# Kadison-Singer Algebras, II ——General Case

**Liming Ge** * † **and Wei Yuan** *

*L. K. Hua Key Laboratory of Mathematics, Chinese Academy of Sciences, Beijing, China, and † University of New Hampshire, Durham, USA

Submitted to Proceedings of the National Academy of Sciences of the United States of America

**A new class of operator algebras, Kadison-Singer algebras, is introduced. These highly noncommutative, non selfadjoint algebras generalize triangular matrix algebras. They are determined by certain minimally generating lattices of projections in the von Neumann algebras corresponding to the commutant of the diagonals of the Kadison-Singer algebras. A new invariant for the lattices is introduced to classify these algebras.**

von Neumann algebra | Kadison-Singer algebra | Kadison-Singer lattice | triangular algebra | reflexive algebra

## Introduction

Kadison-Singer algebras were introduced in [1]. Examples of Kadison-Singer algebras with hyperfinite diagonals were given and studied. In this article, we shall continue our study of Kadison-Singer algebras and mostly deal with the case when the diagonal is a finite von Neumann algebra. We shall use the notation and definitions introduced in [1].

Suppose $\mathcal{H}$ is a Hilbert space and $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded operators on $\mathcal{H}$. Recall that a Kadison-Singer algebra is a maximal reflexive subalgebra of $\mathcal{B}(\mathcal{H})$ with respect to a given von Neumann algebra as its diagonal algebra. In [1], we constructed Kadison-Singer algebras with hyperfinite factors as their diagonal algebras and provided many new reflexive lattices. Our main result in this paper is to prove that the reflexive lattice generated by a double triangle (a special lattice with only three nontrivial projections) is, in general, isomorphic to the two-dimensional sphere $S^2$ (plus two distinct points corresponding to 0 and $I$), and the corresponding reflexive algebra is a Kadison-Singer algebra. In particular, we show that the algebra leave three free projections invariant is a Kadison-Singer algebra. This shows that many factors are (minimally) generated by a reflexive lattice of projections which is topologically homeomorphic to $S^2$. Noncommutative algebraic structures on $S^2$ determined by the projections give rise to non isomorphic factors and Kadison-Singer factors.

The paper contains five sections. In Section 2, maximal triangularity is discussed in different aspects. In Section 3, we describe the reflexive lattice generated by three free projections and show that it is homeomorphic to $S^2$. In Section 4, we show that this lattice is a Kadison-Singer lattice and thus the corresponding algebra is a Kadison-Singer algebra. Certain generalization of the result is also discussed. In Section 5, we introduce a notion of connectedness of projections in a lattice of projections in a finite von Neumann algebra and show that all connected components form another lattice, called a *reduced lattice*. Reduced lattices of most of our examples were computed.

The authors wish to thank J. Shen and M. Ravichandran for many helpful discussions.

## Maximality conditions

In the definition of Kadison-Singer algebras, we require that the algebra be maximal in the class of reflexive algebras with the same diagonal. Our examples of KS-algebras given in [1] are "maximal triangular" in the class of all algebras with the same diagonal, i.e., an algebraic maximality without reflexiveness or closedness assumptions. In general, the algebraic maximality assumption is a much stronger requirement. We call a subalgebra $\mathfrak{A}$ of $\mathcal{B}(\mathcal{H})$ *maximal triangular with respect to its diagonal $C^*$- (or von Neumann) algebra* $\mathfrak{A}\cap\mathfrak{A}^*$ if, for any subalgebra $\mathfrak{B}$ of $\mathcal{B}(\mathcal{H})$, $\mathfrak{B}$ contains $\mathfrak{A}$ and has the same diagonal as $\mathfrak{A}$, then $\mathfrak{B}$ is equal to $\mathfrak{A}$. This may lead to new interesting classes of non selfadjoint algebras. Many similar questions as those in [2], e.g., the closedness of $\mathfrak{A}$, can be asked accordingly.

In the following, we give a canonical method to construct a maximal triangular algebra in the class of weak-operator closed algebras with respect to a given von Neumann algebra as its diagonal.

Suppose $\mathcal{M}$ is a von Neumann algebra acting on a Hilbert space $\mathcal{H}_0$ and $H_1,\dots,H_n$ are positive elements in $\mathcal{M}$ such that $H_1^2,H_2^2,\dots,H_n^2$ generate $\mathcal{M}$ as a von Neumann algebra. From [3], we know that many von Neumann algebras can be generated by such positive elements, especially all type III and properly infinite von Neumann algebras. Let $\mathcal{H}$ be the direct sum of $n+1$ copies of $\mathcal{H}_0$. Then $\mathcal{B}(\mathcal{H})\cong M_{n+1}(\mathcal{B}(\mathcal{H}_0))$. With this identification, we shall view both $M_{n+1}(\mathbb{C})$ and $\mathcal{B}(\mathcal{H}_0)$ as subalgebras of $\mathcal{B}(\mathcal{H})$. Let $E_{ij}$, $i,j=1,\dots,n+1$ be a matrix unit system for $M_{n+1}(\mathbb{C})$. We shall write elements in $\mathcal{B}(\mathcal{H})$ in a matrix form with respect to this unit system (with entries from $\mathcal{B}(\mathcal{H}_0)$).

**Theorem 1.** *Define*

$$\mathfrak{A}=\left\{\begin{pmatrix} A & * & \dots & * \\ 0 & H_1^{-1}AH_1 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & H_n^{-1}AH_n \end{pmatrix} : \quad A\in\mathcal{B}(\mathcal{H}_0)\right\},$$

*where $*$ denotes all possible elements in $\mathcal{B}(\mathcal{H}_0)$. Then $\mathcal{A}$ is maximal upper triangular with respect to the diagonal $M_{n+1}(\mathcal{M})'$ ($\cong\mathcal{M}'\cap\mathcal{B}(\mathcal{H}_0)$).*

**Proof** It is easy to check that $\mathfrak{A}\cap\mathfrak{A}^*=M_{n+1}(\mathcal{M})'$. Similar techniques we used in the proof of Theorem 3 in [1] will show that $\mathfrak{A}$ is maximal upper triangular with the given diagonal. We omit the details here.

With $\mathfrak{A}$ given in Theorem 1, suppose $P$ is a projection in $\mathrm{Lat}(\mathfrak{A})$ with $P=(P_{ij})_{i,j=1}^{n+1}$, $P_{ij}\in\mathcal{B}(\mathcal{H}_0)$. It is easy to see that $P$ must be diagonal and $(I-P_{ii})TP_{jj}=0$ for all $i<j$ and any $T$ in $\mathcal{B}(\mathcal{H}_0)$. Thus $P=\sum_{j=1}^{k}E_{jj}$ for some

---

**Reserved for Publication Footnotes**

$k$. We know that such a $P$ lies in $\mathrm{Lat}(\mathfrak{A})$. This shows that $\mathrm{Lat}(\mathfrak{A}) = \{0, E_{11}, \ldots, \sum_{j=1}^{n} E_{jj}, I\}$. Clearly this implies that $\mathfrak{A}$ is not reflexive. It is interesting to know if $\mathfrak{A}$ contains a subalgebra which is a Kadison-Singer algebra with the same diagonal.

The semidirect product of a von Neumann algebra $\mathcal{M}$ with a semigroup $S$ (embedded in the automorphism group of $\mathcal{M}$) will give us another construction of triangular algebras. In general, such construction will not give us a maximal triangular algebra. Whether there is a KS-algebra containing such a triangular algebra for certain ergodic actions is another interesting question.

On the other hand, if we start with a "minimal" lattice $\mathcal{L}$ of projections in a von Neumann algebra so that the lattice generate the von Neumann algebra, then is $\mathrm{Alg}(\mathcal{L})$ a Kadison-Singer algebra? In general, $\mathcal{L} \neq \mathrm{Lat}(\mathrm{Alg}(\mathcal{L}))$. Thus $\mathcal{L}$ may not be a Kadison-Singer lattice. But we shall see that $\mathrm{Alg}(\mathcal{L})$ is often a Kadison-Singer algebra in next section.

## Reflexive lattices generated by three projections

Lattices generated by two projections are always reflexive [4]. But lattices generated by three projections are complicated. Most of factors acting on a separable Hilbert space are known to be generated by three projections or a projection and a positive operator (see [3]).

**Example 1.** *Suppose $\mathcal{M}$ is a factor acting on $\mathcal{H}$ and write $\mathcal{M} = M_2(\mathcal{N})$, where $\mathcal{N}$ is a subfactor of $\mathcal{M}$. We assume that $\mathcal{N}$ is generated by a projection $P$ and a positive element $H$ with $0 \le H \le I$ and $\mathrm{supp}(H) = \mathrm{supp}(I-H) = I$. Here we view $M_2(\mathbb{C})$ as a subalgebra of $\mathcal{M}$ and $\mathcal{N}$ as the relative commutant of $M_2(\mathbb{C})$ in $\mathcal{M}$. Let $E_{11}, E_{12}, E_{21}$ and $E_{22}$ be the standard matrix unit system for $M_2(\mathbb{C})$. Define $P_1 = E_{11}$, $P_2 = E_{11}H + E_{12}\sqrt{H(I-H)} + E_{21}\sqrt{H(I-H)} + E_{22}(I-H)$, $P_3 = E_{11} + E_{22}P$ and $P_4 = P_2 \wedge P_3$. Assume that $P_1 \wedge P_2 = 0$, $P_1 \vee P_2 = I$. Then $\mathcal{L} = \{0, I, P_1, P_2, P_3, P_4\}$ generated by $\{P_1, P_2, P_3\}$ is a distributive lattice and thus reflexive [5]. From our construction, we know that $\mathcal{M}$ is generated by $\mathcal{L}$ as a von Neumann algebra. One also easily checks that any proper sublattice of $\mathcal{L}$ does not generate $\mathcal{M}$. Thus $\mathcal{L}$ is a Kadison-Singer lattice and $\mathrm{Alg}(\mathcal{L})$ is a Kadison-Singer factor. From this construction, we can realize most of the factors as diagonals of Kadison-Singer algebras. For example, one may choose $\mathcal{N}$ as a factor of type $\mathrm{II}_1$ generated by a projection $P$ of trace $\frac{1}{2}$ and a positive operator $H$ such that $P$ and $H$ are free, and $H$ has the same distribution (with respect to the trace on $\mathcal{N}$) as the function $\cos^2 \frac{\pi}{2}\theta$ on $[0,1]$ (with respect to Lebesgue measure). Let $\tau$ be the trace on $\mathcal{M}$. In this case, $\tau(P_1) = \tau(P_2) = \frac{1}{2}$, $\tau(P_3) = \frac{3}{4}$ and $\tau(P_4) = \frac{1}{4}$. Then $\mathrm{Alg}(\mathcal{L})$ is a Kadison-Singer factor of type $\mathrm{II}_1$.*

It is hard to determine when a lattice is reflexive even for a finite lattice. Finite distributive lattices are reflexive [5]. But most of the lattice are not distributive. The simplest non distributive lattice is a *double triangle* where it contains $0, I$ and three projections $P_1, P_2$ and $P_3$ so that $P_i \vee P_j = I$ and $P_i \wedge P_j = 0$ for any $i \neq j$ and $i, j = 1, 2, 3$. Any lattice that contains a double triangle sublattice is not distributive. Three free projections with trace $\frac{1}{2}$ in a factor of type $\mathrm{II}_1$ (together with $0, I$) form a double triangle lattice. In the following, we first describe factors generated by free projections. For basic theory on freeness and distributions, we refer to [6].

Let $G_n$ be the free product of $\mathcal{Z}_2$ with itself $n$ times, for $n \ge 2$, or $= \infty$. When $n \ge 3$, $G_n$ is an i.c.c. group so its associated group von Neumann algebra $\mathcal{L}_{G_n}$ is a factor of type $\mathrm{II}_1$ acting on $l^2(G_n)$ (see [7]). If $U_1, \ldots, U_n$ are canonical generators for $\mathcal{L}_{G_n}$ corresponding to the generators of $G_n$ with $U_j^2 = I$. Then $\frac{I-U_j}{2}$, $j = 1, \ldots, n$, are projections of trace $\frac{1}{2}$. Let $\mathcal{F}_n$ be the lattice consisting of these $n$ free projections and $0, I$.

Clearly $\mathcal{F}_n$ is a minimal lattice which generates $\mathcal{L}_{G_n}$ as a von Neumann algebra. Is $\mathrm{Alg}(\mathcal{F}_n)$, $n \ge 3$, a Kadison-Singer algebra? What is $\mathrm{Lat}(\mathrm{Alg}(\mathcal{F}_n))$? When $n = 2$, Halmos [4] showed that $\mathcal{F}_2$ is reflexive and thus $\mathrm{Alg}(\mathcal{F}_2)$ is maximal and hence a Kadison-Singer algebra. We shall answer the above questions for the case when $n = 3$ and show that $\mathrm{Alg}(\mathcal{F}_3)$ is a KS-algebra and $\mathrm{Lat}(\mathrm{Alg}(\mathcal{F}_3)) \setminus \{0, I\}$ is homeomorphic to $S^2$, the two-dimensional sphere.

We shall realize $\mathcal{L}_{G_3}$ as the von Neumann algebra generated by $M_2(\mathbb{C})$ and its relative commutant $\mathcal{M}$ in $\mathcal{L}_{G_3}$ and write projection generators of $\mathcal{L}_{G_3}$ in terms of $2 \times 2$ matrices (with respect to the standard matrix units in $M_2(\mathbb{C})$) given by the following equations:

$$P_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, P_2 = \begin{pmatrix} H_1 & \sqrt{H_1(I-H_1)} \\ \sqrt{H_1(I-H_1)} & I-H_1 \end{pmatrix},$$
$$P_3 = \begin{pmatrix} H_2 & \sqrt{H_2(I-H_2)}V \\ V^*\sqrt{H_2(I-H_2)} & V^*(I-H_2)V \end{pmatrix}.$$

The freeness among $P_1, P_2, P_3$ require that $H_1, H_2$ and $V$ be free, $H_1$ and $H_2$ have the same distribution as $\cos^2 \frac{\pi}{2}\theta$ on $[0,1]$ with respect to Lebesgue measure and $V$ a Haar unitary element. Then the subalgebra $\mathcal{M}$ of $\mathcal{L}_{G_3}$ is the von Neumann algebra generated by $H_1, H_2$ and $V$. Now $\mathcal{F}_3 = \{0, I, P_1, P_2, P_3\}$, $\mathcal{H} = l^2(G_3)$.

When $M_2(\mathbb{C})$ is a subalgebra of $\mathcal{L}_{G_3}$, we may also view $\mathcal{B}(\mathcal{H}) = M_2(\mathbb{C}) \otimes \mathcal{B}$, where $\mathcal{B}$ is the commutant of $M_2(\mathbb{C})$ in $\mathcal{B}(\mathcal{H})$. Thus all operators can be written as $2 \times 2$ matrices with entries from $\mathcal{B}$. In fact, when $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1$ for some Hilbert space $\mathcal{H}_1$, then $\mathcal{B} \cong \mathcal{B}(\mathcal{H}_1)$.

Since $P_1 \in \mathcal{F}_3$, any operator $T$ belonging to $\mathrm{Alg}(\mathcal{F}_3)$ must be upper triangular. The following lemma follows from the invariance of $P_2$ and $P_3$ under $T$. The computation is straight forward.

**Lemma 1.** *With notation given above, $\begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \in \mathrm{Alg}(\mathcal{F}_3)$, where $T_1, T_2, T_3 \in \mathcal{B}$, if and only if*

$$\sqrt{I-H_1}T_2\sqrt{I-H_1} = \sqrt{H_1}T_3\sqrt{I-H_1} - \sqrt{I-H_1}T_1\sqrt{H_1};$$
$$\sqrt{I-H_2}T_2V^*\sqrt{I-H_2} = \sqrt{H_2}VT_3V^*\sqrt{I-H_2} - \sqrt{I-H_2}T_1\sqrt{H_2}.$$

Using unbounded operators affiliated with $\mathcal{L}_{G_3}$, one can construct many finite rank operators in $\mathrm{Alg}(\mathcal{F}_3)$. Unbounded operators affiliated with a finite von Neumann form an algebra [7]. Any finitely many unbounded operators have a common dense domain. Let $\xi$ and $\eta$ be vectors in the common domain of $\sqrt{H_1(I-H_1)^{-1}}$, $\sqrt{H_2(I-H_2)^{-1}}V$ and the adjoint $(\sqrt{H_2(I-H_2)^{-1}}V)^*$. We shall use $x \otimes y$ to denote the rank one operator defined by $x \otimes y(z) = \langle z, x\rangle y$, for any $z \in \mathcal{H}$ with $x$ and $y$ arbitrarily given. Now let $T_1 = \xi \otimes (\sqrt{H_1(I-H_1)^{-1}} - \sqrt{H_2(I-H_2)^{-1}}V)\eta$, $T_3 = ((\sqrt{H_1(I-H_1)^{-1}} - \sqrt{H_2(I-H_2)^{-1}}V)^*\xi) \otimes \eta$ and $T_2 = \sqrt{H_1(I-H_1)^{-1}}T_3 - T_1\sqrt{H_1(I-H_1)^{-1}}$ (determined by Lemma 1). Then $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \in \mathrm{Alg}(\mathcal{F}_3)$ is a finite rank operator (at most rank 4). This shows that $\mathrm{Alg}(\mathcal{F}_3)$ contains many finite rank (and thus compact) operators. In fact, Lemma 3 below will show that $\mathrm{Alg}(\mathcal{F}_3)$ contains "almost" a copy of $\mathcal{B}(\mathcal{H})$.

The following is a technical result that will be used frequently. The result might be well known. We only sketch a proof here.

**Lemma 2.** *Suppose $U$ is a Haar unitary element in a factor $\mathcal{M}$ of type $II_1$ and $A$ is an element in (or an unbounded operator affiliated with) $\mathcal{M}$ such that $A$ and $U$ are free with each other. Then any nonzero scalar $\lambda$ can not be a point spectrum of $AU$.*

**Proof** Suppose $\lambda \in \mathbb{C}$ is a nonzero point spectrum of $AU$. By symmetry and freeness of $A$ and $\omega U$, we know that $\lambda$ must be a point spectrum for $A(\omega U)$ for any $\omega \in \mathbb{C}$ with $|\omega| = 1$. This implies that $\omega^{-1}\lambda$ is a point spectrum of $AU$. Suppose $P_\beta$ is the spectral projection of $AU$ supported at $\beta \in \mathbb{C}$. Then $P_\lambda$ is equivalent to $P_{\omega\lambda}$ for any $|\omega| = 1$. A direct computation shows that if $\lambda \neq \lambda_j$ for $j = 1, 2, \ldots, n$, then $P_\lambda \wedge (P_{\lambda_1} \vee \cdots \vee P_{\lambda_n}) = 0$. From finiteness of $\mathcal{M}$, it is easy to conclude that $P_\lambda = 0$ when $\lambda \neq 0$.

In the following, we shall describe all elements in $\mathrm{Lat}(\mathrm{Alg}(\mathcal{F}_3))$. Unbounded operators will be used in our computation. All unbounded operators affiliated with the factor $\mathcal{L}_{G_3} = M_2(\mathcal{M})$ form an algebra. From function calculus, many unbounded operators we encounter in this paper can be viewed as (positive) functions defined on $(0, 1)$ with respect to Lebesgue measure. When $H$ ($= H_1$ or $H_2$) is identified with $\cos^2 \frac{\pi}{2}\theta$. Then $I - H$, $\sqrt{H(I-H)}$, $\sqrt{H^{-1}}$, $\sqrt{(I-H)^{-1}}$, etc., can be viewed as trigonometric functions and they are all determined by any one of them. Lemma 2 also tells that many linear combinations of free (non selfadjoint) operators such as $\sqrt{H_1(I-H_1)^{-1}} - \sqrt{H_2(I-H_2)^{-1}}V = \sqrt{H_1(I-H_1)^{-1}}(I - \sqrt{H_1^{-1}(I-H_1)}\sqrt{H_2(I-H_2)^{-1}}V)$ are invertible (with unbounded inverses).

When $S, T$ are unbounded operators affiliated with $\mathcal{L}_{G_3}$ or a finite von Neumann algebra and $X \in \mathcal{B}(\mathcal{H})$, then $SXT$ is an unbounded operator that can be viewed as the weak-operator limit of bounded operators of the form $SE_\epsilon XF_\epsilon T$ for projections $E_\epsilon$ and $F_\epsilon$ in $\mathcal{L}_{G_3}$ (or the finite von Neumann algebra) so that $SE_\epsilon$ and $F_\epsilon T$ are bounded and $E_\epsilon$, $F_\epsilon$ have strong operator limit $I$ (as $\epsilon \to 0$). Thus, for any operator $X$ in a weak-operator dense subalgebra $\cup_\epsilon E_\epsilon \mathcal{B}(\mathcal{H})F_\epsilon$ of $\mathcal{B}(\mathcal{H})$, the operator $SXT$ is a bounded operator.

Using unbounded operators, we may restate the above lemma in the following form.

**Lemma 3.** *With $H_1, H_2, V \in \mathcal{M} \subset \mathcal{L}_{G_3}$ given above, let $S = \sqrt{H_1(I-H_1)^{-1}} - \sqrt{H_2(I-H_2)^{-1}}V$ be an unbounded operator affiliated with $\mathcal{M}$. If $T \in \mathrm{Alg}(\mathcal{F}_3)$, then there is an $A \in \mathcal{B}$ such that*

$$T = \begin{pmatrix} A & \sqrt{H_1(I-H_1)^{-1}}S^{-1}AS - A\sqrt{H_1(I-H_1)^{-1}} \\ 0 & S^{-1}AS \end{pmatrix}.$$

*Conversely, if $A \in \mathcal{B}$ such that $\sqrt{H_1(I-H_1)^{-1}}S^{-1}AS - A\sqrt{H_1(I-H_1)^{-1}}$ and $S^{-1}AS$ are bounded operators, then the above $T$ belongs to $\mathrm{Alg}(\mathcal{F}_3)$.*

The above lemma shows that $\mathrm{Alg}(\mathcal{F}_3)$ is quite large, in particular, $\mathrm{Alg}(\mathcal{F}_3) \cap \mathcal{L}_{G_3}$ is infinite dimensional. The following result follows easily from the above lemma and shows that all nontrivial projections in $\mathrm{Lat}(\mathrm{Alg}(\mathcal{F}_3))$ have trace $\frac{1}{2}$.

**Corollary 2.** *For any $Q \in \mathrm{Lat}(\mathrm{Alg}(\mathcal{F}_3)) \setminus \{0, I, P_1\}$, we have that $Q \wedge P_1 = 0$, $Q \vee P_1 = I$, and $\tau(Q) = \frac{1}{2}$.*

**Proof** For any $Q \in \mathrm{Lat}(\mathrm{Alg}(\mathcal{F}_3))$, $Q \wedge P_1 \in \mathrm{Lat}(\mathrm{Alg}(\mathcal{F}_3))$. Thus $Q \wedge P_1$ is invariant under all $T$ given in Lemma 3 and thus the $A$ in $(1,1)$ entry of $T$. This implies that $Q \wedge P_1 = P_1$ or 0. Similarly we can show that $Q \vee P_1 = I$ or $P_1$.

This corollary actually shows that for any distinct projections $Q_1, Q_2$ in $\mathrm{Lat}(\mathrm{Alg}(\mathcal{F}_3)) \setminus \{0, I\}$, $Q_1 \wedge Q_2 = 0$, and $Q_1 \vee Q_2 = I$.

For any $X \in \mathcal{B}(\mathcal{H})$ or an unbounded operator $X$ affiliated with a (finite) von Neumann algebra, we shall use $supp(X)$ to denote the support of $X$, i.e., the range projection of $X^*X$. When $supp(X) = supp(X^*) = I$, $X$ has an (unbounded) inverse.

**Theorem 3.** *For any projection $Q$ in $\mathrm{Lat}(\mathrm{Alg}(\mathcal{F}_3)) \setminus \{0, I, P_1\}$, there are $K$ and $U$ in $\mathcal{M}$ such that*

$$Q = \begin{pmatrix} K & \sqrt{K(I-K)}U \\ U^*\sqrt{K(I-K)} & U^*(I-K)U \end{pmatrix},$$

*where $\sqrt{K(I-K)^{-1}}$ (or $K$) and $U$ are determined by the polar decomposition of $(1+a)\sqrt{H_1(I-H_1)^{-1}} - a\sqrt{H_2(I-H_2)^{-1}}V = aS + \sqrt{H_1(I-H_1)^{-1}} = \sqrt{K(I-K)^{-1}}U$ for some $a \in \mathbb{C}$. Moreover for any given $a$ in $\mathbb{C}$, the polar decomposition determines $U$ and $K$ uniquely which give rise to a projection $Q$ (in the above form) in $\mathrm{Lat}(\mathrm{Alg}(\mathcal{F}_3))$.*

**Proof** Suppose $Q$ is given in the theorem. From Corollary 2, we know that $supp(I-K) = I$. From Lemma 3, for any $A \in \mathcal{B}$ (the commutant of $M_2(\mathbb{C})$ in $\mathcal{B}(l^2(G_3))$, $\mathcal{B} \cong \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$) such that $S^{-1}AS$ and $\sqrt{H_1(I-H_1)^{-1}}S^{-1}AS - A\sqrt{H_1(I-H_1)^{-1}}$ are bounded, then

$$T = \begin{pmatrix} A & \sqrt{H_1(I-H_1)^{-1}}S^{-1}AS - A\sqrt{H_1(I-H_1)^{-1}} \\ 0 & S^{-1}AS \end{pmatrix} \in \mathrm{Alg}(\mathcal{F}_3),$$

here $S = \sqrt{H_1(I-H_1)^{-1}} - \sqrt{H_2(I-H_2)^{-1}}V$. Thus $(I-Q)TQ = 0$. This implies that

$$\begin{aligned}(I-K)A\sqrt{K(I-K)}U + (I-K)\Big[&\sqrt{H_1(I-H_1)^{-1}}S^{-1}AS \\ &- A\sqrt{H_1(I-H_1)^{-1}}\Big]U^*(I-K)U \\ &- \sqrt{K(I-K)}US^{-1}ASU^*(I-K)U = 0.\end{aligned}$$

Since $supp(I-K) = I$, $I-K$ is invertible. We have

$$\begin{aligned}&\sqrt{I-K}A\Big[\sqrt{K} - \sqrt{H_1(I-H_1)^{-1}}U^*\sqrt{I-K}\Big] \\ &= \Big[\sqrt{K}U - \sqrt{I-K}\sqrt{H_1(I-H_1)^{-1}}\Big]S^{-1}ASU^*\sqrt{I-K}.\end{aligned}$$

This gives us

$$\begin{aligned}&A[\sqrt{K} - \sqrt{H_1(I-H_1)^{-1}}U^*\sqrt{I-K}](SU^*\sqrt{I-K})^{-1} \\ &= \sqrt{(I-K)^{-1}}[\sqrt{K}U - \sqrt{I-K}\sqrt{H_1(I-H_1)^{-1}}]S^{-1}A.\end{aligned}$$

The above equation holds for all $A$ in a weak-operator dense subalgebra of $\mathcal{B}$ ($\cong \mathcal{B}(\mathcal{H}_1)$). Thus there is an $a \in \mathbb{C}$ such that

$$\begin{aligned}a\sqrt{I-K} &= [\sqrt{K}U - \sqrt{I-K}\sqrt{H_1(I-H_1)^{-1}}]S^{-1}, \\ aSU^*\sqrt{I-K} &= \sqrt{K} - \sqrt{H_1(I-H_1)^{-1}}U^*\sqrt{I-K}.\end{aligned}$$

This implies that

$$\sqrt{K(I-K)^{-1}}U = aS + \sqrt{H_1(I-H_1)^{-1}}.$$

Conversely, when $K$ and $U$ are given by this equation, all above equations hold. From Lemma 3 one checks easily that $Q$ given in the theorem lies in $\mathrm{Lat}(\mathrm{Alg}(\mathcal{F}_3))$.

## $\mathrm{Alg}(\mathcal{F}_3)$ is a Kadison-Singer algebra

In this section, we shall prove that $\mathrm{Lat}(\mathrm{Alg}(\mathcal{F}_3))$ is a Kadison-Singer lattice which implies that $\mathrm{Alg}(\mathcal{F}_3)$ is a Kadison-Singer algebra.

**Lemma 4.** *For any two distinct projections $Q_1$, $Q_2$ in $\mathrm{Lat}(\mathrm{Alg}(\mathcal{F}_3)) \setminus \{0, I, P_1\}$, we have that $\mathrm{Alg}(\{P_1, Q_1, Q_2\}) = \mathrm{Alg}(\mathcal{F}_3)$.*

**Proof** By Theorem 3, we may assume that, for $i = 1, 2$,

$$Q_i = \begin{pmatrix} K_i & \sqrt{K_i(I-K_i)}U_i \\ U_i^*\sqrt{K_i(I-K_i)} & U_i^*(I-K_i)U_i \end{pmatrix} \text{ and}$$

$$\begin{aligned} \sqrt{K_i(I-K_i)^{-1}}U_i &= (1+a_i)\sqrt{H_1(I-H_1)^{-1}} \\ &\quad - a_i\sqrt{H_2(I-H_2)^{-1}}V \\ &= a_iS + \sqrt{H_1(I-H_1)^{-1}}. \end{aligned} \qquad [\mathbf{1}]$$

where $a_1, a_2 \in \mathbb{C}$ and $a_1 \neq a_2$. Then we have $\sqrt{K_1(I-K_1)^{-1}}U_1 - \sqrt{K_2(I-K_2)^{-1}}U_2 = (a_1-a_2)S$. Replacing $S$ by $(a_1-a_2)S$ in Lemma 3, we know that $\mathrm{Alg}(\{P_1, Q_1, Q_2\}) = \mathrm{Alg}(\mathcal{F}_3)$.

**Lemma 5.** *For any three distinct projections $Q_1, Q_2$ and $Q_3$ in $\mathrm{Lat}(\mathrm{Alg}(\mathcal{F}_3)) \setminus \{0, I\}$, we always have that $P_1 \in \mathrm{Lat}(\mathrm{Alg}(\{Q_1, Q_2, Q_3\}))$.*

**Proof** We may assume that $Q_i \in \mathrm{Lat}(\mathrm{Alg}(\mathcal{F}_3)) \setminus \{0, I, P_1\}$ and $Q_i$ is given by [1] in the proof of Lemma 4, for $i = 1, 2, 3$ and $a_1, a_2, a_3$ are distinct scalars.

To prove this lemma, we only need to show that if

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathrm{Alg}(\{Q_1, Q_2, Q_3\}),$$

then $A_{21} = 0$. Now assume that the above $A$ belongs to $\mathrm{Alg}(\{Q_1, Q_2, Q_3\})$. Then $(I - Q_i)AQ_i = 0$, for $i = 1, 2, 3$. The $(1, 2)$ entries of the equation $AQ_i = Q_iAQ_i$ give us that

$$\begin{aligned} &(I-K_i)(A_{11}\sqrt{K_i(I-K_i)}U_i + A_{12}U_i^*(I-K_i)U_i) \\ &= \sqrt{K_i(I-K_i)}U_i(A_{21}\sqrt{K_i(I-K_i)}U_i + A_{22}U_i^*(I-K_i)U_i). \end{aligned}$$

This implies that

$$\begin{aligned} &A_{11}\sqrt{K_i(I-K_i)^{-1}}U_i + A_{12} \\ &\quad = \sqrt{K_i(I-K_i)^{-1}}U_i[A_{21}\sqrt{K_i(I-K_i)^{-1}}U_i + A_{22}]. \end{aligned}$$

Thus, for $i, j = 1, 2, 3$ and from $\sqrt{K_i(I-K_i)^{-1}}U_i = a_iS + \sqrt{H_1(I-H_1)^{-1}}$, we have

$$\begin{aligned} &\sqrt{K_i(I-K_i)^{-1}}U_iA_{21}\sqrt{K_i(I-K_i)^{-1}}U_i \\ &\quad - \sqrt{K_j(I-K_j)^{-1}}U_jA_{21}\sqrt{K_j(I-K_j)^{-1}}U_j \\ &\quad = (a_i - a_j)[A_{11}S - SA_{22}]. \end{aligned}$$

From this, we conclude that

$$\begin{aligned} &(a_1-a_2)\sqrt{K_3(I-K_3)^{-1}}U_3A_{21}\sqrt{K_3(I-K_3)^{-1}}U_3 + \\ &(a_2-a_3)\sqrt{K_1(I-K_1)^{-1}}U_1A_{21}\sqrt{K_1(I-K_1)^{-1}}U_1 + \\ &(a_3-a_1)\sqrt{K_2(I-K_2)^{-1}}U_2A_{21}\sqrt{K_2(I-K_2)^{-1}}U_2 = 0. \end{aligned}$$

Using the relation $\sqrt{K_i(I-K_i)^{-1}}U_i = a_iS + \sqrt{H_1(I-H_1)^{-1}}$ again, we easily get

$$\begin{aligned} &[a_3^2(a_1-a_2) + a_1^2(a_2-a_3) + a_2^2(a_3-a_1)]SA_{21}S \\ &+ [a_3(a_1-a_2) + a_1(a_2-a_3) + a_2(a_3-a_1)] \\ &\times [SA_{21}\sqrt{H_1(I-H_1)^{-1}} + \sqrt{H_1(I-H_1)^{-1}}A_{21}S] \\ &+ [(a_1-a_2) + (a_2-a_3) \\ &\quad + (a_3-a_1)]\sqrt{H_1(I-H_1)^{-1}}A_{21}\sqrt{H_1(I-H_1)^{-1}} = 0, \end{aligned}$$

which implies that

$$(a_1-a_2)(a_1-a_3)(a_2-a_3)SA_{21}S = 0.$$

This gives us that $A_{21} = 0$ and the lemma follows.

Now the follow theorem follows easily from our lemmas and Theorem 3.

**Theorem 4.** *With the above notation, we have that $\mathrm{Alg}(\mathcal{F}_3)$ is a KS-algebra and $\mathrm{Lat}(\mathrm{Alg}(\mathcal{F}_3))$ is determined by the following: $P \in \mathrm{Lat}(\mathrm{Alg}(\mathcal{F}_3))$ and $P \neq 0, I, P_1$, if and only if*

$$P = \begin{pmatrix} K & \sqrt{K(I-K)}U \\ U^*\sqrt{K(I-K)} & U^*(I-K)U \end{pmatrix},$$

*where $K$ and $U$ are uniquely determined by the following polar decomposition with any $a \in \mathbb{C}$: $(a+1)\sqrt{H_1(I-H_1)^{-1}} - a\sqrt{H_2(I-H_2)^{-1}}V = \sqrt{K(I-K)^{-1}}U$. As a consequence, we have $\tau(P) = \frac{1}{2}$ and as $a$ tends to $\infty$, the projection $P$ converges strongly to $P_1$. Thus $\mathrm{Lat}(\mathrm{Alg}(\mathcal{F}_3)) \setminus \{0, I\}$ is homeomorphic to $S^2$.*

**Proof** We only need to show the minimality of $\mathrm{Lat}(\mathrm{Alg}(\mathcal{F}_3))$. Clearly for any sublattice $\mathcal{L}_1$ containing only two projections $Q_1, Q_2$ in $\mathrm{Lat}(\mathrm{Alg}(\mathcal{F}_3))$, $\mathcal{L}_1$ can not generate the type $\mathrm{II}_1$ factor $\mathcal{L}_{G_3}$. Lemma 4 shows that any reflexive sublattice of $\mathrm{Lat}(\mathrm{Alg}(\mathcal{F}_3))$ containing more than two nontrivial projections must agree with $\mathrm{Lat}(\mathrm{Alg}(\mathcal{F}_3))$. Thus $\mathrm{Lat}(\mathrm{Alg}(\mathcal{F}_3))$ is a Kadison-Singer lattice.

Although the above theorem is stated for $\mathcal{F}_3$, similar results hold when $P_1, P_2$ and $P_3$ are not assumed to be free. There are many other factors which can be generated by three projections. Let $\mathcal{M}$ be a factor of type $\mathrm{II}_1$ with trace $\tau$ and $M_2(\mathbb{C}) \subseteq \mathcal{M}$ and $M_2(\mathbb{C})' \cap \mathcal{M} = \mathcal{N}$, a subfactor of $\mathcal{M}$ with generators $H_1$, $H_2$ and $V$, where $0 \leq H_1, H_2 \leq I$ and $V$ is a unitary operator. Suppose $\mathcal{H} = L^2(\mathcal{M}, \tau)$. Let $P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{C})$ be a projection in $\mathcal{M}$. Furthermore, we assume that

$$P_2 = \begin{pmatrix} H_1 & \sqrt{H_1(I-H_1)} \\ \sqrt{H_1(I-H_1)} & I-H_1 \end{pmatrix}$$

$$P_3 = \begin{pmatrix} H_2 & \sqrt{H_2(I-H_2)}V \\ V^*\sqrt{H_2(I-H_2)} & I-H_2 \end{pmatrix}$$

are projections in $\mathcal{M}$ such that $P_2 \vee P_3 = I$ and $P_2 \wedge P_3 = 0$, and $\sqrt{H_1(I-H_1)^{-1}}$, $\sqrt{H_2(I-H_2)^{-1}}$ and $\sqrt{H_1(I-H_1)^{-1}} - \sqrt{H_2(I-H_2)^{-1}}V$ are bounded invertible operators. All above conditions are satisfied when $H_1$, $H_2$ are free positive elements with disjoint spectra in $(0, 1)$ $(V = I)$, and $\sqrt{H_1(I-H_1)^{-1}}$ and $\sqrt{H_2(I-H_2)^{-1}}$ have disjoint spectra.

**Theorem 5.** *With the above notation and assumptions, we have that $\mathrm{Alg}(\{P_1, P_2, P_3\})$ is a Kadison-Singer algebra and $\mathrm{Lat}(\mathrm{Alg}(\{P_1, P_2, P_3\}))$ is determined by the following: $P \in \mathrm{Lat}(\mathrm{Alg}(\{P_1, P_2, P_3\}))$ and $P \neq 0, I, P_1$, if and only if*

$$P = \begin{pmatrix} L & \sqrt{L(I-L)}U \\ U^*\sqrt{L(I-L)} & U^*(I-L)U \end{pmatrix}$$

*where $L$ and $U$ are uniquely determined by the following polar decomposition with any $a \in \mathbb{C}$: $(a+1)\sqrt{H_1(I-H_1)^{-1}} - a\sqrt{H_2(I-H_2)^{-1}}V = \sqrt{L(I-L)^{-1}}U$. As a consequence, we have $\tau(P) = \frac{1}{2}$.*

The proof of this theorem will be the same as that of Theorem 4. Moreover, as $a \to \infty$ in Theorem 4, the projection $P \to P_1$ in strong-operator topology. Thus $\mathrm{Lat}(\mathrm{Alg}(\{P_1, P_2, P_3\})) \setminus \{0, I\}$ is homeomorphic to the one point compactification of $\mathbb{C}$, i.e., homeomorphic to $S^2$.

When a lattice contains four or more projections in a von Neumann algebra, the situation is not clear. Even for $\mathcal{F}_4$ (the lattice generated by four free projections), we know from our above result that $\mathrm{Lat}(\mathrm{Alg}(\mathcal{F}_4))$ contains several copies of $S^2$.

But we do not have a complete characterization of this lattice. The following theorem shows that $\mathrm{Alg}(\mathcal{F}_\infty)$ contains no nonzero compact operators. We believe that $\mathrm{Alg}(\mathcal{F}_n)$, when $n \geq 4$, does not contain any compact operators.

**Theorem 6.** *Let $\mathcal{F}_\infty$ be the lattice generated by countably infinitely many free projections of trace $\frac{1}{2}$ in $L_{G_\infty}$, where $G_\infty$ is the free product of countably infinitely many copies of $\mathbb{Z}_2$. Then $\mathrm{Alg}(\mathcal{F}_\infty)$ does not contain any nonzero compact operators.*

**Proof** Let $U_j$, $j = 1, 2, \ldots$, be the standard generators of $\mathcal{L}_{G_n}$ and $P_j = \frac{I-U_j}{2}$ the free projections in $\mathcal{F}_\infty$. Suppose $T \in \mathrm{Alg}(\mathcal{F}_\infty)$ is a compact operator. From $(I - P_j)TP_j = 0$, we know that $(I + U_j)T(I - U_j) = 0$ for $j = 1, 2, \ldots$. For any $x, y$ in $l^2(G_\infty)$, we have

$$\langle (I + U_j)T(I - U_j)x, y\rangle = \langle T(I - U_j)x, (I + U_j)y\rangle = 0.$$

Since $U_j$ has weak-operator limit zero, as $j \to \infty$, and $T$ compact, we have that $\langle U_j x, y\rangle \to 0$, $\langle Tx, U_j y\rangle \to 0$ and $TU_j x \to 0$ (as $j \to \infty$). This implies that $\langle Tx, y\rangle = 0$ and thus $T = 0$.

## Reduced lattices

Suppose $\mathcal{L}$ is a lattice of projections in a finite von Neumann algebra $\mathcal{M}$ with a faithful normal trace $\tau$. Two projections $P$ and $Q$ in $\mathcal{L}$ are said to be *connected* if, for any $\epsilon > 0$, there are elements $P_1, P_2, \ldots, P_n$ in $\mathcal{L}$ such that $P_1 = P$, $P_n = Q$, $|\tau(P_j - P_{j+1})| < \epsilon$, and either $P_j \leq P_{j+1}$ or $P_j \geq P_{j+1}$, for $j = 1, \ldots, n-1$. Define the connected component $O(P)$ of $P$ to be the set of all projections in $\mathcal{L}$ that are connected with $P$. Let $\Gamma_0(\mathcal{L})$ be the set of all connected components in $\mathcal{L}$. We shall see that $\Gamma_0(\mathcal{L})$ carries an induced lattice structure from $\mathcal{L}$ and we call $\Gamma_0(\mathcal{L})$ the *reduced lattice* of $\mathcal{L}$. It is clear that if $\mathcal{L}$ is a continuous nest, then $\Gamma_0(\mathcal{L})$ contains only one point. A basic fact on connected components is given in the following.

**Proposition 7.** *Suppose $\mathcal{L}$ is a lattice of projections in a finite von Neumann algebra $\mathcal{M}$, $P, Q \in \mathcal{L}$. If $O(P) \neq O(Q)$, then $O(P) \cap O(Q) = \emptyset$.*

The proof of this proposition follows easily from the definition. If $O(P)$ and $O(Q)$ are two elements in $\Gamma_0(\mathcal{L})$, then, for any $Q_1 \in O(Q)$, it is easy to see that $O(P \vee Q) = O(P \vee Q_1)$. Thus $O(P \vee Q)$ depends on the components $O(P)$ and $O(Q)$, not the choices of $P$ and $Q$ in the components. We define $O(P) \vee O(Q) = O(P \vee Q)$. Similarly we define that $O(P) \wedge O(Q) = O(P \wedge Q)$. It is easy to shows that $\Gamma_0(\mathcal{L})$ is a lattice. The following theorem is immediate.

**Theorem 8.** *Suppose $\mathcal{L}$ is a lattice of projections in a finite von Neumann algebra $\mathcal{M}$ with a (faithful normal) trace $\tau$. If $\tau(\mathcal{L})$ contains only finitely many trace values, then $\Gamma_0(\mathcal{L})$ is the same as $\mathcal{L}$ (i.e., every connected component contains only one element in $\mathcal{L}$).*

From this theorem, we know that $\Gamma_0(\mathrm{Lat}(\mathrm{Alg}(\mathcal{F}_3))) = \mathrm{Lat}(\mathrm{Alg}(\mathcal{F}_3))$. Suppose $\mathcal{L}$ is a Kadison-Singer lattice and $\mathcal{L}$ generates a finite von Neumann algebra. If $\Gamma_0(\mathcal{L})$ contains only one point, then we call $\mathcal{L}$ *contractible*. Thus continuous nests are contractible. In a forthcoming paper, we shall construct other contractible Kadison-Singer lattices and also show that manifolds of higher dimensions can appear as the reduced lattices of Kadison-Singer lattices.

Reduced lattices can not contain continuous nests. The following theorem shows that all possible trace values can appear in a reduced lattice.

**Theorem 9.** *Suppose $\mathcal{L}^{(n)}$ is the lattice given in Section 4 in [1]. Then $\Gamma_0(\mathcal{L}^{(n)}) = \mathcal{L}^{(n)}$.*

The proof is a direct computation. We omit the details here.

**ACKNOWLEDGMENTS.** Research supported in part by President Fund of Academy of Mathematics and Systems Science, Chinese Academy of Sciences

1. L. Ge and W. Yuan, Type I C*-algebrasKadison-Singer algebras, I—hyperfinite case, to appear, 2009
2. R. Kadison and I. Singer, Triangular operator algebras. Fundamentals and hyperreducible theory, Amer. J. Math., 82 (1960), 227–259.
3. L. Ge and J. Shen, On the generator problem of von Neumann algebras, Proc. of ICCM 2004 (Hong Kong), Intern. Press, Boston.
4. P. Halmos, Reflexive lattices of subspaces, J. London Math. Soc. 4 (1971), 257–263.
5. K. Harrison, On lattices of invariant subspaces, Doctoral Thesis, Monash University, Melbourne, 1970.
6. D. Voiculescu, K. Dykema and A. Nica, "Free Random Variables," CRM Monograph Series, vol. 1, 1992.
7. R. Kadison and J. Ringrose, "Fundamentals of the Operator Algebras," vols. I and II, Academic Press, Orlando, 1983 and 1986.