# Fundamentals of Electrical Circuits

## VI

## Sinusoidal steady state

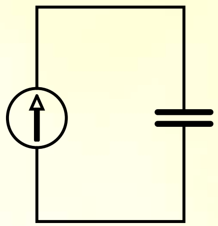
SINUSOIDAL STEADY STATE, REPRESENTATION OF A SINE WAVE AS A PHASOR, CIRCUIT ELEMENTS AT SINUSOIDAL EXCITATION, IMPEDANCE AND ADMITTANCE. PHASOR DIAGRAMS.

# Phasors, Impedance, Admittance

*Till now we were concerned with circuits, supplied from a DC voltage / current sources – circuits in stationary (DC) steady state*

- We know both capacitor and inductor are "inertial" circuit elements; if the structure of the circuit changes (connection or disconnection of a source, part of a circuit, resistivity varies, …), the capacitors and inductors will be charged or discharged to reach another steady state.
- But what if the circuit is not supplied from the DC source, but sinusoidal?
  - In following lectures we will suppose the sources have sinusoidal waveforms.

**Capacitor, supplied from the sinusoidal current source:**



The waveform of current: $i(t) = I_m \sin \omega t$

Charge, stored in the capacitor (DC): $q = Cu = it$

But now charge varies with time: $\mathrm{d}q = C\mathrm{d}\,u = i\mathrm{d}\,t$

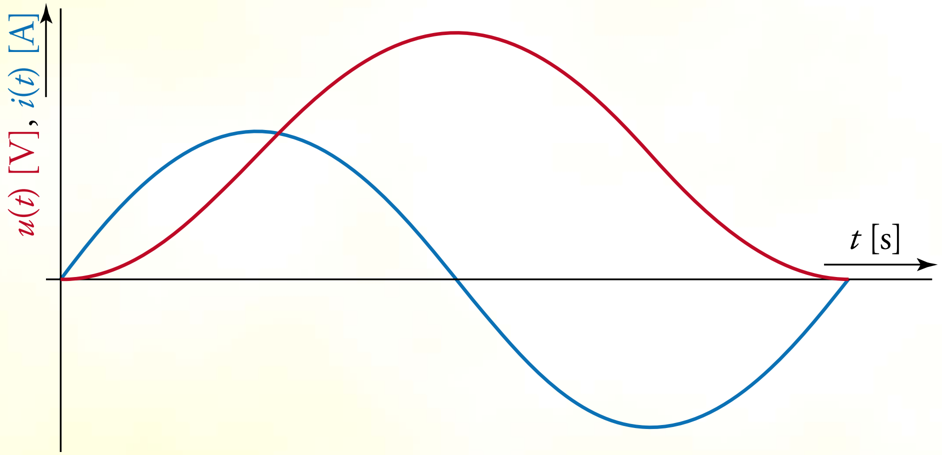
$$u(t) = \frac{q(t)}{C} = \frac{1}{C}\int_0^t i(\tau)\,\mathrm{d}\tau + u_c(0)$$

$$i(t) = C\,\frac{\mathrm{d}u(t)}{\mathrm{d}t}$$

$$u(t) = \frac{1}{C}\int_0^t I_m \sin \omega\tau\,\mathrm{d}\tau + u_c(0) =$$

$$= \frac{I_m}{\omega C}(1 - cos\omega t) + u_c(0) = \frac{I_m}{\omega C}\sin\left(\omega t - \frac{\pi}{2}\right) + \frac{I_m}{\omega C} + u_c(0)$$

$$u(t) = \frac{1}{C}\int_0^t I_m \sin\omega\tau \, \mathrm{d}\tau + u_c(0) = \frac{I_m}{\omega C}(1 - cos\omega t) + u_c(0) = \frac{I_m}{\omega C}\sin\left(\omega t - \frac{\pi}{2}\right) + \frac{I_m}{\omega C} + u_c(0)$$




The waveform doesn't change – even voltage is sinusoidal
Only magnitude was changed (indirect proportional to the frequency and capacity)
The waveform is shifted by a quarter of period ($\frac{\pi}{2}$)

Positive orientation of the current source corresponds to delivery of a charge into the capacitor, and negative opposite, the source draws the charge from the capacitor, (*or transfer electrons from one plate of capacitor on another*) we can see, in the first half of the period the capacitor is charged – the voltage increases, in the second half period the capacitor is discharged – the voltage drops to zero

**Capacitor supplied from the sinusoidal voltage source:**

The waveform of voltage: $u(t) = U_m \sin\omega t$

$$i(t) = C\frac{\mathrm{d}u(t)}{\mathrm{d}t} = C\frac{\mathrm{d}}{\mathrm{d}t}(U_m \sin\omega t) = U_m\omega C\cos\omega t = U_m\omega C\sin\left(\omega t + \frac{\pi}{2}\right)$$

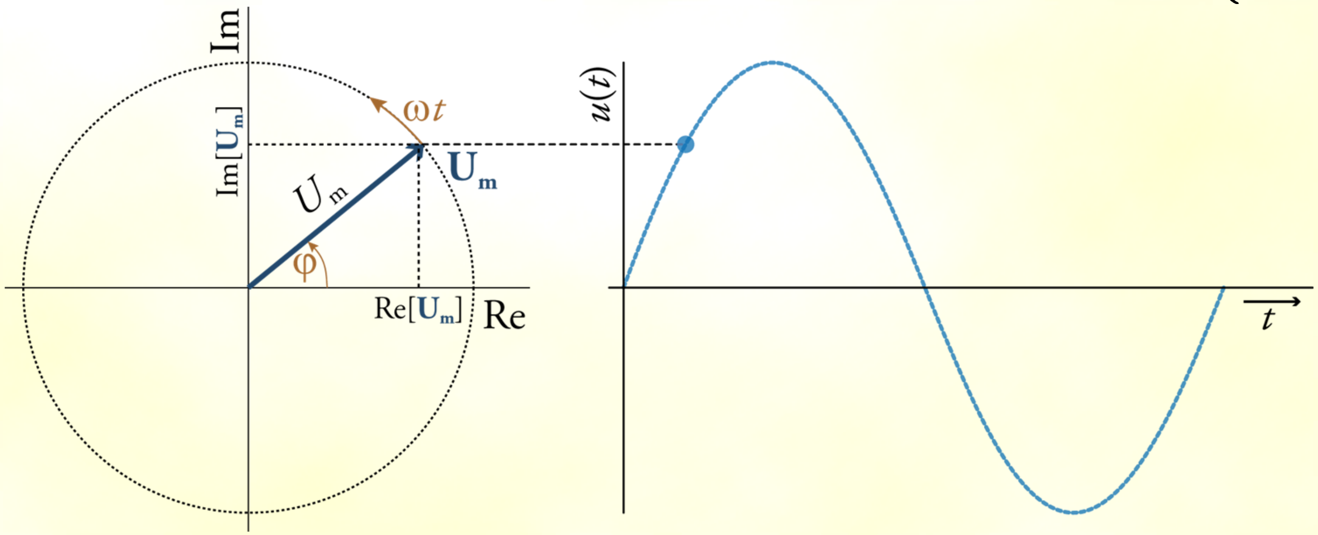
**In both cases the result is phase shifted sinusoidal function, its magnitude varies with frequency and capacity – if we would use some kind of *transformation* to replace sinusoidal function by its *transform*, then it would possible to replace integral and derivative by division or multiplication**

☞ The solution is the Euler's identity: $e^{jx} = \cos x + j \sin x$

☞ We need to replace sinusoidal waveform by complex number $u(t) = U_m \sin(\omega t + \varphi)$

Sine is imaginary part of polar form of a complex number:

$$U_m \sin(\omega t + \varphi) = \operatorname{Im}\left\{U_m \left[\cos(\omega t + \varphi) + j \sin(\omega t + \varphi)\right]\right\} = \operatorname{Im}\left\{U_m e^{j(\omega t + \varphi)}\right\}$$




Hus_demo.exe

- **Sine function is transformed by a complex number, Graphically represented by a vector in complex plane of length $U_m$, with initial phase shift with respect to the real axis φ, rotating counterclockwise at angular velocity ω.**
- Since we cannot add / subtract, … vectors rotating at different angular frequencies, superposition may be applied just at sources with same angular frequency
- Linear circuit elements can not change angular frequency of circuit variables – vectors of all circuit variables rotate at same angular frequency – it makes no sense to copy angular frequency
- **Phasor** of voltage is then defined: $\widehat{U}_m = U_m e^{j\varphi}$ Another symbols: $\mathbf{U}_m, \overline{U}_m, \ldots$
- Steinmetz polar form may be used to express phasors: $\widehat{U}_m = U_m e^{j\varphi} = U_m \underline{/\varphi}$

**Example:** $u(t) = 100\sin\left(1000t + \frac{\pi}{4}\right) \quad \rightarrow \quad \widehat{U}_m = 100\,e^{j\frac{\pi}{4}} = 100\left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right) = 70.71 + 70.71j$

Exponential form – suitable for multiplication and dividing

Rectangular form – suitable for addition and subtraction

**Phasor** may be evaluated from **maximum value, or RMS value** – applicable for power

$$u(t) = 100\sin\left(1000t + \frac{\pi}{4}\right) \rightarrow \widehat{U}_m = 100\,e^{j\frac{\pi}{4}} = 100\left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right) = 70.71 + 70.71j$$

$$\widehat{U} = \frac{100}{\sqrt{2}}\,e^{j\frac{\pi}{4}} = 70.71\left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right) = 50 + 50j$$

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*Note: exponential form of complex number contains both sine and cosine elements – using Euler's identity it is possible to use cosine for the phasor:*

$$U_m\cos(\omega t + \varphi) = \mathrm{Re}\left\{U_m\left[\cos(\omega t + \varphi) + j\sin(\omega t + \varphi)\right]\right\} = \mathrm{Re}\left\{U_m e^{j(\omega t + \varphi)}\right\}$$

*We know, that* $U_m\cos(\omega t) = U_m\sin\left(\omega t + \frac{\pi}{2}\right)$

*Then the sinusoidal waveform may be transformed as*

$$u(t) = 100\sin\left(1000t + \frac{\pi}{4}\right) = 100\cos\left(1000t + \frac{\pi}{4} - \frac{\pi}{2}\right)$$

$$\widehat{U}'_m = 100\,e^{j\frac{-\pi}{4}} = 100\left(\cos\frac{-\pi}{4} + j\sin\frac{-\pi}{4}\right) = 70.71 - 70.71j$$

*Calculations in frequency domain are identical, just when we transform sinusoidal function from time domain into frequency domain we may apply phase shift*

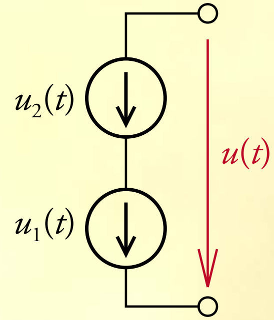
**Example: superposition of sinusoidal waveforms** – since we add rotating vectors, all sources must have the same frequency – or the result of superposition is non-sinusoidal periodical (Fourier series), or even aperiodic waveform.

Two sinusoidal sources of the same frequency are connected in series. Find their total voltage:

$$u_1(t) = 20\sin(100t),\ u_2(t) = 30\sin(100t + \frac{\pi}{4}) \longrightarrow \mathbf{U}_1 = 20\,\mathrm{V},\ \mathbf{U}_2 = 30\underline{/\tfrac{\pi}{4}}\,\mathrm{V}$$

$$\mathbf{U} = \mathbf{U}_1 + \mathbf{U}_2 = 20 + 30\underline{/\tfrac{\pi}{4}} = 41.213 + 21.213j = 46.352\,\underline{/0.475}$$

$$u(t) = 46.352\sin(100t + 0.475)\,\mathrm{V}$$




Back to the relation between voltage across capacitor and passing current – when sine function will be transformed by phasor

$$u(t) = \frac{1}{C}\int_0^t I_m \sin\omega\tau \,\mathrm{d}\tau + u_c(0) =$$

*Superposition of two independent sources– DC a AC*

$$= \frac{I_m}{\omega C}(1 - cos\omega t) + u_c(0) = \frac{I_m}{\omega C}\sin\left(\omega t - \frac{\pi}{2}\right) + \frac{I_m}{\omega C} + u_c(0)$$

$$\widehat{U}_m = \frac{1}{j\omega C}\cdot\widehat{I}_m$$

Why divide by *j*? $\frac{1}{j} = -j = e^{\frac{-\pi}{2}}$

Expression $\frac{1}{j\omega C}$ is called impedance of a capacitor and it has symbol $\widehat{Z}_C$, or $\mathbf{Z}_C$

**Impedance and admittance of fundamental circuit elements:**

| Circuit element | | | | |
|---|---|---|---|---|
| R | resistivity | $R$ | conductance | $G = \frac{1}{R}$ |
| C | impedance | $\frac{1}{j\omega C}$ | admittance | $j\omega C$ |
| L | impedance | $j\omega L$ | admittance | $\frac{1}{j\omega L}$ |

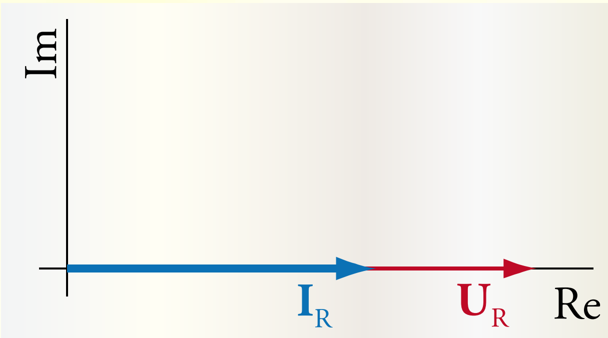
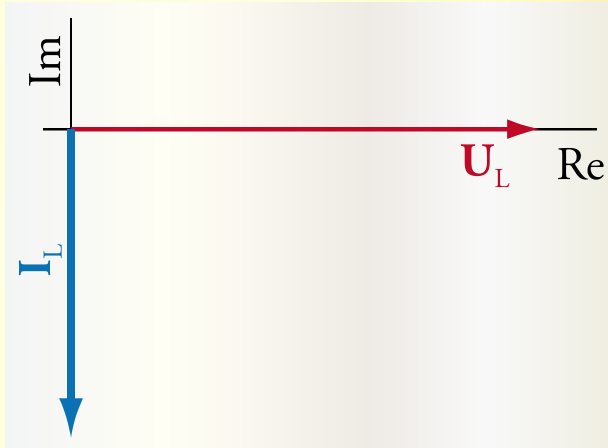
**Phasor relationships for circuit elements:**

| Circuit element | | |
|---|---|---|
| R | $\mathbf{U} = R\,\mathbf{I}$ | $\mathbf{I} = G\,\mathbf{U}$ |
| C | $\mathbf{U} = \frac{1}{j\omega C}\mathbf{I}$ | $\mathbf{I} = j\omega C\,\mathbf{U}$ |
| L | $\mathbf{U} = j\omega L\,\mathbf{I}$ | $\mathbf{I} = \frac{1}{j\omega L}\mathbf{U}$ |

In sinusoidal steady state **generalized Ohm's law** is valid for phasors and impedances

## Phase shift

- In the case of **capacitor** we derived in *time domain*, the **current leads the voltage by a quarter of period**, or by 90°
- Similarly on **inductor** in time domain **voltage leads the current by a quarter of period**, or 90°
- In *frequency domain* the phase shift is represented by imaginary unit *j*: $j = e^{j\frac{\pi}{2}}$
- In complex plane the time (phase) shift after transform is represented by rotation of an vector counterclockwise with respect to positive part of a real axis.

| R | C | L |
|---|---|---|
|  |  |  |

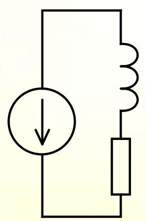
Common mistake is misunderstanding of the difference between waveform and phasor. The transform take us to quite another world. From the world of functions to the world of complex numbers. From time domain to frequency domain.

**Terms, like** $i(t) = j\omega C\,\mathbf{U},\ \mathbf{P} = \dfrac{U_{2m}\sin(\omega t + \varphi_2)}{U_{1m}\sin(\omega t + \varphi_1)}$ **are just insane!!!**

# Reactance

Now, we will study following example:

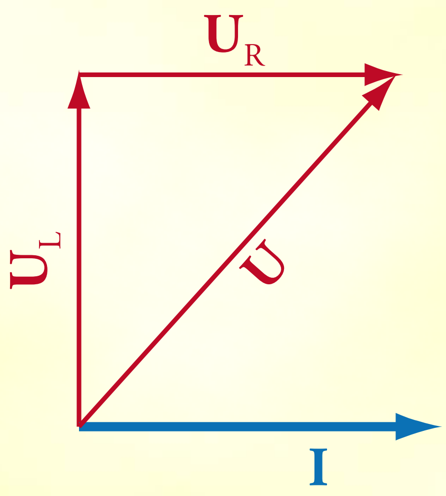
The resistor with resistivity $R$ is connected in series with inductor with inductance $L$. It will be supplied by sinusoidal voltage source $U$ = 230 V. The voltage across resistor is $U_R$ = 120 V. What is the voltage across inductor?



$$U_L = U - U_R = 230 - 120 = 110\,\mathrm{V}?$$

Mistake! So easy it is just with resistors. How about the phase shift?

Both circuit elements are passed by the same current. The voltage across resistor is in phase with current, but on inductor voltage leads the current by $\frac{\pi}{2}$. If the voltage vectors will be drawn such it ties together like elements in the circuit, we get rectangular triangle (phasor diagram):




We know hypotenuse (230 V) and one leg (120 V) of the triangle. Using the Pythagoras' formula we may easy compute the length of remaining leg:

$$U_L = \sqrt{U^2 - U_R^2} = \sqrt{230^2 - 120^2} \doteq 196.2\,\mathrm{V}$$

*Due to the phase shift simple „linear" math is not valid, the voltage across reactive elements (capacitors and inductors) may be quite higher we could expect from experience in resistive circuits.*

Go on with investigation of this circuit. We know the voltage across resistor is $U_R$ = 120 V. Suppose the (heat) power of that resistor is $P_R$ = 600 W. Then the current passing the circuit is $I = \frac{P_R}{U_R} = \frac{600}{120} = 5\,\mathrm{A}$. What inductance has to have the inductor so the voltage across resistor will be required 120 V? Frequency is 50 Hz.

We know the voltage across inductor : $U_L$ = 196.2 V. We know the inductor is passed by the current 5 A. The relation between voltage and current for inductor is $\mathbf{U}_L = j\omega L\,\mathbf{I}$. But in this particular case we are not interested in, the voltage leads the current by $\frac{\pi}{2}$ . Between voltage and current amplitude the relation is

$$U_L = \omega L I = X_L I$$

Term

$$X_L = \omega L$$

is called **reactance** of the inductor. Its unit is $\Omega$.

Now it is easy to compute required inductance $$L = \frac{U_L}{\omega I} = \frac{196.2}{100 \cdot \pi \cdot 5} \doteq 0.125\,\mathrm{H}$$

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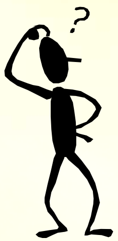


May we write: $U = I(R + X_L)$?

No, this is not resistive circuit!
This equation asserts the voltage across inductor is 110 V. But we already know it is not true. What is the correct form?

$$\mathbf{U} = \mathbf{I}(R + jX_L) = \mathbf{I}(R + \mathbf{Z}_L)$$

*How to transform rectangular form of complex number on exponential form?*

$$\mathbf{U} = \sqrt{(\mathrm{Re}\,\{\mathbf{U}\})^2 + (\mathrm{Im}\,\{\mathbf{U}\})^2} \cdot e^{j\arctan\frac{\mathrm{Im}\{\mathbf{U}\}}{\mathrm{Re}\{\mathbf{U}\}}} = I \cdot \sqrt{R^2 + X_L^2} \cdot e^{j(\varphi_I + \arctan\frac{X_L}{R})}$$

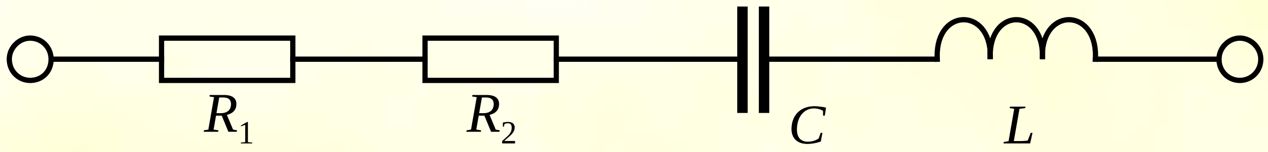
*YES! Here is hidden necessary Pythagoras' formula!*

☞ **Reactance is imaginary part of impedance.** *We may use reactance to evaluate voltage or current just on one reactive element (inductor, capacitor), not in series connection with resistor.*

Terminology:

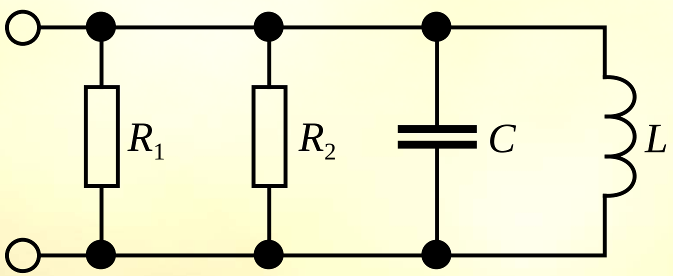
| | | | |
|---|---|---|---|
| $R = \mathrm{Re}\{\mathbf{Z}\}$ | rezistance | $X = \mathrm{Im}\{\mathbf{Z}\}$ | reactance |
| $G = \mathrm{Re}\{\mathbf{Y}\}$ | conductance | $B = \mathrm{Im}\{\mathbf{Y}\}$ | susceptance |

| Circuit element | reactance $X$ | susceptance $B$ |
|---|---|---|
| $C$ | $\frac{-1}{\omega C}$ | $\omega C$ |
| $L$ | $\omega L$ | $\frac{-1}{\omega L}$ |




$$\mathbf{Z} = R_1 + R_2 + \frac{1}{j\omega C} + j\omega L = R_1 + R_2 + j\left(\frac{-1}{\omega C} + \omega L\right) = \underbrace{R_1 + R_2}_{resistance} + j\underbrace{(X_C + X_L)}_{reactance}$$

When the circuit elements are series connected, we may add resistivities and reactances separately, but not resistivity and reactance together




$$\mathbf{Y} = G_1 + G_2 + j\omega C + \frac{1}{j\omega L} = G_1 + G_2 + j\left(\omega C - \frac{1}{\omega L}\right) =$$

$$= \underbrace{G_1 + G_2}_{conductance} + j\underbrace{(B_C + B_L)}_{susceptance}$$

When the circuit elements are connected in parallel, we may add conductances and susceptances separately, but not conductances and susceptances together

The phasor of current, which flows through the circuit, is:

$$\mathbf{I} = \frac{\mathbf{U}}{R + j\omega L} = \frac{230}{24 + j \cdot 100 \cdot \pi \cdot 0.125} = 2.606 - 4.264j = 5\,\underline{/-1.02}\ \mathrm{A}$$

Phasor voltage across resistor and inductor:

$$\mathbf{U}_R = R\,\mathbf{I} = 24 \cdot 5\,\underline{/-1.02} = 120\,\underline{/-1.02}\ \mathrm{V}$$

$$\mathbf{U}_L = j\omega L \cdot \mathbf{I} = j\,100\pi \cdot 0.125 \cdot 5\,\underline{/-1.02} = 196.25\,\underline{/0.549}\ \mathrm{V}$$

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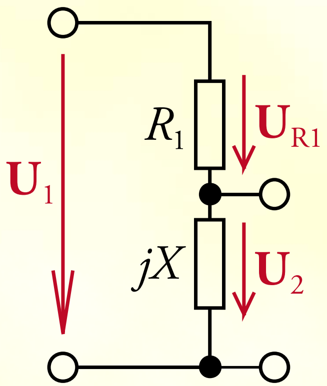
## Fundamental circuits

**Voltage divider**



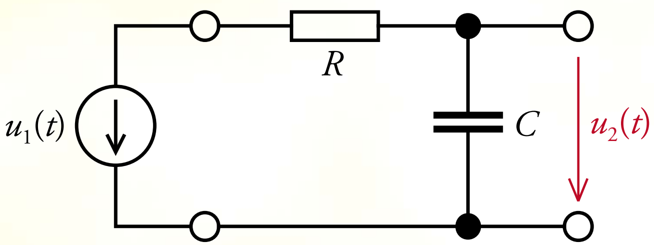

$$\mathbf{I} = \frac{\mathbf{U}_1}{R + jX} \quad \rightarrow \quad \mathbf{U}_2 = jX \cdot \mathbf{I} = jX \frac{\mathbf{U}_1}{R + jX}$$

$$\boxed{\mathbf{U}_2 = \mathbf{U}_1 \frac{jX}{R + jX}}$$

When more circuit elements are in series then voltage across reactive element and across resistor:

$$\mathbf{U}_k = \mathbf{U}_1 \frac{jX_k}{\sum_{m=1}^{M} R_m + \sum_{n=1}^{N} jX_n} \qquad \mathbf{U}_k = \mathbf{U}_1 \frac{R_k}{\sum_{m=1}^{M} R_m + \sum_{n=1}^{N} jX_n}$$

**Example:** circuit in figure (integrating RC circuit) is supplied from the sinusoidal voltage source with amplitude $U_m$ = 20 V. Values of circuit elements are: $R$ = 1 kΩ, $C$ = 1 μF. Compute the waveform of voltage across capacitor $u_2(t)$ for following angular frequencies of the source: ω = 100 s$^{-1}$, ω = 1000 s$^{-1}$ a ω = 10000 s$^{-1}$.




**Voltage phasor – in this example it has the same value for all frequencies**, but, keep in mind, they are still **different phasors** (*they are vectors, rotating with different angular speed!!!*) – in the case of series connected sources we can not add their phasors!!!

$$\mathbf{U}_1 = 20$$

General solution – voltage divider:

$$\mathbf{U}_2 = \mathbf{U}_1 \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \mathbf{U}_1 \frac{1}{j\omega RC + 1} = 20 \cdot \frac{1}{j\omega \cdot 0.001 + 1}$$

1) ω = 100 s$^{-1}$

$$\mathbf{U}_2 = 20 \cdot \frac{1}{j \cdot 100 \cdot 0.001 + 1} = \frac{20}{1.005\, e^{0.1j}} = 19.9 \underline{/-0.1} \qquad u_2(t) = 19.9 \sin\left(100t - 0.1\right) \text{ V}$$

2) ω = 1000 s$^{-1}$

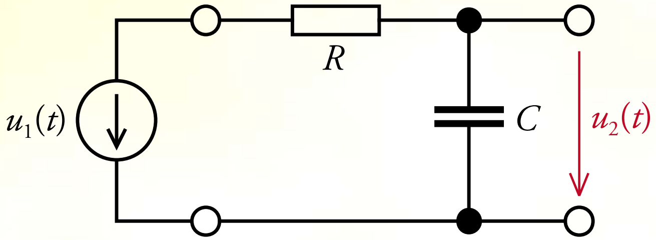
$$\mathbf{U}_2 = 20 \cdot \frac{1}{j \cdot 1000 \cdot 0.001 + 1} = \frac{20}{1.414 \underline{/\frac{\pi}{4}}} = 14.14 \underline{/-\frac{\pi}{4}} \qquad u_2(t) = 14.14 \sin\left(100t - \frac{\pi}{4}\right) \text{ V}$$

3) ω = 10000 s$^{-1}$

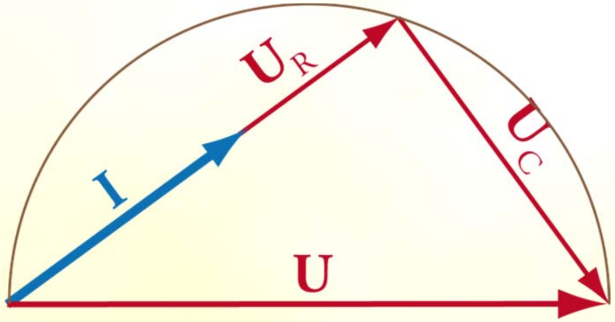
$$\mathbf{U}_2 = 20 \cdot \frac{1}{j \cdot 10000 \cdot 0.001 + 1} = \frac{20}{10.05 \underline{/1.47}} = 1.99 \underline{/-1.47} \qquad u_2(t) = 1.99 \sin\left(100t - 1.47\right) \text{ V}$$

**Since in the circuit is connected reactive circuit element (capacitor ), the amplitude and phase of voltage $u_2(t)$ varies with frequency**

**Phasor diagram** of an integrating RC circuit:

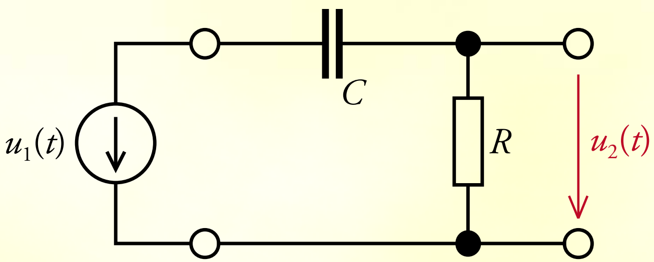


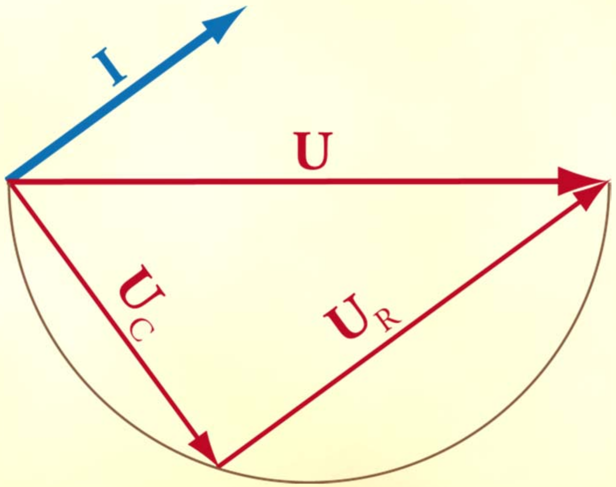

- Voltage and current on resistor are in phase
- On capacitor current leads the voltage by $\frac{\pi}{2}$
- In the circuit in figure all circuit elements are in series – same current flows through them




- Amplitudes of voltages across resistor and capacitor varies with frequency, but $\mathbf{U}_R$ and $\mathbf{U}_C$ are still perpendicular, so vertex of a triangle move along a (half)circle, according to the Thales's theorem
- If the vertex of a triangle is above or under the hypotenuse, corresponding to supplying voltage phasor depends on the order in which the circuit elements are connected – RC, or CR

**Phasor diagram** of a derivative RC circuit:

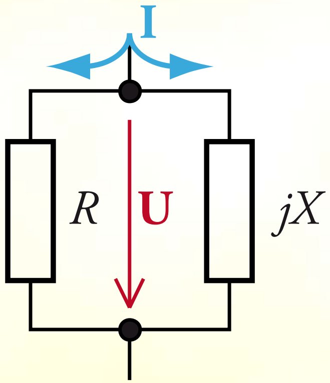



- The current which flows through the circuit is the same like in integrating RC circuit
- But due to opposite order in which the resistor and capacitor are connected in the circuit, amplitude of voltage across resistor $u_2(t)$ increase with frequency, since reactance of a capacitor decrease.

### Current divider

The same relationship, we know from resistive circuit will be valid in slightly modified form in sinusoidal steady state:




$$\mathbf{U} = \mathbf{Z}\mathbf{I} = \frac{R \cdot jX}{R + jX}\mathbf{I} \quad \rightarrow \quad \mathbf{I}_1 = \frac{\mathbf{U}}{R} = \mathbf{I}\frac{R \cdot jX}{R + jX}\frac{1}{R}$$

$$\boxed{\mathbf{I}_1 = \mathbf{I}\frac{jX}{R + jX}} \quad \text{or} \quad \boxed{\mathbf{I}_1 = \mathbf{I}\frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}}$$

More circuit elements:

$$\mathbf{Z} = \left(\sum_{m=1}^{M}\frac{1}{R_m} + \sum_{n=1}^{N}\frac{1}{jX_n}\right)^{-1} = \left(\sum_{m=1}^{M}G_m + \sum_{n=1}^{N}jB_n\right)^{-1}$$

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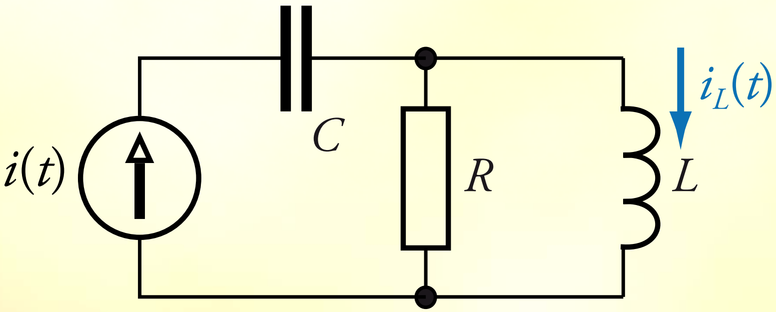
**Example:** circuit in the figure is supplied from sinusoidal current source with amplitude $I_m$ = 10 mA and phase shift $\frac{\pi}{8}$. Values of circuit elements are: $R$ = 1 kΩ, $C$ = 1 μF, $L$ = 0.1 H. Calculate waveform of current $i_L(t)$, which flows through inductor $L$, if the angular frequency of the source is: ω = 10000 s$^{-1}$.




$$\mathbf{I} = 0.01\,\angle\tfrac{\pi}{8}$$

$$\mathbf{I}_L = \mathbf{I}\frac{R}{R + j\omega L} = 0.01\,\angle\tfrac{\pi}{8} \cdot \frac{1000}{1000 + j \cdot 0.1 \cdot 10000}$$

$$= \frac{0.01\,\angle\frac{\pi}{8}}{\sqrt{2}\,\angle\frac{\pi}{4}} = 7.07\,\angle-\tfrac{\pi}{8}\,\text{mA}$$

$$i_L(t) = 7.07\sin\left(10000t - \frac{\pi}{8}\right)\,\text{mA}$$

# Thévenin's and Norton's theorem

It is possible to extend validity of both theorems on sinusoidal steady state:

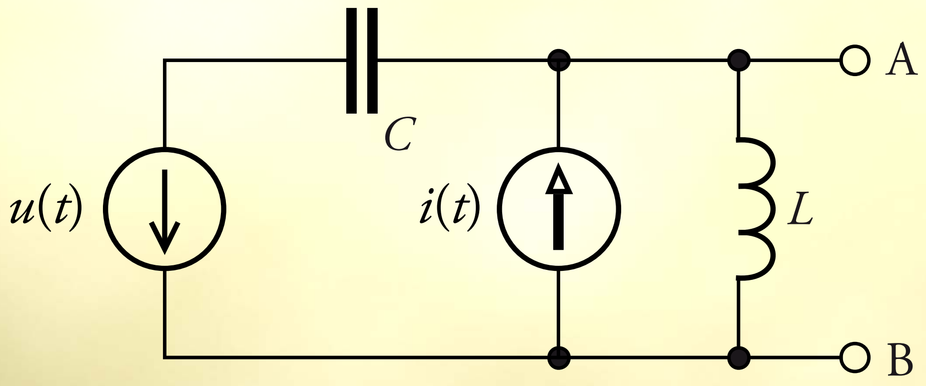
- According to both theorems we may look on any circuit, containing any number of voltage sources, current sources and resistors, as „black box“ with two terminals. Between the terminals we can measure only just open-circuit voltage and internal impedance.

**Limitations:**

- Only **linear circuits**
- All sources in replaced circuit have to have the **same frequency**, or the circuit have to be analyzed step by step for each frequency separately (each group of sources of the same frequency)
- Since power is not linearly dependent on voltage or current, the power dissipation of the Thévenin equivalent $R_i$ is not identical to the power dissipation of the real system
- **The Thévenin's equivalent has an equivalent I-V characteristic only from the point of view of the A, B terminals (the load)!!!**

**Example:**

For the circuit in the figure find the Thévenin's / Norton's equivalent at the terminals A, B. Frequency of **both** sources is $f = 50$ Hz. $C = 3.183$ μF, $L = 1.592$ H, amplitude of voltage source is $U_m = 15$ V, amplitude of current source $I_m = 20$ mA, $\varphi_u = 0$, $\varphi_I = \frac{\pi}{3}$.

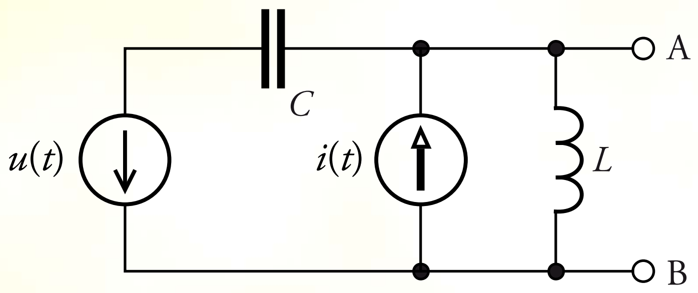



$$u(t) = 15 \sin(100\pi t) \text{ V}$$

$$i(t) = 20 \sin\left(100\pi t + \frac{\pi}{3}\right) \text{ mA}$$

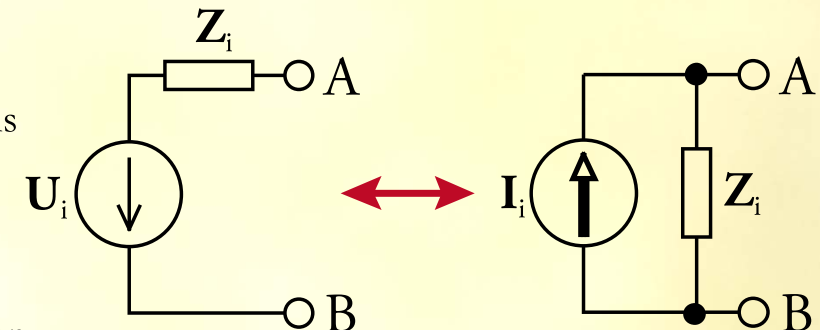
**Example:**

For the circuit in the figure find the Thévenin's / Norton's equivalent at the terminals A, B. Frequency of both sources is $f = 50$ Hz. $C = 3.183$ mF, $L = 1.592$ H, amplitude of voltage source is $U_m = 15$ V, amplitude of current source $I_m = 20$ mA, $\varphi_u = 0$, $\varphi_I = \frac{\pi}{3}$.




$$u(t) = 15\sin(100\pi t)\ \text{V} \longrightarrow \mathbf{U} = 15$$

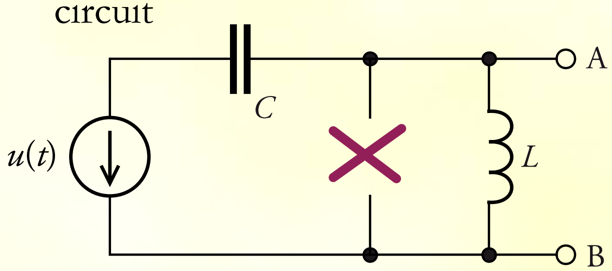
$$i(t) = 20\sin\left(100\pi t + \frac{\pi}{3}\right)\ \text{mA} \qquad \mathbf{I} = 0.02\,\angle\frac{\pi}{3}$$




In the sinusoidal steady state both superposition and equivalence of sources is valid identically as in the sinusoidal steady state. Same are also rules for removing of sources.

To find Thévenin's equivalent voltage we may use superposition:

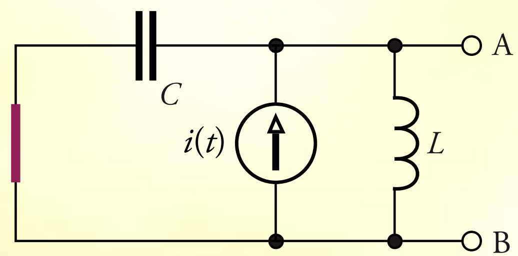
- To remove current source – its current is made zero – we replace it by open circuit




$$\mathbf{U}_i' = \mathbf{U}\frac{j\omega L}{\frac{1}{j\omega C} + j\omega L} = \mathbf{U}\frac{-\omega^2 LC}{1-\omega^2 LC} = 15 \cdot \frac{-314^2 \cdot 1.592 \cdot 3.183 \cdot 10^{-6}}{1 - 314^2 \cdot 1.592 \cdot 3.183 \cdot 10^{-6}} =$$

$$= 15 \cdot -0.51 - 0.5 = -15\,\text{V}$$

- To remove voltage source – it is replaced by short circuit:




$$\mathbf{U}_i'' = \mathbf{I}\frac{j\omega L \cdot \frac{1}{j\omega C}}{j\omega L + \frac{1}{j\omega C}} = \mathbf{I}\frac{j\omega L}{-\omega^2 LC + 1} =$$

$$= 0.02 \cdot \angle\frac{\pi}{3} \cdot \frac{500j}{-0.5+1} = 20\,\angle\left(\frac{\pi}{3} + \frac{\pi}{2}\right) = 20\,\angle\frac{5\pi}{6}\ \text{V}$$

$$\mathbf{U}_i = \mathbf{U}_i' + \mathbf{U}_i'' = -15 + 20\,\angle\frac{5\pi}{6} = -15 - 17.32 + 10j = -32.32 + 10j = \sqrt{32.32^2 + 10^2}\,\angle\tan^{-1}\frac{10}{-32.32} = 33.83\,\angle 2.84\ \text{V}$$

$$\mathbf{Z}_i = \mathbf{Z}_C \parallel \mathbf{Z}_L = 1000j\ \Omega$$

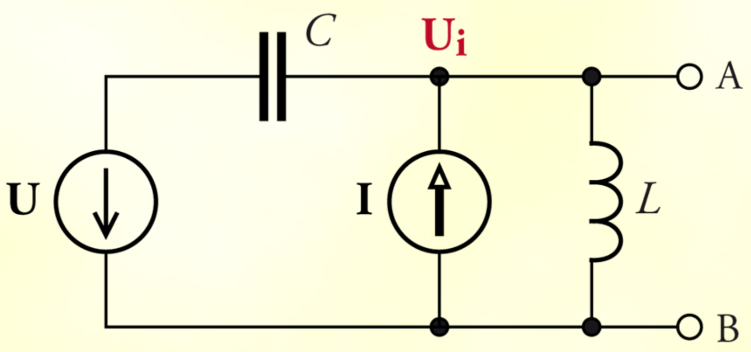
- Norton's equivalent circuit we can simply find by transforming voltage source on current source.
- Or, we can use superposition to find contributions of both sources to short circuit current.
- Or, transform voltage source on current source and then add both current sources.

$$\mathbf{I}_i = \frac{\mathbf{U}_i}{\mathbf{Z}_i} = \frac{33.83\,\underline{/2.84}}{1000j} = 33.83\,\underline{/\left(2.84 - \frac{\pi}{2}\right)} = 33.83\,\underline{/1.27}\,\mathrm{mA} \qquad \mathbf{Z}_i = \mathbf{Z}_C \parallel \mathbf{Z}_L = 1000j\,\Omega$$

## Circuit equations

- Circuit equations may be used also in a sinusoidal steady state
- Using generalized Ohm's law, circuit equations will be written using the same rules as with DC circuits
- When in the circuit are AC sources of different frequencies (and / or DC sources), then one set of circuit equations may be written only with sources of the same frequency, other sources must be removed out of the circuit! – harmonic superposition

### Nodal analysis



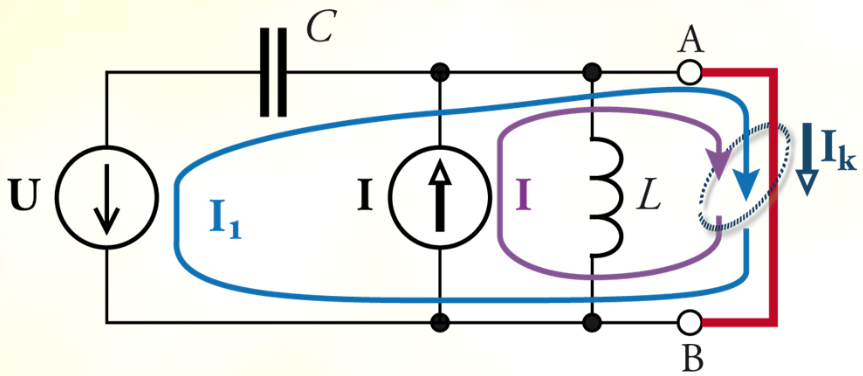

1 equation:

$$-\mathbf{I} + \frac{\mathbf{U}_i - \mathbf{U}}{\frac{1}{j\omega C}} + \frac{\mathbf{U}_i}{j\omega L} = 0$$

or $\left[j\omega C + \frac{1}{j\omega L}\right]\left[\mathbf{U}_i\right] = \left[I + j\omega C\mathbf{U}\right]$

$$\mathbf{U}_i = \frac{I + j\omega C\mathbf{U}}{j\omega C + \frac{1}{j\omega L}} = \frac{0.02\,\underline{/\frac{\pi}{3}} + j\cdot 314\cdot 3.183\cdot 10^{-6}\cdot 15}{j\cdot 314\cdot 3.183\cdot 10^{-6} + \frac{1}{j\cdot 314\cdot 1.592}} = \frac{0.0250 + 0.0173j}{-0.001j} = -32.32 + 10j = 33.83\,\underline{/2.84}\,\mathrm{V}$$

## Mesh analysis



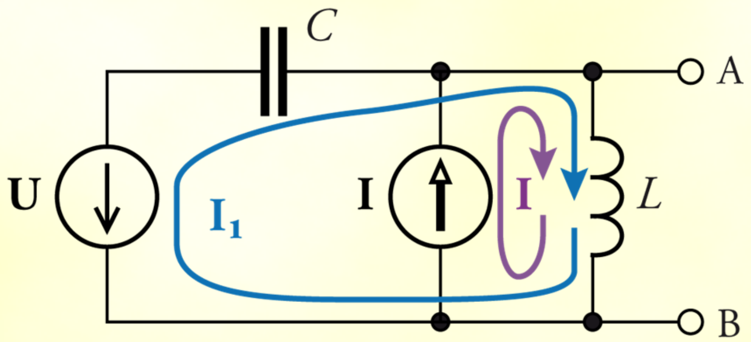

1 equation:

$$-\mathbf{U} + \frac{1}{j\omega C}\mathbf{I}_1 = 0$$

$$\mathbf{I}_1 = \mathbf{U}\, j\omega C = 15 \cdot j \cdot 314 \cdot 3.183 \cdot 10^{-6} = 0.015j \text{ A}$$

$$\mathbf{I}_k = \mathbf{I}_1 + \mathbf{I} = 0.015j + 0.02 \angle \tfrac{\pi}{3} = 0.01 + 0.0323j = 0.0338 \angle 1.27 \text{ A}$$

$$\mathbf{Z}_i = \frac{\mathbf{U}_i}{\mathbf{I}_k} = 1000j\,\Omega$$



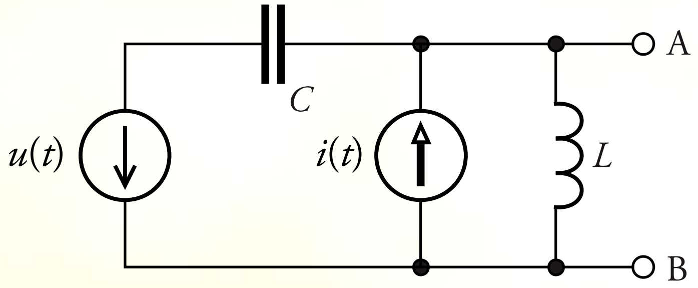

$$-\mathbf{U} + \frac{1}{j\omega C}\mathbf{I}_1 + j\omega L\,(\mathbf{I}_1 + \mathbf{I}) = 0$$

$$\mathbf{U}_i = j\omega L\,(\mathbf{I}_1 + \mathbf{I}) = 0$$

The result is, of course, the same.

**When the frequency of both sources is different, we must write an individual equations** *(compare with previous example)*:

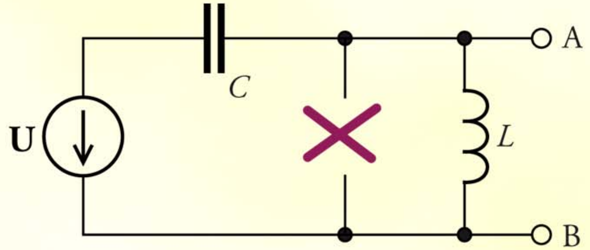
For the circuit in the figure find the Thévenin's / Norton's equivalent at the terminals A, B. Frequency of the voltage source is $f = 50$ Hz, frequency of the current source is $f = 100$ Hz. $C = 3.183$ μF, $L = 1.592$ H, amplitude of voltage source is $U_m = 15$ V, amplitude of current source $I_m = 20$ mA, $\varphi_u = 0$, $\varphi_I = \frac{\pi}{3}$.

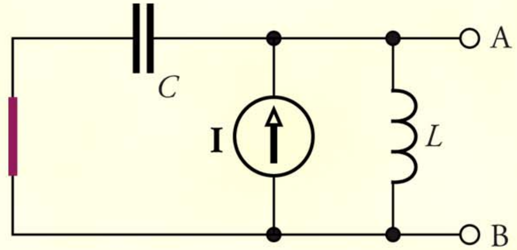



$$u(t) = 15\sin(100\pi t)\ \mathrm{V} \qquad \mathbf{U} = 15$$

$$i(t) = 20\sin\left(200\pi t + \frac{\pi}{3}\right)\ \mathrm{mA} \qquad \mathbf{I} = 0.02\angle\frac{\pi}{3}$$

**Phasors are same, as in the previous example, but, nevertheless, due to different frequency of both sources, they can't be included in the same equation**




$$\frac{\mathbf{U}_1' - \mathbf{U}}{\frac{1}{j\omega_1 C}} + \frac{\mathbf{U}_1'}{j\omega_1 L} = 0 \quad \rightarrow \quad \mathbf{U}_1' = \frac{\mathbf{U} j\omega_1 C}{j\omega_1 C + \frac{1}{j\omega_1 L}} = \frac{15 \cdot j \cdot 314 \cdot 3.183 \cdot 10^{-6}}{j \cdot 314 \cdot 3.183 \cdot 10^{-6} + \frac{1}{j \cdot 1.592 \cdot 314}} = -15\ \mathrm{V}$$

$$-\mathbf{I} + \mathbf{U}_1'' j\omega C + \frac{\mathbf{U}_1''}{j\omega L} = 0 \quad \rightarrow \quad \mathbf{U}_1'' = \frac{\mathbf{I}}{j\omega C + \frac{1}{j\omega L}} = \frac{0.02\angle\frac{\pi}{3}}{j \cdot 628 \cdot 3.183 \cdot 10^{-6} + \frac{1}{j \cdot 628 \cdot 1.592}} = 20\angle-\frac{\pi}{6}\ \mathrm{V}$$

**I can't make a sum of phasors, but waveforms only** (the result is not sinusoidal)

$$u_1(t) = u_1'(t) + u_1''(t) = 15\sin(314t + \pi) + 20\sin\left(628t - \frac{\pi}{6}\right)\ \mathrm{V}$$

# Recapitulation: Problem-Solving strategy

- Express $u(t)$, $i(t)$ as a phasor and determine impedance of each passive element
- Use suitable method of analysis to find phasor of desired circuit variable
- Convert the phasor $\mathbf{U}$, $\mathbf{I}$ to $u(t)$, $i(t)$.

☞ For relatively simple circuits use:

- Ohm's law for ac analysis $\mathbf{U} = \mathbf{ZI}$
- The rules for combining $\mathbf{Z}_s$ and $\mathbf{Y}_p$
- KCL and KVL
- Current and voltage division
- Step by step simplification method
- Delta star transformation ($Y$-$\Delta$ transform)

☞ For more complicated circuits, especially with multiple sources, use:

- Nodal analysis, or
- Mesh / loop analysis (whatever has less number of equations)
- Superposition
- Source transformation method
- Thévenin's and Norton's theorems