On a new class of non selfadjoint algebras

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Notations

- $\mathbb{C}$: the field of complex numbers;
- $\mathscr{H}$: a separable Hilbert spaces, e.g. $l^2(\mathbb{Z})$, $l^2(\mathbb{Z}_n)$
- $\mathscr{B}(\mathscr{H})$: the algebra of all bounded linear operators on $\mathscr{H}$, e.g. $M_n(\mathbb{C})$

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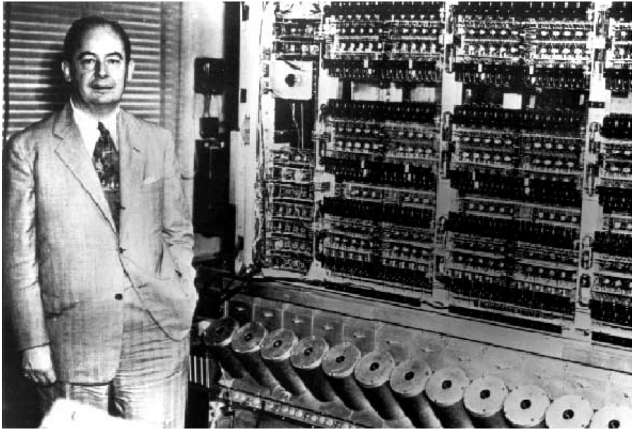
The Legacy of John Von Neumann



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A self adjoint subalgebra $\mathfrak{A}$ of $\mathscr{B}(\mathscr{H})$ is a von Neumann algebra if and only if $\mathfrak{A}'' = \mathfrak{A}$.

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Is there an infinite dimensional "finite" factor?

1934, Murray

Theorem

If $\mathfrak{A}$ *is a factor and* $E_1 \in \mathfrak{A}$ *and* $E_2 \in \mathfrak{A}'$*, then* $E_1 E_2 \neq 0$ *if* $E_1, E_2 \neq 0$.

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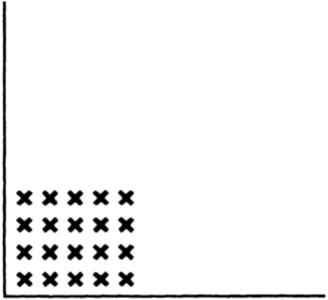

Figure : $l^2(\mathbb{Z}_4) \otimes l^2(\mathbb{Z}_5)$, $\quad \mathfrak{A} \cong M_4(\mathbb{C})$, $\quad \mathfrak{A}' \cong M_5\mathbb{C}$

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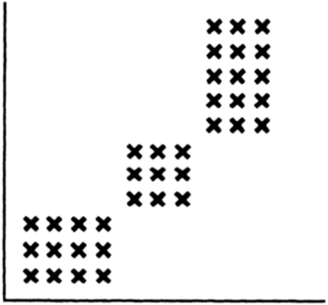

Figure : $l^2(\mathbb{Z}_4) \otimes l^2(\mathbb{Z}_4) \oplus l^2(\mathbb{Z}_3) \otimes l^2(\mathbb{Z}_3) \oplus l^2(\mathbb{Z}_5) \otimes l^2(\mathbb{Z}_3)$

Bernstein's Theorem's analogy

Ranges of projections		Cardinal numbers of sets
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If E and F are projections in a von Neumann algebra $\mathfrak{A}$ such that $E \preceq F$ and $F \preceq E$, then $E \sim F$.

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$\mathrm{I}_n (n < \infty)$	$\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}\}$

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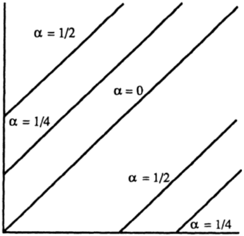
The immediate problem:
obtain an example of a case other than type I.

Exmaple of II_1 factors:



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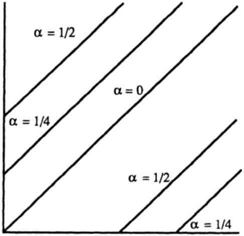

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$$\begin{array}{cc} \mathfrak{A} & \mathfrak{A}' \\ M_x : (M_x f)(x,y) = xf(x,y) & M_y : (M_y f)(x,y) = yf(x,y) \\ L_\alpha : (L_\alpha f)(x,y) = f(x+\alpha, y) & R_\alpha : (R_\alpha f)(x,y) = f(x, y+\alpha) \end{array}$$

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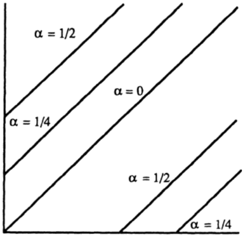

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$$Tr_{\mathfrak{A}}(A) = \langle A\xi,\xi\rangle, \qquad \xi(x,y) = \delta(x,y).$$

$$
\begin{aligned}
Tr_{\mathfrak{A}}(M_x L_\alpha) &= \sum_\beta \int x\delta(x+\alpha, x+\beta)\delta(x, x+\beta)dx \\
&= \int x\delta(x+\alpha, x)dx = 0 = Tr_{\mathfrak{A}}(L_\alpha M_x)
\end{aligned}
$$

Group factors

G : a discrete group, e.g. $\Pi = \cup_n S_n$.
$l^2(G) = \{\sum_{g\in G} a_g e_g | \sum_{g\in G} |a_g|^2 < \infty\}$

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$L_h(\sum_{g\in G} a_g e_g) = \sum_{g\in G} a_g e_{hg}$	$R_h(\sum_{g\in G} a_g e_g) = \sum_{g\in G} a_g e_{hg^{-1}}$

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$$Tr_{\mathfrak{A}}(A) = \langle Ae_g, e_g\rangle.$$

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Proof.

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Fact:
Suppose $\mathfrak{A}$ is a type II_1 factor, then for any $n > 0$, there is a subalgebra $\mathfrak{M}$ of $\mathfrak{A}$ such that $\mathfrak{M} \cong M_n(\mathbb{C})$.

Definition (Hyperfinite II_1 factor)

The hyperfinite II_1 factor R is a countably generated factor of type II_1 such that $\forall \varepsilon > 0$ and $\{A_1, \ldots, A_n\} \subset (R)_1$(unit ball of R), there is a finite type I subfactor $\mathscr{N}$ of R and $\{B_1, \ldots, B_n\} \subset (\mathscr{N})_1$ such that $\|A_i - B_i\|_2 < \varepsilon$ when $i \in \{1, \ldots, n\}$.

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Theorem

L_Π is the only hyperfinite II_1 factor.

Constructions of hyperfinie II_1 factor

- For any amenable i.c.c group G, $L_G \cong R$;
- $R \cong M_{n_1}(\mathbb{C}) \bar{\otimes} M_{n_2}(\mathbb{C}) \bar{\otimes} M_{n_3}(\mathbb{C}) \bar{\otimes} \cdots$

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$$M_{n_1}(\mathbb{C}) \otimes M_{n_2}(\mathbb{C}) \otimes \cdots \otimes M_{n_k}(\mathbb{C}) \otimes \cdots \quad \Leftrightarrow \quad \Pi_p p^{a_p}$$

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$$\begin{aligned} \mathfrak{A}_1 &= M_{n_1}(\mathbb{C}) \otimes M_{n_2}(\mathbb{C}) \otimes \cdots \otimes M_{n_k}(\mathbb{C}) \otimes \cdots \quad &\Leftrightarrow \quad \Pi_p p^{a_p} \\ \mathfrak{A}_2 &= M_{m_1}(\mathbb{C}) \otimes M_{m_2}(\mathbb{C}) \otimes \cdots \otimes M_{m_k}(\mathbb{C}) \otimes \cdots \quad &\Leftrightarrow \quad \Pi_p p^{b_p} \end{aligned}$$

Theorem (J. Glimm)

The C^* algebras A_1 and A_2 are isomorif and only if $\Pi_p p^{a_p} = \Pi_p p^{b_p}$.

Some results about hyperfinite factor of type II_1

Alain Connes (1976)

All subfactors of R are isomorto R or finite dimensional.

V. Jones (1983)
The possible values of the index for subfactors of R:

$$\{4\cos^2(\frac{\pi}{n})|n=3,4,\dots\}$$
$$\cup\{r\in\mathbb{R}|r\geq 4\}\cup\{\infty\}$$

Non-selfadjoint algebras

Definition (Kadison, Singer (1960))

If $\mathfrak{M}$ is a factor and $\mathfrak{A}$ a maximal abelian (selfadjoint) subalgebra of $\mathfrak{M}$, a subalgebra $\mathscr{T}$ of $\mathfrak{M}$ will be said to be “triangular in $\mathfrak{M}$ with diagonal $\mathfrak{A}$ when $\mathscr{T} \cap \mathscr{T}^* = \mathfrak{A}$.

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Theorem

If $\mathfrak{A}$ is a maximal triangular algebra on a finite dimensional space, then there is an orthonormal basis such that $\mathfrak{A}$ is the algebra of all operators with upper triangular matrices relative to this basis.

Some Problems

Operator Theory: Invariant Subspace Problem asks whether every bounded operator on a (separable) Hilbert space has a (nontrivial) invariant subspace.

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The invariant subspace problem relative to a factor: If every operator in the factor $\mathfrak{A}$ has a non-trivial invariant subspace affiliated with $\mathfrak{A}$.

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Haagerup and H. Schultz(2006): If T in a II_1 factor $\mathfrak{A}$ has a non-Dirac Brown measure, then T has a non-trivial hyperinvariant subspace.

$$d_{\mu_T}(x+iy) = \frac{1}{2\pi}\nabla^2 log\Delta[T-(x+iy)I]dxdy, \Delta(T) = exp\tau(log|T|).$$

if $T \in M_n(\mathbb{C})$, $\Delta(T) = |detT|^{\frac{1}{n}}$ and $\mu_T = \frac{1}{n}\sum_{i=1}^{n}\delta_{\lambda_i}$.

- $\mathscr{L}$: a set (lattice) of (orthogonal) projections in $\mathscr{B}(\mathscr{H})$.

$$\mathrm{Alg}(\mathscr{L}) = \{T \in \mathscr{B}(\mathscr{H}) : (I - P)TP = 0, \text{for all } P \in \mathscr{L}\}$$

  – a weak-operator closed subalgebra of $\mathscr{B}(\mathscr{H})$.

- $\mathscr{S}$: a subset of $\mathscr{B}(\mathscr{H})$,

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Example

$$\mathscr{L} = \{0, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, I\} \qquad \mathrm{Alg}(\mathscr{L}) = \{\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}\}$$

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A nest is a totally ordered subspace lattice.
Let $\mathscr{H} = L^2([0;1])$ with Lebesgue measure. For each $t \in [0;1]$. Let $N_t = \{f | f(x) = 0; x \in [t;1]\}$, then $\mathscr{N} = \{N_t | t \in [0;1]\}$ is a "continuous" nest.

If $\mathscr{L}$ is a nest, then $\mathrm{Alg}(\mathscr{L})$ is called a nest algebra.

Kadison and Singer: Nest algebras are the only maximal triangular refexive algebra.

The algebras we are looking for

A subalgebra $\mathfrak{A}$ of $\mathscr{B}(\mathscr{H})$ which is refexive and maximal with respect to the diagonal subalgebra $\mathfrak{A} \cap \mathfrak{A}^*$ of $\mathfrak{A}$, in the sense that if there is another refexive subalgebra $\mathfrak{B}$ of $\mathscr{B}(\mathscr{H})$ such that $\mathfrak{A} \subset \mathfrak{B}$ and $\mathfrak{B} \cap \mathfrak{B}^* = \mathfrak{A} \cap \mathfrak{A}^*$, then $\mathfrak{A} = \mathfrak{B}$.

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$$\text{Alg}(\mathscr{L}) \text{ is a KS-algebra} \quad \Leftrightarrow \quad \begin{array}{l} \mathscr{L} \text{ is a minimal refexive lattice} \\ \text{that generates } \mathscr{L}''. \end{array}$$

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Alg$(\mathscr{L})$ is a KS-algebra $\Leftrightarrow$ $\mathscr{L}$ is a minimal refexive lattice that generates $\mathscr{L}''$.

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The algebras we are looking for

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Kadison-Singer lattice

Theorem

*Suppose $\mathscr{L}$ is a KS-lattice for $\mathfrak{A}\subset\mathscr{B}(\mathscr{H})$, if α is a *-isomorof $\mathfrak{A}$, and $\alpha(\mathfrak{A})\subset\mathscr{B}(\mathscr{K})$, then $\alpha(\mathscr{L})$ is also a KS-lattice for $\alpha(\mathfrak{A})$.*

Exmaple

Let $\mathscr{L} = \{0, I, P_1, P_2\}$.

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad P_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$\mathrm{Alg}(\mathscr{L}) = \{ \begin{pmatrix} x+y & -y \\ 0 & x \end{pmatrix} | x, y \in \mathbb{C}\}$

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Are there abelian KS-algebras (of dimension ≥ 3) ?

Exmaples of KS-lattice in $M_n(\mathbb{C})$

For any given $N = n_1 n_2 \cdots n_m$, $n_i \geq 2$, consider

$$M_N(\mathbb{C}) \cong M_{n_1} \otimes M_{n_2} \otimes \cdots \otimes M_{n_m}(\mathbb{C}).$$

$E_{ij}^{(k)}$, $i, j = 1, \ldots, n_k$, the standard matrix unit system for $M_{n_k}(\mathbb{C})$, for $k = 1, 2, \ldots, m$.

For $j = 1, \ldots, n_1 - 1$, define

$$P_{1j} = \sum_{i=1}^{j} E_{ii}^{(1)}, \qquad P_{1n_1} = \frac{1}{n_1} \sum_{s,t=1}^{n_1} E_{st}^{(1)}$$

For $1 < k \leq m$, $j = 1, \ldots, n_k$.

$$P_{kj} = P_{k-1,n_{k-1}} + (I - P_{k-1,n_{k-1}}) \sum_{i=1}^{j} E_{ii}^{(k)}, (j = 1, \ldots, n-1)$$

$$P_{kn_k} = P_{k-1,n_{k-1}} + (I - P_{k-1,n_{k-1}})(\frac{1}{n_k} \sum_{s,t=1}^{n_k} E_{st}^{(k)})$$

Let $\mathscr{L}(n_1, n_2, \ldots, n_m)$ be the lattice generated by $\{P_{kj} | 1 \leq k \leq m, 1 \leq j \leq n_k\}$, then $\mathscr{L}(n_1, n_2, \ldots, n_m)$ is a KS-lattice.

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$$\frac{N^2}{4} < dim\mathrm{Alg}(\mathscr{L}(n_1, n_2, \ldots, n_m)) = \frac{1}{2}N^2[\frac{n_1 - 1}{n_1} + \frac{n_2 - 1}{n_1^2 n_2} + \cdots + \frac{n_m - 1}{\Pi_{i=1}^{m-1} n_i^2 m}] + 1 \leq \frac{N(N-1)}{2} + 1$$

Let $\mathfrak{A} = M_2^{(1)}(\mathbb{C}) \otimes M_2^{(2)} \otimes \cdots$.

$$P_{11} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \qquad P_{12} = \begin{pmatrix} \frac{1}{2}I & \frac{1}{2}I \\ \frac{1}{2}I & \frac{1}{2}I \end{pmatrix}$$

$$P_{21} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad P_{22} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & \frac{1}{2}I & \frac{1}{2}I \\ 0 & 0 & \frac{1}{2}I & \frac{1}{2}I \end{pmatrix}$$

$$\mathrm{Alg}(P_{11}, P_{12}) = \{\begin{pmatrix} A_1 & -A_1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & A_2 \\ 0 & A_2 \end{pmatrix}\}$$

$$\mathrm{Alg}(P_{11}, P_{12}, P_{21}, P_{22}) = \{ \begin{pmatrix} A_1 & -A_1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \begin{pmatrix} A_2 & A_3 - A_2 \\ 0 & A_3 \end{pmatrix} \\ 0 & \begin{pmatrix} A_2 & A_3 - A_2 \\ 0 & A_3 \end{pmatrix} \end{pmatrix} \}$$

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$$\mathrm{Alg}(P_{11}, P_{12}, P_{21}, P_{22}) \cap \mathrm{Alg}(P_{11}, P_{12}, P_{21}, P_{22})^* = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & A \end{pmatrix}$$

Let $\mathscr{L}_\infty$ be the lattice generated by $\{P_{kj} | k \geq 1; 1 \leq j \leq 2\}$.

Theorem

Let τ be the trace state and $\mathscr{L}$ be the strong-operator closure of L_∞, $\mathscr{H}$ the Hilbert space obtained by GNS construction on $(\mathfrak{A}, \tau)$. Then we have that $\mathscr{L} = \mathrm{Lat}(\mathrm{Alg}(\mathscr{L}_\infty))$ is a KS-lattice. For any $r \in (0, 1)$, if there are $a, l \in \mathbb{N}$ such that $r = \frac{a}{2^l}$, then there are two distinct projections in $\mathscr{L}$ with trace value r ; otherwise there is only one projection in $\mathscr{L}$ with trace r.

S^2 as a Kadison-Singer lattice

Let L_{G_3} be the group von Neumann algebra acting on $l^2(G_3)$(it is a factor of type II$_1$ with a unique normal faithful trace state), where $G_3 = Z_2 * Z_2 * Z_2$. If U_1, U_2, U_3 are canonical generators for L_{G_3} corresponding to the generators of G_3 with $U_j^2 = 1$. Then $\frac{I-U_j}{2}$, $j = 1, \ldots, 3$, are free projections.
Let

$$Q_\infty = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, Q_0 = \begin{pmatrix} H_1 & \sqrt{H_1(I-H_1)} \\ \sqrt{H_1(I-H_1)} & I-H_1 \end{pmatrix},$$

$$Q_{-1} = \begin{pmatrix} H_2 & \sqrt{H_2(I-H_2)}V \\ V^*\sqrt{H_2(I-H_2)} & V^*(I-H_2)V \end{pmatrix}.$$

be three free projections. Let F_3 be the lattice generated by $\{Q_\infty, Q_0, Q_{-1}\}$.

Theorem

For any projection Q *in* $\mathrm{Lat}(\mathrm{Alg}(F_3)) \setminus \{0, I, Q_\infty\}$*, there are* K_z *and* U_z *in* $Q_\infty L_{G_3} Q_\infty$ *such that*

$$Q = Q_z = \begin{pmatrix} K_z & \sqrt{K_z(I - K_z)}U_z \\ U_z^z\sqrt{K_z(I - K_z)} & U_z^*(I - K_z)U_z \end{pmatrix},$$

where

$$\sqrt{K_z(I - K_z)^{-1}}U_z = (1 - z)\sqrt{H_1(I - H_1)} - \sqrt{H_2(I - H_2)}V,$$

for some $z \in \mathbb{C}$*. Moreover for any given z in* $\mathbb{C}$*, the polar decomposition determines* U_z *and* K_z *uniquely which give rise to a projection* Q_z *(in the above form) in* $\mathrm{Lat}(\mathrm{Alg}(F3))$.

Let $\mathscr{L} = \mathrm{Lat}(\mathrm{Alg}(F_3))$ and $G[\mathscr{L}] = \{\alpha \in Aut(L_{G_3}) | \alpha(\mathscr{L}) = \mathscr{L}\}$.

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Theorem

If $\alpha \in G[\mathscr{L}]$ such that $\alpha(Q_0) = Q_{z_1}$, $\alpha(Q_\infty) = Q_{z_2}$, $\alpha(Q_{-1}) = Q_{z_3}$, then $\forall z \in \mathbb{C}$, we have $\alpha(Q_z) = Q_{f(z)}$, where

$$f(z) = \frac{zz_2(z_3 - z_1) + z_1(z_3 - z_2)}{z(z_3 - z_1) + (z_3 - z_2)},$$

the unique Möobius transformation satisfies $f(0) = z_1, f(1) = z_2, f(\infty) = z_3$.

$\alpha_1(Q_\infty) = Q_\infty$, $\alpha_1(Q_0) = Q_{-1}$, $\alpha_1(Q_{-1}) = Q_0$, then

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$\alpha_2(Q_\infty) = Q_0$, $\alpha_1(Q_0) = Q_{-1}$, $\alpha_1(Q_{-1}) = Q_\infty$, then

$$\alpha_1(Q_z) = Q_{\frac{-1}{1+z}}$$

Basic facts about Möbius transformation

For any $A \in GL_2(\mathbb{C})$, the Möbius transformation is defined as

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- g is elliptic if and only if $g \sim m_k$, $m_k(z) = kz$, $|k| = 1$, $k \neq 1$.

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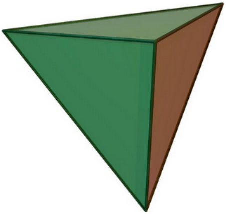

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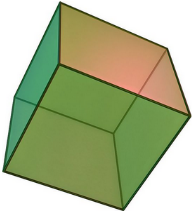

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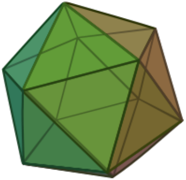

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and two infinite closed subgroups :

- $C_\infty \approx SO(2)$ generated by an arbitrary rotation around an axis,
- D_∞ which is generated by C_∞ and a rotation π around an axis orthogonal to the axis of rotation of C_∞.

Distance between Q_z and Q_∞

$$\|Q_z - Q_\infty\|_2^2 = 1 - 2tr(Q_z Q_\infty),$$

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$$\|Q_z - Q_\infty\|_2 = \sqrt{\frac{1}{1 + |z| + |z+1|}}$$

$$\|Q_z - Q_0\|_2 = \sqrt{\frac{|z|}{1 + |z| + |z+1|}}$$

$$\|Q_z - Q_{-1}\|_2 = \sqrt{\frac{|z+1|}{1 + |z| + |z+1|}}$$

A distance formula

$$dist(z_1, z_2) = \sqrt{\frac{2|z_1 - z_2|}{(1+|z_1|+|z_1+1|)(1+|z_2|+|z_2+1|)}},$$

and

$$dist(\infty, z_2) = \sqrt{\frac{1}{(1+|z_2|+|z_2+1|)}},$$

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- $z_1 = -1$ and $z_2 \in [0, +\infty]$

Fuglede-Kadison-determinant

Let $\mathfrak{M}$ be a II_1 factor, and τ be the unique normal faithful tracial state.
For $T\eta\mathfrak{M}$(T is a closed, densely defined operators that affiliated with $\mathfrak{M}$), if

$$\tau(log^+|T|) = \int_0^\infty log^+(t)d\mu_{|T|} < \infty, log^+(t) = max(0, log(t)),$$

the Fuglede-Kadison determinant Δ is defined by:

$$\Delta(T) = exp(\int_0^\infty log(t)d_{\mu_{|T|}}(t)),$$

where $T = U|T| = U\int_0^\infty tdE_{|T|}(t)$.

Δ has the following properties: $\Delta(ST) = \Delta(S)\Delta(T)$,
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Theorem

$G[\mathscr{L}] \approx S_3$.

Thank You!