Fundamentals of Electrical Circuits

IX

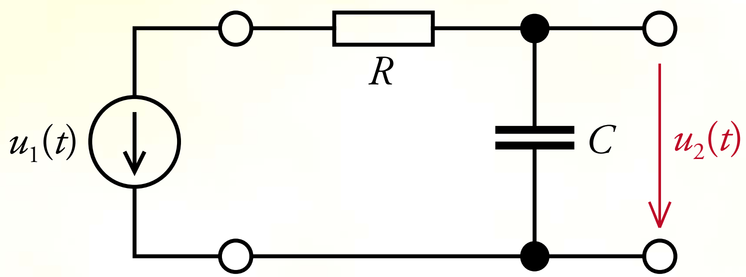
Frequency response

FREQUENCY DEPENDENCY OF CIRCUIT VARIABLES (IMPEDANCE, ADMITTANCE, TRANSFER FUNCTION). FREQUENCY RESPONSE, GRAPHICAL REPRESENTATION, BODE PLOT.

Graphical representation of frequency response

- In previous lectures we already learned, the impedance / admittance of a capacitor and inductor is frequency-dependent
- Now we will consider, how such frequency dependency affects total impedance, currents and voltages in the circuits, and, how represent such frequency dependency graphically
- What is the significance of frequency responses?
  - ✓ We want to choose suitable components into new home cinema – what is the frequency response of AV receiver, loudspeakers and another components – so how deep and high sounds such system can play?
  - ✓ We should construct amplifier for deflection circuit in oscilloscope – if the amplifier should transfer undistorted sawtooth signal, it have to have proper frequency response.
  - ✓ We should design filter, which block out mains disturbance – we have to understand to frequency response.
  - ✓ May given bus operate at given frequency? – even this is related to frequency response…
  - ✓ With which kind of signals (frequency, waveform) can use given gauging device (voltmeter, ammeter, wattmeter)
  - ✓ … and many other applications…

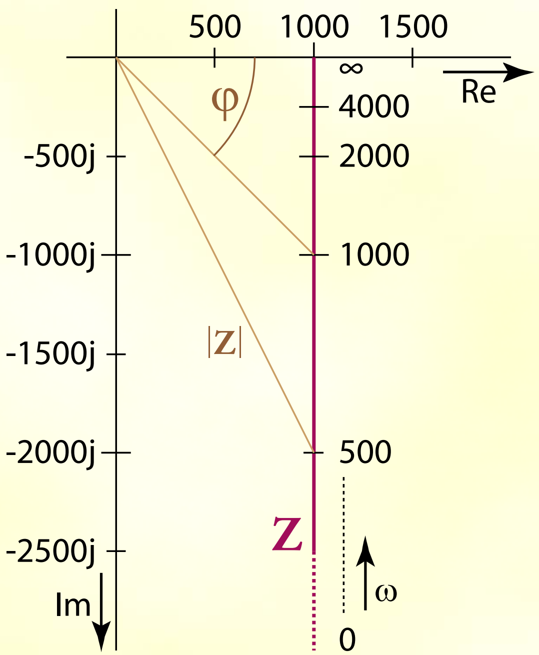
- Consider series RC circuit (integrating circuit):



R = 1 kΩ, C = 1 μF, U_{1m} = 1V

Frequency f varies from 0 to ∞

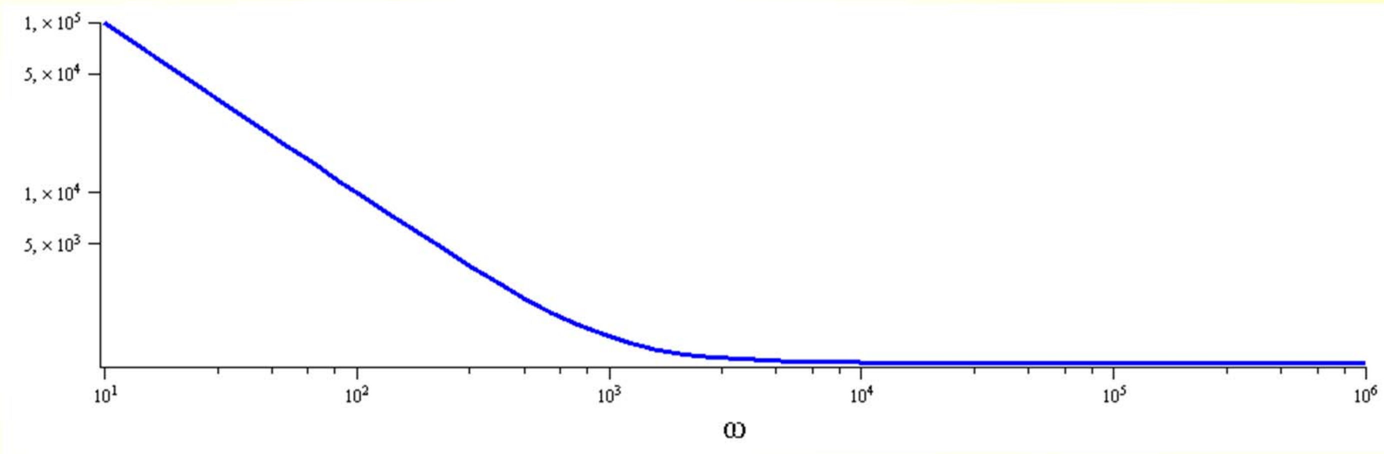
- Impedance of the circuit: $$\mathbf{Z} = R + \frac{1}{j\omega C} = 1000 - \frac{10^6}{\omega} j$$
- While real part of impedance of this circuit is constant, imaginary part varies with frequency, from -∞ to 0
- With different frequencies we obtain different complex numbers, which we can draw in complex plane:



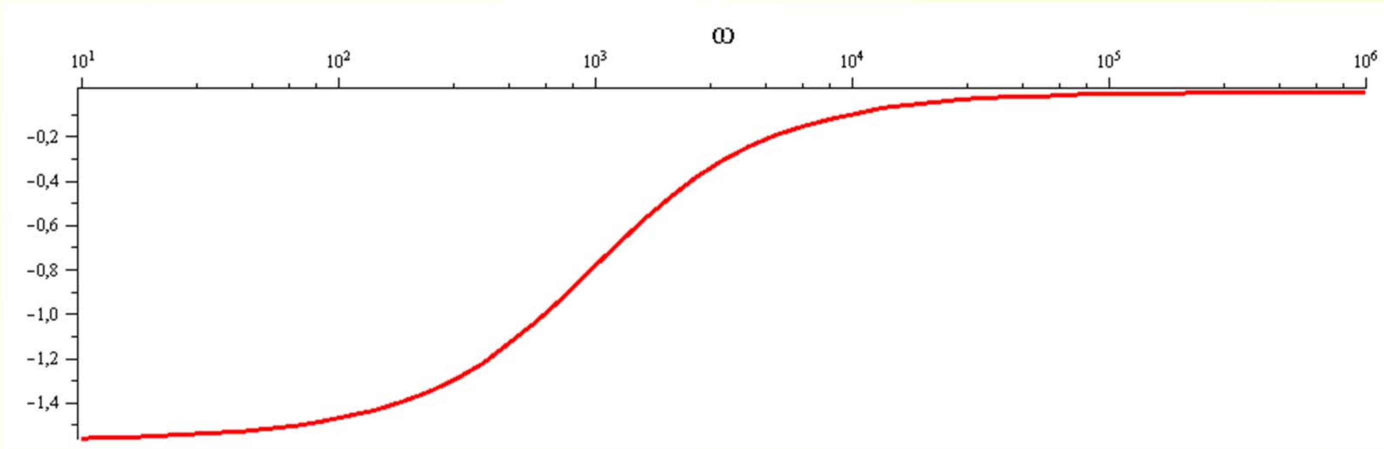
- Here the impedance is drawn as red line in complex plane (generally it is curve)
- Impedance curve has scale from 0 to ∞
- The value of impedance can be read out as distance of selected point on impedance curve from coordinate origin (at distinct frequency)
- Phase shift is angle between real half-axis and join of distinct point on impedance curve and origin.
- It is related as Nyquist plot; Nyquist plot of transfer function is uses especially in control engineering, since it is possible to determine, if the circuit is *stable,* but, generally the reading of this graph is … somewhat complicated

- From this reason frequency response is usually separated into two distinct graphs as **modulus (magnitude) frequency response** and **phase frequency response**.

Magnitude frequency response of impedance



Phase frequency response of impedance



- Frequency axis is always logarithmic
- Modulus axis is logarithmic, in the case of impedance its magnitude is plotted in its actual value; phase axis is linear
- In contrast to Nyquist plot two frequency regions are notable, with distinct break frequency at 1000 s^{-1} – at low frequency dominates the impedance of capacitor, but at high frequencies it is negligible in contrast to resistivity

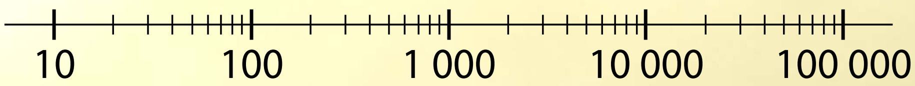
- To find frequency response of a circuit most frequently transfer function is used : $\mathbf{P} = \dfrac{\mathbf{U}_2}{\mathbf{U}_1}$
- The transfer function of a circuit is the frequency dependent ratio of a phasor output to a phasor input – it represents circuit properties, but its value is independent on input voltage amplitude and phase
- The transfer function may be defined only with *transforms* – phasors, Fourier transform, Laplace transform, …, but never in time domain (with waveforms)!!!
- In given integrating circuit:

$$\mathbf{U}_2 = \mathbf{U}_1 \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \mathbf{U}_1 \frac{1}{1 + j\omega RC}$$

- The transfer function is:

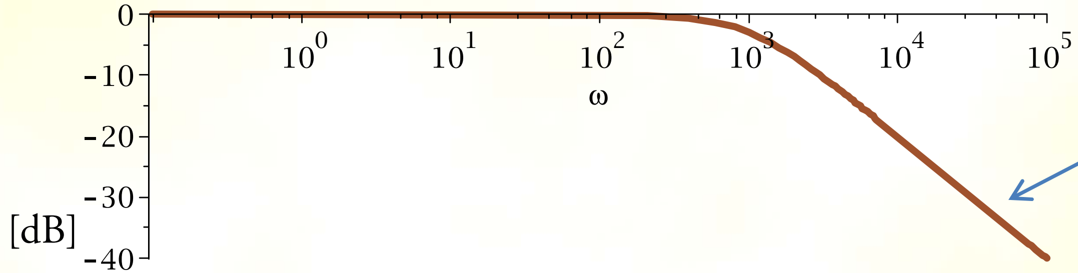
$$\mathbf{P} = \frac{\mathbf{U}_1 \frac{1}{1+j\omega RC}}{\mathbf{U}_1} = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j\omega \cdot 1000 \cdot 10^{-6}} = \frac{1}{1 + j\omega \cdot 0.001}$$

- Consider ω << 1000. Then ω · 0.001<< 1 and $\mathbf{P} \rightarrow \frac{1}{1} = 1, \quad \log(|\mathbf{P}|) = 0, \ \arg(\mathbf{P}) = 0$
- Consider ω >> 1000. Then ω · 0.001 >> 1 and $\mathbf{P} \rightarrow \frac{1}{j\omega \cdot 0.001} = \frac{-1000j}{\omega}, \quad \arg(\mathbf{P}) = \frac{-\pi}{2}$
  - How the modulus varies? If the angular frequency is increased ten times, modulus of transfer function in this example drops ten times; in logarithmic scale tenfold values are equidistant – as result it is decreasing line



  - The values on vertical axis of magnitude plot are multiplied by 20 the unit is decibel [dB]

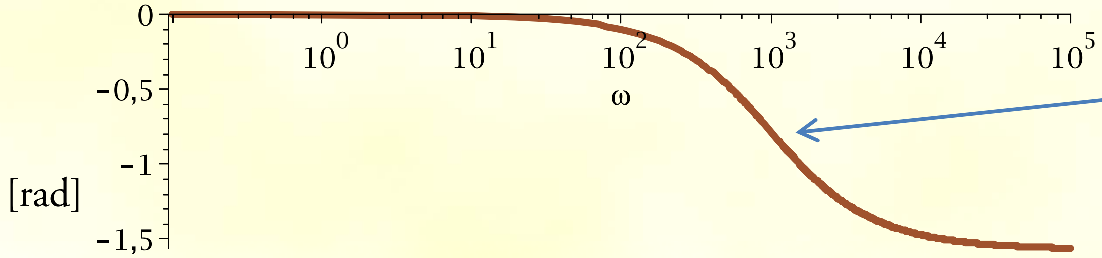
Frequency response is separated as **magnitude plot and phase plot**



Slope -20dB/decade

decade = frequency changes tenfold

- Modulus frequency response is plotted as $20\log(|\mathbf{P}(j\omega)|)$
- **Both axes** of modulus frequency response are **logarithmic**
- The unit is decibel [dB]



Slope –π/4 /decade

- Phase frequency response is plotted as $\arg(\mathbf{P}(j\omega))$
- **Frequency axis** is **logarithmic**, **phase axis** is **linear**
- The unit is radian [rad]

$$\mathbf{P} = \frac{1}{1 + j\omega RC}$$

As it was mentioned on previous slide, break frequency is given by condition $\omega RC = 1$, which gives an angular frequency:

$$\omega_p = \frac{1}{RC}$$

This circuit is referred as integrating circuit – why?

☞ The transform of an integral in frequency domain is jω

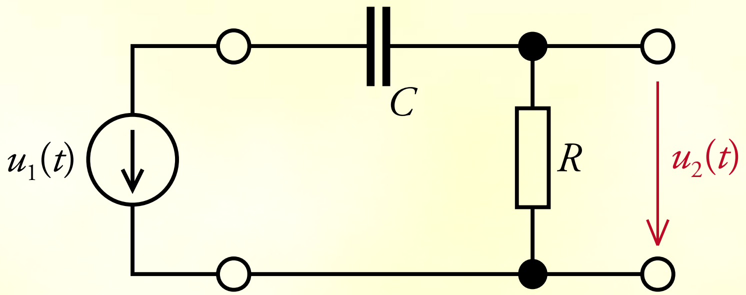
Recall the capacitor: $u(t) = \frac{q(t)}{C} = \frac{1}{C}\int_0^t i(\tau)\,\mathrm{d}\tau \quad \rightarrow \quad \mathbf{U} = \frac{\mathbf{I}}{j\omega C}$

The transfer function is $\mathbf{P} = \frac{1}{1+j\omega RC}$, but when $\omega \gg \frac{1}{RC}$ it is possible omit 1 in denominator, so

$$\mathbf{P} \rightarrow \frac{1}{j\omega RC} \quad \Rightarrow \quad u_2(t) = \frac{1}{RC}\int_0^t u_1(\tau)\,\mathrm{d}\tau$$

The waveform of output voltage is an integral of input voltage; but, the circuit is not ideal, because such statement is valid only when $\omega \gg \frac{1}{RC}$ *(it is possible to implement "ideal" integrating circuit using operating amplifier)*

- Consider series-connected RC circuit (derivative circuit):



R = 1 kΩ, C = 1 μF, U_{1m} = 1V

Frequency f varies from 0 to ∞

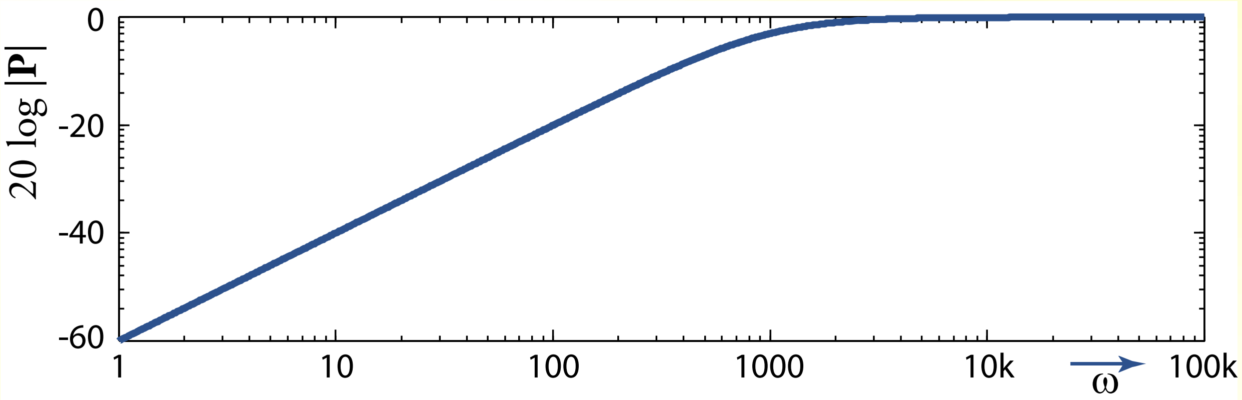
- Impedance of the circuit: $\mathbf{Z} = R + \frac{1}{j\omega C} = 1000 - \frac{10^6}{\omega}j$
- Impedance of the circuit is the same as impedance of integrating circuit, the same is also passing current
- Since the KVL must be satisfied and sequence of circuit elements is opposite to integrating circuit, frequency response is inverse of frequency response of integrating circuit.

Output voltage phasor and transfer function of the circuit:

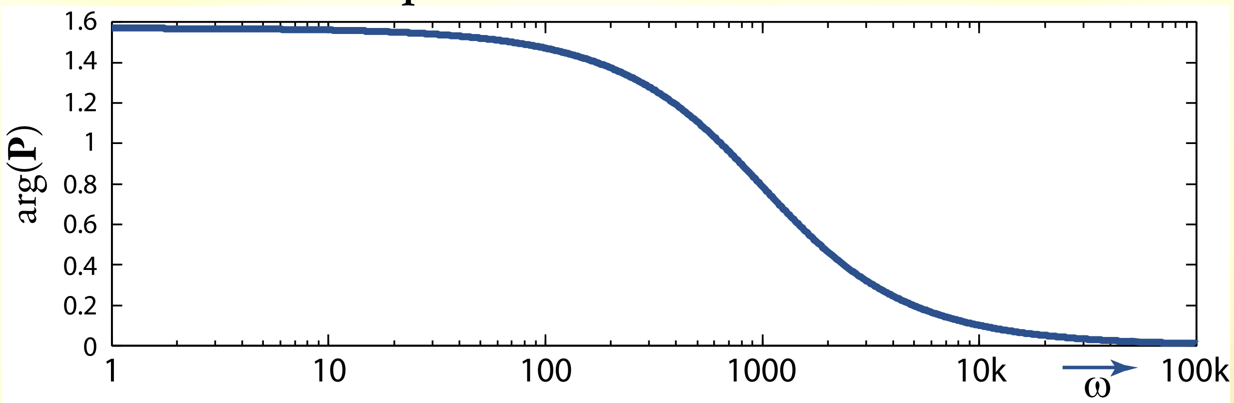
$$\mathbf{U}_2 = \mathbf{U}_1 \frac{R}{R + \frac{1}{j\omega C}} = \mathbf{U}_1 \frac{j\omega RC}{1 + j\omega RC}$$

$$\mathbf{P} = \frac{\mathbf{U}_1 \frac{j\omega RC}{1+j\omega RC}}{\mathbf{U}_1} = \frac{j\omega RC}{1 + j\omega RC} = \frac{j\omega \cdot 1000 \cdot 10^{-6}}{1 + j\omega \cdot 1000 \cdot 10^{-6}} = \frac{j\omega \cdot 0.001}{1 + j\omega \cdot 0.001}$$

Magnitude plot of derivative RC circuit



Phase plot of derivative RC circuit



This circuit is referred as derivative RC circuit – why?

☞ The transform of derivative in in frequency domain is multiplication by jω

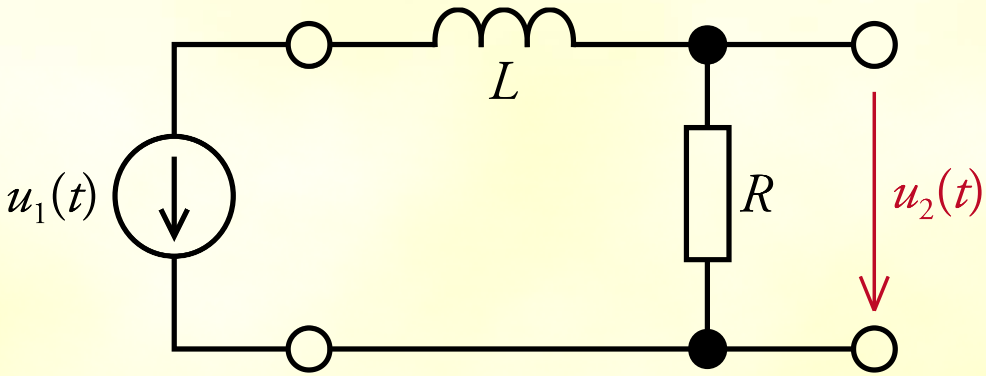
Recall capacitor: $i(t) = C\dfrac{\mathrm{d}u(t)}{\mathrm{d}t} \quad \Rightarrow \quad \mathbf{I} = j\omega C\,\mathbf{U}$

The transfer function is $\mathbf{P} = \dfrac{j\omega RC}{1 + j\omega RC}$, but if $\omega \ll \dfrac{1}{RC}$ it is possible omit $j\omega RC$ in denominator, so

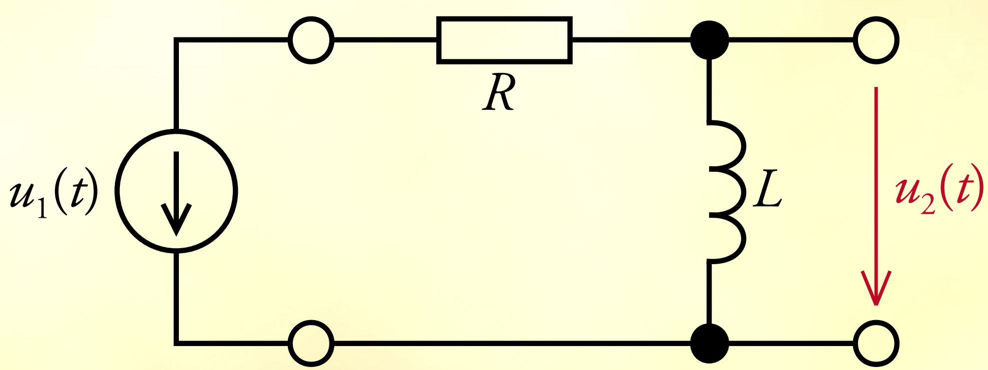
$$\mathbf{P} \to \frac{j\omega RC}{1} \quad \Rightarrow \quad u_2(t) = RC\frac{\mathrm{d}u_1(t)}{\mathrm{d}t}$$

The waveform of output voltage is then derivative of an input voltage; but, the circuit is not ideal, because such statement is valid only when $\omega \ll \frac{1}{RC}$ as we can see on magnitude plot *(again, an ideal derivative circuit would be possible to implement with ideal operating amplifier)*

Integrating and derivative LR circuit



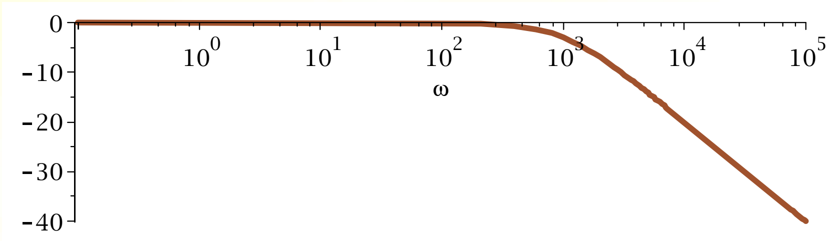
$$\mathbf{P} = \frac{R}{R + j\omega L}$$

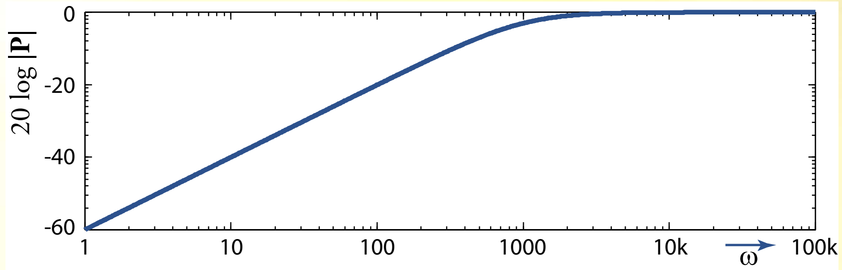


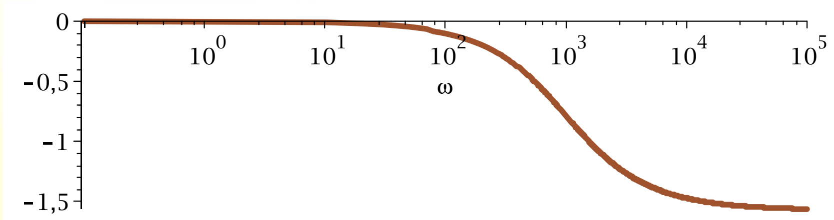
$$\mathbf{P} = \frac{j\omega L}{R + j\omega L}$$

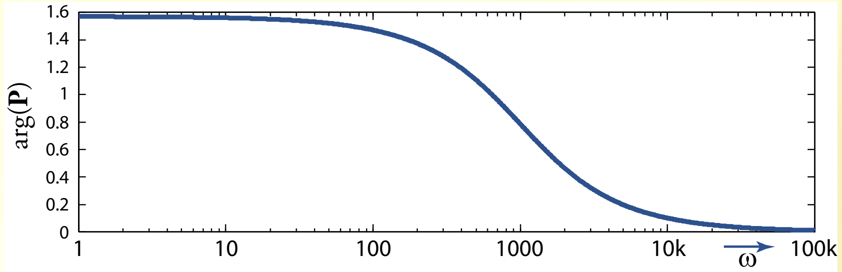
R = 1 kΩ, L = 1 H, U_{1m} = 1V, frequency f varies from 0 to ∞

Amplitude and phase plot of integrating and derivative LR circuit









If we normalize denominator such we obtain an expression $1 + j\omega\ldots$, it results in:

$$\mathbf{P} = \frac{1}{1 + j\omega\frac{L}{R}} = \frac{1}{1 + j\omega \cdot 0.001}$$

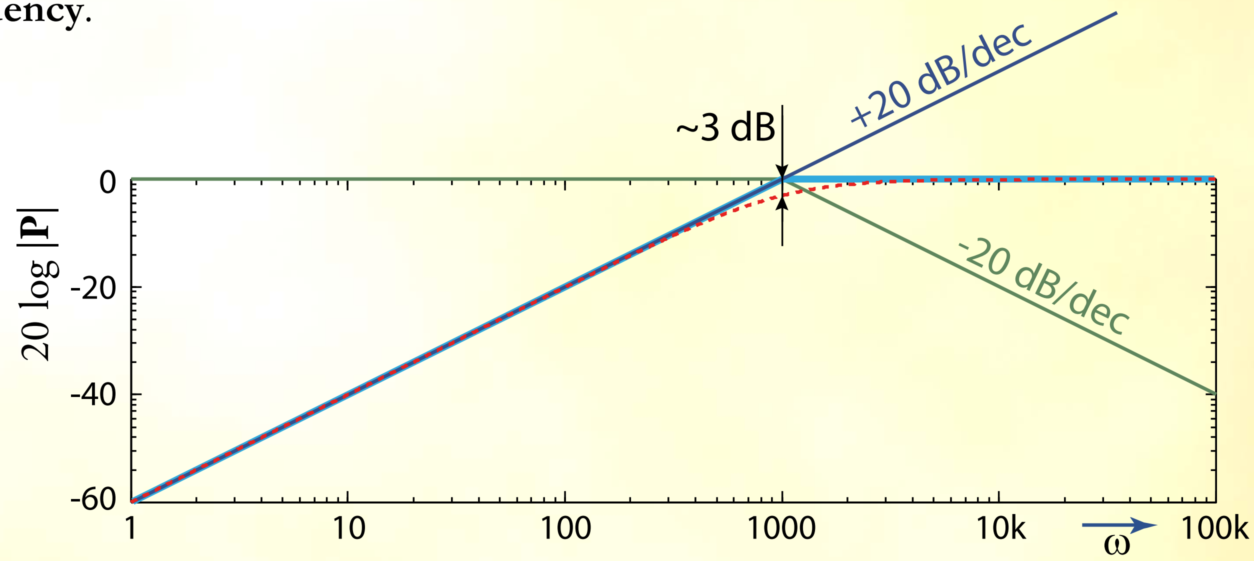
$$\mathbf{P} = \frac{j\omega\frac{L}{R}}{1 + j\omega\frac{L}{R}} = \frac{j\omega \cdot 0.001}{1 + j\omega \cdot 0.001}$$

We got expressions equivalent to transfer functions of RC circuits, with respect to selected value of resistivity and inductance the break frequency is:

$$\omega_p = \frac{R}{L} = 1000\,\mathrm{s}^{-1}$$

BODE PLOT

- In previous section we consider frequency response of circuits, which contains just one reactive circuit element; from graphs it was clear, in logarithmic scale it is possible to **approximate** amplitude plot of such circuit **by two lines** which intersects in distinct **breaking frequency**.
- Recall derivative circuit:
  - RC: $\mathbf{P} = \frac{j\omega RC}{1+j\omega RC}$
  - RL: $\mathbf{P} = \frac{j\omega \frac{L}{R}}{1+j\omega \frac{L}{R}}$

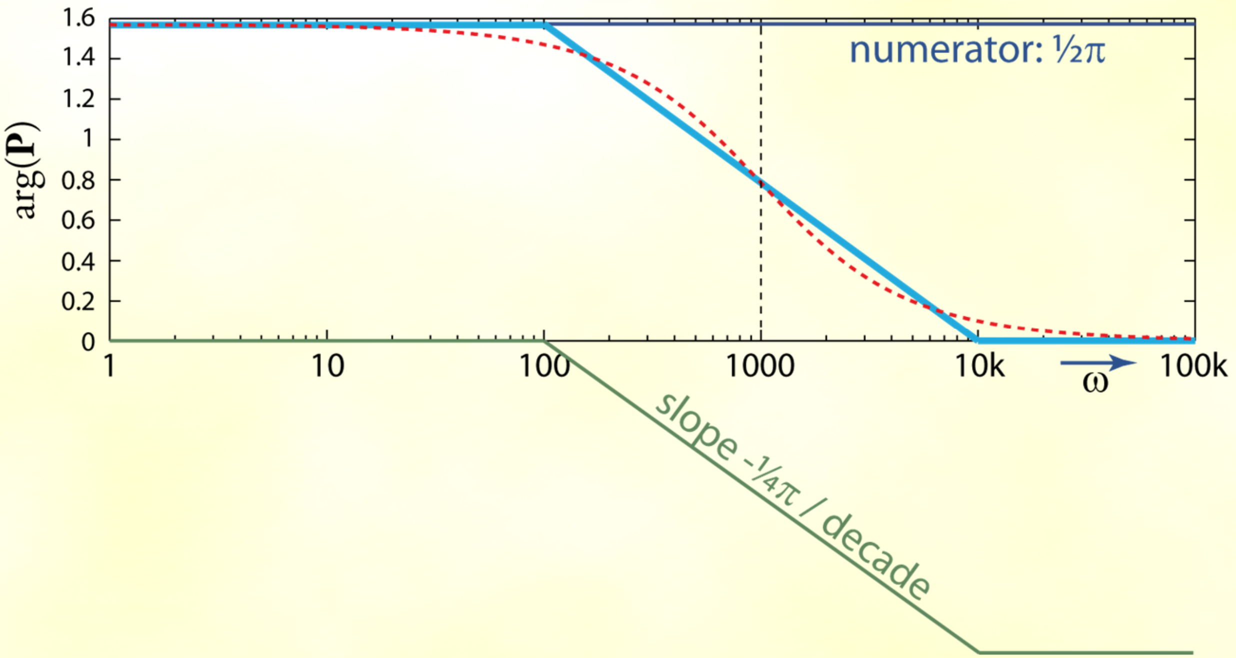


- In numerator is the term $j\frac{\omega}{\omega_0}$: when the frequency increases, its amplitude growths; if the frequency increase 10x, its amplitude also increases 10x, or by 20 dB in logarithmic scale: in amplitude plot it is blue **line, which intersects axis at frequency of ω_0**
- In denominator is the term $1 + j\frac{\omega}{\omega_0}$: when the frequency is $\omega \ll \omega_0$ imaginary part can be omitted, in amplitude scale it is constant line with zero slope (left green in our example); if the frequency is $\omega \gg \omega_0$ it is possible to omit real part, the result is increasing (if the term is in numerator), or decreasing (in denominator) line with slope 20 dB / decade the largest deviation from actual plot is:

$$20\log(|1+j|) = 20\log(\sqrt{2}) = 3.01\,\text{dB}$$

Accordingly we can draw asymptotic phase plot:

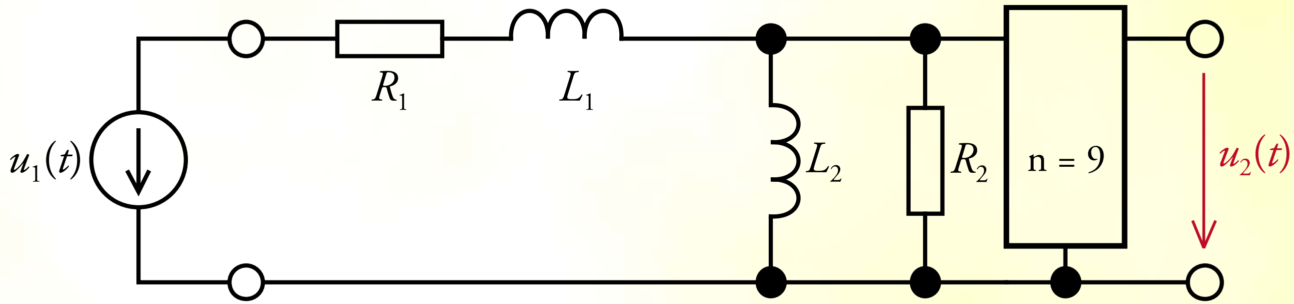
- In numerator is the term $j\frac{\omega}{\omega_0}$: it has constant phase shift, which doesn't vary with frequency, $\frac{\pi}{2}$
- In denominator is the term $1 + j\frac{\omega}{\omega_0}$: for small values of ω it tends to the real number of zero phase shift; for high frequencies it tends to the term $j\frac{\omega}{\omega_0}$ and its phase shift is $\frac{\pi}{2}$
  But now just one breaking frequency is not enough – the result would be step function, which is not consistent with actual characteristics – it is continuous ⇒ phase plot has two breaking frequencies. It is possible to prove, the minimal deviation of straight-line approximation is when breaking frequencies are 0.1 ω_0 and 10 ω_0.



Finally we add both factors (that of numerator and denominator) graphically

Frequency response of 2^{nd} order circuit:

Consider following equivalent circuit of a transformer:



The values of circuit elements are: R_1 = 2 kΩ, L_1 = 1.9 mH, L_2 = 8.1 mH, R_2 = 8.1 kΩ

(the transformer has resistivity of primary winding 2 kΩ, secondary winding 100 Ω, inductance of primary winding 10 mH, inductance of secondary winding0.1 mH and coupling factor 0.9)

Because we evaluate frequency response of the circuit, we have to find the transfer function:

$$\mathbf{P} = \frac{\mathbf{U}_2}{\mathbf{U}_1} = \frac{1}{9}\frac{\frac{j\omega L_2 R_2}{j\omega L_2 + R_2}}{R_1 + j\omega L_1 + \frac{j\omega L_2 R_2}{j\omega L_2 + R_2}} = \frac{1}{9}\frac{j\omega L_2 R_2}{(j\omega)^2 L_1 L_2 + j\omega\left(L_2 R_1 + L_1 R_2 + L_2 R_2\right) + R_1 R_2} =$$

$$= \frac{1}{9}\frac{j\omega \cdot 65.61}{(j\omega)^2 \cdot 1.539 \cdot 10^{-5} + j\omega \cdot 97.2 + 1.62 \cdot 10^{7}}$$

- Compared to investigated elementary RC / RL circuits there are another two terms:
  - Constant term
  - In denominator we have 2^{nd} order polynomial 2. (quadratic equation)

- Certainly, we can draw frequency response directly on PC (Maple, Matlab, …) using any form of transfer function
- But what is valid for distinct breaking frequencies?

- In the 1930s Hendrik Wade Bode proposed simple, but accurate method for graphing modulus and phase responses
- By this method is possible draw accurate responses without use of computer
- Frequency response has information about time constants (in transients), quality factor of resonant circuits etc.

➢ Transfer function is generally

$$P(j\omega) = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k} = \frac{b_0 + b_1(j\omega) + b_2(j\omega)^2 + \cdots + b_M(j\omega)^M}{a_0 + a_1(j\omega) + a_2(j\omega)^2 + \cdots + a_N(j\omega)^N}$$

$$= K\frac{\prod_{k=1}^{M}(j\omega - z_k)}{\prod_{k-1}^{N}(j\omega - p_k)} = \frac{b_M}{a_N}\cdot\frac{(j\omega - z_1)(j\omega - z_2)\cdots(j\omega - z_M)}{(j\omega - p_1)(j\omega - p_2)\cdots(j\omega - p_N)}$$

z_k roots of the polynomial in nominator – zeros

p_k roots of the polynomial in denominator – poles – here are "hidden" time constants of the transient

Note: we can also write the transfer function using Laplace transform – it is the most versatile description:

$$P(p) = \frac{\sum_{k=0}^{M} b_k p^k}{\sum_{k=0}^{N} a_k p^k} = \frac{b_0 + b_1 p + b_2 p^2 + \cdots + b_M p^M}{a_0 + a_1 p + a_2 p^2 + \cdots + a_N p^N}$$

$$= K\frac{\prod_{k=1}^{M}(p - z_k)}{\prod_{k-1}^{N}(p - p_k)} = \frac{b_M}{a_N}\cdot\frac{(p - z_1)(p - z_2)\cdots(p - z_M)}{(p - p_1)(p - p_2)\cdots(p - p_N)}$$

- Keep in mind the **variable** is not just frequency ω, but **complex frequency** $j\omega$!!!

Bode plots are based on logarithms, namely on following properties of logarithms:

- Logarithm of the product is the sum of logarithms $\log P_1 P_2 = \log P_1 + \log P_2$
- Logarithm of the quotient is the difference of logarithms $\log \frac{P_1}{P_2} = \log P_1 - \log P_2$
- $\log P^n = n \log P$
- $\log 1 = 0$

- With respect to the last property it is necessary normalize terms in factorization:

$$j\omega - z_k = -z_k \left(j\frac{\omega}{-z_k} + 1 \right)$$

Then, if $\frac{\omega}{-z_k} \ll 1,\ \log\left(\left|j\frac{\omega}{-z_k} + 1\right|\right) \to 0$

$$\frac{\omega}{-z_k} \gg 1,\ \log\left(\left|j\frac{\omega}{-z_k} + 1\right|\right) \to \log\left(\frac{\omega}{-z_k}\right)$$

Graphically
– straight line with slope 20 db / decade

$$\mathbf{P}'(j\omega) = K' \frac{\prod_{k=1}^{M}(j\frac{\omega}{-z_k} + 1)}{\prod_{k-1}^{N}(j\frac{\omega}{-p_k} + 1)} = \frac{b_M(-z_1)(-z_2)\cdots(-z_M)}{a_N(-p_1)(-p_2)\cdots(-p_N)} \cdot \frac{(j\frac{\omega}{-z_1} + 1)(j\frac{\omega}{-z_2} + 1)\cdots(j\frac{\omega}{-z_M} + 1)}{(j\frac{\omega}{-p_1} + 1)(j\frac{\omega}{-p_2} + 1)\cdots(j\frac{\omega}{-p_N} + 1)}$$

Magnitude plot:

$$F_{dB}(\omega) = 20\log(|\mathbf{P}'(j\omega)|) = 20\log(K') + \sum_{k=1}^{M} 20\log\left(\left|j\frac{\omega}{-z_k} + 1\right|\right) - \sum_{k=1}^{N} 20\log\left(\left|j\frac{\omega}{-p_k} + 1\right|\right)$$

Phase plot:

$$\varphi(\omega) = \arg(\mathbf{P}'(j\omega)) = \sum_{k=1}^{M} \arg\left(j\frac{\omega}{-z_k} + 1\right) - \sum_{k=1}^{N} \arg\left(j\frac{\omega}{-p_k} + 1\right)$$

Back to our example – in denominator we have to find roots of quadratic equation and factorize it:

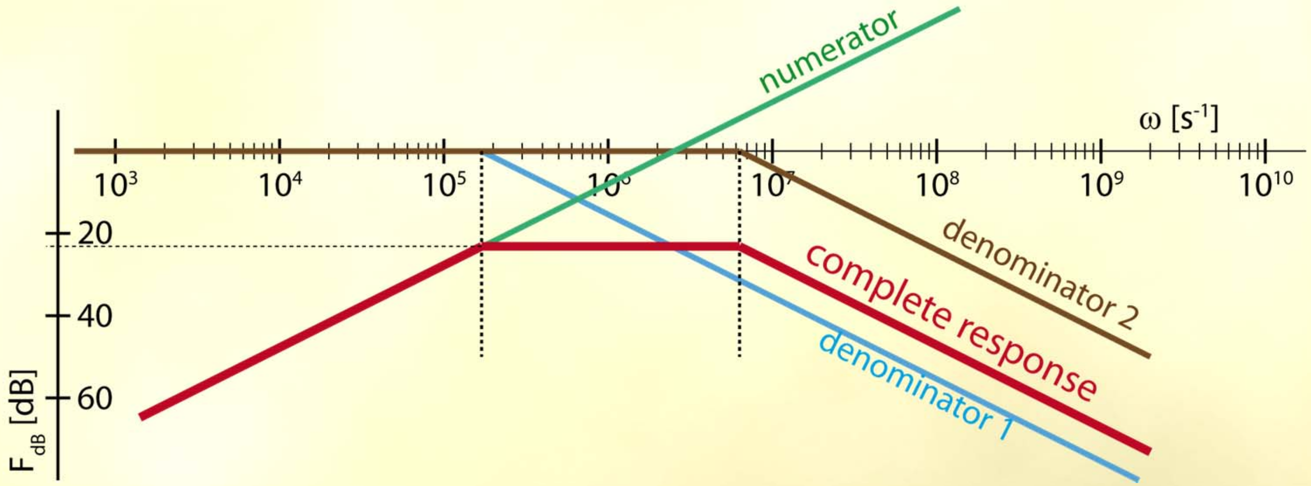
$$\mathbf{P} = \frac{1}{9}\frac{j\omega \cdot 65.61}{(j\omega)^2 \cdot 1.539 \cdot 10^{-5} + j\omega \cdot 97.2 + 1.62 \cdot 10^7} = \frac{1}{9}\frac{j\omega \cdot 65.61}{1.539 \cdot 10^{-5} \cdot (j\omega + 1.71 \cdot 10^5)(j\omega + 6.144 \cdot 10^6)}$$

- Now we know breaking frequencies – *poles* of transfer function $1.71 \cdot 10^5$,and $6.144 \cdot 10^6$, but we still doesn't know frequency response amplitude – we have to **normalize** both terms:

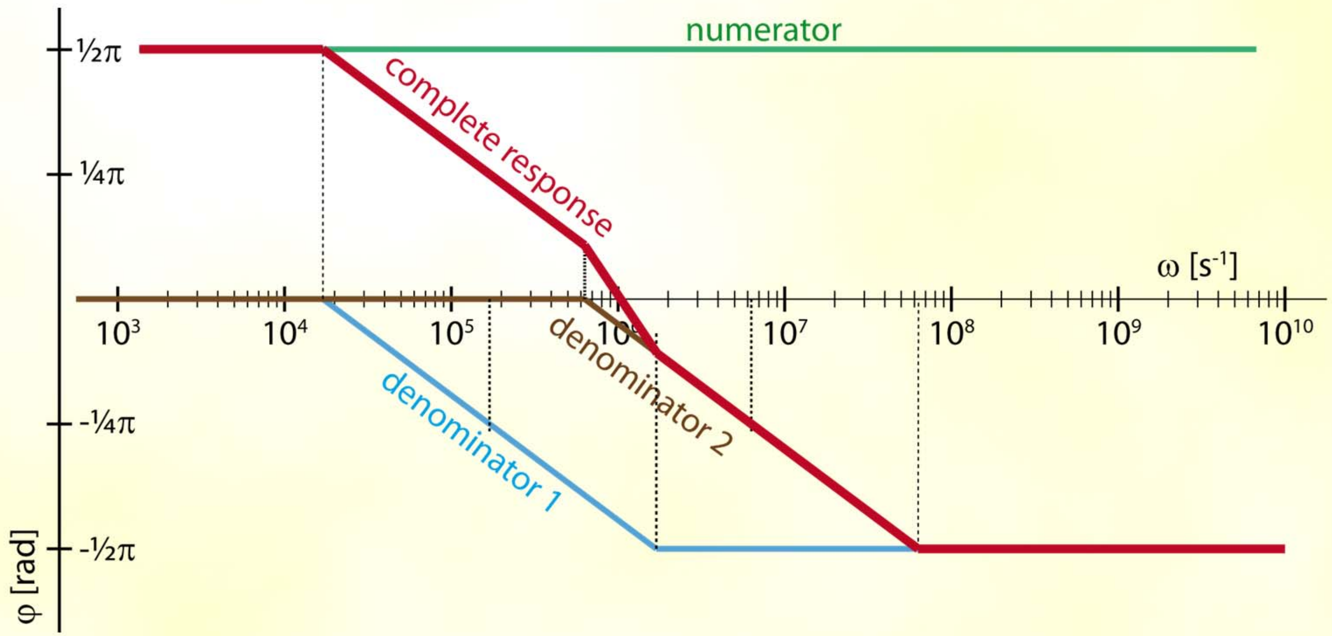
$$\mathbf{P} = \frac{1}{9}\frac{j\omega \cdot 65.61}{1.539 \cdot 10^{-5} \cdot 1.71 \cdot 10^5 \cdot 6.144 \cdot 10^6 \cdot (j\frac{\omega}{1.71 \cdot 10^5} + 1)(j\frac{\omega}{6.144 \cdot 10^6} + 1)} =$$

$$= \frac{j\omega \cdot \frac{65.61}{9 \cdot 1.539 \cdot 10^{-5} \cdot 1.71 \cdot 10^5 \cdot 6.144 \cdot 10^6}}{(j\frac{\omega}{1.71 \cdot 10^5} + 1)(j\frac{\omega}{6.144 \cdot 10^6} + 1)} = \frac{j\frac{\omega}{2.22 \cdot 10^6}}{(j\frac{\omega}{1.71 \cdot 10^5} + 1)(j\frac{\omega}{6.144 \cdot 10^6} + 1)}$$

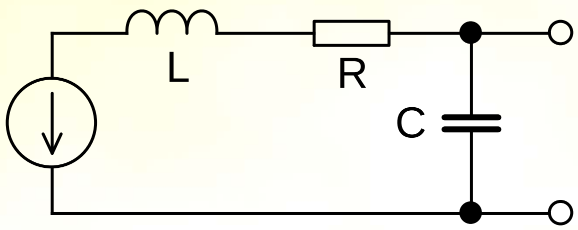
- Amplitude is a part of the term in numerator – this line feeds down complete plot:



Complete phase plot has three parts – numerator jω advance phase by $\frac{\pi}{2}$, the term 1 + jω in denominator has again two breaking frequency – at 0.1ω and 10ω, where ω is breaking frequency of magnitude plot; since this term is in denominator, phase shift is negative



Now consider frequency response of RLC circuit



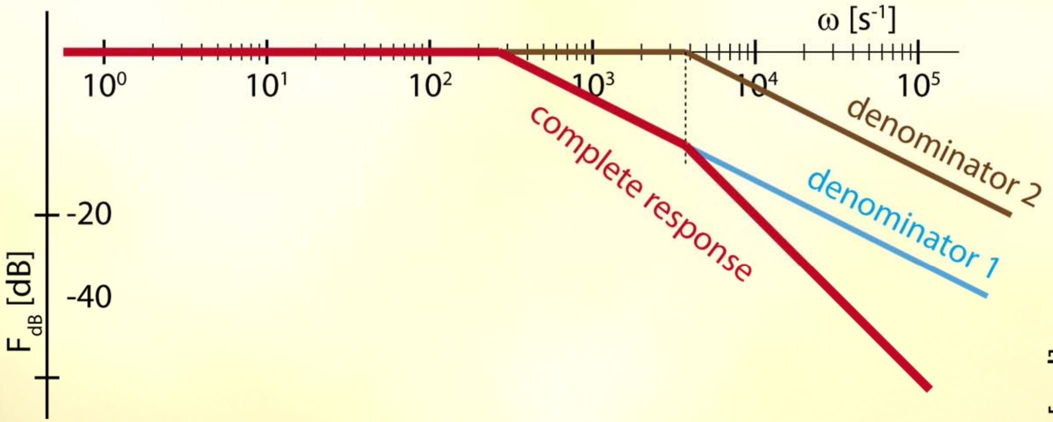
L = 1 H
C = 1 μF
R = 4 kΩ, 2 kΩ a 1 kΩ

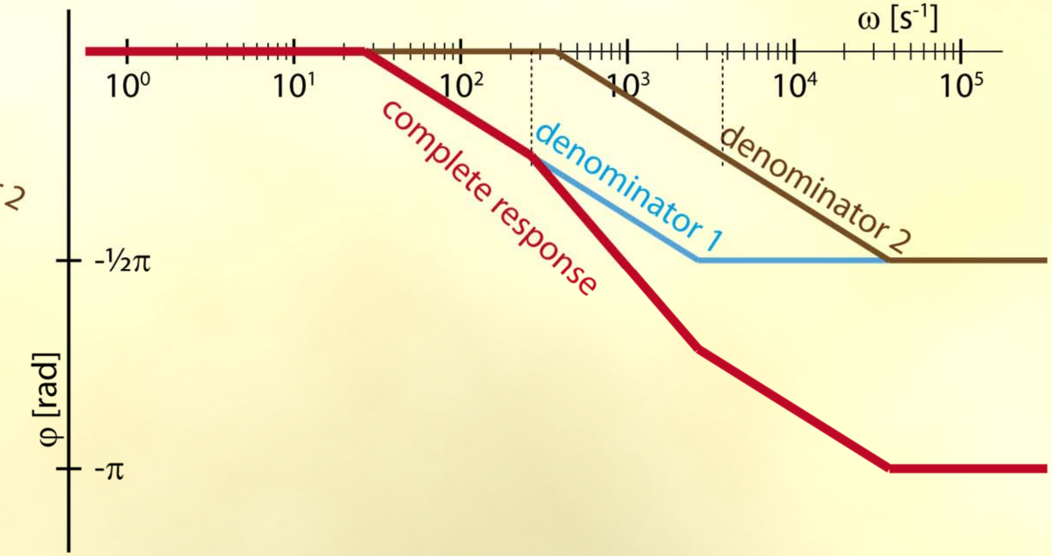
$$P(j\omega) = \frac{\frac{1}{j\omega C}}{j\omega L + R + \frac{1}{j\omega C}} = \frac{1}{(j\omega)^2 LC + j\omega RC + 1} = \frac{\frac{1}{LC}}{(j\omega)^2 + (j\omega)\frac{R}{L} + \frac{1}{LC}}$$

1. R = 4 kΩ

$$\mathbf{P}(j\omega) = \frac{10^6}{(j\omega)^2 + 4000(j\omega) + 10^6} \qquad (j\omega)_{1,2} = -2000 \pm \sqrt{2000^2 - 10^6} = -2000 \pm 1732.1 \begin{array}{l} = -3732.1 \\ = -267.9 \end{array}$$

$$\mathbf{P}'(j\omega) = \frac{10^6}{3732.1 \cdot 267.9} \cdot \frac{1}{(j\frac{\omega}{3732.1} + 1)(j\frac{\omega}{267.9} + 1)} = \frac{1}{(j\frac{\omega}{3732.1} + 1)(j\frac{\omega}{267.9} + 1)}$$

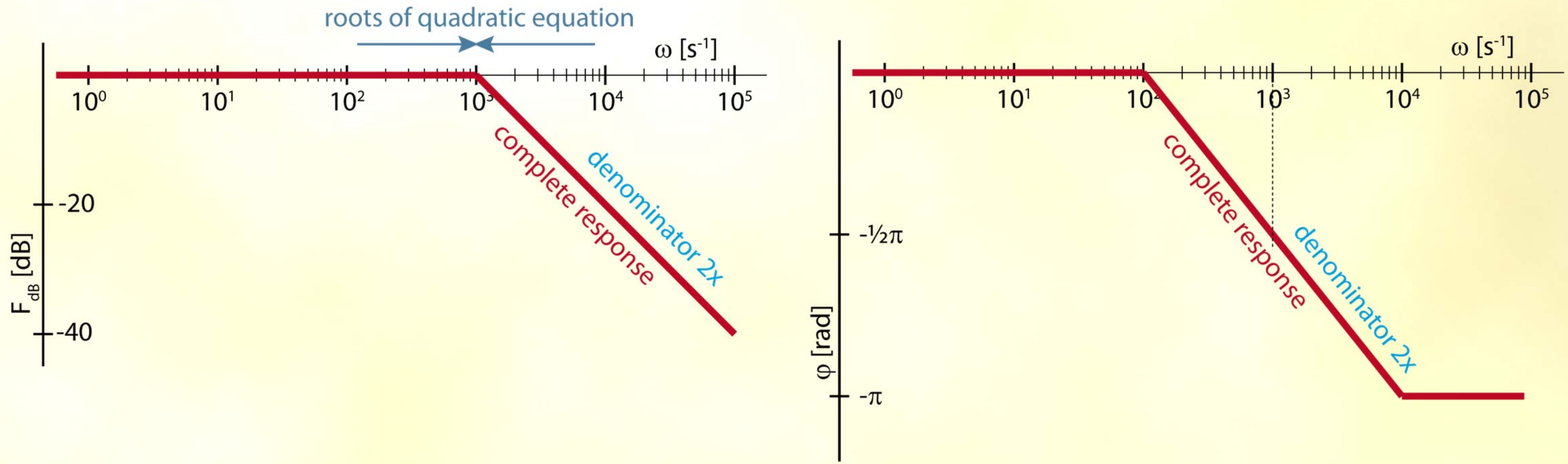




2. R = 2 kΩ

$$\mathbf{P}(j\omega) = \frac{10^6}{(j\omega)^2 + 2000(j\omega) + 10^6} \qquad (j\omega)_{1,2} = -1000 \pm \sqrt{1000^2 - 10^6} = -1000$$

$$\mathbf{P}'(j\omega) = \frac{10^6}{1000 \cdot 1000} \cdot \frac{1}{(j\frac{\omega}{1000} + 1)^2} = \frac{1}{(j\frac{\omega}{1000} + 1)^2}$$!!!!!

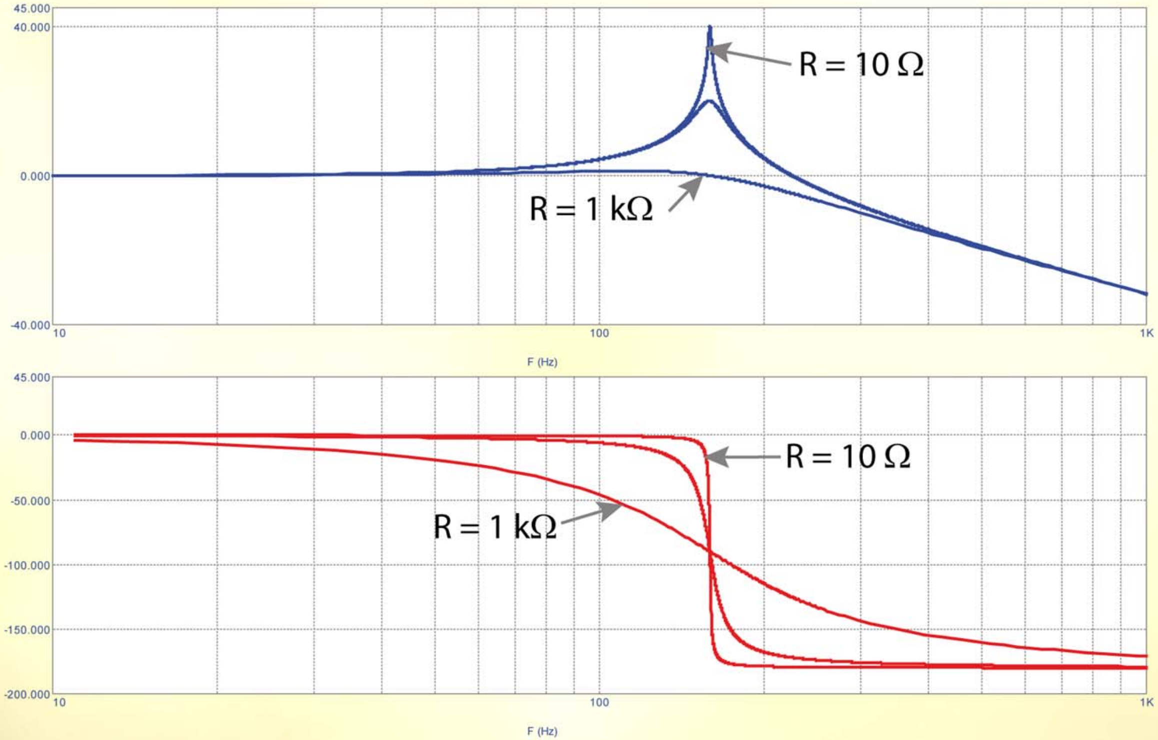


- With decreasing resistivity both roots of quadratic equation go near to each other and at distinct *critical frequency* merge together – both magnitude and phase plot are similar to frequency response of integrating circuit, but slope and phase shift are double the values of an integrating circuit – 40 dB/decade slope, -π phase shift – this is due to second power of the term in denominator
- *In the case of transients we will refer this frequency as critical frequency*

3. R = 1 kΩ

$$\mathbf{P}(j\omega) = \frac{10^6}{(j\omega)^2 + 1000(j\omega) + 10^6} \qquad (j\omega)_{1,2} = -500 \pm \sqrt{500^2 - 10^6} = -500 \pm 866j$$

$$\mathbf{P}'(j\omega) = \frac{10^6}{1000^2} \cdot \frac{1}{(j\frac{\omega}{1000})^2 + j\frac{\omega}{1000} + 1} = \frac{1}{(j\frac{\omega}{\underbrace{1000}_{\omega_r}})^2 + \underbrace{1}_{\frac{1}{Q}} \cdot j\frac{\omega}{1000} + 1}$$



- When the resistivity further decrease, the roots of an quadratic equation will be complex conjugated
- On magnitude plot it result in *resonance* peak over (when the quadratic equation is in denominator), or (theoretically) under the axis
- In such case we will talk about *voltage* resonance of RLC circuit
- In the time domain the same roots results in *sinusoidal oscillations with exponentially decreasing amplitudes*, as we will see later, when we will deal with 2nd order transients

Resonance

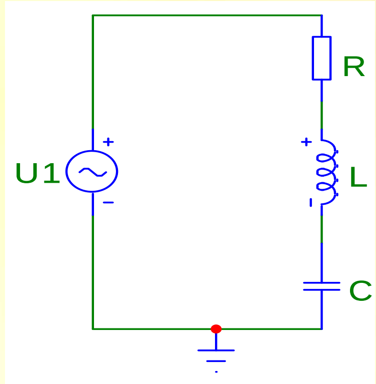
First some physical analogies…

- In physics, resonance is the tendency of a system to oscillate at maximum amplitude at certain frequencies, when small periodic driving forces can produce large amplitude vibrations.
- The reason is the system **accumulates energy**. The source successively delivers an energy and **damping is small**; to raise resonant oscillations two kind of energy are necessary in that system, e.g. potential and kinetic (pendulum, swing), or electric and magnetic – in electrical circuits.
- Most of musical instruments are based on resonance.
- Sometimes resonance could have destructive effects – we can break a glass by the sound of distinct frequency.
- The resonance is crucial in architecture. Wrong designed buildings and other structures can collapse. An example would be Angers Bridge, that collapsed in 1850, when vibrations from marching French soldiers causes bridge collapse. Another example is collapse of American Tacoma Narrows Bridge (1940), when steady wind (!!!) causes bridge oscillations – the oscillations were initiated by wind whirls (aerolastic flutter).

Authentic image from the film „The Tacoma Narrows Bridge Collapse“

- In electrical circuits an example of resonant circuits would be band pass filters, or power factor correction (power factor 100%)

In the next section we will consider series RLC circuit, where R = 10 Ω, L = 1 H, C = 1 μF, if not otherwise specified.



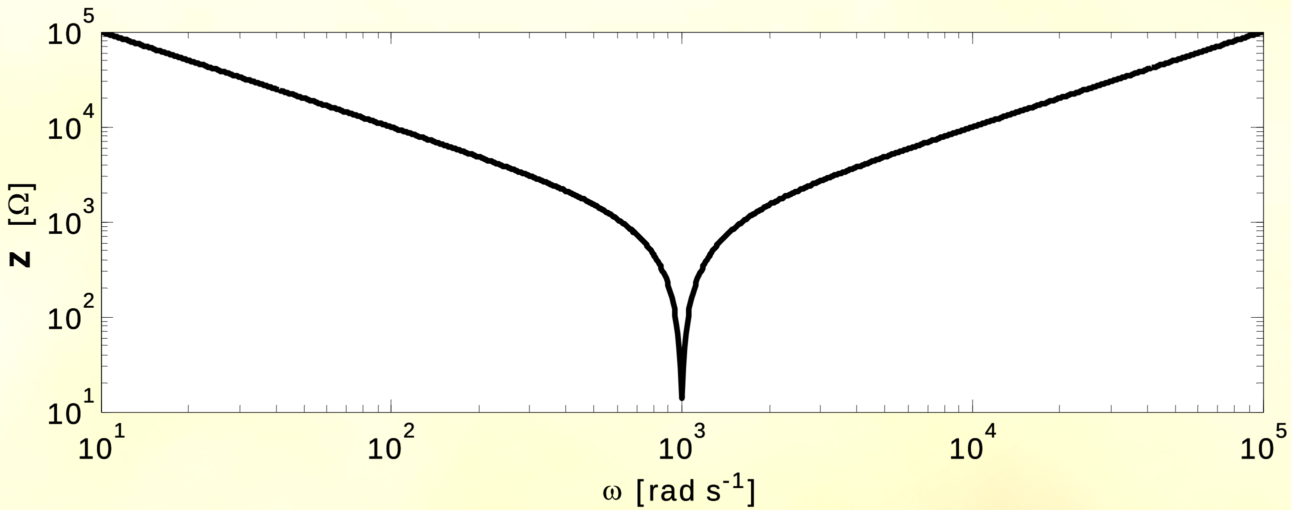
The total impedance of this circuit is:

$$\mathbf{Z} = R + j\omega L + \frac{1}{j\omega C} = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

The magnitude of impedance is:

$$|\mathbf{Z}| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

It is obvious the magnitude of impedance is frequency dependent, with strict minimum at frequency 1000 s^{-1}.



- In this minimum following condition is met:

$$\mathbf{Z} = R + j\underbrace{\left(\omega L - \frac{1}{\omega C}\right)}_{=0} = R$$

Impedance at such frequency is **real**

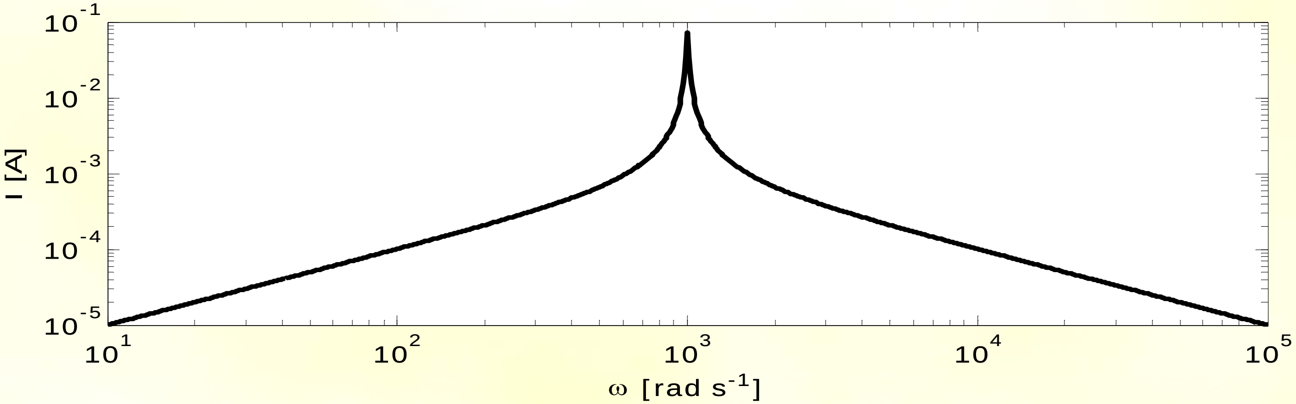
From this condition (or, by differentiation of the impedance with respect to the frequency) we can define:

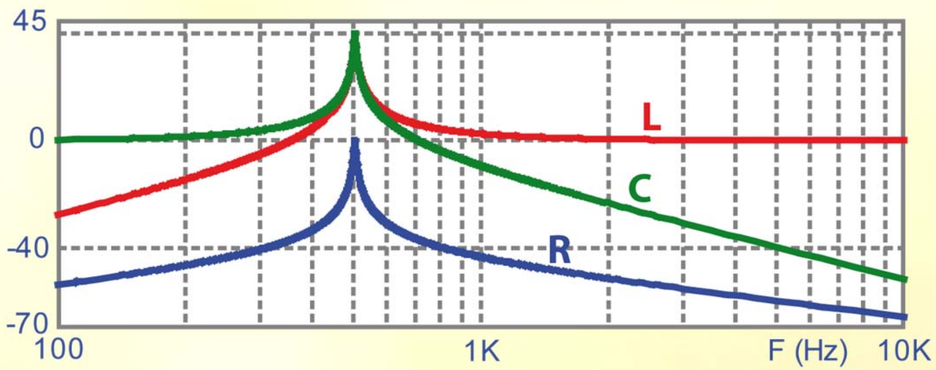
Thomson's formula:

$$\boxed{\omega_r = \frac{1}{\sqrt{LC}}}$$

This formula defines **resonant frequency** of **series RLC circuit** (keep in mind it isn't generally valid for arbitrary circuit!)

Compare frequency dependence of current magnitude and voltage magnitude:



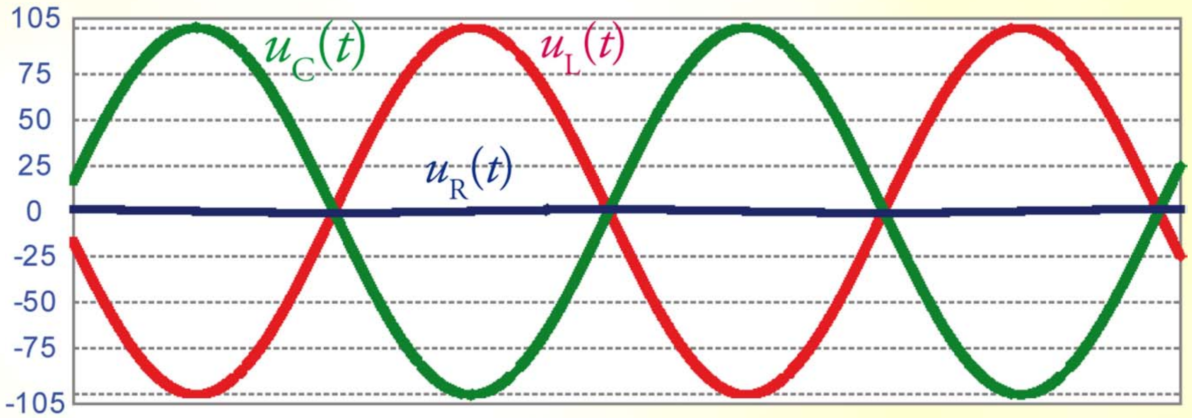


Green is capacitor voltage.
Red is inductor voltage.
Blue is resistor voltage.

Both axes are logarithmic, the units on vertical axis are dB, thus
$20\log\frac{U_C}{U}$, $20\log\frac{U_L}{U}$, a $20\log\frac{U_R}{U}$

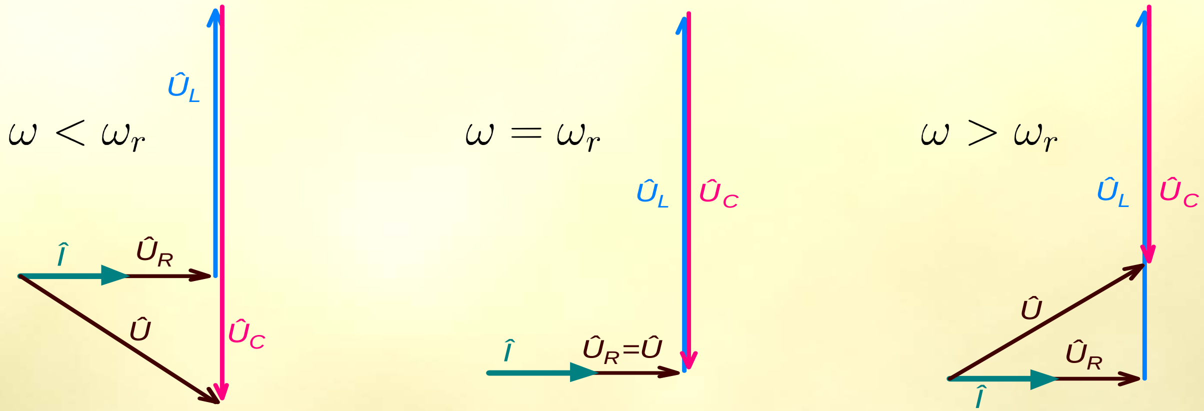
When $U_1 = 1$ V, then 40 dB ≈ 100 V.

Why voltage across capacitor and inductor can be 100 × greater than source voltage?



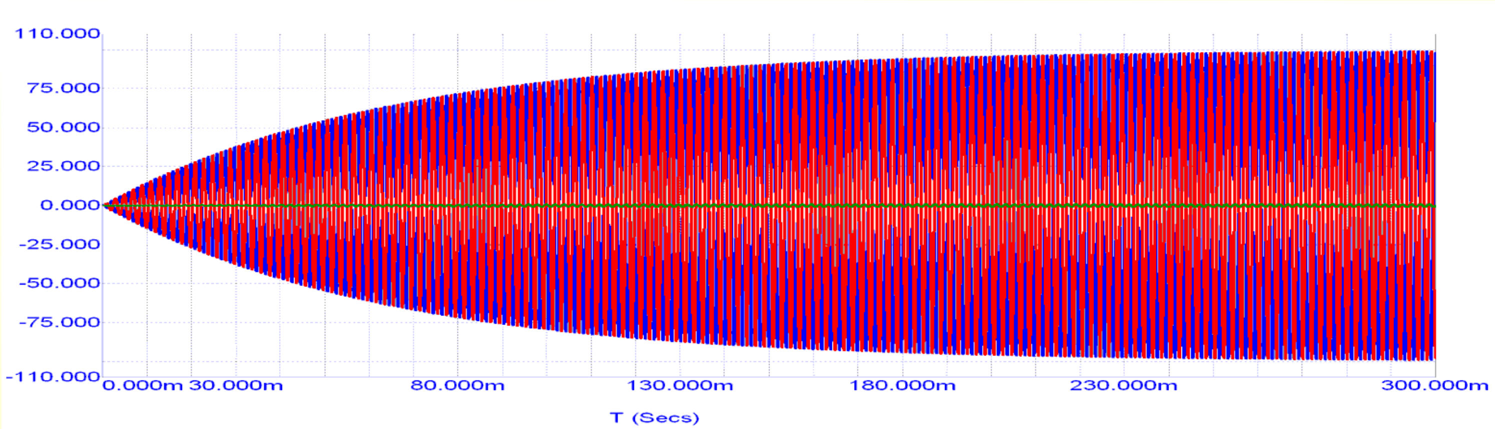
Voltage across inductor leads the current by 90º, voltage across capacitor lags the current by 90º, so **these voltages are shifted by 180º** – they have same (absolute) magnitude, but with opposite sign, so **they cancels.** Then the voltage across resistor is the same voltage as the source voltage.

- Such kind of resonance is called **voltage resonance**
- Voltage magnitude on capacitor and inductor can be much times greater, then the magnitude of the source
- **The condition of voltage resonance:** $Im\{\mathbf{Z}\} = 0$
- In order to be in resonance, the circuit have to have at least 2 reactive circuit elements – but not 2 capacitors, or 2 inductors; to mutual **exchange of energy both L and C must be present**
- Phasor diagrams:

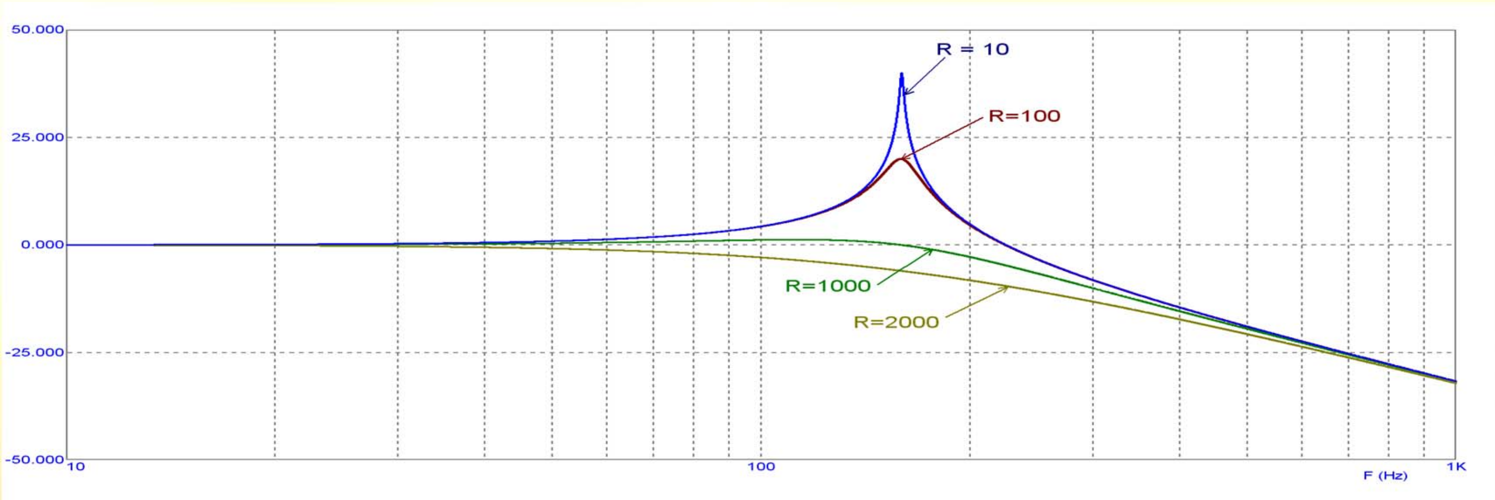


But the steady state in the circuit is not reached immediately, it takes some time before the maximum voltage across capacitor / inductor is reached, settling time is inverse proportional to $\frac{R}{2L}$

Lesser R means longer time to reach steady state. Recall the first lecture – if an ideal inductor is connected to the ideal voltage source, infinite current should flow through it (but in steady state! - such state is achieved not until infinitely long period). The current passing an inductor, according to Faraday's law rise up linearly, $I = \frac{Ut}{L}$. Current in the circuit (and so the voltage) could not reach its maximum value (determined by resistivity in the circuit) immediately.



Since the impedance of series RLC circuit is real at resonant frequency, it is resistivity *R*, the resistor is circuit element, which limits the current, passing the circuit – the voltage across capacitor and inductor is related to multiple of their reactance and passing current ⇒ lesser *R* means greater voltage across *C* and *L*.



Not each series RLC circuit is resonant circuit. Beside the presence of reactive circuit elements of both kinds (L, C), the second necessary condition that resonant oscillations rise in the circuit is „sufficiently“ small damping – so resistivity. And the third – sinusoidal source of resonant frequency.
Resonant RLC circuit could be identified in that way, the connection of DC voltage source to resonant circuit invoke exponentially dumped oscillations.

Quality factor

One of most important applications of resonant circuits are filters. For example, in radio and TV receivers filters are used to tune in stations and in TV receivers to separate audio signal from video signal. The frequency response of such filters must be so sharp-cutting and narrow as possible to suppress neighboring stations. In the case of resonant LC filter the voltages U_L, U_C should be so large as possible. The large the voltage is, the filter is „better quality“.

In this way the **quality factor** is defined for voltage resonance as :

$$\boxed{Q = \frac{U_C}{U} = \frac{U_L}{U}}$$

Since the voltage in resonance is:

$$\widehat{\mathbf{U}}_C = \frac{1}{j\omega_r C}\widehat{\mathbf{I}}, \quad \widehat{\mathbf{U}}_L = j\omega_r L\widehat{\mathbf{I}}, \quad \widehat{\mathbf{U}} = \widehat{\mathbf{Z}}_r\widehat{\mathbf{I}} = R\widehat{\mathbf{I}}$$

We can define quality factor by one of an equations:

$$\boxed{Q = \frac{1}{\omega_r C R} = \frac{\omega_r L}{R} = \frac{\sqrt{\frac{L}{C}}}{R}}$$

Maximum (critical) resistivity, at which the circuit still exhibits resonant properties:

$$R = \sqrt{\frac{L}{C}}$$

Resonant curve

If we normalize current, which flows in the circuit, we got:

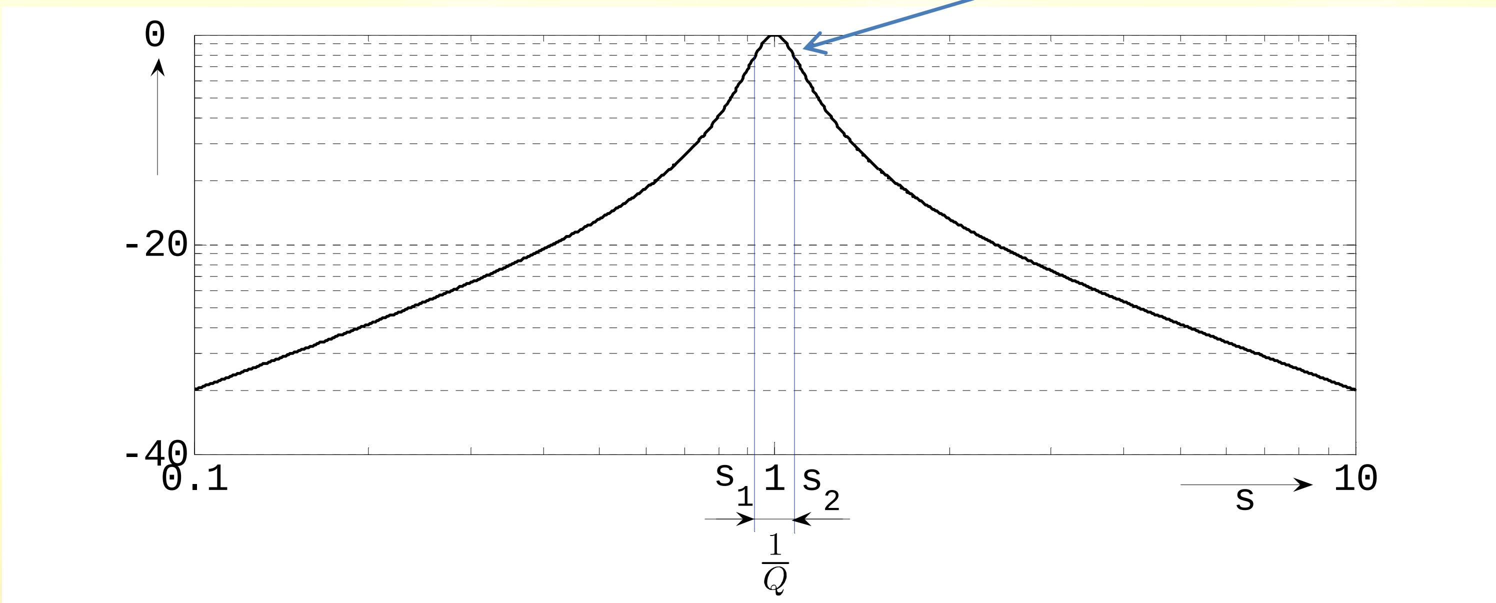
$$\frac{\mathbf{I}}{\mathbf{I}_r}=\frac{\frac{\mathbf{U}}{\mathbf{Z}}}{\frac{\mathbf{U}}{\mathbf{Z}_r}}=\frac{\mathbf{Z}_r}{\mathbf{Z}}=\frac{R}{R+j\left(\omega L-\frac{1}{\omega C}\right)}=\frac{1}{1+j\left(\underbrace{\frac{\omega L}{R}}_{=\frac{1}{Q}|_{\omega=\omega_r}}-\underbrace{\frac{1}{\omega C R}}_{=Q|_{\omega=\omega_r}}\right)}$$

Since the frequency in the equation can be arbitrary, not only resonant, we will introduce relative frequency

$$s=\frac{\omega}{\omega_r}$$

$$\frac{\mathbf{I}}{\mathbf{I}_r}=\frac{1}{1+j\left(\frac{\omega}{\omega_r}\frac{\omega_r L}{R}-\frac{\omega_r}{\omega}\frac{1}{\omega_r C R}\right)}=\frac{1}{1+jQ\left(s-\frac{1}{s}\right)}$$

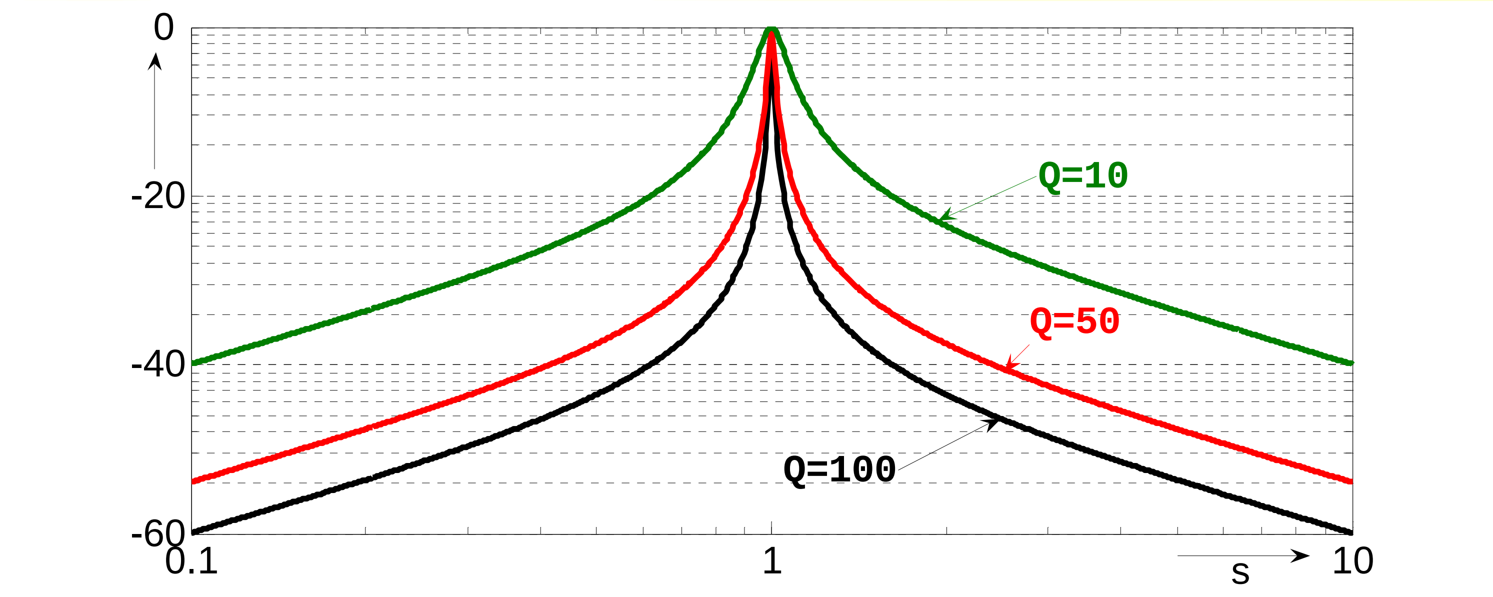
Drop by 3 dB



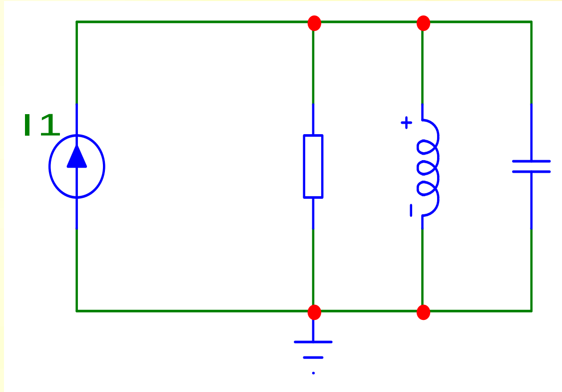
The resonant curve is thus narrower (and higher), the higher the quality factor is. Mathematically the bandwidth of resonant curve can be defined as:

$$\frac{1}{Q} = s_2 - s_1$$

In following figure are resonant curves when Q = 10, Q = 50 and Q = 100



Parallel resonant circuit – current resonance



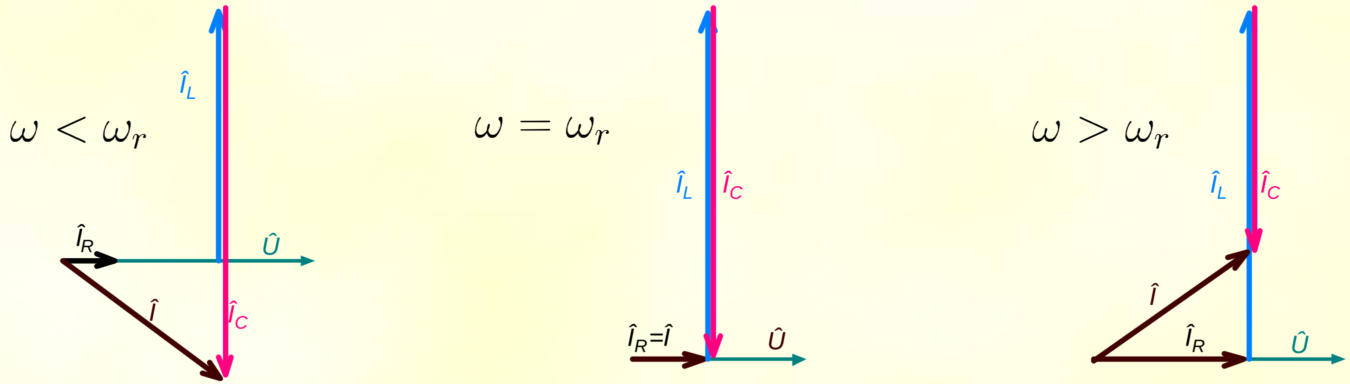
In series RLC circuit the voltage resonance has been described.

An ideal (however physically unrealizable) parallel RLC circuit (see figure) is the best suitable to clarify the current resonance:

The voltage across all circuit elements is the same; currents, which passes through distinct circuit elements are different.

If in the circuit are present circuit elements that stores both kinds of energy (electric - C and magnetic - L), they can mutually exchange an energy – an electrical current which flows through them increase (and so energy).

The relationship among all currents show phasor diagrams:



Then it is valid: $|I_L| = |I_C|$, but both currents are mutually shifted by 180°, so they subtract. Since:

$$\mathbf{I}_L = \mathbf{Y}_L\mathbf{U} = \frac{1}{j\omega L}\mathbf{U}, \qquad \mathbf{I}_C = \mathbf{Y}_C\mathbf{U} = j\omega C\mathbf{U} \quad \Rightarrow \quad \frac{1}{\omega_r L} = \omega_r C$$

…and resonant frequency of an **ideal** parallel RLC circuit is:

$$\omega_r = \frac{1}{\sqrt{LC}}$$

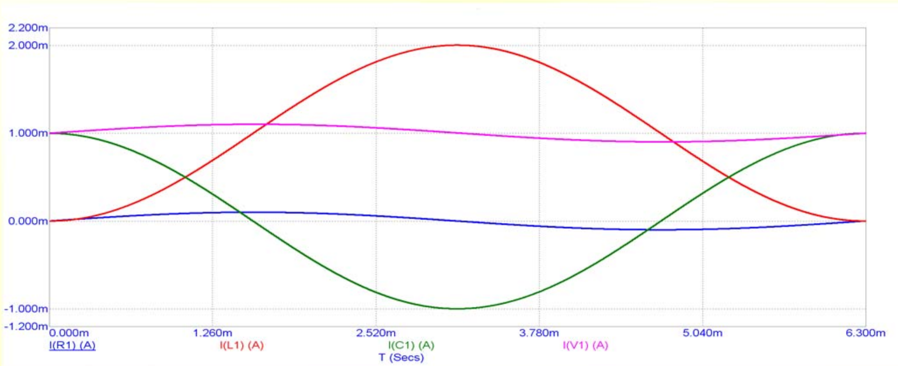
The condition of current resonance is given for addmittance:

$$\mathbf{Y} = G + j\underbrace{\left(\omega C - \frac{1}{\omega L}\right)}_{=0} = G$$

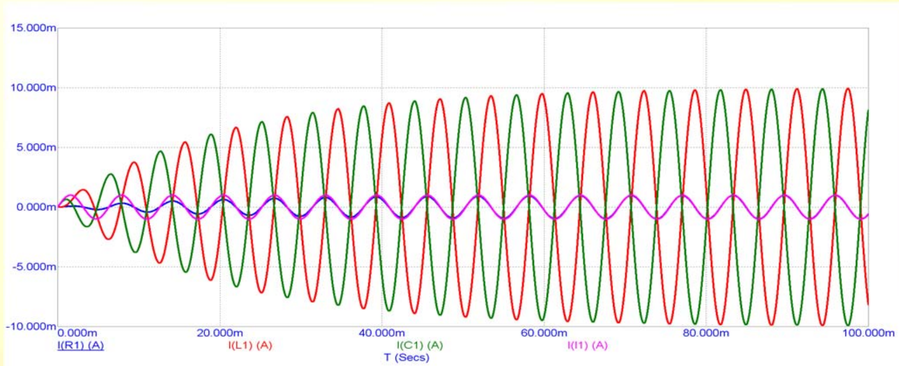
Quality factor in this circuit is defined by ratio of currents:

$$\boxed{Q = \frac{I_C}{I} = \frac{I_L}{I}} \longrightarrow \boxed{Q = \frac{\omega_r C}{G} = \frac{1}{\omega_r L G}}$$

- The current resonance occurs when the **circuit is supplied by current source**
- In contrast to the voltage resonance, **the quality factor is larger, as the resistivity is greater**
- When the circuit is supplied from voltage source, there is no accumulation of energy (the voltage source doesn’t allow it as it controls the voltage – and when capacitor and inductor accumulates an energy in the form of increasing current, the voltage would also increase):

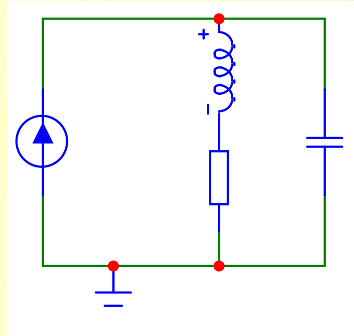


Voltage source excitation



Current source excitation

Actually, we can't implement such circuit. Actual parallel resonant circuit is in the figure:

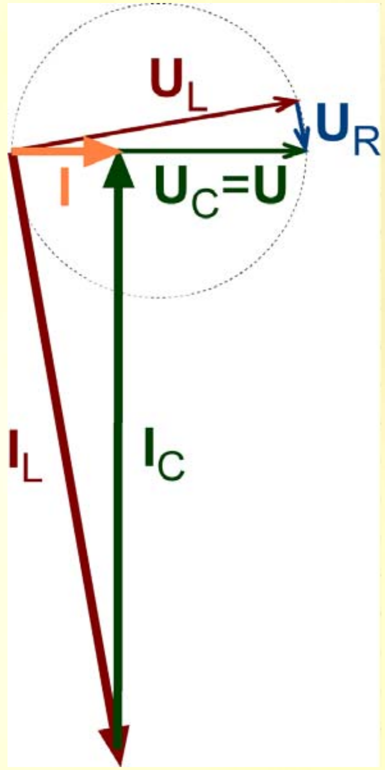


- Phase shift between currents, that passes inductor and capacitor, is less than 180°.
- Admittance (and so impedance) is real (condition of resonance)

$$\mathbf{Y} = j\omega C + \frac{1}{R + j\omega L} = \frac{R}{R^2 + \omega^2 L^2} + j\underbrace{\left(\omega C - \frac{\omega L}{R^2 + \omega^2 L^2}\right)}_{=0}$$

- Now the resonant frequency is different from that given by Thomson formula:

$$\omega_r = \sqrt{\frac{1}{LC} - \left(\frac{R}{L}\right)^2}$$



- Now, since resistor is in series with inductor and not in parallel, the resistivity should be as small as possible
- Thompson formula is just special case (but very important), in more complex circuits it is necessary to compute resonant frequency from condition of resonance
- The circuit which has more reactive circuit elements can have more resonant frequencies
- Here we cannot use definition of quality factor as ratio of currents (since the currents in actual parallel RLC resonant circuit are different), general is energetic definition:

$$Q = 2\pi \frac{\text{energy sored in resonant circuit}}{\text{energy converted in heat within one period}} \qquad W_s(t) = \frac{1}{2}Li_L^2(t) + \frac{1}{2}Cu_C^2(t)$$