A Necessary and Sufficient Condition for The Riemann Hypothesis

February 19, 2014

Notations and preliminaries

Denote $z = x + iy \in \mathbb{C}$.

Gamma Function:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, x > 0.$$

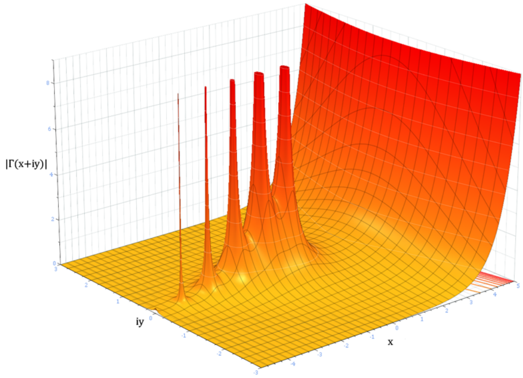


Figure : $|\Gamma(z)|$, $-4 < x < 4$

It's known that Γ can be extended to a meromorphic function on $\mathbb{C}$ with simple poles at $0, -1, -2, -3 \ldots$.
Later we will use the fact that :
$\Gamma(1-z)$ has simple poles at z $=1$, 2, $\ldots$; the residue of $\Gamma(1-z)$ at $z=k$ is equal to

$$\frac{(-1)^{k-1}}{(k-1)!}.$$

The Riemann zeta-function $\zeta(z)$ has its origin in the following identity:

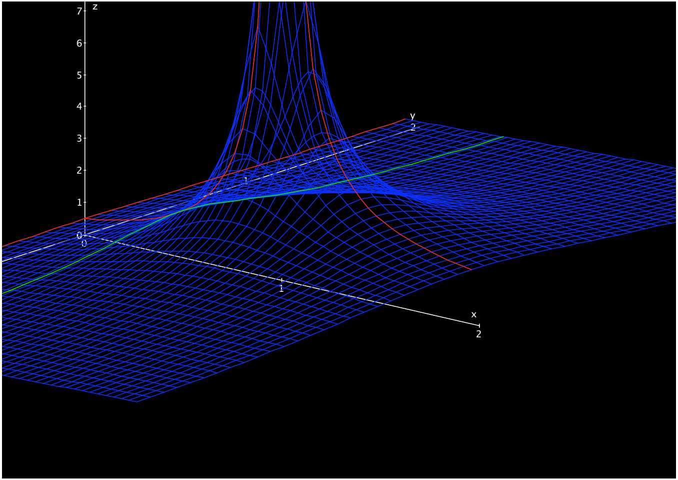
$$\sum_{n=1}^{\infty} \frac{1}{n^z} = \zeta(z) = \prod_{p} (1 - \frac{1}{p^z})^{-1}, x > 1$$

By this equation, we know that $\forall \delta > 0$, there exists a number $B(\delta) > 0$, such that if $x > 1 + \delta$, $|\zeta(z)| < B(\delta)$.

$$\frac{1}{\zeta(z)} = \prod_p (1 - \frac{1}{p^z}) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^z}, x > 1,$$

where μ is möbius function, defined as

$$\mu(n) = \begin{cases} 1 & n = 1 \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ different primes} \\ 0 & \text{otherwise} \end{cases}$$


z
7
6
5
4
3
2
1
0
0
y
2
1
1
x
2

In the paper **On the Number of Primes Less Than a Given Magnitude**(1859), Bernhard Riemann extend the function $\zeta(z)$ to all complex values of z different from 1, and give the following functional equation

$$\zeta(z) = 2^z \pi^{z-1} sin(\frac{1}{2} z\pi)\Gamma(1-z)\zeta(1-z), \forall z \in \mathbb{C}.$$

Because $\sin(\frac{1}{2}z\pi)$ has zero at $-2, -4, -6, \ldots$, $\zeta(z)$ has zero at the negative even integers. These are called the **trivial zeros**.
Let

$$\xi(z) = \frac{1}{2} z(z-1)\pi^{-\frac{1}{2}z}\Gamma(\frac{1}{2}z)\zeta(z),$$
$$\Xi(z) = \xi(\frac{1}{2} + iz).$$

The function equation can be write as $\Xi(z) = \Xi(-z)$.

In the very same paper, Riemann states the **Riemann hypothesis** in terms of roots of $\Xi(z)$:

......it is very probable that all roots are real. Of course one would wish for a stricter proof here; I have for the time being, after some fleeting futile attempts, provisionally put aside the search for this, as it appears unnecessary for the next objective of my investigation.

Riemann hypothesis

All (nontrival) zero of $\zeta(z)$ have real part $\frac{1}{2}$.

A Necessary and Sufficient Condition for RH

Bernoulli's number B_k is given by:

$$\frac{z}{e^z - 1} + \frac{1}{2}z = 1 + B_1 \frac{z^2}{2!} - B_2 \frac{z^4}{4!} + \cdots .$$

$$\zeta(2k) = 2^{2k-1}\pi^{2k} \frac{B_k}{(2k)!} (k = 1, 2, \ldots).$$

$$A_k = \frac{(2\pi)^2}{(2k+2)(2k+1)} \frac{B_{k+1}}{B_k} = \frac{\zeta(2k+2)}{\zeta(2k)}.$$

Write

$$f(t) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} A_k}{(k-1)!} t^k.$$

The Riemann Hypothesis is equivalent to the following statement: For any $\epsilon > 0$, there exists a constant $C(\epsilon) > 0$ such that for any $t > 1$,

$$|f(t)| \leq C(\epsilon) t^{\frac{1}{4}+\epsilon}.$$

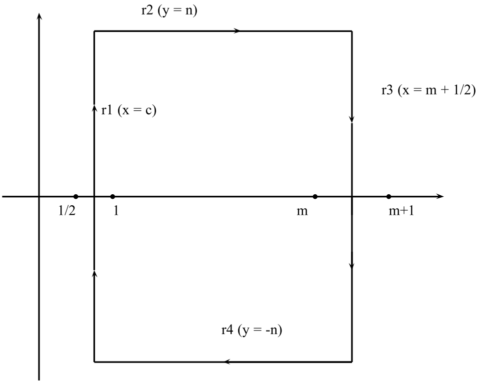
For $c \in \mathbb{R}$, $c \notin \mathbb{Z}$ and $t > 0$ define

$$I(c,t) = \frac{1}{2\pi i}\int_{c-\infty}^{c+\infty} \frac{\zeta(2z+2)\Gamma(1-z)}{\zeta(2z)} t^z \, dz.$$

Notice if $c > \frac{1}{2}$, and $c \neq 1, 2, \ldots$, then the integrand is analytic on the line $x = c$.
We first show that for $\frac{1}{2} < c < 1$,

$$I(c,t) = I(m + \frac{1}{2}, t) + \sum_{1 \le k \le m} \frac{(-1)^{k-1} A_k}{(k-1)!} t^k.$$

Consider the contour:



We only need to show that

$$\lim_{n\to\infty}\int_{c+in}^{m+\frac{1}{2}+in}\frac{\zeta(2z+2)\Gamma(1-z)}{\zeta(2z)}t^z\,dz$$
$$=\lim_{n\to\infty}\int_{c-in}^{m+\frac{1}{2}-in}\frac{\zeta(2z+2)\Gamma(1-z)}{\zeta(2z)}t^z\,dz=0$$

Notice that there exists a constant C_1 such that

$$\left|\frac{\zeta(2z+2)}{\zeta(2z)}\right| < C_1,\ x \geq c.$$

Also for $c \leq x \leq m + \frac{1}{2}$, $|t^z| \leq max\{1, |t|^{m+\frac{1}{2}}\}$.
For $x > \frac{1}{4}, |y| \geq 1$, there exists a constant C_2 such that

$$|\Gamma(1-z)| \leq C_2 e^{\frac{-\pi}{2}|y|}.$$

On r1(or r4),

$$\left|\frac{\zeta(2z+2)\Gamma(1-z)}{\zeta(2z)}t^z\right| \leq C_1C_2max\{1,|t|^{m+\frac{1}{2}}\}e^{\frac{-\pi}{2}n}.$$

$$\lim_{n\to\infty}\left|\int_{c+in}^{m+\frac{1}{2}+in}\frac{\zeta(2z+2)\Gamma(1-z)}{\zeta(2z)}t^z dz\right|$$
$$\leq \lim_{n\to\infty} C_1C_2max\{1,|t|^{m+\frac{1}{2}}\}e^{\frac{-\pi}{2}n}\int_c^{m+\frac{1}{2}} dx = 0$$

On $x = m + \frac{1}{2}$, there is a constant C_3 such that

$$|\Gamma(1-z)| \leq \frac{C_3}{(m-1)!} e^{-(\frac{\pi}{2})|y|}.$$

Then by Stirling's formula:

$$(m-1)! \approx \sqrt{2\pi(m-1)}(\frac{m-1}{e})^{(m-1)}.$$

$$\lim_{m\to\infty} \left| I(m+\frac{1}{2}, t) \right| \leq \lim_{m\to\infty} \frac{C_1 C_3 t^{m+\frac{1}{2}}}{(m-1)!} \int_{-\infty}^{\infty} e^{\frac{-\pi}{2}|y|} dy = 0.$$

By letting $m \to \infty$, for any $\frac{1}{2} < c < 1$, $t > 0$,

$$I(c,t) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} A_k}{(k-1)!} t^k = f(t).$$

Assume now that Riemann Hypothesis is true, then we know that for any $\epsilon > 0$, $\frac{\zeta(2z+2)\Gamma(1-z)}{\zeta(2z)} t^z$ is analytic for $\frac{1}{4} + \epsilon \le x < 1$. Furthermore, for any $\epsilon > 0$, there is a $C_1(\epsilon) > 0$ such that if $x \ge \frac{1}{4} + \epsilon$,

$$\frac{1}{|\zeta(2z)|} \le C_1(\epsilon)(1 + y^2).$$

Similarly, we have

$$
\begin{aligned}
|f(t)| &= \left| I(\frac{1}{4}+\epsilon, t) \right| \\
&= \left| \frac{1}{2\pi i} \int_{\frac{1}{4}+\epsilon-\infty}^{\frac{1}{4}+\epsilon+\infty} \frac{\zeta(2z+2)\Gamma(1-z)}{\zeta(2z)} t^z dz \right| \\
&\leq \frac{C_1(\frac{\epsilon}{2}) C_2 \zeta(2) t^{\frac{1}{4}+\epsilon}}{2\pi} \int_{-\infty}^{+\infty} (1+y^2) e^{-\frac{\pi}{2}|y|} dy.
\end{aligned}
$$

Mellin Transform

If $F(z)$ is analytic for $\frac{1}{2} < x < 1$, and if it tends to zero uniformly with increasing y for any real value c between $\frac{1}{2}$ and 1, with its integral along such a line converging absolutely, then if for $t > 0$, $\frac{1}{2} < c < 1$,

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(z) t^z dz,$$

we have that

$$F(z) = \int_0^\infty f(t) t^{-z-1} dt.$$

For any $\epsilon > 0$, $\frac{1}{2} + \epsilon < x < 1$,

$$\left| \frac{\zeta(2z+2)\Gamma(1-z)}{\zeta(2z)} \right| < C_1 C_2 e^{-\frac{\pi}{2}|y|}.$$

By Mellin's transform, for all $\frac{1}{2} < x < 1$,

$$\begin{aligned} \frac{\zeta(2z+2)\Gamma(1-z)}{\zeta(2z)} &= \int_0^\infty f(t) t^{-z-1} dt \\ &= \int_0^1 f(t) t^{-z-1} dt + \int_1^\infty f(t) t^{-z-1} dt \\ &= I_1 + I_2 \end{aligned}$$

Since $f(t) = O(t)$ as $t \to 0^+$, for any $0 < x < 1$, there exists a constant $C_4(x)$ such that

$$\begin{aligned} |I_1| &\leq \int_0^1 |f(t)t^{-z-1}| dt \\ &\leq C_4(x) \int_0^1 t^{-x} dt < +\infty. \end{aligned}$$

If for any $\epsilon > 0$, there is a constant $C(\epsilon)$ such that

$$|f(t)| \leq C(\epsilon) t^{\frac{1}{4}+\epsilon},\ t > 1.$$

Assume $\frac{1}{4} < x < 1$, denote $x = \frac{1}{4} + \epsilon$ $(0 < \epsilon < \frac{3}{4})$,

$$\begin{aligned} |I_2| &\leq \int_1^\infty |f(t) t^{-z-1}| dt \\ &\leq C(\frac{\epsilon}{2}) \int_1^\infty t^{-1-\frac{\epsilon}{2}} dt < \infty. \end{aligned}$$

Therefore $\int_0^\infty f(t)t^{-z-1}dt$ converges absolutely and represents an analytic function in the strip $\frac{1}{4} < x < 1$. In other words, $\frac{\zeta(2z+2)\Gamma(1-z)}{\zeta(2z)}$ is an analytic function in this strip. Further, since $\zeta(2z+2)\Gamma(1-z)$ is analytic and non-vanishing for $\frac{1}{4} < x < 1$, it follows that $\zeta(2z)$ is analytic and non-vanishing for $\frac{1}{4} < x < 1$. This implies the Riemann Hypothesis.

A Generalization

Let $\mathcal{G}$ be the family of functions G of the complex variables z such that $G \in \mathcal{G}$ iff the following hold:

1. G is analytic on the half-plane $x > \frac{1}{4}$ and it satisfies

$$|G(z)| \leq \frac{e^{(\frac{\pi}{2})|z|}}{|z|^4},$$

2. $G(z) \neq 0$ if $\frac{1}{4} < x < \frac{1}{2}$.

$$\Phi_G(t) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} A_k G(k)}{(k-1)!} t^k.$$

Condition (C): For any $\epsilon > 0$, there exists a $C(\epsilon) > 0$ depending on ϵ such that

$$|\Phi_G(t)| \leq C(\epsilon) t^{\frac{1}{4}+\epsilon}, \text{if } x > 1.$$

Theorem (A Generalization)

If there exists a function $G \in \mathcal{G}$ satisfying the condition (C), then the Riemann Hypothesis is true. Conversely, if the Riemann Hypothesis is true, then Condition (C) is satisfied by any $G \in \mathcal{G}$.