Geometry of Sphere of Projections

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ABSTRACT. Enter abstract here

INTRODUCTION

Introduction here!

1. PRELIMINARY

Definition 1.1. *Let $\mathcal{M}$ be a manifold with a complex structure. A rank n* ***Hermitian holomorphic vector bundle*** *over $\mathcal{M}$ consists of a manifold $\mathcal{E}$ with a complex structure together with a holomorphic map π from $\mathcal{E}$ onto $\mathcal{M}$ such that each fibre $\mathcal{E}_x = \pi^{-1}(x)$ is isomorphic to a Hilbert space and such that for each x_0 there exists a neighborhood Δ of x_0 and holomorphic functions $\xi_i(x)$, $i = 1, \ldots, n$, form Δ to $\mathcal{E}$ whose values form a basis for $\mathcal{E}_x$.*

For a separable Hilbert space $\mathcal{H}$, let

$$\mathcal{G}r(n, \mathcal{H}) = \{\text{n-dimensional subspaces of } \mathcal{H}\}.$$

1.1. **Curvature Matrix.** Let $E \to X$ be a $\mathbb{C}$-vector bundle. $\mathcal{E}(X, E)$ denotes the sections of E.

Definition 1.2. *Let $E \to X$ be a $\mathbb{C}$-vector bundle. Then a connection D on $E \to X$ is a $\mathbb{C}$-linear mapping*

$$D : \mathcal{E}(X, E) \to \mathcal{E}(X, T^*(X) \otimes_{\mathbb{C}} E)$$

which satisfies

$$D(f\xi) = df \cdot \xi + fD\xi,$$

where $f \in C^\infty(X)$ and $\xi \in \mathcal{E}(X, E)$.

Suppose that $E \to X$ is a $\mathbb{C}$-vector bundle. Let $\{e_1, \ldots, e_n\}$ be a frame for E over a open subset U. Then locally, $\xi \in \mathcal{E}(U, E)$ can be write as

$$\begin{pmatrix} e_1 & e_2 & \ldots & e_n \end{pmatrix} \times \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix} = \sum^k \xi_k e_k.$$

If

$$De_j = \sum_i^n \theta_{ij} e_i,$$

2010 *Mathematics Subject Classification.* Primary 47L75; Secondary 15A30.

where $\theta_{ij} \in \mathcal{E}(U, T^*(X) \otimes E)$, then

$$D\left(\begin{pmatrix} e_1 & e_2 & \dots & e_n \end{pmatrix} \times \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix}\right) =$$

$$\begin{pmatrix} e_1 & e_2 & \dots & e_n \end{pmatrix} \times \left(\begin{pmatrix} d\xi_1 \\ d\xi_2 \\ \vdots \\ d\xi_n \end{pmatrix} + \begin{pmatrix} \theta_{11} & \dots & \theta_{1n} \\ \vdots & \ddots & \vdots \\ \theta_{n1} & \dots & \theta_{nn} \end{pmatrix} \times \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix} \right).$$

$\theta(e) = \begin{pmatrix} \theta_{11} & \dots & \theta_{1n} \\ \vdots & \ddots & \vdots \\ \theta_{n1} & \dots & \theta_{nn} \end{pmatrix}$ will be called the associated connection matrix.

If

$$\begin{pmatrix} e_1' & e_2' & \dots & e_n' \end{pmatrix} = \begin{pmatrix} e_1 & e_2 & \dots & e_n \end{pmatrix} \times \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{n1} & \cdots & g_{nn} \end{pmatrix}$$

is another frames over U, then

$$\begin{pmatrix} e_1' & e_2' & \dots & e_n' \end{pmatrix} \times \begin{pmatrix} \xi_1' \\ \xi_2' \\ \vdots \\ \xi_n' \end{pmatrix} = \xi = \begin{pmatrix} e_1 & e_2 & \dots & e_n \end{pmatrix} \times \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix}$$

implies that

$$\begin{pmatrix} e_1' & e_2' & \dots & e_n' \end{pmatrix} \times \begin{pmatrix} \xi_1' \\ \xi_2' \\ \vdots \\ \xi_n' \end{pmatrix} = \begin{pmatrix} e_1' & e_2' & \dots & e_n' \end{pmatrix} \times \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{n1} & \cdots & g_{nn} \end{pmatrix}^{-1} \times \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix}$$

or

$$\begin{pmatrix} e_1 & e_2 & \dots & e_n \end{pmatrix} \times \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix} = \begin{pmatrix} e_1 & e_2 & \dots & e_n \end{pmatrix} \times \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{n1} & \cdots & g_{nn} \end{pmatrix} \times \begin{pmatrix} \xi_1' \\ \xi_2' \\ \vdots \\ \xi_n' \end{pmatrix}.$$

Then

$$D(\begin{pmatrix} e_1 & e_2 & \dots & e_n \end{pmatrix} \times \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix}) = D(\begin{pmatrix} e_1 & e_2 & \dots & e_n \end{pmatrix} \times \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{n1} & \cdots & g_{nn} \end{pmatrix} \times \begin{pmatrix} \xi_1' \\ \xi_2' \\ \vdots \\ \xi_n' \end{pmatrix})$$

$$= \begin{pmatrix} e_1 & e_2 & \dots & e_n \end{pmatrix} \begin{pmatrix} dg_{11} & \dots & dg_{1n} \\ \vdots & \ddots & \vdots \\ dg_{n1} & \dots & dg_{nn} \end{pmatrix} \times \begin{pmatrix} \xi_1' \\ \xi_2' \\ \vdots \\ \xi_n' \end{pmatrix} + \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{n1} & \cdots & g_{nn} \end{pmatrix} \times \begin{pmatrix} d\xi_1' \\ d\xi_2' \\ \vdots \\ d\xi_n' \end{pmatrix}$$

$$+ \begin{pmatrix} \theta_{11} & \dots & \theta_{1n} \\ \vdots & \ddots & \vdots \\ {}_{n1} & \dots & \theta_{nn} \end{pmatrix} \times \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{n1} & \cdots & g_{nn} \end{pmatrix} \times \begin{pmatrix} \xi_1' \\ \xi_2' \\ \vdots \\ \xi_n' \end{pmatrix})$$

$$= \begin{pmatrix} e_1' & e_2' & \dots & e_n' \end{pmatrix} (\begin{pmatrix} d\xi_1' \\ d\xi_2' \\ \vdots \\ d\xi_n' \end{pmatrix} + \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{n1} & \cdots & g_{nn} \end{pmatrix}^{-1} \times \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{n1} & \cdots & g_{nn} \end{pmatrix} \times \begin{pmatrix} d\xi_1' \\ d\xi_2' \\ \vdots \\ d\xi_n' \end{pmatrix}$$

$$+ \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{n1} & \cdots & g_{nn} \end{pmatrix}^{-1} \times \begin{pmatrix} \theta_{11} & \dots & \theta_{1n} \\ \vdots & \ddots & \vdots \\ \theta_{n1} & \dots & \theta_{nn} \end{pmatrix} \times \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{n1} & \cdots & g_{nn} \end{pmatrix} \times \begin{pmatrix} \xi_1' \\ \xi_2' \\ \vdots \\ \xi_n' \end{pmatrix}).$$

Therefore

$$\begin{pmatrix} \theta_{11}' & \dots & \theta_{1n}' \\ \vdots & \ddots & \vdots \\ \theta_{n1}' & \dots & \theta_{nn}' \end{pmatrix} = \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{n1} & \cdots & g_{nn} \end{pmatrix}^{-1} \times \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{n1} & \cdots & g_{nn} \end{pmatrix}$$
$$+ \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{n1} & \cdots & g_{nn} \end{pmatrix}^{-1} \times \begin{pmatrix} \theta_{11} & \dots & \theta_{1n} \\ \vdots & \ddots & \vdots \\ \theta_{n1} & \dots & \theta_{nn} \end{pmatrix} \times \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{n1} & \cdots & g_{nn} \end{pmatrix}$$

Definition 1.3. *Let $E \to X$ be a $\mathbb{C}$-vector bundle with a connection D and $\theta(e)$ be the associated connection matrix for a frame $e = \begin{pmatrix} e_1 & e_2 & \dots & e_n \end{pmatrix}$. We define the curvature matrix associated with the connection matrix $\theta(e)$ to be*

$$\Theta(D, e) = d\theta(e) + \theta(e) \wedge \theta(e),$$

an $n \times n$ matrix of 2-forms.

Lemma 1.1. *Let g be a change of frame, i.e.*

$$\begin{pmatrix} e_1' & e_2' & \dots & e_n' \end{pmatrix} = \begin{pmatrix} e_1 & e_2 & \dots & e_n \end{pmatrix} \cdot g.$$

Then

$$\Theta(D, e \cdot g) = g^{-1} \Theta(D, e) g.$$

Lemma 1.2. $[d + \theta(e)][d + \theta(e)]\xi = \Theta(D, e)\xi.$

Remark 1.1. *Let*

$$\mathcal{E}^p(X,E) = \mathcal{E}(X, \wedge^p T^*(X) \otimes_{\mathbb{C}} E)$$

be the differential forms of degree p on X with coefficients in E. We can extend D to a linear map

$$\begin{aligned} &D : \mathcal{E}^p(X,E) \to \mathcal{E}^{p+1}(X,E) \\ &D\xi = d\xi + \theta(e) \wedge \xi \quad \text{for any } \xi \in \mathcal{E}^p(X,E) \end{aligned}$$

Then, we have $D^2\xi = \Theta(D,e)\xi$.

1.2. **The Canonical Connection and Curvature of a Hermitian Holomorphic Vector Bundle.**

Definition 1.4. *Let* $E \to X$ *be a* $\mathbb{C}$*-vector bundle. A Hermitian metric h on E is an assignment of a Hermitian inner product* $\langle , \rangle_x$ *to each fibre* E_x *of E such that for any open set* $U \subset X$ *and* $\xi, \beta \in \mathcal{E}(U,E)$ *the function*

$$\langle \xi, \beta \rangle : U \to \mathbb{C}$$

given by

$$\langle \xi, \beta \rangle(x) = \langle \xi(x), \beta(x) \rangle_x$$

is C^∞.

Remark 1.2. *If E is a Hermitian vector bundle over X. Then we can extend the metric on E in a natrual manner to act on E-valued covectors in the following*

$$\langle \omega \otimes \xi, \omega' \otimes \xi' \rangle_x = \omega \wedge \omega' \langle \xi, \xi' \rangle_x$$

for $\omega \in \wedge^p T_x^*(X)$, $\omega' \in \wedge^q T_x^*(X)$, *and* $\xi, \xi' \in E$, *for* $x \in X$.

Definition 1.5. *Let* $E \to X$ *be a holomorphic vector bundle over a complex manifold X. If E, as a differentiable bundle, is equipped with a differentiable Hermitian metric, then E is called a Hermitian holomorphic vector bundle.*

If X is a complex manifold,

$$\sum_r \mathcal{E}^r(X,E) = \sum_{p,q} \mathcal{E}^{p,q}(X,E),$$

where $\mathcal{E}^{p,q}(X,E) = \mathcal{E}(X \wedge^{p,q} T^*(X) \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathcal{E}(X,\mathbb{C})} \mathcal{E}(X,E)$. If D is a connection, then $D = D' + D''$ and

$$\begin{aligned} D' &= \partial + \theta : \mathcal{E}(X,E) \to \mathcal{E}^{1,0}(X,E) \\ D'' &= \bar{\partial} : \mathcal{E}(X,E) \to \mathcal{E}^{0,1}(X,E). \end{aligned}$$

Note that $H = (\langle e_i, e_j \rangle)_{i,j}$ is a Hermitian matrix, where $(e_1, \dots, e_n)$ is a holomorphic frame. Then

$$\theta = H^{-1}\partial H$$

give a connection D such that

- $d\langle \xi, \beta \rangle = \langle D\xi, \beta \rangle + \langle \xi, D\beta \rangle$ for any ξ, $beta \in \mathcal{E}(X,E)$;
- if ξ is a holomorphic section of E, then $D''\xi = 0$.

Remark 1.3. *If $(e_1, \ldots, e_n)$ is a holomorphic frame, then*

$$D^{'} = \partial + \theta$$
$$D^{''} = \bar{\partial}.$$

We also have

- *θ is of type $(1,0)$, and $\partial\theta = -\theta \wedge \theta$.*
- *$\Theta(D, e) = \bar{\partial}\theta$ and $\Theta(D, e)$ is of type $(1,1)$.*
- *$\bar{\partial}\Theta(D, e) = 0$, and $\partial\Theta(D, e) = [\Theta(D, e), \theta]$.*

1.3. **Chern Classes.** Let $\widetilde{I}_k(M_n(\mathbb{C}))$ be the $\mathbb{C}$-vector space of all invariant $\mathbb{C}$-linear forms on $M_n(\mathbb{C})$, i.e. $\widetilde{\phi} : M_n(\mathbb{C}) \times \cdots \times M_n(\mathbb{C}) \to \mathbb{C}, \in \widetilde{I}_k(M_n(\mathbb{C})$ if

$$\widetilde{\phi}(TA_1T^{-1}, \ldots, TA_kT^{-1}) =$$
$$widetildephi(A_1, \ldots, A_k)$$

for and invertible element $T \in M_n(\mathbb{C})$.

Suppose $\widetilde{\phi} \in \widetilde{I}_k(M_n(\mathbb{C})$. Let $\phi : M_n(\mathbb{C}) \to \mathbb{C}$ be

$$\phi(A) = \widetilde{\phi}(A, \ldots, A).$$

Then ϕ is a homogeneous polynomial of degree k in the entries of A. Let

$$I_k(\mathcal{M}_n) = \{\phi : \phi(gAg^{-1} = \phi(A),\ \phi \text{ is homogeneous polynomials of degree } k\}.$$

If $\phi \in I_k$, then

$$\widetilde{\phi}(A_1, \ldots, A_k) = \frac{(-1)^k}{k!} \sum_{j=1}^{k} \sum_{i_1 < \cdots < i_j} (-1)^j \phi(A_{i_1} + \cdots + A_{i_j})$$

is a element in $\widetilde{I}_k(M_n(\mathbb{C})$.

Note that we can extend $\phi \in \widetilde{I}_k(M_n(\mathbb{C})$ to $\mathcal{E}^*(X, Hom(E, E))$ in the following way,

$$\phi(A_1 \cdot \omega_1, \ldots, A_k \cdot \omega_k) = \omega_1 \wedge \cdots \wedge \omega_k \phi(A_1, \ldots, A_k).$$

Theorem 1.1. *Let $E \to X$ be a differentiable $\mathbb{C}$-vector bundle, let D be a connection on E, and suppose that $\phi \in I_k(M_n(\mathbb{C}))$. Then*

- *$\phi(\Theta(D, E))$ is closed.*
- *$\phi(\Theta(D, E))$ in the de Rham group $H^{2k}(X, \mathbb{C})$ is independent of the connection D.*

Definition 1.6. *Let $E \to X$ be a differentiable $\mathbb{C}$-vector bundle equipped with a connection D. Then the kth Chern form of E relative to the connection D is defined to be*

$$c_k(E, D) = \Phi_k(\frac{i}{2\pi}\Theta(D, E)) \in \mathcal{E}^{2k}(X, E),$$

where $\Phi_k \in I_k(M_n(\mathbb{C}))$ are invariant polynomials defined by

$$det(I + A) = \sum_{k=0}^{n} \Phi_k(A), \quad A \in M_n(\mathbb{C}).$$

The total Chern form of E relative to D is defined to be

$$c(E, D) = \sum_{k=0}^{n} c_k(E, D), \quad n = rankE.$$

The kth Chern class of the vector bundle E, denoted by $c_k(E)$, is the cohomology class of $c_k(E, D)$ in the de Rham group $H^{2k}(X, \mathbb{C})$, and the total Chern class of E, denoted by $c(E)$, is the cohomology class of $c(E, D)$ in $H^*(X, \mathbb{C})$; i.e., $c(E) = \sum_{k=0}^{n} c_k(E)$.

1.4. **Complex Line Bundles.**

Proposition 1.1. *Let X be a (paracompact) differentiable manifold with topological dimension n. Suppose $E \to X$ is a differentiable vector bundle. Then there is $n+1$ open covers $\{U_i\}_{i=0}^n$ of X such that $E|_{U_i}$ is trivial.*

References

[1] M. J. Cowen and R. G. Douglas, *Complex geometry and operator theory*, Acta Math. 141 (1978) 187–261.

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