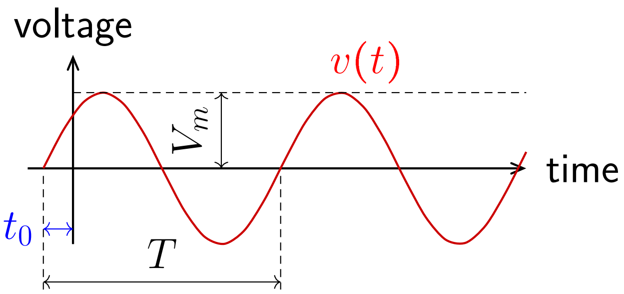
AC circuits

In the coming weeks, we will address a new topic – circuits that are excited by harmonic sources of voltage and current.



This waveform can be mathematically described by the equation:

$$v(t) = V_m \sin(\omega t + \varphi), \qquad \omega = 2\pi f, \qquad f = \frac{1}{T} \tag{1}$$

where T is period in s; f is frequency in Hz; ω is angular frequency in rad s^{-1}; V_m is amplitude, maximum value, or peak value in V; $\varphi = -t_0\omega$ is phase shift in rad (for all calculations in AC, therefore, do not forget to switch the calculator from degrees to radians).

In DC circuits, we considered the capacitor as an open circuit, and the inductor as a short circuit. In AC circuits, however, neither voltage nor current is constant, and therefore these elements will affect the voltages and currents in the circuit. Recall the basic relationships between voltage and current on a capacitor and an inductor. The capacitor accumulates charge, $v_C(t) = \frac{q(t)}{C}$ and so:

$$v_C(t) = \frac{1}{C}\int_0^t i(\tau)\,\mathrm{d}\tau + v_C(0) \tag{2}$$

$$i_C(t) = C\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} \tag{3}$$

If we connect a sinusoidal current source to the capacitor, the voltage will be:

$$
\begin{aligned}
v(t) =& \frac{1}{C}\int_0^t I_m \sin\omega\tau \,\mathrm{d}\tau + v_c(0) = \\
=& \frac{I_m}{\omega C}(1-\cos\omega t) + v_c(0) = \frac{I_m}{\omega C}\sin\left(\omega t - \frac{\pi}{2}\right) + \frac{I_m}{\omega C} + v_c(0)
\end{aligned} \tag{4}
$$

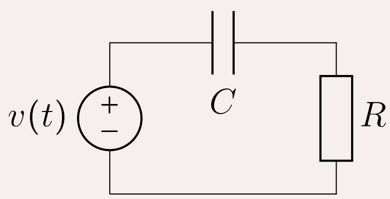
The voltage is a superposition of the DC and AC components. Within the superposition we can calculate the AC component separately. From equation 4 is evident that:

1. $V_m = \frac{1}{\omega C} I_m$
2. The voltage is phase shifted by $\frac{-\pi}{2}$. In other words, the current leads to voltage; respectively, the voltage lags the current by $\frac{\pi}{2}$.

If we were only interested in the relationship between the amplitude of voltage and current on the capacitor itself, then the first point would be enough – this is the same relationship as the Ohm's law, where instead of the resistance R is the frequency-dependent capacitive reactance $X_C = \frac{1}{\omega C}$. However we must not forget the phase shift between voltage and current.

Example 1

Consider the following circuit:



$v(t) = V\sqrt{2}\sin(\omega t + \varphi)$ V
$V = 230$ V, $V_R = 120$ V. What voltage is on the capacitor? Is it $230 - 120 = 110$ V?

Mistake! This is true for resistors. However on the capacitor there is a phase shift of $\frac{\pi}{2}$. For a resistor, the phase shift is 0, and the inductor behaves the opposite of the capacitor – the voltage precedes the current (by $\frac{\pi}{2}$). This phase shift can be represented graphically for individual elements:

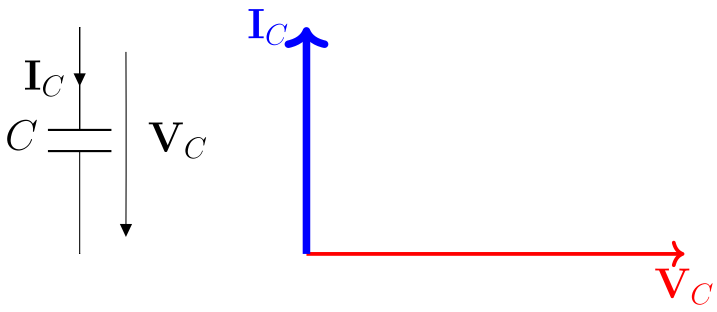


Figure 1

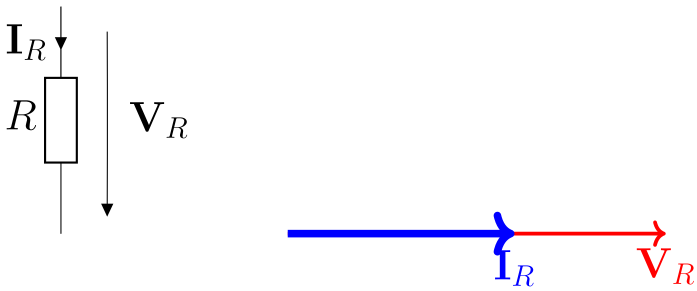


Figure 2

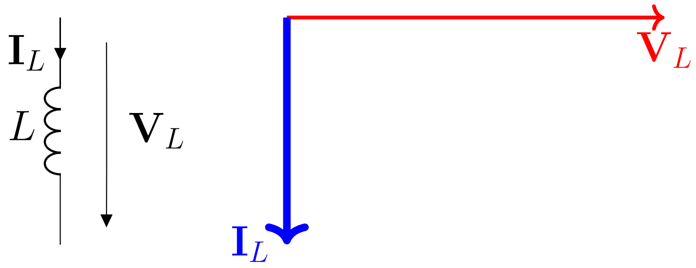
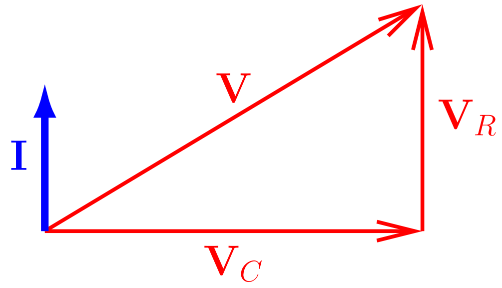
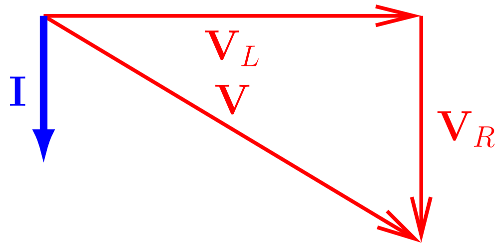


Figure 3

The second Kirchhoff's law (KVL) must apply to the RC circuit, ie $\mathbf{V}_C + \mathbf{V}_R - \mathbf{V} = 0$. We can draw this equation graphically. All elements are in series in the circuit, so the same current flows through them. On the capacitor, there is a phase shift between voltage and current $\frac{\pi}{2}$, on the resistor, there are voltage and current in phase – and therefore, the phase shift between the voltage on the capacitor and the resistor is also $\frac{\pi}{2}$ and their graphical representations are perpendicular lines. We work with lines as with vectors. Just as we add voltages, we add these vectors. For a clockwise loop, a capacitor is connected first; its voltage is plotted first. A voltage across the resistor follows it. Finally, the source's voltage returns to the beginning; in the loop, we go against its arrow (negative sign).



If there were an inductor in the circuit instead of a capacitor, then the diagram would look like this:



Since it is a right triangle, we can use the Pythagorean theorem to calculate the magnitude (amplitude, or RMS value, we use it for both) of the voltage on the capacitor:

$$V_C = \sqrt{V^2 - V_R^2} = \sqrt{230^2 - 120^2} = 196.214\,\text{V} \tag{5}$$

This is a very different result from 110 V, which would come out of KVL without taking into account the phase shift. Therefore, we cannot write:

$$\xcancel{V = RI + \frac{1}{\omega C} I} \tag{6}$$

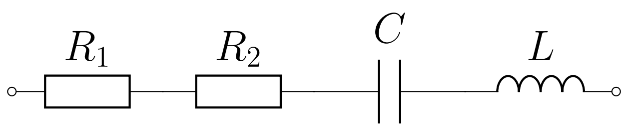
We, therefore, need a mathematical notation that includes the phase shift. Fortunately, there is such a thing – the complex numbers. The trigonometric and exponential shape is essential to us. You can recall basic operations with complex numbers in this separate document. If we add an imaginary unit to the reactance, which we denote by j in electrical engineering so that it does not confuse with current, we get *impedance*:

element	reactance X	impedance $\mathbf{Z}$	admittance $\mathbf{Y}$
resistor	R	R	$G = \frac{1}{R}$
capacitor	$\frac{-1}{\omega C}$	$\frac{1}{j\omega C} = \frac{1}{\omega C}\, e^{-j\frac{\pi}{2}}$	$j\omega C = \omega C\, e^{j\frac{\pi}{2}}$
inductor	ωL	$j\omega L = \omega L\, e^{j\frac{\pi}{2}}$	$\frac{1}{j\omega L} = \frac{1}{\omega L}\, e^{-j\frac{\pi}{2}}$

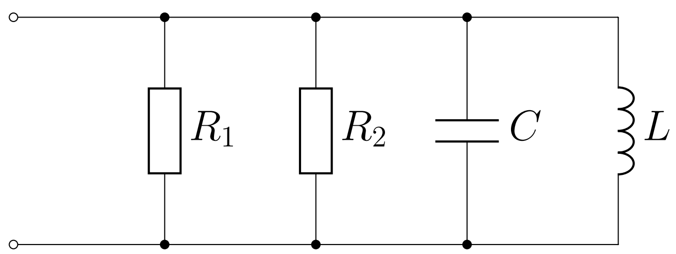
The relationship between impedance and reactance is as follows:

$$\mathbf{Z} = R + jX \tag{7}$$

Thus reactance is an imaginary part of impedance, and because $\mathbf{Z}_C = \frac{1}{j\omega C} = -j\frac{1}{\omega C}$, the reactance of the capacitor is negative. The previously used concept of capacitive reactance is its modulus. Applies to:



$$\mathbf{Z} = R_1 + R_2 + \frac{1}{j\omega C} + j\omega L = R_1 + R_2 + j\left(\frac{-1}{\omega C} + \omega L\right) = \underbrace{R_1 + R_2}_{resistance} + j\underbrace{\left(X_C + X_L\right)}_{reactance} \tag{8}$$

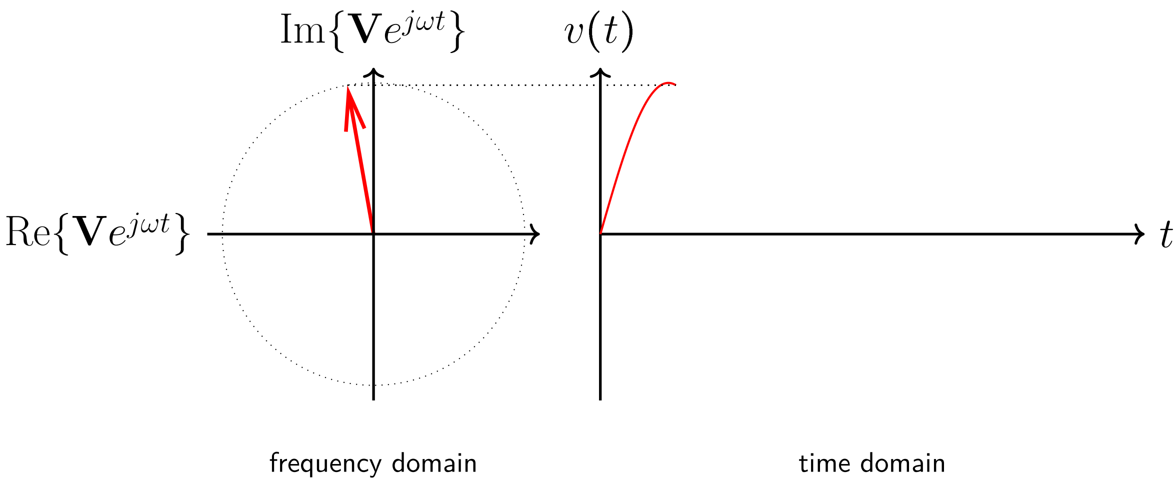


$$\begin{aligned}\mathbf{Y} &= G_1 + G_2 + j\omega C + \frac{1}{j\omega L} = G_1 + G_2 + j\left(\omega C - \frac{1}{\omega L}\right) = \\ &= \underbrace{G_1 + G_2}_{conductance} + j\underbrace{\left(B_C + B_L\right)}_{susceptance}\end{aligned} \tag{9}$$

In addition to impedance, which is the transformation of the integral/derivative into the space of complex numbers, we must also introduce the transformation of the function $\sin$. Here we will again use the properties of complex numbers:

$$\begin{aligned} V_m \sin(\omega t + \varphi) &= \mathrm{Im}\left\{V_m \left[\cos(\omega t + \varphi) + j \sin(\omega t + \varphi)\right]\right\} \\ &= \mathrm{Im}\left\{V_m e^{j(\omega t + \varphi)}\right\} = \mathrm{Im}\left\{\underbrace{V_m e^{j\varphi}}_{\mathbf{V}}\ e^{j\omega t}\right\} \end{aligned} \tag{10}$$

The waveform is, therefore, an imaginary part of a complex number, which can be divided into two parts: constant *phasor* $\mathbf{V} = V_m e^{j\varphi}$ and the term $e^{j\omega t}$, which represents the angular speed of rotation of the phasor. Since the phasors of all quantities in the circuit (voltages, currents) rotate at the same angular velocity, we ignore this term.



With phasors we can write a generalized ohm's law for AC circuits:

$$\mathbf{V} = \mathbf{Z}\mathbf{I} \tag{11}$$

where $\mathbf{V}$ is the voltage phasor, $\mathbf{I}$ the current phasor, and $\mathbf{Z}$ is the impedance from table.

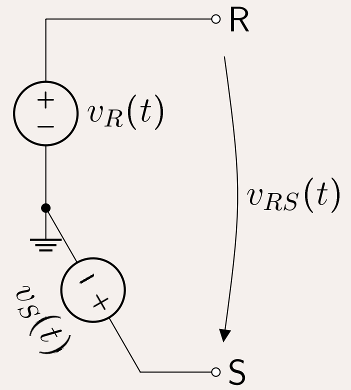
As a symbol for phasors and impedance, in printed form, we usually use the bold font $\mathbf{V}$, $\mathbf{Z}$. In the written form we usually use the accents $\widehat{V}$, $\overline{V}$,

.... We can also use angle notation (Steinmetz's polar form): $V\underline{/\varphi}$ – which is another form of notation $V\,e^{j\varphi}$.

We can substitute the maximum or effective value of voltages / currents into the equations, depending on how the assignment is formulated. If it is written in the assignment that the RMS value of voltage is $V = 230\,\mathrm{V}$, we can use a phasor in the equations in the scale of RMS values $\mathbf{V} = 230\,\mathrm{V}$. If it is written in the assignment that the time waveform of the voltage is $v(t) = 325.269\,\sin(\omega t)$ [V], we can substitute a phasor into the equations in the scale of maximum values $\mathbf{V}_m = 325.269\,\mathrm{V}$.

Example 2

We have two sinusoidal voltage sources, $v_R(t) = 230\sqrt{2}\,\sin(\omega t)$ [V] and $v_S(t) = 230\sqrt{2}\,\sin(\omega t - \frac{2\pi}{3})$ [V] connected antiseries. Calculate terminal voltage $v_{RS}(t)$. Frequency of both sources is $f = 50\,\mathrm{Hz}$.



Solution:

1. We transform both time waveforms into phasors. We can calculate in the scale of maximum or effective values.
   The transformation is simple – we copy the amplitude from the time waveform into the phasor, add the term e^{j} and copy the phase. Is therefore:
   $v_R(t) = 230\sqrt{2}\,\sin(\omega t)$ [V] $\Rightarrow$ $\mathbf{V}_{R_m} = 230\sqrt{2}\,e^{0j} = 230\sqrt{2}\,\mathrm{V}$ (or $\mathbf{V}_R = 230\,\mathrm{V}$).
   $v_S(t) = 230\sqrt{2}\,\sin(\omega t - \frac{2\pi}{3})$ [V] $\Rightarrow$ $\mathbf{V}_{S_m} = 230\sqrt{2}\,e^{\frac{-2\pi}{3}j}\,\mathrm{V}$ (or $\mathbf{V}_S = 230\,e^{\frac{-2\pi}{3}j}\mathrm{V}$)

2. We subtract the phasors (for simplicity I will use the scale of effective

values)

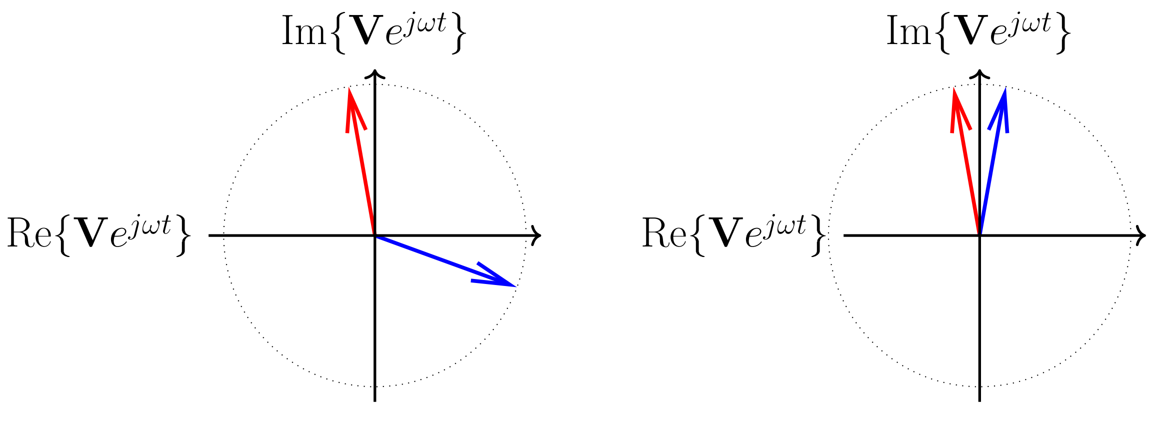
$$\mathbf{V}_{RS} = \mathbf{V}_R - \mathbf{V}_S = 230 - 230\,e^{j\frac{-2\pi}{3}} = 230 - \left[\cos\left(\frac{-2\pi}{3}\right) + \sin\left(\frac{-2\pi}{3}\right)\right]$$
$$= 230 - (-115 - 115\sqrt{3}) = 345 + 199.186j =$$
$$= \sqrt{345^2 + 199.186^2}\,e^{j\arctan\frac{199.186}{345}} = 398.37\,e^{j\frac{\pi}{6}}$$

During the calculation, we converted the exponential form of the complex number to algebraic for addition, and then the result back to exponential for inverse transformation.

3. We transform the phasor back into a time waveform.
   We copy the amplitude, the expression e^j replace by $\sin(\omega t)$ and copy the phase shift:
   $\mathbf{V}_{RS} = 398.37\,e^{j\frac{\pi}{6}} \Rightarrow v_{RS}(t) = 398.37\sqrt{2}\,\sin(100\pi t + \frac{\pi}{6})$.

What if the voltage $v_R(t) = 230\sqrt{2}\,\sin(\omega t)$ [V] has frequency $f_R = 50\,\text{Hz}$ and the voltage $v_S(t) = 230\sqrt{2}\,\sin(\omega t - \frac{2\pi}{3})$ [V] has frequency $f_S = 100\,\text{Hz}$?

In this case, both phasors are the same as in the previous example – $\mathbf{V}_R = 230\,\text{V}$ and $\mathbf{V}_S = 230\,e^{\frac{-2\pi}{3}j}\text{V}$, because the phasors do not include the angular frequency. However, we must not add up both phasors, because they rotate in a complex plane, and due to the different angular frequencies, their position relative to each other is not constant – we would get a different result at each time instant.



the same frequency | different frequencies

We have to solve this case by the method of superposition – the periodic non-harmonic steady state.

Let us now return to the example 1. Let's say that the power dissipation on the resistor is e.g. $P = 60\,\mathrm{W}$ and the voltage source has frequency $f = 50\,\mathrm{Hz}$. What capacity must the capacitor have?
$I = \frac{P}{V} = \frac{60}{120} = 0.5\,\mathrm{A}$. Using reactance (we know both the voltage on the capacitor and the flowing current) $V_C = \frac{1}{\omega C} I \Rightarrow C = \frac{I}{\omega V_C} = \frac{0.5}{100\pi 196.214} = 8.11\,\mu\mathrm{F}$.

Now that we know both the resistance $R = \frac{V}{I} = \frac{120}{0.5} = 240\,\Omega$ and capacitance, we can now calculate the current phasor and the voltage phasors on the capacitor and resistor. For such calculation, we already need the impedance.

The source does not have the phase shift specified. For us, it will be a reference with a phase shift of 0 (we must always select a reference in the circuit, whether it is a reference node in the nodal analysis or a zero phase shift in the AC analysis).

$$\mathbf{I} = \frac{\mathbf{V}}{R + \mathbf{Z}_C} = \frac{\mathbf{V}}{R + \frac{1}{j\omega C}} \cdot \frac{j\omega C}{j\omega C} = \mathbf{U}\frac{j\omega C}{j\omega RC + 1} \tag{12}$$

$$= 230 \cdot \frac{j \cdot 100 \cdot \pi \cdot 8.11 \cdot 10^{-6}}{j \cdot 100 \cdot \pi \cdot 240 \cdot 8.11 \cdot 10^{-6} + 1} =$$

$$= 0.261 + 0.427j = 0.5e^{1.022j}\ \mathrm{A}$$

$$\mathbf{V}_C = \mathbf{V}\frac{\mathbf{Z}_C}{\mathbf{Z}_C + R} = \mathbf{V}\frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + R} = \mathbf{V}\frac{1}{1 + j\omega RC} = \tag{13}$$

$$= 230 \cdot \frac{1}{1 + j \cdot 100 \cdot \pi \cdot 240 \cdot 8.11 \cdot 10^{-6}} =$$

$$= 167.39 - 102.37j = 196.214\,e^{-0.549j}\ \mathrm{V}$$

$$\mathbf{V}_R = \mathbf{V}\frac{R}{\mathbf{Z}_C + R} = \mathbf{V}\frac{R}{\frac{1}{j\omega C} + R} = \mathbf{V}\frac{j\omega RC}{1 + j\omega RC} = \tag{14}$$

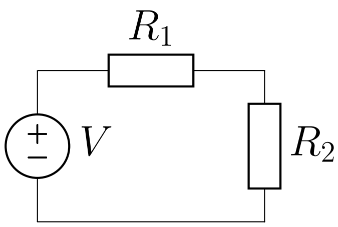
$$= 230 \cdot \frac{j \cdot 100 \cdot \pi \cdot 240 \cdot 8.11 \cdot 10^{-6}}{1 + j \cdot 100 \cdot \pi \cdot 240 \cdot 8.11 \cdot 10^{-6}} =$$

$$= 62.61 + 102.37 = 120\, e^{1.022j} \text{ V}$$

From these calculations, it is clear that $\mathbf{V}_C + \mathbf{V}_R = 230\,\text{V}$ and that Kirchhoff's law works as it should if we include phase shift.

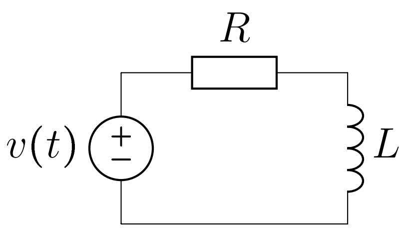
In Equations 13 . . . 15 I used Ohm's law and the voltage divider in the same way as in the resistive circuits, with the only difference – phasors and impedances are complex numbers. In the AC circuits we use the same rules (dividers, superposition, circuit equations . . .) as in the DC circuits.

In DC circuits, the voltage divider was:



$$V_2 = V\,\frac{R_2}{R_1 + R_2} \tag{15}$$

In the AC circuits it is:



$$\mathbf{V}_2 = \mathbf{V}\,\frac{j\omega L}{R_1 + j\omega L} \tag{16}$$

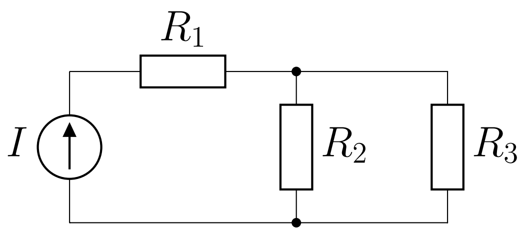
In general, for multiple resistors and reactance elements (inductor / capacitor) connected in series, for the k-th element:

$$\mathbf{V}_k = \mathbf{V}\frac{R_k}{\sum_{m=1}^{M} R_m + \sum_{n=1}^{N} jX_n} \tag{17}$$

(voltage on the resistor R_k), or in general

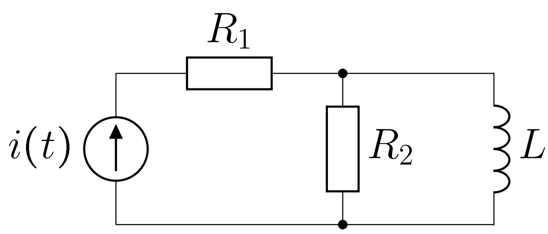
$$\mathbf{V}_k = \mathbf{V}\frac{\mathbf{Z}_k}{\sum_{j=1}^{M+N} \mathbf{Z}_j} \tag{18}$$

In DC, the following applied for the current divider:



$$I_{R_2} = I\,\frac{G_2}{G_2 + G_3} \tag{19}$$

where $G = \frac{1}{R}$ is conductance. Resistor R_1 does not affect the current through the resistor R_2 because it is connected in series with the current source.
In the AC circuits it is:



$$\mathbf{I}_{R_2} = \mathbf{I}\,\frac{G_2}{G_2 + \mathbf{Y}_L} \tag{20}$$

where $\mathbf{Y}_L = \frac{1}{\mathbf{Z}_L} = \frac{1}{j\omega L}$ is admittance of the inductor. In general, for multiple resistors and reactance elements (inductor / capacitor) connected in series, for the k-th element:

$$\mathbf{I}_j = \mathbf{I}\frac{\mathbf{Y}_j}{\sum_{i=1}^{N}\mathbf{Y}_i} \tag{21}$$

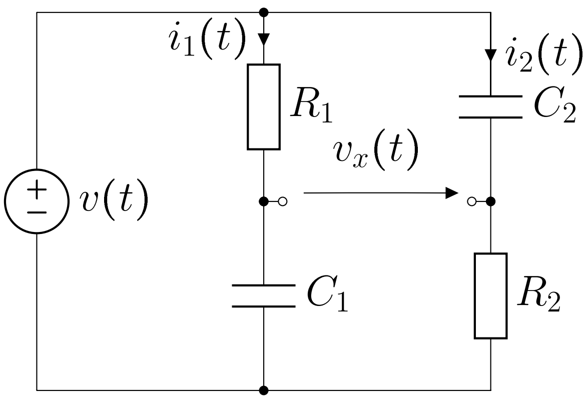
and applies to the sum of admittances $\mathbf{Y} = \sum_{m=1}^{M}\frac{1}{R_m} + \sum_{n=1}^{N}\frac{1}{jX_n} = \sum_{m=1}^{M} G_m + \sum_{n=1}^{N} jB_n$

I will illustrate the effect of the phase shift on another example:

Example 3

Design a bridge (see figure below) with two RC networks so that the bridge voltage is 0 V

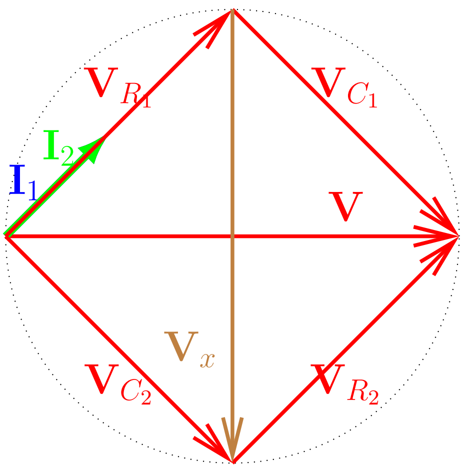
First of all, we have to ask ourselves the question – is the order of the two elements important? The answer is given by phasor diagrams – we have already drawn it for RC network based on rules.



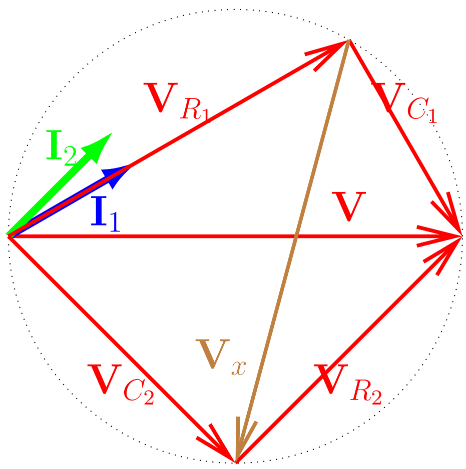
The phasors of currents $\mathbf{I}_1$ and $\mathbf{I}_2$ will both lead the voltage V. The phase shift is $\varphi \in (0, \frac{\pi}{2})$ – see the angle between $\mathbf{V}$ and $\mathbf{I}$ in the phasor diagram of the RC network. In Matlab, you can run simulation (due to the limitations of Acrobat Reader compressed to 7 zip format).

Since we now have two triangles, we can start from the common supply voltage $\mathbf{V}$. From it we derive the currents $\mathbf{I}_1$ and $\mathbf{I}_2$ using the rules on figures 1 and 2. To show the behavior of the bridge, I will draw two phasor diagrams. In the first case, I choose an angle for both currents 45°, in the second e.g.

$\varphi_{I_1} = 30°$, $\varphi_{I_2} = 45°$. In the enclosed program for Matlab (start it from HUS_RC.m script) you can display a phasor diagram for any other case. To construct a right triangle, draw a Thalet circle. Voltage $\mathbf{V}_{R_1}$ is in the phase with current, the voltage $\mathbf{V}_{C_2}$ lag the current by $\frac{\pi}{2}$. In the left branch we start from voltage $\mathbf{V}_{R_1}$, in the right branch with voltage $\mathbf{V}_{C_2}$ – Kirchhoff's law must respect the order of the elements! The resulting phasor diagram is shown in the figure:



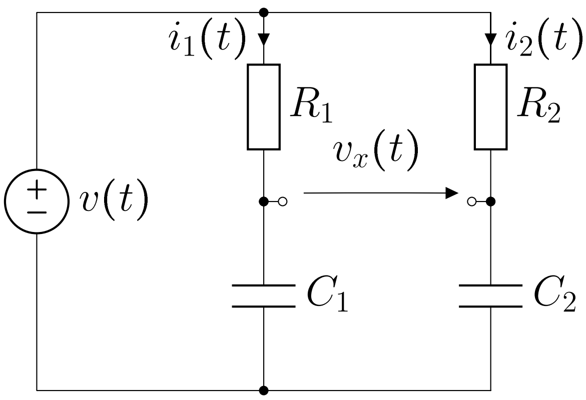
$\varphi_{I_1} = 45°$, $\varphi_{I_2} = 45°$



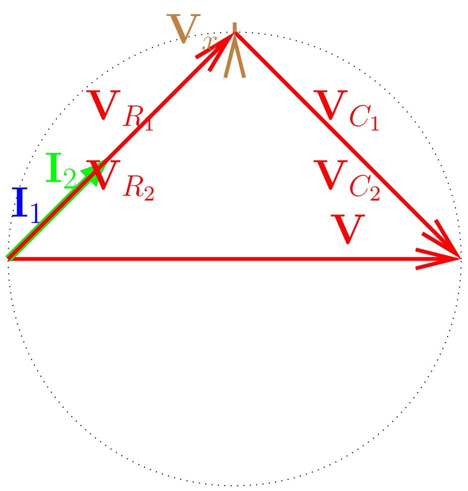
$\varphi_{I_1} = 30°$, $\varphi_{I_2} = 45°$

Since the vertices of triangles can only move along the Thales's circle with changes in the values of R, C, and f, it is obvious that in this arrangement, $\mathbf{V}_x$ cannot be zero.

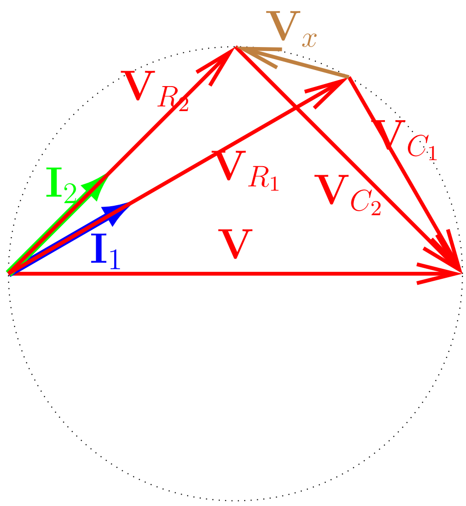
For the circuit



is the phasor diagram:



$\varphi_{I_1} = 45°$, $\varphi_{I_2} = 45°$

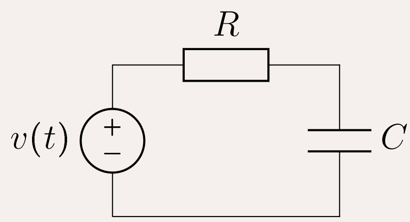


$\varphi_{I_1} = 30°$, $\varphi_{I_2} = 45°$

It is obvious that the voltage $\mathbf{V}_x$ can be zero if the phase shift of both currents is the same, ie $\varphi_{I_1} = \varphi_{I_2}$ and so the both triangles are identical. A special case is the phase shift 45°, thus under conditions $|\mathbf{V}_{R_1}| = |\mathbf{V}_{C_1}|$ and $|\mathbf{V}_{R_2}| = |\mathbf{V}_{C_2}|$, ie $R_1 I_1 = \frac{1}{\omega C_1} I_1$ and $R_2 I_2 = \frac{1}{\omega C_2} I_2$. So the conditions for this special case are $R_1 = \frac{1}{\omega C_1}$ and $R_2 = \frac{1}{\omega C_2}$. If no other boundary conditions are specified (maximum power dissipation, etc., we can choose any value, eg R and calculate the value of C from the condition.

Example 4

In the circuit



find the condition at which there will be a phase shift of the voltage on the capacitor to the voltage of the source −60°.

This example is important, for example, for the design of an RC oscillator, where it is necessary to create a phase shift between the output and input voltage −180°. This is achieved by connecting 3 RC circuits in a row, because one RC circuit can shift the voltage in the interval $\varphi \in (0, -90°)$. Thus $-60 = \frac{-180}{3}$.

The voltage on the capacitor is:

$$\mathbf{V}_C = \mathbf{V}\,\frac{\mathbf{Z}_C}{\mathbf{Z}_C + R} = \mathbf{V}\,\frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + R} = \mathbf{V}\,\frac{1}{1 + j\omega RC} \tag{22}$$

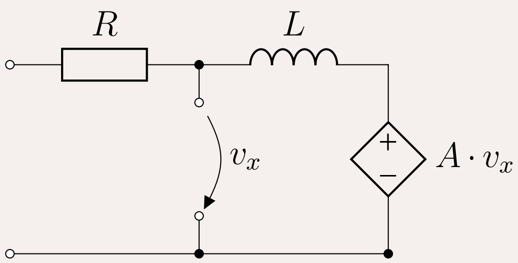
We rewrite this equation into an exponential form:

$$\mathbf{V}_C = \mathbf{V}\,\frac{1}{1 + j\omega RC} = \mathbf{V}\,\frac{1}{\sqrt{1 + (\omega RC)^2}}\,e^{-j\arctan(\omega RC)} \tag{23}$$

From this equation we get $-\arctan(\omega RC) = -60°$, from where we get the condition $\omega RC = \sqrt{3}$. By substitution of this condition into the relation $\frac{1}{\sqrt{1+(\omega RC)^2}}$ we get solution for the magnitude $\frac{1}{\sqrt{1+(\sqrt{3})^2}} = \frac{1}{2}$.

Example 5

Calculate the input impedance of the circuit in the figure.



$R = 1\,\mathrm{k\Omega}$, $L = 0.1\,\mathrm{H}$, $A = -10$, $\omega = 10\,000\,\mathrm{rad\,s^{-1}}$.

To calculate the input impedance, connect a voltage source of any value $\mathbf{V}$ to the input terminals, and calculate the current in the circuit. The input impedance is defined by Ohm's law, $\mathbf{Z}_{in} = \frac{\mathbf{V}}{\mathbf{I}}$. It is the total impedance of the circuit between the input terminals. For the circuit in the figure, we can write the circuit equation:

$$R\mathbf{I} + j\omega L\mathbf{I} + A\mathbf{V}_x - \mathbf{V} = 0 \tag{24}$$

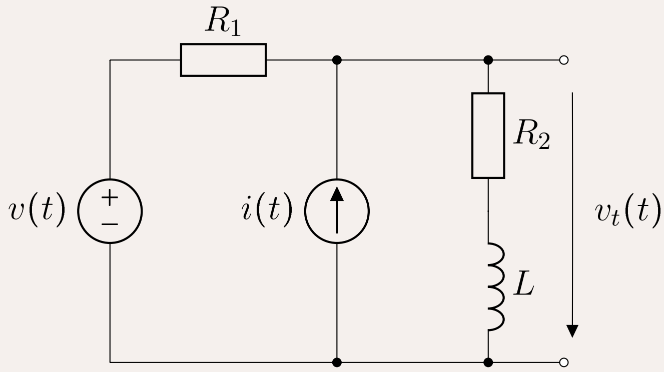
In the equation we must evaluate $\mathbf{V}_x$. We can use the Kirchhoff's voltage law (KVL) – we can run the loop either to the left via the voltage source, then $-\mathbf{V} + R\mathbf{I} + \mathbf{V}_x = 0$, or to the right, and then $j\omega L\mathbf{I} + A\mathbf{V}_x - \mathbf{V}_x = 0$. From one or the other equation we express $\mathbf{V}_x$ and substitute into the equation 24.

After the rearrangement we get $\mathbf{I}[j\omega L + R(1 - A)] = \mathbf{V}(1 - A)$, so

$$\mathbf{Z}_{in} = R + \frac{j\omega L}{1 - A} = 1000 - 111.1j\,\Omega \tag{25}$$

Example 6

Draw equivalent Thévenin and Norton circuit and calculate their parameters for the following network:



$v(t) = 100\sin(10000t + \frac{\pi}{4})$ V, $i(t) = 0.1\sin(10000t)$ A, $R_1 = 1\,\text{k}\Omega$, $R_2 = 3\,\text{k}\Omega$, $L = 0.3\,\text{H}$.

Among other methods, we can use superposition here. We convert voltage and current waveforms into phasors: $\mathbf{V} = 100\,e^{j\frac{\pi}{4}}$, $\mathbf{I} = 0.1$.

1. Disconnect the current source. We get a voltage divider where

$$\mathbf{V}_t' = \mathbf{V}\,\frac{R_2 + j\omega L}{R_1 + R_2 + j\omega L} = 100\,e^{j\frac{\pi}{4}}\,\frac{3000 + 3000j}{4000 + 3000j} = 59.91 + 67.88j$$

2. Replace voltage source by the short circuit. R_1 is parallel to $R_2 + j\omega L$,

$$\mathbf{V}_t'' = \mathbf{I}\,\frac{R_1(R_2 + j\omega L)}{R_1 + R_2 + j\omega L} = 84 + 12j$$

3. The Thévenin's voltage is:

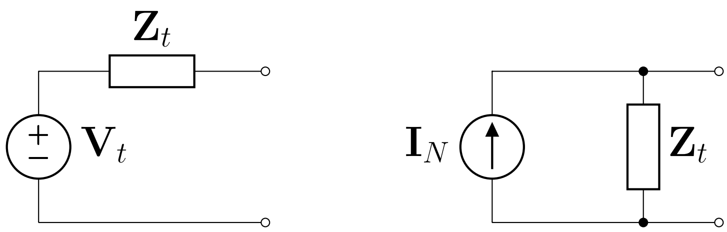
$$\mathbf{V}_t = V_t' + V_t'' = 59.91 + 67.88j + 84 + 12j = 134.91 + 79.88j = 156.79\,e^{0.53j}\,\text{V}$$

4. To calculate Thévenin's equivalent impedance, remove the current source and replace the voltage source by a short circuit,

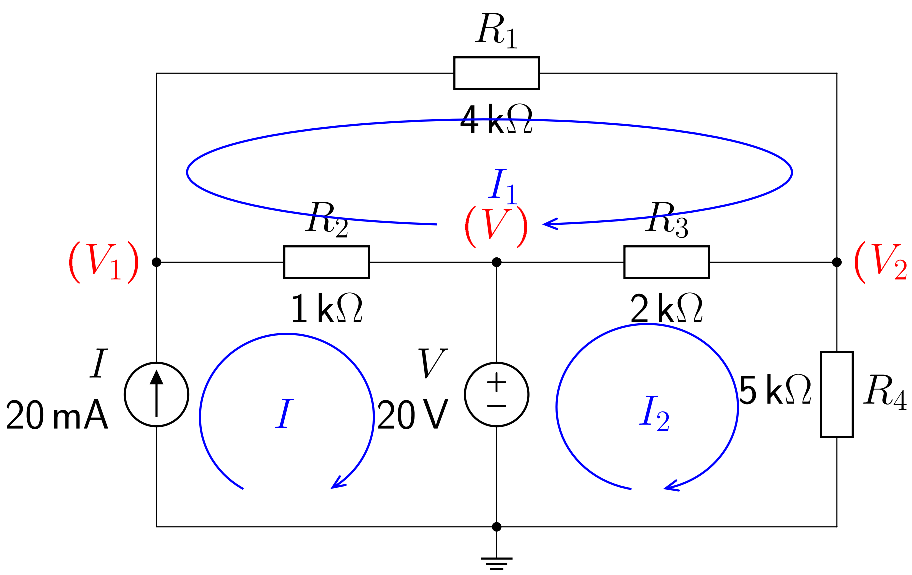
$$\mathbf{Z}_t = \frac{R_1(R_2 + j\omega L)}{R_1 + R_2 + j\omega L} = 840 + 120j\,\Omega$$

5. Norton equivalent current source:

$$\mathbf{I}_N = \frac{\mathbf{V}_t}{\mathbf{Z}_t} = 0.17 + 0.07j \text{ A}$$



Circuit equations – the rules for writing the equations are the same as in DC or general waveforms. As in DC, there are no initial conditions in AC. Let's compare the solution in DC and AC. Let's have a circuit:



The circuit has four nodes. The bottom one is a reference node – the ground, its potential is 0 V. We know the voltage in the middle node – it is the source voltage, $V = 20$ V. So there are two nodes left with unknown voltages V_1 and V_2. For those nodes, we write the equations:

$$-I + \frac{V_1 - V}{R_2} + \frac{V_1 - V_2}{R_1} = 0$$

$$\frac{V_2 - V_1}{R_1} + \frac{V_2 - V}{R_3} + \frac{V_2}{R_4} = 0$$

We can write them in matrix form as:

$$\begin{bmatrix} \frac{1}{R_1} + \frac{1}{R_2} & \frac{-1}{R_1} \\ \frac{-1}{R_1} & \frac{1}{R_1} + \frac{1}{R_3} + \frac{1}{R_4} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} I + \frac{V}{R_2} \\ \frac{V}{R_3} \end{bmatrix}$$

In Matlab, we solve this system of equations very easily:

```
>> R1=4000;
>> R2=1000;
>> R3=2000;
>> R4=5000;
>> I=0.02;
>> V=20;
>> G=[1/R1+1/R2, -1/R1; -1/R1, 1/R1+1/R3+1/R4];
>> IM=[I+V/R2; V/R3];
>> VM=inv(G)*IM

VM =

    36
    20
```

In the circuit are three meshes (it is the loop which does not enclose any circuit element; the general loop, not a mesh, would be, e.g., the loop through I, R_1 and R_4). However, one of the loops goes through the current source. For that reason, we already know this current, so we do not write an equation for the mesh (and we can't; we don't know the value of the voltage at the current source). So we have just two equations as well:

$$R_1 I_1 + R_3(I_1 - I_2) + R_2(I_1 - I) = 0$$

$$R_4 I_2 - V + R_3(I_2 - I_1) = 0$$

We can write them in matrix form as:

$$\begin{bmatrix} R_1 + R_2 + R_3 & -R_3 \\ -R_3 & R_3 + R_4 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} R_2 I \\ V \end{bmatrix}$$

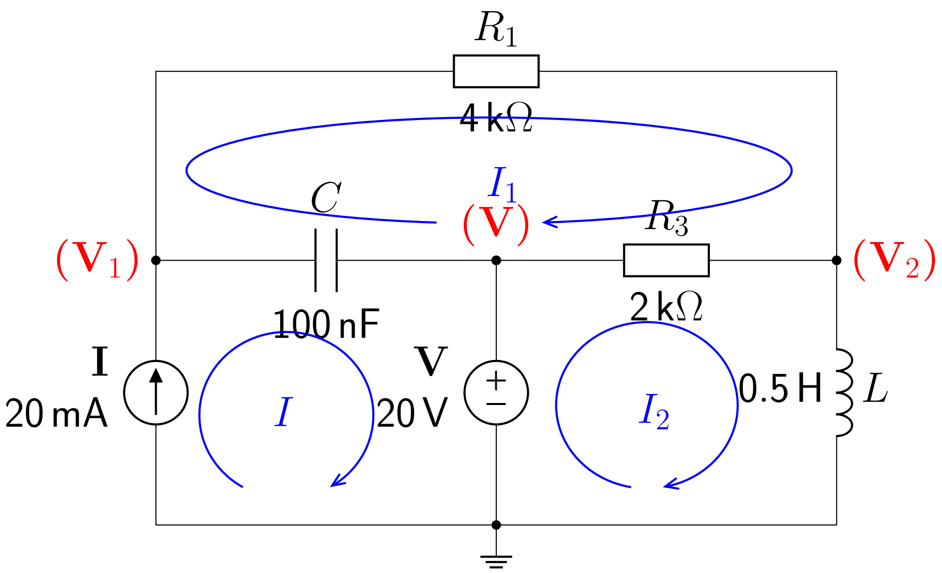
In Matlab, we solve this system of equations very easily:

```
>> RM=[R1+R2+R3,  -R3;  -R3,  R3+R4];
>> VM=[R2*I;  V];
>> IM=inv(RM)*VM

IM =

    0.0040
    0.0040
```

We have the AC circuit:



The voltage and current in the circuit are their rms values, the frequency of both sources is $\omega = 10\,000\,\mathrm{rad\,s^{-1}}$.

In such a case the equations are:

$$-\mathbf{I} + \frac{\mathbf{V}_1 - \mathbf{V}}{\frac{1}{j\omega C}} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{R_1} = 0$$

$$\frac{\mathbf{V}_2 - \mathbf{V}_1}{R_1} + \frac{\mathbf{V}_2 - \mathbf{V}}{R_3} + \frac{\mathbf{V}_2}{j\omega L} = 0$$

We can write them in matrix form as:

$$\begin{bmatrix} \frac{1}{R_1} + \frac{1}{\frac{1}{j\omega C}} & \frac{-1}{R_1} \\ \frac{-1}{R_1} & \frac{1}{R_1} + \frac{1}{R_3} + \frac{1}{j\omega L} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} I + \frac{\mathbf{V}}{\frac{1}{j\omega C}} \\ \frac{\mathbf{V}}{R_3} \end{bmatrix}$$

In Matlab, we solve this system of equations very easily:

```
>> w=10000;
>> R1=4000;
>> C=100E-9;
>> R3=2000;
>> L=0.5;
>> I=0.02;
>> V=20;
>> ZC=1/(j*w*C);
>> ZL=j*w*L;
>> YM=[1/R1+1/ZC, -1/R1; -1/R1, 1/R1+1/R3+1/ZL];
>> IM=[I+V/ZC; V/R3];

>> VM=inv(YM)*IM

VM =

  24.659 -19.265i
  21.721 - 0.63i
```

Now we convert the results in the exponential form $\mathbf{V}_1 = 31.293\, e^{-0.663j}$, $\mathbf{V}_2 = 21.73\, e^{-0.029j}$ and finally into the time domain $v_1(t) = 31.293 \sin(10000t - 0.663)$, $v_2(t) = 21.73 \sin(10000t - 0.029)$.

The mesh analysis:

$$R_1\mathbf{I}_1 + R_3(\mathbf{I}_1 - \mathbf{I}_2) + \frac{1}{j\omega C}(\mathbf{I}_1 - \mathbf{I}) = 0$$

$$j\omega L\mathbf{I}_2 - \mathbf{V} + R_3(\mathbf{I}_2 - \mathbf{I}_1) = 0$$

We can write them in matrix form as:

$$\begin{bmatrix} R_1 + \frac{1}{j\omega C} + R_3 & -R_3 \\ -R_3 & R_3 + j\omega L \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{j\omega C}\mathbf{I} \\ \mathbf{V} \end{bmatrix}$$

In Matlab, we solve this system of equations very easily:

```
>> w=10000;
>> R1=4000;
>> C=100E-9;
>> R3=2000;
>> L=0.5;
>> I=0.02;
>> V=20;
>> ZC=1/(j*w*C);
>> ZL=j*w*L;
>> ZM=[R1+R3+ZC, -R3; -R3, R3+ZL];
>> VM=[ZC*I; V];
>> IM = inv(ZM)*VM

IM =

   0.000735 - 0.004659i
  -0.000126 - 0.004344i
```

The exponential form: $\mathbf{I}_1 = 0.004716\, e^{-1.414j}$ A, $\mathbf{I}_2 = 0.004346\, e^{-1.6j}$ A. The waveforms: $i_1(t) = 4.716\sqrt{2}\,\sin(10000t - 1.414)$ mA, $i_2(t) = 4.346\sqrt{2}\,\sin(10000t - 1.6)$ mA.