Fundamentals of Electrical Circuits

VIII

AC Power Analysis

AC POWER. MAXIMUM POWER TRANSFER THEOREM IN SINUSOIDAL STEADY STATE. METHODS OF ANALYSIS IN SINUSOIDAL STEADY STATE.

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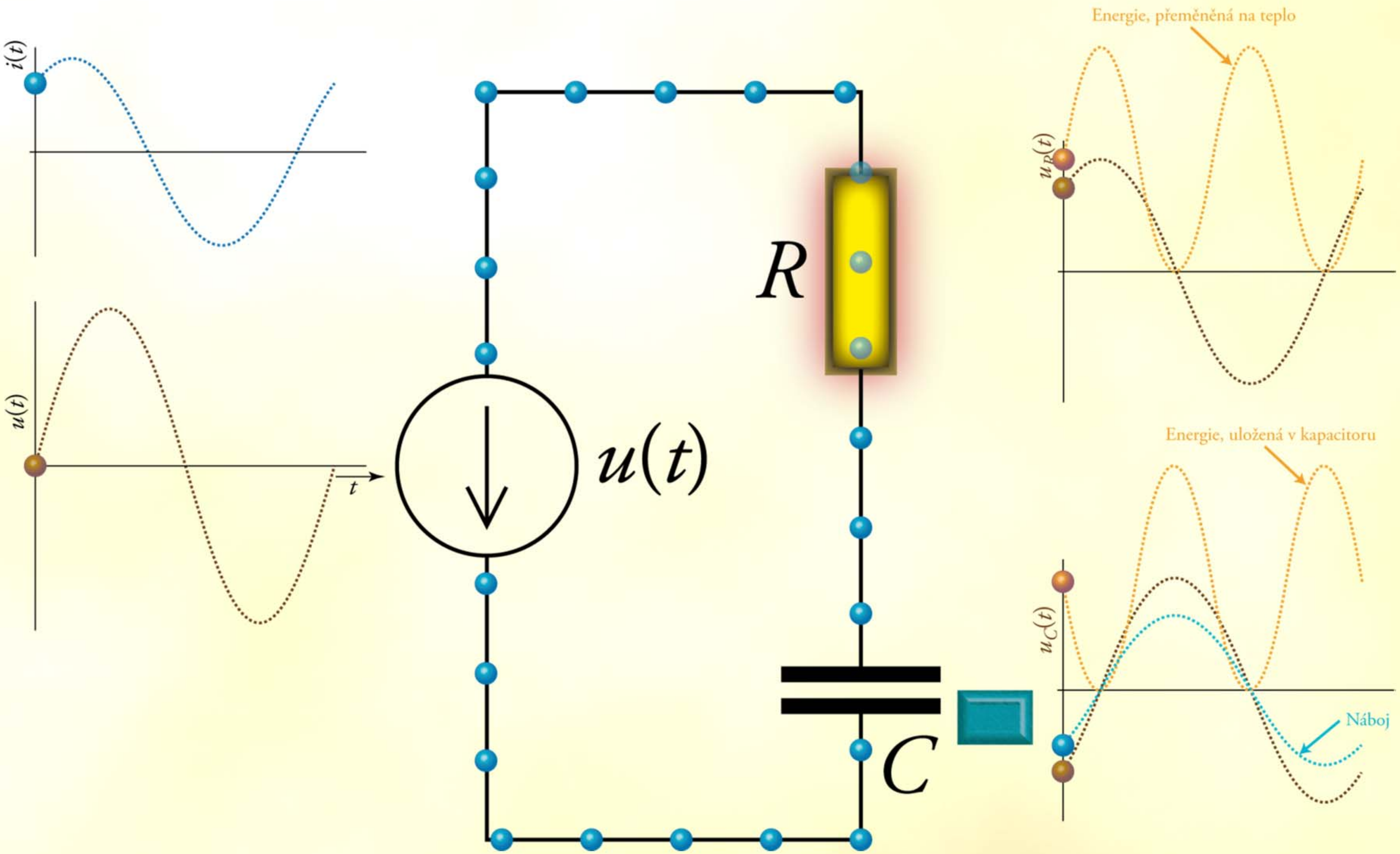


i(t)
u(t)
t
u(t)
R
C
Energie, přeměněná na teplo
$u_R(t)$
Energie, uložená v kapacitoru
$u_C(t)$
Náboj

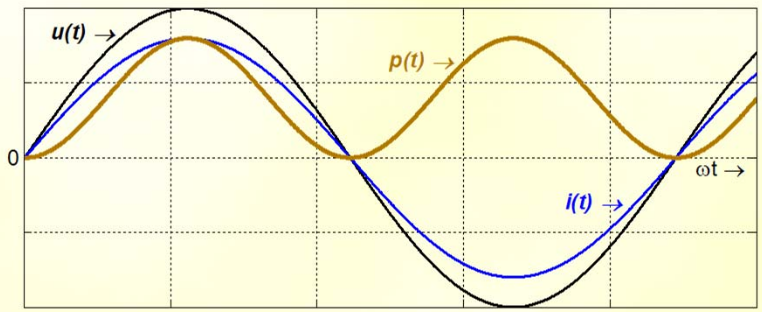
AC Power

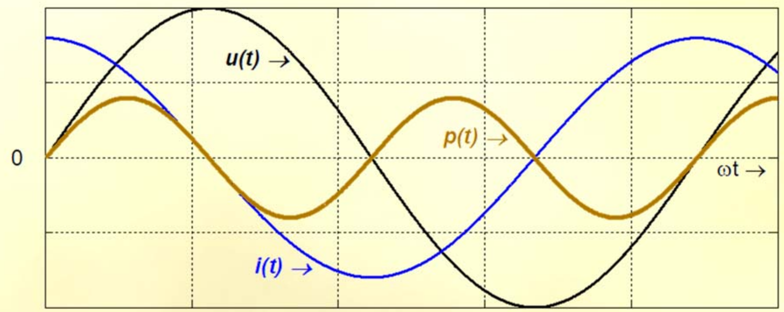
- **Resistor**: current, which flows through the resistor, heats it up, so the energy, delivered to the resistor, is irreversibly converted on heat
- **Capacitor**: electric current, which flows into the capacitor, delivers electric charge – capacitor stores the energy (in its electric field), when we change orientation of electric current (polarity of source, connected into the circuit), it draws back the charge, and capacitor returns previously stored energy $W_c = \frac{1}{2}CU^2$
- **Inductor**: it stores an energy in its magnetic field, it does not dissipate energy $W_L = \frac{1}{2}LI^2$

⇒ **The only circuit element, which give off useful energy, is resistor (it can be even model of mechanical load of motor or loudspeaker), capacitors and inductors in AC circuits just periodically exchange energy with sources or among each other**

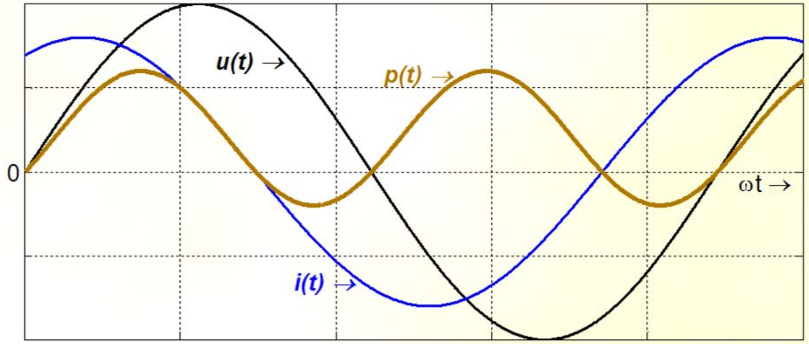
- **Instantaneous power** *(general definition of power)*: $p(t) = u(t)i(t)$
- Unlike the DC, in AC both voltage and current are sinusoidal, so when the circuit contains some inductors and capacitors, passing current can be phase shifted with respect to the voltage: $u(t) = U_m \sin(\omega t)$

$$i(t) = I_m \sin(\omega t + \varphi)$$





An examples of instantaneous power waveform when current has different phase shift with respect to the voltage – in the left figure the voltage and current are in phase (it contains just resistors, the value of instantaneous power is always positive), in the right figure current leads the voltage by $\frac{\pi}{2}$ (capacitor, average value of instantaneous power is zero)



An example of instantaneous power waveform when current leads the voltage by $\frac{\pi}{3}$ - the circuit may be resistor and capacitor connected in series; in such case the source delivers an energy to the circuit just within a part of the period, another part of the period capacitor returns stored energy back; but portion of an energy delivered by the source is irreversibly converted to heat

- Instantaneous power is a time-varying quantity, It may have both + sign (power is being delivered to or absorbed by the circuit), or - sign (power is being supplied by the element / circuit, or, is being returned back), the angular frequency is twice the frequency of a voltage / current.
- But we are interested in total energy or, in average energy delivered each second ⇒ mean (average) value

Derivation:

$$P = \frac{1}{T}\int_0^T p(t)\,\mathrm{d}t = \frac{1}{T}\int_0^T u(t)i(t)\,\mathrm{d}t$$

$$P = \frac{1}{T}\int_0^T U_m \sin(\omega t)\cdot I_m \sin(\omega t+\varphi)\,\mathrm{d}t = \frac{U_m I_m}{2T}\left[\int_0^T \cos(\varphi)\,\mathrm{d}t - \underbrace{\int_0^T \cos(2\omega t+\varphi)\,\mathrm{d}t}_{=0}\right] = \frac{U_m I_m}{2T}\cos(\varphi)\,[t]_0^T =$$

$$= \frac{1}{2}U_m I_m \cos(\varphi)$$

- Mean value of power, delivered to the circuit, is proportional to the cosine of **phase shift** between voltage and current

Mean value of instantaneous power is power, which is irreversibly delivered to the circuit, is called

Active (average) power

$$\boxed{P = \frac{1}{2}U_m I_m \cos\varphi = UI\cos\varphi}$$

its unit is Watt [W]

- Maximum average power is delivered to the circuit, if $\varphi = 0$, so voltage and current are in phase. Resistive power absorbs power at all times, while a reactive (L, C) load absorbs zero average power.
- When the phase shift between voltage and current is $\pm\frac{\pi}{2}$, the average power, delivered to the circuit, is zero. No heat, no light, no work. But it is clear, the **current still flows through the wires**, and distribution network has to be **sized up proportionally to this total current**!!! That is the reason why to define **apparent power**, which is proportional to the **total energy, transferred through the wires each second in both directions**. It is defined by RMS values of voltage and current as

Apparent power

$$\boxed{S = UI = \frac{1}{2}U_m I_m}$$

its unit is Volt Ampere [VA]

- To evaluate power, exchanged among sources, inductors and capacitors, see apparent (total) power and average power – it in addition has the cosine term. Since $1 - \cos^2(\varphi) = \sin^2(\varphi)$

Reactive power

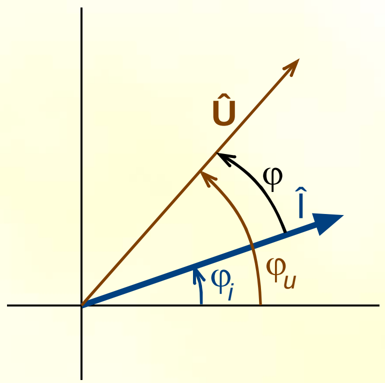
$$\boxed{Q = \frac{1}{2}U_m I_m \sin\varphi = UI\sin\varphi}$$

its unit is volt-ampere reactive [var]

With regard to the phase shift and ever-present trigonometric functions the Pythagorean theorem is “hidden” in power relationship:

$$\boxed{S = \sqrt{P^2 + Q^2}}$$

Even the base units are same, to distinguish among active, reactive and apparent power, each power has its own unit (active power has Watts, reactive has volt-ampere reactive, apparent has volt ampere).



Keep in mind! The angle φ is phase shift between voltage and current, from the voltage to the current, phase shift of current is subtracted from phase shift of voltage $\varphi = \varphi_u - \varphi_i$

AC power and phasors

- Since in sinusoidal steady state analysis harmonic functions may be transformed on phasors, it is possible to define power directly using phasors. Negative sign of current phase shift is qualified by complex conjugate because $\mathbf{I}^* = I\underline{/-\varphi_i}$
- In this way we can define **complex power**

$$\mathbf{S} = \frac{1}{2}\mathbf{U}_m\mathbf{I}_m^* = \frac{1}{2}U_m \cdot \underline{/\varphi_u} \cdot I_m\underline{/-\varphi_i} = \frac{1}{2}U_m I_m \cdot \underline{/\varphi_u - \varphi_i} = \frac{1}{2}U_m I_m \cdot \underline{/\varphi}$$

- When we use Euler's identity $e^{j\varphi} = \cos\varphi + j\sin\varphi$, we obtain

$$\mathbf{S} = \frac{1}{2}U_m I_m \cdot e^{j\varphi} = \frac{1}{2}U_m I_m\underline{/\varphi} = \frac{1}{2}U_m I_m(\cos\varphi + j\sin\varphi) = P + jQ$$

$$\boxed{\mathbf{S} = \mathbf{U}\mathbf{I}^* = \frac{1}{2}\mathbf{U}_m\mathbf{I}_m^* = P + jQ}$$

$$\boxed{P = \mathrm{Re}\{\mathbf{S}\} = \mathrm{Re}\{\mathbf{U}\mathbf{I}^*\} = \mathrm{Re}\left\{\frac{1}{2}\mathbf{U}_m\mathbf{I}_m^*\right\}}$$

$$\boxed{Q = \mathrm{Im}\{\mathbf{S}\} = \mathrm{Im}\{\mathbf{U}\mathbf{I}^*\} = \mathrm{Im}\left\{\frac{1}{2}\mathbf{U}_m\mathbf{I}_m^*\right\}}$$

$$\boxed{S = |\mathbf{S}|}$$

AC power on distinct circuit elements

- Finally, third way, how to calculate AC power, arise from its physical principle. We can determine AC power on each circuit element separately:
- R – heat, dissipated on resistor

$$\boxed{P_R = UI = RI^2 = \frac{U^2}{R}}$$

- **C** – again, power (average exchanged energy!!!) on capacitor is a product of voltage and current (on capacitor!!!): current leads the voltage by $\frac{\pi}{2} \Rightarrow$ always **negative sign** $\sin(0 - \frac{\pi}{2}) = -1$

$$\boxed{Q_C = -UI = X_C \cdot I \cdot I = \frac{-1}{\omega C} I^2 = -U^2 \omega C}$$

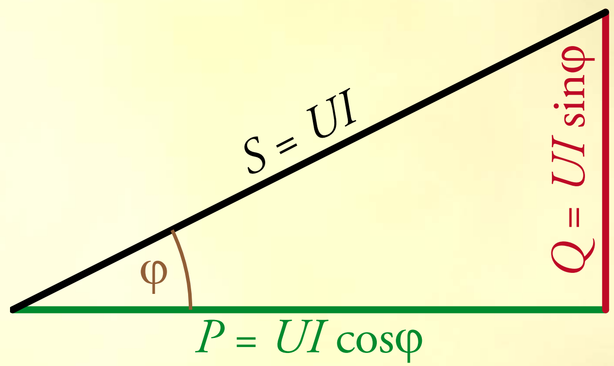
- **L** – current lags the voltage by $\frac{\pi}{2} \Rightarrow$ always **positive sign** $\sin(0 - \frac{-\pi}{2}) = 1$

$$\boxed{Q_L = UI = X_L \cdot I \cdot I = \omega L I^2 = \frac{U^2}{\omega L}}$$

- Using this approach we need to know voltage and current on each circuit element (*R*, *L*, *C*) separately! From this point of view it is useful just in a very simple circuits.

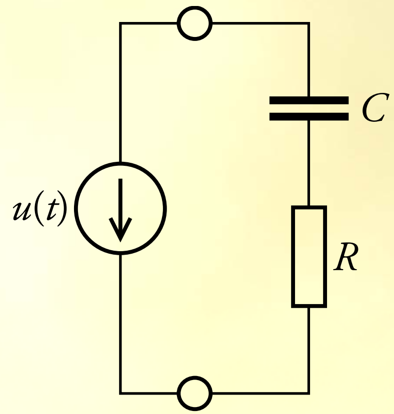
Power triangle – phasor diagram of powers

Recall the active power is cosine term of the apparent power (or real part of complex power) and reactive power is its sine term (or imaginary part of complex power).
Similar to the case of voltages and currents, it is possible represent powers with oriented vectors. Vectors representing active and reactive powers are perpendicular so they form a triangle, known as the power triangle.



The angle φ between active and apparent power is the difference between voltage and current phase shift.

Example:
The circuit is supplied by sinusoidal voltage source: $u(t) = 10\,\sin(1000t - \frac{\pi}{3})$
$R = 1\,\mathrm{k}\Omega$, $C = 1\,\mu\mathrm{F}$. Calculate active, reactive and apparent power.



The impedance of the circuit is:

$$\mathbf{Z} = \frac{1}{j\omega C} + R = \frac{-j}{1000\cdot 10^{-6}} + 1000 = 1000 - 1000j = 1000\sqrt{2}\,\underline{/-\frac{\pi}{4}}$$

Phasor of current which passes the circuit and its waveform is:

$$\mathbf{I}_m = \frac{\mathbf{U}_m}{\mathbf{Z}} = \frac{10\,\underline{/-\frac{\pi}{3}}}{1000\sqrt{2}\,\underline{/-\frac{\pi}{4}}} = \frac{0.01}{\sqrt{2}}\underline{/-\frac{\pi}{12}} \quad \rightarrow \quad i(t) = \frac{0.01}{\sqrt{2}}\sin(1000t - \frac{\pi}{12})$$

First, we can use sin / cos form related to waveforms (remember we know amplitudes!):

$$P = \frac{1}{2}U_m I_m \cos(\varphi_u - \varphi_i) = \frac{1}{2}\cdot 10\cdot\frac{0.01}{\sqrt{2}}\cdot\cos\left(\frac{-\pi}{3} - \frac{-\pi}{12}\right) = 25\,\mathrm{mW}$$

$$Q = \frac{1}{2}U_m I_m \sin(\varphi_u - \varphi_i) = \frac{1}{2}\cdot 10\cdot\frac{0.01}{\sqrt{2}}\cdot\sin\left(\frac{-\pi}{3} - \frac{-\pi}{12}\right) = -25\,\mathrm{mvar}$$

$$S = \sqrt{P^2 + Q^2} = \sqrt{0.025^2 + 0.025^2} = 35.35\,\mathrm{mVA}$$

Another way, how to determine AC powers, is use phasors to evaluate complex power:
Complex power is:

$$\mathbf{S} = \frac{1}{2}\mathbf{U}_m\mathbf{I}_m^* = \frac{1}{2}10\,\angle{-\tfrac{\pi}{3}}\cdot\frac{0.01}{\sqrt{2}}\,\angle{\tfrac{\pi}{12}} = \frac{0.01}{2\sqrt{2}}\left[\cos\left(\frac{-\pi}{3}+\frac{\pi}{12}\right)+j\sin\left(\frac{-\pi}{3}+\frac{\pi}{12}\right)\right] = \underbrace{0.025}_{P}\,\underbrace{-0.025}_{Q}\,j$$

Using this method we got both active and reactive power in single step. Apparent power would be again evaluated by means of Pythagorean theorem.

Finally, we can determine powers directly on distinct circuit elements:

$$P = \frac{1}{2}RI^2 = \frac{1}{2}\cdot 1000\cdot\left(\frac{0.01}{\sqrt{2}}\right)^2 = 25\,\mathrm{mW} \qquad Q = \frac{1}{2}\frac{-1}{\omega C}I^2 = \frac{1}{2}\cdot\frac{-1}{1000\cdot 10^{-6}}\cdot\left(\frac{0.01}{\sqrt{2}}\right)^2 = -25\,\mathrm{mvar}$$

Note that in this case **there are no phase shifts** – why? In this case we use voltage and current on distinct circuit elements, so that phase shift is 0, or 90º and their cos / sin values are in both cases 1, or in other worlds, we calculate power on lags of impedance triangle, not on its hypotenuse (equivalent to power triangle). In first two methods we used different (total) voltage and current.

Finally recall phase shift in first two methods, between voltage and current:
Phase shift of voltage was $\frac{-\pi}{3}$, phase shift of current $\frac{-\pi}{12}$, the angle of impedance was $\frac{-\pi}{4}$ and phase difference between voltage and current was – again $\frac{-\pi}{4}$. It is not just coincidence. Generally, the current and complex power are given by:

$$\mathbf{I} = \frac{\mathbf{U}}{\mathbf{Z}} = \frac{U\angle\varphi_u}{Z\angle\varphi} = \frac{U}{Z}\angle\varphi_u-\varphi$$

$$\mathbf{S} = \mathbf{U}\mathbf{I}^* = U\angle\varphi_u\cdot\frac{U}{Z}\angle{-\varphi_u+\varphi} = \frac{U^2}{Z}\angle\varphi_u-\varphi_u+\varphi = \frac{U^2}{Z}\angle\varphi$$

The phase shift between voltage and current, when we evaluate total AC power, delivered to the circuit, is the angle of total impedance of the circuit.

Power factor

Since only active power perform useful work, but apparent power is related to total energy transferred through the wires per second, the power factor may be considered as rate of utilization of electrical device. 1 implies that total transferred energy is dissipated in the load, 0 implies, the energy is uselessly transferred from the source to the load and back. It is dimensionless quantity, but sometimes is multiplied by 100 and expressed as a percentage %.

$$\boxed{\lambda = \frac{P}{S} = \cos\varphi}$$

Note: The power factor is sometimes referred to as φ, „cosine of phi“, or just „cosine“. But it is valid only with harmonic waveforms, so in linear circuits. In nonlinear circuits (e.g. switched sources of present-day appliances – computers, monitors and TV flat panels, DVD players and recorders, …) such term could not be valid anymore. Circuit variables are in such case described by Fourier series and there is no unique phase shift φ. Likewise in three phase circuits there is no unique phase shift.

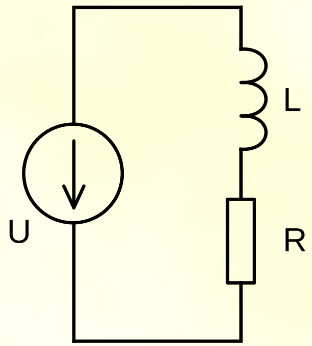
Power factor correction

Seeing that the electricity transmission lines must be designed to transfer apparent power S (greater passing current!), the power factor should be as greatest as possible (usually > 0.90, ideally 0.95). Big energy consumers have to either correct reactive power of their appliances (usually motors), or to pay for "consumed" apparent power, qualified from the power factor. Consumer sector still pay just for consumed active power - fortunately, because power factor of consumer electronic appliances in stand-by mode is usually less than 0.1. For example flat TV panel may have active power $P = 3$ W, but $|Q| > 50$ var. The nature of this reactive power is capacitive (capacitors in switched power source) so it may partially correct reactive power of motors or other inductive appliances.

There are two different methods, how to correct power factor:

a) Right in the appliance – to the appliance is in parallel connected correction element. It has same reactive power as the corrected appliance, but opposite signed. Then the correction element interchange energy with the circuit (they are near, but not in resonance); example include fluorescent tube
b) Whole factory building is corrected at once – blocks of capacitors are connected or disconnected from the distribution net (smaller powers), or over-excited unloaded synchronous motor ("synchronous condenser "); disadvantages of the capacitors – they are susceptible to failure caused by non-linear distortion (in non-linear circuits currents or even voltages has high-frequency components – high currents passing capacitors → overheat ⇒ destruction), there are transients when connected to the circuit, it is necessary discharge capacitors when disconnected, limited power

Example: Appliance consist of series connected inductor *L* and resistor *R*. The objective is to raise power factor of the circuit. L = 0.42 H, R = 68 Ω, U = 230 V, 50 Hz



Current, drawn from the source:

$$\mathbf{I} = \frac{\mathbf{U}}{R + j\omega L} = \frac{230}{68 + j \cdot 100 \cdot \pi \cdot 0.42} = 0.71 - 1.377j = 1.55\underline{/-1.095}$$

Complex power:

$$\mathbf{S} = \mathbf{U} \cdot \mathbf{I}^* = 230 \cdot 1.55\underline{/1.095} = 356.5\underline{/1.095} = 163.37 + 316.71j$$

Active power can be determined in different ways:

$$\begin{aligned} P &= \mathrm{Re}\{\mathbf{S}\} = 163.37 \text{ W} \\ &= UI\cos(\varphi_u - \varphi_i) = 230 \cdot 1.55 \cdot \cos(1.095) = 163.37 \text{ W} \\ &= RI^2 = 68 \cdot 1.55^2 = 163.37 \text{ W} \end{aligned}$$

… as well as reactive power

$$\begin{aligned} Q &= \mathrm{Im}\{\mathbf{S}\} = 316.71 \text{ var} \\ &= UI\sin(\varphi_u - \varphi_i) = 230 \cdot 1.55 \cdot \sin(1.095) = 316.71 \text{ var} \\ &= \omega L \cdot I^2 = 100 \cdot \pi \cdot 0.42 \cdot 1.55^2 = 316.71 \text{ var} \end{aligned}$$

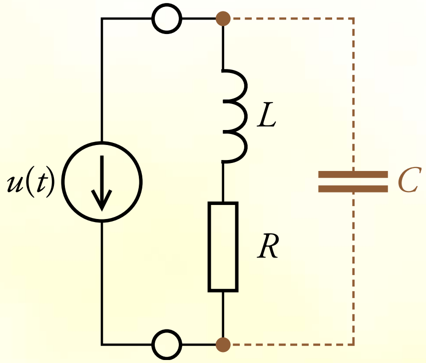
Apparent power:

$$S = |\mathbf{S}| = 356.5 \text{ VA}$$

Power factor of this circuit is:

$$\lambda = \frac{P}{S} = \frac{163.7}{356.5} = 0.458 = 45.8\ \%$$

To achieve power factor correction the circuit element with negative reactive power has to be connected – the capacitor. It will be connected in parallel:



If we would rise power factor to 1, reactive power of capacitor would be:

$$Q_C \overset{!}{=} -316.71 \text{ var} = -U^2 \cdot \omega C$$

Then: $$C = \frac{-Q_C}{U^2 \cdot \omega} = \frac{316.71}{230^2 \cdot 100 \cdot \pi} = 19.07\ \mu\text{F}$$

Total impedance of the circuit $\mathbf{Z} = \dfrac{\frac{1}{j\omega C} \cdot (R + j\omega L)}{\frac{1}{j\omega C} + (R + j\omega L)} \doteq 324\ \Omega$ **is real.**

Total reactive power is 0 var, $P = S = 163.37$ [W / VA], power factor $\lambda \to 1$

Total current, loaded from the source is **I** = 0.71 A - **wires are loaded by less than half of the primary current – or, RMS value of the current, loaded from the source after power factor correction is in this circuit 45.8 % of the RMS value of the current, loaded from the source by the same, but uncorrected circuit.**

The power factor can be also defined as

$$\lambda = \frac{I_k}{I}$$

where I_k is the current, loaded from the source by circuit, where $\lambda = 1$
I is the circuit without power correction

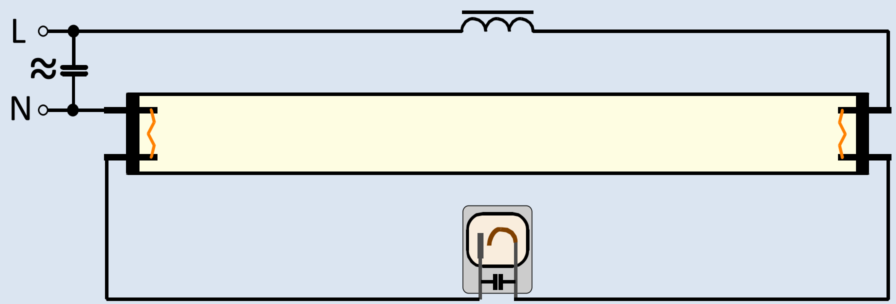
Note: raising of power factor to ***λ = 1 is never used****, because such circuit would be in* ***resonance****; λ = 0.95* is usually used

Note:

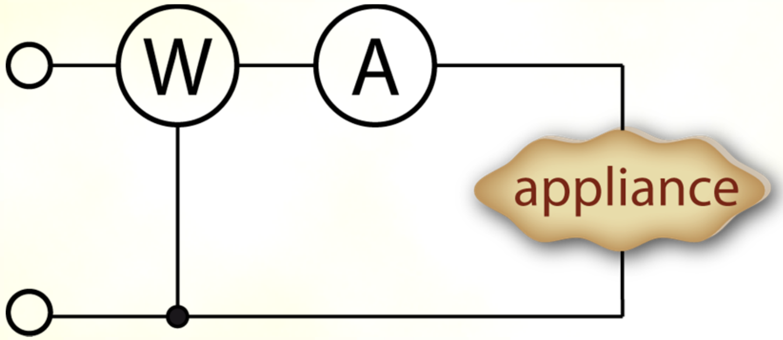
Circuit alignment arises from the circuit of the fluorescent tube. It contains 4 circuit elements:

- Fluorescent tube – fluorescent tube is the glass tube, which has on each side in the socket embedded heated tungsten filament, covered by the barium oxide, strontium oxide and calcium oxide. The electrode, which is connected through the choke coil to the live wire, emits at 700 °C electrons. The pipe is filled up by the mixture of argon (400 Pa) and mercury vapor (0.6 Pa). Flying electrons collides with an atom in the gas and gives them part of its kinetic energy. The collisions transfer energy to the atom's outer electron, causing that electron to temporarily jump up to a higher energy level. When it jumps back, it emits ultraviolet light. These photons are absorbed by electrons in the atoms of the lamp's fluorescent coating, which emits visible light. *(From the electrons jump arises major disadvantage of all energy saving fluorescent lamps – the light has discrete spectrum, similar to the "white" light emitted by a TV screen).*

  The discharge voltage is typically 100 – 160 V depending on the length of the tube. Then the voltage has to be limited. Secondly, the resistivity of the plasma discharge in the tube is negative (the more the current is, the more ionization of the gas and less resistivity, decreasing resistivity means increasing current and so forth (positive feedback) – without some limiting element the current would be growing and could destroy the tube). The choking coils are used as the ballast (the limiting element). To initial ionization much greater voltage is required.
- Starter – To start the fluorescent lamp the starter is required. It consists of a small glass cell, filled by en argon and xenon, and bi-metallic thermostat electrode. In ordinary circumstances it is disconnected. On the instant when the supplying voltage is connected the fluorescent lamp is not lighting, its resistivity is huge. In the starter cell glow-discharge is set. It heat up the bi-metallic electrode, it bends and after some time it switch on and short the fluorescent lamp. The rather hi current passes the choking coil and fluorescent tube electrodes and heat it up. The gas in the starter cool down, bi-metallic switcher switches off. But the current has to be constant and the resistivity is now much higher than the impedance of the coil inductance, so the voltage jumps. High voltage starts the discharge, the fluorescent lamp resistivity decreases and less voltage could not set another glow-discharge in the starter. The fluorescent lamp is now lighting.
- The ballast – choking coil – has two purposes – to limit voltage across fluorescent lamp and to set discharge in the lamp, see the starter section.
- Capacitor – the only purpose is the power factor correction.



Example: Unknown circuit is supplied from electrical distribution network U = 230 V, 50 Hz. The active power loaded from network P =1200 W, and passing current I = 9.5 A were measured. Current lags the voltage. Design appropriate power factor correction..



Apparent power: $S = UI = 230 \cdot 9.5 = 2185\ \text{VA}$

Power factor: $\lambda = \dfrac{P}{S} = \dfrac{1200}{2185} = 0.549 = 54.9\ \%$

Reactive power: $Q = \sqrt{S^2 - P^2} = \sqrt{2185^2 - 1200^2} = 1826\ \text{var}$

Since the current lags the voltage, the nature of the circuit is inductive so the correction element will be capacitor. It will be connected in parallel to the load. **Power factor will be raised to λ = 95%.**
First we have to calculate remaining reactive power. From power triangle:

$$Q = S\sin\varphi = \frac{P}{\lambda}\sin\varphi = \frac{P}{\cos\varphi}\sin\varphi = P\tan\varphi = 1200{\cdot}\tan(\arccos 0.95) = 394.42\,\text{var}$$

So capacitor has reactive power Q_C = –(1826 – 394) = – 1432 var. Now we can easy evaluate capacitance:

$$C = \frac{Q_C}{-U^2\omega} = \frac{-1432}{-230^2 \cdot 100 \cdot \pi} = 86.17\ \mu\text{F}$$

Current, loaded from the source:

$$I = \frac{S}{U} = \frac{\sqrt{1200^2 + 394^2}}{230} = 5.49\ \text{A}$$

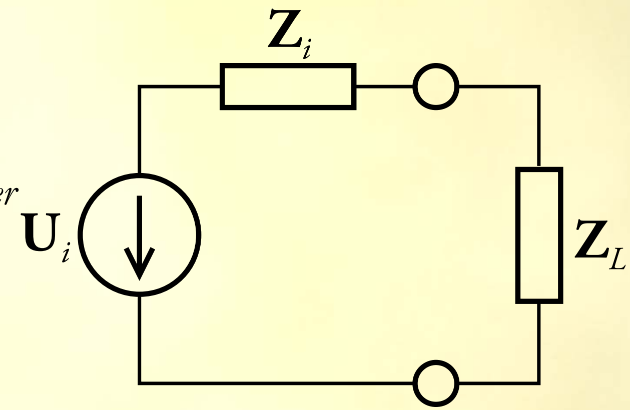
so 57.8% of initially loaded current…

The current, which flows through the capacitor $I_C = \frac{U}{X_C} = U\omega C = 6.27\ \text{A}$

… and since no capacitor is ideal, it must be sized up to such current.

Maximum average power transfer theorem

In resistive circuits we solved the problem of maximizing the power delivered by Thévenin equivalent circuit to a load – we deal with maximum power transfer theorem. But in sinusoidal steady state we have also reactive power (*the power we would like to eliminate, since it is not useful, on the contrary, it gains the current, loaded from the source and so power dissipation in wires*).



- **The objective is to deliver maximum possible active power, and minimize reactive one.**

Derivation:

Active power, delivered to the load $\mathbf{Z}_l$:

$$P_L = \mathrm{Re}\{\mathbf{U}_L\mathbf{I}_L^*\} = \mathrm{Re}\left\{\mathbf{U}_i\frac{\mathbf{Z}_L}{\mathbf{Z}_i+\mathbf{Z}_L}\cdot\left(\frac{\mathbf{U}_i}{\mathbf{Z}_i+\mathbf{Z}_L}\right)^*\right\} = \mathrm{Re}\left\{\frac{\mathbf{U}_i\mathbf{U}_i^*\mathbf{Z}_L}{(\mathbf{Z}_i+\mathbf{Z}_L)(\mathbf{Z}_i+\mathbf{Z}_L)^*}\right\} = \mathrm{Re}\left\{|\mathbf{U}_i|^2\frac{\mathbf{Z}_L}{|\mathbf{Z}_i+\mathbf{Z}_L|^2}\right\} =$$

$$= U_i^2\frac{R_L}{(R_i+R_L)^2+(X_i+X_L)^2}$$

where $\mathbf{Z}_i = R_i + jX_i,\ \mathbf{Z}_S = R_S + jX_S$

Reactive terms decrease delivered active power $\Rightarrow$ **it must be valid** $X_i + X_S = 0$

When we qualify partial derivative and set it to zero we find extremum of the function

$$\frac{\partial P_S}{\partial R_S} = U_i^2\frac{(R_i+R_S)^2 - 2R_S(R_i+R_S)}{(R_i+R_S)^4} = U_i^2\frac{R_i^2-R_S^2}{(R_i+R_S)^4} \overset{!}{=} 0$$

From here comes: $\boxed{\mathbf{Z}_i = \mathbf{Z}_S^*}$

Note it is series correction of power factor.

Note

In derivation of active power (slide 5) were used following identities:

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\cos(-\alpha) = \cos(\alpha)$$

$$\begin{aligned}
\int_0^T \cos(2\omega t + \varphi)\,\mathrm{d}t &= \int_0^T \left[\cos(2\omega t)\cos(\varphi) - \sin(2\omega t)\sin(\varphi)\right]\,\mathrm{d}t = \\
&= \cos(\varphi)\int_0^T \cos(2\omega t)\,\mathrm{d}t - \sin(\varphi)\int_0^T \sin(2\omega t)\,\mathrm{d}t = \\
&= \cos(\varphi)\left[\frac{\sin(2\omega t)}{2\omega}\right]_0^T - \sin(\varphi)\left[\frac{-\cos(2\omega t)}{2\omega}\right]_0^T = \\
&= \frac{\cos(\varphi)}{2\omega}\left[\sin\left(2\frac{2\pi}{T}T\right) - \sin(0)\right] + \frac{\sin(\varphi)}{2\omega}\left[\cos\left(2\frac{2\pi}{T}T\right) - \cos(0)\right] = 0
\end{aligned}$$