

Electrical Circuits

III

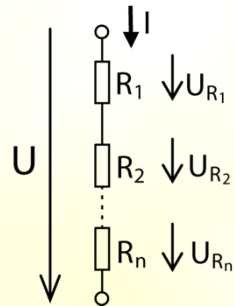
Basic laws and theorems

BASIC LAWS AND THEOREMS (KIRCHHOFF'S CIRCUIT LAWS, THÉVENIN'S AND NORTON'S THEOREM), EXAMPLES OF APPLICATION (EQUIVALENCE OF CIRCUIT ELEMENTS, VOLTAGE DIVIDER, CURRENT DIVIDER, ACTUAL SOURCES).

Equivalence of circuit elements

... or series and parallel connection of circuit elements ☺

➤ Resistor

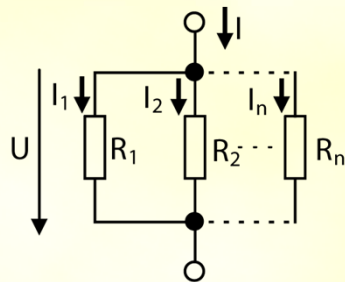


series connection – common current

the voltages across distinct resistors are summed

$$U = R_1 I + R_2 I + \cdots + R_n I \\ = I(R_1 + R_2 + \cdots + R_n) = IR$$

$$R = \sum_{i=1}^n R_i$$



parallel connection – common voltage

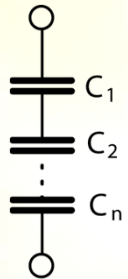
the currents flowing through distinct resistors are summed

$$I = \frac{U}{R_1} + \frac{U}{R_2} + \cdots + \frac{U}{R_n} \\ = U\left(\frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_n}\right) = UG$$

$$G = \sum_{i=1}^n G_i$$

$$\frac{1}{R} = \sum_{i=1}^n \frac{1}{R_i}$$

➤ Capacitor



series connection – common current

current passing series connection of capacitors is same
 \Rightarrow charges stored in distinct capacitors are all the same

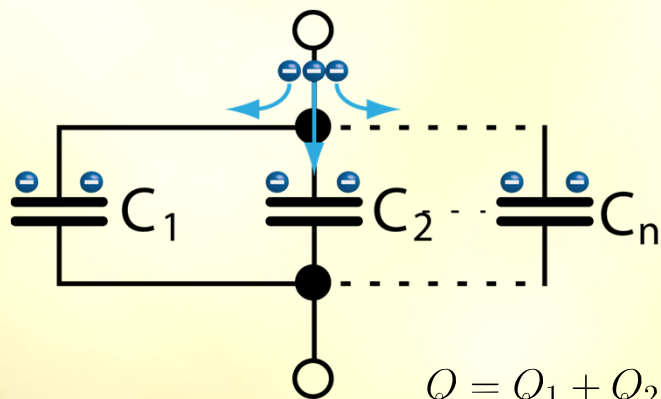
$$\frac{1}{C} = \sum_{i=1}^n \frac{1}{C_i}$$

$$U = \frac{Q}{C} \quad U = U_1 + U_2 + \dots + U_n =$$

$$= \left(\frac{Q_1}{C_1} + \frac{Q_2}{C_2} + \dots + \frac{Q_n}{C_n} \right) = Q \left(\frac{1}{C_1} + \frac{1}{C_2} + \dots + \frac{1}{C_n} \right) = Q \frac{1}{C}$$

mnemonic: in the case of parallel-plate capacitor is approximately valid $C \approx \varepsilon \frac{S}{d}$ where S is the area of plates and d distance between the plates; when capacitors are series connected, then the total distance is greater and total capacitance less

advantage – real capacitor has limited maximum allowed voltage; when more capacitors are series connected, then total voltage is divided among all capacitors; e.g. if I connect three same capacitors with nominal voltage 100 V in series, I can connect up to 300V across series of capacitors



parallel connection – common voltage

Charge (delivered in form of current) is divided among all capacitors
 \Rightarrow charges stored in distinct capacitors adds together

$$C = \sum_{i=1}^n C_i$$

$$Q = Q_1 + Q_2 + \dots + Q_n = CU$$

$$Q = UC_1 + UC_2 + \dots + UC_n = U(C_1 + C_2 + \dots + C_n) = UC$$

Inductor

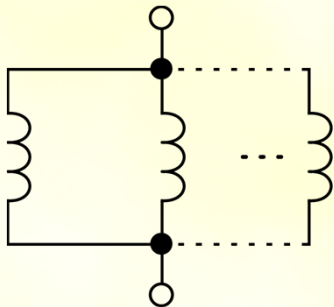


series connection – common current

\Rightarrow Magnetic fluxes of distinct inductors are summed

$$\begin{aligned}\Phi &= \Phi_1 + \Phi_2 + \cdots + \Phi_n = \\ &= L_1 I + L_2 I + \cdots + L_n I = I(L_1 + L_2 + \cdots + L_n) = IL\end{aligned}$$

$$L = \sum_{i=1}^n L_i$$



parallel connection – common voltage

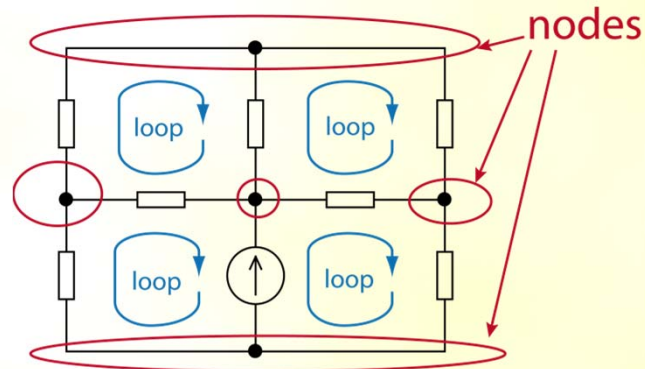
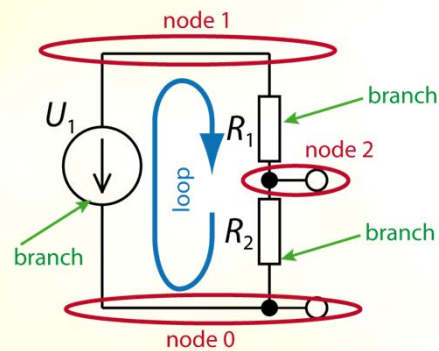
Since $u = \frac{\Delta\Phi}{\Delta t}$ and voltage u across all inductors is the same, and difference of time Δt is also the same, **all inductors have same magnetic flux Φ**

Influent electric current is distributed among all inductors, so

$$I = \frac{\Phi}{L} = I_1 + I_2 + \cdots + I_n = \frac{\Phi}{L_1} + \frac{\Phi}{L_2} + \cdots + \frac{\Phi}{L_n} = \Phi \left(\frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_n} \right)$$

$$\frac{1}{L} = \sum_{i=1}^n \frac{1}{L_i}$$

Basic terms



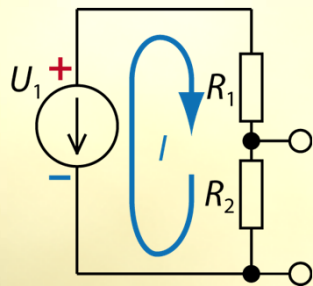
- **Node:** conductive connection of two, or more circuit elements; it can be but not have to be marked by dot (in the case of exactly two circuit elements only connecting wire will be drawn)
- **Loop:** any closed path in circuit diagram, it mustn't intersect itself
- **Branch:** circuit element, connected between two nodes

2nd Kirchhoff's law (voltage, KVL)

From the previous lecture we know its basic definition; now we introduce it in actual circuit: **the sum of all voltages in any loop in the circuit is 0**

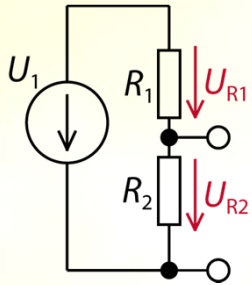
Convention:

- We consider voltages across all passive circuit elements are positive
- The signs of source voltages are determined by their orientation – when the loop enters negative source terminal, the sign will be negative, when enters into positive terminal, then the sign will be positive
- It is impossible directly evaluate voltage across terminals of current source



$$U_{R_1} + U_{R_2} - U_1 = 0 \quad \rightarrow \quad R_1 I + R_2 I - U_1 = 0$$

Voltage divider



From the KVL we know the current passing the loop (*so from **voltage** Kirchhoff's law we evaluate current*), then using Ohm's law we can evaluate voltage across both resistors:

$$I = \frac{U_1}{R_1 + R_2} \quad U_2 = R_2 I = R_2 \frac{U_1}{R_1 + R_2}$$

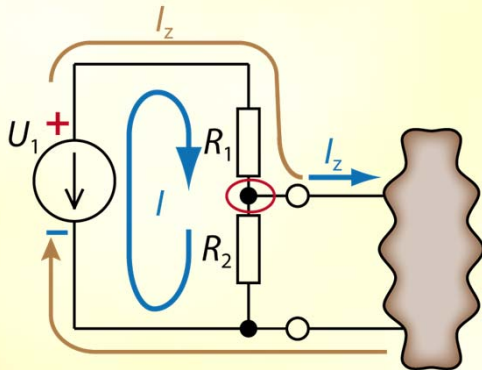
$$U_2 = U_1 \frac{R_2}{R_1 + R_2}$$

N resistor generalization:

$$U_j = U_1 \frac{R_j}{\sum_{i=1}^N R_i}$$

1st Kirchhoff's law (current, KCL)

Again, basic definition we know from previous lecture; **the sum of currents in any node is 0**



In marked node according to the KCL is valid:

$$-I_{R_1} + I_z + I_{R_2} = 0 \quad \rightarrow \quad I_{R_1} = I_{R_2} + I_z = I + I_z$$

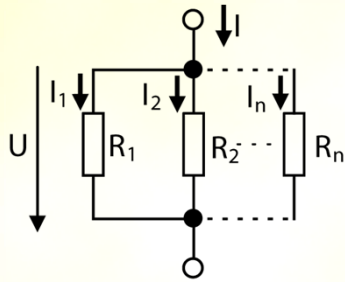
Now we use KVL:

$$U_{R_1} + U_{R_2} - U_1 = 0 \quad \rightarrow \quad R_1(I + I_z) + R_2 I - U_1 = 0$$

And finally output voltage of **voltage divider with load**:

$$U_2 = R_2 I = R_2 \frac{U_1 - R_1 I_z}{R_1 + R_2}$$

Current divider



When we analyze some circuit, a simplification using **current divider** may be used:

First, assume two resistor connected in parallel.

- The common circuit variable is voltage

$$U = RI = \frac{R_1 R_2}{R_1 + R_2} I$$

- According to the Ohm's law

$$I_2 = \frac{U}{R_2} = I \frac{R_1 R_2}{R_1 + R_2} \frac{1}{R_2}$$

$$I_2 = I \frac{R_1}{R_1 + R_2}$$

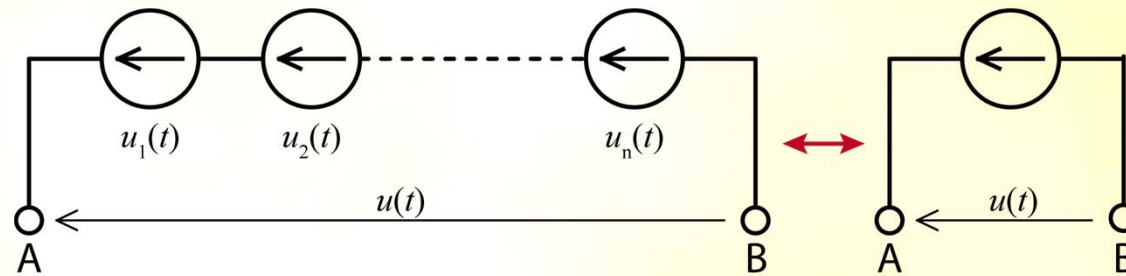
- When 3 resistors are connected in parallel:

$$R = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right)^{-1} = \frac{R_1 R_2 R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3}$$

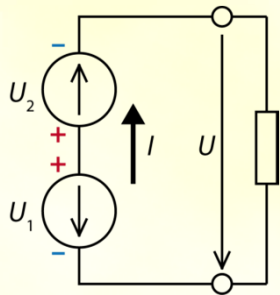
General case

$$I_j = I \frac{G_j}{\sum_{i=1}^N G_i}$$

➤ Equivalence of voltage sources



Series connection of voltage sources $u(t) = \sum_{k=1}^n u_k(t)$



$$U = U_1 - U_2$$

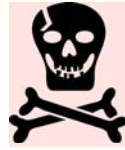
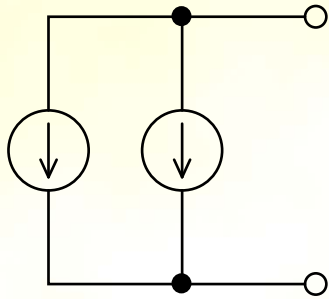
When we connect actual sources in this way we may experience serious problems:

What are the powers in such circuit?

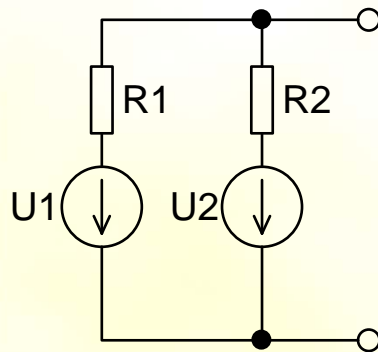
- According to the marked orientation of current is $U_1 > U_2$
- Current I leaves the positive terminal of $U_1 \Rightarrow$ this source delivers power to the circuit
 $P_1 = U_1 I$
- Current I enter the **positive terminal** of the source $U_2 \Rightarrow$ this source absorbs power
 $P_2 = -U_2 I$, **it is negative and it acts like power consumer**
- (Rechargeable) secondary cell would be charged, alternator mechanically loaded, primary cell (as well as overloaded secondary) could be destroyed (in some cases, e.g. primary Li-SOCl₂ or secondary Li-ion in very destructive way)



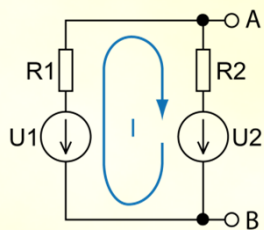
Illustrative photo, source: Sandia National Labs



Parallel connection of an ideal voltage sources is not possible, with exception of absolutely identical voltages; if one source would have higher voltage, it would deliver infinite power into the second one



Parallel connection of actual voltage sources is possible, (and is often used), but still, both sources **should have same voltage** (*or the lower voltage source acts like power consumer*) – for charging of secondary cells balancers are used

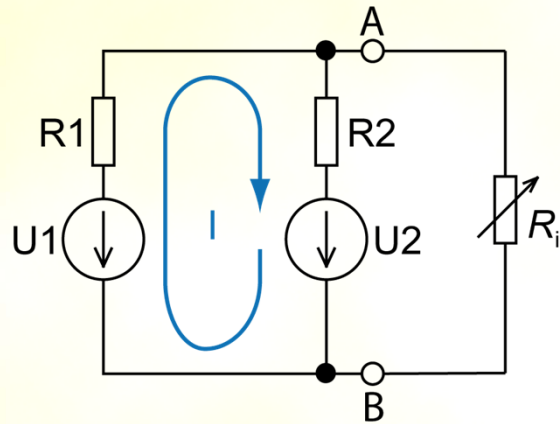


To analyze this circuit we may use KVL:

$$-U_1 + R_1 I + R_2 I + U_2 = 0 \quad \Rightarrow \quad I = \frac{U_1 - U_2}{R_1 + R_2}$$

$$U_{AB} = U_2 + R_2 I \quad \text{or} \quad U_{AB} = U_1 - R_1 I$$

... it is **maximum possible voltage**, which can be across terminals of this circuit



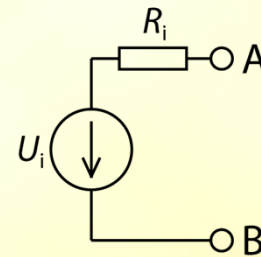
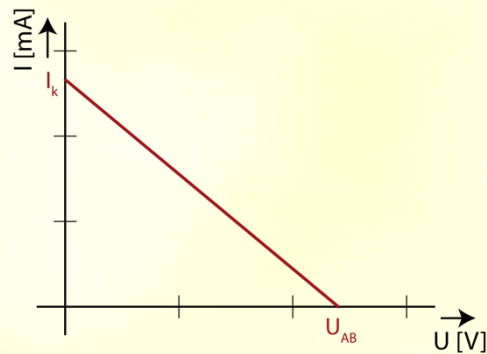
Maximum possible current, we may load from this circuit: suppose we connect some resistor between terminals A and B. As the resistance is decreased and becomes smaller and smaller, we finally reach a point when the resistance is 0 = short circuit – we will call this current (maximum possible) **short-circuit current**

$$I_k = \frac{U_1}{R_1} + \frac{U_2}{R_2}$$

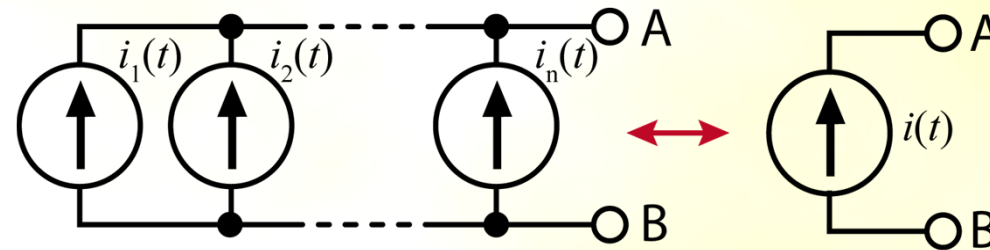
The voltage U_{AB} will be denoted **open-circuit voltage** $U_p (U_i)$

Current–voltage characteristic (I-V characteristic) on the picture below is the same as current-voltage characteristic of the voltage source with internal resistivity

$$R_i = \frac{U_p}{I_k}$$

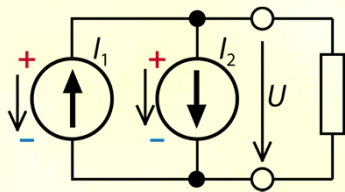


Equivalence of current sources



Parallel connection of current sources

$$i(t) = \sum_{k=1}^n i_k(t)$$

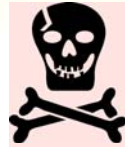
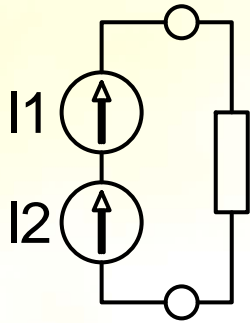


$$I = I_1 - I_2$$

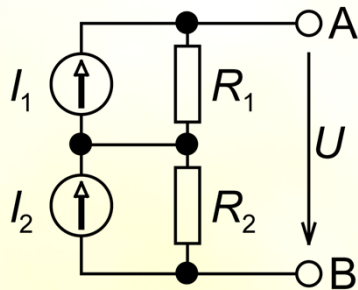
- For limited time inductor function as a current source (in the sense it maintains constant current)
- Permanent operating current source has to be implemented electronically

What are the powers in such circuit?

- According to the marked orientation of voltage across source's terminals it has to be valid $I_1 > I_2$, since the first source enforces its voltage orientation (current source when delivers power to the circuit has positive voltage on that terminal, which the current is leaving)
- With respect to voltage direction across terminals of current source $I_1 \Rightarrow$ that source delivers power to the circuit, $P_1 = I_1 U$
- **The terminal of the source I_2 , which the current is leaving, has negative voltage \Rightarrow The source absorbs power $P_2 = -I_2 U$, the power is **negative**, thus the current source **acts like power consumer****



Series connection of an ideal current sources is not possible, except absolutely same current values



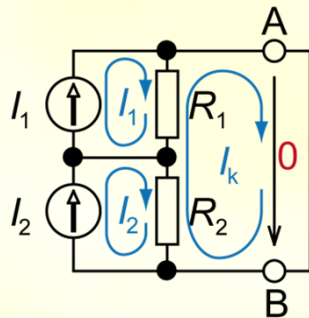
Series connection of non-ideal (actual) current sources is possible

Open-circuit voltage:

The current I_1 may flow through resistor R_1 only (*the current source I_2 maintains its own value of current, so that we could not count this branch, and then resistor R_2 is not connected into any closed loop*); so, the voltage across resistor R_1 is $U_1 = R_1 I_1$

Likewise the current I_2 may flow through resistor R_2 only (*the current source I_1 maintains its own value of current, and then resistor R_1 is not connected in to any closed loop*); so, the voltage across the resistor R_2 is $U_2 = R_2 I_2$

In total $U_{AB} = R_1 I_1 + R_2 I_2$



Short-circuit current:

We will use Kirchhoff's laws:

according to KCL in upper node is valid $-I_1 + I_z - I_{R1} = 0$

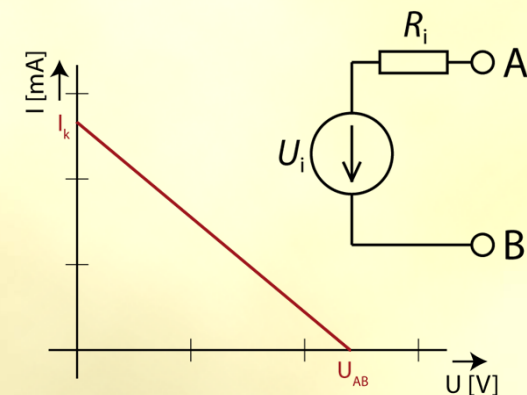
in the lower node accordingly $I_2 - I_z + I_{R2} = 0$

And with reference to KVL $R_1 I_{R1} + R_2 I_{R2} = 0$

By merging of the equations we got:

$$R_1 (I_k - I_1) + R_2 (I_k - I_2) = 0 \Rightarrow I_k = \frac{R_1 I_1 + R_2 I_2}{R_1 + R_2}$$

$$R_i = \frac{U_p}{I_k} = R_1 + R_2$$



Thévenin's and Norton's theorem

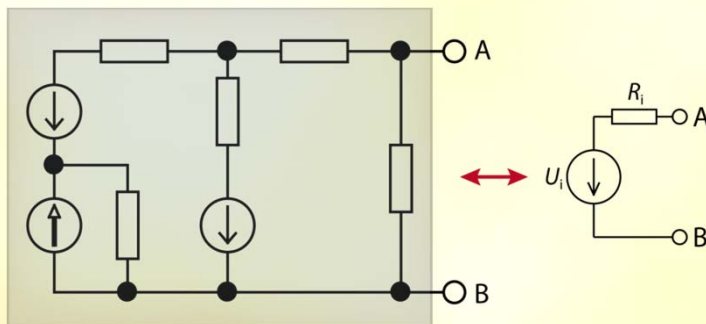
In preceding examples we've already seen, in the circuit, in which two sources (voltage / current) and two resistors were connected, it was possible to evaluate maximum value of voltage, which may be between the terminals – open-circuit voltage, and maximum possible current, that may be (theoretically) drawn from the circuit, short-circuit current.

I-V characteristic of investigated circuits was the same as that of single voltage source with single series connected resistor.



Léon Charles Thévenin published in 1883 this principle and in honor of him this technique take his name, **Thévenin's theorem**

According to that theorem we may look on any circuit, containing any number of voltage sources, current sources and resistors (*later we extend this principle on capacitors and inductors*), as „black box“ with two terminals. Between the terminals we can measure only just open-circuit voltage and internal resistivity. Based on these two values we cannot distinguish between original circuit and single voltage source with single series connected resistor.



Limitations:

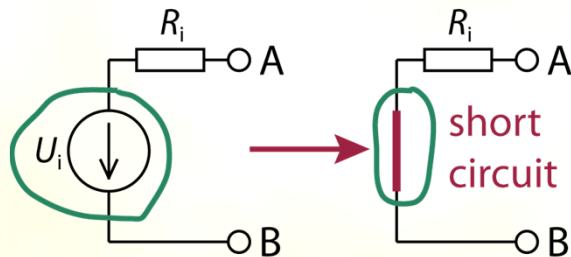
- Only **linear circuits**.
- Since power is not linearly dependent on voltage or current, the **power dissipation of the Thévenin equivalent R_i is not identical to the power dissipation of the real system.**
- The Thévenin's equivalent has an **equivalent I-V characteristic only from the point of view of the A, B terminals** (the load)!!!

Rules for removing of sources

What is internal resistivity of an ideal voltage source?

☞ The current of any size may flow through an **ideal voltage source** with constant voltage across its terminals – its **internal resistivity has to be zero**

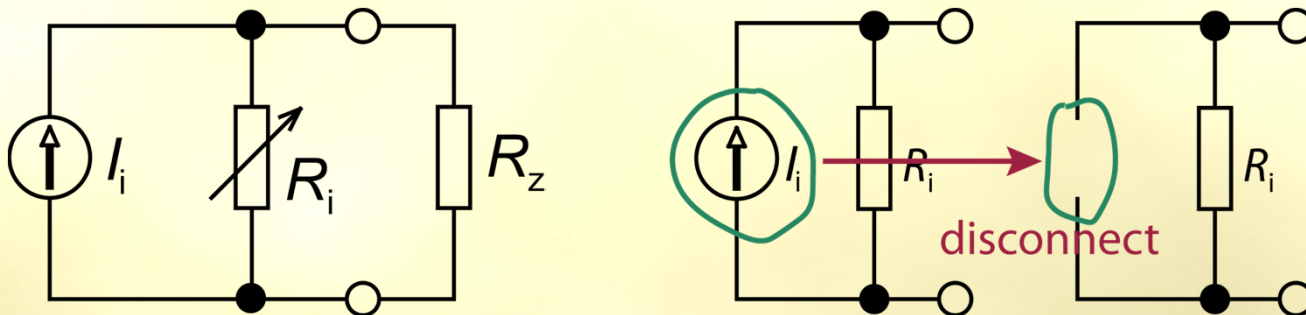
♦ **If we remove an ideal voltage source from the circuit we replace it by short-circuit**



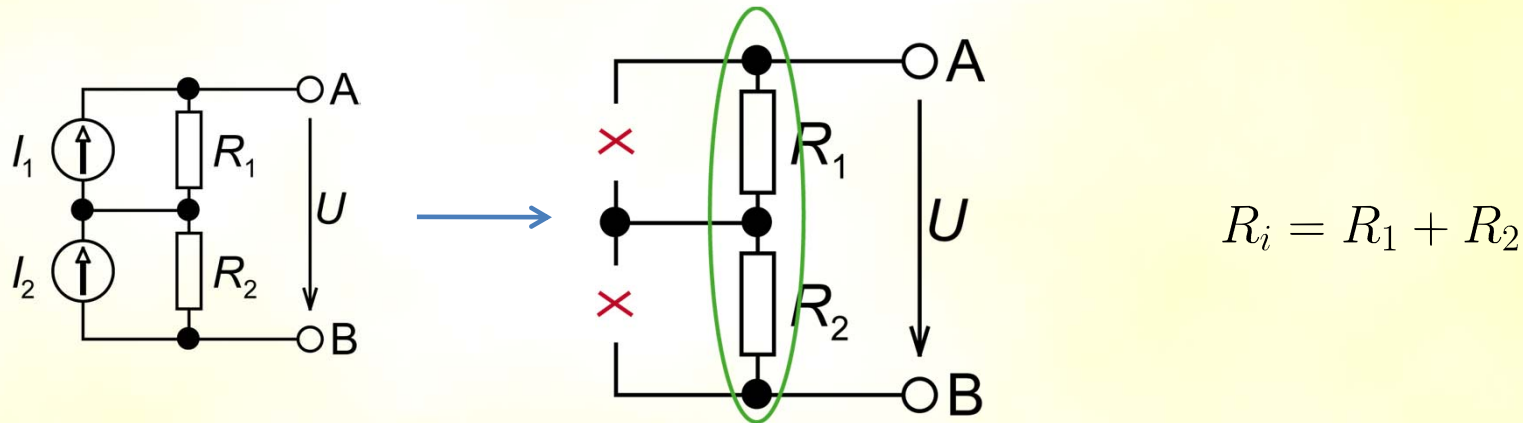
What is internal resistivity of an ideal current source?

☞ An ideal current source has to deliver all its current into load; see circuit diagram of non-ideal current source – a portion of a current is flowing into the load, portion of current is flowing through internal resistor R_i , which represents all the power dissipation in the source; the bigger the resistor R_i is, the more current is flowing into the load; **when R_i will be infinite, the current source will be ideal**

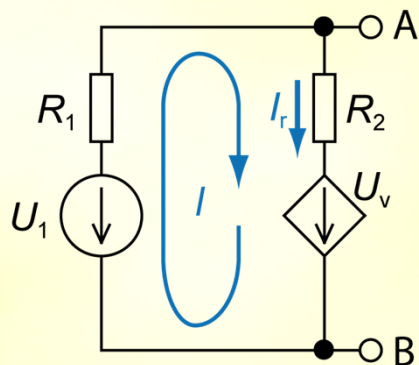
♦ **If we remove an ideal current source from the circuit we replace it by an open circuit**



Back to the problem of series connected non-ideal current sources; when we use rule for removing of current sources, calculation of internal resistivity will be much more easier:



BUT – in general is not possible to remove controlled sources, since controlled variable can affect controlling variable, and, by return, itself (feedback)



If we remove sources the internal resistivity is $R_i = \frac{R_1 R_2}{R_1 + R_2}$

But – using KCL

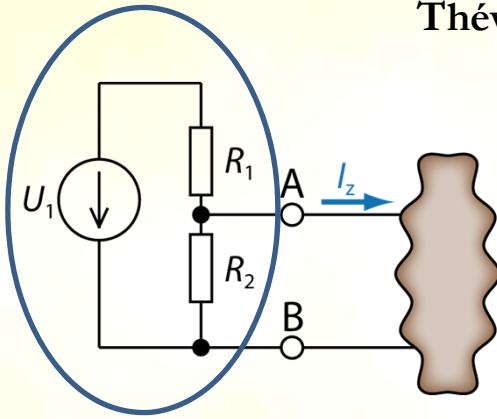
$$-U_1 + R_1 I + R_2 I + R I_r = 0, \quad I_r = I$$

$$I = \frac{U_1}{R_1 + R_2 + R}, \quad U_p = U_v + R_2 I = (R + R_2) I = \frac{U_1 \cdot (R + R_2)}{R_1 + R_2 + R}$$

$$I_k = \frac{U_1}{R_1} \Rightarrow R_i = \frac{U_p}{I_k} = \frac{R_1 \cdot (R + R_2)}{R_1 + R_2 + R}$$

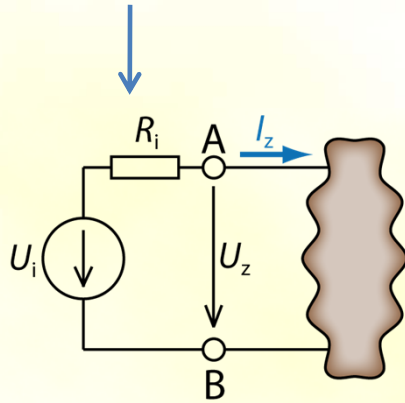
When circuit contains controlled sources we should use this procedure

Thévenin's equivalent circuit – example of application



With this circuit we saw how to use Kirchhoff's laws

Another way how to analyze this circuit is Thévenin's equivalent circuit:



$$U_i = U_1 \frac{R_2}{R_1 + R_2}$$

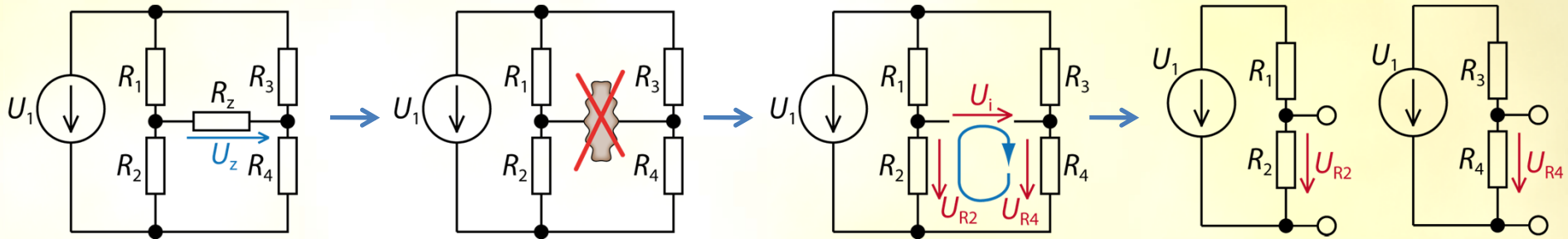
$$R_i = \frac{R_1 R_2}{R_1 + R_2}$$

If the load draw from the circuit current I_z , then this current flow through Thévenin's equivalent resistivity R_i ; the voltage drop across this resistivity is:

$$U_{R_i} = R_i I_z$$

$$U_z = U_i - U_{R_i} = U_i - R_i I_z$$

Thévenin's equivalent circuit – example of application 2



The objective in this example is to find the voltage on diagonal of Wheatstone's bridge R_z (it is used for *compensation measurement of capacitance, inductance, as well as for truck weight measurement and overload warning systems*)

The first (but not the last) way how to analyze this circuit is Thévenin's equivalent circuit

- First remove the resistor R_z – then the analysis is significantly simplified on calculation of Thévenin's equivalent voltage **from the point of view of the terminals of the resistor R_z** – and, it is just calculation of two voltage dividers

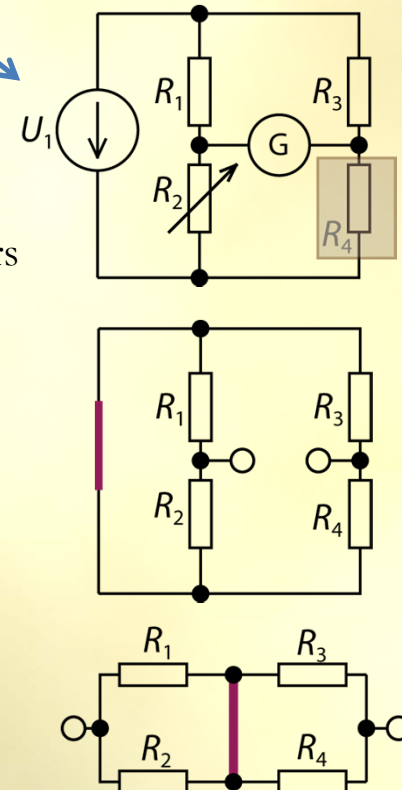
$$U_{R_2} = U \frac{R_2}{R_1 + R_2} \quad U_{R_4} = U \frac{R_4}{R_3 + R_4}$$

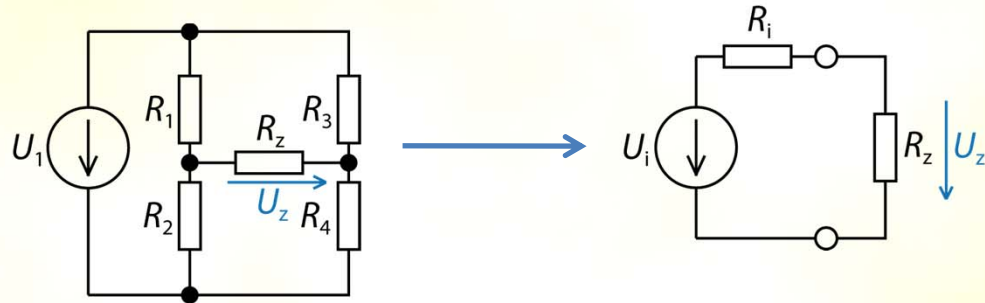
- Now we use KVL:

$$U_i + U_{R_4} - U_{R_2} = 0 \quad \rightarrow \quad U_i = U \left(\frac{R_2}{R_1 + R_2} - \frac{R_4}{R_3 + R_4} \right)$$

- And the last step of simplification is calculation of Thévenin's equivalent resistivity

$$R_i = R_1 \parallel R_2 + R_3 \parallel R_4 = \frac{R_1 R_2}{R_1 + R_2} + \frac{R_3 R_4}{R_3 + R_4}$$





The calculation of the voltage on diagonal of Wheatstone's bridge R_z was changed to calculation of voltage on voltage divider

So, we've calculated:

$$U_i = U \left(\frac{R_2}{R_1 + R_2} - \frac{R_4}{R_3 + R_4} \right), \quad R_i = R_1 \parallel R_2 + R_3 \parallel R_4$$

And finally,

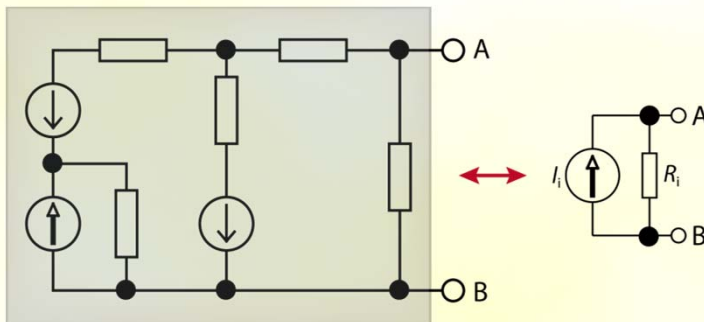
$$U_z = U_i \frac{R_z}{R_i + R_z}$$

Norton's theorem



Edward Lawry Norton and Hans Ferdinand Mayer introduced in 1926 separately this principle, but it was just named after Norton: **Norton's theorem**

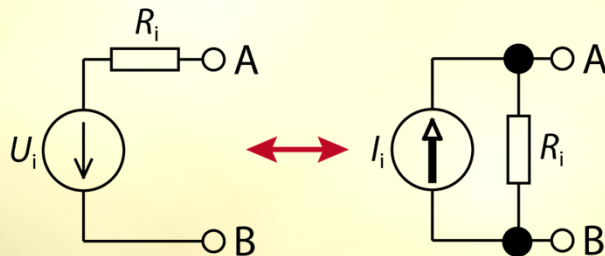
According to that theorem we may look on any circuit, containing any number of voltage sources, current sources and resistors (*later we extend this principle on capacitors and inductors*), as „black box“ with two terminals – the same, as Thévenin's theorem. But in contrast to Thévenin's theorem, according to Norton we think of that box as **single current source with single parallel connected resistor**.



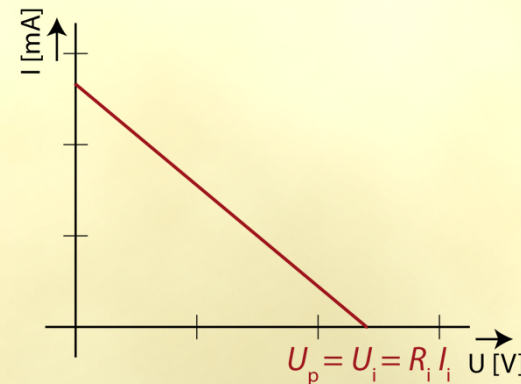
Limitations:

- Only **linear circuits**
- Since power is not linearly dependent on voltage or current, the **power dissipation of the Norton's equivalent R_i is not identical to the power dissipation of the real system**.
- The Norton's equivalent has an **equivalent I-V characteristic only from the point of view of the A, B terminals** (the load)!!!

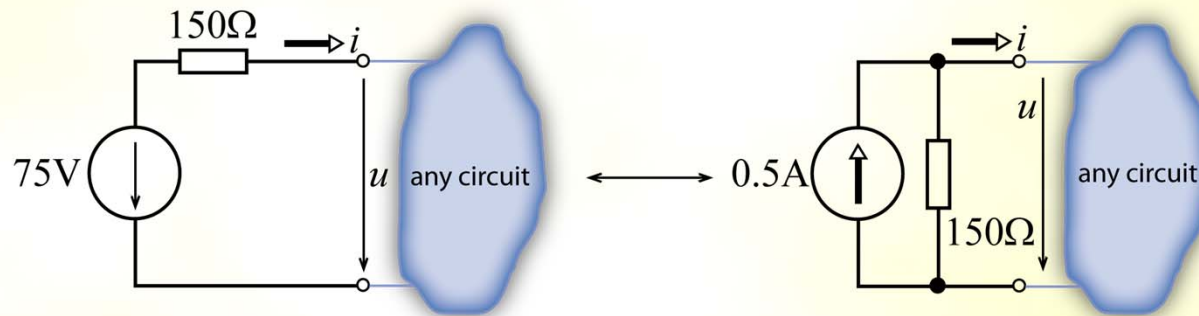
Equivalence of both theorems



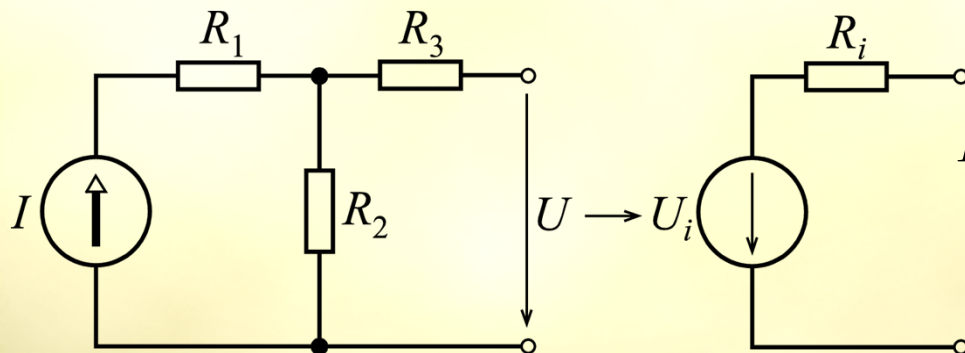
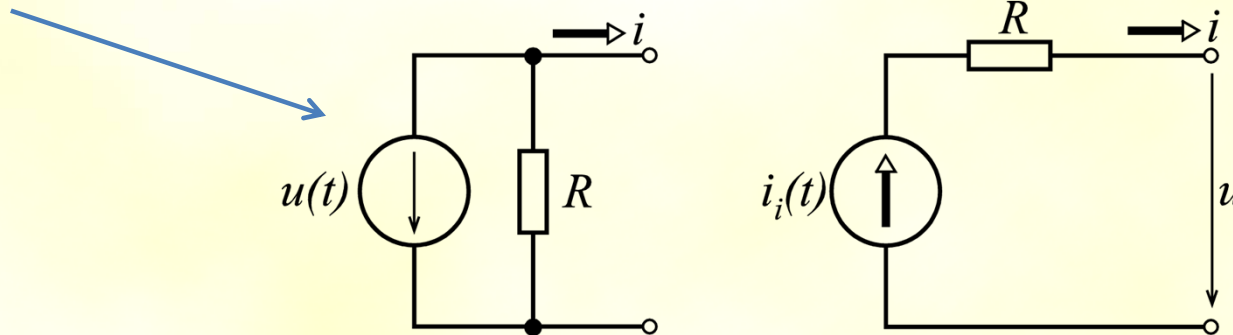
$$I_k = I_i = \frac{U_i}{R_i}$$



Example:



These circuits are not equivalent – from the point of view of output terminal these are still ideal sources!

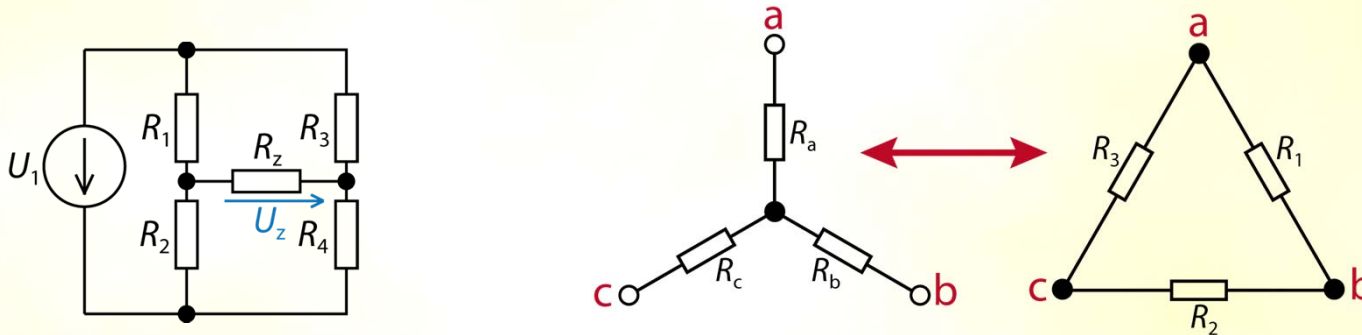


$$I = 1 \text{ A}, R_1 = 100 \Omega, R_2 = 200 \Omega, R_3 = 300 \Omega$$

$$U_i = I R_2 = 200 \text{ V}$$

$$R_i = R_2 + R_3 = 500 \Omega$$

Delta star transformation



Also called $Y-\Delta$ transform, π -delta, $T-\Pi$ transform, ...

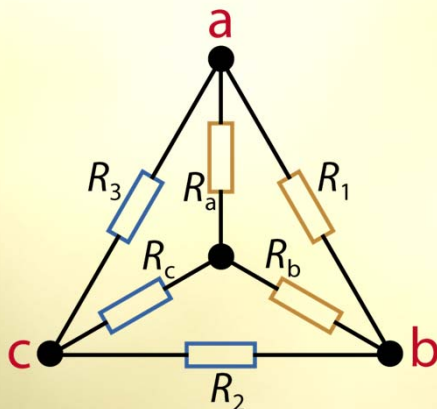
We already analyzed Wheatstone's bridge, using Thévenin's equivalent circuit. First, try to find total resistivity of Wheatstone's bridge – it is not possible, since any of the resistors are neither in series nor in parallel. But, it is possible using delta star transform. So, another possible way, how to analyze this circuit, is delta star transform.

- If in the circuit is delta connection of 3 resistors, then we may replace it with star connected resistors with following parameters:

$$R_a = \frac{R_1 R_3}{R_1 + R_2 + R_3}$$

$$R_b = \frac{R_1 R_2}{R_1 + R_2 + R_3}$$

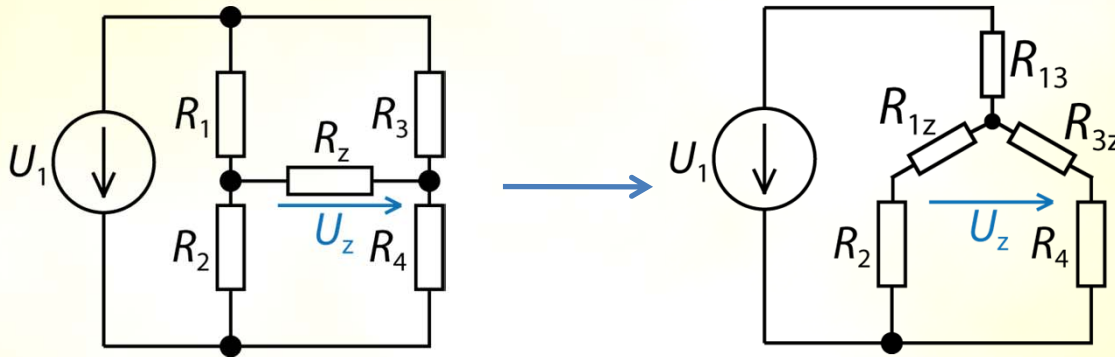
$$R_c = \frac{R_2 R_3}{R_1 + R_2 + R_3}$$



Hint:

It is not necessary to memorize these equations – it even has nearly no sense as in the circuit transformed resistors have out of doubt completely different indices.

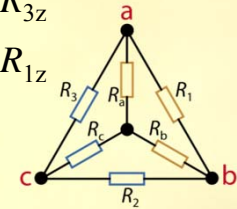
Note the blue highlighted resistors, connected into the c node – When we calculate resistivity of the R_c resistor, then in numerator is always product of resistivity of that resistors, which are connected into given node (c in this case) in initial delta configuration, in denominator is the sum of resistivity of all three resistors.



Resistors R_1 and $R_3 \rightarrow R_{13}$

Resistors R_3 and $R_z \rightarrow R_{3z}$

Resistors R_1 and $R_z \rightarrow R_{1z}$



$$R_{13} = \frac{R_1 R_3}{R_1 + R_3 + R_z}$$

$$R_{3z} = \frac{R_3 R_z}{R_1 + R_3 + R_z}$$

$$R_{1z} = \frac{R_1 R_z}{R_1 + R_3 + R_z}$$

Now we can calculate U_z using step-by-step simplification:

$$R = R_{13} + (R_{1z} + R_2) \parallel (R_{3z} + R_4)$$

$$I = \frac{U}{R}$$

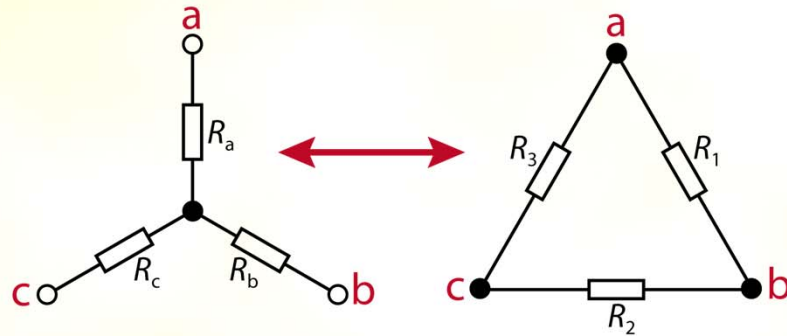
$$U_x = U - R_{13}I$$

$$U_{R2} = U_x \frac{R_2}{R_{1z} + R_2}$$

$$U_{R4} = U_x \frac{R_4}{R_{3z} + R_4}$$

$$U_Z = U_{R2} - U_{R4}$$

Star → delta



$$R_1 = R_a + R_b + \frac{R_a R_b}{R_c}$$

$$R_2 = R_b + R_c + \frac{R_b R_c}{R_a}$$

$$R_3 = R_a + R_c + \frac{R_a R_c}{R_b}$$

Hint:

Again, it isn't necessary memorize these equations.

Note the brown highlighted resistors, connected between a, b nodes – we calculate resistivity of R_1 resistor; resulting resistivity is the sum of resistivity of resistors, connected between the nodes a, b plus fraction, where in the numerator is its product, in denominator resistivity of third resistor in star connection.