

Electrical Circuits

V

Circuit Equations

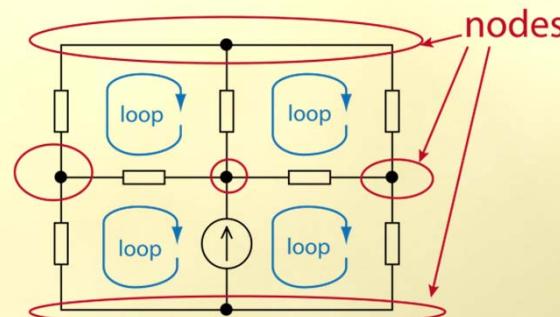
METHODS OF ANALYSIS – NODAL ANALYSIS, MESH ANALYSIS

Nodal analysis

- At previous lectures we studied two fundamental laws of circuit analysis – Kirchhoff's laws.
- Nodal analysis provides general procedure for analyzing circuits **using KCL** and introduce simple, algorithmic procedure to write out set of equations, necessary to obtain all unknown node voltages (and branch currents of them) in the circuit.
- Steps to determine node voltages:

1. Select a node as the reference node (ground, zero voltage potential).
We will not study generalized nodal analysis, where the reference node is chosen out of the circuit.
2. Assign voltages U_1, U_2, \dots, U_{n-1} (U_A, U_B, \dots , or another unique notation) to the remaining $n - 1$ nodes. The voltages in each of $n - 1$ nodes are referenced with respect to the reference node.
3. Apply KCL to each of the $n - 1$ nonreference nodes. Currents passing distinct circuit elements (branch currents) are expressed using Ohm's law in terms of node voltages. As a result we get system of $n - 1$ linear equations.
4. Using any of the methods of linear algebra (Gaussian elimination algorithm, Cramer's rule, inverse matrix), solve the set of equations to obtain $n - 1$ unknown node voltages.

Repetition – node = conductive connection of 2 or more circuit elements



1) Selecting a reference node:

- Usually (amplifiers, etc.) the reference node is determined by a construction of given circuit, in the circuit diagram is indicated by any of the symbols in figure below – ground, chassis ground, common ground, ...



ground



chassis



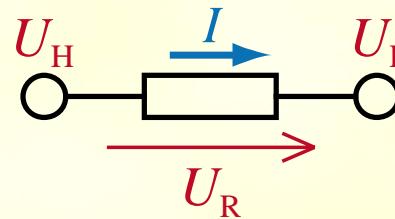
common ground

- If the reference node is not determined by circuit construction then is appropriate to select that node where maximum number of voltage sources is connected – it simplifies resulting equations
- In most cases the reference node is in bottom part of circuit diagram

2) Apply KCL to set up equations in each of distinct nodes:

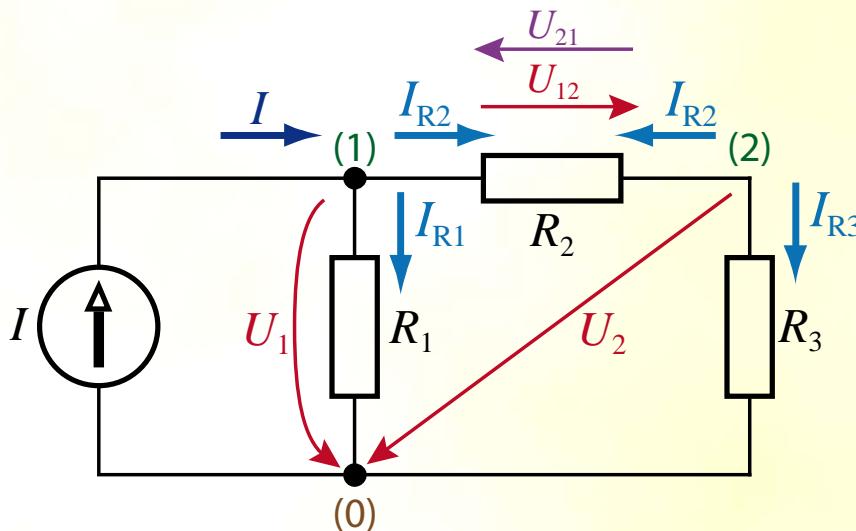
- The current flows from a higher potential to a lower potential in a resistor:

$$I = \frac{U_H - U_L}{R}, \quad U_H > U_L$$



- Can we determine, in which node is higher potential?
 - we cannot! It isn't possible, since we only want to compute the unknown nodal voltages. But, it is enough to assume the same voltage / current orientation in all nodes. The polarity of unknown voltages is given by a convention:

In every circuit node, where we write circuit equation using nodal analysis method we suppose, the potential in that node is higher than in neighbor node, (or, all currents, in each nodes, are leaving the nodes – except current sources, which impose orientation of their currents).



- In circuit in figure the current I is enforced by current source, it enters the node (1)
- All other currents (flowing through passive circuit elements) in both nonreference nodes leaving the nodes
- Currents, passing through distinct circuit elements:
 - At node (1):

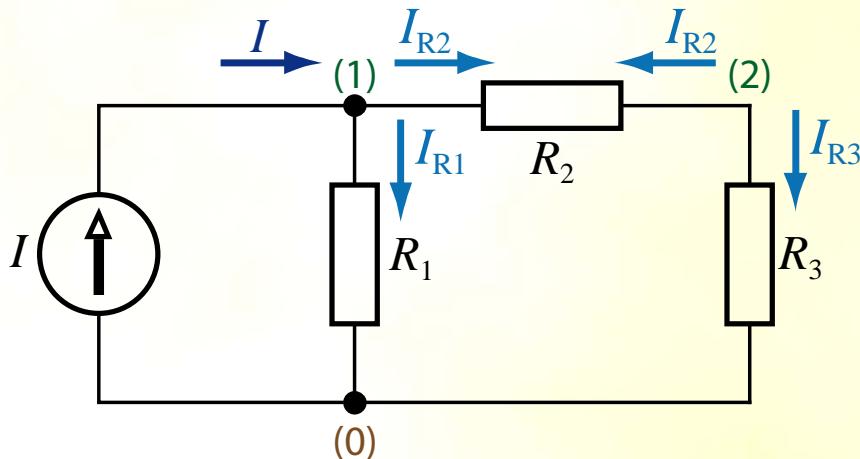
$$R_1: \quad I_{R_1} = \frac{U_1 - 0}{R_1}, \quad \text{or } I_{R_1} = G_1 U_1$$

$$R_2: \quad I_{R_2} = \frac{U_1 - U_2}{R_2}, \quad \text{or } I_{R_2} = G_2(U_1 - U_2)$$

- At node (2):

$$R_2: \quad I_{R_2} = \frac{U_2 - U_1}{R_2}, \quad \text{or } I_{R_2} = G_2(U_2 - U_1)$$

$$R_3: \quad I_{R_3} = \frac{U_2 - 0}{R_3}, \quad \text{or } I_{R_3} = G_3 U_2$$



- Resulting set of equations:

$$-I + \frac{U_1}{R_1} + \frac{U_1 - U_2}{R_2} = 0$$

$$\frac{U_2 - U_1}{R_2} + \frac{U_2}{R_3} = 0$$

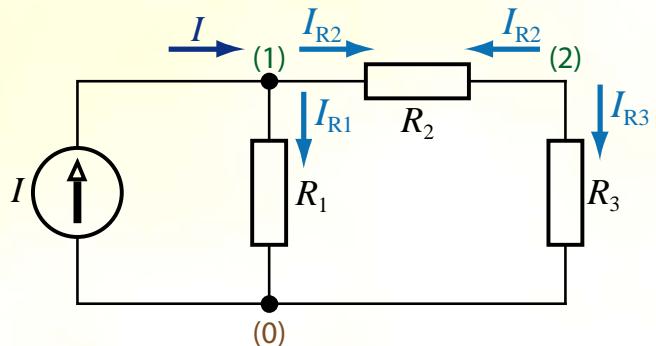
- The set of equations we may put in matrix form as:

$$\begin{bmatrix} \frac{1}{R_1} + \frac{1}{R_2} & \frac{-1}{R_2} \\ \frac{-1}{R_2} & \frac{1}{R_2} + \frac{1}{R_3} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} G_1 + G_2 & -G_2 \\ -G_2 & G_2 + G_3 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} \quad \text{or } \mathbf{GU} = \mathbf{I}$$

- We may use the solution, e.g.: $\mathbf{U} = \mathbf{G}^{-1}\mathbf{I}$

Or, use Cramer's rule (if we need just certain node voltage):

$$U_i = \frac{\Delta_i}{\Delta}$$



Example:

$$R_1 = 1 \text{ k}\Omega, R_2 = 1 \text{ k}\Omega, R_3 = 3 \text{ k}\Omega$$

$$I = 10 \text{ mA}$$

$$\begin{bmatrix} \frac{1}{1000} + \frac{1}{1000} & \frac{-1}{1000} \\ \frac{-1}{1000} & \frac{1}{1000} + \frac{1}{3000} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0.01 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0.002 & -0.001 \\ -0.001 & 0.001\bar{3} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0.01 \\ 0 \end{bmatrix}$$

Cramer's rule: if we need voltage U_1 , we replace the first column of conductance matrix \mathbf{G} with current matrix \mathbf{I}

$$\Delta = \begin{vmatrix} 0.002 & -0.001 \\ -0.001 & 0.001\bar{3} \end{vmatrix} = 0.002 \cdot 0.001\bar{3} - (-0.001)^2 = 1.\bar{6} \cdot 10^{-6}$$

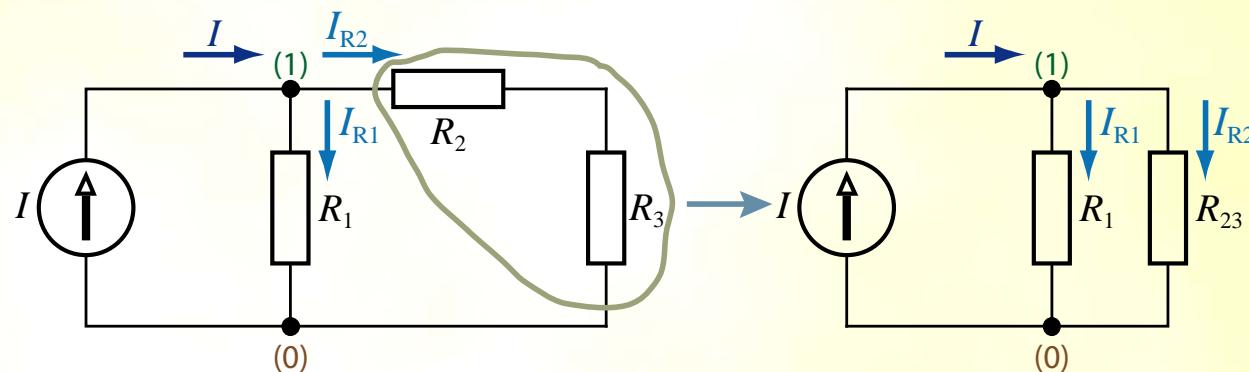
$$\Delta_1 = \begin{vmatrix} 0.01 & -0.001 \\ 0 & 0.001\bar{3} \end{vmatrix} = 0.01 \cdot 0.001\bar{3} - (-0.001) \cdot 0 = 1.\bar{3} \cdot 10^{-5}$$

$$U_1 = \frac{\Delta_1}{\Delta} = \frac{1.\bar{3} \cdot 10^{-5}}{1.\bar{6} \cdot 10^{-6}} = \underline{\underline{8 \text{ V}}}$$

Matrix inversion: in Matlab:

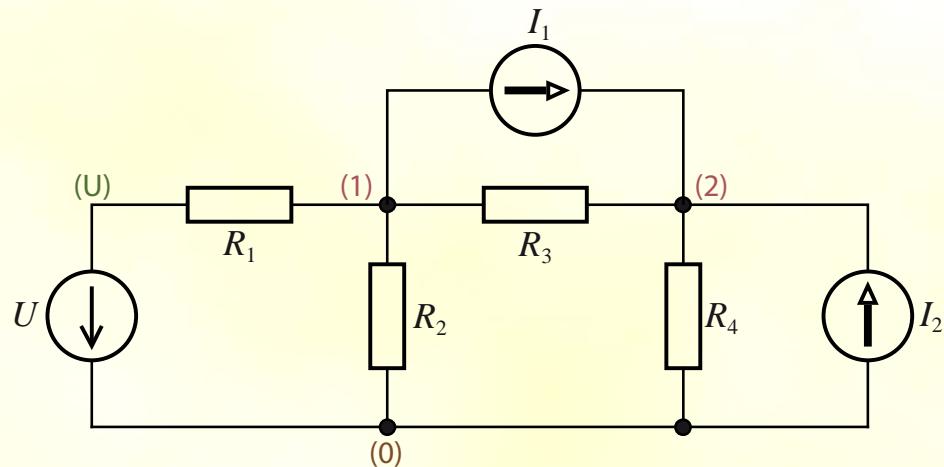
```
>> G = [0.002, -0.001; -0.001, 0.001 + 1/3000];
>> I = [0.01; 0];
>> U = G^-1 * I
```

Or $\mathbf{U} = \text{inv}(\mathbf{G}) * \mathbf{I}$, or $\mathbf{U} = \mathbf{G} \setminus \mathbf{I}$



In the case, when the voltage in the node between series connected resistors is not important (in this example in the node (2) between resistors R_2 and R_3), is possible to combine the resistors and simplify the circuit...

Voltage source, connected between reference and nonreference node



$$R_1 = 8 \Omega, R_2 = 8 \Omega, R_3 = 10 \Omega, R_4 = 10 \Omega$$

$$U = 40 \text{ V}, I_1 = 3 \text{ A}, I_2 = 2 \text{ A}$$

If a voltage source is connected between reference and nonreference node, we simply set the voltage at the nonreference node equal to the voltage of that voltage source (*moreover we could not evaluate KCL in that node – which current flows through that voltage source???*) \Rightarrow **the number of equations is decreased by one per each voltage source connected to the circuit**

$$I_1 + \frac{U_1 - U}{R_1} + \frac{U_1}{R_2} + \frac{U_1 - U_2}{R_3} = 0$$

$$-I_1 - I_2 + \frac{U_2 - U_1}{R_3} + \frac{U_2}{R_4} = 0$$

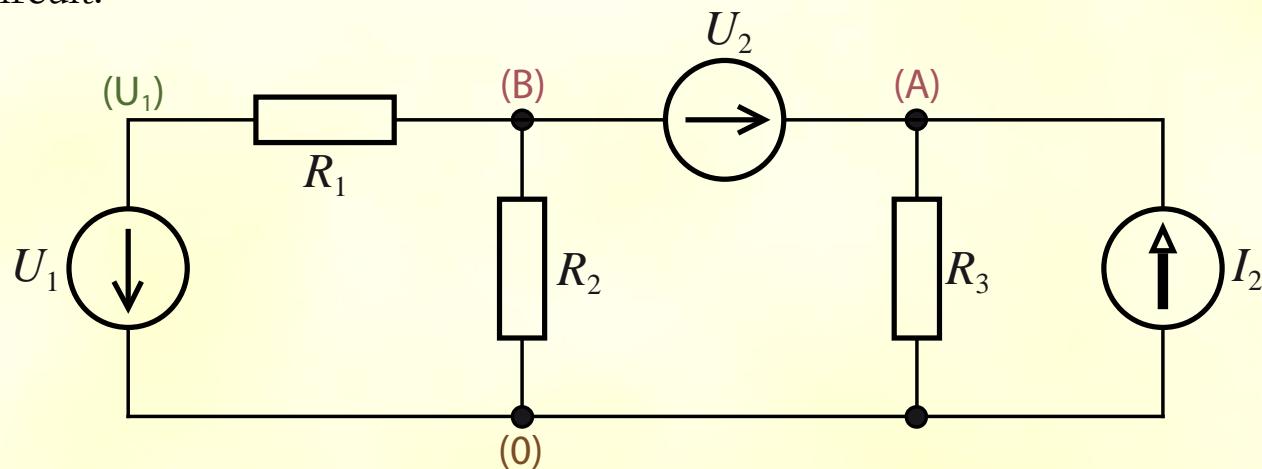
or
$$\begin{bmatrix} G_1 + G_2 + G_3 & -G_3 \\ -G_3 & G_3 + G_4 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} \frac{U}{R_1} - I_1 \\ I_1 + I_2 \end{bmatrix}$$

Matlab:

```
>> G = [1/8 + 1/8 + 1/10, -1/10; -1/10, 1/10 + 1/10];  
>> I = [40/8 - 3; 3 + 2];  
>> U = G \ I  
ans =  
15  
32.5
```

Voltage source connected between two nonreference nodes

We know the number of circuit equations is decreased in the case of nodal analysis method by one per each voltage source connected to the circuit and we are not able do evaluate current passing voltage source directly. Suppose following circuit:

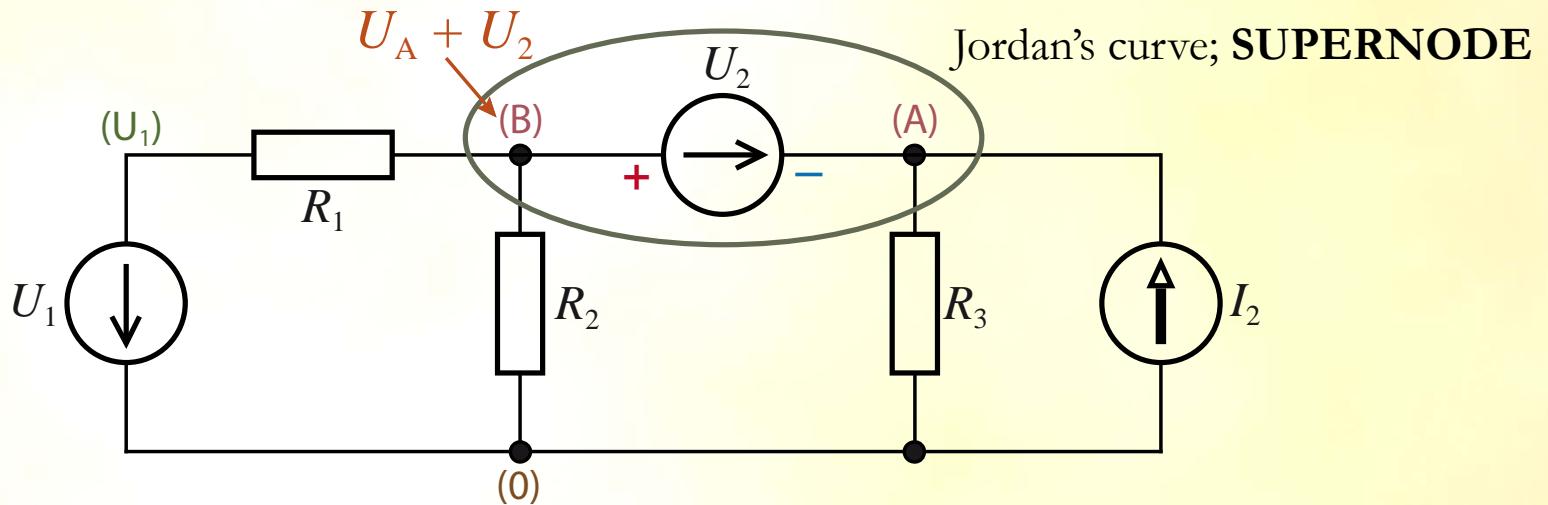


In the circuit are 4 nodes.

1 of them is reference node

⇒ the circuit describes just 1 circuit equation

In the circuit are 2 voltage sources



At node (A) we may write equation:

$$-I_2 + I_{R_3} + I_{U_2} = 0$$

At node (B):

$$I_{R_1} + I_{R_2} - I_{U_2} = 0$$

The currents, flowing through the resistors, will be evaluated using Ohm's law; but the current, which flows through the voltage source U_2 we could not evaluate directly, but – that current is present in both equations
 \Rightarrow we may combine both equations:

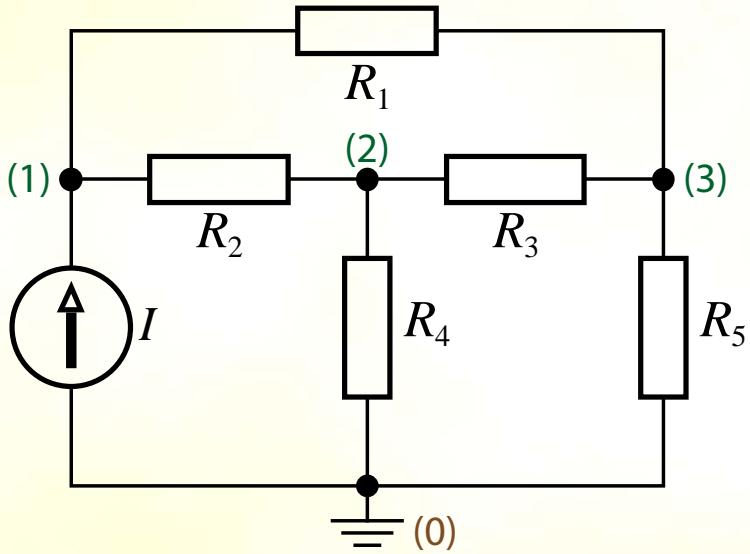
$$-I_2 + I_{R_3} + I_{R_1} + I_{R_2} = 0$$

If the voltage source (dependent or independent) is connected between two nonreference nodes, we may (or we **MUST) both its terminals enlace by „Jordan's curve“, so the two nonreference nodes form a generalized node or supernode, at which we write just one equation. We apply both KCL and KVL since the voltage at both nonreference nodes differs just by voltage of that voltage source.**

$$-I_2 + G_3 U_A + G_2(U_A + U_2) + G_1(U_A + U_2 - U_1) = 0$$

$$U_A = \frac{I_2 - G_2 U_2 - G_1 U_2 + G_1 U_1}{G_3 + G_2 + G_1}$$

Steps to write equations directly in matrix form



Circuit describes 3 equations: (1):

$$-I + \frac{U_1 - U_3}{R_1} + \frac{U_1 - U_2}{R_2} = 0$$

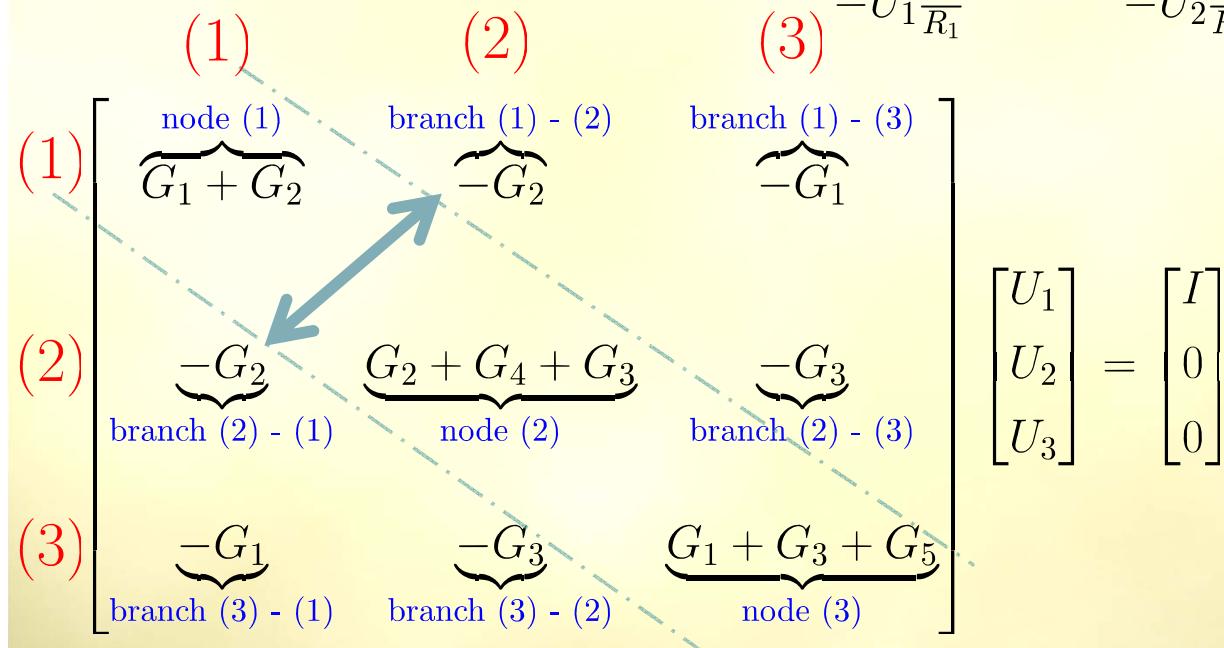
$$(2): \quad \frac{U_2 - U_1}{R_2} + \frac{U_2}{R_4} + \frac{U_2 - U_3}{R_3} = 0$$

$$(3): \quad \frac{U_3 - U_1}{B_1} + \frac{U_3 - U_2}{B_3} + \frac{U_3}{B_5} = 0$$

$$U_1 \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \quad -U_2 \frac{1}{R_2} \qquad \qquad \qquad -U_3 \frac{1}{R_1} \qquad \qquad = I$$

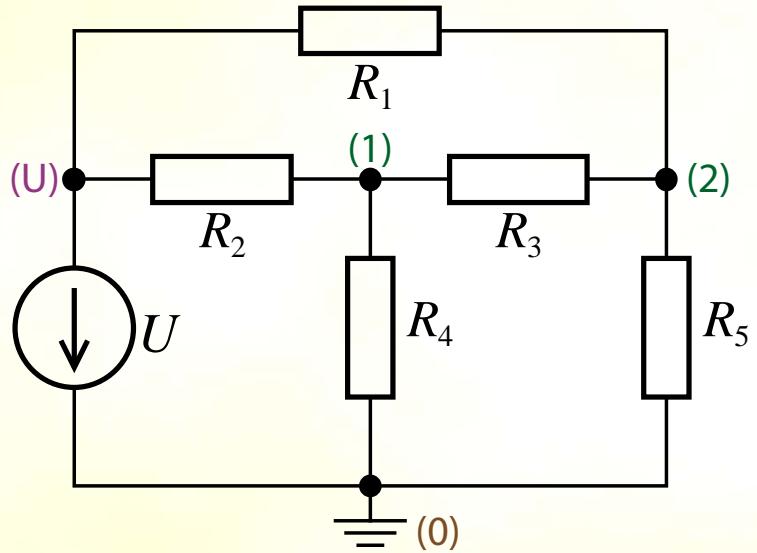
$$-U_1 \frac{1}{R_2} + U_2 \left(\frac{1}{R_2} + \frac{1}{R_4} + \frac{1}{R_3} \right) - U_3 \frac{1}{R_3} = 0$$

$$-U_2 \frac{1}{R_3} + U_3 \left(\frac{1}{R_1} + \frac{1}{R_3} + \frac{1}{R_5} \right) = 0$$



- Matrix is symmetrical with respect to the main diagonal
 - The terms on main diagonal – G_{ii} is the sum of all conductances, connected to node i
 - Other terms – $G_{ij} = G_{ji}$ is negative of the sum of the conductances, directly connecting nodes i and j , $i \neq j$
 - Right-hand side – I_i is the sum of all independent current sources, connected to node i

Steps to write equations directly in matrix form – if the circuit contains voltage source



Circuit describes just 2 equations:

$$(1): \frac{U_1 - U}{R_2} + \frac{U_1}{R_4} + \frac{U_1 - U_2}{R_3} = 0$$

$$(2): \frac{U_2 - U}{R_1} + \frac{U_2 - U_1}{R_3} + \frac{U_2}{R_5} = 0$$

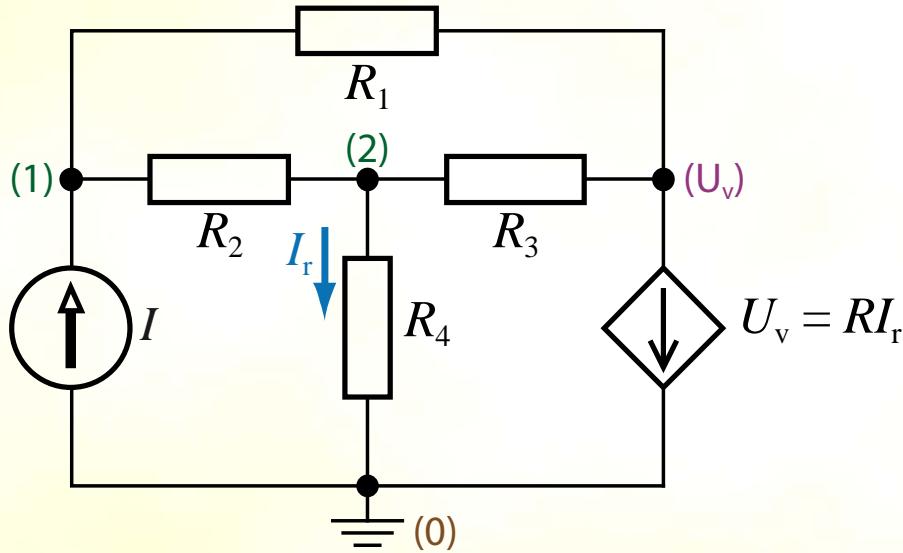
$$U_1 \left(\frac{1}{R_2} + \frac{1}{R_4} + \frac{1}{R_3} \right) - U_2 \frac{1}{R_3} = \frac{U}{R_2}$$

$$-U_1 \frac{1}{R_3} + U_2 \left(\frac{1}{R_1} + \frac{1}{R_3} + \frac{1}{R_5} \right) = \frac{U}{R_1}$$

$$\begin{bmatrix} (1) & \text{node (1)} & \text{branch (1) - (2)} \\ (1) & \overbrace{G_2 + G_4 + G_3}^{\text{node (1)}} & \overbrace{-G_3}^{\text{branch (1) - (2)}} \\ (2) & \overbrace{-G_3}^{\text{branch (2) - (1)}} & \overbrace{G_1 + G_3 + G_5}^{\text{node (2)}} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} \frac{U}{R_2} \\ \frac{U}{R_1} \end{bmatrix}$$

Voltage sources transformed on current sources,
connected in series with resistor to i^{th} node

Circuits with dependent sources



$$(1): -I + \frac{U_1 - U_v}{R_1} + \frac{U_1 - U_2}{R_2} = 0$$

$$(2): \frac{U_2 - U_1}{R_2} + \frac{U_2}{R_4} + \frac{U_2 - U_v}{R_3} = 0$$

$$U_v = RI_r = R \frac{U_2}{R_4}$$

$$U_1 \left(\frac{1}{R_1} + \frac{1}{R_2} \right) - U_2 \frac{1}{R_2} - \frac{R}{R_4} \frac{1}{R_1}$$

$$-U_1 \frac{1}{R_2} + U_2 \left(\frac{1}{R_2} + \frac{1}{R_4} + \frac{1}{R_3} - \frac{R}{R_4} \frac{1}{R_1} \right) = 0$$

$= I$

$$\begin{bmatrix} \text{node (1)} & \text{branch (1) - (2)} \\ \underbrace{G_+ G_2}_{\text{branch (2) - (1)}} & \underbrace{-G_2 - \frac{R}{R_4 R_1}}_{\text{node (2)}} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

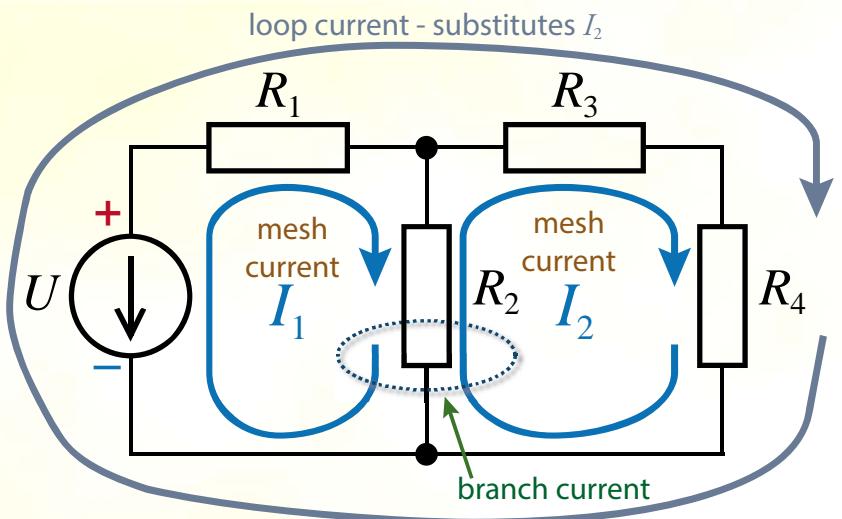
Symmetry of conductance matrix is corrupted – it is not possible write conductance matrix directly
(without introducing new rules)

Mesh analysis / loop analysis

- While the nodes in the circuit can be easily and uniquely determined, with loops this is true only in very simple circuits. In the case of more complex circuits, especially nonplanar (where it is not possible to draw circuit diagram without crossing branches), graph theory is necessary to identification of the loops.
- The number of linear independent equations can be determined by:
where b is the number of branches
 n is the number of nodes
 N_i is the number of current sources (both independent and dependent)
- Mesh analysis provides general procedure for analyzing circuits **using KVL** and introduce simple, algorithmic procedure to write out set of equations, necessary to obtain all unknown mesh currents (and eventually voltages across circuit elements of them) in the circuit.
- Steps to determine mesh currents:

- Identify all independent meshes / loops in the circuit. Assign currents I_1, I_2, \dots, I_n (I_A, I_B, \dots , or another unique notation) to that n meshes / loops.
- Apply KVL to each of the n independent loop. The voltages across distinct circuit elements are expressed using Ohm's law in terms of currents, that passes through the circuit elements. As a result we get system of n linear equations.
- Using any of the methods of linear algebra (Gaussian elimination algorithm, Cramer's rule, inverse matrix), solve the set of equations to obtain n unknown mesh currents.

① **Mesh vs. loop:** sometimes there is a confusion between terms mesh and loop, but, generally, mesh is just kind of a simple loop. Loop may be any closed path which doesn't intersect itself either in complex circuit, even nonplanar, mesh may not. Mesh just circumscribes neighboring circuit elements. Mesh is a loop, which does not contain any other loops within it.



$$X_i = b - n + 1 - N_i = 5 - 4 + 1 - 0 = 2 \text{ equations}$$

- Branch current is the sum of mesh currents, passing that branch
- In given circuit we mark both mesh currents; the equations are:

$$(1): \quad R_1 I_1 + R_2(I_1 - I_2) - U = 0$$

$$(2): \quad R_2(I_2 - I_1) + R_3 I_2 + R_4 I_2 = 0$$

Convention:

- In each loop we suppose positive orientation of that loop current, for which we currently write the equation**
- The voltage of voltage source has negative sign, if the loop enters its negative terminal and positive, if loop enters its positive terminal**

- Putting equations (1) and (2) in matrix form it yields:

$$\begin{bmatrix} R_1 + R_2 & -R_2 \\ -R_2 & R_2 + R_3 + R_4 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} U \\ 0 \end{bmatrix} \quad \text{or } \mathbf{RI} = \mathbf{U}$$

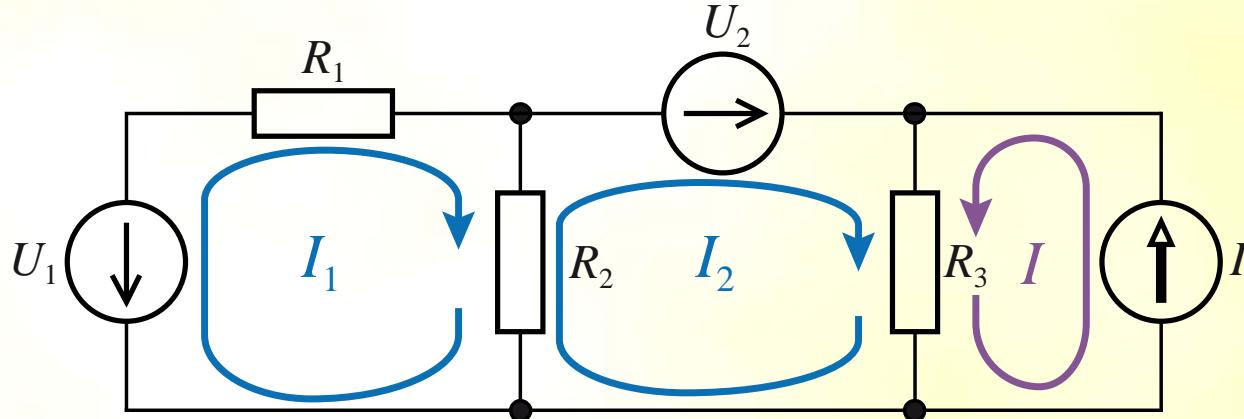
Solution:

$$\mathbf{I} = \mathbf{R}^{-1}\mathbf{U}$$

$$I_i = \frac{\Delta_i}{\Delta}$$

...

Circuits with current sources



The circuit has 3 loops, but we obtain just 2 equations:

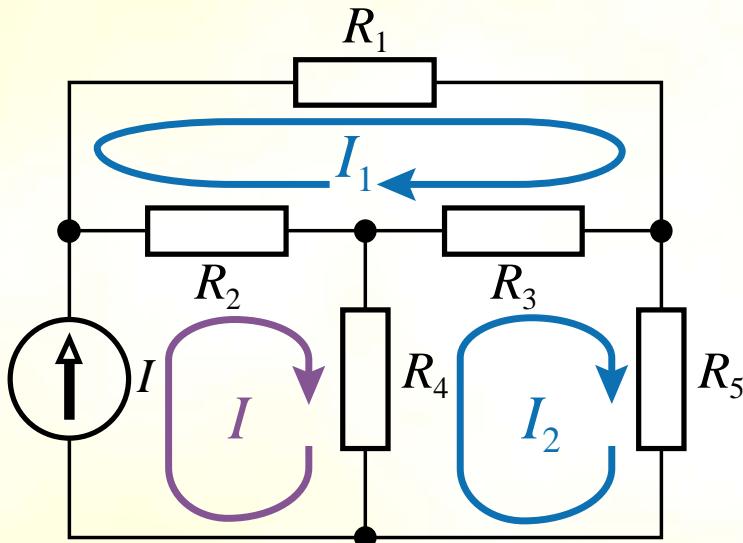
- We can't write an equation in loop passing through current source – we don't know the value of voltage across its terminals and we can't evaluate it (directly)
- We just set the value of loop current passing the current source to its value (and direction) and write mesh / loop equations for the other mesh / loop in the usual way.

$$(1): \quad R_1 I_1 + R_2(I_1 - I_2) - U_1 = 0 \quad \begin{bmatrix} R_1 + R_2 & -R_2 \\ -R_2 & R_2 + R_3 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} U_1 \\ -U_2 - R_3 I \end{bmatrix}$$

$$(2): \quad R_2(I_2 - I_1) + U_2 + R_3(I_2 + I) = 0$$

- A mesh current may be assigned to each mesh in an arbitrary direction; but the direction of a mesh current which passes current source is enforced by that source and we have to respect its orientation

Steps to write equations directly in matrix form – writing an equations by inspection



$$\begin{bmatrix} (1) & (2) \\ (1) & \begin{bmatrix} R_1 + R_3 + R_2 & -R_3 \\ -R_3 & R_4 + R_3 + R_5 \end{bmatrix} \\ (2) & \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} R_2 I \\ R_3 I \end{bmatrix} \end{bmatrix}$$

The diagram shows the formation of a 2x2 resistance matrix from the circuit. The diagonal elements are the sum of resistances in each mesh: $R_1 + R_3 + R_2$ for mesh (1) and $R_4 + R_3 + R_5$ for mesh (2). The off-diagonal elements are negative resistances common to both meshes: $-R_3$ for the (1,2) position and $-R_3$ for the (2,1) position. Arrows indicate the transformation of the circuit's components into the matrix entries.

The circuit describes 2 equations:

$$(1): R_1 I_1 + R_3(I_1 - I_2) + R_2(I_1 - I) = 0$$

$$(2): R_4(I_2 - I) + R_3(I_2 - I_1) + R_3 I_2 = 0$$

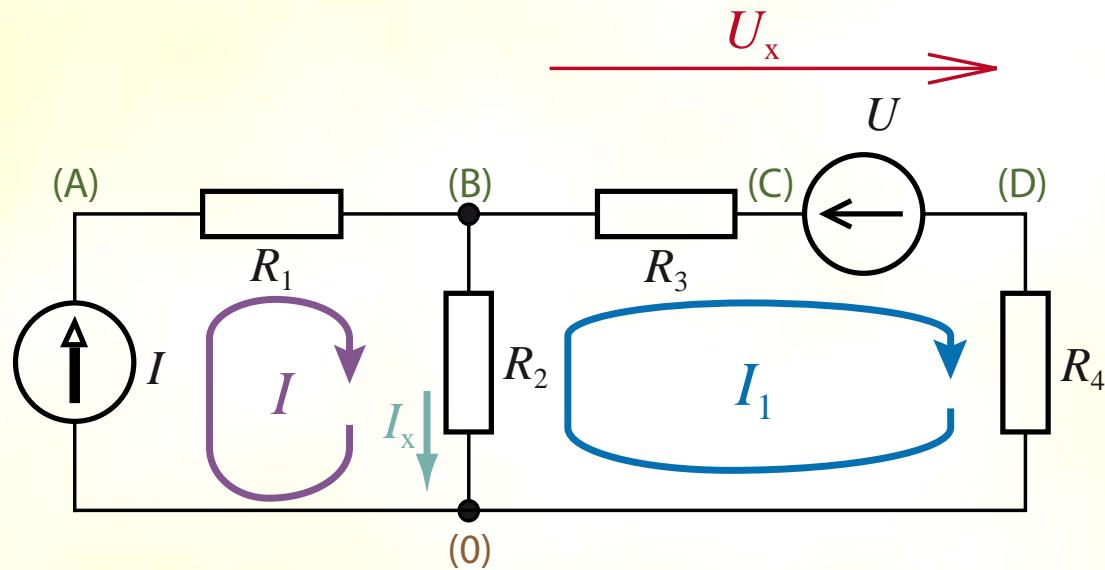
$$\begin{aligned} I_1(R_1 + R_3 + R_2) - I_2 R_3 \\ -I_1 R_3 \end{aligned} = R_2 I$$

$$+I_2(R_4 + R_3 + R_5) = R_4 I$$

- Matrix is symmetrical with respect to the main diagonal
- The terms on main diagonal $-R_{ii}$ is the sum of all resistances in the related mesh i
- Other terms $-R_{ij} = R_{ji}$ is negative of the sum of the resistances, common to meshes i and j , $i \neq j$
- Right-hand side – the sum of independent voltage sources in related mesh i and current sources transformed on voltage sources

- Current source transformed on voltage source has positive sign in the voltage matrix, when the orientation of its mesh on „transformation“ resistor is opposite to related mesh i
- Since the voltage sources are moved to the right side of the equation, its sign is opposite with respect to standalone equations – so positive, if mesh enters negative terminal of voltage source, positive in opposite case

As well as in the case of nodal analysis dependent sources corrupts symmetry of resistance matrix R – it is not possible to write resistance matrix directly by inspection without introducing additional rules



Example:

$$R_1 = 1 \text{ k}\Omega, R_2 = 2 \text{ k}\Omega, R_3 = 3 \text{ k}\Omega, R_4 = 1 \text{ k}\Omega$$

$$I = 10 \text{ mA}, U = 12 \text{ V}$$

In the circuit in figure calculate current I_x and voltage U_x .

Number of equations using nodal analysis: $X_u = n - 1 - N_u = 5 - 1 - 1 = 3$

Since the resistor R_1 is connected in series with current source, 2 equations are enough:

$$\begin{bmatrix} G_2 + G_3 & -G_3 \\ -G_3 & G_3 + G_4 \end{bmatrix} \begin{bmatrix} U_B \\ U_C \end{bmatrix} = \begin{bmatrix} I \\ -UG_4 \end{bmatrix} \quad \begin{bmatrix} 0.0008\bar{3} & -0.000\bar{3} \\ -0.000\bar{3} & 0.001\bar{3} \end{bmatrix} \begin{bmatrix} U_B \\ U_C \end{bmatrix} = \begin{bmatrix} 0.01 \\ -0.012 \end{bmatrix}$$

Matlab:

$$G = [1/R2 + 1/R3, -1/R3; -1/R3, 1/R3 + 1/R4];$$

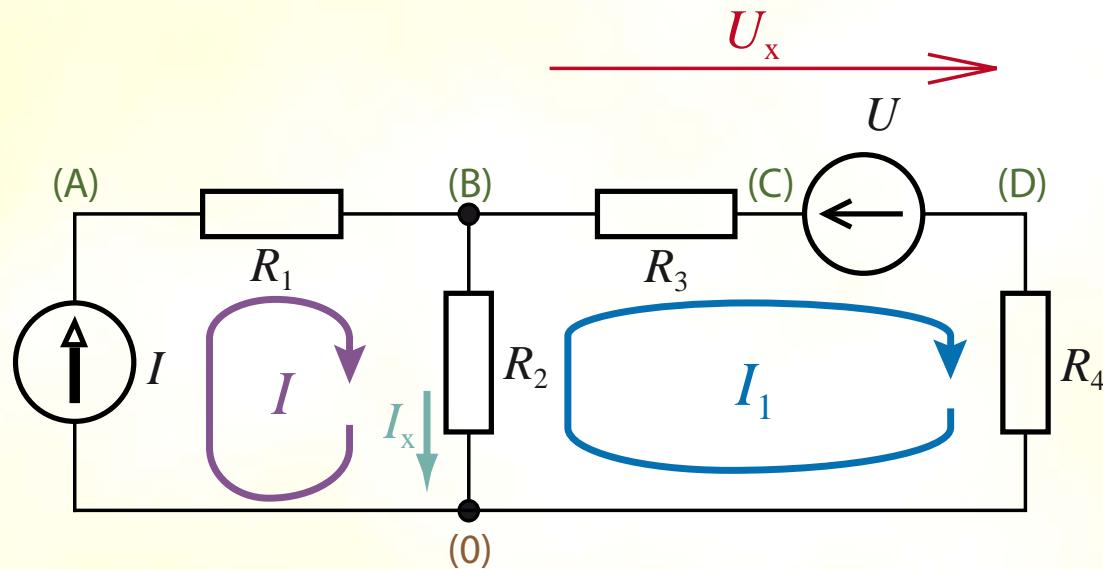
$$Im = [I; -U/R4];$$

$$Um = G^{\wedge}(-1) * Im$$

$$\begin{bmatrix} U_B \\ U_C \end{bmatrix} = \begin{bmatrix} 9.\bar{3} \\ -6.\bar{6} \end{bmatrix}$$

$$I_x = \frac{U_B}{R_2} = \frac{9.\bar{3}}{2000} = \underline{\underline{4.6 \text{ mA}}}$$

$$U_x = U_B - U_C - U = 9.\bar{3} - (-6.\bar{6}) - 12 = \underline{\underline{4 \text{ V}}}$$



Example:

$$R_1 = 1 \text{ k}\Omega, R_2 = 2 \text{ k}\Omega, R_3 = 3 \text{ k}\Omega, R_4 = 1 \text{ k}\Omega \\ I = 10 \text{ mA}, U = 12 \text{ V}$$

In the circuit in figure calculate current I_x and voltage U_x .

Number of equations using mesh analysis: $X_i = b - n + 1 - N_i = 6 - 5 + 1 - 1 = 1$

$$[R_2 + R_3 + R_4] [I_1] = [U + R_2 I] \\ [6000] [I_1] = [12 + 0.01 \cdot 2000]$$

$$[I_1] = [0.005\bar{3}]$$

$$I_x = I - I_1 = 10 - 5.\bar{3} = \underline{\underline{4.6 \text{ mA}}}$$

$$U_x = R_3 I_1 - U = 3000 \cdot \frac{32}{6000} - 12 = 3000 \cdot 0.005\bar{3} - 12 = \underline{\underline{4 \text{ V}}}$$

The current I_x is the branch current – it is expressed in meaning of difference of a mesh currents

The voltage U_x is a voltage across series connection of two circuit elements – according to KVL it is expressed in meaning of the sum of distinct voltages

Recapitulation – number of equations and independent loops

We learned, that when we need to find necessary number of equations we have to determine:

- Number of nodes: n
- Number of branches: b
- Number of voltage sources: N_u
- Number of current sources: N_i
- Number of separate parts of the circuit c – *in the nodal voltages method one of the nodes was denoted as reference node, or zero voltage node; in that node we didn't write any equation; in general case, imagine circuit, where e.g. the transformer with separate primary and secondary winding is connected – in such circuit there are two distinct reference nodes – one for primary winding, the second one for secondary, there are two separate parts.*

In section describing nodal analysis 1 was used.

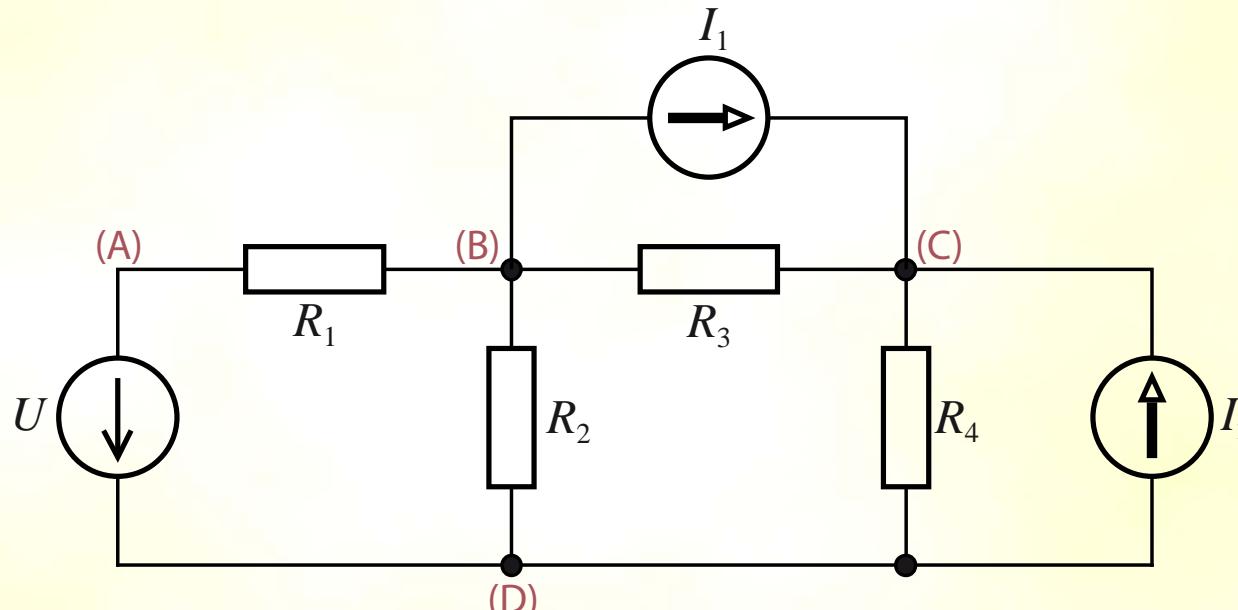
- The number of equations:

- Nodal analysis: $X_u = n - 1 - N_u$, or $X_u = n - c - N_u$

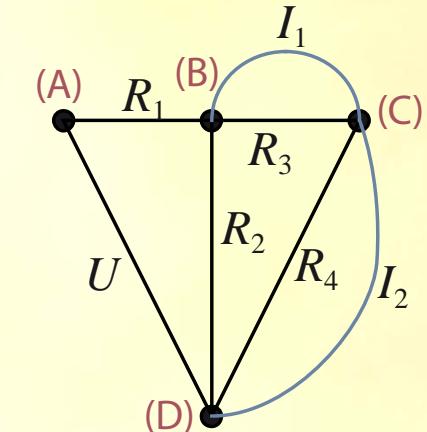
- Mesh / loop analysis: $X_i = b - n + 1 - N_i$, or $X_i = b - n + c - N_i$

With nodal analysis, selection of nodes is unique, in the case of simple circuits it's not a problem to find meshes – but what if the circuit is more complicated?

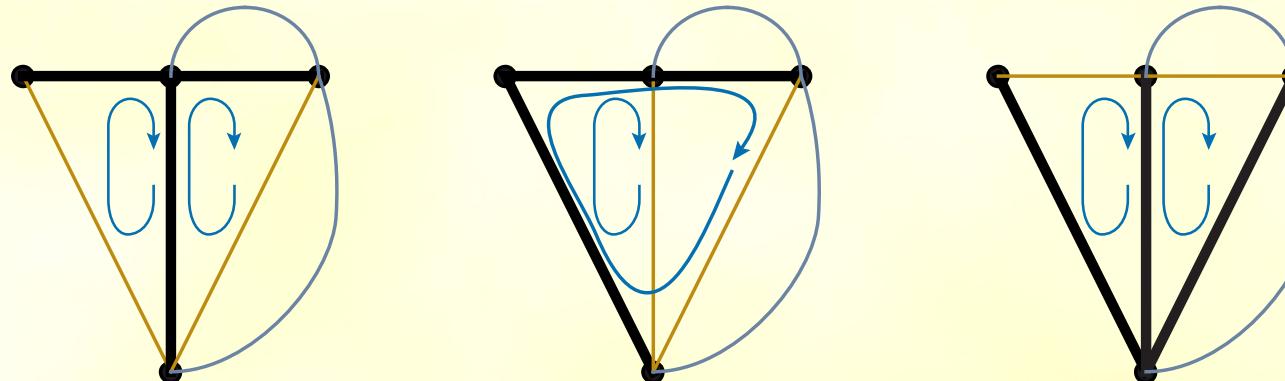
In such case it is necessary to introduce two new terms – the graph and a tree (*these terms are introduced just for completeness' sake, circuits, analyzed in this subject / and even subsequent/ is possible to write circuit equations without this analysis*)



circuit diagram



graph of the circuit



examples of trees and selection of loops

The tree: minimum number of branches, connecting all nodes in the circuit

Choice of loops: select one branch of the graph, which is not part of the tree, or another loop – combining with neighboring branches of a tree we got closed loop