Unbounded Operator With Trivial Relative Commutant

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$$U\mathfrak{D}(T) = \mathfrak{D}(T)$$

Lemma 16.4.4. *Let* $X_1, X_2, \cdots$ *be a (finite or infinite) sequence of linear, closed operators, each one having an everywhere dense domain. Assume that all* $X_i \,\eta\, \mathbf{M}$. *Let* $p(x_1, y_1, x_2, y_2, \cdots)$, $q(x_1, y_1, x_2, y_2, \cdots)$, $r(x_1, y_1, x_2, y_2, \cdots)$ *be (non-commutative) polynomials of the symbolic variables* x_i, y_i *corresponding to* X_i, $i = 1, 2, \cdots$.

Then we have

(i) $[p(X_1, X_1^*, X_2, X_2^*, \cdots)]$ *can be formed, it is linear, closed, has an everywhere dense domain, and it is* $\eta\, \mathbf{M}$ *too.*

(ii) *If* ${}^{(r)}p(x_1, y_1, x_2, y_2, \cdots) = {}^{(r)}q(x_1, y_1, x_2, y_2, \cdots)$ *then* $[p(X_1, X_1^*, X_2, X_2^*, \cdots)] = [q(X_1, X_1^*, X_2, X_2^*, \cdots)]$; *that is* $[p(X_1, X_1^*, X_2, X_2^*, \cdots)]$ *depends on* ${}^{(r)}p(x_1, y_1, x_2, y_2, \cdots)$ *only.*

(iii) *If* $p(x_1, y_1, x_2, y_2, \cdots)^+ = q(x_1, y_1, x_2, y_2, \cdots)$ *then* $[p(X_1, X_1^*, X_2, X_2^*, \cdots)]^* = [q(X_1, X_1^*, X_2, X_2^*, \cdots)]$.

(iv) *If* $ap(x_1, y_1, x_2, y_2, \cdots) = q(x_1, y_1, x_2, y_2, \cdots)$ *then* $[a[p(X_1, X_1^*, X_2, X_2^*, \cdots)]] = [q(X_1, X_1^*, X_2, X_2^*, \cdots)]$. *(If* $a \neq 0$ *then the first* [ ] *is obviously unnecessary).*

(v) *If* $p(x_1, y_1, x_2, y_2, \cdots) + q(x_1, y_1, x_2, y_2, \cdots) = r(x_1, y_1, x_2, y_2, \cdots)$ *then* $[[p(X_1, X_1^*, X_2, X_2^*, \cdots)] + [q(X_1, X_1^*, X_2, X_2^*, \cdots)]] = [r(X_1, X_1^*, X_2, X_2^*, \cdots)]$.

(vi) *If* $p(x_1, y_1, x_2, y_2, \cdots) \cdot q(x_1, y_1, x_2, y_2, \cdots) = r(x_1, y_1, x_2, y_2, \cdots)$ *then* $[[p(X_1, X_1^*, X_2, X_2^*, \cdots)] \cdot [q(X_1, X_1^*, X_2, X_2^*, \cdots)]] = [r(X_1, X_1^*, X_2, X_2^*, \cdots)]$.

Figure : On Rings of Operators(1936)

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- Is there a $T\eta\mathfrak{A}$ such that T does not commute with any non trivial normal operator in $\mathfrak{A}$?

Fuglede-Putnam Theorem

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$$TN \subset MT$$

$$\Rightarrow TN^* \subset M^*T$$

Lemma

Let $\mathfrak{A}$ be a finite von Neumann algebra. If N is a normal operator and P is a projection such that $(I-P)NP = 0$, then $PN = NP$.

Lemma

Let $\mathfrak{A}$ be a finite von Neumann algebra, and H is a closed positive operator affiliated with $\mathfrak{A}$. If M and N are normal operators in $\mathfrak{A}$ and $NH = HM$, then $NH = HN$, $MH = HM$ and $M^*H = HN^*$.

Proof.

We assume that $Ker(H) = \{0\}$ and

$$H = \int_0^\infty \lambda dE_\lambda$$

Proof.

$$P = E_\lambda = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \quad H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}$$

$$N = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}, \quad M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

$H_1 = HE_\lambda$, $\|H_1\| \leq \lambda$ and $H_2 = H(I - E_\lambda)$, $\|H_2^{-1}\| \leq \frac{1}{\lambda}$

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$$\Rightarrow \begin{pmatrix} H_1^{-1}N_{11}H_1 & H_1^{-1}N_{12}H_2 \\ H_2^{-1}N_{21}H_1 & H_2^{-1}N_{22}H_2 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

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$$\Rightarrow \tau|_{P\mathfrak{A}P}(H_1 N_{21}^* H_2^{-2} N_{21} H_1) = \tau|_{P\mathfrak{A}P}(H_1^{-1} N_{12} H_2^2 N_{12}^* H_1^{-1})$$

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$$\tau\left(\begin{pmatrix} H_1 N_{21}^* H_2^{-2} N_{21} H_1 & 0 \\ 0 & 0 \end{pmatrix}\right) \leq \frac{1}{\lambda^2}\tau\left(\begin{pmatrix} H_1 N_{21}^* N_{21} H_1 & 0 \\ 0 & 0 \end{pmatrix}\right)$$

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$$= \frac{1}{\lambda^2}\tau\left(\begin{pmatrix} 0 & H_1 N_{21}^* \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ N_{21} H_1 & 0 \end{pmatrix}\right)$$

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Let $Q = E_\beta - E_\lambda$, $\beta > \lambda$

$$\begin{aligned} \frac{\beta^2}{\lambda^2}\tau\left(\begin{pmatrix} N_{12}(I-Q)N_{12}^* & 0 \\ 0 & 0 \end{pmatrix}\right) &+ \tau\left(\begin{pmatrix} N_{12} Q N_{12}^* & 0 \\ 0 & \end{pmatrix}\right) \\ &\leq \tau\left(\begin{pmatrix} H_1^{-1} N_{12} H_2^2 N_{12}^* H^{-1} & 0 \\ 0 & 0 \end{pmatrix}\right) \end{aligned}$$

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$$E_\lambda N = NE_\lambda \text{ and } NH = HN$$

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Corollary

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Corollary

If $\mathfrak{A}$ be a separable II$_1$ factor, then there exists a closed operator T affiliated with $\mathfrak{A}$ such that $NT \neq NT$ for any nontrivial normal operator affiliated with $\mathfrak{A}$.

Is there a $T\eta\mathfrak{A}$ such that $T' \cap \mathfrak{A} = \{\mathbb{C}I\}$?

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$$\sum_n \alpha(-n)(h) \otimes E_n, \qquad I \otimes l_n \qquad h \in L^\infty(\mathbb{T})$$

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$AT \neq TA$ for each $A \in \mathcal{R} \setminus \{\mathbb{C}I\}$

$$h(\theta) = \frac{e^{2\pi i\theta} + 1}{e^{2\pi i\theta} - 1}$$

$$A = \sum_m \sum_n \alpha(-n)(f_n) \otimes E_n l_m$$

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$$\frac{\alpha(1)(f_n)}{f_n} = \frac{\alpha(n)(h)}{h}$$

$$k_m = \begin{cases} h\alpha(1)(h)\cdots\alpha(m-1)(h) & \text{if } m > 0 \\ 1 & \text{if } m = 0 \\ \alpha(-1)(\frac{1}{h})\alpha(-2)(\frac{1}{h})\cdots\alpha(m)(\frac{1}{h}) & \text{if } m < 0 \end{cases}$$

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$$f_m = \lambda k_m$$

Thank You!