1. Isomorphism fix the sphere

Let

$$Q_\infty = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, Q_0 = \begin{pmatrix} H_1 & \sqrt{H_1(I-H_1)} \\ \sqrt{H_1(I-H_1)} & I-H_1 \end{pmatrix},$$
$$Q_{-1} = \begin{pmatrix} H_2 & \sqrt{H_2(I-H_2)}V \\ V^*\sqrt{H_2(I-H_2)} & V^*(I-H_2)V \end{pmatrix}.$$

be three free projections. Let $\mathcal{F}_3$ be the lattice generated by $\{Q_\infty, Q_0, Q_{-1}\}$, and $\mathcal{M}\otimes M_2(\mathbb{C}) = \mathcal{F}_3''$, the von Neumann algebra generated by $\mathcal{F}_3$. In [**?**], we have the following result:

Theorem 1.1. *For any projection Q in $\mathrm{Lat}(\mathrm{Alg}(\mathcal{F}_3)) \setminus \{0, I, Q_\infty\}$, there are K_z and U_z in $\mathcal{M}$ such that*

$$Q = Q_z = \begin{pmatrix} K_z & \sqrt{K_z(I-K_z)}U_z \\ U_z^*\sqrt{K_z(I-K_z)} & U_z^*(I-K_z)U_z \end{pmatrix},$$

where

$$\begin{aligned}\sqrt{K_z(I-K_z)^{-1}}U_z =& (1+z)\sqrt{H_1(I-H_1)^{-1}} - z\sqrt{H_2(I-H_2)^{-1}}V \\ =& zS + \sqrt{H_1(I-H_1)^{-1}},\end{aligned} \tag{1}$$

and $S = \sqrt{H_1(I-H_1)^{-1}} - \sqrt{H_2(I-H_2)^{-1}}V$ for some $z \in \mathbb{C}$. Moreover for any given z in $\mathbb{C}$, the polar decomposition determines U_z and K_z uniquely which give rise to a projection Q_z (in the above form) in $\mathrm{Lat}(\mathrm{Alg}(\mathcal{F}_3))$.

In another word, by fixing Q_∞, we give a coordinate chart of $\mathrm{LatAlg}(\mathcal{F}_3) \setminus \{0, I, Q_\infty\}$.

Lemma 1.1. *Let $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ be two reflexive lattices, $\mathfrak{A} = \mathcal{L}''$ and $\widetilde{\mathfrak{A}} = \widetilde{\mathcal{L}}''$. If $\{P_i\}_{i\in I}$ and $\{Q_i\}_{i\in I}$ are in $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ respectively, and $\mathrm{Lat}(\mathrm{Alg}(\{P_i\}_{i\in I})) = \mathcal{L}$, $\mathrm{Lat}(\mathrm{Alg}(\{Q_i\}_{i\in I})) = \widetilde{\mathcal{L}}$. Let $\varphi \in Aut(\mathfrak{A})$, if $\varphi(\{P_i\}_{i\in I}) = \{Q_i\}_{i\in I}$, then $\varphi(\mathcal{L}) = \widetilde{\mathcal{L}}$.*

Proof. Since we have already showed that if $\mathcal{L}$ is a reflexive lattice, and $\varphi \in Aut(\mathcal{L}'')$, then $\varphi(\mathcal{L})$ is also reflexive. So by the fact that $\varphi(\mathcal{L}) \supseteq \varphi(\{P_i\}_{i\in I}) = \{Q_i\}_{i\in I}$, it is not hard to see

$$\varphi(\mathcal{L}) \supseteq \mathrm{Lat}(\mathrm{Alg}(\{Q_i\}_{i\in I})) = \widetilde{\mathcal{L}}.$$

By considering φ^{-1}, we also have

$$\varphi^{-1}(\widetilde{\mathcal{L}}) \supseteq \mathcal{L},$$

which implies $\widetilde{\mathcal{L}} \supseteq \varphi(\mathcal{L})$, thus $\widetilde{\mathcal{L}} = \varphi(\mathcal{L})$. □

By the symmetry of the three free projections Q_∞, Q_0, Q_{-1}, any permutation of $\{Q_\infty, Q_0, Q_{-1}\}$ will give a new coordinate chart of $\mathcal{L} = \mathrm{LatAlg}(\mathcal{F}_3) \setminus \{0, I\}$. In another word, if $\varphi \in Aut(\mathcal{L}'')$ such that $\varphi(\{Q_\infty, Q_0, Q_{-1}\}) = \{Q_\infty, Q_0, Q_{-1}\}$, then by the above lemma, there exists $f : \mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}$ such that

$$\varphi(Q_z) = Q_{f(z)}.$$

For the rest of this note, we will show what this f is. Before that, we first rewrite the equation (1) as following:

$$
\begin{aligned}
&(1+z)(I-Q_\infty Q_0 Q_\infty)^{-1}[Q_\infty Q_0(I-Q_\infty)]\\
&-z(I-Q_\infty Q_{-1}Q_\infty)^{-1}[Q_\infty Q_{-1}(I-Q_\infty)]\\
&=(I-Q_\infty Q_z Q_\infty)^{-1}[Q_\infty Q_z(I-Q_\infty)].
\end{aligned}\tag{2}
$$

Lemma 1.2. *With the notations above, if* $\varphi(Q_\infty)=Q_\infty$, $\varphi(Q_0)=Q_{-1}$, *and* $\varphi(Q_{-1})=Q_0$, *then* $f(z)=-1-z$.

Proof. By applying φ on both sides of (2), and the definition of f we have

$$
\begin{aligned}
&(1+z)(I-Q_\infty Q_{-1} Q_\infty)^{-1}[Q_\infty Q_{-1}(I-Q_\infty)]\\
&-z(I-Q_\infty Q_0 Q_\infty)^{-1}[Q_\infty Q_0(I-Q_\infty)]\\
&=(I-Q_\infty Q_{f(z)} Q_\infty)^{-1}[Q_\infty Q_{f(z)}(I-Q_\infty)].
\end{aligned}\tag{3}
$$

Compare (2), (3), we have $f(z)=-1-z$. □

Now if we consider $\varphi\in Aut(\mathcal{L}'')$ such that $\varphi(Q_{-1})=Q_{-1}$, $\varphi(Q_\infty)=Q_0$, $\varphi(Q_0)=Q_\infty$, we know that

$$
f(-1)=-1,\qquad f(0)=\infty,\qquad f(\infty)=0.\tag{4}
$$

By the results above, we will expect that f gives a automorphism of the Riemann sphere, thus is a *Möbius* transform, and the only *Möbius* transforms satisfies (4) is $f(z)=\frac{1}{z}$. Next we will show that this is indeed the case.

Lemma 1.3. *With the notations above, if* $\varphi\in Aut(\mathcal{L}'')$ *such that* $\varphi(Q_{-1})=Q_{-1}$, $\varphi(Q_\infty)=Q_0$, $\varphi(Q_0)=Q_\infty$, *then* $f(z)=\frac{1}{z}$.

Proof. Again by applying φ on the both side of (2), we have

$$
\begin{aligned}
&(1+z)(I-Q_0 Q_\infty Q_0)^{-1}[Q_0 Q_\infty(I-Q_0)]\\
&-z(I-Q_0 Q_{-1}Q_0)^{-1}[Q_0 Q_{-1}(I-Q_0)]\\
&=(I-Q_0 Q_{f(z)} Q_0)^{-1}[Q_0 Q_{f(z)}(I-Q_0)].
\end{aligned}\tag{5}
$$

Let $W=\begin{pmatrix}\sqrt{H_1} & \sqrt{I-H_1}\\ \sqrt{I-H_1} & -\sqrt{H_1}\end{pmatrix}(\in\mathcal{L}'')$, $WQ_0W=Q_\infty$ and $WQ_\infty W=Q_0$. Thus

$$
\begin{aligned}
&(1+z)(I-Q_\infty Q_0 Q_\infty)^{-1}[Q_\infty Q_0(I-Q_\infty)]\\
&-z(I-Q_\infty WQ_{-1}WQ_\infty)^{-1}[Q_\infty WQ_{-1}W(I-Q_\infty)]\\
&=(I-Q_\infty WQ_{f(z)}WQ_\infty)^{-1}[Q_\infty WQ_{f(z)}W(I-Q_\infty)].
\end{aligned}\tag{6}
$$

Remember that

$$
Q_z=\begin{pmatrix}K_z & \sqrt{K_z(I-K_z)}U_z\\ U_z^*\sqrt{K_z(I-K_z)} & U_z^*(I-K_z)U_z\end{pmatrix},
$$

By direct computation we know for any $z\in\mathbb{C}$,

$$
\begin{aligned}
(WQ_zW)_{1,1}=&\sqrt{H_1}K_z\sqrt{H_1}+\sqrt{I-H_1}U_z^*\sqrt{K_z(I-K_z)}\sqrt{H_1}\\
&+\sqrt{H_1}\sqrt{K_z(I-K_z)}U_z\sqrt{I-H_1}+\sqrt{I-H_1}U_z^*(I-K_z)U_z\sqrt{I-H_1}.
\end{aligned}
$$

So by (2) we have

$$
\begin{aligned}
I-(WQ_zW)_{1,1} &= \sqrt{H_1}(I-K_z)\sqrt{H_1}-\sqrt{I-H_1}U_z^*\sqrt{K_z(I-K_z)}\sqrt{H_1}\\
&\qquad -\sqrt{H_1}\sqrt{K_z(I-K_z)}U_z\sqrt{I-H_1}+\sqrt{I-H_1}U_z^*K_zU_z\sqrt{I-H_1}\\
&= \sqrt{H_1}(I-K_z)(\sqrt{\frac{H_1}{I-H_1}}-\sqrt{\frac{K_z}{I-K_z}}U_z)\sqrt{I-H_1}\\
&\qquad -\sqrt{I-H_1}U_z^*\sqrt{K_z(I-K_z)}(\sqrt{\frac{H_1}{I-H_1}}-\sqrt{\frac{K_z}{I-K_z}}U_z)\sqrt{I-H_1}\\
&= -z[\sqrt{H_1}(I-K_z)-\sqrt{I-H_1}U_z^*\sqrt{K_z(I-K_z)}]S\sqrt{I-H_1}\\
&= -z\sqrt{I-H_1}(\sqrt{\frac{H_1}{I-H_1}}-U_z^*\sqrt{\frac{K_z}{I-K_z}})(I-K_z)S\sqrt{I-H_1}\\
&= |z|^2\sqrt{I-H_1}S^*(I-K_z)S\sqrt{I-H_1}.
\end{aligned}
$$

Similarly, direct computation and (2) will give

$$
\begin{aligned}
(WQ_zW)_{1,2} &= \sqrt{H_1}K_z\sqrt{I-H_1}+\sqrt{I-H_1}U_z^*\sqrt{K_z(I-K_z)}\sqrt{I-H_1}\\
&\qquad -\sqrt{H_1}\sqrt{K_z(I-K_z)}U_z\sqrt{H_1}-\sqrt{I-H_1}U_z^*(I-K_z)U_z\sqrt{H_1}\\
&= -\sqrt{H_1}(I-K_z)\sqrt{I-H_1}+\sqrt{I-H_1}U_z^*\sqrt{K_z(I-K_z)}\sqrt{I-H_1}\\
&\qquad -\sqrt{H_1}\sqrt{K_z(I-K_z)}U_z\sqrt{H_1}+\sqrt{I-H_1}U_z^*K_zU_z\sqrt{H_1}\\
&= \sqrt{I-H_1}(U_z^*\sqrt{\frac{K_z}{I-K_z}}-\sqrt{\frac{H_1}{I-H_1}})(I-K_z)\sqrt{I-H_1}\\
&\qquad +\sqrt{I-H_1}(U_z^*\sqrt{\frac{K_z}{I-K_z}}-\sqrt{\frac{H_1}{I-H_1}})\sqrt{K_z(I-K_z)}U_z\sqrt{H_1}\\
&= \overline{z}\sqrt{I-H_1}S^*[(I-K_z)\sqrt{I-H_1}+\sqrt{K_z(I-K_z)}U_z\sqrt{H_1}]\\
&= \overline{z}\sqrt{I-H_1}S^*(I-K_z)(\sqrt{I-H_1}+\sqrt{\frac{K_z}{I-K_z}}U_z\sqrt{H_1})\\
&= \overline{z}\sqrt{I-H_1}S^*(I-K_z)(zS\sqrt{H_1}+\frac{I}{\sqrt{I-H_1}}).
\end{aligned}
$$

Thus

(7)

$$
\begin{aligned}
[I-(WQ_zW)_{1,1}]^{-1}(WQ_zW)_{1,2} &= (\frac{1}{|z|^2}\frac{I}{\sqrt{I-H_1}}S^{-1}\frac{I}{I-K_z}S^{*-1}\frac{I}{\sqrt{I-H_1}})\\
&\quad\times\overline{z}\sqrt{I-H_1}S^*(I-K_z)(zS\sqrt{H_1}+\frac{I}{\sqrt{I-H_1}})\\
&= \sqrt{\frac{H_1}{I-H_1}}+\frac{1}{z}\frac{1}{\sqrt{I-H_1}}S^{-1}\frac{1}{\sqrt{I-H_1}}.
\end{aligned}
$$

Let

$$
WQ_{-1}W=\begin{pmatrix} H_2' & \sqrt{H_2'(I-H_2')}V' \\ V'^*\sqrt{H_2'(I-H_2')} & V'^*(I-H_2')V' \end{pmatrix},
$$

by (7) we have

$$\sqrt{\frac{H_2'}{I-H_2'}}V' = \sqrt{\frac{H_1}{I-H_1}} - \frac{1}{\sqrt{I-H_1}}S^{-1}\frac{1}{\sqrt{I-H_1}}.$$

So we may rewrite the equation in (7),

(8)

$$[I-(WQ_{f(z)}W)_{1,1}]^{-1}(WQ_{f(z)}W)_{1,2} = (\frac{1}{f(z)}+1)\sqrt{\frac{H_1}{I-H_1}} - \frac{1}{f(z)}\sqrt{\frac{H_2'}{I-H_2'}}V'.$$

Since Q_∞, Q_0 and $WQ_{-1}W$ are also free, we know that

$$[I-(WQ_{f(z)}W)_{1,1}]^{-1}(WQ_{f(z)}W)_{1,2} = (z+1)\sqrt{\frac{H_1}{I-H_1}} - z\sqrt{\frac{H_2'}{I-H_2'}}V'. \tag{9}$$

(This fact can also be deduced from (6)). Compare (8) and (9), we know $f(z) = \frac{1}{z}$. □

Now we introduce some notations.

Let $\varphi_1 \in Aut(\mathcal{L}'')$ be the automorphism such that $\varphi_1(Q_\infty) = Q_\infty$, $\varphi_1(Q_0) = Q_{-1}$, and $\varphi_1(Q_{-1}) = Q_0$, and f_1 be the map such that $\varphi_1(Q_z) = Q_{f_1(z)}$, then $f_1(z) = -1-z$.

Let $\varphi_2 \in Aut(\mathcal{L}'')$ be the automorphism such that $\varphi_2(Q_{-1}) = Q_{-1}$, $\varphi_2(Q_0) = Q_\infty$, and $\varphi_2(Q_\infty) = Q_0$, and f_2 be the map such that $\varphi_2(Q_z) = Q_{f_2(z)}$, then $f_2(z) = \frac{1}{z}$.

Corollary 1.1. *Let $\varphi_3 \in Aut(\mathcal{L}'')$ be the automorphism such that $\varphi_3(Q_0) = Q_0$, $\varphi_3(Q_\infty) = Q_{-1}$, and $\varphi_3(Q_{-1}) = Q_\infty$, and f_3 be the map such that $\varphi_3(Q_z) = Q_{f_3(z)}$. Since $\varphi_3 = \varphi_1 \circ \varphi_2 \circ \varphi_1$, then $f_3(z) = \frac{-z}{z+1}$.*

Corollary 1.2. *Let $\varphi_4 \in Aut(\mathcal{L}'')$ be the automorphism such that $\varphi_4(Q_0) = Q_{-1}$, $\varphi_4(Q_\infty) = Q_0$, and $\varphi_4(Q_{-1}) = Q_\infty$, and f_4 be the map such that $\varphi_4(Q_z) = Q_{f_4(z)}$. then $f_4(z) = \frac{-1}{z+1}$.*

Remark 1.1. *The results (except lemma 1) in this note depend on the fact that $\{Q_\infty, Q_0, Q_{-1}\}$ are free, because otherwise the automorphism which permutes these three projections may not exist.*

Since $\{I-Q_\infty, I-Q_0, I-Q_{-1}\}$ are three free projections, then the map

$$Q_\infty \to I-Q_\infty, \qquad Q_0 \to I-Q_0, \qquad Q_{-1} \to I-Q_{-1},$$

induce a *-isomorphism of $\mathcal{L}''$. Also we know that $\mathrm{LatAlg}(\{I-Q_\infty, I-Q_0, I-Q_{-1}\}) = \{I-Q | Q \in \mathcal{L}\}$.

Lemma 1.4. *Let φ be the *-isomorphism of $\mathcal{L}''$ such that $\varphi(Q_\infty) = I-Q_\infty$, $\varphi(Q_0) = I-Q_0$, $\varphi(Q_{-1}) = I-Q_{-1}$, then $\varphi(Q_z) = I-Q_{\overline{z}}$.*

Proof. By the discussion above, we know there exists a function $f : \mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}$ such that $\varphi(Q_z) = I-Q_{f(z)}$. Applying φ on both side of the equation (2), we have

$$\begin{aligned}&(1+z)[I-(I-Q_\infty)(I-Q_0)(I-Q_\infty)]^{-1}[(I-Q_\infty)(I-Q_{-1})Q_\infty]\\ &-z[I-(I-Q_\infty)(I-Q_{-1})(I-Q_0)]^{-1}[(I-Q_\infty)(I-Q_{-1})Q_\infty]\\ &=[I-(I-Q_\infty)(I-Q_{f(z)})(I-Q_\infty)]^{-1}[(I-Q_\infty)(I-Q_{f(z)})Q_\infty],\end{aligned}$$

which implies

$$(1+z)\sqrt{H_1(I-H_1)^{-1}} - zV^*\sqrt{H_2(I-H_2)^{-1}} = U^*_{f(z)}\sqrt{K_{f(z)}(I-K_{f(z)})^{-1}}.$$

Thus by (1), we have $f(z) = \overline{z}$. $\square$

Corollary 1.3. $\|Q_{z_1} - Q_{z_2}\|_2 = \|Q_{\overline{z}_1} - Q_{\overline{z}_2}\|_2$.

Proof. By the notations in the above lemma, we have

$$\|Q_{z_1} - Q_{z_2}\|_2 = \|\varphi(Q_{z_1}) - \varphi(Q_{z_2})\|_2 = \|(I - Q_{\overline{z}_1}) - (I - Q_{\overline{z}_2})\|_2 = \|Q_{\overline{z}_1} - Q_{\overline{z}_2}\|_2.$$

$\square$

2. DISTANCE FORMULA

In this section, we will compute the distance between $\| Q_z - Q_\infty \|_2$.Note that

$$\| Q_z - Q_\infty \|_2^2 = 1 - 2tr(Q_z Q_\infty) = 1 - \tau(Q_\infty Q_z Q_\infty|_{Q_\infty \mathcal{H}}),$$

where tr is the trace on $\mathcal{F}_3''$, and τ is the trace on $Q_\infty \mathcal{F}_3'' Q_\infty$, such that

$$tr(\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}) = \frac{1}{2}(\tau(A_{11}) + \tau(A_{22})).$$

Remember that

$$Q_z = \begin{pmatrix} K_z & \sqrt{K_z(I-K_z)}U_z \\ U_z^*\sqrt{K_z(I-K_z)} & U_z^*(I-K_z)U_z \end{pmatrix},$$

where

$$\sqrt{K_z(I-K_z)^{-1}}U_z = (1+z)\sqrt{H_1(I-H_1)^{-1}} - z\sqrt{H_2(I-H_2)^{-1}}V.$$

In order to compute $\tau(K_z)$, we will use the technique developed by Haagerup and Schultz in [**?**].

We first introduce some notations.

Let $\mathfrak{A}$ be a finite von Neumann algebra with faithful, tracial state τ, we will use $\widetilde{\mathfrak{A}}$ to denote the set of closed, densely defined operators affiliated with $\mathfrak{A}$. Recall that every operator $T \in \widetilde{\mathfrak{A}}$ has a polar decomposition

$$T = U|T| = U\int_0^\infty t dE_{|T|}(t),$$

where $U \in \mathfrak{A}$ is a unitary, and the spectral measure $E_{|T|}$ takes values in $\mathfrak{A}$. In particular, for $T \in \widetilde{\mathfrak{A}}$ we may define $\mu_{|T|} \in Prob(\mathbb{R})$ by

$$\mu_{|T|}(B) = \tau(E_{|T|}(B)), (B \in \mathbb{B}),$$

where (B) is set of Borel subset of $\mathbb{R}$.

Definition 2.1. *For $\mu \in Prob(\mathbb{R}, \mathbb{B})$ let $\widetilde{\mu}$ denote the symmetrization of μ. That is $\widetilde{\mu} \in Prob(\mathbb{R}, \mathbb{B})$ is given by*

$$\widetilde{\mu}(B) = \frac{1}{2}(\mu(B) + \mu(-B)), (B \in \mathbb{B}).$$

Proposition 2.1 (Proposition 3.11 in [**?**])**.** *Let S, $T \in \widetilde{\mathfrak{A}}$ be *-free R-diagonal elements. Then*

$$\widetilde{\mu}_{|S+T|} = \widetilde{\mu}_{|S|} \boxplus \widetilde{\mu}_{|T|}.$$

Since we may treat K_z as the summation of two *-free R-diagonal elements, then by the above proposition, we can compute the distribution of K_z.

Lemma 2.1. *By the notation above, we have*

$$d\mu_{\sqrt{\frac{K_z}{I-K_z}}}(t) = \frac{2}{\pi}\frac{|z|+|z+1|}{t^2+(|z|+|z+1|)^2}1_{(0,\infty)}(t)dt.$$

Proof. Since the distribution of $H_i (i=1,\,2)$ has the same distribution as $cos^2\frac{\pi}{2}\theta$ on $[0,1]$, thus $\sqrt{\frac{H_i}{I-H_i}}$ has the same distirubtion as $cot\frac{\pi}{2}\theta$ on $[0,1]$. We have

$$d\mu_{\sqrt{\frac{H_i}{I-H_i}}}(t) = \frac{2}{\pi}\frac{1}{1+t^2}1_{(0,\infty)}dt.$$

Then for any $r \in \mathbb{R}^+$,

$$d\mu_r(t) = d\mu_{(r\sqrt{\frac{H_i}{I-H_i}})}(t) = \frac{2}{\pi}\frac{r}{r^2+t^2}1_{(0,\infty)}dt.$$

Thus we have

$$d\widetilde{\mu_r}(t) = \frac{1}{\pi}\frac{r}{r^2+t^2}1_{(-\infty,+\infty)}dt.$$

By the definition and Corollary 5.8 in [**?**],

$$\begin{aligned} G_{\widetilde{\mu_r}}(z) &= \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{1}{z-t}\frac{r}{r^2+t^2}dt, (Imz>0) \\ &= \frac{1}{z+ri}, \end{aligned}$$

and $F_{\widetilde{\mu_r}}(z) = \frac{1}{G_{\widetilde{\mu_r}}(z)} = z+ri$, $\varphi_{\widetilde{\mu_r}}(z) = F^{-1}_{\widetilde{\mu_r}}(z) - z = -ri$. Then by proposition 2.1, we have for any $z \in \mathbb{C}$,

$$\begin{aligned} \varphi_{\widetilde{\mu_{|z|}}}(t) + \varphi_{\widetilde{\mu_{|z+1|}}}(t) &= -i(|z|+|z+1|), \\ F_{\widetilde{\mu_{|z|}}\boxplus\widetilde{\mu_{|z+1|}}}(t) &= t + i(|z|+|z+1|), \\ G_{\widetilde{\mu_{|z|}}\boxplus\widetilde{\mu_{|z+1|}}}(t) &= \frac{1}{t+i(|z|+|z+1|)}, \end{aligned}$$

therefore

$$\begin{aligned} d\widetilde{\mu_{\sqrt{\frac{K_z}{I-K_z}}}}(t) = d\widetilde{\mu_{|z|}} \boxplus \widetilde{\mu_{|z+1|}}(t) &= -\frac{1}{\pi}\lim_{u\to 0+} ImG_{\widetilde{\mu_{|z|}}\boxplus\widetilde{\mu_{|z+1|}}}(t+iu) \\ &= \frac{1}{\pi}\lim_{u\to 0+}\frac{|z|+|z+1|+u}{t^2+(|z|+|z+1|+u)^2}1_{(-\infty,+\infty)}dt \\ &= \frac{1}{\pi}\frac{|z|+|z+1|}{t^2+(|z|+|z+1|)^2}1_{(-\infty,+\infty)}dt, \end{aligned}$$

and

$$d\mu_{\sqrt{\frac{K_z}{I-K_z}}}(t) = \frac{2}{\pi}\frac{|z|+|z+1|}{t^2+(|z|+|z+1|)^2}1_{(0,+\infty)}dt.$$

□

Corollary 2.1. *Q_z is free with Q_∞ if and only if $z \in [-1,0]$.*

Lemma 2.2. *With the notations above,* $\| Q_z - Q_\infty \|_2 = \sqrt{\frac{1}{1+|z|+|z+1|}}$.

Proof. By the above lemma, let $\Delta_z = |z| + |z+1|$,

$$\begin{aligned}\tau(K_z) &= \frac{2}{\pi}\int_0^{+\infty}\frac{t^2}{t^2+1}\frac{|z|+|z+1|}{t^2+(|z|+|z+1|)^2}dt\\ &= \frac{\Delta_z}{\pi}\int_{-\infty}^{+\infty}(1-\frac{1}{t^2+1})\frac{1}{t^2+\Delta_z^2}dt\\ &= \frac{\Delta_z}{2i\pi}[\int_{-\infty}^{+\infty}\frac{2i}{t^2+\Delta_z^2}dt - \int_{-\infty}^{+\infty}\frac{2i}{(t^2+1)(t^2+\Delta_z^2)}dt]\\ &= \Delta_z[\frac{1}{\Delta_z} - \frac{1}{\Delta_z(1-\Delta_z^2)} - \frac{1}{\Delta_z^2-1}] = 1 - \frac{1}{1+\Delta_z}.\end{aligned}$$

Thus

$$\| Q_z - Q_\infty \|_2^2 = 1 - \tau(K_z) = \frac{1}{1+|z|+|z+1|}.$$

□

Corollary 2.2. *For any $z \in \mathbb{C}$, we have $\| Q_z - Q_0 \|_2 = \sqrt{\frac{|z|}{1+|z|+|z+1|}}$, and $\| Q_z - Q_{-1} \|_2 = \sqrt{\frac{|z+1|}{1+|z|+|z+1|}}$.*

Proof. Let $\varphi_2 \in Aut(\mathcal{L}'')$ be the automorphism such that $\varphi_2(Q_{-1}) = Q_{-1}$, $\varphi_2(Q_0) = Q_\infty$, and $\varphi_2(Q_\infty) = Q_0$, and f_2 be the map such that $\varphi_2(Q_z) = Q_{f_2(z)}$, then $f_2(z) = \frac{1}{z}$. Thus we have

$$\sqrt{\frac{1}{1+|z|+|z+1|}} = \| Q_{f_2(z)} - Q_0 \|_2 = \| Q_{\frac{1}{z}} - Q_0 \|_2,$$

which implies that $\| Q_z - Q_0 \|_2 = \sqrt{\frac{|z|}{1+|z|+|z+1|}}$. Similarly we can prove the other equation. □

3. METRIC

By the information we have, it's tempting to give the following distance formula for any two points on the Riemann sphere. For any z_1, $z_2 \in \mathbb{C}$,

$$dist(z_1, z_2) = \sqrt{\frac{2|z_1 - z_2|}{(1+|z_1|+|z_1+1|)(1+|z_2|+|z_2+1|)}},$$

and

$$dis(\infty, z_2) = \sqrt{\frac{1}{(1+|z_2|+|z_2+1|)}}.$$

Lemma 3.1. *dist is a metric on the Riemann sphere, i.e. $\mathbb{C} \cup \{\infty\}$.*

Proof. We will show that for any z_1, z_2, $z_3 \in \mathbb{C}$, we have

$$dist(z_1, z_3) \leq dist(z_1, z_2) + dist(z_2, z_3).$$

By the definition of *dist*, we only need to show that

(10)
$$\sqrt{|z_1 - z_3|(1+|z_2|+|z_2+1|)} \leq \sqrt{|z_1 - z_2|(1+|z_3|+|z_3+1|)} + \sqrt{|z_2 - z_3|(1+|z_1|+|z_1+1|)}.$$

Since

$$|z_1 - z_3| \le |z_1 - z_2| + |z_2 - z_3|;$$
$$|z_2||z_1 - z_3| \le |z_3||z_1 - z_2| + |z_1||z_2 - z_3|;$$
$$|z_2 + 1||z_1 - z_3| \le |z_3 + 1||z_1 - z_2| + |z_1 + 1||z_2 - z_3|,$$

we have

$|z_1 - z_3|(1 + |z_2| + |z_2 + 1|) \le |z_1 - z_2|(1 + |z_3| + |z_3 + 1| + |z_2 - z_3|(1 + |z_1| + |z_1 + 1|,$

which implies (10). Similarly we can show the cases involve ∞. □

Lemma 3.2. *Let (x_1, y_1) be the point on the ellipse $\frac{x^2}{a_1^2} + \frac{y^2}{a_1^2-1} = 1$, and (x_2, y_2) be the point on the ellipse $\frac{x^2}{a_2^2} + \frac{y^2}{a_2^2-1} = 1$, where $a_2 > a_1 > 1$, then $max(\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}) = a_2 + a_1$, when $(x_1, y_1) = (-a_1, 0)$ and $(x_2, y_2) = (a_2, 0)$.*

Proof. Although you may compute this value by the Lagrange Multipliers, we will show this fact directly. Let $(x_1, y_1) = (a_1 cos\theta_1, b_1 sin\theta_1)$, and $(x_2, y_2) = (a_2 cos\theta_2, b_2 sin\theta_2)$, where $b_i = \sqrt{a_i^2 - 1}$ and $\theta_i \in [0, 2\pi)$ $(i = 1, 2)$. It is not hard to see that

$$\begin{aligned}&(a_2 cos\theta_2 - a_1 cos\theta_1)^2 + (b_2 sin\theta_2 - b_1 sin\theta_1)^2\\ &= a_2^2 cos\theta_2{}^2 + b_2^2 sin\theta_2{}^2 + a_1^2 cos\theta_1{}^2 + b_1^2 sin\theta_1{}^2 - 2a_1a_2 cos\theta_2 cos\theta_1 - 2b_1b_2 sin\theta_2 sin\theta_1\\ &\le (a_1 + a_2)^2.\end{aligned}$$

□

Lemma 3.3. *$dist(z_1, z_2) \le \frac{1}{\sqrt{2}}$.*

Proof. We need to show

$$4|z_1 - z_2| \le (1 + |z_1| + |z_1 + 1|)(1 + |z_2| + |z_2 + 1|).$$

Assume z_1 and z_2 are on the ellipses $|z| + |z + 1| = r_1$, and $|z| + |z + 1| = r_2$ respectively, where $r_2 \ge r_1 \ge 1$. By the above lemma, we have

$$\begin{aligned}&LHS = 4|z_1 - z_2| \le 4|\frac{r_2 - 1}{2} + \frac{r_1 + 1}{2}| = 2(r_1 + r_2)\\ &RHS = 1 + r_1r_2 + r_1 + r_2\\ &RHS - LHS \ge (r_1 - 1)(r_2 - 1) \ge 0.\end{aligned}$$

□

Remark 3.1. *By the proof of the lemma above, it is not hard to see that $dist(z_1, z_2) = \frac{1}{\sqrt{2}}$ if and only if $z_1 = \infty$ and $z_2 \in [-1, 0]$, or $z_1 = 0$ and $z_2 \in [-\infty, -1]$ or $z_1 = -1$ and $z_2 \in [0, +\infty]$.*

A metric d is called a Ptolemaic metric if

$$d(z_1, z_2)d(z_3, z_4) \le d(z_2, z_4)d(z_1, z_3) + d(z_1, z_4)d(z_2, z_3).$$

For example the Chordal metric on S^2 given by:

$$d(z_1, z_2) = \frac{2|z_1 - z_2|}{\sqrt{1 + |z_1|^2}\sqrt{1 + |z_2|^2}}$$

is a Ptolemaic metric.

Lemma 3.4. *With the notation above, $dist(z_1, z_2)$ is a Ptolemaic metric.*

Let (M, d) be a metric space. We may define a new metric d_I on M, known as the induced intrinsic metric, as follows: $d_I(x, y)$ is the infimum of the lengths of all paths form x to y. Here a path form x to y is a continuous map $\gamma : [0, 1] \to M$ with $\gamma(0) = x$ and $\gamma(1) = y$. We set $d_I(x, y) = \infty$ if there is no path of finite length from x to y.

If$d_I(x, y) = d(x, y)$ for all points x and y in M, we say that (M, d) is a length space or a path metric space and the metric d is intrinsic.

Lemma 3.5. *dist is not intrinsic.*

Proof. A metric d is intrinsic if it has approximate midpoints, i.e. for any $\epsilon > 0$ and any pair of points x and y in M there exists c in M such that $d(x, c)$ and $d(c, y)$ are both smaller than $d(x, y)/2 + \epsilon$. We will show that *dist* do not has approximate midpoints for $z_1 = -1$ and $z_2 = \infty$.

If there exists z such that (note $\frac{1}{6} > \frac{1}{(2\sqrt{2})^2}$)

$$dist^2(-1, z) = \frac{|z+1|}{1 + |z| + |z+1|} \leq \frac{1}{6},$$
$$dist^2(\infty, z) = \frac{1}{1 + |z| + |z+1|} \leq \frac{1}{6},$$

then we have

$$5|z| - 5 \leq 5|z+1| \leq 1 + |z| \Longrightarrow |z| \leq \frac{3}{2},$$
$$6 \leq 1 + |z| + |z+1| \leq 2 + 2|z| \to |z| \geq 2,$$

thus we have a contradiction. ☐

4. GENERAL CASE

Actually the proof in section 1 works for general case.

Lemma 4.1. *With the notations in section 1, if $\varphi \in Aut(\mathcal{L}'')$ such that $\varphi(Q_0) = Q_{z_1}$, $\varphi(Q_\infty) = Q_{z_2}$, $\varphi(Q_{-1}) = Q_{z_3}$, then $f(z) = \frac{zz_2(z_3-z_1)+z_1(z_3-z_2)}{z(z_3-z_1)+(z_3-z_2)}$, the unique Möbius transform satisfies $f(0) = z_1$, $f(\infty) = z_2$, $f(-1) = z_3$.*

Proof. Again by applying φ on the both side of (2), we have

$$\begin{aligned}(11)\qquad &(1+z)(I - Q_{z_2}Q_{z_1}Q_{z_2})^{-1}[Q_{z_2}Q_{z_1}(I - Q_{z_2})] \\ &- z(I - Q_{z_2}Q_{z_3}Q_{z_2})^{-1}[Q_{z_2}Q_{z_3}(I - Q_{z_2})] \\ &= (I - Q_{z_2}Q_{f(z)}Q_{z_2})^{-1}[Q_{z_2}Q_{f(z)}(I - Q_{z_2})].\end{aligned}$$

Let $W = \begin{pmatrix} \sqrt{K_{z_2}} & \sqrt{I - K_{z_2}}U_{z_2} \\ U_{z_2}^*\sqrt{I - K_{z_2}} & -U_{z_2}^*\sqrt{K_{z_2}}U_{z_2} \end{pmatrix} (\in \mathcal{L}'')$, $WQ_{z_2}W = Q_\infty$ and $WQ_\infty W = Q_{z_2}$. Thus

$$\begin{aligned}(12)\qquad &(1+z)(I - Q_\infty WQ_{z_1}WQ_\infty)^{-1}[Q_\infty WQ_{z_1}W(I - Q_\infty)] \\ &- z(I - Q_\infty WQ_{z_3}WQ_\infty)^{-1}[Q_\infty WQ_{z_3}W(I - Q_\infty)] \\ &= (I - Q_\infty WQ_{f(z)}WQ_\infty)^{-1}[Q_\infty WQ_{f(z)}W(I - Q_\infty)].\end{aligned}$$

Remember that

$$Q_z = \begin{pmatrix} K_z & \sqrt{K_z(I-K_z)}U_z \\ U_z^*\sqrt{K_z(I-K_z)} & U_z^*(I-K_z)U_z \end{pmatrix},$$

By direct computation we know for any $z \in \mathbb{C}$,

$$\begin{aligned}(WQ_zW)_{1,1} = &\sqrt{K_{z_2}}K_z\sqrt{K_{z_2}} + \sqrt{I-K_{z_2}}U_{z_2}U_z^*\sqrt{K_z(I-K_z)}\sqrt{K_{z_2}} \\ &+ \sqrt{K_{z_2}}\sqrt{K_z(I-K_z)}U_zU_{z_2}^*\sqrt{I-K_{z_2}} + \sqrt{I-K_{z_2}}U_{z_2}U_z^*(I-K_z)U_zU_{z_2}^*\sqrt{I-K_{z_2}}.\end{aligned}$$

So by (2) we have

$$\begin{aligned}I-(WQ_zW)_{1,1} &= \sqrt{K_{z_2}}(I-K_z)\sqrt{K_{z_2}} - \sqrt{I-K_{z_2}}U_{z_2}U_z^*\sqrt{K_z(I-K_z)}\sqrt{K_{z_2}} \\ &\quad - \sqrt{K_{z_2}}\sqrt{K_z(I-K_z)}U_zU_{z_2}^*\sqrt{I-K_{z_2}} + \sqrt{I-K_{z_2}}U_{z_2}U_z^*K_zU_zU_{z_2}^*\sqrt{I-K_{z_2}} \\ &= \sqrt{K_{z_2}}(I-K_z)(\sqrt{\frac{K_{z_2}}{I-K_{z_2}}}U_{z_2} - \sqrt{\frac{K_z}{I-K_z}}U_z)U_{z_2}^*\sqrt{I-K_{z_2}} \\ &\quad - \sqrt{I-K_{z_2}}U_{z_2}U_z^*\sqrt{K_z(I-K_z)}(\sqrt{\frac{K_{z_2}}{I-K_{z_2}}}U_{z_2} - \sqrt{\frac{K_z}{I-K_z}}U_z)U_{z_2}^*\sqrt{I-K_{z_2}} \\ &= (z_2-z)[\sqrt{K_{z_2}}(I-K_z) - \sqrt{I-K_{z_2}}U_{z_2}U_z^*\sqrt{K_z(I-K_z)}]SU_{z_2}^*\sqrt{I-K_{z_2}} \\ &= (z_2-z)\sqrt{I-K_{z_2}}U_{z_2}(U_{z_2}^*\sqrt{\frac{K_{z_2}}{I-K_{z_2}}} - U_z^*\sqrt{\frac{K_z}{I-K_z}})(I-K_z)SU_{z_2}^*\sqrt{I-H_1} \\ &= |z_2-z|^2\sqrt{I-K_{z_2}}U_{z_2}S^*(I-K_z)SU_{z_2}^*\sqrt{I-K_{z_2}}.\end{aligned}$$

Similarly, direct computation and (2) will give

$$\begin{aligned}(WQ_zW)_{1,2} &= \sqrt{K_{z_2}}K_z\sqrt{I-K_{z_2}}U_{z_2} + \sqrt{I-K_{z_2}}U_{z_2}U_z^*\sqrt{K_z(I-K_z)}\sqrt{I-K_{z_2}}U_{z_2} \\ &\quad - \sqrt{K_{z_2}}\sqrt{K_z(I-K_z)}U_zU_{z_2}^*\sqrt{K_{z_2}}U_{z_2} - \sqrt{I-K_{z_2}}U_{z_2}U_z^*(I-K_z)U_zU_{z_2}^*\sqrt{K_{z_2}}U_{z_2} \\ &= -\sqrt{K_{z_2}}(I-K_z)\sqrt{I-K_{z_2}}U_{z_2} + \sqrt{I-K_{z_2}}U_{z_2}U_z^*\sqrt{K_z(I-K_z)}\sqrt{I-K_{z_2}}U_{z_2} \\ &\quad - \sqrt{K_{z_2}}\sqrt{K_z(I-K_z)}U_zU_{z_2}^*\sqrt{K_{z_2}}U_{z_2} + \sqrt{I-K_{z_2}}U_{z_2}U_z^*K_zU_zU_{z_2}^*\sqrt{K_{z_2}}U_{z_2} \\ &= \sqrt{I-K_{z_2}}U_{z_2}(U_z^*\sqrt{\frac{K_z}{I-K_z}} - U_{z_2}^*\sqrt{\frac{K_{z_2}}{I-K_{z_2}}})(I-K_z)\sqrt{I-K_{z_2}}U_{z_2} \\ &\quad + \sqrt{I-K_{z_2}}U_{z_2}(U_z^*\sqrt{\frac{K_z}{I-K_z}} - U_{z_2}^*\sqrt{\frac{K_{z_2}}{I-K_{z_2}}})\sqrt{K_z(I-K_z)}U_zU_{z_2}^*\sqrt{K_{z_2}}U_{z_2} \\ &= \overline{(z-z_2)}\sqrt{I-K_{z_2}}U_{z_2}S^*[(I-K_z)\sqrt{I-K_{z_2}}U_{z_2} + \sqrt{K_z(I-K_z)}U_zU_{z_2}^*\sqrt{K_{z_2}}U_{z_2}] \\ &= \overline{(z-z_2)}\sqrt{I-K_{z_2}}U_{z_2}S^*(I-K_z)(\sqrt{I-K_{z_2}}U_{z_2} + \sqrt{\frac{K_z}{I-K_z}}U_zU_{z_2}^*\sqrt{K_{z_2}}U_{z_2}) \\ &= \overline{(z-z_2)}\sqrt{I-K_{z_2}}U_{z_2}S^*(I-K_z)[(z-z_2)SU_{z_2}^*\sqrt{K_{z_2}}U_{z_2} + \frac{I}{\sqrt{I-K_{z_2}}}U_{z_2}].\end{aligned}$$

Thus

(13)

$$[I-(WQ_zW)_{1,1}]^{-1}(WQ_zW)_{1,2} =$$
$$(\frac{1}{|z-z_2|^2}\frac{I}{\sqrt{I-K_{z_2}}}U_{z_2}S^{-1}\frac{I}{I-K_z}S^{*-1}U^*_{z_2}\frac{I}{\sqrt{I-K_{z_2}}}$$
$$\times\overline{(z-z_2)}\sqrt{I-K_{z_2}}U_{z_2}S^*(I-K_z)[(z-z_2)SU^*_{z_2}\sqrt{K_{z_2}}U_{z_2}+\frac{I}{\sqrt{I-K_{z_2}}}U_{z_2}]$$
$$=\sqrt{\frac{K_{z_2}}{I-K_{z_2}}}U_{z_2}+\frac{1}{(z-z_2)}\frac{I}{\sqrt{I-K_{z_2}}}U_{z_2}S^{-1}\frac{I}{\sqrt{I-K_{z_2}}}U_{z_2}.$$

by (12) and (13) we have

$$(1+z)[\sqrt{\frac{K_{z_2}}{I-K_{z_2}}}U_{z_2}+\frac{1}{z_1-z_2}\frac{I}{\sqrt{I-K_{z_2}}}U_{z_2}S^{-1}\frac{I}{\sqrt{I-K_{z_2}}}U_{z_2}]$$
$$-z[\sqrt{\frac{K_{z_2}}{I-K_{z_2}}}U_{z_2}+\frac{1}{z_3-z_2}\frac{I}{\sqrt{I-K_{z_2}}}U_{z_2}S^{-1}\frac{I}{\sqrt{I-K_{z_2}}}U_{z_2}]$$
$$=\sqrt{\frac{K_{z_2}}{I-K_{z_2}}}U_{z_2}+\frac{1}{f(z)-z_2}\frac{I}{\sqrt{I-K_{z_2}}}U_{z_2}S^{-1}\frac{I}{\sqrt{I-K_{z_2}}}U_{z_2}$$

thus

$$\frac{1}{f(z)-z_2}=\frac{1+z}{z_1-z_2}-\frac{z}{z_3-z_2},$$

which implies that

$$f(z)=\frac{zz_2(z_3-z_1)+z_1(z_3-z_2)}{z(z_3-z_1)+(z_3-z_2)}.$$

□

Let G be the group generated by $\{\varphi_1,\varphi_2\}$. We want to determine the group $G[\mathcal{L}]=\{\varphi\in Aut(\mathcal{L}'')|\varphi(\mathcal{L})=\mathcal{L}\}$.

By lemma 4.1, we know that each $\varphi\in Aut(\mathcal{L}'')\setminus G$ satisfying $\varphi(\mathcal{L})=\mathcal{L}$ is associate with a *Möbius* transformation f such that $\varphi(Q_z)=Q_{f(z)}$. For the rest of this paper, we will denote the set of *Möbius* transformation by $\mathfrak{M}$, and $\mathfrak{M}[\mathcal{L}]=\{f\in\mathfrak{M}|\exists\varphi\in G[\mathcal{L}]$ such that $\varphi(Q_z)=Q_{f(z)}\}$.

Recall some basic facts about *Möbius* transformation.

For any $A\in GL_2(\mathbb{C})$, the *Möbius* transformation is defined as

$$A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\longrightarrow g_A(z)=\frac{az+b}{cz+d}.$$

And we can extend g_A to $\widehat{\mathbb{R}}^3=\{z+tj|z\in\mathbb{C},t\in\mathbb{R}\}\cup\{\infty\}(H^3=\{z+tj|z\in\mathbb{C},t>0\})$ by *Poincaré* extension:

$$g_A(z+tj)=\frac{(az+b)\overline{(cz+d)}+a\bar{c}t^2+|ad-bc|tj}{|cz+d|^2+|c|^2t^2}. \tag{14}$$

Definition 4.1. *Let $g(\neq I)$ be any Möbius transformation. We say*

(1) *g is parabolic if and only if g has a unique fixed point in $\mathbb{C}\cup\{\infty\}$;*

(2) *g is loxodromic if and only if g exactly two fixed points in $\widehat{\mathbb{R}}^3$;*

(3) *g is elliptic if and only if g has infinitely many fixed points in $\widehat{\mathbb{R}}^3$.*

We have the following theorem ([**?**])

Theorem 4.1. *Let $g(\neq I)$ be any Möbius transformation. Then*

(1) *if g is parabolic with fixed point α. Then for all z in $\mathbb{C}\cup\{\infty\}$, we have $g^n(z)\to\alpha$ as $n\to+\infty$. The convergence being uniform on compact subsets of $\mathbb{C}\setminus\{\alpha\}$.*
(2) *if g is loxodromic, then the fixed points α and β of g can be labeled so that $g^n(z)\to\alpha$ as $n\to+\infty$. The convergence being uniform on compact subsets of $\mathbb{C}\setminus\{\beta\}$.*
(3) *if g is elliptic with fexed points α and β, then g leaves invariant each circle for which α and β are inverse points.*

Lemma 4.2. *With the notations above, if $\varphi\in Aut(\mathcal{L}'')\setminus G$ such that $\varphi(\mathcal{L})=\mathcal{L}$. Then the corresponding Möbius transformation f must be elliptic.*

Proof. If f is not elliptic by the lemma above, let $z_1=f(\infty)$, $z_2=f(0)$, $z_3=f(-1)$. Two of $f^n(z_1)$, $f^n(z_2)$ and $f^n(z_3)$ will converge to the same point. Assume $f^n(z_1)$, $f^n(z_2)$ converge to the same point. Thus $\parallel Q_{f^n(z_1)}-Q_{f^n(z_2)}\parallel_2\to 0$, but $Q_{f^n(z_1)}$ and $Q_{f^n(z_2)}$ are free, so $\parallel Q_{f^n(z_1)}-Q_{f^n(z_2)}\parallel_2=\frac{1}{\sqrt{2}}$. $\square$

Theorem 4.2 ([**?**])**.** *A subgroup G of Möbius transformation contains only elliptic elements (and I) if and only if the elements of G have a common fixed point in H^3.*

The common fixed point of f_1 and f_2:

Let $A_1=\begin{pmatrix}1&1\\0&-1\end{pmatrix}$, then $g_{A_1}=f_1$. By (14), we have the fixed points of f_1 in H^3 satisfies:

$$-z-1+tj=z+tj,$$

which implies the fixed points of f_1 are $\{t|\frac{-1}{2}+tj\}$.

Let $A_2=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, then $g_{A_2}=f_2$. By (14), we have the fixed points of f_2 in H^3 satisfies:

$$\frac{\overline{z}+tj}{|z|^2+t^2}=z+tj,$$

this implies $|z|^2+t^2=1(t>0)$ and $z=\overline{z}$, thus the fixed points of f_2 in H^3 are $\{x+tj|x\in\mathbb{R},x^2+t^2=1\}$.

Combine these facts we deduce that the common fixed point of f_1 and f_2 are $\frac{-1}{2}+\frac{\sqrt{3}}{2}j$.

So the group we are looking for is a subgroup of the subgroup of *Möbius* transformations that fixed $\frac{-1}{2}+\frac{\sqrt{3}}{2}j$. We denote $\{g|g\in\mathfrak{M},g(\frac{-1}{2}+\frac{\sqrt{3}}{2}j)=\frac{-1}{2}+\frac{\sqrt{3}}{2}j\}$ by $\mathcal{G}$. Then $\mathfrak{M}[\mathcal{L}]\subseteq\mathcal{G}$.

Next we shall map H^3 onto $B^3=\{(x,y,t)|x^2+y^2+t^2<1\}$ and $\frac{-1}{2}+\frac{\sqrt{3}}{2}j$ to $(0,0,0)$.

Let

$$\phi((x,y,t))=(\frac{2x}{x^2+y^2+(t+1)^2},\frac{2y}{x^2+y^2+(t+1)^2},\frac{x^2+y^2+t^2-1}{x^2+y^2+(t+1)^2}).$$

For the points in $\mathbb{R}^2$ i.e $t = 0$,

$$\phi((x,y,0)) = (\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}).$$

$$(\phi^{-1}((x,y,t)) = (\frac{x}{1-t}, \frac{y}{1-t}, 0), \qquad \forall (x,y,t) \in S^2)$$

Thus ϕ map H^3 onto $B^3 = \{(x,y,t)|x^2+y^2+t^2 = 1\}$, and $\mathbb{C} \cup \{\infty\}$ onto S^2 ($\phi(\infty) = (0,0,1)$).

Let $g \in \mathfrak{M}$ be

$$g(z) = \frac{z-1}{\sqrt{3}z+\sqrt{3}}, \qquad g^{-1}(z) = \frac{\sqrt{3}z+1}{-\sqrt{3}z+1},$$

then

$$f_1'(z) = gf_1g^{-1}(z) = \frac{-z+\sqrt{3}}{\sqrt{3}z+1}, \qquad f_2'(z) = gf_2g^{-1}(z) = -z.$$

We also denote the *Poincaré* extension of g by g. It is not hard to check that $g(\frac{-1}{2} + \frac{\sqrt{3}}{2}j) = j$, and $\phi((0,0,1)) = (0,0,0)$. Thus $\phi' = \phi \circ g$ map H^3 onto B^3 and $\frac{-1}{2} + \frac{\sqrt{3}}{2}j$ to $(0,0,0)$, and $\phi'\mathcal{G}\phi'^{-1} = SO(3)$ ([**?**] Theorem 3.4.1). Also note that $\phi'(\{(x,0,0)|x \in \mathbb{R}\}) = \{(x,0,z)|x^2+z^2 = 1\}$.

We have

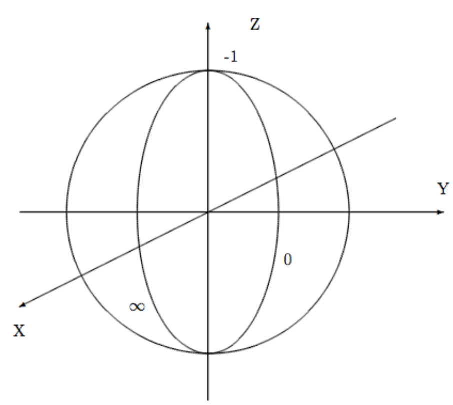

FIGURE 1

$$\phi'(\infty) = (\frac{\sqrt{3}}{2}, 0, -\frac{1}{2}), \qquad \phi'(0) = (\frac{-\sqrt{3}}{2}, 0, -\frac{1}{2}), \qquad \phi'(-1) = (0,0,1),$$

$$\phi'(\frac{-1}{2} + \frac{\sqrt{3}}{2}i) = (0,1,0), \qquad \phi'(\frac{-1}{2} - \frac{\sqrt{3}}{2}i) = (0,-1,0), \qquad \phi'(1) = (0,0,-1),$$

then $\phi' f_4 \phi'^{-1}$ is a counterclockwise rotation about the positive y-axis by angle $\frac{2\pi}{3}$, and $\phi' f_2 \phi'^{-1}$ is a counterclockwise rotation about the positive z-axis by angle π.

Lemma 4.3. *If $\varphi \in G[\mathcal{L}] \setminus \{I\}$ be the automorphism such that $\varphi(Q_{-1}) = Q_{-1}$, then $\varphi = \varphi_2$.*

Proof. Let $f \in \mathfrak{M}$ be the *Möbius* transform such that $\varphi(Q_z) = Q_{f(z)}$. Then $\phi' f \phi'^{-1} (\in SO(3))$ will fix $(0,0,0)$ and $(0,0,1)$, thus $\phi' f \phi'^{-1}$ is a rotation about the z-axis. Let $z_1 = f(\infty)$, $z_2 = f(0)$. Since $\{Q_{-1}, Q_{z_1}, Q_{z_2}\}$ must be free, we have z_1, z_2 are in $[0, +\infty)$, then $\phi'(z_1)$ and $\phi'(z_2)$ are in xz-Plane. and the only non-trivial rotation about the z-axis that map $(\frac{\sqrt{3}}{2}, 0, -\frac{1}{2}) (= \phi'(\infty))$ onto some point in xz-Plane is $\phi' f_2 \phi'^{-1}$, thus $\varphi = \varphi_2$. $\square$

Let

$$\mathcal{D} = \{z | \exists z_1, z_2 \in \mathbb{C} \cup \{\infty\}, \text{ such that } \{Q_z, Q_{z_1}, Q_{z_2}\} \text{ are 3 free projections}\}.$$

Corollary 4.1. *For any $z \in \mathcal{D}$, there exist one and only one $\varphi \in G[\mathcal{L}] \setminus \{I\}$ such that $\varphi(Q_z) = Q_z$.*

Corollary 4.2. $\phi' \mathfrak{M}[\mathcal{L}] \phi'^{-1} \subsetneq SO(3)$.

Proof. If $\phi' \mathfrak{M}[\mathcal{L}] \phi'^{-1} = SO(3)$, then $\mathcal{D} = \mathbb{C} \cup \{\infty\}$, but $\frac{-1}{2} + \frac{\sqrt{3}}{2} i \notin \mathcal{D}$ since φ_4 and φ_4^2 are fix $Q_{\frac{-1}{2} + \frac{\sqrt{3}}{2} i}$. $\square$

Lemma 4.4. *$\mathfrak{M}[\mathcal{L}]$ is a closed subgroup of $\mathcal{G}$. Thus $\phi' \mathfrak{M}[\mathcal{L}] \phi'^{-1}$ is a closed subgroup of $SO(3)$.*

Proof. The topology on $\mathcal{G}$ is induce by the following metric

$$\sigma(f, g) = sup_{z \in \widehat{\mathbb{C}}} d(f(z), g(z)),$$

where d is the chordal metric on $\widehat{\mathbb{C}}$. Let g_1, $g_2 \ldots$ be *Möbius* transformations in $\mathfrak{M}[\mathcal{L}]$ such that $g_n \to g$. Let $z_1^n = g_n(\infty)$, $z_2^n = g_n(0)$, $z_3^n = g_n(-1)$, we know that $\{Q_{z_1^n}, Q_{z_2^n}, Q_{z_3^n}\}$ are free projection for each n. Also denote $g(\infty)$, $g(0)$, $g(-1)$ by z_1, z_2, z_3 respectively. If we can show $\{Q_{z_1}, Q_{z_2}, Q_{z_3}\}$ are free projections, then $g \in \mathfrak{M}[\mathcal{L}]$ since there will be an automorphism ϕ, such that $\phi(Q_\infty) = Q_{z_1}$, $\phi(Q_0) = Q_{z_2}$, $\phi(Q_{-1}) = Q_{z_3}$.

Because $\sigma(g_n, g) \to 0$, we have $d(z_i, z_i^n) \to 0$, $i = 1, 2, 3$, this implies

$$\|Q_{z_i} - Q_{z_i^n}\|_2 \to 0, \qquad i = 1, 2, 3.$$

Then by Cauchy-Schwarz inequality (also note that the product of projections has norm less then 1), it is not hard to see that each joint moment of $\{Q_{z_1^n}, Q_{z_2^n}, Q_{z_3^n}\}$ converges towards the corresponding joint moment of $\{Q_{z_1}, Q_{z_2}, Q_{z_3}\}$ (we only need to consider the joint moment because the random variables here are all projections). Thus $\{Q_{z_1^n}, Q_{z_2^n}, Q_{z_3^n}\}$ converges in distribution towards $\{Q_{z_1}, Q_{z_2}, Q_{z_3}\}$, and $\{Q_{z_1}, Q_{z_2}, Q_{z_3}\}$ are free. $\square$

The closed subgroups of $SO(3)$ are cyclic groups C_n, dihedral groups D_n, tetrahedral group T, octahedral group O, the icosahedral group Y, and two infinite closed subgroups $C_\infty \approx SO(2)$ generated by an arbitrary rotation around an axis and D_∞ which is generated by C_∞ and a rotation π around an axis orthogonal to the axis of rotation of C_∞.

Since $D_3 \subset \phi' \mathfrak{M}[\mathcal{L}] \phi'^{-1}$, so $\phi' \mathfrak{M}[\mathcal{L}] \phi'^{-1}$ can not be C_n and C_∞, which are both Abelian.

Lemma 4.5. $\phi' \mathfrak{M}[\mathcal{L}] \phi'^{-1} \neq D_n$, *for $n \geq 4$.*

Proof. Assume that $\phi'\mathfrak{M}[\mathcal{L}]\phi'^{-1} = D_n$ for $n \geq 4$. Since $D_3 \subset \phi'\mathfrak{M}[\mathcal{L}]\phi'^{-1}$, then n must divide 3 or $n = \infty$. For any $\theta \in [0, 2\pi]$, let $T_\theta = \phi'^{-1}T'_\theta\phi' \in \mathfrak{M}[\mathcal{L}]$, where $T'_\theta \in SO(3)$ is the counterclockwise rotation about the positive y-axis by angle θ:

$$T'_\theta((x,y,t)) = \begin{pmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & 0 & cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \\ t \end{pmatrix}.$$

Direct computation will give

$$T_\theta(\infty) = \frac{sin(\frac{\theta}{2} + \frac{2\pi}{3})}{sin(\frac{\theta}{2})}, T_\theta(0) = \frac{sin(\frac{\theta}{2})}{sin(\frac{\theta}{2} - \frac{2\pi}{3})}, T_\theta(-1) = \frac{sin(\frac{\theta}{2} + \frac{2\pi}{3}) + sin(\frac{\theta}{2})}{sin(\frac{\theta}{2} - \frac{2\pi}{3}) + sin(\frac{\theta}{2})}.$$

Next we will show that for any $\theta \in (\frac{2\pi}{3}, \frac{4\pi}{3})$, $T_\theta \notin \mathfrak{M}[\mathcal{L}]$. This will end the proof, since for any $\theta \notin \{\frac{2\pi}{3}, \frac{4\pi}{3}, 0\}$, there exist $n \in \mathbb{N}$, such that $n\theta \ mod\ 2\pi \in (\frac{2\pi}{3}, \frac{4\pi}{3})$.

Fix any $\theta \in (\frac{2\pi}{3}, \frac{4\pi}{3})$, if $T_\theta \in \mathfrak{M}[\mathcal{L}]$, let $T_\theta(0) = z_1$, $T_\theta(\infty) = z_2$, $T_\theta(-1) = z_3$. By the computation in the proof of Lemma 4.1, we have

$$\begin{aligned}
&(I - Q_\infty W Q_{z_1} W Q_\infty)^{-1}[Q_\infty W Q_{z_1} W(I - Q_\infty)] \\
&\quad - (I - Q_\infty W Q_{z_3} W Q_\infty)^{-1}[Q_\infty W Q_{z_3} W(I - Q_\infty)] \\
&\quad = (\frac{1}{z_1 - z_2} - \frac{1}{z_3 - z_2})\frac{I}{\sqrt{I - K_{z_2}}}U_{z_2}S^{-1}\frac{I}{\sqrt{I - K_{z_2}}}U_{z_2} \\
&\quad = -\frac{4}{3}sin^2(\frac{\theta}{2})\frac{I}{\sqrt{I - K_{z_2}}}U_{z_2}S^{-1}\frac{I}{\sqrt{I - K_{z_2}}}U_{z_2},
\end{aligned}$$

where W is defined in the proof of Lemma 4.1. Since $\{WQ_{z_2}W = Q_\infty, WQ_{z_1}W, WQ_{z_3}W\}$ are free, we must have

$$L(-\frac{4}{3}sin^2(\frac{\theta}{2})\frac{I}{\sqrt{I - K_{z_2}}}U_{z_2}S^{-1}\frac{I}{\sqrt{I - K_{z_2}}}U_{z_2}) = L(S),$$

where

$$L(a) = log\Delta(a) = \int_0^\infty logtd_{\mu_{|a|}(t)} \in [-\infty, \infty[$$

(Δ is the Fuglede-Kadison-determinant on $(\mathcal{L}'', \tau)$). Thus we have

$$\begin{aligned}
L(S) &= L(-\frac{4}{3}sin^2(\frac{\theta}{2})) + 2L(\frac{I}{\sqrt{I - K_{z_2}}}U_{z_2}) + L(S^{-1}) \\
&= log(\frac{4}{3}sin^2(\frac{\theta}{2})) + 2L(\frac{I}{\sqrt{I - K_{z_2}}}) - L(S),
\end{aligned}$$

which implies that

$$2L(S) = log(\frac{4}{3}sin^2(\frac{\theta}{2})) + 2L(\frac{I}{\sqrt{I - K_{z_2}}}).$$

We already know for all $\theta \in [\frac{2\pi}{3}, \frac{4\pi}{3}]$,

$$L(\frac{I}{\sqrt{I - K_{z_2}}}) = L(\frac{I}{\sqrt{I - K_0}}),$$

and if $\theta = \frac{2\pi}{3}$, $\frac{4}{3}sin^2(\frac{\theta}{2}) = 1$,

$$L(S) = L(\frac{I}{\sqrt{I - K_0}}),$$

thus we have $log(\frac{4}{3}sin^2(\frac{\theta}{2})) = 0$, and this is impossible for $\theta \in (\frac{2\pi}{3}, \frac{4\pi}{3})$. $\square$

Remark 4.1. *Actually, we can check $L(S) = L(\frac{I}{\sqrt{I-K_{-1}}})$ explicitly. By the same method in the proof of lemma 2.1, we have*

$$d_{\mu_{|S|}}(t) = \frac{4}{\pi}\frac{1}{t^2+4}1_{(0,+\infty)}(t)dt.$$

So a application of residue formula([?] Chapter3 Exercises 10) will give

$$L(S) = \frac{4}{\pi}\int_0^\infty \frac{logt}{t^2+4}dt = log2.$$

Now we need to show $L(\frac{I}{\sqrt{I-K_0}}) = log2$. Because $d_{\mu_{\sqrt{\frac{K_{-1}}{I-K_0}}}}(t) = \frac{2}{\pi}\frac{1}{t^2+1}1_{(0,+\infty)}(t)dt$, we have

$$d_{\mu_{\sqrt{\frac{I}{I-K_0}}}}(t) = \frac{2}{\pi}\frac{1}{t\sqrt{t^2-1}}1_{(1,+\infty)}(t)dt.$$

Then we know

$$\begin{aligned}
L(\frac{I}{\sqrt{I-K_0}}) &= \frac{2}{\pi}\int_1^{+\infty}\frac{logt}{t\sqrt{t^2-1}}1_{(1,+\infty)}(t)dt \\
&= \frac{1}{\pi}\int_0^{+\infty}\frac{log(x^2+1)}{x^2+1}dx, \qquad (t=\sqrt{x^2+1}), \\
&= -\frac{2}{\pi}\int_0^{\frac{\pi}{2}} log(cos\theta)d\theta. \qquad (x = tg\theta).
\end{aligned}$$

The last integral is computed in [?] by rewriting it as

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} log(cos\theta)d\theta &= -\frac{1}{4}\int_0^1 logt\frac{1}{\sqrt{t}\sqrt{1-t}} \\
&= \frac{1}{4}\frac{\partial}{\partial\beta}\int_0^1 t^{\beta-\frac{1}{2}}(1-t)^{\alpha-\frac{1}{2}}|_{\alpha=\beta=0} \\
&= \frac{\pi}{4}\frac{\partial}{\partial\beta}[\frac{\Gamma(\frac{1}{2}+\alpha)\Gamma(\frac{1}{2}+\beta)}{\Gamma(1+\alpha+\beta)\Gamma^2(\frac{1}{2})}]|_{\alpha=\beta=0} = -\frac{\pi}{2}log2.
\end{aligned}$$

Remark 4.2 (Shortcut)**.** *Since*

$$\begin{aligned}
L(I-K_z) &= \frac{1}{\pi}\int_0^1 log(t)\frac{\Delta}{\sqrt{t(1-t)}(1+(\Delta^2-1)t)}dt \\
&= -\frac{2}{\pi}\int_0^{+\infty} log(t^2+1)\frac{\Delta}{t^2+\Delta^2}dt = -2log(\Delta+1),
\end{aligned}$$

where $\Delta = |z| + |z+1|$. Then

$$\begin{aligned}
L(I-(WQ_zW)_{1,1}) &= 2log(|z_2-z|) + L(I-K_z) + L(I-K_{z_2}) + 2L(S) \\
&= 2log(|z_2-z|) + 2log(2) - 2log(1+|z|+|z+1|) - 2log(1+|z_2|+|z_2+1|) \\
&= 2log(\frac{2|z_2-z|}{(1+|z|+|z+1|)(1+|z_2|+|z_2+1|)}).
\end{aligned}$$

Note by the facts in section 3, we will know the fact we want immediately.

Lemma 4.6. $\phi'\mathfrak{M}[\mathcal{L}]\phi'^{-1} \neq T$.

Proof. Since $T \approx A_4$, the alternating subgroup of S_4, and A_4 has no subgroup of order 6. This contradict with the fact that $S_3 \subset \phi' \mathfrak{M}[\mathcal{L}] \phi'^{-1}$. $\square$

Because both O and Y contains S_3 as subgroup, we need to find out if $\phi' \mathfrak{M}[\mathcal{L}] \phi'^{-1}$ could be one of them.

First consider O. O is the symmetry group of the cube. We can write down a permutation representation of the symmetric group on four letters that indicates the group action on the vertices of the cube. If we label the vertices of the cube as in the picture below, we have

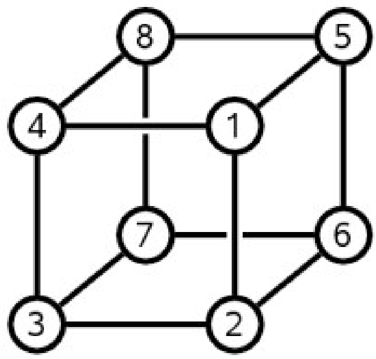

FIGURE 2

$$r = (1,2)(3,5)(4,6)(7,8), \qquad s = (1,2,3,4)(5,6,7,8),$$

and $r^2 = id$, $s^4 = id$, $(rs)^3 = (2,5,4)(3,6,8) = id$. By the symmetry, we know that a group element of O is a rotation by $120°$, if and only if it is a rotation about the line passing through two opposite vertices, for instance vertices 1 and 7 (and there are 8 rotations by $120°$). Also there are six rotation by $180°$ about a 2-fold axis, i.e. the rotation by $180°$ about the line passing through the midpoint of opposite edges (for example edge (1,2) and edge (8,7)).

Next we assume the coordinate of the vertices of the cube in Figure 1 is

$$(1) = (\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}); \qquad (4) = (\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3});$$
$$(5) = (-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}); \qquad (8) = (-\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3});$$
$$(2) = (\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}); \qquad (3) = (\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3});$$
$$(6) = (-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}); \qquad (7) = (-\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}).$$

We will map $(0,1,0)$ to $(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$.

Let

$$T = \begin{pmatrix} -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \end{pmatrix}, \qquad A_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

then

$$T^{-1}A_xT \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{6}}{3} \\ 0 \end{pmatrix}, T^{-1}A_xT \begin{pmatrix} \frac{\sqrt{3}}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}, T^{-1}A_xT \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{6} \\ \frac{\sqrt{6}}{3} \\ -\frac{1}{2} \end{pmatrix}.$$

Let

$$\begin{aligned} z_1 &= \phi'^{-1}(\frac{\sqrt{3}}{3}, -\frac{\sqrt{6}}{3}, 0) = -1 - \sqrt{2}i; \\ z_2 &= \phi'^{-1}((\frac{-\sqrt{3}}{2}, 0, \frac{1}{2})) = -\frac{1}{2}; \\ z_3 &= \phi'^{-1}((\frac{\sqrt{3}}{6}, \frac{\sqrt{6}}{3}, -\frac{1}{2})) = \sqrt{2}i. \end{aligned}$$

By the same method we used in the proof of Lemma 4.5, the following fact tells us that $\phi'\mathfrak{M}[\mathcal{L}]\phi'^{-1} \neq O$.

$$|\frac{1}{z_1 - z_2} - \frac{1}{z_3 - z_2}| = |\frac{1}{-\frac{1}{2} - \sqrt{2}i} - \frac{1}{\frac{1}{2} + \sqrt{2}i}| = |\frac{2}{\frac{1}{2} + \sqrt{2}i}| = \frac{4}{3} \neq 1.$$

Next we will consider Y. Y is the symmetry group of the regular icosahedron.

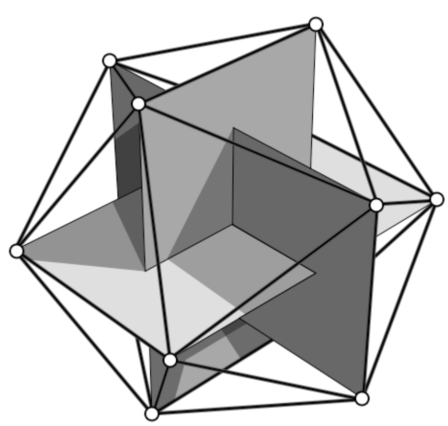

FIGURE 3

In the group above, we use Cartesian coordinates define the vertices of an icosahedron:

$$(\pm 1, 0, \pm\frac{1+\sqrt{5}}{2}); \qquad (0, \pm\frac{1+\sqrt{5}}{2}, \pm 1); \qquad (\pm\frac{1+\sqrt{5}}{2}, \pm 1, 0).$$

Let

$$T = \begin{pmatrix} \frac{-2-\sqrt{5}}{\sqrt{42+18\sqrt{5}}} & \frac{1+\sqrt{5}}{\sqrt{42+18\sqrt{5}}} & \frac{-2-\sqrt{5}}{\sqrt{14+6\sqrt{5}}} \\ \frac{1+\sqrt{5}}{2\sqrt{42+18\sqrt{5}}} & \frac{4+2\sqrt{5}}{\sqrt{42+18\sqrt{5}}} & \frac{1+\sqrt{5}}{2\sqrt{14+6\sqrt{5}}} \\ \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \end{pmatrix},$$

we have

$$T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{6}(3-\sqrt{5})} \\ \sqrt{\frac{1}{6}(3+\sqrt{5})} \\ 0 \end{pmatrix}, \qquad T\begin{pmatrix} \frac{\sqrt{3}}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

If we assume that $\phi'\mathfrak{M}[\mathcal{L}]\phi'^{-1} \approx Y$, we know the rotation by 180° about $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and the rotation by 120° about $\begin{pmatrix} \sqrt{\frac{1}{6}(3-\sqrt{5})} \\ \sqrt{\frac{1}{6}(3+\sqrt{5})} \\ 0 \end{pmatrix}$ are in $T\phi'\mathfrak{M}[\mathcal{L}]\phi'^{-1}T^{-1}$, thus

$$A_x = T^{-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} T = \begin{pmatrix} \frac{-9+\sqrt{5}}{12} & -\frac{1}{3} & \sqrt{\frac{7}{24}+\frac{\sqrt{5}}{8}} \\ -\frac{1}{3} & -\frac{\sqrt{5}}{3} & -\frac{1}{\sqrt{3}} \\ \sqrt{\frac{7}{24}+\frac{\sqrt{5}}{8}} & -\frac{1}{\sqrt{3}} & \frac{-1+\sqrt{5}}{4} \end{pmatrix} \in \phi'\mathfrak{M}[\mathcal{L}]\phi'^{-1}.$$

Thus

$$A_x\begin{pmatrix} \frac{\sqrt{3}}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}, A_x\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{7}{24}+\frac{\sqrt{5}}{8}} \\ -\frac{1}{\sqrt{3}} \\ \frac{-1+\sqrt{5}}{4} \end{pmatrix}, A_x\begin{pmatrix} -\frac{\sqrt{3}}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{3-\sqrt{5}}{4\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{-1-\sqrt{5}}{4} \end{pmatrix}.$$

Then

$$z_1 = \phi'^{-1}(\begin{pmatrix} \sqrt{\frac{7}{24}+\frac{\sqrt{5}}{8}} \\ -\frac{1}{\sqrt{3}} \\ \frac{-1+\sqrt{5}}{4} \end{pmatrix}) = -\frac{1}{2} - \frac{1}{2}(\sqrt{5}+2i);$$

$$z_2 = \phi'^{-1}(\begin{pmatrix} -\frac{\sqrt{3}}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}) = -\frac{1}{2};$$

$$z_3 = \phi'^{-1}(\begin{pmatrix} \frac{3-\sqrt{5}}{4\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{-1-\sqrt{5}}{4} \end{pmatrix}) = -\frac{1}{2} + \frac{1}{2}(\sqrt{5}+2i),$$

and we also have

$$|\frac{1}{z_1-z_2} - \frac{1}{z_3-z_2}| = |\frac{1}{-\frac{1}{2}(\sqrt{5}+2i)} - \frac{1}{\frac{1}{2}(\sqrt{5}+2i)}| = |\frac{4}{\sqrt{5}+2i}| = \frac{4}{3} \neq 1.$$

The conclusions of the above argument are summarized in the following result.

Theorem 4.3. $\phi'\mathfrak{M}[\mathcal{L}]\phi'^{-1} \approx S_3$

5. DERIVATIVE

Remember that

$$\begin{aligned}\sqrt{K_z(I-K_z)^{-1}}U_z =&(1+z)\sqrt{H_1(I-H_1)^{-1}}-z\sqrt{H_2(I-H_2)^{-1}}V\\ =& zS+\sqrt{H_1(I-H_1)^{-1}},\end{aligned}$$

where $S=\sqrt{\frac{H_1}{I-H_1}}-\sqrt{\frac{H_2}{I-H_2}}V$. Let

$$\begin{aligned}F(z)&=|z|^2SS^*+zS\sqrt{\frac{H_1}{I-H_1}}+\overline{z}\sqrt{\frac{H_1}{I-H_1}}S^*+\frac{H_1}{I-H_1};\\ G(z)&=|z|^2S^*S+z\sqrt{\frac{H_1}{I-H_1}}S+\overline{z}S^*\sqrt{\frac{H_1}{I-H_1}}+\frac{H_1}{I-H_1},\end{aligned}$$

then we have

$$K_z=I-(I+F(z))^{-1};\qquad U_z^*(I-K_z)U_z=(I+G(z))^{-1}.$$

And simple calculation will give (below we treat K_z as operator valued function and $z=x+iy$):

$$\begin{aligned}
\frac{\partial K_z}{\partial x}&=(I+F(z))^{-1}(2xSS^*+S\sqrt{\frac{H_1}{I-H_1}}+\sqrt{\frac{H_1}{I-H_1}}S^*)(I+F(z))^{-1};\\
\frac{\partial K_z}{\partial y}&=(I+F(z))^{-1}[2ySS^*+i(S\sqrt{\frac{H_1}{I-H_1}}-\sqrt{\frac{H_1}{I-H_1}}S^*)](I+F(z))^{-1};\\
\frac{\partial\sqrt{K_z(I-K_z)}U_z}{\partial x}&=-\frac{\partial K_z}{\partial x}\sqrt{\frac{K_z}{(I-K_z)}}U_z+(I-K_z)S;\\
\frac{\partial\sqrt{K_z(I-K_z)}U_z}{\partial y}&=-\frac{\partial K_z}{\partial y}\sqrt{\frac{K_z}{(I-K_z)}}U_z+i(I-K_z)S;\\
\frac{\partial U_z^*\sqrt{K_z(I-K_z)}}{\partial x}&=-U_z^*\sqrt{\frac{K_z}{(I-K_z)}}\frac{\partial K_z}{\partial x}+S^*(I-K_z)=(\frac{\partial\sqrt{K_z(I-K_z)}U_z}{\partial x})^*;\\
\frac{\partial U_z^*\sqrt{K_z(I-K_z)}}{\partial y}&=-U_z^*\sqrt{\frac{K_z}{(I-K_z)}}\frac{\partial K_z}{\partial y}-iS^*(I-K_z)=(\frac{\partial\sqrt{K_z(I-K_z)}U_z}{\partial y})^*;\\
\frac{\partial U_z^*(I-K_z)U_z}{\partial x}&=-(I+G(z))^{-1}(2xS^*S+\sqrt{\frac{H_1}{I-H_1}}S+S^*\sqrt{\frac{H_1}{I-H_1}})(I+G(z))^{-1};\\
\frac{\partial U_z^*(I-K_z)U_z}{\partial y}&=-(I+G(z))^{-1}[2yS^*S+i(\sqrt{\frac{H_1}{I-H_1}}S-S^*\sqrt{\frac{H_1}{I-H_1}})](I+G(z))^{-1}.
\end{aligned}$$

Since

$$\frac{\partial}{\partial z}=\frac{1}{2}[\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}];\qquad \frac{\partial}{\partial\overline{z}}=\frac{1}{2}[\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}],$$

we have

$$\frac{\partial Q_z}{\partial z} = \begin{pmatrix} (I+F(z))^{-1}(\overline{z}SS^* & (I-K_z)S-(I+F(z))^{-1}(\overline{z}SS^* \\ +S\sqrt{\frac{H_1}{I-H_1}})(I+F(z))^{-1} & +S\sqrt{\frac{H_1}{I-H_1}})(I+F(z))^{-1}\sqrt{\frac{K_z}{(I-K_z)}}U_z \\ -U_z^*\sqrt{\frac{K_z}{(I-K_z)}}(I+F(z))^{-1}(\overline{z}SS^* & -(I+G(z))^{-1}(\overline{z}S^*S \\ +S\sqrt{\frac{H_1}{I-H_1}})(I+F(z))^{-1} & +\sqrt{\frac{H_1}{I-H_1}}S)(I+G(z))^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} (I-K_z)(\overline{z}SS^* & (I-K_z)S-(I-K_z)(\overline{z}SS^* \\ +S\sqrt{\frac{H_1}{I-H_1}})(I-K_z) & +S\sqrt{\frac{H_1}{I-H_1}})\sqrt{K_z(I-K_z)}U_z \\ -U_z^*\sqrt{K_z(I-K_z)}(\overline{z}SS^* & -U_z^*(I-K_z)U_z(\overline{z}S^*S \\ +S\sqrt{\frac{H_1}{I-H_1}})(I-K_z) & +\sqrt{\frac{H_1}{I-H_1}}S)U_z^*(I-K_z)U_z \end{pmatrix}$$

$$= \begin{pmatrix} (I-K_z) & 0 \\ 0 & U_z^*\sqrt{K_z(I-K_z)} \end{pmatrix} \times \begin{pmatrix} S & S \\ -S & -S \end{pmatrix} \times \begin{pmatrix} U_z^*\sqrt{K_z(I-K_z)} & 0 \\ 0 & U_z^*(I-K_z)U_z \end{pmatrix}$$

$$\frac{\partial Q_z}{\partial \overline{z}} = \begin{pmatrix} (I+F(z))^{-1}(zSS^* & -(I+F(z))^{-1}(zSS^*+\sqrt{\frac{H_1}{I-H_1}}S^*) \\ +\sqrt{\frac{H_1}{I-H_1}}S^*)(I+F(z))^{-1} & \times(I+F(z))^{-1}\sqrt{\frac{K_z}{(I-K_z)}}U_z \\ -U_z^*\sqrt{\frac{K_z}{(I-K_z)}}(I+F(z))^{-1}(zSS^* & -(I+G(z))^{-1}(zS^*S \\ +\sqrt{\frac{H_1}{I-H_1}}S^*)(I+F(z))^{-1}+S^*(I-K_z) & +S^*\sqrt{\frac{H_1}{I-H_1}})(I+G(z))^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} (I+K_z)(zSS^* & -(I+K_z)(zSS^*+\sqrt{\frac{H_1}{I-H_1}}S^*) \\ +\sqrt{\frac{H_1}{I-H_1}}S^*)(I+K_z) & \times\sqrt{K_z(I-K_z)}U_z \\ -U_z^*\sqrt{K_z(I-K_z)}(zSS^* & -U_z^*(I-K_z)U_z(zS^*S \\ +\sqrt{\frac{H_1}{I-H_1}}S^*)(I+K_z)+S^*(I-K_z) & +S^*\sqrt{\frac{H_1}{I-H_1}})U_z^*(I-K_z)U_z \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{K_z(I-K_z)}U_z & 0 \\ 0 & U_z^*(I-K_z)U_z \end{pmatrix} \times \begin{pmatrix} S^* & -S^* \\ S^* & -S^* \end{pmatrix} \times \begin{pmatrix} I-K_z & 0 \\ 0 & \sqrt{K_z(I-K_z)}U_z \end{pmatrix}$$

Note $F(z)$ and $G(z)$ are self joint, we have the following lemma,

Lemma 5.1. $(\frac{\partial Q_z}{\partial z})^* = \frac{\partial Q_z}{\partial \overline{z}}$.

Example 5.1.

$$\frac{\partial Q_z}{\partial z}|_{z=0} = \begin{pmatrix} (I-H_1)S\sqrt{H_1(I-H_1)} & (I-H_1)S(I-H_1) \\ -\sqrt{H_1(I-H_1)}S\sqrt{H_1(I-H_1)} & \sqrt{H_1(I-H_1)}S(I-H_1) \end{pmatrix}$$

$$\frac{\partial Q_z}{\partial \overline{z}}|_{z=0} = \begin{pmatrix} \sqrt{H_1(I-H_1)}S^*(I-H_1) & -\sqrt{H_1(I-H_1)}S^*\sqrt{H_1(I-H_1)} \\ (I-H_1)S^*(I-H_1) & (I-H_1)S^*\sqrt{H_1(I-H_1)} \end{pmatrix}$$

If the projections Q_∞, Q_0, Q_{-1} in $M_{2n}(\mathbb{C})$ are in general position, then the map $\varphi(z) = Q_z$ gives a holomorphic curve in complex Grassmann manifolds $\mathrm{G}_{n,2n}$.

Let $U(2n)$ be the unitary group. A smooth map $g : S^2 \to U(2n)$ is a harmonic map if and only if it satisfies the following equation ([**?**]):

$$\frac{\partial A_z}{\partial \overline{z}} = [A_z, A_{\overline{z}}],$$

where $A_z = \frac{1}{2}g(z)^{-1}\frac{\partial g(z)}{\partial z}$ and $A_{\overline{z}} = \frac{1}{2}g(z)^{-1}\frac{\partial g(z)}{\partial \overline{z}}$.

Let $g(z) = 2Q_z - I$. Then we have

$$A_z = (2Q_z - I)\frac{\partial Q_z}{\partial z}$$
$$= \begin{pmatrix} \sqrt{K_z} & \sqrt{I-K_z}U_z \\ U_z^*\sqrt{I-K_z} & -U_z^*\sqrt{K_z}U_z \end{pmatrix} \times \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \times \begin{pmatrix} \sqrt{K_z} & \sqrt{I-K_z}U_z \\ U_z^*\sqrt{I-K_z} & -U_z^*\sqrt{K_z}U_z \end{pmatrix}$$
$$\times \begin{pmatrix} (I-K_z) & 0 \\ 0 & U_z^*\sqrt{K_z(I-K_z)} \end{pmatrix} \times \begin{pmatrix} S & S \\ -S & -S \end{pmatrix} \times \begin{pmatrix} U_z^*\sqrt{K_z(I-K_z)} & 0 \\ 0 & U_z^*(I-K_z)U_z \end{pmatrix}$$
$$= \begin{pmatrix} -(I-K_z) & 0 \\ U_z^*\sqrt{K_z(I-K_z)} & 0 \end{pmatrix} \times \begin{pmatrix} S & S \\ -S & -S \end{pmatrix} \times \begin{pmatrix} U_z^*\sqrt{K_z(I-K_z)} & 0 \\ 0 & U_z^*(I-K_z)U_z \end{pmatrix},$$

$$A_{\overline{z}} = (2Q_z - I)\frac{\partial Q_z}{\partial \overline{z}}$$
$$= \begin{pmatrix} \sqrt{K_z} & \sqrt{I-K_z}U_z \\ U_z^*\sqrt{I-K_z} & -U_z^*\sqrt{K_z}U_z \end{pmatrix} \times \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \times \begin{pmatrix} \sqrt{K_z} & \sqrt{I-K_z}U_z \\ U_z^*\sqrt{I-K_z} & -U_z^*\sqrt{K_z}U_z \end{pmatrix}$$
$$\times \begin{pmatrix} \sqrt{K_z(I-K_z)}U_z & 0 \\ 0 & U_z^*(I-K_z)U_z \end{pmatrix} \times \begin{pmatrix} S^* & -S^* \\ S^* & -S^* \end{pmatrix} \times \begin{pmatrix} I-K_z & 0 \\ 0 & \sqrt{K_z(I-K_z)}U_z \end{pmatrix}$$
$$= \begin{pmatrix} 0 & \sqrt{K_z(I-K_z)}U_z \\ 0 & U_z^*(I-K_z)U_z \end{pmatrix} \times \begin{pmatrix} S^* & -S^* \\ S^* & -S^* \end{pmatrix} \times \begin{pmatrix} I-K_z & 0 \\ 0 & \sqrt{K_z(I-K_z)}U_z \end{pmatrix}.$$

Therefore

$$A_zA_{\overline{z}} =$$
$$\begin{pmatrix} -(I-K_z) & 0 \\ U_z^*\sqrt{K_z(I-K_z)} & 0 \end{pmatrix} \times \begin{pmatrix} S & S \\ -S & -S \end{pmatrix} \times \begin{pmatrix} U_z^*\sqrt{K_z(I-K_z)} & 0 \\ 0 & U_z^*(I-K_z)U_z \end{pmatrix}$$
$$\times \begin{pmatrix} 0 & \sqrt{K_z(I-K_z)}U_z \\ 0 & U_z^*(I-K_z)U_z \end{pmatrix} \times \begin{pmatrix} S^* & -S^* \\ S^* & -S^* \end{pmatrix} \times \begin{pmatrix} I-K_z & 0 \\ 0 & \sqrt{K_z(I-K_z)}U_z \end{pmatrix},$$

$$A_{\overline{z}}A_z =$$
$$\begin{pmatrix} 0 & \sqrt{K_z(I-K_z)}U_z \\ 0 & U_z^*(I-K_z)U_z \end{pmatrix} \times \begin{pmatrix} S^* & -S^* \\ S^* & -S^* \end{pmatrix} \times \begin{pmatrix} I-K_z & 0 \\ 0 & \sqrt{K_z(I-K_z)}U_z \end{pmatrix}$$
$$\times \begin{pmatrix} -(I-K_z) & 0 \\ U_z^*\sqrt{K_z(I-K_z)} & 0 \end{pmatrix} \times \begin{pmatrix} S & S \\ -S & -S \end{pmatrix} \times \begin{pmatrix} U_z^*\sqrt{K_z(I-K_z)} & 0 \\ 0 & U_z^*(I-K_z)U_z \end{pmatrix},$$

$$\frac{\partial A_z}{\partial \overline{z}} =$$
$$= \begin{pmatrix} \sqrt{K_z(I-K_z)}U_zS^*(I-K_z) & 0 \\ U_z^*(I-K_z)U_zS^*(I-K_z) & 0 \end{pmatrix} \times \begin{pmatrix} S & S \\ -S & -S \end{pmatrix}$$
$$\times \begin{pmatrix} U_z^*\sqrt{K_z(I-K_z)} & 0 \\ 0 & U_z^*(I-K_z)U_z \end{pmatrix} +$$
$$\begin{pmatrix} -(I-K_z) & 0 \\ U_z^*\sqrt{K_z(I-K_z)} & 0 \end{pmatrix} \times \begin{pmatrix} S & S \\ -S & -S \end{pmatrix}$$
$$\times \begin{pmatrix} U_z^*(I-K_z)U_zS^*(I-K_z) & 0 \\ 0 & -U_z^*(I-K_z)U_zS^*\sqrt{K_z(I-K_z)}U_z \end{pmatrix}$$
$$= A_zA_{\overline{z}} - A_{\overline{z}}A_z.$$

So g is harmonic.

$$(I-Q_z)\frac{\partial Q_z}{\partial \overline{z}} =$$

$$\begin{pmatrix} \sqrt{K_z} & \sqrt{I-K_z}U_z \\ U_z^*\sqrt{I-K_z} & -U_z^*\sqrt{K_z}U_z \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} \sqrt{K_z} & \sqrt{I-K_z}U_z \\ U_z^*\sqrt{I-K_z} & -U_z^*\sqrt{K_z}U_z \end{pmatrix}$$
$$\begin{pmatrix} \sqrt{K_z(I-K_z)}U_z & 0 \\ 0 & U_z^*(I-K_z)U_z \end{pmatrix}\begin{pmatrix} S^* & -S^* \\ S^* & -S^* \end{pmatrix}\begin{pmatrix} I-K_z & 0 \\ 0 & \sqrt{K_z(I-K_z)}U_z \end{pmatrix} = 0.$$

This implies $\varphi(z) = Q_z$ is a holomorphic curve. Let $\omega = g^{-a}dg$ be the Maurer-Cartan form on $U(2n)$, and $ds^2_{U(2n)} = \frac{1}{8}tr\omega\omega^*$ be the metric on $U(2n)$. Then the metric induced by s on S^2 is given by

$$ds^2 = -trA_zA_{\overline{z}}dzd\overline{z} = \frac{1}{2}\tau(U_z^*(I-K_z)U_zS^*(I-K_z)S)(dx^2+dy^2).$$

The Riemannian metric if $g_{11} = g_{22} = f$, $g_{12} = g_{21} = 0$, where $f(z) = \frac{1}{2}\tau(U_z^*(I-K_z)U_zS^*(I-K_z)S)$. By simple computation, we have

$$\Gamma_{122} = \Gamma_{221} = -\Gamma_{212} = \frac{1}{2}\frac{\partial f}{\partial x}, \qquad \Gamma_{222} = \Gamma_{211} = -\Gamma_{121} = \frac{1}{2}\frac{\partial f}{\partial y},$$
$$\Gamma^1_{11} = \Gamma^2_{12} = \frac{1}{2}f^{-1}\frac{\partial f}{\partial x}, \qquad \Gamma^1_{12} = -\Gamma^2_{11} = \frac{1}{2}f^{-1}\frac{\partial f}{\partial y}.$$

Thus

$$\begin{aligned} R_{1212} &= \frac{\partial \Gamma_{122}}{\partial x} - \frac{\partial \Gamma_{121}}{\partial y} + \Gamma^h_{11}\Gamma_{2h2} - \Gamma^h_{12}\Gamma_{2h1} \\ &= \frac{1}{2}\left[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} - \frac{1}{2}f^{-1}(\frac{\partial f}{\partial x})^2 - \frac{1}{2}f^{-1}(\frac{\partial f}{\partial y})^2 - \frac{1}{2}f^{-1}(\frac{\partial f}{\partial x})^2 - \frac{1}{2}f^{-1}(\frac{\partial f}{\partial y})^2\right] \\ &= \frac{1}{2f}\left[f\Delta f - (\frac{\partial f}{\partial x})^2 - (\frac{\partial f}{\partial y})^2\right], \end{aligned}$$

and the Gauss curvature K is

$$K = -\frac{R_{1212}}{f^2} = \frac{-1}{2f^3}\left[f\Delta f - (\frac{\partial f}{\partial x})^2 - (\frac{\partial f}{\partial y})^2\right].$$

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