Electrical Circuits

Pavel Máša

March 3, 2018

FEE CTU Prague

1. Introduction – Basic Concepts

1.1. Electric charge

symbol:	Q
unit:	Coulomb (C)
SI standard system units:	$\mathrm{A} \cdot \mathrm{s}$

Elementary charge:
$e \doteq 1.602176487 \cdot 10^{-19}\ \mathrm{C}$

It is an unremovable and uncreatable property of elementary particles. If an electric charge is placed in an electromagnetic field, it experiences a force. There are two types of electric charges: *positive* and *negative*. The negative charge is carried by electrons, positive one by protons. Like charges repel and unlike attract. The motion of the electric charge a specific direction is known as *electric current*. In metals, it is the flow of electrons, in electrolytes (present e.g. in accumulators) it is the flow of ions.

By means of the electric charge is also given the capacity of accumulators (see Section 1.2). The capacitor is an electric device, which stores an electric charge. On capacitor, an electric voltage and stored energy is related to the electric charge.

The Law of Conservation of Charge

Since the electric charge can neither be created nor destroyed, in an isolated system the total electric charge remains constant. The only way to change the total charge of a system is to deliver charge from elsewhere or remove charge from the system. However, then such system is not isolated anymore ☺. Note that this law also results in Kirchhoff's current law.

If you are good in physics, you may remember the equation

$$\nabla \cdot \boldsymbol{i} = -\frac{\partial \rho}{\partial t}$$

This equation tells us, that the current flow through closed area is the same as the time rate of the charge, inside the volume, enclosed by this area. However, in electric circuit analysis, we will rather use Kirchhoff's current law ☺.

1.2. Electric current

symbol:	I
unit:	Ampere (A)
SI standard system units:	A

Definition

Electric current is an equivalent amount of electric charge, transferred per second.

Convention

When the quantity has a *constant value*, we will write it in the upper case, e.g. Q; when it is a *function of time* (and it *varies* in time), we will use lower case. Eventually, we will write time in round brackets, e.g. $q(t)$.

The charge, delivered by *constant* electric current I within *constant* time period T can be evaluated according to equation (1.0a). However, the delivered charge is in general case the function of time, and even if the electric current I is still *constant*, the time t is variable, and we will then use *lower case*, as in the equation (1.0b).

$$Q = I \cdot T \tag{1.0a}$$
$$q(t) = I \cdot t. \tag{1.0b}$$

Except for the DC, the electric current varies in time, and then we should use the integral to calculate the amount of delivered charge. In opposite, an electric current is a time rate of change of the charge.

$$q(t) = \int_0^t i(\tau)\,\mathrm{d}\tau \tag{1.1}$$

$$i(t) = \frac{\mathrm{d}q(t)}{\mathrm{d}t} \tag{1.2}$$

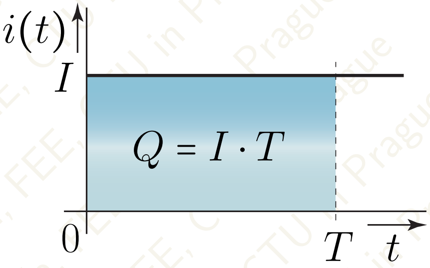


Figure 1.1

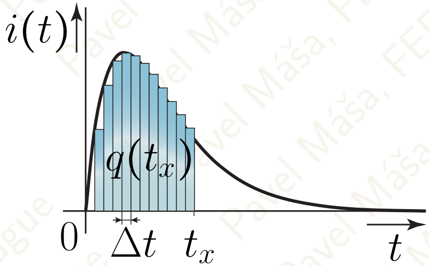


Figure 1.2

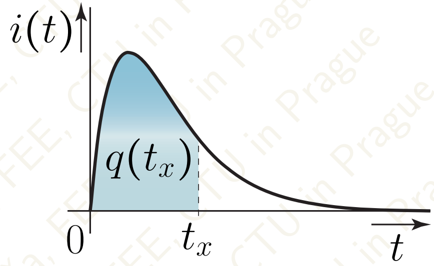


Figure 1.3

The total value of the charge Q, at time instant t_x is the area, bounded by axis and the function value, see Figure 1.3.

The upper limit of the integral in the equation (1.1) is time *variable* and for this reason the calculated charge is a *function*. If we want to calculate a specific charge value at time t_x, the upper limit is *the value* t_x.

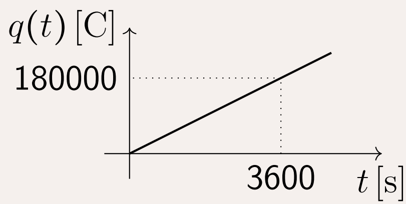
Example 1.1

Assignment: A rechargable battery has a rated capacity of $50\,\mathrm{Ah}$. Calculate total charge, stored in the accumulator.

Solution: Because the rated capacity of a recharchable battery indicates what constant current the battery is able to supply to the circuit for one hour, we will use equation (1.0a). The solution is then:

$$Q = I \cdot T = 50 \cdot 3600 = 180000\,\mathrm{C}.$$

The equation (1.0b) results in a function. In this case, $q(t) = 50t$, which is the equation of the line. If we substitute 3600 for a time variable, we get the same solution as before.



Example 1.2

Assignment: The waveform of current is given as $i(t) = 0.1\sin(100t)\,[\mathrm{A}]$.

- Find the function, describing the charge at time instant t.
- Calculate total charge transferred within a time interval of $t_x = 10\,\mathrm{ms}$.

Solution:

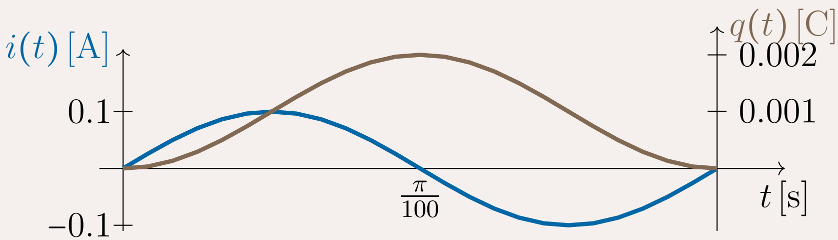
- The function, decribing the charge at time instant t results from the equation (1.1):

$$q(t) = \int_0^t 0.1\sin(100\tau)\,\mathrm{d}\tau = \left[\frac{-0.1\cos(100\tau)}{100}\right]_0^t = 0.001\,[1-\cos(100t)]\ \ [\mathrm{C}].$$

- If we substitute $t_x = 10\,\mathrm{ms}$ for a time variable in the result above, or if the upper limit in the equation (1.1) will be *constant* value 0.01, instead of *variable* t, we got:

$$q(0.01) = \int_0^{0.01} 0.1\sin(100t)\,\mathrm{d}t = \left[\frac{-0.1\cos(100t)}{100}\right]_0^{0.01} =$$

$$= 0.001\,[-\cos(100\cdot 0.01)+\cos(0)] \doteq \underline{\underline{459.7\ \mu\mathrm{C}}}.$$



1.2.1. Direction of current – Conventional Versus Electron Flow

Consider a device, which delivers an electrical charge to other parts of an electrical circuit. It acts as a source. Assume that our device has two *terminals*. An electrical charge leaves one terminal, it flows through the electrical circuit, and the same amount of charge enters the other terminal of our device. We will distinguish both terminals by + and – symbols. The flow of an electrical charge is an electric current. Some electrical devices tolerate electrical currents of either direction with no difference in operation. However, other devices function differently on currents of a different direction. For example, LED is lighting only if the electric current flows from anode to cathode. For this reason, it is essential to know the direction of current flow.

When pioneers of electrical engineering studied the phenomenon of electricity they believed, that some "electrical fluid" flows from positive matter to the negative one. Using this idea, we define *positive current direction as the current flow from positive to negative terminal*. This is called *conventional current direction*. Since 1897, thanks to British physicist Thompon, we know that the electric current in metals is the flow of electrons. The electrons (which, due to the convention have assigned negative charge) flows from negative terminal of our source (where is the surplus of the electrons) through the electric circuit to the positive terminal of the source (where is the lack of the electrons). In this case, we define *positive current direction as the current flow from* *negative to positive terminal.*

Conventional Current Flow positive current direction is defined as the current flow from positive to negative terminal.

Electron Current Flow positive current direction is defined as the current flow from negative to positive terminal.

It does not matter which convention we use – in most equations (except special cases, which will be mentioned in the next lectures), if we use e.g. conventional current flow convention, and we multiply the equation by -1, we got the electron current flow convention. However, it is important to use the same one to prevent the mistakes. *In this course, we will use the conventional current flow direction.*

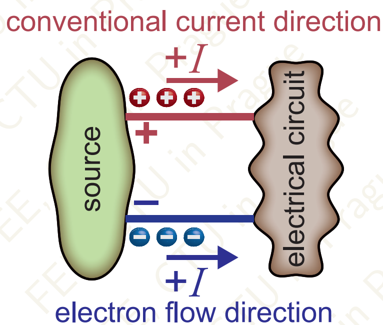


Figure 1.4: Current flow conventions

1.2.2. Kirchhoff's current law (KCL)

..., or 1st Kirchhoff's law.

Imagine two pipes that join. The first of these passes through one liter of water per second, a second flow two liters of water per second. How much water flows out of their joints?

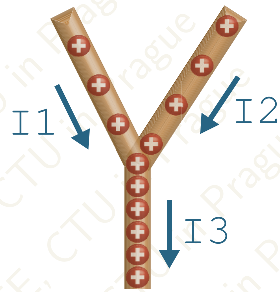


Figure 1.5: Concept of KCL

For the volume flow rate of the water we can write $Q_l = Q_{e1} + Q_{e2} = 1 + 2 = 3\,\mathrm{l\,s^{-1}}$. We can also think of the opposite case when one pipe divides into two. Once again, one liter of water flows away per second through the first one, through the other two liters. We can again write an equation $Q_e = Q_{l1} + Q_{l2} = 1 + 2 = 3\,\mathrm{l\,s^{-1}}$. Both equations are formally the same, except I use symbol Q_l if the water leaves the joint and Q_e if it enters the joint. *We must know the direction, in which the water flows.*

It is much better to adopt sign convention, which clearly defines the flow direction. We will use the following one:

- $+$ if the flow *leaves* the joint.

- – if the flow *enters* the joint.

The same idea is, of course, valid also for electric current, so we can write for Figure 1.5 $-I_1 - I_2 + I_3 = 0$. We can extend this idea to any number of wires which connects in the same joint, or even device terminals, and write:

$$\sum_{k=1}^{n} I_k = 0 \tag{1.3}$$

The equation (1.3) is generally valid, so instead of I_k for a constant (DC) current, we can use $i(t)$ for general time functions, $\mathbf{I}$ for AC phasors, $I(p)$ for Laplace transforms etc.

The equation (eq:KCL) is valid not only for connections of wires (nodes), but it is general law (resulting from law of conservation of charge. So it is valid also for the N-terminal on the Figure 1.6. Using passive sign convention all currents enters the N-terminal, which means, that some current will be positive and others negative. Negative value means, that actual direction is opposite.

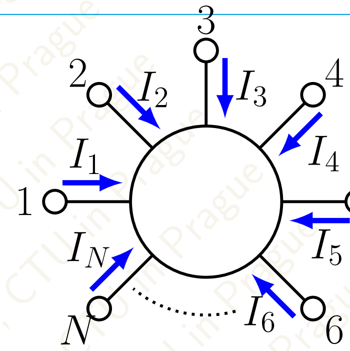


Figure 1.6: N-terminal

1.3. Electric voltage

symbol:	V or U
unit:	Volt (V)
SI standard system units:	$\mathrm{m}^2 \cdot \mathrm{kg} \cdot \mathrm{s}^{-3} \cdot \mathrm{A}^{-1}$

Definition

Electric voltage is the difference in electric potential $\varphi\,[\mathrm{V}]$ which can be expressed from the known distribution of the electric field intensity $\vec{E}\,[\mathrm{V\cdot m^{-1}}]$ ($\vec{E} = -\operatorname{grad}\varphi$) along path $\vec{l}\,[\mathrm{m}]$. *The voltage between points A and B is equal to the work $W\,[\mathrm{J}]$ which would have to be done, per unit charge $Q\,[\mathrm{C}]$, to move the charge from A to B.*

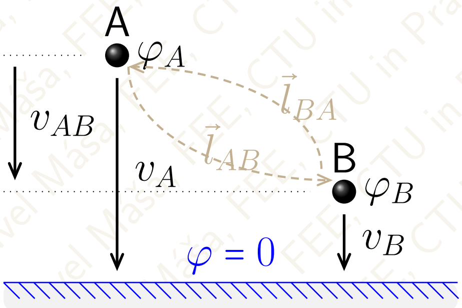


Figure 1.7

$$v_{AB} = \varphi_A - \varphi_B = \int_A^B \vec{E}\,\mathrm{d}\vec{l} = \frac{W}{Q}. \tag{1.4}$$

It does not matter, along which path the charge is moved. Important is the ponential of points A and B. If we move unit charge along path $\vec{l}_{AB}$ in the figure 1.7, where the potential energy in the point A is greater than in the point B, the voltage v_{AB} is positive, and the work $W = v_{AB}\cdot Q\,[\mathrm{J}]$ is done. If we move the same charge along arbitrary path $\vec{l}_{BA}$ from the point B to the point A, we must deliver the same work W and the voltage v_{AB} is negative. The sum of both voltages, along path $\vec{l}_{AB}$ and $\vec{l}_{BA}$ is then zero. We can extend it to any number of points:

- if we move along some closed path (from given point into the same one) in the electric field, with arbitrary number of points, the voltage between two subsequent points is generally nonzero (depending on eletric potential in both points), but total sum is zero.

This rule is called:

1.3.1. Kirchhoff's voltage law (KVL)

$$\boxed{\sum_{k=1}^{n} V_k = 0 \qquad (1.5)}$$

In electric circuits, the equation 1.5 is usually applied on circuit elements - resistors, capacitors, inductors, voltage and current sources, semiconductors, etc., however, since the voltage is the difference between electric potential in two points, we can evaluate KVL even between circuit terminals between which no circuit element is connected.

Figure 1.9 demonstrates passive and active sign convention. In the *passive sign convention* the positive current has the same direction as the positive voltage. In the *active sign convention* the positive current has opposite direction. However, in circuits with more sources the stronger source will force the direction of current (see example 3.6).

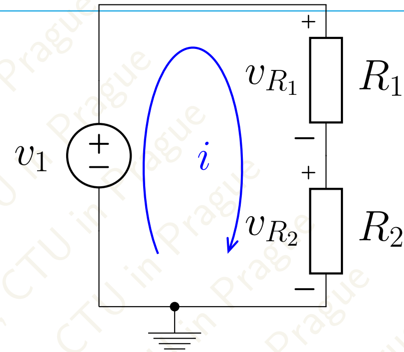
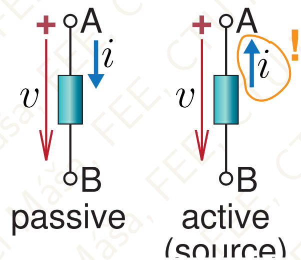


Figure 1.8

First, see the circuit in the Figure 1.8. In this circuit we have voltage source and two resistors. I *choose* direction of the loop i. The KVL will be of the form:

$$v_{R_1} + v_{R_2} - v_1 = 0.$$



The signs in the equation are given *by the convention*.

Convention

- Voltages across **passive circuit elements** (resistors, capacitors, inductors . . .) have always positive sign.
- Voltages of **active circuit elements** (voltage and current sources) can be both positive and negative. If the loop enters the negative node of the voltage source, the sign is minus, otherwise the sign is plus.

Compare now circuits in Figures 1.8 and 1.10. First of all, you can notice, that in the Figure 1.10 is missing the ground. It is not problem, many electrical devices are not connected to ground. Your mobile phones, notebooks, for example. In such a case, we can choose *any* node as the reference (ground) one. So, we will declare the lower node as the reference one, and we will assume it to be connected to ground again. Next, you can notice, that the loop is now anticlockwise. No problem, I *may* choose it like that. You can notice, that voltages across resistors are now opposite (+,- symbols, marking voltage direction). It is exactly according the convention - the voltage across passive circuit element has positive sign with respect the loop direction (in other words, loop always enters positive terminal on passive circuit element). The equation is now:

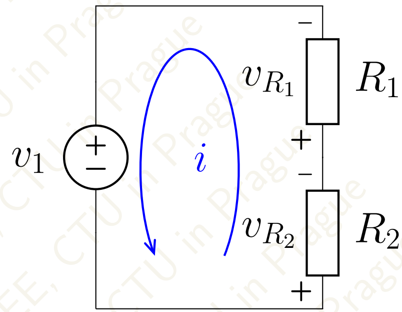


Figure 1.10

$$v_{R_1} + v_{R_2} + v_1 = 0.$$

The sign of the soure is now positive, because we enter the positive node of the source. If we rewrite this equation, we got $v_{R_1} + v_{R_2} = -v_1$. Well, the negative sign means, that we choosen wrong direction and current flows clockwise, as well as the voltages across resistors are oriented upside down.

Now, see Figure 1.11. There are two paths - the blue one goes through resistors R_1 and R_2, voltage controlled voltage source Av_d and voltage source v_1. KVL for this loop has the form:

$$v_{R_1} + v_{R_2} + Av_d - v_1 = 0.$$

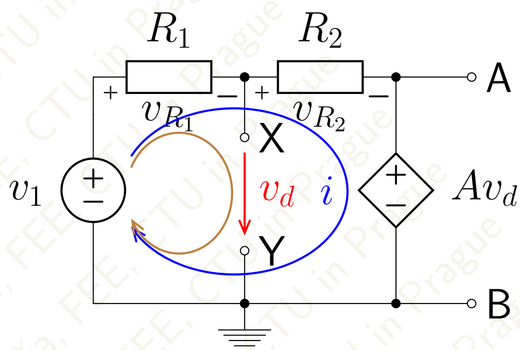


Figure 1.11

It is passed by the electric current i; However, there is also the brown path. It passes the terminals X and Y and so no electric current flows in this path. Still, we can write KVL for this path:

$$v_{R_1} + v_d - v_1 = 0.$$

1.4. Power

symbol:	P (constant) or $p(t)$
unit:	Watt (W)
SI standard system units:	$\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-3}$, derived units $\mathrm{J} \cdot s^{-1}$

Definition

Power is rate of energy transfer.
First, we define the *instantaneous power* as:

$$p(t) = \frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\mathrm{d}W}{\mathrm{d}q}\frac{\mathrm{d}q}{\mathrm{d}t} = v(t)i(t) \tag{1.6}$$

Instantaneous power is a function of time, and it generally varies with time, as you can see in the Figure 1.12. The voltage and current are sinusoidal, the instantaneous power is also sinusoidal, but of twice frequency.

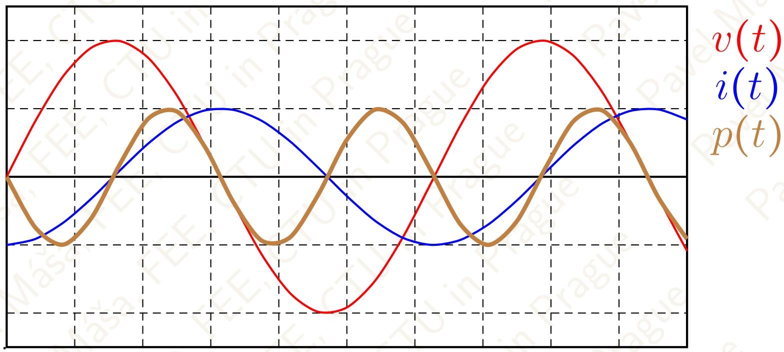


Figure 1.12: Instantaneous power - animation demontrates how it varies, if the phase of current (or voltage) varies

However, while instantaneous power give us information about peak values etc., common type of rate is "per unit of time".

Definition

The power is defined as:

$$P = \frac{1}{T}\int_0^T p(t)\,\mathrm{d}t \tag{1.7}$$

In the case of DC (constant) voltage and current the power can be written as:

$$P = VI. \tag{1.8}$$

In the case of AC (sinudoidal) voltage and current the power can be zero, even if both voltage and current are nonzero - see animation in the Figure 1.12. If the phase shift is $\pm 90\,^\circ$, the power sinusoidal is symmetrical to x axis, and its mean value is zero. It is clear, that energy, which is transfered in the circuit is nonzero. In the lecture **??** we will learn about another power definitions, which covers this energy transfer.
The important is that P **represents electric energy, which is irreversibly converted to another kind of energy** - usually heat, but eventually also light one, kinetic in motors...

Example 1.3

Assignment: The battery A has a rated capacity of $C_{QA} = 52\,\mathrm{Ah}$ at the battery voltage of $12\,\mathrm{V}$. The battery B has a rated capacity of $C_{QB} = 2\,\mathrm{Ah}$ at the battery voltage of $403.2\,\mathrm{V}$. Which battery can deliver more energy?

Solution:

$$Q_A = C_{QA}t = 52 \cdot 3600 = 187.2\,\mathrm{kC}, \qquad W_A = Q_A V_A = 187200 \cdot 12 \doteq 2.25\,\mathrm{MJ}$$
$$Q_B = C_{QB}t = 2 \cdot 3600 = 7.2\,\mathrm{kC}, \qquad W_B = Q_B V_B = 7200 \cdot 403.2 \doteq 2.9\,\mathrm{MJ}$$

The battery B can deliver more energy, than the battery A.

Example 1.4

Assignment: The EV car battery has rated capacity of $C_W = 24\,\mathrm{kWh}$ at voltage $V = 403.2\,\mathrm{V}$. What is the ampere-hour capacity of the battery?

Solution: The kilowatt-hour capacity is an energy, given as the power, delivered to the load within one hour, $W = 3600\,C_W$. The ampere-hour capacity is an electrical current, which delivers a charge within one hour, $Q = 3600\,C_Q$. It is related to the delivered energy by the relation $W = QV$.

$$C_Q = \frac{3600 C_W}{3600 V} = \frac{C_W}{V} = \frac{24000}{403.2} = 59.5\,\mathrm{Ah}.$$

Example 1.5

Assignment: The source has DC voltage $V = 230\sqrt{2}\,\mathrm{V}$ and to the circuit delivers electrical current $I = 0.5\sqrt{2}\,\mathrm{A}$. Calculate delivered power.

Solution:

$$P = VI = 230\sqrt{2} \cdot 0.5\sqrt{2} = 230\,\mathrm{W}.$$

Example 1.6

Assignment: The source has AC voltage $v(t) = 230\sqrt{2}\sin(314t)\,\mathrm{V}$ and to the circuit delivers electrical current $i(t) = 0.5\sqrt{2}\sin(314t)\,\mathrm{A}$. Calculate delivered power.

Solution:

$$\begin{aligned} P &= \frac{1}{T}\int_0^T 230\sqrt{2}\sin(314t)\cdot 0.5\sqrt{2}\sin(314t)\,\mathrm{d}t \\ &= \frac{230\sqrt{2}\cdot 0.5\sqrt{2}}{2T}\left[\int_0^T \cos(0)\,\mathrm{d}t - \underbrace{\int_0^T \cos(2\cdot 314t)\,\mathrm{d}t}_{=0}\right] = \frac{230}{2T}\cos(0)\,[t]_0^T = 115\,\mathrm{W}. \end{aligned}$$

In the Example 1.6 we use following identities:

$$
\begin{aligned}
\sin(\alpha)\sin(\beta) &= \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right] \\
\cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta) \\
\cos(-\alpha) &= \cos(\alpha)
\end{aligned}
$$

and the integral of cosine of double angle is:

$$
\begin{aligned}
\int_0^T \cos(2\omega t+\varphi)\,\mathrm{d}t &= \int_0^T \left[\cos(2\omega t)\cos(\varphi)-\sin(2\omega t)\sin(\varphi)\right]\mathrm{d}t = \\
&= \cos(\varphi)\int_0^T \cos(2\omega t)\,\mathrm{d}t - \sin(\varphi)\int_0^T \sin(2\omega t)\,\mathrm{d}t = \\
&= \cos(\varphi)\left[\frac{\sin(2\omega t)}{2\omega}\right]_0^T - \sin(\varphi)\left[\frac{-\cos(2\omega t)}{2\omega}\right]_0^T = \\
&= \frac{\cos(\varphi)}{2\omega}\left[\sin\left(2\frac{2\pi}{T}T\right)-\sin(0)\right] + \frac{\sin(\varphi)}{2\omega}\left[\cos\left(2\frac{2\pi}{T}T\right)-\cos(0)\right] = 0
\end{aligned}
$$

Example 1.7

Assignment: The source has AC voltage $v(t) = 230\sqrt{2}\sin(314t)\,\mathrm{V}$ and to the circuit delivers electrical current $i(t) = 0.5\sqrt{2}\sin(314t + \frac{\pi}{2})\,\mathrm{A}$. Calculate delivered power.

Solution:

$$P = \frac{1}{T}\int_0^T 230\sqrt{2}\sin(314t)\cdot 0.5\sqrt{2}\sin(314t + \frac{\pi}{2})\,\mathrm{d}t$$

$$= \frac{230\sqrt{2}\cdot 0.5\sqrt{2}}{2T}\left[\int_0^T \cos\left(\frac{\pi}{2}\right)\mathrm{d}t - \underbrace{\int_0^T \cos\left(2\cdot 314t + \frac{\pi}{2}\right)\mathrm{d}t}_{=0}\right] =$$

$$= \frac{230}{2T}\cos\left(\frac{\pi}{2}\right)[t]_0^T = 115\cos\left(\frac{\pi}{2}\right) = 0\,\mathrm{W}.$$

In the Example 1.7 the voltage and current have the same magnitudes as in the Example 1.6. The only difference is the phase shift of current. And while in the Example 1.6 the power was $115\,\mathrm{W}$, in the Example 1.7 the power is $0\,\mathrm{W}$, due to the phase shift.

1.5. Node, loop

Sometimes some students misunderstand the term *node*. They consider as a node only black circles in the circuit diagram. It is not true - every connection of two or more circuit elements are considered as a node, even if the black circle is missing (in the case of connection of two circuit elements only). On the other hand, sometimes more circuit elements are connected in the same point, however, it is better arranged if the circuit elements are drawn parallel and then we have more black circles representing node on the same line. However, such line represents just a connection, and we must consider all such node symbols as just single node. See Figure 1.13.

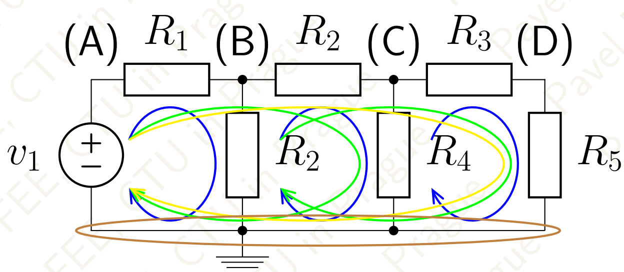


Figure 1.13

In this circuit, we have five nodes - (A), (B), (C), (D) and ground. Nodes (A) and (D) have no black circle symbol, and ground have two black circle symbols and four circuit elements connected.

Definition

Node is connection of two or more circuit elements. The small black circle is used as a symbol of a node, but only if more than two circuit elements are connected together. If just two circuits elements are connected, there is usually no black circle symbol. If a line, which represents an ideal wire connects more black circle symbols, it is still considered as single node (see brown ellipse in the Figure 1.13, surrounding ground node.

Definition

Loop is closed path, which does not intersect itself, and go through circuit elements in the circuit.
Mesh is simple loop, which connects just neighboring circuit elements.

In the Figure 1.13 six loops are drawn. Three of them, the blue one, are meshes.

1.6. Node and device voltage

Recall that the voltage is a difference of two electric potentials φ_A and φ_B.

Definition

Device voltage is a voltage between terminals of a single circuit device. It can be voltage across a resistor, capacitor, inductor, voltage or current source, two terminals of a transistor... We use device voltages in the [KVL].

Definition

Node voltage is a voltage between distinct node and the ground. We use node voltages in the nodal analysis.

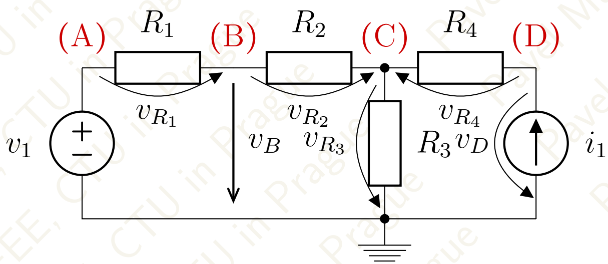


Figure 1.14

In the Figure 1.14 we have six *device* voltages, v_{R_1}, v_{R_2}, v_{R_3}, v_{R_4}, v_1 and , v_{i_1}. Or, we can define four nodal voltages v_A, v_B, v_C, v_D.

Example 1.8

Assignment: In circuit in the figure 1.14 we know device voltages $v_{R_1} = 10\,\mathrm{V}$, $v_{R_2} = 5\,\mathrm{V}$, $v_{R_3} = 10\,\mathrm{V}$, $v_{R_4} = 20\,\mathrm{V}$. Find other device voltages, and all nodal voltages.

Solution:

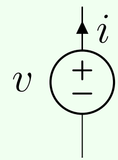
$$
\begin{aligned}
v_B &= v_{R_3} + v_{R_2} = 10 + 5 = 15\,\mathrm{V} \\
v_A &= v_{R_3} + v_{R_2} + v_{R_1} = v_B + R_1 = 15 + 10 = 25\,\mathrm{V} \\
v_D &= v_{R_3} + v_{R_4} = 10 + 20 = 30\,\mathrm{V} \\
v_C &= v_{R_3} = 10\,\mathrm{V} \\
v_1 &= v_A = 25\,\mathrm{V} \\
v_{i_1} &= v_D = 30\,\mathrm{V}
\end{aligned}
$$

In the Example 1.8 we can also write, that $v_{R_1} = v_A - v_B$, $v_{R_2} = v_B - v_C$, $v_{R_4} = v_D - v_C$.

1.7. Independent Voltage Source

Definition

Ideal independent voltage source is *hypothetical* device, which maintains between its terminals constant voltage, no matter what circuit we connect to the source. It is able deliver to the connected circuit any current, even infinite. We will use symbol:



The label i is not part of the symbol, it demonstrates positive direction of the current.
Power delivered by the source is defined as:

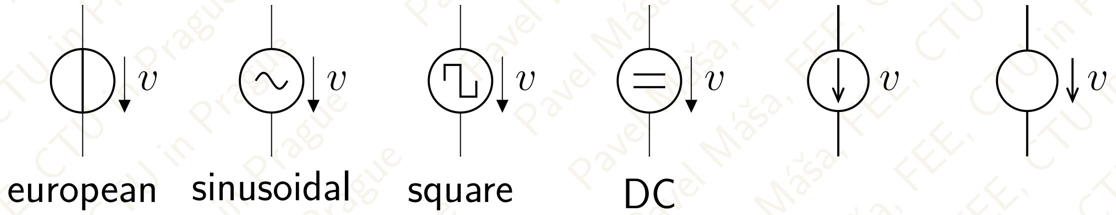
$$P_v = vi \tag{1.9}$$

If the current flows in the opposite direction than indicated in the figure, the power supplied by the source will be negative (the source behaves as an appliance). We also use terms:

Source if an electric current leaves the terminal.
Sink if an electric current enters the terminal.

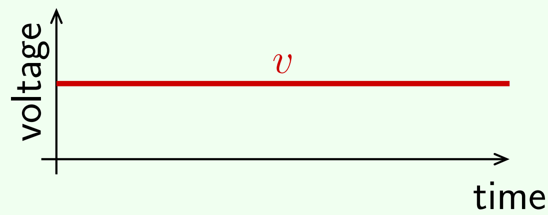
The terms source and sink are widely used, e.g. in IC circuits.

Other voltage source symbols:



Definition

DC stands for *Direct Current*. DC current has constant direction.
DC voltage is a voltage which has constant orientation.

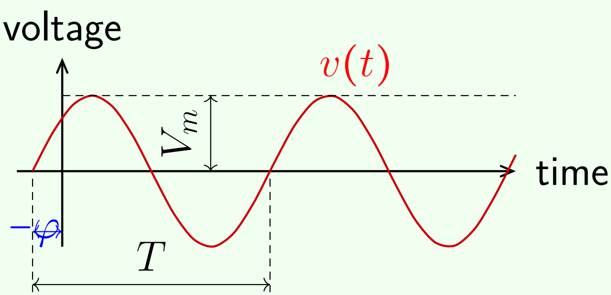


DC voltage may sound somewhat confusing. It is gramatically incorrect, but it is set phrase. DV should be the correct term, however, is is not widely used. DC voltage originates in resistive circuits. There, with constant load, if the current is constant, also voltage must be, according to Ohm's law, constant.

Definition

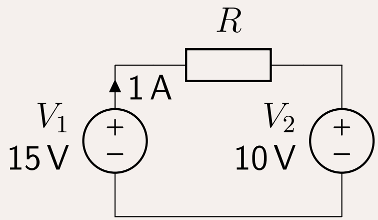
AC stands for *Alternating Current* - it changes its direction in time. **AC voltage** is alternating voltage.

$$v(t) = V_m \sin(\omega t + \varphi), \qquad \omega = 2\pi f, \qquad f = \frac{1}{T}$$



Example 1.9

Assignment: For the circuit in the figure calculate power, delivered by both DC voltage sources.



Solution: According to Equation (1.9), power, delivered by the source is given as:

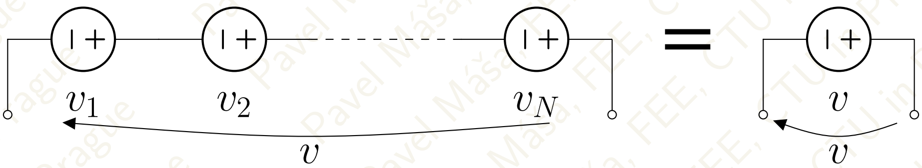
$$P_{V_1} = V_1 I = 15 \cdot 1 = 15\,\mathrm{W}$$
$$P_{V_2} = -V_2 I = -10 \cdot 1 = -10\,\mathrm{W}$$

Notice the negative sign of power at the second source. It is because current *enters* positive terminal of the voltage source. The source with greater voltage forces current direction and source with lower voltage acts as an electric appliance - it does not deliver but consumes energy. For example, a rechargeable battery during charging is a case.
Note, that resistor in the circuit consumes power $P_R = P_{V_1} + P_{V_2} = 15 - 10 = 5\,\mathrm{W}$.
Resistor voltage (it's device voltage) is $V_R = V_1 - V_2 = 15 - 10 = 5\,\mathrm{V}$.

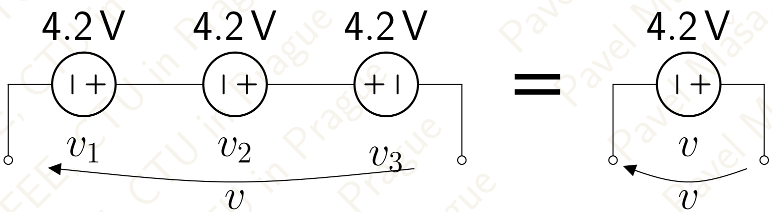
Definition

Conservation of power: Total power, delivered by all sources in the circuit is the same, as total power, consumed by all appliances.

If we connect two or more voltage sources in series, we can replace them, according to the KVL, by single voltage source of voltage $v = v_1 + v_2 + \cdots + v_N$.



In the figure below we have three voltage sources. The third one is connected in opposite direction. The total voltage is then $v = 4.2 + 4.2 - 4.2 = 4.2\,\mathrm{V}$.



Keep in mind, that anti-series connection is possible with ideal sources. It is possible even with practical sources, like batteries, but some caution is needed. Lithium polymer battery can have maximum charging current $1.2\,\mathrm{A}$, but maximum discharge current e.g. $60\,\mathrm{A}$. The first two batteries of the same polarity are discharged if we connect some appliance, but the third one is charged. Then the current in the circuit can easily exceed maximum charging current. The consequences may be very dramatic.

We can even add voltages of different time functions, e.g. DC and AC.

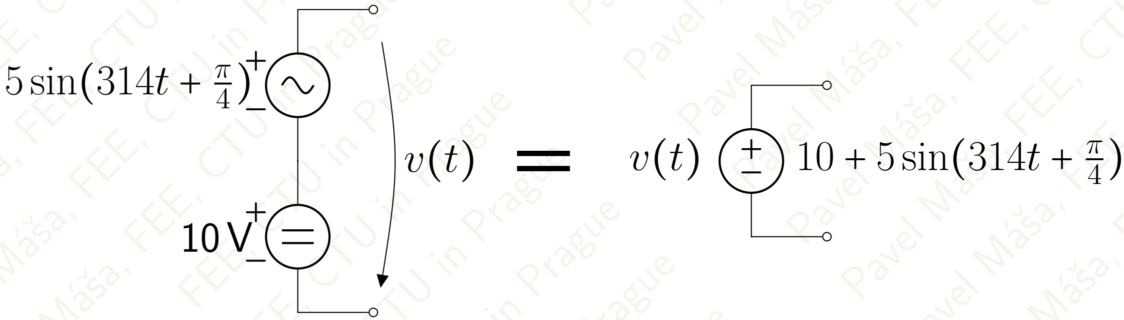
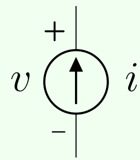


Figure 1.15: AC and DC sources in series

1.8. Independent Current Source

Definition

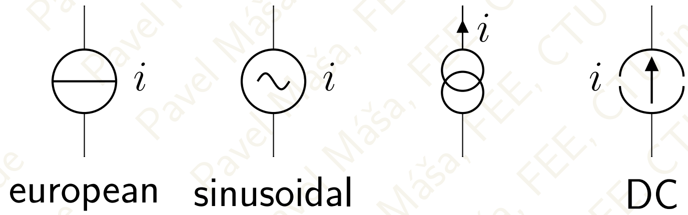
Ideal independent current source is *hypothetical* device, which maintains between its terminals constant current, no matter what circuit we connect to the source (if any). It is able maintain between its terminals any voltage, even infinite. We will use symbol:



The label v is not part of the symbol, it demonstrates the positive direction of the voltage across power source (required for determination of the sign of delivered power).
Power delivered by the source is defined as:

$$P_i = vi \tag{1.10}$$

Other current source symbols:

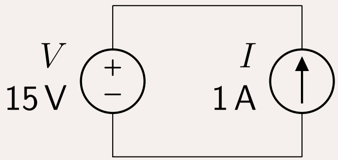


Note that conventional batteries, generators and so act as a voltage source, not the current one. Practical current sources are electronic circuits, consisting of at least one resistor and two transistors, in practice rather IC circuits, like Texas Instruments or ST Microelectronic LM134/LM234/LM334, Linear Technology LT3092, Nexperia PSSI2021SAY... The current source is significant electronic device because without it could not operate high-speed serial buses such as USB 2.0 and later.

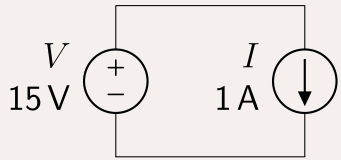
Of course, the current source, just as the voltage source, can be DC, AC, or of any other time function.

Example 1.10

Assignment: For the circuits in the figure calculate power, delivered by the voltage and the current sources.



Circuit A



Circuit B

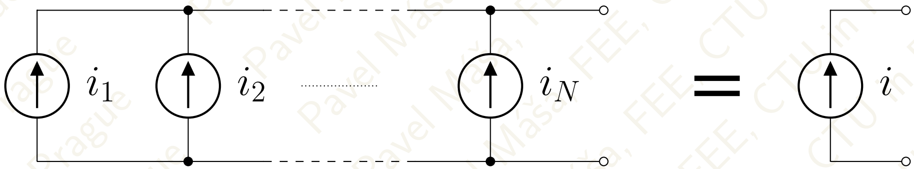
Solution: In both circuits, the voltage source forces the voltage of $15\,\mathrm{V}$, while the current source forces the current of $1\,\mathrm{A}$. It means that the voltage source does not deliver any current, but on the other hand, it is passed by the current from the current source, so the voltage source acts as a short circuit (an ideal wire) concerning the current source.
In the Circuit A, electric current enters positive terminal of the voltage source, which is opposite to source convention, so the sign of power is negative. On the current source, the positive voltage is on the source terminal, so the power has positive sign.

$$P_V = -VI = -15 \cdot 1 = -15\,\mathrm{W} \qquad P_I = VI = 15 \cdot 1 = 15\,\mathrm{W}.$$

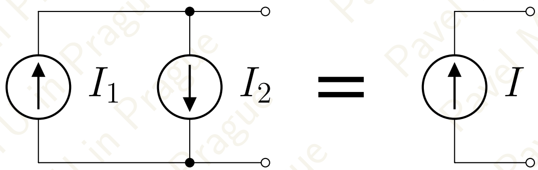
In the Circuit B, electric current leaves positive terminal of the voltage source, so the sign of power is positive. On the current source, the positive voltage is on the sink terminal, so the power has negative sign.

$$P_V = VI = 5 \cdot 1 = 15\,\mathrm{W} \qquad P_I = -VI = -15 \cdot 1 = -15\,\mathrm{W}.$$

If we connect two or more current sources in parallel, we can replace them, according to the KCL, by single current source of current $i = i_1 + i_2 + \cdots + i_N$.



For example, let in the figure below $I_1 = 4\,\mathrm{A}$ and $I_2 = 5\,\mathrm{A}$.



Recall sign convention - current entering the node is negative, current leaving the node is positive. For the upper node we can write an equation, where on the left handed side is circuit on the left, and on the right handed side resulting current: $-I_1 + I_2 = -I$. Changing the signs we get $I = I_1 - I_2 = 4 - 5 = -1\,\mathrm{A}$.

1.8.1. Dependent sources

Definition

Dependent source is a voltage source or a current source whose value depends on a voltage or current somewhere else in the network.

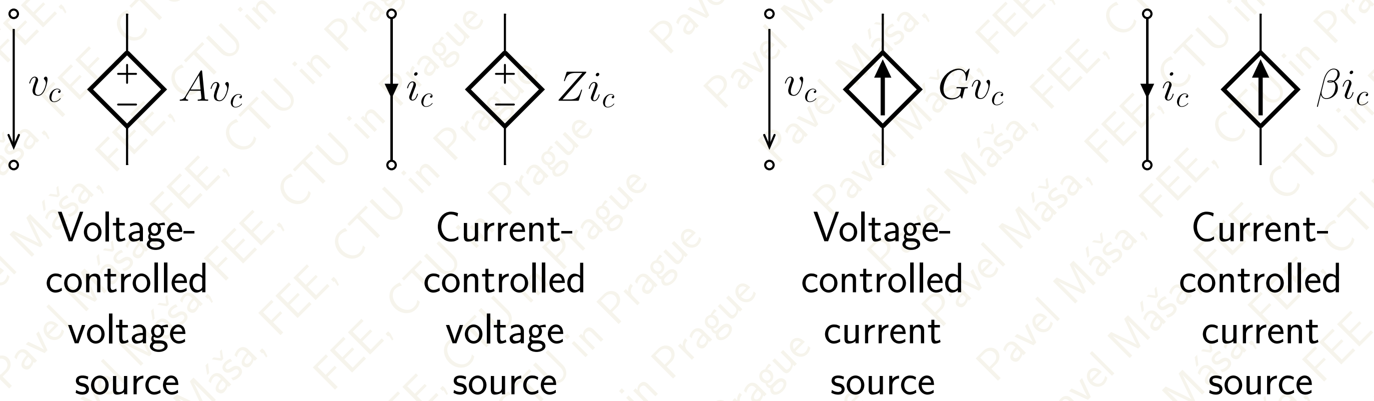


Figure 1.16: Symbols of dependent sources

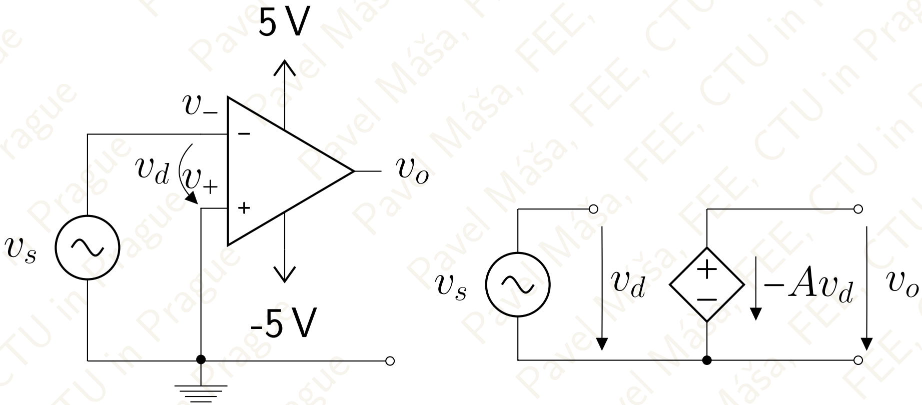
Voltage-controlled voltage source | Current-controlled voltage source | Voltage-controlled current source | Current-controlled current source

Figure 1.17: Alternative symbols of dependent sources (most in Europe)

Proportionality constants A, Z, G, β between dependent and independent variables defines the value of dependent variable. They are dimensionless, if both dependent and independent variable is voltage, or current. In such a case they are usually called (voltage or current) *gain*. In the case of voltage-controlled current source the proportionality factor is expressed in units of conductance (Siemens) and it is called *transconductance*. In the case of current-controlled voltage source the proportionality factor is expressed in units of resistance (Ohms) and it is called *transresistance*.

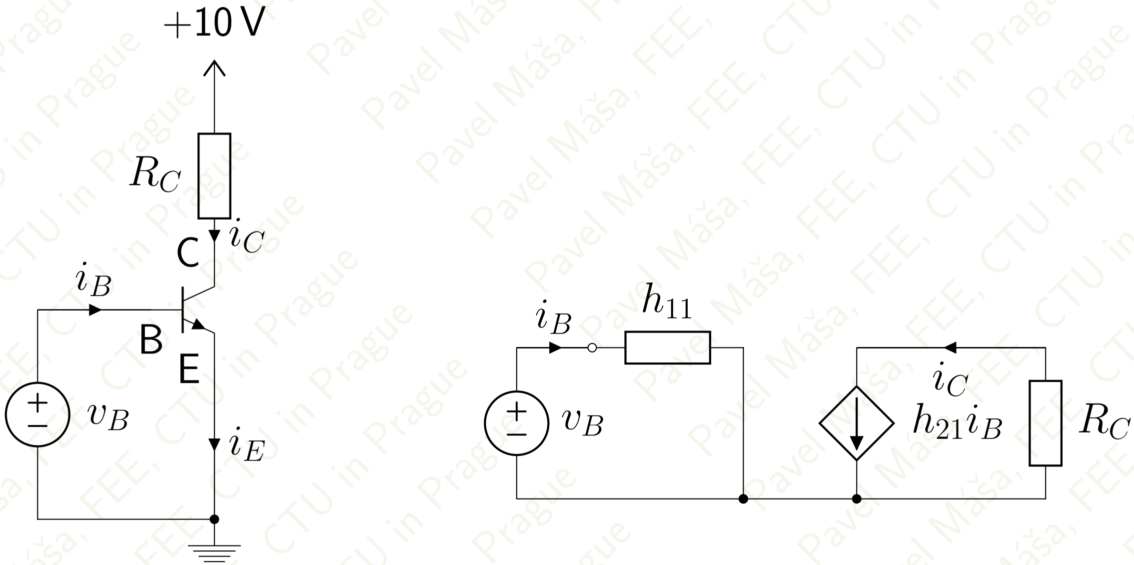
Dependent source itself does not deliver energy. It is not device, like a battery. It requires auxiliary source of energy, and dependent source controls current or voltage of such auxiliary source. Controlled sources are usefull in modelling of electronic devices.

For example, we use voltage-controlled voltage source in circuit model of operational amplifier.



On the left part of the figure is an operational amplifier. It is electronic circuit with very large voltage gain. It has two input terminals, inverting (–), and non-inverting (+). The output voltage v_o is A times greater, than differential voltage - the voltage between terminals (–) and (+). On the right part of the figure is an ekvivalent circuit with voltage-controlled voltage source. If, e.g., $v_d = 1\,\mu\mathrm{V}$, and the voltage gain will be $A = 250000\,[-]$, than output voltage will be $v_o = -Av_d = -250000 \cdot 10^{-6} = -0.25\,[V]$. The operational amplifier is supplied by two voltage sources, $5\mathrm{V}$ and $-5\mathrm{V}$. Note, that equivalent cicrcuit of practical OA contains input and ouput resistors, as well as some capacitors. The OA also in practical applications use some feedback network, which also is not for simplicity included.

Current-controlled current source is useful in modelling of bipolar transistor, and voltage-controlled current source in modeling of MOSFETs.



On the left part of the figure is bipolar transistor with collector transistor ans $+10\,\mathrm{V}$ voltage source as auxiliary power supply. On the right handed part is simplified model of the transistor (parameters h_{12} and h_{22} are omitted). There is current-controlled current source. The collector current $i_C = h_{21} i_B$. For example, if base of the transistor sinks $100\,\mu\mathrm{A}$ and current gain is $h_{21} = 200\,[-]$, collector sinks $20\,\mathrm{mA}$. And, according to KCL, $i_E = i_B + i_C = 20.1\,\mathrm{mA}$. Note that the $+10\,\mathrm{V}$ voltage source was replaced by short circuit according to the superposition principle.

1.9. Classification of time functions of circuit variables

Circuit variable is general term, used for current, voltage, power, and so on. Circuit variables may vary with time. The basic classification is as follows:

Circuit variables

- **Determined**
  Can be expressed by mathematical functions
  - **Stationary (DC)**
    Does not change its magnitude and direction during the time interval of observation – DC analysis
  - **Periodic**
    from time interval of length T (period) repeat their instantaneous values, (special case – AC); $u(t + kT) = u(t)$
    AC analysis – Complex numbers; Rectangular, triangular, . . . – Periodic steady state – Fourier series and AC analysis.
  - **Non-periodic**
    Transients – Integral differential equations or Laplace transform.
- **Non-determined (stochastic)**
  Random processes (probability density function, . . .) - noise analysis

Although all laws and methods of analysis are universally valid, depending on the time waveforms of the circuit variables we use different mathematical apparatuses in the analysis. If circuit variables are DC, we use DC analysis, and math apparatus includes basic math operations in the real domain. If circuit variables are AC, we use AC analysis, and math apparatus includes basic math operations in the complex domain. If circuit variables are not AC, but periodical, we first find the Fourier series expansion of the time function, and then we calculate the AC analysis for each harmonic component. If the time functions of source variables are not even periodical, we have to use Laplace transform. As a result of changes in the circuit (connection or disconnection of the source or part of the circuit), there are an aperiodic time waveforms of the circuit variables (transients). We can use Integral differential equations, or Laplace transform.

In this course we will target on DC, AC and transient analysis.

1.9.1. Steady state

Definition

Steady state is an equilibrium condition of a circuit or network, that occurs in a moment, when transients are no longer important.

When there is a change in the circuit (e.g., connection/disconnection of the power supply), capacitors and inductors in the circuit must first be charged or discharged -

this is a transient. After charging or discharging, there is a balance in the circuit – a steady state.

1.10. Special values

The determined circuit variables are completely described by their time functions. On the other hand, in many circumstance use of time functions is not necessary, and it would be also quite uncomfortable. For example, if we are going to calculate power consumption of kitchen kettle, we do not need to know time function of voltage - just RMS. Voltmeters and ammeters also does not measure time functions, but average or RMS values of time functions.

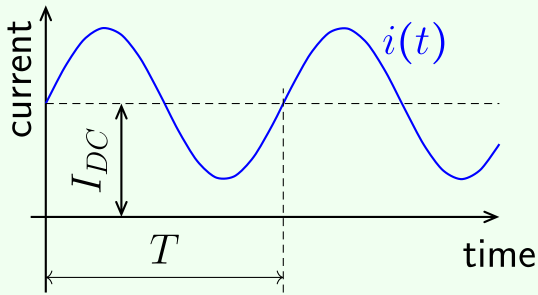
1.10.1. Mean and average value

Definition

Mean value (or average value) of electric current is defined by:

$$I_{DC} = \frac{1}{T}\int_{t_0}^{t_0+T} i(t)\,\mathrm{d}t \tag{1.11}$$

The same definition is valid for any circuit variable.



The equation (1.11) defines DC component of the waveform. In this case, we can write the time function of current as $i(t) = I_{DC} + I_m \sin(\omega t + \varphi)$. DC component represents independent part of the equation and we can represent this waveform as DC and AC current sources connected in parallel, or, in the case of voltage, DC and AC voltage sources, connected in series.

Let's think now about magneto-electric (moving-coil) meter. Because it is based on magnetic deflection of a coil and permanent magnet, it is sensitive to direction of passing current. If we connect AC current, measured value is zero - DC component. In order to use this voltmeter for AC circuit variables, we need to rectify the circuit variable. Mathematically, this reflects an *average* value:

Definition

Average value is defined by:

$$I_{AVG} = \frac{1}{T} \int_{t_0}^{t_0+T} |i(t)| \,\mathrm{d}t \tag{1.12}$$

Integral calculates the area under the circuit variable waveform. After dividing by period, we got the height of a rectangle of the same area, see Figure 1.18. It is clear, that the area under the rectified sine function is the same in both halves of period. For this reason, we can use alternative equation:

$$I_{AVG} = \frac{2}{T} \int_{0}^{\frac{T}{2}} i(t) \,\mathrm{d}t. \tag{1.13}$$

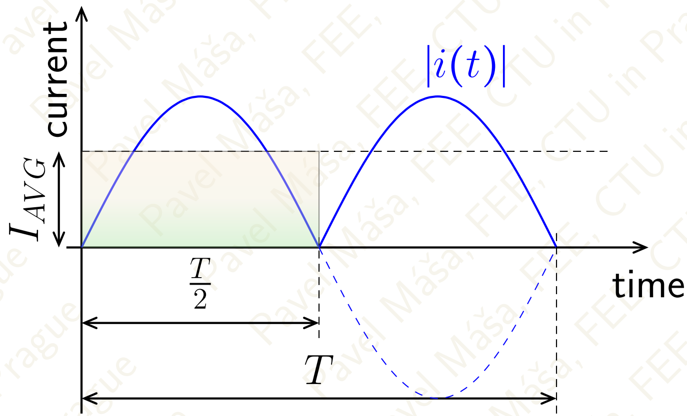


Figure 1.18: Average value

Example 1.11

Assignment: Calculate average value of voltage $v(t) = 325\sin(314t)\,\mathrm{V}$.

Solution:

$$V_{AVG} = \frac{1}{\frac{T}{2}}\int_0^{\frac{T}{2}} V_m \sin(\omega t)\,\mathrm{d}t = \frac{2V_m}{T}\left[\frac{-\cos(\omega t)}{\omega}\right]_0^{\frac{T}{2}} = \frac{2V_m}{T}\left[\frac{-\cos(\frac{2\pi}{T}t)}{\frac{2\pi}{T}}\right]_0^{\frac{T}{2}} =$$

$$= \frac{V_m}{\pi}\left(-\cos(\pi) + \cos(0)\right) = \frac{2V_m}{\pi} = 0.636 \cdot 325 = 206.9\,\mathrm{V}$$

1.10.2. RMS value

In electrical engineering we often need to calculate power. We already know, that power is the mean value of instantaneous power – see Equation (1.7). We will now see, that it is not necessary every time calculate Equation (1.7). The power is often evaluated as the value of DC current I, that converts an equivalent amount of heat in the same linear conducting medium of resistance R within a period T.

For a resistance R, DC power is defined as:

$$P = RI^2.$$

According to Equation (1.7), an average power for any time function on resistor is:

$$P = \frac{1}{T}\int_0^T p(t)\,\mathrm{d}t = \frac{1}{T}\int_0^T v(t)i(t)\,\mathrm{d}t = \frac{1}{T}\int_0^T Ri^2(t)\,\mathrm{d}t$$

Combining both equations into one we get:

$$RI^2 = \frac{1}{T}\int_0^T Ri^2(t)\,\mathrm{d}t$$

And from this relation we finally get

Definition

RMS value

$$I = \sqrt{\frac{1}{T}\int_0^T i^2(t)\,\mathrm{d}t} \tag{1.14}$$

Definition

AC power can be evaluated in terms of RMS values of voltage V and current I:

$$P = V\,I \tag{1.15}$$

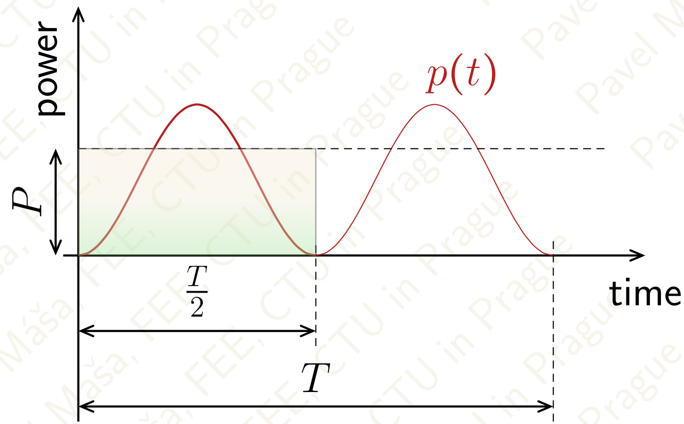


Figure 1.19: RMS value

Example 1.12

Assignment: Calculate RMS value of voltage $v(t) = 325\sin(314t)\,\mathrm{V}$.

Solution:

$$V = \sqrt{\frac{1}{T}\int_0^T (V_m \sin(\omega t))^2 \,\mathrm{d}t} = \left|\sin^2(\omega t) = \frac{1-\cos(2\omega t)}{2}\right| =$$

$$= \sqrt{\frac{V_m^2}{2T}\int_0^T (1-\cos(2\omega t))\,\mathrm{d}t} = V_m\sqrt{\frac{1}{2T}\left\{[t]_0^T - \frac{1}{2\omega}\left[\sin(2\omega t)\right]_0^T\right\}} =$$

$$= V_m\sqrt{\frac{1}{2T}\left[T - \frac{1}{2\cdot\frac{2\pi}{T}}(\sin(4\pi) - sin(0))\right]} = \frac{V_m}{\sqrt{2}} = 0.707\cdot 325 = 229.8\,\mathrm{V}$$

1.10.3. Form Factor

Definition

Form factor is defined as

$$k_f = \frac{RMS}{AVG} \tag{1.16}$$

RMS and AVG in the Equation (1.16) can be any circuit variable, usually it is voltage, or current. For sinusoidal voltage the form factor is:

$$k_f = \frac{\frac{V_m}{\sqrt{2}}}{\frac{2V_m}{\pi}} = \frac{\pi}{2\sqrt{2}} = 1.11\,[-].$$

If meter measures average value, it is often scaled to display the RMS value of a sine wave. The measured value is then multiplied by the form factor of a sinusoid – 1.11. However, it can be a little troublesome, because the value of 1.11 is valid only for sine wave as other waveforms have different form factors. But meter multiplies result by 1.11 regardless of the true time function, so we read wrong result for other time functions!

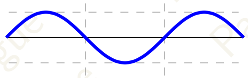
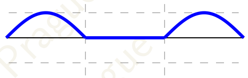
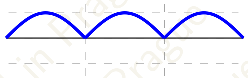
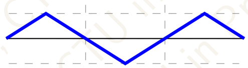
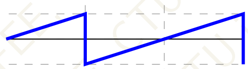
1.10.4. Crest Factor

Definition

Crest factor is defined as

$$k_c = \frac{peak}{RMS} \tag{1.17}$$

Meters lost their accuracy for high crest factors. In audio engineering, it gives an information about a safe zone between maximum amplitude and nominal signal value which is required to prevent signal distorsion due to clipping (signal value exceeds maximum possible value in the system and is limited), or even destruction of the device.

waveform		V_{RMS}	V_{AVG}	k_f
Sine wave		$V_m \frac{1}{\sqrt{2}}$	$V_m \frac{2}{\pi}$	$\frac{\pi}{2\sqrt{2}} \approx 1.11$
Half-wave rectified sine		$V_m \frac{1}{2}$	$V_m \frac{1}{\pi}$	$\frac{\pi}{2} \approx 1.57$
Full-wave rectified sine		$V_m \frac{1}{\sqrt{2}}$	$V_m \frac{2}{\pi}$	$\frac{\pi}{2\sqrt{2}}$
Square wave		V_m	V_m	1
Triangle wave		$V_m \frac{1}{\sqrt{3}}$	$V_m \frac{1}{2}$	$\frac{2}{\sqrt{3}} \approx 1.155$
Sawtooth wave		$V_m \frac{1}{\sqrt{3}}$	$V_m \frac{1}{2}$	$\frac{2}{\sqrt{3}}$

2. Basic Passive Circuit Elements

Physical note . . . Before we introduce resistors

From quantum theory (Schrödinger's wave equation) it follows, that the electrons:

- can occupy only *discrete energy levels*
- their energies falls into bands:
  - the highest range of electron energies in which electrons are normally present at absolute zero temperature is called the *valence band*
  - above the valence band is a *conduction band*;
    the electrons, which energy falls into the conduction band moves freely within the atomic lattice
- between the valence band and conduction band is a *band gap (forbidden band)*;
  the electrons can't have energy which falls into the band gap.

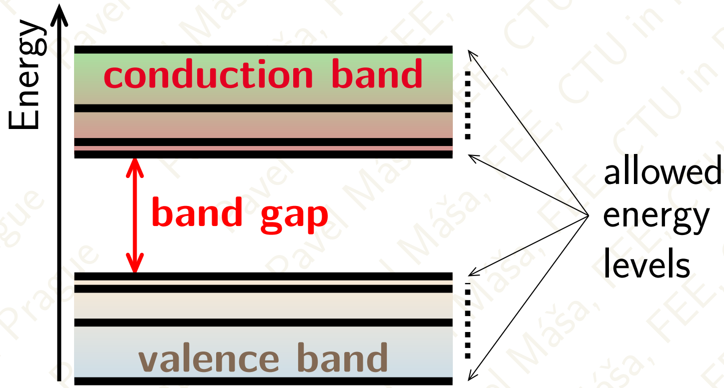


Figure 2.1: Atom energy bands

- insulators have very large forbidden band. At normal conditions, there are virtually no conductive electrons.
- in metals is no forbidden band, some valence electrons moves freely within atomic lattice; for example. in copper the number of conduction electrons is the same as the number of atoms. We can calculate it from molar mass of copper $M = 63.546\,\mathrm{g\,mol^{-1}}$, Avogadro's constant $N_A = 6.022\,140\,857 \times 10^{23}\,\mathrm{mol^{-1}}$ and copper's density $\rho = 8.96\,\mathrm{g\,cm^{-3}}$. $n = \frac{N_A}{M}\rho = 8.49 \times 10^{22}\,\mathrm{cm^{-3}}$, or $n = 8.49 \times 10^{28}\,\mathrm{m^{-3}}$.
- semiconductors have narrow conductive band, heat or radiation can give to a valence electron energy sufficient to jump from valence to conductive band. Concentration of conduction electrons is much less than in the metals, e.g. in silicon it is $n_i = 1.56 \times 10^{16}\,\mathrm{m^{-3}}$.

The electrons in the conductor are not steady, but they moves very fast, due to thermal kinetic energy. *Thermal velocity* is about $1 \times 10^5\,\mathrm{m\,s^{-1}}$. However, the electrons are scattered readily by each other or by vibrating atoms in the conductor and therefore they tend to perform a random motion. The average value of this thermal volocity $\langle v_T \rangle = 0$.

If we apply an electric field $\vec{E}$ to the conductor, the electrons start to move in opposite direction to the electric field. If an electric field would be applied on free electron of the charge e, the electric force acting on the electron $\vec{F}_e = -e\vec{E}$ would cause constant acceleration $\vec{a} = \frac{\vec{F}_e}{m_e} = -\frac{e\vec{E}}{m_e}$. However, because the electrons in a conductor are scattered each other and by the atoms, they can not accelerate forever, but after each scatter they start to accelerate again. *In scatters, the electrons gives their energy to the atoms, which increase total thermal energy, so an electrical energy is converted to heat.*

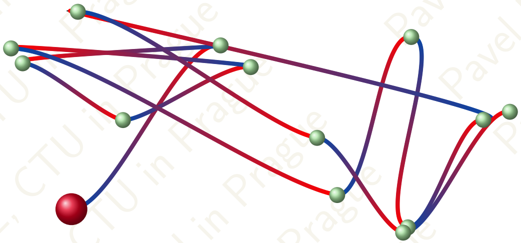


Figure 2.2: Electon motion in conductor

The mean value of the speed, at which electrons moves in an electric field, is called *drift velocity* $\vec{v}_d$. Assume now a conductor with conduction electrons density of n. Let the wire have a cylindrical shape with a cross-section area A, see Figure 2.3.

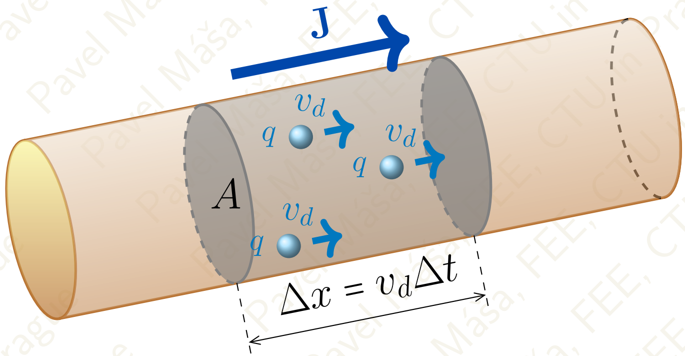


Figure 2.3: Moving charge in a conductor - drift in an electric field

The electrons travels within the time Δt along the path $\Delta x = v_d \Delta t$. Their total charge is $\Delta Q = nVq = nA\Delta xq = nAv_d\Delta tq$ and it carries an electric current $I = \frac{\Delta Q}{\Delta t} = nAv_dq$. If the mean time between two scatters is τ the drift velocity is $v_d = \vec{a}t = -\frac{q\vec{E}}{m_e}\tau$. The current density in an conductor is

$$\vec{J} = \frac{\vec{I}}{A} = -nqv_d = nq\left(-\frac{q\vec{E}}{m_e}\tau\right) = \frac{nq^2\tau}{m_e}\vec{E} = \sigma\vec{E}. \tag{2.1}$$

We call the term σ in the Equation (2.1) *conductivity*. It follows, that:

- Conductivity increases with concentration of conduction electrons n and depends on mean time τ between two scatters.
- In metals, n is constant. However, with increasing temperature, τ decreases and so the conductivity decreases.

- In semiconductors, with increasing temperature, τ decreases but n increases more, so the conductivity increases.

If an electric field is homogenous, an electric voltage is

$$v_{AB} = \varphi_A - \varphi_B = \int_A^B \vec{E}\,\mathrm{d}\vec{l} = El.$$

The current density is then

$$J = \sigma E = \sigma \left(\frac{v_{AB}}{l} \right). \tag{2.2}$$

Since $J = \frac{I}{A}$ where I is an electric current, we can write:

$$v_{AB} = \frac{l}{\sigma} J = \left(\frac{l}{\sigma A} \right) i = Ri. \tag{2.3}$$

Now we can finally introduce few important definitions:

Definition

Resistance of a conductor is defined as:

$$R = \frac{l}{\sigma A} = \rho \frac{l}{A} \tag{2.4}$$

where
σ is conductivity $[\mathrm{S\,m^{-1}}]$
ρ is resistivity $[\Omega\,\mathrm{m}]$
l is wire length $[\mathrm{m}]$
A is cross-section area of the conductor $[\mathrm{m^2}]$
The unit of resistance is Ohm $[\mathrm{ohm}]$, in SI base units $\mathrm{kg\,m^2\,s^{-3}\,A^{-2}}$

Definition

Temperature dependence of resistivity can be defined as linear approximation:

$$\rho = \rho_0 \left[1 + \alpha(T - T_0)\right] \tag{2.5}$$

where α is temperature coefficient of resistivity $[\mathrm{K^{-1}}]$ and ρ_0 is the resistivity at temperature T_0.

Definition

Ohm's law:

$$v = Ri \tag{2.6}$$

Example 2.1

Assignment: Find the drift speed of electrons in copper wire, if the wire has cross-section $1\,\mathrm{mm}^2$ and the wire is passed by the current $1\,\mathrm{A}$. The conduction electrons concentration is $n = 8.49 \times 10^{28}\,\mathrm{m}^{-3}$. Charge of the electron is $1.602 \times 10^{-19}\,\mathrm{C}$.

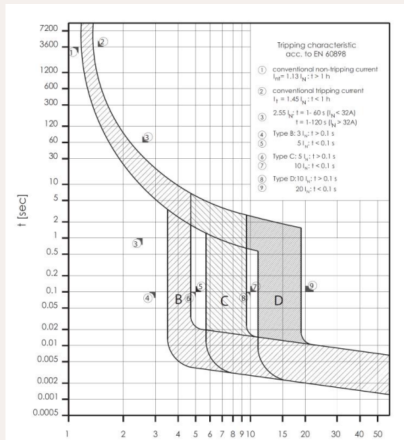
Solution:

$$v_d = \frac{I}{nAq} = 73.5\,\mu\mathrm{m}.$$

Example 2.2

Assignment: How many light bulbs rated at voltage of 230 V and power of 60 W, connected into the circuit causes that the circuit breaker interrupts current flow, if the rated current of the circuit breaker is 10 A?

Solution: Using Equation (1.15) we can find current $I_{lb} = \frac{P}{V} = \frac{60}{230} = 0.2608\,\text{A}$. If the current in one light bulb is 0.2608 A, 10 A will flow through $N = \frac{I_{cb}}{I_{lb}} = \frac{10}{0.26} = 38.46$ light bulbs. And circuit breaker should interrupt current flow, if we will connect 39 light bulbs. Is it the answer? Not exactly. When we switch on light bulb, it is cold. Cold bulb, according to Equation (2.5) will have less resistivity, and resistance. From Equation (2.6) resistance of shining light bulb is $R = \frac{V}{I} = \frac{230}{0.26} = \frac{V^2}{P} = 881.7\,\Omega$. Temperature coefficient of tungsten is $\alpha = 0.0045\,\text{K}^{-1}$. The temperature of shining light bulb can be $\approx 2000 \ldots 3300$K. Let's use $T = 2800\,\text{K}$. The resistance of cold bulb is then $R_0 = \frac{R}{1+\alpha(T-T_0)} = \frac{881.7}{1+0.0045\cdot(2800-300)} = 71.98\,\Omega$. If we switch-on light bulbs all at once, the total current will be $I_t = \frac{230}{72} = 3.19\,\text{A}$ and number of light bulbs $N^{‘} = 3.13$. However, it is not correct answer, because the circuit breaker does not interrupt current at 10 A. . . Typical type B characteristic curve circuit breaker rated at 10 A switches current at ≈ 30 ampere, so the answer would be 10 light bulbs.

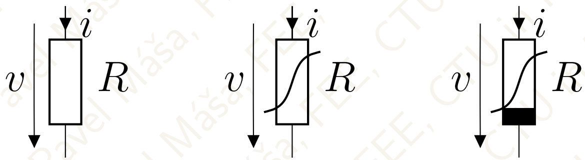


2.1. Resistor

Definition

Resistor is circuit element, which converts electrical energy to heat. It is non-inertial element (the voltage-current relation is not time dependent).

(a) Examples of resistor packages

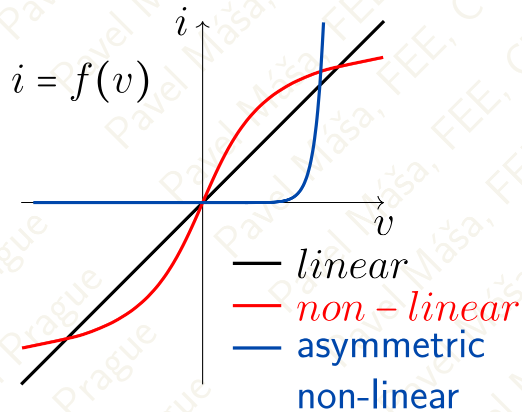


linear

non-linear

assymetric non-linear

(b) Resistor symbols



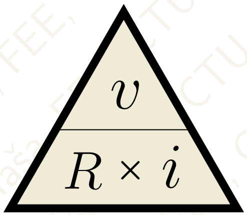
(c) Resistor V-A characteristic

Figure 2.4

On linear resistor, an electric current is related to voltage by Ohm's law:

$$i = \frac{v}{R} = Gv \tag{2.7}$$

where $G = \frac{1}{R}$ is *conductance* in Siemens [S]. The Ohm's law on linear resistor can be also expressed in the form of the triangle:



Cover the variable you want to find and perform calculation between remaining two variables as indicated.

Example 2.3

Assignment: For the circuit in the Figure 1.8 find current in the circuit, and voltage on both resistors, if $v_1 = 12\,\mathrm{V}$, $R_1 = 4\,\mathrm{k}\Omega$ and $R_2 = 2\,\mathrm{k}\Omega$.

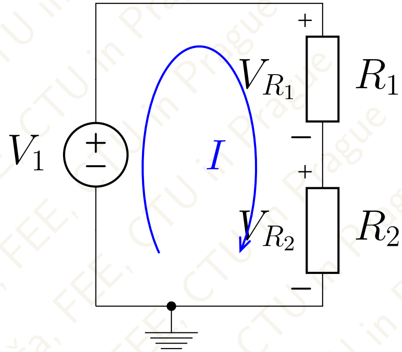
Solution: Using KVL and Ohm's law we can write $v_{R_1} + v_{R_2} - v_1 = 0$ and $R_1 i + R_2 i - v_1 = 0$. We can reqrite this equation as:

$$i = \frac{v_1}{R_1 + R_2} = \frac{12}{4000 + 2000} = 2\,\mathrm{mA}$$

Using Ohm's law, we get resistor voltages as $v_{R_1} = R_1 i = 8\,\mathrm{V}$ and $v_{R_2} = R_2 i = 4\,\mathrm{V}$.

If we substitute Ohm's law 2.6 into Equation (1.8), we get for power on resistor:

$$P = VI \tag{2.7a}$$
$$= RI^2 \tag{2.7b}$$
$$= \frac{V^2}{R}. \tag{2.7c}$$



The power calculation on resistors is very important. Assume, for example circuit in the Figure, where $V = 24\,\mathrm{V}$, $R_1 = 180\,\Omega$ and $R_1 = 180\,\Omega$. First we calculate current in the circuit, $I = \frac{V}{R_1+R_2} = \frac{24}{205} = 123\,\mathrm{mA}$. Using Equation (2.7b):

$$P_{R_1} = R_1 I^2 = 180 \cdot 0.123^2 = 2.727\,\mathrm{W} \qquad P_{R_2} = R_2 I^2 = 15 \cdot 0.123^2 = 0.227\,\mathrm{W}$$

Note, that power, delivered by the voltage source is $P_V = VI = P_{R_1} + P_{R_2} = 2.954\,\mathrm{W}$. If we would like to use Equation (2.7a), or (2.7c), we must first to calculate voltage on both resistors. $V_{R_1} = R_1 I = 22.154\,\mathrm{V}$, and $P_{R_1} = 22.154 \cdot 0.123 = \frac{22.154^2}{180}$, $V_{R_2} = R_2 I = V - V_{R_1} = 1.85\,\mathrm{V}$, and $P_{R_1} = 1.85 \cdot 0.123 = \frac{1.85^2}{15}$. We must use resistors with power rating greater, than calculated power. Resistor R_2 can have power rating 330 mW, while the resistor R_1 requires power rating at least 3 W. If we would use resistor, rated at 330 mW for R_1, it would just burn.

An example of non-linear V-A function can be the Shockley equation,

$$I_D = I_S\left(e^{\frac{V_D}{nV_T}} - 1\right), \tag{2.8}$$

where I_D is current in non-linear circuit element, V_D is voltage accross it, n is emission coefficient (production parameter, the value typically falls into interval 1. . . 2), I_S so called saturation current (given by temperature and fabrication process, very roughly $\approx 10^{-12}$, depends on kind of device) and V_T temperature voltage ($\approx$ 26 mV). Note, that Shockley equation describes voltage-current relation on semiconductor diode (however, this course doesn't deal with semiconductors, just application of KVL is important to us).

We can write KVL equation even for this non-linear terms. Assume that in the Figure 1.8 instead of resistor R_2 is connected a diode with V-A relation according to Equation (2.8). We can rewrite the Shockley equation in the form $V_D = nV_T \ln\left(\frac{I_D}{I_S} + 1\right)$. Then the KVL would be:

$$R_1 i + nV_T \ln\left(\frac{i}{I_S} + 1\right) - v_1 = 0. \tag{2.9}$$

We can, of course solve Equation (2.9), for example, using numerical methods, however, this is beyond the scope of this course. Here I will mention just *linearisation*, because it use only the methods we deal with. It is *piecewise linearization*, see Figure 2.5.

We can make a tangent line at some point of the V-A characteristic of a non-linear resistor. The point is called *operating point*. How to find it, see Thévenin theorem. The tangent line is the blue line in the Figure 2.5. It crosses voltage axis in the point V_T. If we move current axis to this point, the tangent line becomes the V-A characteristic of the resistor (see Figure 2.4c). So, we have voltage V_T – and it can be represented by voltage source, and resistor. Its resistance is inverse slope of the tangent line, $R_D = \frac{\Delta V}{\Delta I} = \frac{V_2 - V_1}{I_2 - I_1}$, using any two points on the tangent line with coordinates (V_1, I_1) and (V_2, I_2). Note, that we can not use this method to large magnitude signals. The voltage can only move in a small area around the operating point, where the difference between the tangent and the actual V-A characteristic is small, see the red voltage waveform and according blue current waveform. It means, that the voltage must have two components - auxiliary DC, which defines operating point, and AC, which is usefull signal (speech, music . . .).

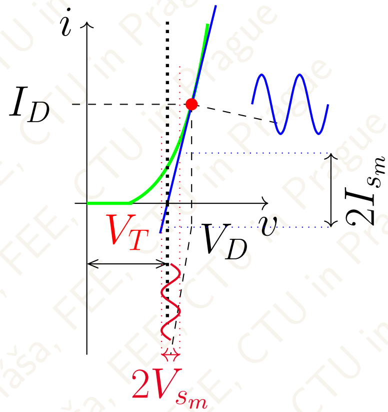


Figure 2.5: Piecewise linearization

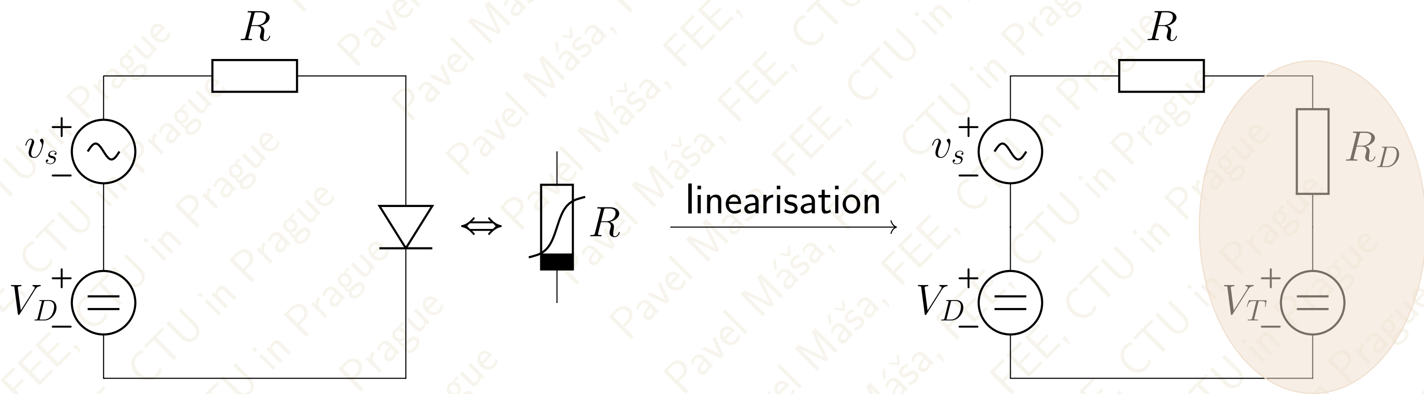


Figure 2.6: Example - linearisation of a diode

In the Figure 2.4a are examples of packages of linear resistors - through hole and SMD. The other two packages contains multiple resistors. In te Figure 2.7 is american symbol of resistor, as well as a few examples of other resistors and resistive devices.

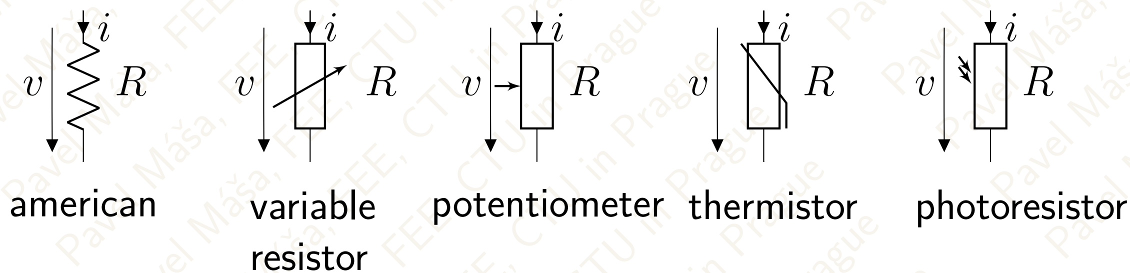
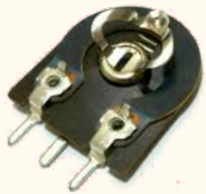
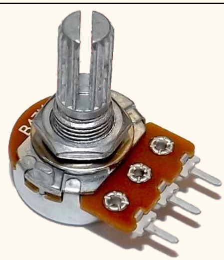
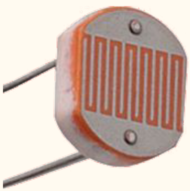


Figure 2.7: Other resistor symbols

variable resistor		Variable resistor has three terminals, instead of two. Between the outer terminals A and C is resistive path with rated, *fixed* resistance R. The middle terminal B is connected to the slider (wiper, brush), which can move between both ends of the resistive path. Between terminals A and B is resistance R_1 and between terminals B and C is resistance $R_2 = R - R_1$. Terminal B acts like output of adjustable voltage divider. Variable resistor is designed to use a screwdriver to set the resistance (usually) one-time. It is used for one-time setting of voltage or resistance in the circuit. It can have more turns between both end for accurate resistance setting.
poten-tiometer		It is similar to the variable resistor, except it does not require the screwdriver. The wiper is connected to the shaft, to which the control knob can be mounted. We can found it for example on AV receivers for volume control.

thermistor		A thermistor is a type of resistor whose resistance is significantly dependent on temperature. There are two kinds of thermistors. With NTC (Negative Temperature Coefficient) its resistance decreases as temperature rises. The NTC resistor is used especially as a temperature sensor. The resistance of PTC (Positive Temperature Coefficient) increases as temperature increases. The PTC thermistors are used as overcurrent protection - if the temperature (which depends on power loss and so on passing current) exceeds given critical value, PTC resistor increases its resistance many times and limits current in the circuit.
photoresistor		A photoresistor is a type of resistor whose resistance is dependent on light intensity. In the dark, it can have a resistance as high as several megohms, in the light just a few hundreds of Ohms. A photoresistor can be applied in light-sensitive detector circuits like in the streetlights to control when the light is on, camera light meters...

2.1.1. Resistors in series

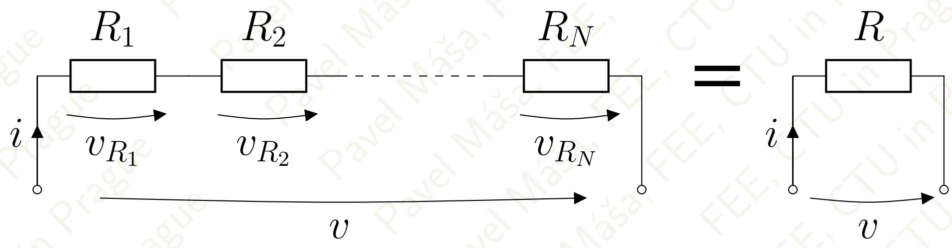


Figure 2.8: Resistors in series

Resistors are in series when the same current flows through them, and if just one terminal connects them. In the Figure 2.8 the voltages across distinct resistors are, according to the Ohm's law $v_1 = R_1 i$, $v_2 = R_2 i \ldots v_N = R_N i$, so we can write:

$$v = R_1 i + R_2 i + \cdots + R_n i = i(R_1 + R_2 + \cdots + R_n) = iR \tag{2.10}$$

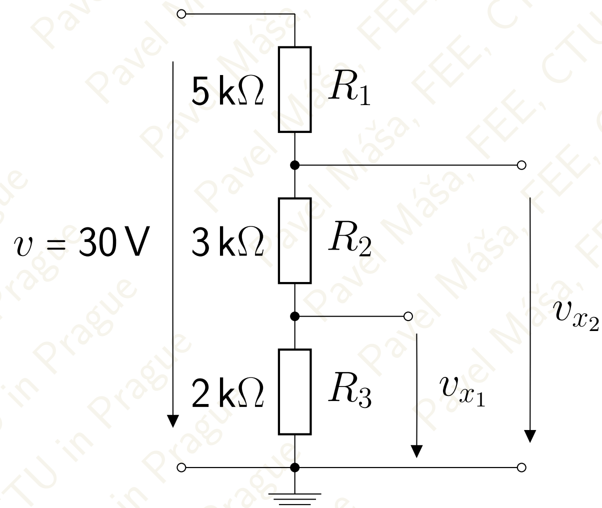
$$\boxed{R = \sum_{i=1}^{n} R_i} \tag{2.11}$$

2.1.1.1 Voltage divider

In the Example 2.3 we use the KVL to find voltage across two resistors in series. Assume that we have N resistors in series, as in the Figure 2.8, the total voltage is v and we need to calculate voltage across one or more resistors. In such a case, it is not necessary to calculate current in the circuit, but we can just use *voltage divider rule*. Using the Equation (2.10),

$$v_{R_j} = R_j i = R_j \frac{v}{R} = v \frac{R_j}{\sum_{i=1}^{N} R_i}. \tag{2.12}$$

Example: For the circuit in the figure find voltages v_{x_1} and v_{x_2}.



Solution:

$$v_{x_1} = v \frac{R_3}{R_1 + R_2 + R_3} = 30 \cdot \frac{2000}{10000} = 6\,\mathrm{V}$$

$$v_{x_2} = v \frac{R_2 + R_3}{R_1 + R_2 + R_3} = 30 \cdot \frac{5000}{10000} = 15\,\mathrm{V}$$

2.1.1.2 Loaded voltage divider

Example: in the Figure 2.9 calculate voltage across resistor R_2 (nodal voltage v_C), if the current $i_{uc} = 5\,\text{mA}$.

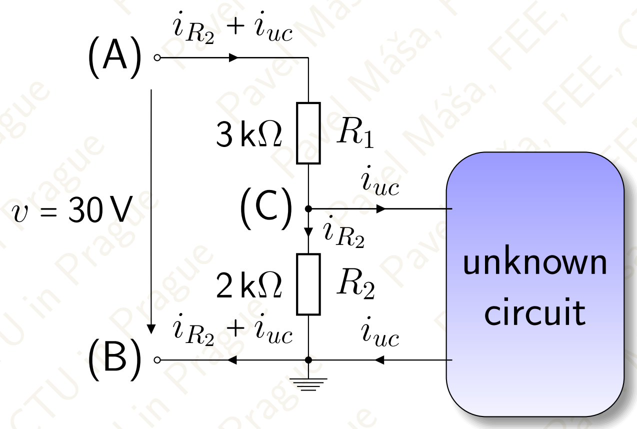


Figure 2.9: Loaded voltage divider

Note: Although the voltage source is not indicated by the symbol in the Figure 2.9, the arrow between terminals A and B shows that a 30 V source is connected there.

Using the Equation (2.12) the solution would be $v_C = v\frac{R_2}{R_1+R_2} = 12\,\text{V}$. However, this value is *not* correct. The voltage divider rule is based on the assumption that the current in the resistors R_1 and R_2 is the same. But it is not the case of this circuit.

It is not important, what the “unknown circuit” actually is, it can be single resistor, or something more complicated. Important is, that in the node (C) the KCL equation is valid:

$$-i_{R_1} + i_{R_2} + i_{uc} = 0 \qquad \Rightarrow \qquad i_{R_1} = i_{R_2} + i_{uc}.$$

This is indicated in the circuit diagram. Of course, also the KVL must be valid, so:

$$v_{R_1} + v_{R_2} - v = 0 \qquad \Rightarrow \qquad R_1(i_{R_2} + i_{uc}) + R_2 i_{R_2} - v = 0.$$

Thus we can find current i_{R_2}:

$$i_{R_2} = \frac{v - R_1 i_{uc}}{R_1 + R_2},$$

and, using Ohm’s law, we get:

$$v_{R_2} = R_2 i_{R_2} = (v - R_1 i_{uc})\frac{R_2}{R_1 + R_2} = (30 - 3000 \cdot 0.005)\frac{2000}{5000} = 6\,\mathrm{V}. \tag{2.13}$$

2.1.2. Resistors in parallel

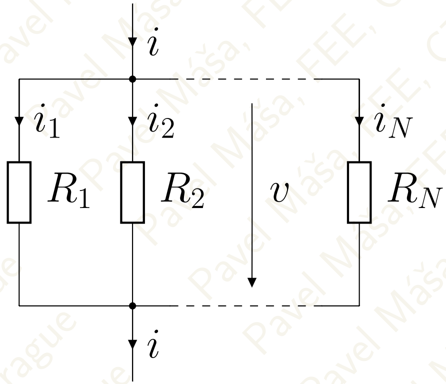


Figure 2.10: Resistors in parallel

Resistors are in parallel when the same voltage is accross all of them, and they are connected by both terminals. In the Figure 2.10 the current i, which enters the node above, divides among all resistors. We can write the KCL $-i + i_1 + i_2 + \cdots + i_N = 0$. All resistors have the same voltage, and each current can be evaluated according to the Ohm's law as $i_1 = \frac{v}{R_1}$, $i_2 = \frac{v}{R_2} \ldots$, which results in:

$$i = i_1 + i_2 + \cdots + i_N = \left(\frac{v}{R_1} + \frac{v}{R_2} + \cdots + \frac{v}{R_N}\right) = \frac{v}{R}. \tag{2.14}$$

From Equation (2.14) we get:

$$\frac{1}{R} = \sum_{i=1}^{n} \frac{1}{R_i} \tag{2.15}$$

or, in terms of conductances

$$G = \sum_{i=1}^{n} G_i \tag{2.16}$$

2.1.2.1 Current divider

In the case of resistors in parallel, we often know the total current flowing into them, and we need to calculate the current flowing through one resistor. This is called the *current divider*. The total voltage in the circuit in the Figure 2.10 is $v = Ri$, and, for example, current $i_1 = \frac{v}{R_1} = i\frac{R}{R_1} = i\frac{G_1}{G}$. Thus, the current divider equation is:

$$i_j = i\,\frac{G_j}{\sum_{i=1}^{N} G_i} = i\,\frac{G_j}{G} \tag{2.17}$$

Note, that in the case of two parallel resistors only, sometimes the direct calculation is used:

$$i_{R_1} = i\,\frac{G_1}{G_1+G_2} = i\,\frac{\frac{1}{R_1}}{\frac{1}{R_1}+\frac{1}{R_2}} = i\,\frac{\frac{1}{R_1}}{\frac{R_1+R_2}{R_1R_2}} = i\frac{R_2}{R_1+R_2}. \tag{2.18}$$

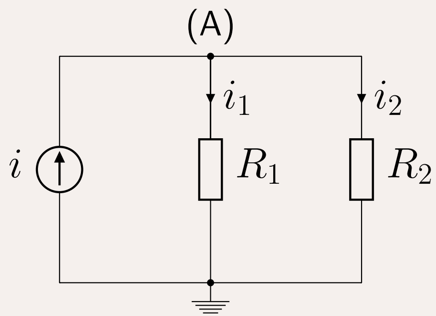
However, this rule is a risky, because in the numerator of the fraction is a *resistor, whose current we do not calculate*. In the case of three resistors in parallel, the direct rule is more complicated, since

$$i_{R_1} = i\frac{R_2R_3}{R_1R_2+R_1R_3+R_2R_3}. \tag{2.19}$$

With increasing number of resistors, the rule is more and more complicated. For this reason I recommend to use conductances, and Equation (2.17).

Example 2.4

Assignment: For the circuit in the figure below find nodal voltage v_A and currents i_1 and i_2, if $i = 10\,\mathrm{mA}$, $R_1 = 3\,\mathrm{k\Omega}$ and $R_2 = 7\,\mathrm{k\Omega}$.



Solution:

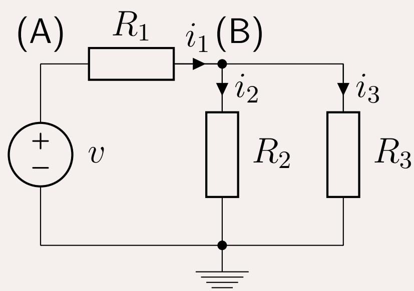
$$i_1 = i\,\frac{G_1}{G_1 + G_2} = i\,\frac{\frac{1}{R_1}}{\frac{1}{R_1} + \frac{1}{R_2}} = i\,\frac{R_2}{R_1 + R_2} = 0.01 \cdot \frac{7000}{10000} = 7\,\mathrm{mA}.$$

$$i_2 = i - i_1 = 10 - 7 = 3\,\mathrm{mA}.$$

$$v_A = Ri = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} = \frac{R_1 R_2}{R_1 + R_2} = \frac{21 \cdot 10^6}{10000} \cdot 0.01 = 21\,\mathrm{V}.$$

Example 2.5

Assignment: For the circuit in the figure below find nodal voltage v_B and currents i_1 and i_2, if $v = 20\,\mathrm{V}$, $R_1 = 800\,\Omega$, $R_2 = 2\,\mathrm{k}\Omega$ and $R_3 = 3\,\mathrm{k}\Omega$.



Solution: First we have to calculate the total resistance of the circuit. Resistors R_2 and R_3 are in parallel, and their total resistance R_{23} is in series with resistor R_1. *Resistors R_1 and R_2, or resistors R_1 and R_3 are* not *in series, even they are connected just by single terminal, because the currents passing these resistors is different*. Total resistance is:

$$R = R_1 + \frac{R_2 R_3}{R_2 + R_3} = 800 + \frac{2000 \cdot 3000}{5000} = 2\,\mathrm{k}\Omega.$$

Now we can calculate currents:

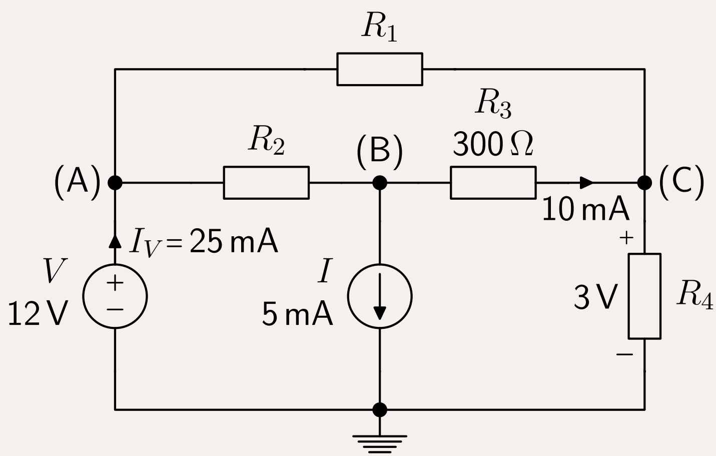
$$i_1 = \frac{v}{R} = \frac{20}{2000} = 10\,\mathrm{mA}, \qquad i_2 = i_1 \frac{G_2}{G_2 + G_3} = 6\,\mathrm{mA}, \qquad i_3 = i_1 - i_2 = 4\,\mathrm{mA}.$$

$$v_B = v - R_1 i_1 = 20 - 800 \cdot 0.01 = 12\,\mathrm{V}, \qquad v_B = R_2 i_2 = R_3 i_3$$

Example 2.6

Assignment: In circuit in the figure below were measured currents I_3 and I_V, as indicated in the figure.

1. Compute currents I_1, I_2 and I_4, which flows through resistors R_1, R_2 and R_4.
2. Voltage source has value $V = 12\,\mathrm{V}$, $R_3 = 300\,\Omega$. The voltage across resistor R_4 was measured as $V_4 = 3\,\mathrm{V}$. Compute resistance of other resistors in the circuit and voltage at node (2).
3. Determine powers, delivered to the circuit by both sources.



Solution: Using the KCL in the node (2):

$$-I_2 + I + I_3 = 0 \quad \rightarrow \quad I_2 = I + I_3 = 5 + 10 = \underline{\underline{15\,\mathrm{mA}}}.$$

Now we can use KCL in node (1):

$$-I_U + I_2 + I_1 = 0 \quad \rightarrow \quad I_1 = I_U - I_2 = 25 - 15 = \underline{\underline{10\,\mathrm{mA}}},$$

and finally in the node (3):

$$-I_1 - I_3 + I_4 = 0 \quad \rightarrow \quad I_4 = I_1 + I_3 = 10 + 10 = \underline{\underline{20\,\mathrm{mA}}}.$$

We know nodal valtages in nodes (1) and (3), so we can find device voltage on resistor R_1 and, using Ohm's law,

$$R_1 = \frac{V_1 - V_3}{I_1} = \frac{12 - 3}{0.01} = \underline{\underline{900\,\Omega}}. \qquad R_4 = \frac{V_3}{I_4} = \frac{3}{0.02} = \underline{\underline{150\,\Omega}}$$

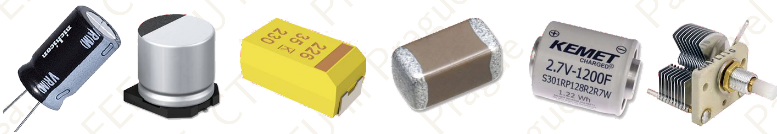
$$V_2 = V_3 + R_3 I_3 = 3 + 300 \cdot 0.01 = \underline{\underline{6\,\mathrm{V}}} \qquad R_2 = \frac{V_1 - V_2}{I_2} = \frac{12 - 6}{0.015} = \underline{\underline{400\,\Omega}}$$

$$P_V = V I_V = 12 \cdot 0.025 = \underline{\underline{300\,\mathrm{mW}}}, \qquad P_I = -I V_2 = 0.005 \cdot 6 = \underline{\underline{-30\,\mathrm{mW}}}$$

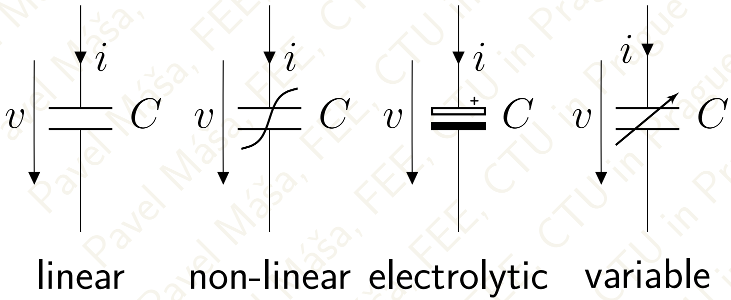
2.2. Capacitor

Definition

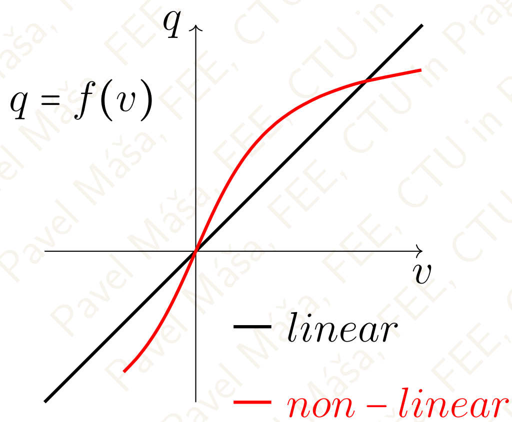
A capacitor is a passive two-terminal electrical component that stores electrical energy in an electric field. It is inertial circuit element, because voltage-current relation is time dependent.



(a) Examples of capacitor packages



(b) Capacitor symbols



(c) Capacitor V-Q characteristic

Figure 2.11

Capacitor consists of two parallel conductive plates of the area A, between them is dielectric - an insulator with high permitivity ϵ. Thickness of dielectric is d. Electrons or other charge carriers can not pass through the dielectric.

Let's do now again a short excursion into physics. It brings to us neat explanation of capacitor principle. However, the physical principle is beyond the scope of this course and not required on the exam.

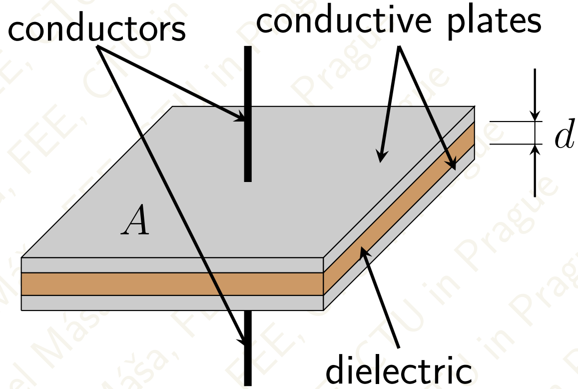


Figure 2.12: Construction of capacitor

The Gauss' law states, that the net electric flux through any hypothetical closed surface is equal to $\frac{1}{\epsilon}$ times the net electric charge within that closed surface. It has integral form

$$\Phi_E = \oiint_S \vec{E}\vec{n}\,\mathrm{d}A = \frac{Q}{\epsilon}, \tag{2.20}$$

and differential form $\nabla\vec{E} = \frac{\rho}{\epsilon}$, where ρ is electric charge density (charge per volume). In the case of two parallel planes as in the case of the capacitor, where the vector of electric field intensity $\vec{E}$ is oriented (in ideal case) perpendicularly to the planes and solution of the integral in the equation 2.20 is simply $EA + EA = \frac{\rho A}{\epsilon} = \frac{Q}{\epsilon}$, where ρ is area charge density (charge per area). In the equation are two terms EA, because the electric flux passes two planes. Now recall our definition of electric voltage, $V = \varphi_1 - \varphi_2$. On planes will be charges of the same size, but opposite sign, so also area charge densities are $+\rho$ and $-\rho$. The voltage is then

$$V = Ed = \frac{\rho}{\epsilon} d = \frac{d}{\epsilon A} Q. \tag{2.21}$$

From this equation we can write for the charge:

$$\boxed{Q = CV \qquad (2.22)}$$

where

$$\boxed{C = \frac{\epsilon A}{d} \qquad (2.23)}$$

is *capacitance*.

symbol:	C
unit:	Farad (F)
SI standard system base units:	kg^{-1} m^{-2} s^{4} A^{2}

$\epsilon = \epsilon_r \epsilon_0$ is *permitivity*.
$\epsilon_0 = \frac{1}{4\pi \cdot 10^{-7} c^2} \approx 8.85410^{-12}$F m^{-1} is the physical constant, called *vacuum permitivity*, or permittivity of free space or electric constant.
ϵ_r is relative static permittivity (of the real media). Capacitor dielectric has large relative permitivity.

Consider the following circuit:

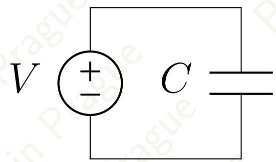


Figure 2.13

In this circuit, we have DC voltage source connected to the capacitor. Since no free charge carriers flows in dielectric, and the voltage is constant, no current flows in the circuit. The capacitor acts like *open circuit* (no closed loop). Of course, before the circuit reached this *steady state*, voltage source delivered to the capacitor charge $Q = CV$.

Now consider following circuit in the Figure 2.14. In this circuit, we have DC current source connected to the capacitor. Even the dielectric does not carry any free charge carriers, in the circuit flow electric current I, which delivers charge $Q = It$ to the capacitor. *The capacitor stores charge*. The voltage in the circuit increases:

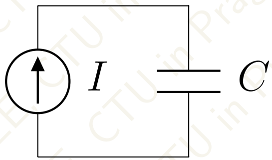


Figure 2.14

$$v(t) = \frac{It}{C} \qquad (2.24)$$

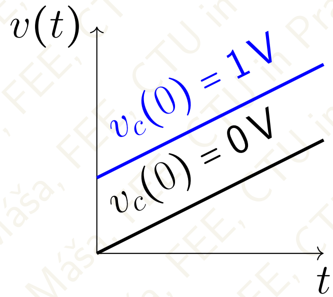


Figure 2.15

Before the current source was connected to the capacitor, a charge could be stored in the capacitor. In such a case the charging would not start from zero voltage, but initil voltage $v_C(0) = \frac{Q}{C}$. In the Figure 2.15 the blue line represents example of charging from initial voltage 1 V.

Of course, the current can vary in time. In such a case, the charge, delivered in the capacitor, is defined by the Equation (1.1), and we got an essential equation, defining relation between voltage and current on capacitor:

$$v(t) = \frac{q(t)}{C} = \frac{1}{C}\int_0^t i(\tau)\,\mathrm{d}\tau + v_c(0) \tag{2.25}$$

From equation 2.25 we can write for current on capacitor:

$$i(t) = C\frac{\mathrm{d}v(t)}{\mathrm{d}t} \tag{2.26}$$

Example 2.7

Assignment: The capacitor is excited from the AC current source. Circuit is the same as in the Figure 2.14, except the current source is sinusoidal $i(t) = I_m \sin(\omega t)$. Calculate the voltage on capacitor.

Solution:

$$v(t) = \frac{1}{C}\int_0^t I_m \sin\omega\tau\,\mathrm{d}\tau + v_c(0) =$$

$$= \frac{I_m}{\omega C}(1 - cos\omega t) + v_c(0) = \frac{I_m}{\omega C}\sin\left(\omega t - \frac{\pi}{2}\right) + \frac{I_m}{\omega C} + v_c(0)$$

Solution of the Example 2.7 will be important to us.

- We can see, that the waveform is the same sine function, but it is *phase shifted* by $\frac{-\pi}{2}$.
- The magnitude is frequency-dependent, by the term $\frac{1}{\omega C}$.
- The solution contains DC component $\frac{I_m}{\omega C}$ and, if the capacitor had some charge - and so voltage before connection to the current source, there will be also this initial voltage $v_C(0)$ component.

If we would ignore phase shift and DC component, we could write $V_m = I_m \frac{1}{\omega C}$. The waveform represents two voltage sources – see Figure 1.15 and in the section superposition we will see, that we can remove DC source and solve AC component independently. In the section AC analysis we will introduce the name reactance for the term $\frac{1}{\omega}C$ and impedance will allow calculations of the phase shift without integrating and differentiating.

From the solution of the example 2.7 we can introduce important rule:

Rule: 2.1

- If $\omega \to 0$, the current in a capacitor $\to 0$, even if the voltage is *not* zero, so the capacitor acts as resistor with resistance $\to \infty$; *the open circuit*.
- If $\omega \to \infty$, then $\frac{1}{\omega C} \to 0$ and so $v \to 0$, even if the passing current is *not* zero, so the capacitor acts as ideal wire – *the short circuit*.

The Equations (2.25) and (2.26) lead to a very important conclusion:

Rule: 2.2

- The voltage on the capacitor is *always continuous*.
- The current flowing through the capacitor is generally *not continuous*.

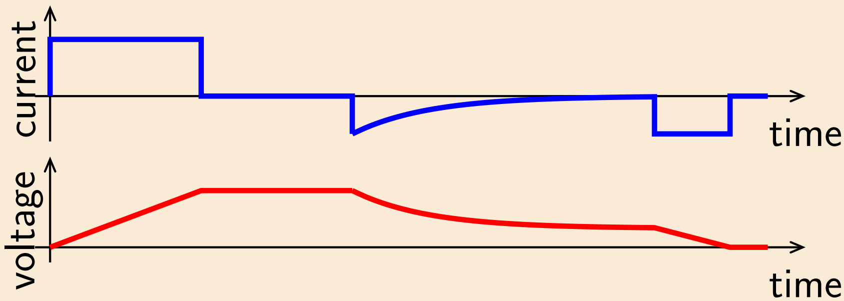


Figure 2.16: Example of current and voltage on a capacitor

The current delivers charge to the capacitor – and stores the energy in it. We can disconnect the source, and capacitor keeps the charge (and energy) – and keeps constant voltage. If we then connect, e.g., resistor to the capacitor, the capacitor acts as the source (it has voltage – and energy) and delivers current to the resistor – the energy is converted to heat in the resistor. So, the voltage, related to the charge, must be continuous (an integral in 2.25), but current not (time rate of voltage in 2.26).

2.2.1. Energy stored in the capacitor

Recall the definition of the power – power is rate of energy transfer – and, also, the electric voltage – the voltage between points A and B is equal to the work W which would have to be done, per unit charge Q, to move the charge from A to B. The energy, stored in the capacitor, is the same, as the work that the source did by supplying the charge Q to the capacitor.

$$\begin{aligned} W_C(t) &= \int_0^t v(\tau) i(\tau)\, \mathrm{d}\tau = \int_0^{q(t)} v(q)\, \mathrm{d}q \\ &= \int_0^{q(t)} \frac{q}{C}\, \mathrm{d}q = \frac{1}{2C} q^2(t) = \frac{1}{2} C v^2(t) \end{aligned} \tag{2.27}$$

For constant voltage, the Equation (2.27) has the form:

$$\boxed{W_C = \frac{1}{2} C V^2} \tag{2.28}$$

Example 2.8

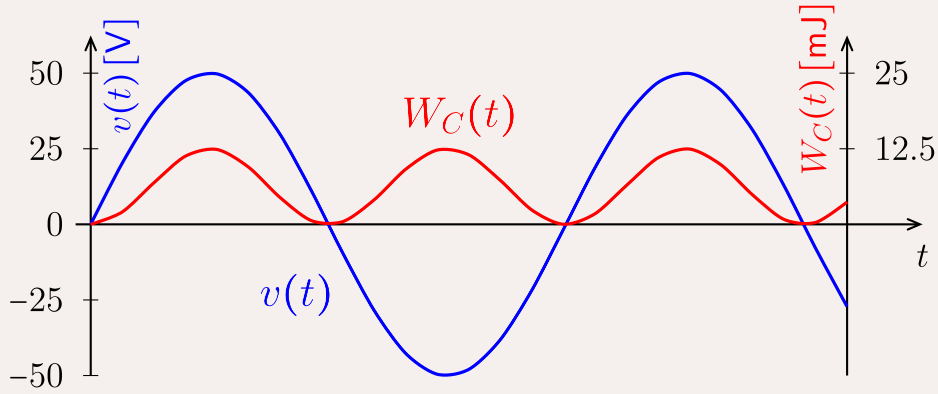
Assignment: Calculate the energy, stored in the capacitor $C = 10\,\mu\text{F}$, if the voltage across capacitor $V = 50\,\text{V}$. **Solution:**

$$W_C = \frac{1}{2} \cdot 10^{-5} \cdot 50^2 = 12.5\,\text{mJ}$$

Example 2.9

Assignment: Find the energy, stored in the capacitor $C = 10\,\mu\text{F}$, if the voltage across capacitor $V = 50\sin(314t)\text{V}$. **Solution:**

$$W_C(t) = \frac{1}{2} \cdot 10^{-5} \cdot 50^2 \sin^2(314t) = 12.5\sin^2(314t)\text{mJ}$$



2.2.2. Capacitors in parallel

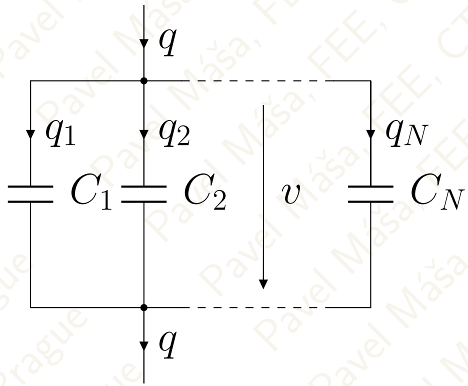


Figure 2.17: Capacitors in parallel

The charge Q divides among all capacitors, so $Q = Q_1 + Q_2 + \cdots + Q_n = CV$. All capacitors have the same voltage V. We can write:

$$Q = VC_1 + VC_2 + \cdots + VC_n = V(C_1 + C_2 + \cdots + C_n) = VC. \tag{2.29}$$

And, the total capacitance C of parallel connection is:

$$C = \sum_{i=1}^{n} C_i \tag{2.30}$$

We can also simply find the same result from the Equation (2.23). If we connect capacitors in parallel, and we assume the distances of plates are the same, the areas of distinct plates are added together, so the total capacitance would be $C = \epsilon \frac{A_1 + A_2 + \cdots + A_N}{d}$.

2.2.3. Capacitors in series

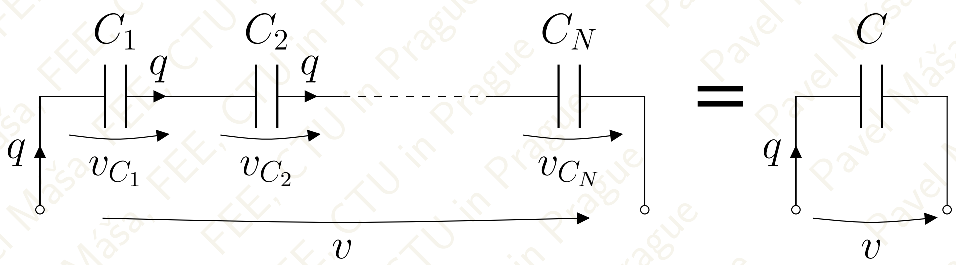


Figure 2.18: Capacitors in series

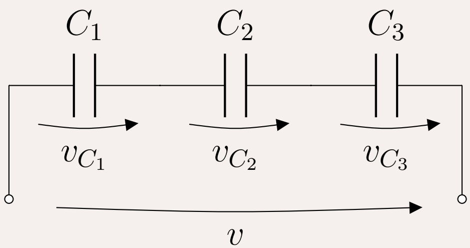
Capacitor constists of two conductive plates, which has the same charge of opposite sign. So, if current delivers negative charge (distinct number of electrons) to the left plate of the capacitor C_1, the same number of electrons must leave the right plate of the capacitor C_1, and the same charge, which charges capacitor C_1 charges also capacito $C_2, \ldots, C_N$. So, all capacitors have the same charge Q. However, since the capacitors are in series, the total voltage divides among all capacitors. We can write:

$$
\begin{aligned}
V &= V_1 + V_2 + \cdots + V_n = \\
&= \left(\frac{Q_1}{C_1} + \frac{Q_2}{C_2} + \cdots + \frac{Q_n}{C_n}\right) = Q\left(\frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}\right) = Q\frac{1}{C}
\end{aligned}
$$

$$
\boxed{\frac{1}{C} = \sum_{i=1}^{n} \frac{1}{C_i} \qquad (2.31)}
$$

Example 2.10

Assignment: In the circuit in the figure calculate coltages across all capacitors, if $v = 100\,\mathrm{V}$, $C_1 = 1\,\mu\mathrm{F}$, $C_2 = 2\,\mu\mathrm{F}$ and $C_3 = 3\,\mu\mathrm{F}$.



Solution: Since the capacitors have the same charge, $Q_1 = Q_2 = Q_3 = Q$, and $Q = VC$, we need just total capacitance:

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \qquad C = 0.\overline{54}\,\mu\mathrm{F}.$$

$$V_1 = \frac{CV}{C_1} = 54.\overline{54}\,\mathrm{V}, \qquad V_2 = \frac{CV}{C_2} = 27.\overline{27}\,\mathrm{V}, \qquad V_3 = \frac{CV}{C_3} = 18.\overline{18}\,\mathrm{V}.$$

2.2.4. Applications of capacitors

2.2.4.1 Coupling capacitor

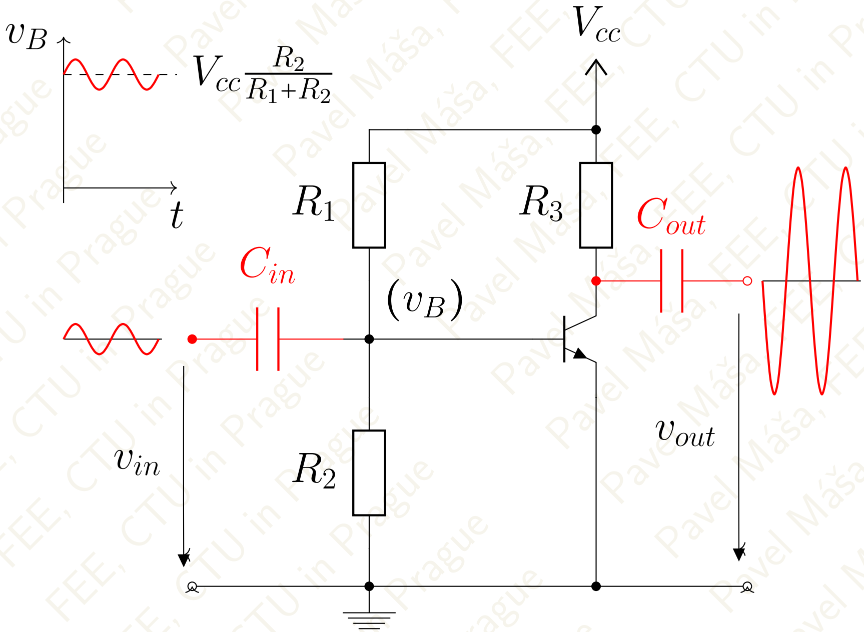


Figure 2.19: Example of coupling capacitors

Resistors R_1 and R_2 makes a DC voltage $V_{cc}\frac{R_2}{R_1+R_2}$. However, source of input signal voltage v_{in} must be detached from this voltage. Capacitor C_{in} makes exactly this thing. The DC current can not pass capacitor dielectric, but AC current can (the reactance is $\frac{1}{\omega C}$). The same function has the capacitor C_{out}.

2.2.4.2 Decoupling capacitor

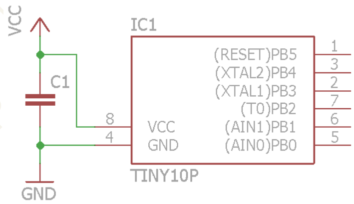


Figure 2.20: Example of decoupling capacitors

It is connected between power supply pins of the digital ICs. It does not pass DC between power supply and ground. However, digital ICs produce high-frequency noise, and the capacitor has very low reactance at high frequencies. In this way, it shorts the high-frequency noise, so it can not spread away from the chip. Decoupling capacitors are connected as close as possible to minimize high-frequency current loops. The decoupling capacitors of typical capacitance 100 nF are combined with a larger one which are reservoirs of charge to supply the instantaneous power requirements of the chips.

2.2.4.3 Energy reservoirs

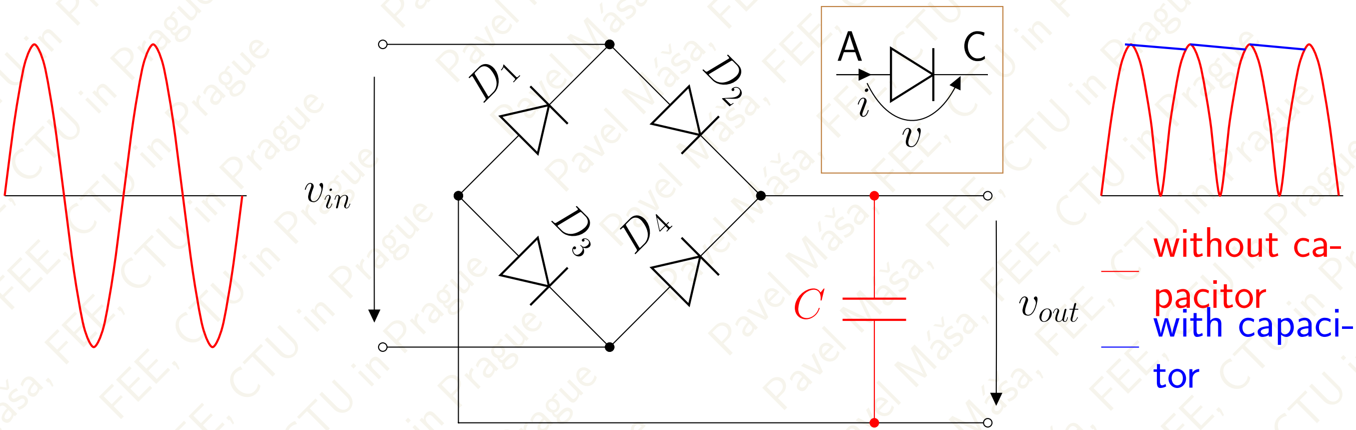


Figure 2.21: Example of capacitor as energy reservoar

For this example we simplify the diode in Figure 2.21 such that the diode is electrically conductive, if the positive voltage is applied to the anode (A), and it does not conduct if we apply the negative voltage to the anode. *Assymetric nonlinear characteristic in Figure 2.4c is example of V-A characteristic, and Shockley equation 2.8 describes exponential part, but for the sake of this example it is not important.*

In the figure is Graetz bridge - a simple rectifier. If the positive voltage v_{in} is in the marked direction at the input terminals, diodes D2 and D3 are electrically conductive, while the diodes D1 and D4 do not conduct. The positive voltage is on the output terminals. In the second half of period, when the negative voltage v_{in} is at the input terminals, diodes D1 and D4 conduct, and diodes D2 and D3 does not. However; the negative voltage in the marked direction means, that the potential is greater at the lower input terminal – and on output is again positive voltage.

Without capacitor, the voltage is pulsing, according to the red waveform. However, if we connect capacitor C, the capacitor charges, and it delivers energy to the load, connected to output terminals, while the source voltage is lower than the peak one.

The accumulated energy in the capacitor is $W_C = \frac{1}{2}CV^2$. The energy, delivered into the load between two voltage peaks (half of sine period) is $W_l = VI\frac{T}{2}$. Of course, as the energy in the capacitor decreases, voltage drops. For a given circuit it has the certain allowable ripple, expressed in a per cents. For example, if the ripple $r = 1\,\%$, we get an equation $0.01W_C = W_l$. The voltage is known, as well as current $(I = \frac{V}{R})$ and period (10 ms for 50 Hz voltage), so we can find required capacitance.

2.2.4.4 Switched capacitors

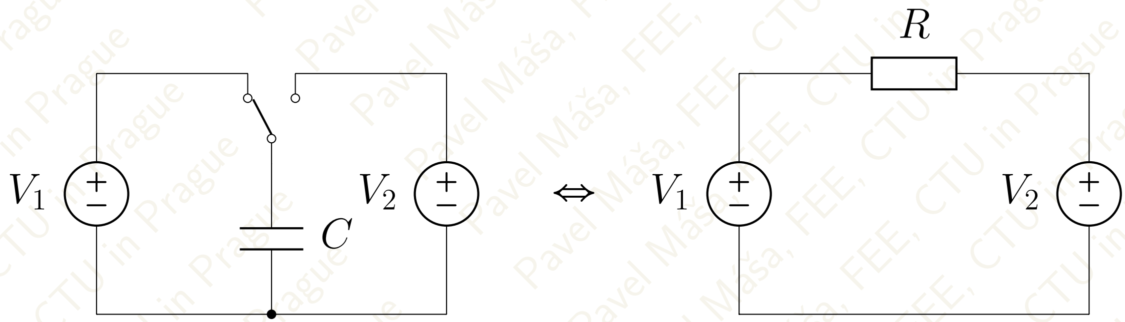


Figure 2.22: Principle of the switched capacitor

If the capacitor in the Figure 2.22 is connected by the switch to the voltage source V_1, it has charge $Q_1 = CV_1$. If it is connected to the voltage source V_2, it has charge $Q_2 = CV_2$. If the difference between both voltages is $\Delta V = V_1 - V_2$, the charge difference is $\Delta Q = Q_1 - Q_2 = C\Delta V$. Assume that the swicth periodically switches between V_1 and V_2, with switching period T. Each second it transfers the charge $Q_s = \frac{\Delta Q}{T}$. In the circuit in the right side of the Figure 2.22 flows the current $I = \frac{V_1 - V_2}{R}$. The current I is the charge, which is transferred from the source V_1 to the source V_2 each second, so if

$\frac{\Delta Q}{T} = \frac{C(V_1-V_2)}{T} \frac{V_1-V_2}{R}$,

$$R = \frac{T}{C} = \frac{1}{fC}.$$

It means that we can replace the resistor with the capacitor. It makes sense in integrated circuits where it is a problem to produce a resistor with an exact resistance value. The deviation is many tens of percent. Therefore, the resistance value in the IC filters must be set by the laser trimming, which is an expensive process. On the other hand, it is possible to produce capacitors with small value deviation.

2.2.4.5 Frequency filtering

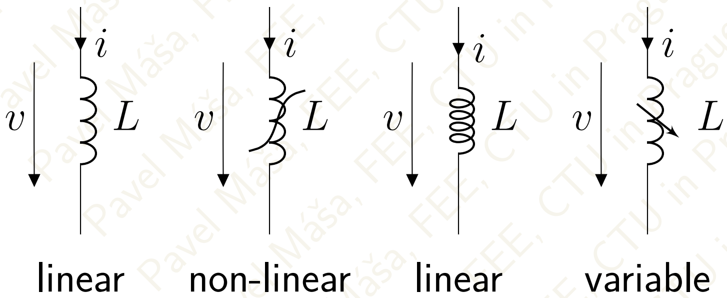
In the section Frequency response we will see, that circuits, which contains capacitors and inductors, pass voltage of distinct frequencies, but blocks other. Low pass, high pass, pass band and stop band filters are significant in radio electronic, biomedical engineering, and many other branches.

2.3. Inductor

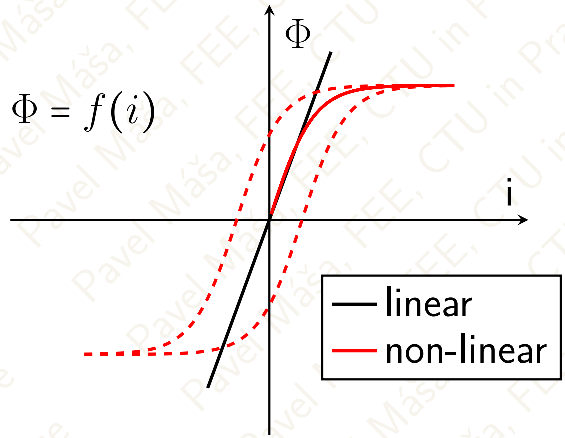
Definition

An inductor is a passive two-terminal electrical component that stores electrical energy in a magnetic field. It is inertial circuit element, because voltage-current relation is time dependent.

(a) Examples of inductor packages



(b) Inductor symbols



(c) Inductor A-W characteristic

Figure 2.23

An inductor consists of a wire. It is, usually, arranged in turns, as can be seen in the first and the fourth package body in the Figure 2.23a. Here I recapitulate a few basic terms from physics, necessary to explain what the inductance is.

The Lorentz force in physics defines a force, affecting the charge q in the presence of an electric field $\vec{E}$, and magnetic field $\vec{B}$, as

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) = q\vec{E} + q|\vec{v}||\vec{B}|\sin(\theta)\vec{n}. \tag{2.32}$$

where $\times$ is the vector cross product and $\vec{n}$ is the normal vector – of unit length and direction of the force.

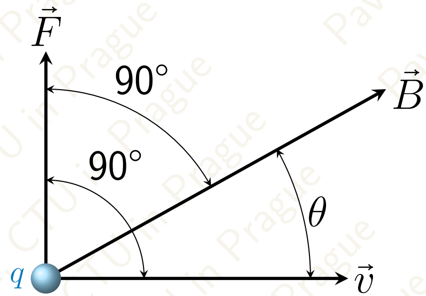


Figure 2.24: Lorentz force – magnetic component

The Equation (2.32) defines *vector of magnetic induction* (magnetic flux density) $\vec{B}$. The unit is tesla (Tesla), or $\mathrm{Wb\,m^{-2}}$, in SI base units $\mathrm{kg\,s^{-2}\,A^{-1}}$.

However, we are more interested in the case, when a stright conductor carry a current i. If we place a freely-suspended magnet in the vicinity of the conductor, the Lorentz force will acting on the conductor and the magnet. Or, if we place two conductors, carrying the current i close to each other, the force will act on both conductors.

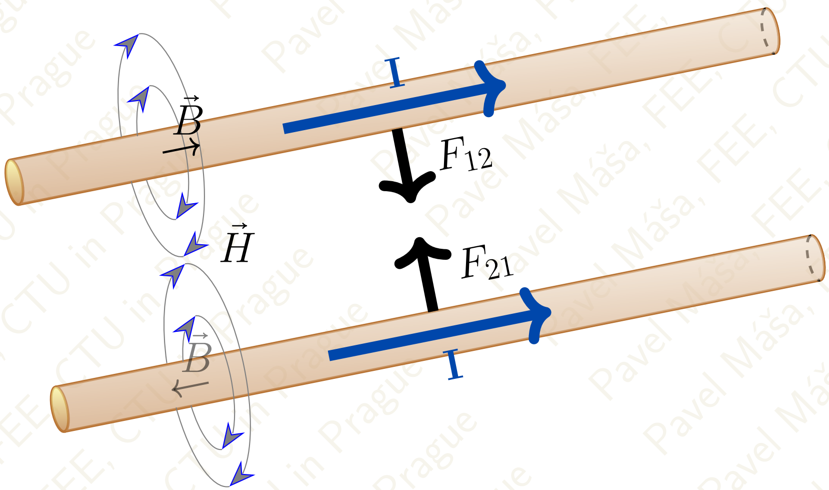


Figure 2.25: Force between two conductors

How are two conductors related to the Lorentz force, and the Equation (2.32), where is no current, but charge q and velocity v? In the Figure 2.3 we have illustration of electric current as a moving charge in the conductor. In the case of conductor, we must sum up effect all conductance electrons. The total charge of conductance electrons is in some volume V is $Q = nVq$, where n is conduction electrons density and q is the unit charge. Putting it into the Equation (2.32) we get for force of magnetic field

$$\vec{F}_m = nVq\vec{v} \times \vec{B}. \tag{2.33}$$

However, the term $nq\vec{v}_d = \vec{J}$, the current density 2.1. So, we can write $F_m = V\vec{J} \times \vec{B}$. Considering that $V = Al$, where l is a length of conductor and $\vec{J} = \frac{\vec{I}}{A}$, we can finally write

$$\vec{F}_m = l\vec{I} \times \vec{B}. \tag{2.34}$$

The vector of magnetic induction $\vec{B}$ is related to *magnetic field intensity* $\vec{H}$ by the relation

$$\vec{B} = \mu\vec{H}, \tag{2.35}$$

where $\mu = \mu_r\mu_0$ is *permeability*. The unit of permeability is $\mathrm{H\,m^{-1}}$. $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{H\,m^{-1}}$, $\mu_0 = \frac{1}{\epsilon_0 c^2}$. The relative permeability μ_r is dimensionless quantity determining the ratio between permeability of the material and permeability of space. It can be less than 1, in the case of diamagnetic materials (e.g. copper, silicon, gold, zinc, brass, graphite, …), greater than one in the case of paramagnetic materials (e.g. platinum, alkali metals), or many times greater than one (even ¿ 10000) in the case of ferromagnetic and ferrimagnetic materials (e.g. iron, nickel, cobalt, ferrite materials, …).

Assume the case, when we arrange the conductor into the loop with N turns – *the coil*. The Ampere's law in such a case describes the relationship between electric current and magnetic field intensity as

$$\oint_C \vec{H}\,\mathrm{d}l = Ni. \tag{2.36}$$

The direction of $\vec{H}$ can be determined by the right-hand rule – if you take the wire in such a way, that the thumb points in the direction of current, and other fingers curl around the wire, than the magnetic field circles in direction of curled fingers.

The vector of magnetic induction $\vec{B}$ is related to the *magnetic flux* by the equation:

$$\vec{\Phi} = \vec{B} \cdot \vec{A}, \tag{2.37}$$

where $\vec{A}$ is the area.

However, to us is most important Faraday's law of induction. Faraday discovered, that if we place an inductor into the magnetic field, which *varies in time*, then an electromotive force is produced:

$$e = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}, \tag{2.38}$$

If we arrange the conductor into N turns (we make the coil), than $e = -N\frac{\mathrm{d}\Phi}{\mathrm{d}t}$. This is fundamental operating principle of inductors (and transformers as their special case), electrical motors, generators and others. The product $N\Phi$ is called *flux linkage* λ and it represents the superposition rule – the same flux Φ passes all turns of the coil, and in each turn is induced voltage, according to the Exuation (2.38), so we sum up voltages from all turns. We can rewrite the Faraday's equation for a coil in the form:

$$e = -\frac{\mathrm{d}\lambda}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}i}\frac{\mathrm{d}i}{\mathrm{d}t}. \tag{2.39}$$

Assuming the sign convention according to the Figure 2.23b, we can finally introduce the fundamental equation, describing relation between voltage and current on inductor:

$$\boxed{v = L\frac{\mathrm{d}i}{\mathrm{d}t},} \tag{2.40}$$

where

$$L = \frac{\lambda}{i} \tag{2.41}$$

is self *inductance*. The unit is Henry [H].

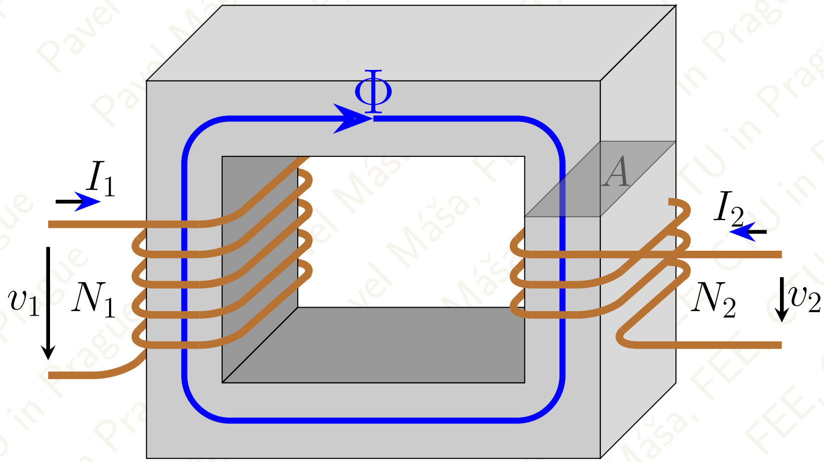


Figure 2.26: Two windings with the same flux

In the Figure 2.26 is an example of a coil with two windings. The silvered structure is the core, made from material with high relative permeability, for example Si iron. The magnetic flux, in ideal case, flows in the core and is the same in both windings (in practice, part of the flux leaves the core – leakage). The flux linkeages are $\lambda_1 = N_1\Phi$ and $\lambda_2 = N_2\Phi$. If the coil on the left – primary winding – carry current i_1, the Faraday's equation for the secondary winding will be:

$$e_2 = -\frac{\mathrm{d}\lambda_2}{\mathrm{d}t} = -\frac{\mathrm{d}\lambda_2}{\mathrm{d}i_1}\frac{\mathrm{d}i_1}{\mathrm{d}t} \tag{2.42}$$

and, assuming the sign convention, we can write:

$$v_2 = M_{21}\frac{\mathrm{d}i_1}{\mathrm{d}t}, \tag{2.43}$$

where M_{21} is *mutual inductance*. If the secondary winding carries current i_2, the voltage, induced on the primary winding is:

$$v_1 = M_{12}\frac{\mathrm{d}i_2}{\mathrm{d}t}. \tag{2.44}$$

The total voltage on primary and secondary sides will be superposition of self-induced voltage and mutual-induced voltage, so:

$$\boxed{v_1 = L_1\frac{\mathrm{d}i_1}{\mathrm{d}t} \pm M_{12}\frac{\mathrm{d}i_2}{\mathrm{d}t}, \qquad v_2 = L_2\frac{\mathrm{d}i_2}{\mathrm{d}t} \pm M_{21}\frac{\mathrm{d}i_1}{\mathrm{d}t}. \qquad (2.45)}$$

The sign in the Equation (2.45) can be both plus and minus – if the secondary winding would be wound in opposite direction, than the sign would be minus. Or, if the current i_2, forced by external source would be opposite, the sign would be also minus.

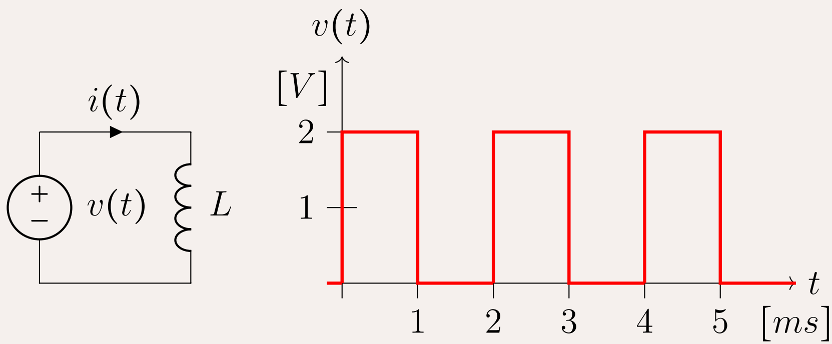
Equation (2.40) defines, how to calculate voltage on inductor, if the current is known. If we know the voltage, we can calculate current using the following equation:

$$i = \frac{1}{L}\int_0^t v(\tau)\,\mathrm{d}\tau + i(0), \qquad (2.46)$$

where $i(0)$ is *initial* current in the inductor - at zero time, before we apply voltage on the inductor.

Example 2.11

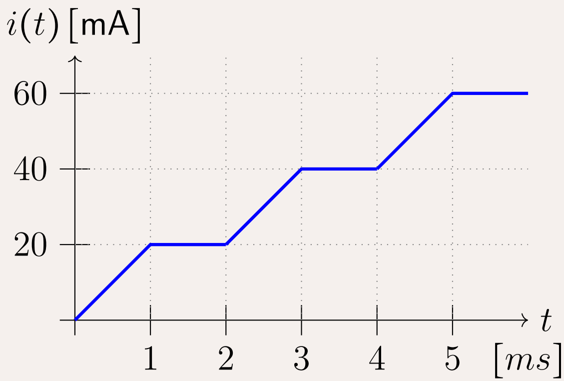
Assignment: To the inductor L we connect voltage source $v(t)$ with waveform according to the figure below. Find current, which flows in the circuit, if $L = 0.1\,\mathrm{H}$ and $i(0) = 0\,\mathrm{A}$.



Solution: If $t \in (0,1)$ms, $v = 2$ V, and current

$$i(t) = \frac{1}{0.1}\int_0^t 2\,\mathrm{d}\tau = 20\,[\tau]_0^t = 20t\ \mathrm{A}.$$

Note that the current is function of time. At $t = 1$ ms, $i = 20 \cdot 0.001$ A. In the interval $t \in (1,2)$ms, $v = 0$ V. However, the current is *not* zero. The term $i(0)$ in the Equation (2.46) is the initial current in this time interval. We can move origin to the time instant of 1 ms and $i(t) = \frac{1}{0.1}\int_0^t 0\,\mathrm{d}\tau + 0.02 = 0.02$ A. The current is *constant*. In the next time interval $t \in (1,2)$ms the initial current is again $i(0) = 20$ mA and $i(t) = \frac{1}{0.1}\int_0^t 2\,\mathrm{d}\tau + 0.02 = 0.02 + 20t$ A, $t \in (0,1)$ ms.



Example 2.12

Assignment: The inductor is excited from the AC voltage source. Circuit is the same as in the Figure 2.11, except the voltage source is sinusoidal $v(t) = V_m \sin(\omega t)$. Calculate the current in the inductor.

Solution:

$$i(t) = \frac{1}{L}\int_0^t V_m \sin\omega\tau \, \mathrm{d}\tau + i_L(0) =$$

$$= \frac{V_m}{\omega L}(1 - cos\omega t) + i_L(0) = \frac{V_m}{\omega L}\sin\left(\omega t - \frac{\pi}{2}\right) + \frac{V_m}{\omega L} + i_L(0)$$

Solution of the Example 2.12 will be important to us, like in the case of Example 2.7 in the case of capacitor.

- We can see, that the waveform of the current is the same sine function, but it is *phase shifted* by $\frac{-\pi}{2}$.
- The magnitude is frequency-dependent, by the term $\frac{1}{\omega L}$.
- The solution contains DC component $\frac{I_m}{\omega L}$ and, if the inductor had some flux - and so current before connection to the voltage source, there will be also this initial current $i_L(0)$ component.

If we would ignore phase shift and DC component, we could write $I_m = V_m \frac{1}{\omega L}$, or

$V_m = I_m \omega L$. The waveform represents two components – AC and DC – and according to the superposition we can remove DC component and solve AC component independently. In the section AC analysis we will introduce the name reactance for the term ωL and impedance will allow calculations of the phase shift without integrating and differentiating.

From the solution of the example 2.12 we can introduce important rule:

Rule: 2.3

- If $\omega \to 0$, the voltage on inductor $\to 0$, even if the current is *not* zero, so the inductor acts as resistor with resistance $\to 0$ – *the short circuit*.
- If $\omega \to \infty$, then $\omega L \to \infty$ and so $i \to 0$, even if the voltage is *not* zero, so the inductor acts as *the open circuit*.

2.3.1. Energy stored in the inductor

$$W_L = \frac{1}{2} L I^2 \qquad (2.47)$$

The Equations (2.46) and (2.40) lead to a very important conclusion:

Rule: 2.4

- The current flowing through the inductor is *always continuous*.
- The voltage on the inductor is generally *not continuous*.

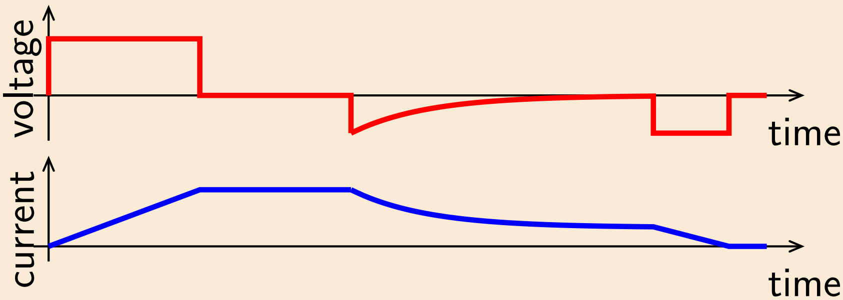


Figure 2.27: Example of current and voltage on an inductor

The voltage on the inductor is a time rate of current. It means, that if the current does not change, the voltage must be zero, even the current is not zero – see Example 2.11. If we connect the voltage source to the inductor as in the Example 2.11, current increases. The energy, stored in the inductor increases. However, we can not simply disconnect the source, because the current flowing through the inductor is continuous – and there is not any other circuit element. So, the resistance between inductor terminals would $\rightarrow \infty$ and the voltage, according to the Ohm's law $V = RI = \infty \times I$. In practice, a spark, or arc would appear between inductor's terminals.

2.3.2. Inductors in series

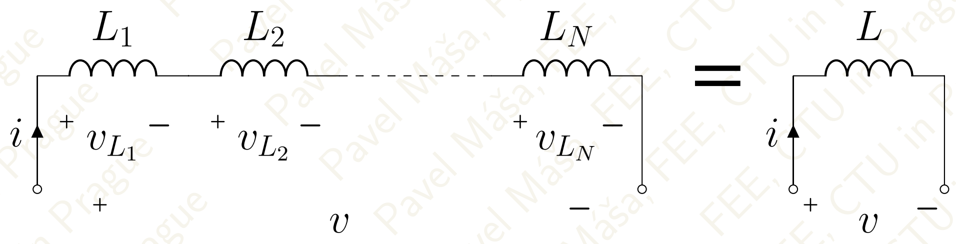


Figure 2.28: Inductors in series

All inductors have the same current, the total voltage is, according to the KVL the sum of voltages across distinct inductors. Using the Equation (2.40),

$$L\frac{\mathrm{d}i}{\mathrm{d}t} = L_1\frac{\mathrm{d}i}{\mathrm{d}t} + L_2\frac{\mathrm{d}i}{\mathrm{d}t} + \cdots + L_N\frac{\mathrm{d}i}{\mathrm{d}t} = (L_1 + L_2 + \cdots + L_N)\frac{\mathrm{d}i}{\mathrm{d}t}. \tag{2.48}$$

From this Equation:

$$\boxed{L = \sum_{i=1}^{n} L_i.} \tag{2.49}$$

2.3.3. Inductors in parallel

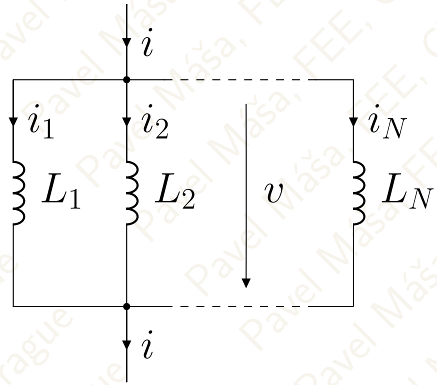


Figure 2.29: Inductors in parallel

All inductors have the same voltage. The total current, according to the KCL is $i = i_1 + i_2 + \cdots + i_N$ and, using the Equation (2.46):

$$\frac{1}{L}\int_0^t v(\tau)\,\mathrm{d}\tau = \frac{1}{L_1}\int_0^t v(\tau)\,\mathrm{d}\tau + \frac{1}{L_2}\int_0^t v(\tau)\,\mathrm{d}\tau + \cdots + \frac{1}{L_N}\int_0^t v(\tau)\,\mathrm{d}\tau = \left(\frac{1}{L_1} + \frac{1}{L_2} + \cdots \frac{1}{L_N}\right)\int_0^t v(\tau)\,\mathrm{d}\tau. \quad (2.50)$$

From this Equation:

$$\frac{1}{L} = \sum_{i=1}^{n} \frac{1}{L_i}. \tag{2.51}$$

2.3.4. Applications of inductors

2.3.4.1 Transformer

The operating principle of transformer is indicated in the Figure 2.26. It has two or more windings with common magnetic flux. The windings have N_1, N_2, ... turns. Because the induced voltage is related to the flux linkage $\lambda = N\Phi$, and Φ is the same, then if the number of turns in distinct windings is different, the transformer works as *AC / AC voltage converter*, which usually changes one AC voltage level to another. If N_1 is number of turns in primary winding, and N_2 number of turns in the secondary one, and we connect voltage v_1 to the primary winding, the voltage v_2 in the case of ideal and lossless transformer would be

$$v_2 = v_1 \frac{N_2}{N_1} = \frac{v_1}{n},$$

where $n = \frac{N_1}{N_2}$ is the *turns ratio*. The transformer is an essential part of power grid network for delivering electricity from producers to consumers, because it is not possible to deliver high power to high distances at low voltages. Imagine, for example, that the power consumption of a city is 1 GW. If the net voltage is 230 V, the current in electrical network would be 4 347 826 A. If a wire, used for electric power distribution has resistance $R = 23.1\,\mathrm{m}\Omega/\mathrm{km}$ and the length of the wire would be 100 km, the power loss in the wire would be $P = RI^2 = 4.35 \times 10^{13}$ W. If the voltage is 400 kV, the current decreases at 2500 A and power loss on 1.44×10^{7} W.

Since the power conservation law must be valid, and so $V_1 I_1 = V_2 I_2$ and

$$i_2 = i_1 \frac{N_1}{N_2} = n i_1.$$

If we connect resistor to the secondary winding, $R_2 = \frac{V_2}{I_2} = \frac{V_1 \frac{1}{n}}{I_1 n} = \frac{1}{n^2}\frac{V_1}{I_1} = \frac{R_1}{n^2}$. So, the transformer also converts a resistance. This is used for impedance matching of antennas, tube amplifiers, and so on.

2.3.4.2 Safety transformer

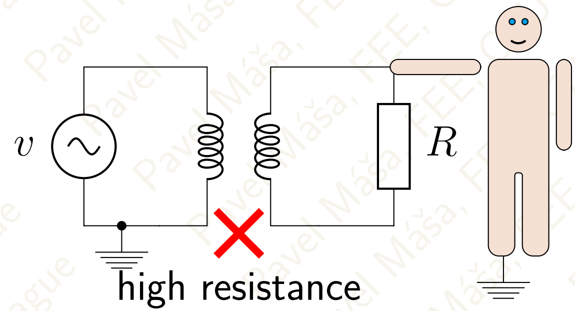


Figure 2.30: Safety (isolation) transformer

Isolation transformers provide galvanic isolation. The primary and secondary side is not galvanically connected, and there is very high resistance between them. DC can not pass from primary to the secondary side, as well as voltage spikes (sudden increase in voltage that lasts for less than three nanoseconds) and surges (sudden increase in voltage that lasts for three nanoseconds or more), which can have the magnitude of thousands of volts. It also blocks interference caused by ground loops (if we connect ground terminals of two electrical devices, they can have different electric potential due to voltage drops on resistances, which causes noise, hum, and interference, e.g., in audio, video, and computer systems). Since primary and secondary side is not connected, the isolation transformer protects against electric shock, because loop, in which the current would flow through the human body is cut off by the transformer. So, if human touches only single wire on the isolated side, no current flows, so it protects against electrical shock. For this reason, the isolation transformer is used in hospital isolated power systems.

2.3.4.3 Energy reservoar – DC/DC voltage converters and constant-current regulators

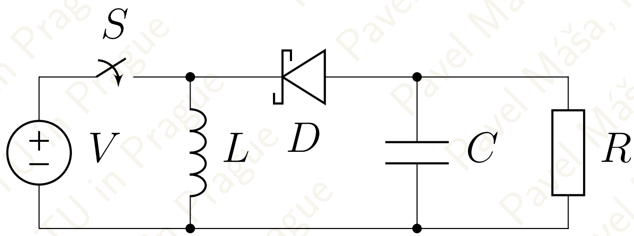


Figure 2.31: Principle of operation of the inverting DC/DC converter

The "S" switch is periodically switched on and off in the circuit. When the switch is on, the voltage source is connected in parallel to the inductor L, exactly as in the Example 2.11. The current in the inductor increases. Diode D is connected in the reverse direction, so it does not conduct, and effectively disconnects capacitor C and resistor R from the source. When the switch is off, it disconnects voltage source from the inductor. We know from the Rule 2.4, that the current flowing through the inductor is continuous. Because it can not flow through the switch, it must flow through the diode. The induced voltage, according to the Faraday's equation 2.38 is negative, so the diode is in the forward direction and conducts. So, the current, which flows through the diode, and charges capacitor C, which works as energy reservoar, when the switch is on, and flows through the resistor R, where makes the voltage Ri_L.

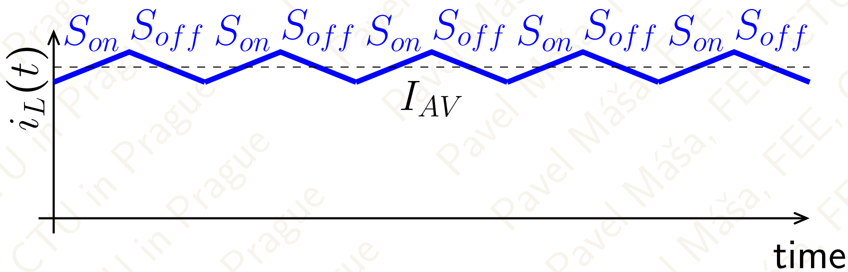


Figure 2.32: Current in the inductor (steady state)

2.3.4.4 Frequency filtering

2.4. Electrical device

An electrical device is any equipment used to electric power generation, transformation, transport, distribution or utilization. We use this term for complex devices like mobile phones, televisions, motors, and generators, as well as relatively simple things as antennas, batteries, but even simple circuit elements like resistors, capacitors, transistors, etc. We use this term even for bimetallic strips, jacks, plugs, connectors.

2.5. Electrical circuit

An electronic circuit is composed of individual electronic components, such as resistors, transistors, capacitors, inductors, and diodes, connected by conductive wires. So, an electronic circuit is a more complex electrical device, and consist of the elementary electrical devices (resistors . . .). We can categorize electrical circuits on analog circuits, digital circuits, or mixed-signal circuits (a combination of analog circuits and digital circuits). The connection of electronic components is given in the circuit diagrams.

The *functional circuit diagram* describes the connection of individual components. For example, the Figure 2.33 instructs us, that we should take a light bulb of given parameters, battery, few wires and a switch – and if we connect it as indicated, we will have a flashlight. On the other hand, from this functional circuit diagram, we can not directly calculate current in the circuit, and other circuit variables. For this purpose, we replace functional circuit diagram by its *circuit model*.

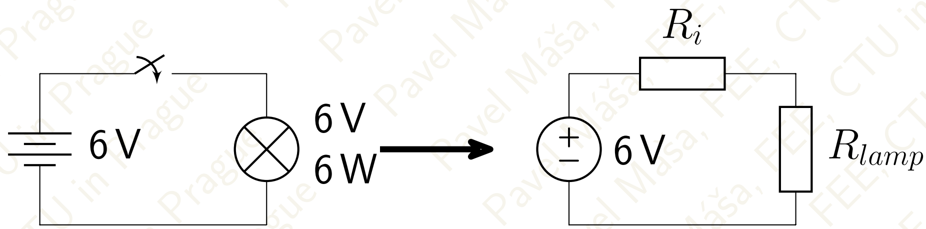


Figure 2.33: Example of functional circuit diagram and its model

2.6. Circuit model of electrical device

The (functional) circuit diagram instruct us, how to connect circuit elements to make some electrical circuit. We use given electrical devices. However, such devices may be quite complex, and it is not possible to find circuit variables easily. Instead of describing an electrical device with often quite complex equations, we replace this device with a circuit, that contains ideal circuit elements and behaves as closely as the original device.

In the Figure 2.33 the battery is represented by an ideal voltage source and resistor

R_i, and light bulb by resistor R_{lamp}. Since resistance is temperature dependent, R_{lamp} is different for a model of the cold lamp and the shining one.

In the Figure 2.34 is circuit diagram of LED constant current regulator. It instructs us, that we should take a few capacitors – an electrical devices, like on the Figure 2.11a, one inductor – like on the Figure 2.23a, resistors – like on the Figure 2.4a and IC circuit, and we make the regulator. And, if we obey the design rules of the designer, it will work. However, the analysis of the circuit can be a little bit complicated. Fortunatelly, software packages like Cadence or Mentor Graphics can help us with this task.

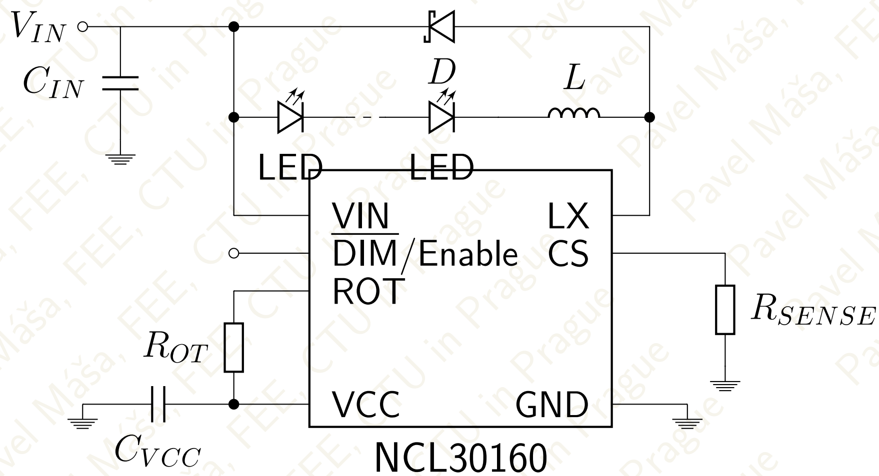


Figure 2.34: LED constant current regulator

What can be a little surprising, the symbols of capacitor, inductor and resistor in (functional) circuit diagram represents distinct electrical devices, however, a model of, e.g. inductor device, consists of *ideal* inductor, capacitor and resistor. When we calculate circuit variables, we always consider the symbols like 2.4b, 2.11b and 2.23b as ideal circuit elements, as defined in sections 2.1, 2.2 and 2.3.

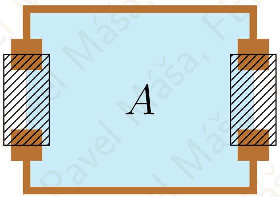
2.6.1. Circuit model of a wire

An ideal wire has zero resistance, zero inductance, zero capacitance.

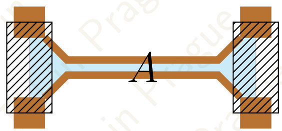
However, in a real world, if the wire is not a superconductor, it has some resistance. Trained electricians, when installs electrical sockets, are usually not interested in wire resistance. They obey the standards, which requires, e.g., for 10 A circuit breaker at least 1.5 mm^2 cross-section of the wire and for 16 A circuit breaker at least 2.5 mm^2 cross-section of the wire. If the wire with less cross-section would be used, it could lead to the fire hazard.

When the charge moves, it makes magnetic field – so each conductor, even a superconductor, is tied to inductance. It has two consequences. The noise can be induced from other wires – and the wire can emits strong electromagnetic interferences. Also, the Equation (2.40) tells us, that the voltage on inductor (and even on a wire) is related to the time rate of the current, $\frac{\mathrm{d}i}{\mathrm{d}t}$. Large $\frac{\mathrm{d}i}{\mathrm{d}t}$ (e.g., rectangular current, which raises very fast) can cause large voltage induced just on a few meters of a wire, which can even destroy sensitive electronic devices. For this reason, inputs of electronic devices should be protected from overvoltage.

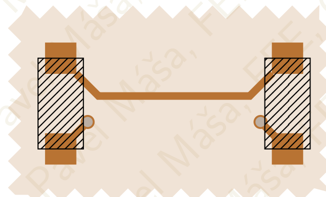
In the Figure 2.35a is an example of the wrong arrangement of conductors on a PCB. The conductors make a loop with a large area A – the inductance (and induced voltage) is related to the magnetic flux – and magnetic flux on the area – see Equations (2.37), (2.38). If the wires are close to each other, it decreases electromagnetic radiation and interferences 2.35b. The best solution is to make a two layer PCB, and have a ground plane (large area of copper foil) on one side, and another conductor on the other side 2.35c.



(a) Single layer PCB - large area

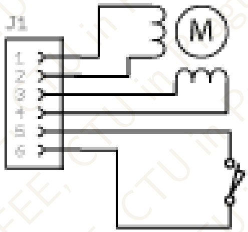


(b) Single layer PCB - small area

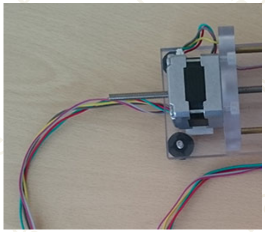


(c) Two layer PCB – ground plane

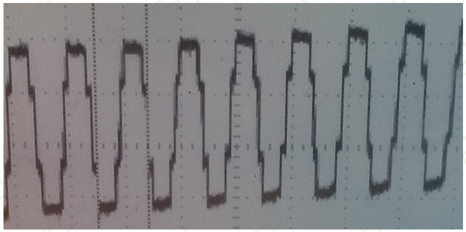
In the Figure 2.36b is another example of problems, related to wiring. There is a device, fitted with the stepper motor and a limit switch. Into the stepper motor flows large current of an almost rectangular waveform. The power lines, as well as the wires from the limit switch, are guided in one bundle of parallel wires. Into wire from a limit switch is induced the voltage from power lines – see Figure 2.36c. The connected CPU has evaluated this interference as a valid limit switch signal, which leads to fault operation.



(a) Connection of stepper motor



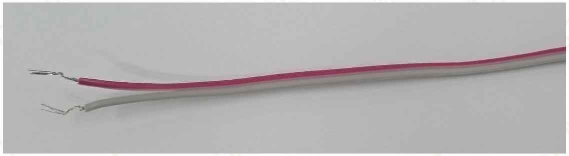
(b) Realization - how not to do it



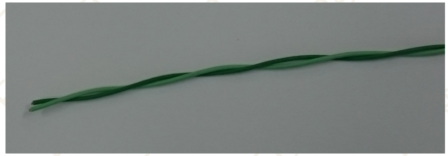
(c) Open – circuit waveform on limit switch

In the Figure 2.37a is a parallel wire, and in the Figure 2.37b twisted-pair line. Although they appear to be almost the same, there is one big difference – twists on twisted-pair. The parallel wire has a relatively large area, exposed to external magnetic field, so it is very susceptible to disturbances from other wires and other sources of a magnetic field (induced voltage is directly proportional to affected area).

We can rewrite the Equation (2.37) to the form $\Phi = BA\cos\theta$, where θ is the angle between the magnetic field lines and the normal (perpendicular) to area A. In the case of a twisted-pair line, the angle varies periodically between 0 and $180°$. So $\cos\theta$ ranges from 1 to -1 and so any disturbance from other wires and magnetic field sources is canceled very effectively.



(a) Parallel wire



(b) Twisted-pair line

In addition to using a twisted pair, spatial separation of power and signal wires, and shielding can solve interference in Figure 2.36b.

Another problem is frequency dependence of the resistance of the wire – not related to wire inductance and impedance. The Equation (2.4) is valid only at very low frequencies, as for alternating current there is a phenomenon of skin effect. An alternating electric current in the conductor causes an alternating magnetic field, which, in the opposite, creates an electric field, which opposes the applied electric current in the conductor. This opposing electric field is called "counter-electromotive force" (back EMF). The back EMF is not constant in the whole cross sectional area. It opposes most strongly in the center of the wire, and this effect decreases to the surface of the wire. In result, the current density J drops exponentially from the surface towards the center. There is the skin depth parameter, which defines the depth below the surface of the conductor at which the current density has fallen to approx. 37 %. The skin depth is defined as

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}}. \tag{2.52}$$

where

f	is frequency of the AC current
μ	magnetic permeability
σ	$\sigma = \frac{1}{\rho}$ electrical conductivity

and, e.g. for the circle-shaped copper wire it is $\delta_{75\,\mathrm{Hz}} = 7.8\,\mathrm{mm}$, $\delta_{1\,\mathrm{kHz}} = 2.1\,\mathrm{mm}$. The resistance of wire increases with frequency of the AC current and the wire impedance (complex resistance) per unit length is

$$Z_i = R_{dc} \frac{m r_0 I_0(m r_0)}{2 I_1(m r_0)} \tag{2.53}$$

where $m = \sqrt{j\omega\mu_0\sigma} = \frac{1+j}{\delta}$, I_0 and I_1 are Bessel functions of the first kind and order of 0 and 1. R_{dc} is a DC resistance according to the Equation (2.4).

Figure 2.38: Skin effect - AC resistance dependence of 2.83 mm diameter wire

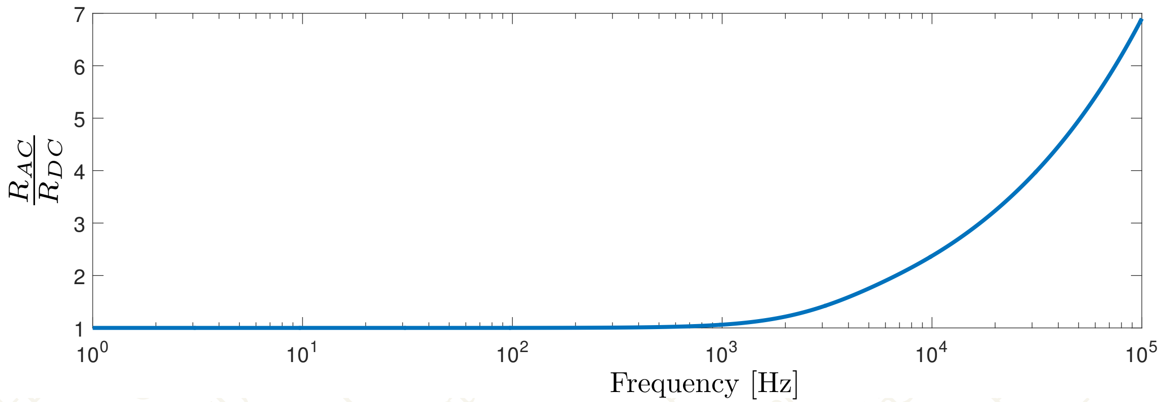
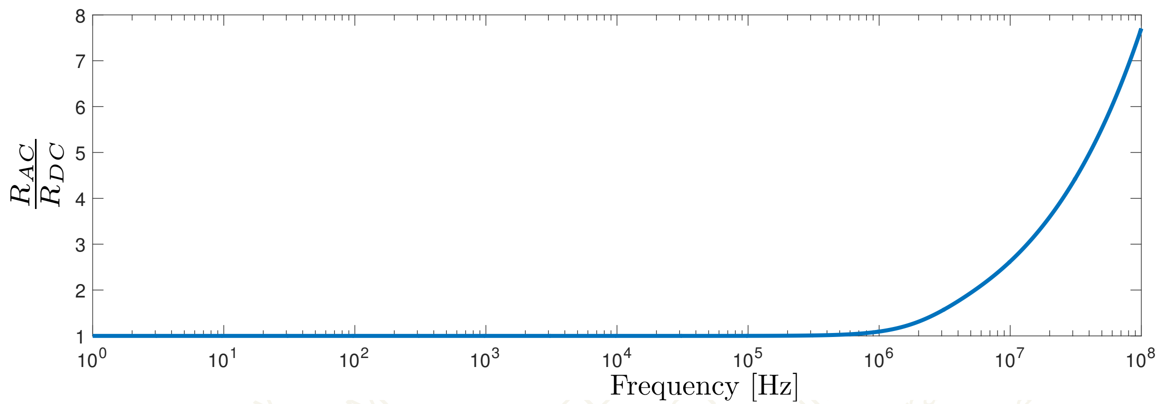


Figure 2.39: Skin effect - AC resistance dependence of 0.1 mm diameter wire



It is evident, that as wire diameter decreases, a critical frequency at which the AC resistance start to raise, also increases. This phenomenon is the reason, why stranded wires, consisting of many thin wires, are used.

Figure 2.40: Stranded wire

So, if we can not consider wire as an ideal one, we can use model in the Figure 2.41. It represents Joule heating, and voltage induction due to $\frac{\mathrm{d}i}{\mathrm{d}t}$.

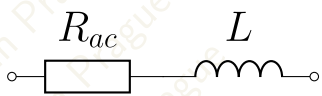


Figure 2.41: Wire model – low frequency

However, the model in the Figure 2.41 does not consider, that between pair of wires is also capacitance, and the insulation is not always perfect, so it can have some non zero conductance G. So, the model of a wire is modified':

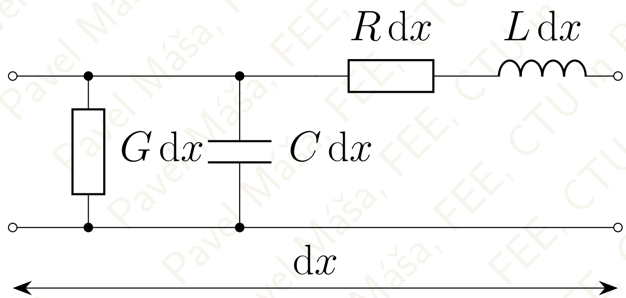


Figure 2.42: Wire model – one element of the length $\mathrm{d}x$

Since the velocity of electricity is limited,

$$v = \frac{c}{\sqrt{\mu_r \epsilon_r}} \tag{2.54}$$

we can not use single element from the Figure 2.42. In the Figure 2.43 we can see simulation of propagation of the rectangular pulses in a wire, modeled by ten elements.

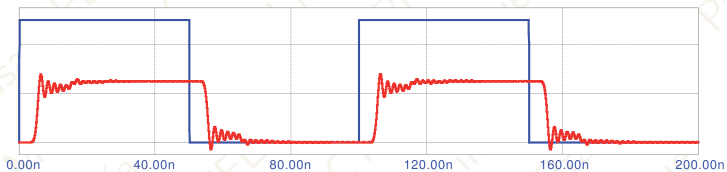


Figure 2.43: Simulation - 10 MHz pulses at wire loaded by 50 Ω resistors on both sides - 10 elements model

The wire causes time delay

$$t_d = \sqrt{LC} \tag{2.55}$$

Inductance and capacitance are related to magnetic and electric field, and energy is transfered as an electromagnetic wave. It can be proved, that the ratio of voltage and current in the wire – which is resistance – at lossless ($R = 0\,\Omega$, $G = 0\,\mathrm{S}$) wire is

$$R_0 = \sqrt{\frac{L}{C}}. \tag{2.56}$$

TV antenna coaxial cable has $R_0 = 75\,\Omega$, twisted-pair cables (UTP internet, USB, ...) have $R_0 = 90\,\Omega \ldots 120\,\Omega$. When on the end of the cable is connected resistor R_L of different resitance than R_0, part of energy reflects back, just like the echo in the valley. The amount of reflected voltage is defined by the voltage reflection coefficient

$$\rho_V = \frac{R_L - R_0}{R_L + R_0}. \tag{2.57}$$

An example of reflections on a wire is in the Figure 2.44.

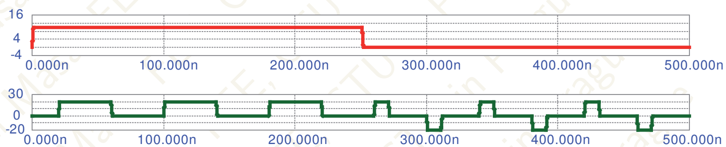


Figure 2.44: Reflections on the transmission line

To prevent reflections, it is necessary to connect resistor on the receiver side, like in the Figure 2.45. This is necessary on computer busses, if the pulse rising time $t_r < 2t_d l$, where l is length of the bus wire. On TV antenna cables, terminating resistor prevents TV ghosting.

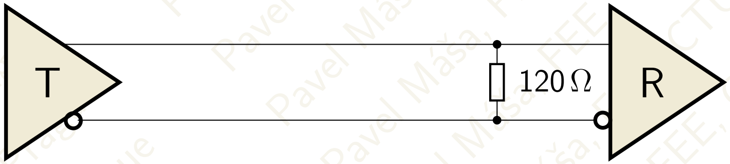


Figure 2.45: Terminating resistor

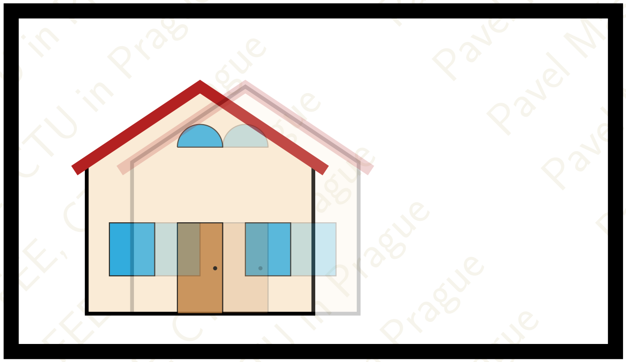


Figure 2.46: TV Ghosting

2.6.2. Circuit model of a capacitor

2.6.2.1 Ideal capacitor – DC steady state

In DC circuits, *if the circuit is in a steady state*, an ideal capacitor can be simply removed from the circuit, because a dielectric can not conduct an electric current.

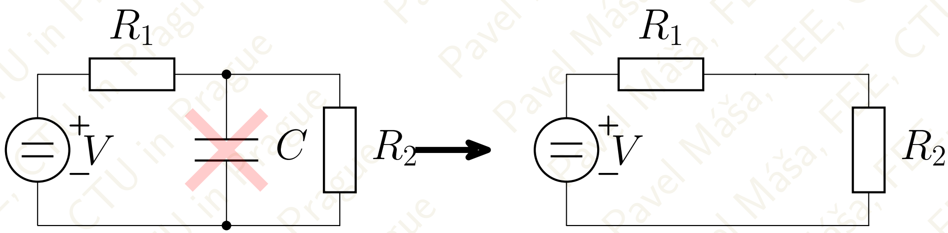


Figure 2.47: Ideal capacitor in DC steady state can be removed from the circuit

An exception is charging the capacitor from the current source – see Figure 2.14. However, this is true only in the circuit according to the Figure 2.14. If in the circuit will be a resistor, connected in parallel to the capacitor, the current in the steady state flows through the resistor and the voltage on the capacitor will not grow infinitely long, but it will settle on the value $v_C = Ri$. The capacitor does not affect this voltage (however, it is charged at this voltage – and stores energy, according to the Equation (2.28)).

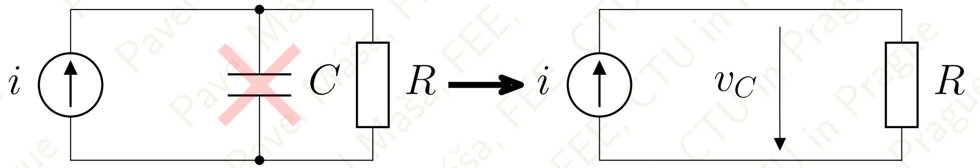


Figure 2.48: Ideal capacitor in DC steady state

However, if the circuit is not in a steady state, the capacitor in given time instant acts as a voltage source. Suppose, for example, that we turn on the switch in the circuit. In the moment, when we turn on the switch, there is still zero charge in the capacitor (since the capacitor was discharged in the resistor R_2) – and so in the node (A) is zero voltage. The current $i_1 = \frac{V - v_C}{R_1} = \frac{V}{R_1}$ is not affected by the resistor R_2, because the nodal voltage is determined by the capacitor voltage. However, we use this analogy only at the beginning of the transient process – and during charging resistor R_2 *must* be included in calculation (see transients.

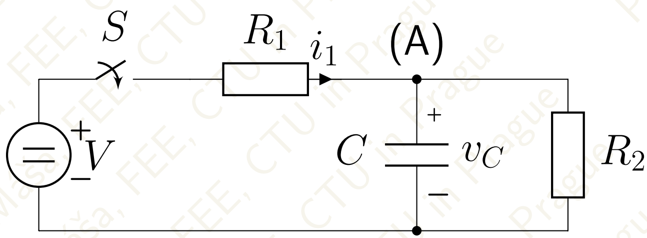


Figure 2.49: Ideal capacitor in a transient condition

2.6.2.2 Practical capacitor

The practical capacitor has leads – conductors, which have some resistance, and some inductance. So, in many circumstances, the symbol of the capacitor as a practical device (which we mount into our circuit), must be replaced by a combination of ideal circuit elements for analysis purposes.

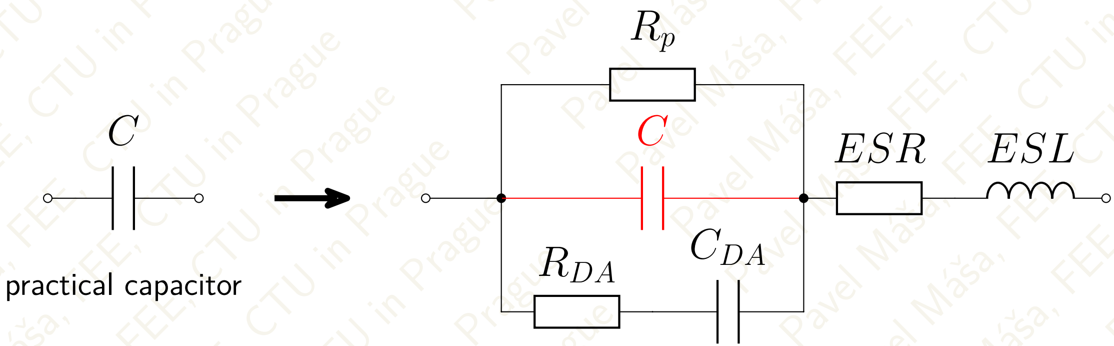


Figure 2.50: Circuit model of a practical capacitor

where

C is capacitance of the capacitor

ESR **Equivalent series resistance** – important in switched sources applications, where large currents flows through the capacitor. It produce Ohm's losses – heat. Ceramic capacitors may have ESR less than 10 mΩ, low ESR electrolytic and tantalum less than 100 mΩ, general aluminum few times higher.

ESL **equivalent series inductance** – inductance of the leads and plates. At high frequencies the capacitor acts like an inductor. The cricital frequency depends on capacitance C and ESL.

R_p insulation resistance (leakage)

R_{DA}, C_{DA} dielectric absorption

In the Figure 2.51 is an example of frequency dependence of capacitor reactance. It should decrease continuously in the case of the ideal capacitor, however, due to ESL it increases again at high frequencies. The minimum is ESR. Decoupling capacitors should have such capacitance, so that the minimum of reactance lay in the noise frequencies range (often 100 nF).

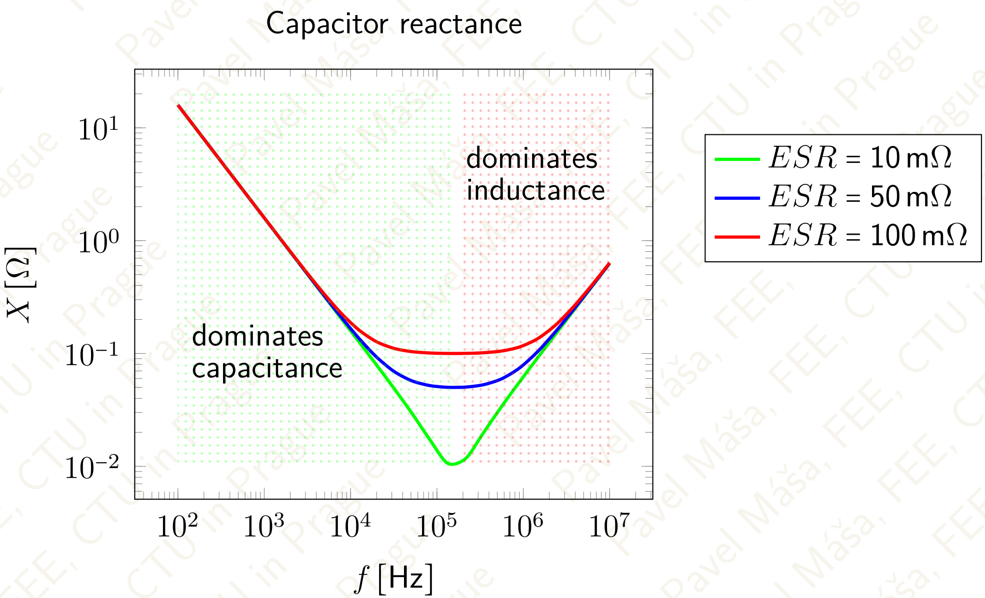


Figure 2.51: The reactance of a practical capacitor

2.6.3. Circuit model of an inductor

2.6.3.1 Ideal inductor – DC steady state

In DC circuits, *if the circuit is in a steady state*, an ideal inductor can be replaced by short circuit, because an ideal inductor would be made of a superconductor.

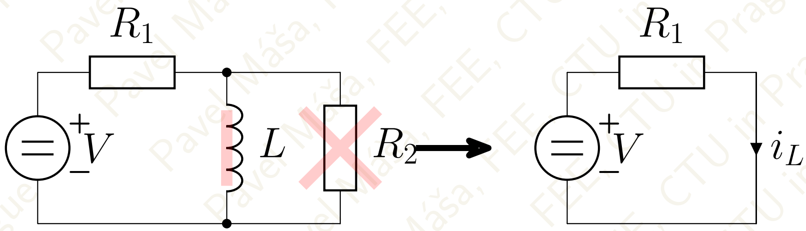


Figure 2.52: Ideal inductor in DC steady state can be replaced by short circuit

Since the resistance of an ideal inductor → 0, in the Figure 2.52 the current flows through the inductor, and not the resistor R_2 – it is current divider, see Equation (2.17). The current is $i_L = \frac{V}{R_1}$.

An exception is a case, when we connect ideal voltage source in parallel to the ideal inductor, see Example 2.11. In such a case the current in the inductor is growing steadily to infinity.

2.6.3.2 Practical inductor

Practical inductor is made of some conductor – and it has some resistance. The coils have many turns of insulated wire, which forms structure conductor – dielectric – conductor — which is the capacitor. Also leads can have some capacitance. So the model of a practical inductor can look as in the Figure 2.53.

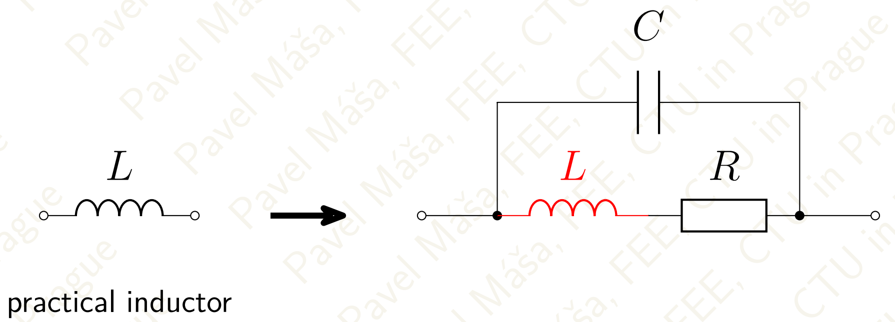


Figure 2.53: Example of circuit model of a practical inductor

The model can be more complicated, and contain other elements, modeling losses in the core . . . .

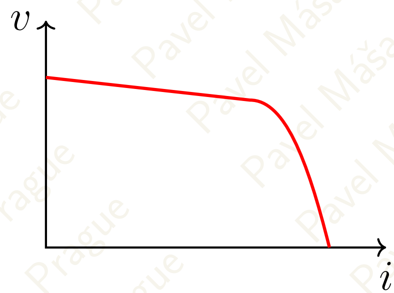
3. DC analysis

3.1. Fundamental methods

3.1.1. Practical voltage source

In the real world, nothing is perfect – nor the voltage sources. If we connect some load to the voltage source terminals, the voltage drops. It drops less or more, depending on the current, drawn from source terminals. The relationshop between terminal voltage and current can be linear, or non-linear, see Figures 3.1a and 3.1b.

(a) Practical source – linear



(b) Practical source – non-linear

Consider linear circuit in the Figure 3.2. To ideal voltage source, V_T is connected in series ideal resistor R_T. To the terminals of that circuit, we connect a variable load, represented by the green marked resistor R_L. In the circuit flows current i_L. On the resistor R_T is the voltage $v_{R_T} = R_T i_L$ and on the resistor R_L is the voltage

$$v_L = v_T - v_{R_T} = v_T - R_T i_L. \tag{3.1}$$

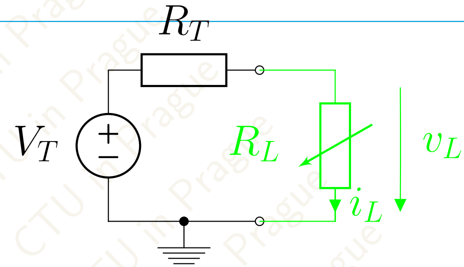


Figure 3.2: Practical voltage source

There are two uttermost cases:

- If $R_L \to \infty$, than $i_L \to 0$ and $v_L = v_T$. It is the same case, as if we disconnect load resistor – this is called *open circuit*, and v_T is *open circuit voltage*.
- If $R_L \to 0$, than $i_L = \frac{v_T}{R_T}$. This is the same case, as if we replace load resistor by ideal wire – the *short circuit*. The current is marked as i_S – the *short-circuit current*. It is the largest possible current that can flow from the source terminals. The voltage $v_L = 0\,\mathrm{V}$.

We can rewrite the Equation (3.1) in the form $i_L = \frac{v_T - v_L}{R_T} = i_S - \frac{v_L}{R_T}$, which is an equation of the line, connecting two points with coordinates $(v_T, 0)$ and $(0, i_S)$. This is the *load line*.

Rule: 3.1

If any load R_T is attached to the source in Figure 3.2, the terminal voltage along with the current must lie on the volt-ampere characteristic on the load line. The point with coordinates (v, i), laying on the load line is referred as *operating point*.

However, the operating point cannot be an arbitrary point on the load line. The voltage v_L can be also expressed as

$$v_L = R_L i_L \tag{3.2}$$

– which is the volt-ampere characteristic of the attached resistor R_L. We can rewrite it in the form $i_L = \frac{v_L}{R_L}$ – an equation of a line passing through the origin. This line and the load line have single intersection – the operating point of that circuit, see Figure 3.3. This is graphical solution of the set of equations 3.1 and 3.2.

Of course, you can argue that there are more natural ways to calculate circuit current and voltage on the resistor R_L – e.g. Ohm's law and voltage divider rule. However, there are many more challenging circuits than the voltage divider. If we replace resistor R_L by a diode, the current can be calculated from logarithmic equation 2.9. However, the solution of logarithmic equations is not so easy as the voltage divider rule. A graphical solution may simplify the process (see Figure 3.4).

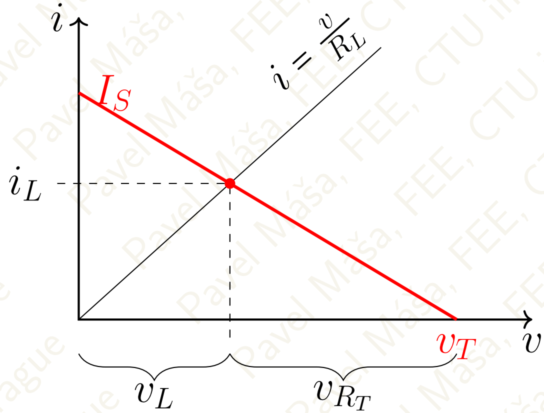
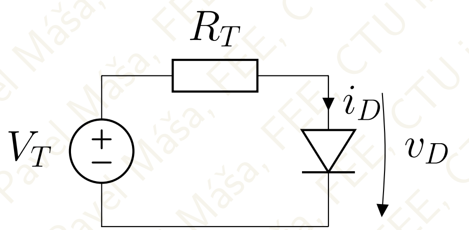


Figure 3.3: Load line and linear load resistor



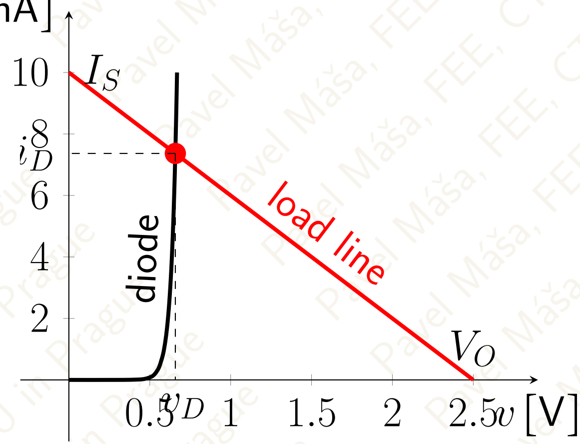


Figure 3.4: Load line – graphical solution of a circuit with a diode

In the Figure 3.4 $V_O = V_T$, $I_S = \frac{V_T}{R_T}$. The black curve is a V-A characteristics of the diode.

The load line is also commonly used for solving transistor amplifiers as in the Figure 3.5.

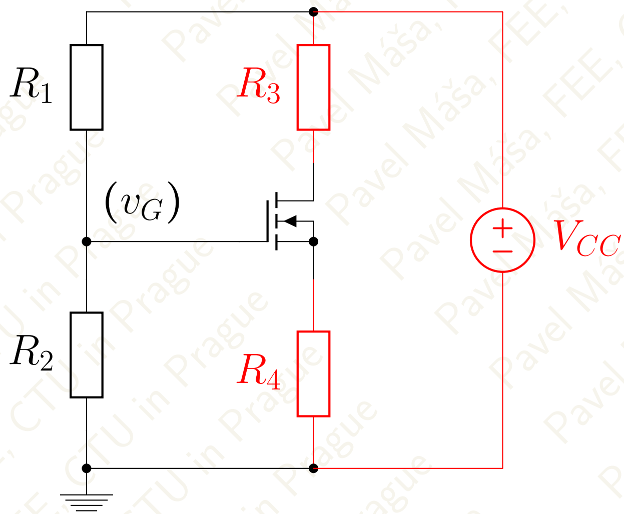


Figure 3.5: MOSFET amplifier

The circuit in Figure 3.5 looks much more complicated, than we did untill now. However. is is not. The black-marked resistors R_1 and R_2 are in series, and connected directly to the voltage source V_{CC}. The gate of the MOSFET transistor is a metal electrode on a silicon oxide layer – which is very good insulator. So, the gate current is in DC virtually zero, and resistors R_1 and R_2 are not affected by resistors R_3, R_4 and the transistor. For analysis purposes, we can consider it as an independent part of the circuit. The voltage v_G is given as $v_G = V_{CC}\frac{R_2}{R_1+R_2}$ (voltage divider rule).

The red-marked part of the circuit can be also solved separately. It consists of two resistors, R_3 and R_4 in series, voltage source V_{CC} and the transistor. The resistors, since they are in series, may be combined into single one. Again, we can draw V-I characteristic of transistor, and the load line.

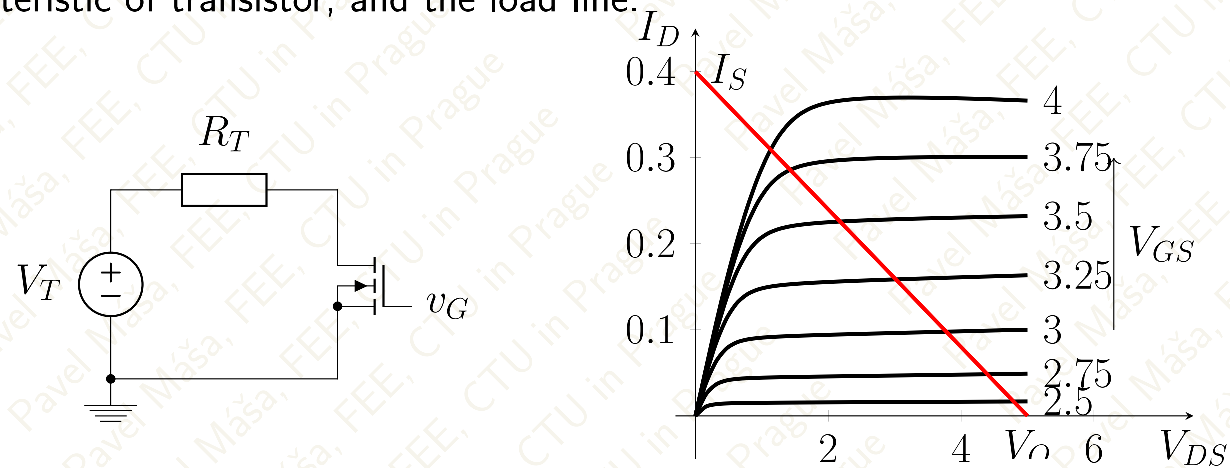


Figure 3.6: Load line – graphical solution of a circuit with a transistor

$$V_T = V_{CC},\ R_T = R_3 + R_4,\ I_S = \frac{V_T}{R_T},\ V_O = V_T.$$

3.1.2. Thévenin equivalent circuit

In Section Practical Voltage Source we saw that the idea of combination of an ideal voltage source and single resistor is very useful for solution of many circuits, using the load line. However, what if the circuit contains more circuit elements?

Consider now linear circuit in the Figure 3.7.

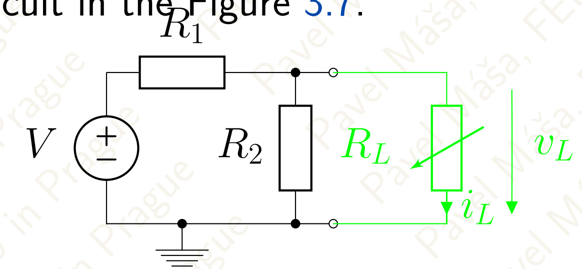


Figure 3.7: Voltage divider with a load resistor

There are again two uttermost cases:

- If $R_L \to \infty$, than $i_L \to 0$ and $v_L = V \frac{R_2}{R_1+R_2} = V_T$. The *open circuit voltage* is a terminal voltage with no load connected – it is maximum possible terminal voltage.
- If $R_L \to 0$, it is a *short circuit*, the current will not flow through the resistor R_2, as the short-circuit represents an infinitely easier path to flow. $i_L = \frac{V}{R_1} = I_S$. This is maximum current, which can be drawn from the circuit terminals. The voltage $v_L = 0\,\mathrm{V}$.

We have two uttermost points in the V-I plane, with coordinates $(V_T, 0)$ and $(0, I_S)$. For linear circuit, the points must lie on a line – we have the same line as in the Figure 3.3. However, it is the same load line, as for the circuit 3.2. For the sake of circuit simplification, *we can replace voltage divider by single voltage source with single resisor in series*.

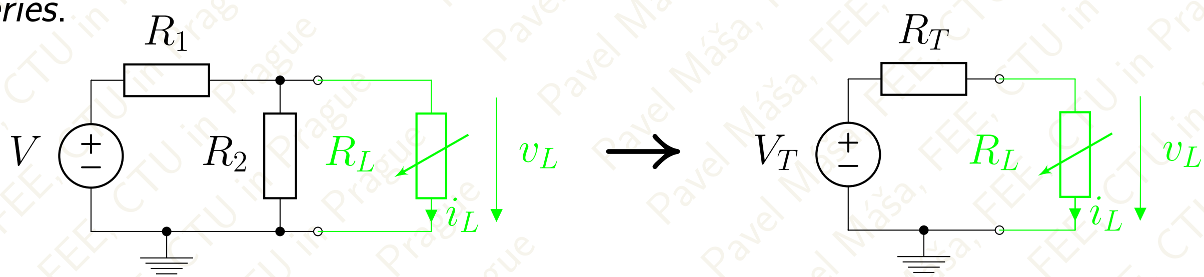


Figure 3.8: Thévenin simplification of a circuit

From the Equation 3.1 for the point $(0, I_S)$ we get

$$0 = V_T - R_T I_S \qquad \Rightarrow \qquad R_T = \frac{V_T}{I_S} \tag{3.3}$$

For circuit in the Figure 3.8 the values are:

$$V_T = V\frac{R_2}{R_1 + R_2}$$
$$I_S = \frac{V}{R_1}$$
$$R_T = \frac{V_T}{I_S} = \frac{V\frac{R_2}{R_1+R_2}}{\frac{V}{R_1}} = \frac{R_1 R_2}{R_1 + R_2}$$

We can simply verify that the voltage V_L is the same. In original circuit, using voltage divider rule:

$$V_L = V\frac{\frac{R_2 R_L}{R_2+R_L}}{R_1 + \frac{R_2 R_L}{R_2+R_L}} = V\frac{R_2 R_L}{R_1 R_2 + R_1 R_L + R_2 R_L}$$

In simplified circuit:

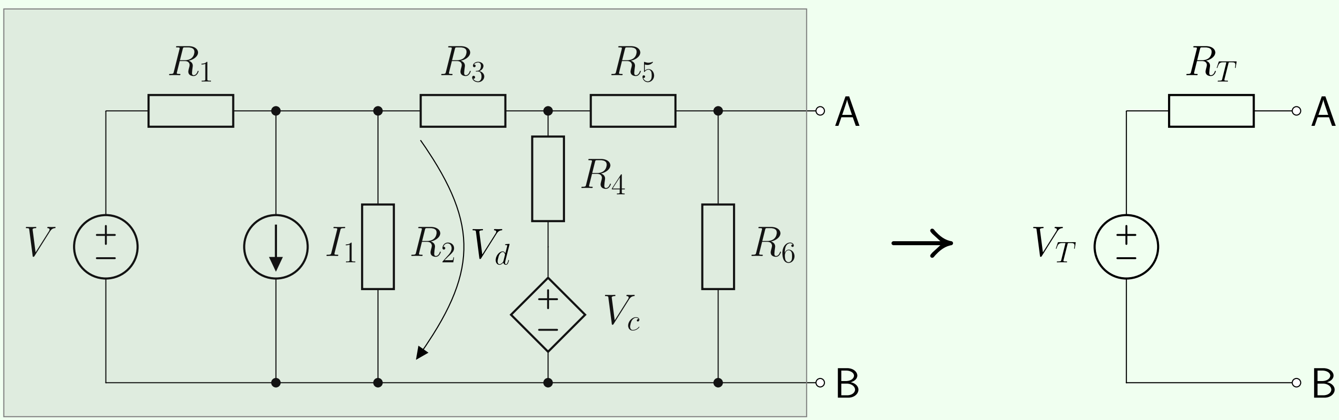
$$V_L = V_T\frac{R_L}{R_T + R_L} = V\frac{R_2}{R_1 + R_2}\cdot\frac{R_L}{\frac{R_1 R_2}{R_1+R_2} + R_L} = V\frac{R_2 R_L}{R_1 R_2 + R_1 R_L + R_2 R_L}$$

French telegraph engineer Léon Charles Thévenin derived that this idea can be extended to any linear electrical network, for any pair of terminals in the electrical network.

Definition

Thévenin theorem

- Any *linear electrical network* can be replaced at terminals A and B by equivalent series connection of single voltage source V_T and single resistor with resistance R_T
- The equivalent voltage V_T is the open circuit voltage at terminals A and B.
- The equivalent resistance R_T is total resistance, which exhibits the circuit between terminals A and B.
- The theorem is valid both for DC and AC circuits.



Rule: 3.2

Thévenin equivalent resistance can be calculated from the open circuit voltage V_T and short circuit current at terminals A and B

$$R_T = \frac{V_T}{I_S}$$

This definition is valid even for circuits, which contains controlled sources.

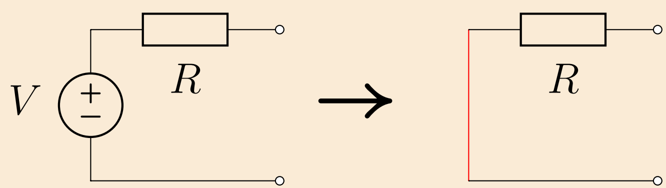
Rule: 3.3

Thévenin equivalent resistance can be calculated as a total resistance between terminals A and B, if we remove all sources.
This definition can fail, if the circuit contains controlled sources.

Rule: 3.4

Voltage source removal

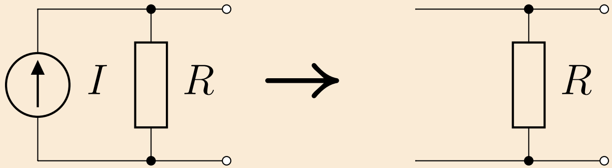
The voltage source is replaced by short circuit.



Rule: 3.5

Current source removal

The current source is replaced by open circuit.



Remember that:

- The Thévenin equivalent has an equivalent V-I characteristic only from the point of view of the load (the A and B terminals).
- The Thévenin equivalent can have different power dissipation from original electrical network. However, the load have still the same power dissipation.

Return back to the problem of loaded voltage divider. We solved this circuit using KCL and KVL rules. However, we can use Thévenin equivalent to simplify the solution, see Figure 3.9. The Thévenin equivalent voltage is the open circuit voltage at terminals, where "unknown circuit"is connected – the terminals of the resistor R_2. To find open circuit voltage, we temporary disconnect "unknown circuit". Then $i_{uc} = 0$ and the circuit is voltage divider, so

$$V_T = V\frac{R_2}{R_1 + R_2} = 30 \cdot \frac{2000}{5000} = 12\,\mathrm{V}.$$

If we remove voltage source (replace it by short circuit), the Thévenin equivalent resistance is:

$$R_T = \frac{R_1 R_2}{R_1 + R_2} = \frac{3000 \cdot 2000}{5000} = 1200\,\Omega.$$

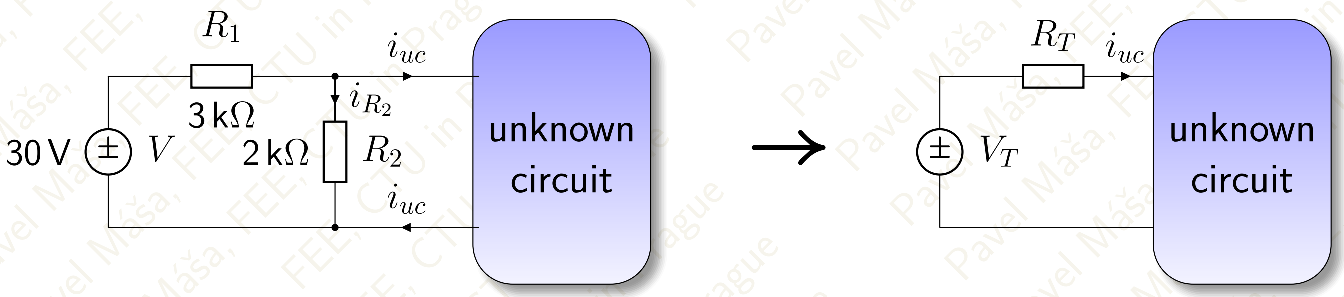


Figure 3.9: Simplification of a loaded voltage divider with Thévenin equivalent

Now we return back the “unknown circuit”. If the current $i_{uc} = 5\,\mathrm{mA}$, the voltage at terminals of the “unknown circuit” is

$$V_{uc} = V_T - R_T I_{uc} = 12 - 1200 \cdot 0.005 = 6\,\mathrm{V}.$$

Thévenin equivalent is also useful for Wheatstone bridge analysis. The Wheatstone bridge is very important circuit used to measure an unknown electrical resistance, and quantities related to resistance of a sensor, connected in the bridge (temperature, weight. . .).

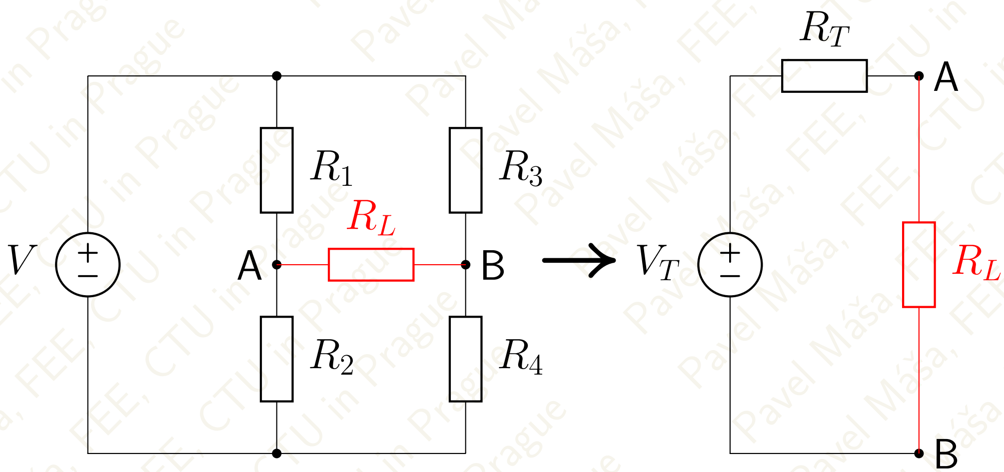


Figure 3.10: Wheatsone bridge

In the Wheatstone bridge (Figure 3.10), we can not combine resistors in series or parallel and simplify circuit in this way. However, if we disconnect the load resistor temporarily, the remaining part of the circuit can be replaced by Thévenin equivalent easily, see Figures 3.11 and 3.12.

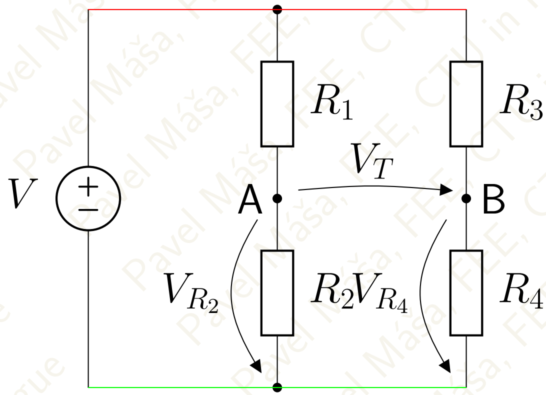


Figure 3.11: Wheatsone bridge simplification using Thévenin equivalent – step 1

The green and red marked rails are connected directly to an ideal voltage source, so there is the source voltage V regardless of what is connected between both rails. For this reason, the series connected pair of resistors R_3 and R_4 does not affect resistors R_1 and R_2 and vice versa. The series connected resistors forms voltage dividers. We can make a loop around resistors R_2 and R_2 and the terminals A and B, and write the KVL equation for it. We can choose both clockwise and anticlockwise direction, in the Equation 3.4 the clockwise direction is used.

$$
\begin{aligned}
V_{R_2} &= V\frac{R_2}{R_1+R_2}\\
V_{R_4} &= V\frac{R_4}{R_3+R_4}\\
V_T + V_{R_4} - V_{R_2} &= 0\\
V_T &= V\left(\frac{R_2}{R_1+R_2}-\frac{R_4}{R_3+R_4}\right)
\end{aligned}
\tag{3.4}
$$

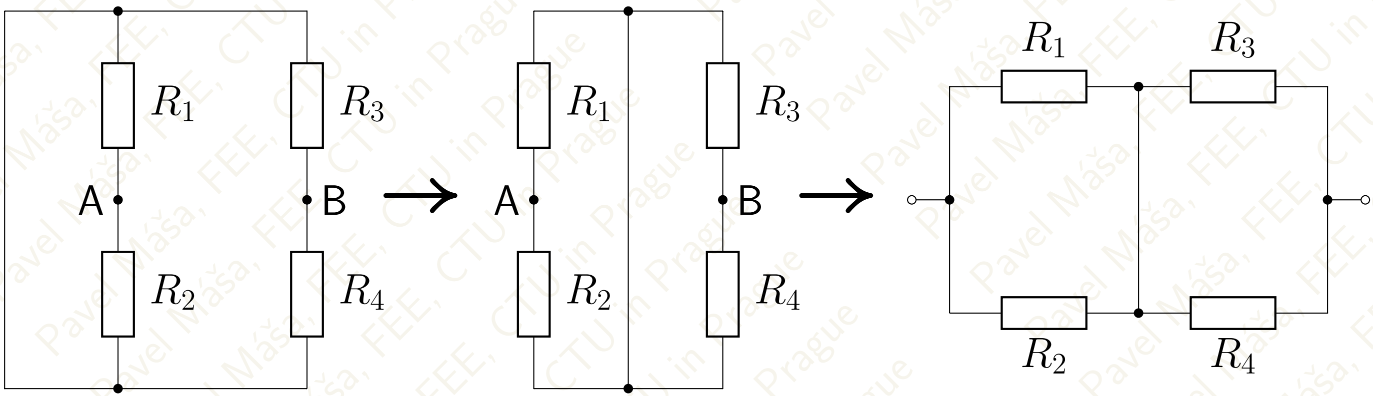


Figure 3.12: Wheatsone bridge simplification using Thévenin equivalent – step 2

In the Figure 3.12 the procedure of Thévenin equivalent resistance calculation is indicated. After we remove the voltage source (by replacing it with short circuit), we can rearrange the resistors so we can see that R_1 is parrallel with R_2 and R_3 is in parallel with R_4, and these two parallel combinations are in the series, so

$$R_T = \frac{R_1 R_2}{R_3 + R_4} + \frac{R_3 R_4}{R_3 + R_4}. \tag{3.5}$$

Finally,

$$V_L = V_T \frac{R_L}{R_L + R_T}. \tag{3.6}$$

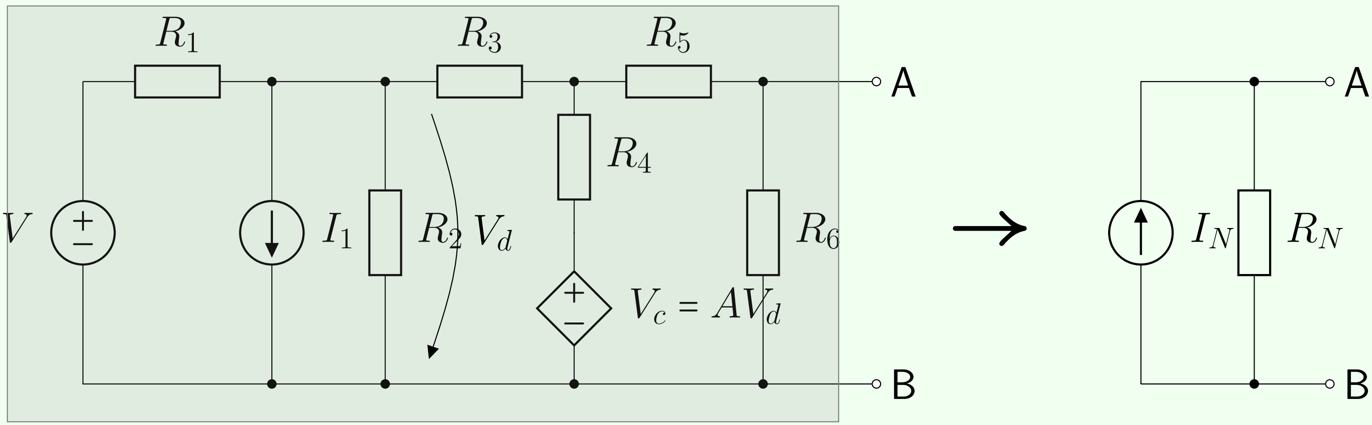
3.1.3. Norton equivalent circuit

Norton equivalent circuit, or Mayer–Norton theorem was independently derived in 1926 by Siemens & Halske researcher Hans Ferdinand Mayer and Bell Labs engineer Edward Lawry Norton. It is a dual to the Thévenin equivalent circuit.

Definition

Norton theorem

- Any *linear electrical network* can be replaced at terminals A and B by equivalent parallel connection of single current source I_N and single resistor with resistance R_N
- The equivalent current I_N is the short circuit current at terminals A and B.
- The equivalent resistance R_T is total resistance, which exhibits the circuit between terminals A and B.
- The theorem is valid both for DC and AC circuits.



3.1.4. Norton – Thévenin equivalent circuit duality

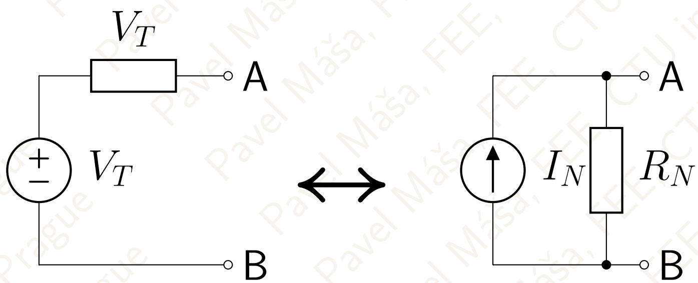


Figure 3.13: Thévenin – Norton equivalent circuit duality

The Thévenin equivalent circuit has the open terminal voltage V_T, and, if we connect terminals A and B by an ideal wire, the short circuit current $I_S = \frac{V_T}{R_T}$. In the Norton equivalent circuit, the open terminal voltage is the voltage drop on the resistor R_N, so $V_O = I_N R_N$, and the short circuit current, when we connect terminals A and B by an ideal wire, is I_S, as all current will flow through zero resistance, and not non-zero resistance R_N (see the current divider).

Rule: 3.6

The Thévenin equivalent circuit and Norton equivalent circuit can be arbitrarily interchanged, according to equations:

$$\begin{aligned} V_T &= I_N R_N \\ R_T &= R_N \\ I_N &= \frac{V_T}{R_T} \end{aligned} \tag{3.7}$$

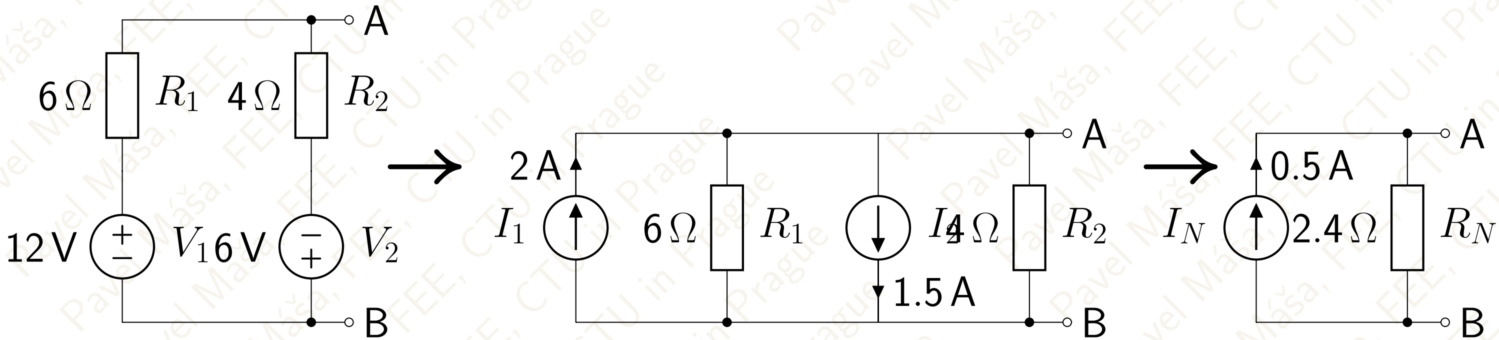


Figure 3.14: Example – Thévenin – Norton equivalent duality application