta\mathbb{N}$ to $\beta\mathbb{N}$ induced by the shift on $\mathbb{N}$: $\sigma(n)=n+1$. Throughout this section, we will use σ to denote this continuous map.

Let V be the unilateral shift, i.e., $Ve_i=e_{\sigma(i)}$. Then

$$\Sigma: M_f\to V^*M_fV=M_{f\circ\sigma}, \text{ for all } f\in L^\infty(\mathbb{N}),$$

is a homomorphism form $L^\infty(\mathbb{N})$ onto $L^\infty(\mathbb{N})$.

Note that

$$V^*M_{f_1}VV^*M_{f_2}V=V^*M_{f_1f_2}V,$$

where f_1 and f_2 are in $L^\infty(\mathbb{N})$. It is easy to see that $\sigma^*=\Sigma$ and $\Sigma^*=\sigma$.

Let $f\in L^\infty(\mathbb{N})$ and $C(f)$ be the C*-algebra generated by $M_{f\circ\sigma^{(n)}}$, $n=0,1,\ldots$ and I. This is an abelian C*-algebra. By Gelfand transform, $C(f)$ can be identified as $\simeq C(\Omega)$, where Ω is a compact Hausdorff space. Ω is metrizable since $C(f)$ is separable.

Remark 8.1. *Let $C(f)\simeq C(\Omega)$ where Ω is the spectrum of $C(f)$. $\{M_{f\circ\sigma^{(n)}}: n=0,1,2\ldots\}$ separates points in Ω.*

Theorem 8.1 (Tietze extension theorem)**.** *If X is a normal topological space (e.g., every compact Hausdorff space is normal) and $f : A \to \mathbb{R}$ is a continuous map from a closed subset A of X into the real numbers carrying the standard topology, then there exists a continuous map $F : X \to \mathbb{R}$ with $F(x) = f(x)$ for all $x \in A$. Moreover, F may be chosen such that $\sup\{|f(a)| : a \in A\} = \sup\{|F(x)| : x \in X\}$, i.e., if f is bounded, F may be chosen to be bounded (with the same bound as f). F is called a continuous extension of f.*

Definition 8.1. *Let $T : X \to X$ and $S : Y \to Y$ be continuous maps of compact (metric) spaces (that is, topological dynamical systems). Then a homeomorphism $\theta : X \to Y$ with $\theta \circ T = S \circ \theta$ is called a topological conjugacy, and if there such a conjugacy then T and S are topologically conjugate. A continuous surjective map $\varphi : X \to Y$ with $\varphi \circ T = S \circ \varphi$ is called a topological factor map, and in this case S is said to be a factor of T.*

Lemma 8.1. *Let $\mathfrak{A} \simeq C(Y)$ be a C^*-subalgebra of $C(X)$, and φ be the continuous surjective from X onto Y induced by the embedding of $C(Y)$ into $C(X)$. Suppose Φ is a homomorphism from $C(X)$ to $C(X)$ such that Φ fixes $\mathfrak{A}$, i.e., $\Phi(\mathfrak{A}) \subset \mathfrak{A}$. Let $\Psi = \Phi|_{\mathfrak{A}}$. Then Ψ^* is a factor of Φ^*.*

Proof. We only need to show that $\varphi \circ \Phi^* = \Psi^* \circ \varphi$. Let ρ in X and $A \in \mathfrak{A}$. Then

$$\varphi \circ \Phi^*(\rho)(A) = \rho(\Phi(A)) = \Psi^* \circ \varphi(\rho)(A).$$

□

Lemma 8.2. *Let $T : X \to X$ and $S : Y \to Y$ be continuous maps of compact (metric) spaces. Let $\varphi : X \to Y$ be a topological factor map. Then φ^* is injective. And $\{f \circ \varphi : f \in C(Y)\}$ is invariant under the map $g \to g \circ T$, $g \in C(X)$.*

Lemma 8.3. *Let $T : X \to X$ and $S : Y \to Y$ be continuous maps of compact (metric) spaces. If S is a factor of T, then $h_{top}(T) \geq h_{top}(S)$.*

Proof. Let φ be a factor map. Suppose that $\{U_1, \dots, U_n\}$ be a open cover of Y. Let $V_i = \varphi^{-1}(U_i)$. Then $\{V_1, \dots, V_n\}$ is a open cover of X. Let $U_i^n = S^{-n}(U_i)$ and $V_i^n = T^{-n}(V_i)$. We claim that $V_i^n = \varphi^{-1}(U_i^n)$. Indeed

$$\begin{aligned} V_i^n &= \{x \in X : T^n(x) \in V_i\} \\ &= \{x \in X : \varphi \circ T^n(x) \in U_i\} \\ &= \{x \in X : S^n \circ \varphi(x) \in U_i\} \\ &= \{x \in X : \varphi(x) \in U_i^n\} = \varphi^{-1}(U_i^n). \end{aligned}$$

This implies that

$$\begin{aligned} V_{i_1} \cap V_{i_2}^2 \cap \cdots \cap V_{i_n}^n &= \varphi^{-1}(U_{i_1}) \cap \varphi^{-1}(U_{i_2}^2) \cap \cdots \cap \varphi^{-1}(U_{i_n}^n) \\ &= \varphi^{-1}(U_{i_1} \cap U_{i_2}^2 \cap \cdots \cap U_{i_n}^n). \end{aligned}$$

Thus

$$N(\bigvee_{i=0}^{m} T^{-i}\{V_1, \dots, V_n\}) = N(\bigvee_{i=0}^{m} S^{-i}\{U_1, \dots, U_n\}).$$

This clearly implies that $h_{top}(T) \geq h_{top}(S)$. □

Lemma 8.4. *Let $f \in L^\infty(\mathbb{N})$ and (Y, φ) be a topological dynamical system. Assume that $C(f) \simeq C(X)$. If there exists $h \in C(Y)$ such that $f(n) = h(\varphi^{(n)}(x))$, and $\{h \circ \varphi^{(n)} : n = 0, 1, \dots\} \cup \{I\}$ generate $C(Y)$, then there is a continuous injective map form X into Y. Furthermore, the range of the continuous map is the closure of $\{x, \varphi(x), \varphi^{(2)}(x), \dots\}$ in Y.*

Proof. Let Ω be the closure of $\{x, \varphi(x), \varphi^{(2)}(x), \ldots\}$ in Y. By the Tietze extension theorem, we have

$$g \to g|_{\Omega}, \qquad \text{for any } g \in C(Y),$$

is a surjective map form $C(Y)$ to $C(\Omega)$. Therefore $\{h \circ \varphi^{(n)}|_{\Omega} : n = 0, 1, \ldots\} \cup \{I\}$ also generate $C(\Omega)$. It is not hard to see that

$$\Phi : h \circ \varphi^n|_{\Omega} \to f \circ \sigma^{(n)}, \qquad n = 0, 1, \ldots$$

induce a *-isomorphism form $C(\Omega)$ to $C(f) \simeq C(X)$. Then

$$\Phi^* : \rho \to \rho \circ \Phi,$$

where $\rho \in X$, is a homeomorphism form X to Ω.

Note that

$$\Phi(g \circ \varphi) = \Phi(g) \circ \sigma, \qquad g \in C(Y).$$

Therefore, for $g \in C(Y)$ and $\rho \in X$, we have

$$g(\varphi(\Phi^*(\rho))) = g \circ \varphi(\Phi^*(\rho)) = \Phi(g \circ \varphi)(\rho) = (\Phi(g) \circ \sigma)(\rho) = \Phi(g)(\sigma(\rho)) = g(\Phi^*(\sigma(\rho))).$$

This means that $\varphi \circ \Phi^* = \Phi^* \circ \sigma$. □

Corollary 8.1. *Let $f \in L^\infty(\mathbb{N})$ and (Y, φ) be a topological dynamical system. Assume that $C(f) \simeq C(X)$. If there exists $h \in C(Y)$ such that $f(n) = h(\varphi^{(n)}(x))$, $\{h \circ \varphi^{(n)} : n = 0, 1, \ldots\} \cup \{I\}$ generate $C(Y)$ and $\{x, \sigma(x), \sigma^{(2)}(x), \ldots\}$ is dense in Y. Then σ and φ are topologically conjugate.*

Lemma 8.5. *Let $f \in L^\infty(\mathbb{N})$ and (Y, φ) be a topological dynamical system. Assume that $C(f) \simeq C(X)$. If there exists $h \in C(Y)$ such that $f(n) = h(\varphi^{(n)}(x))$, then $h_{top}(\varphi) \geq h_{top}(\sigma)$.*

Proof. By Variational principle for the topological entropy. □

Example 8.1 (Theorem 8 in [3])**.** *Let $\Omega^{(2)} = \{0, 1\}^{\mathbb{N}}$. The square-free flow S is the subflow of the full shift on $\Omega^{(2)}$ by the point $\theta = (\mu^2(1), \mu^2(2), \ldots)$, that is $S = (X_S, T)$ with X_S the closure of the T-orbit of θ in $\Omega^{(2)}$.*

- *$T : X_S \to X_S$ is surjective, it is topologically ergodic and $h_{top}(S) = \frac{6}{\pi^2} \log 2$.*
- *S is proximal (i.e. for any x, $y \in X_S$, $\inf_{n \geq 1} d(T^{(n)}(x), T^{(n)}(y)) = 0$) and $\{(0, 0, 0 \ldots)\}$ is the unique T-minimal subset of X_S.*

Remark 8.2. *Let h be the characteristic function on of the subset $\{(a_1, a_2, \ldots,) : a_1 = 1\}$ of $\Omega^{(2)}$. It is clear that $\{h, h \circ T, h \circ T^{(2)}, \ldots\}$ separates points. By Stone–Weierstrass theorem, we have $\{h, h \circ T, h \circ T^{(2)}, \ldots\}$ generates the C^*-algebra $C(X_S)$. By corollary 8.1, $C(\mu^2) \simeq C(X_S)$. Let $\rho \in X$ where X is the spectrum of $C(\mu^2)$. Then*

$$\rho \to (\rho(\mu^2), \rho(\mu^2 \circ \sigma), \rho(\mu^2 \circ \sigma^{(2)}), \ldots)(\in \Omega^{(2)})$$

is the homeomorphism from X to X_S (X endowed with weak toplogy and X_S endowed with the product topology).*

In general, we have the following fact.

Proposition 8.1. *Suppose that f is a function in $L^\infty(\mathbb{N})$. Let Ω be the spectrum of M_f, and S be the left shift map : $\Omega^{\mathbb{N}} \to \Omega^{\mathbb{N}}$ defined by*

$$T(a_1, a_2, \ldots) = (a_2, a_3, \ldots).$$

Note that $\Omega^{\mathbb{N}}$ is a compact Hausdroff space endowed with the product topology. Then the spectrum of $C(f)$ is homeomorphic to the closure of $\{\omega, T(\omega), T^{(2)}(\omega), \dots\}$ in $\Omega^{\mathbb{N}}$, where $\omega = (f(1), f(2), \dots)$. And the homeomorphism is given by

$$\rho \to (\rho(f), \rho(f \circ \sigma), \rho(f \circ \sigma^{(2)}), \dots),$$

where ρ is a point in the spectrum of $C(f)$.

Proof. By corollary 8.1, we only need to show that there exists a continous function $h \in C(\Omega^{\mathbb{N}})$ such that $h(T^{(n)}(\omega)) = f(n)$ and h separates points. It is easy to check that the function given by the projection to the first corrodinate, i.e. $h((a_1, a_2, \dots)) = a_1$, satisfies all conditions. $\square$

We will consider the function $f(n) = e^{2\pi i \sqrt{n}}$.

Lemma 8.6. *$\{e^{2\pi i \sqrt{n}}\}_{n \in \mathbb{N}}$ is dense in the unit circle S^1.*

Proof. Only need to show that $\{\frac{k}{\sqrt{n^2+k}+n} : k = 0, 1, \dots, 2n, n \in \mathbb{N}\}$ is dense in $[0, 1]$. If $\frac{k}{2n} \to \theta$, then

$$\frac{\frac{k}{n}}{\sqrt{1 + \frac{k}{n^2}} + 1} \to \theta.$$

It is clear now that $\{e^{2\pi i \sqrt{n}}\}_{n \in \mathbb{N}}$ is dense in S^1. $\square$

9. $Stone - \check{C}ech$ COMPACTIFICATION

Definition 9.1. *A filter on a set X is a collection $\mathcal{F}$ of subsets of X satisfying:*

(1) $X \in \mathcal{F}$, but $\varnothing \notin \mathcal{F}$.
(2) If $A \in \mathcal{F}$ and $A \subset B \subset X$, then $B \in \mathcal{F}$.
(3) A finite intersection of sets in $\mathcal{F}$ is in $\mathcal{F}$.

A filter is a ultrafilter if

(4) For every set $A \subset X$ either $A \in \mathcal{F}$ or $A^c = X \setminus A \in \mathcal{F}$.

or

(4)' For every finite cover $\{A_i\}_{i=1}^n$ of a set $A \in \mathcal{F}$, $A_i \in \mathcal{F}$ for some i.

It is well-known that the $Stone - \check{C}ech$ Compactification $\beta\mathbb{N}$ of $\mathbb{N}$ can be identified with the set of all ultrafilters on $\mathbb{N}$. The topology of $\beta\mathbb{N}$ is given by the basis $\mathcal{B} = \{U_A : A \subset \mathbb{N}\}$, where for any set $A \subset \mathbb{N}$,

$$U_A = \{\mathcal{F} \in \beta\mathbb{N} : A \in \mathcal{F}\}.$$

Example 9.1. *Let σ be a continuous map from $\mathbb{N}$ to $\mathbb{N}$ defined by $\sigma(n) = n + 1$. By the universal property of $\beta\mathbb{N}$, σ lifts uniquely to a continuous map $\sigma : \beta\mathbb{N} \to \beta\mathbb{N}$.Then the möbius function μ can be viewed as a continuous function on $\beta\mathbb{N}$ and $\sum_{k=1}^n \mu(k)^2$ is the number of square-free numbers below n. It is well-known*

$$\frac{1}{n}\sum_{k=0}^{n-1} \mu(n)^2 = \frac{6}{\pi^2} + o(1).$$

Therefore $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \mu(n)^2 = \frac{6}{\pi^2} \neq 0$. However, this is not a counter-example of sarnak's conjecture, since $h_{top}(\sigma) = \infty$. Indeed, for any $N \in \mathbb{N}$, let

$$A_N = \cup_{k=1}^{\infty}\{kn + 1 + \frac{k(k-1)}{2}, \dots, kn + 1 + \frac{k(k-1)}{2} + k\}.$$

It is easy to see that for any $n_1, \ldots, n_m$*, we have* $\sigma^{n_1}(A_N) \cap \cdots \cap \sigma^{n_m}(A_N) \neq \varnothing$*, and* $\cup_{i=0}^{N-1} \sigma^i(A_N) = \mathbb{N}$*. Therefore,* $\mathcal{U} = \{U_{\sigma^i(A_N)}\}_{i=0}^{N-1}$ *is an open over of* $\beta\mathbb{N}$ *and* $h_{cover}(\sigma, \mathcal{U}) = logN$*.*

Remark 9.1. *Recall that a thick set is a set of integers that contains arbitrarily long intervals. And a syndetic set is a subset of* $\mathbb{N}$*, having the property of "bounded gaps", i.e. the sizes of the gaps in the sequence of natural numbers is bounded. It is easy to see that* A_N *in the example 9.1 is thick and syndetic.*

Definition 9.2. *A dynamical system* (X, σ) *is call minimal if X does not contain any non-empty, proper, closed* σ*-invariant subset, i.e. every orbit is dense in X.*

Remark 9.2. *If* (X, σ) *is a minimal dynamical system and X is a compact Hausdorff space, then* σ *must be surjective.*

Lemma 9.1. $\beta\mathbb{N} \setminus \mathbb{N}$ *is not separable.*

Proof. Let $\{\mathcal{F}_n\}_{n=1}^{\infty}$ be a sequence of ultrafilters. To prove the result, we will construct a subset A of $\mathbb{N}$ recursively such that $A \notin \mathcal{F}_n$ for any n. Let A_1 be a infinite set that $A_1 \notin \mathcal{F}_1$ and $a_1 = minA_1$. Assume that we have $A_n \notin \mathcal{F}_n$. Let $a_n = minA_n$. Choose a infinite subset A_{n+1} of A_n such that $a_{n+1} = minA_{n+1} > a_n$ and $A_{n+1} \notin \mathcal{F}_{n+1}$. If $A \in \mathcal{F}_n$, then $A \cap A_n = A \cap \{1, \ldots, a_n - 1\}^c \in \mathcal{F}_n$. Since $A_n^c \in \mathcal{F}_n$, we have $\mathcal{A} \cap A_n \cap A_n^c = \varnothing \in \mathcal{F}_n$. It is a contradiction. □

Corollary 9.1. *Let* $(\beta\mathbb{N}, \sigma)$ *be the dynamical system where* σ *is the map induced by the shift on* $\mathbb{N}$*, i.e.* $\sigma(n) = n + 1$*. Then* $(\beta\mathbb{N}, \sigma)$ *is not minimal.*

Question 9.1.
(1) Is there a compactification of $\mathbb{N}$ *such that möbius function is continuous and the shift is continuous map?*
(2) Let $\sigma : \mathbb{N} \to \mathbb{N}$*. If* $h_{top}(\sigma) = 0$*, do we have* $\mathbb{N}^-$ *is "countable" in some sense.*

10. AF ALGEBRAS

Definition 10.1. *Let* $\mathfrak{A}$ *be a unital C*-algebra and* $Inn(\mathfrak{A}) = \{AdU : U \in \mathcal{U}(\mathfrak{A})\}$*. A automorphism* α *on* $\mathfrak{A}$ *is called approximately inner if, for every finite subset F of* $\mathfrak{A}$ *and for every* $\varepsilon > 0$*, there is a unitary U such that* $\|\alpha(A) - U^*AU\| < \varepsilon$ *for all* $A \in F$*.*

Remark 10.1. *One can check that if* $\mathfrak{A}$ *is separable and if* α *is an automorphism on* $\mathfrak{A}$*, then* α *is approximately inner if and only if there exists a sequence* $\{U_n\}$ *in* $\mathcal{U}(\mathfrak{A})$ *such that*

$$\lim_{n\to\infty} U_n^* A U_n = \alpha(A), \forall A \in \mathfrak{A}.$$

Definition 10.2. *Let* $\mathfrak{A}$ *be a unital separable C*-algebra. Denote by AInn(A) the group of all asymptotically inner automorphisms. An automorphism* α *is said to be strongly asymptotically inner if there is a continuous path of unitaries* $\{U(t) : t \in [0, \infty)\}$ *of* $\mathfrak{A}$ *such that*

$$U(0) = I \text{ and } \lim_{t\to\infty} U(t)^* A U(t) = \alpha(A) \text{ for all } T \in \mathfrak{A}.$$

Question 10.1. *Consider the* C^**-algebra generated by* μ *and* $\alpha^n(\mu)$ *in* $L^\infty(\mathbb{N})$ *where* α *is the unilateral shift of* $\mathbb{N}$*. What is this algebra?*

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