# Frequency Response, Bode Plot

## Frequency dependence of impedance

Since $\mathbf{Z}_C = \frac{1}{j\omega C} = -j\frac{1}{\omega C}$, and $\mathbf{Z}_L = j\omega L$, the impedance of the capacitor and the inductor is frequency dependent, unlike the resistor – see Figure 1, 2, and 3.

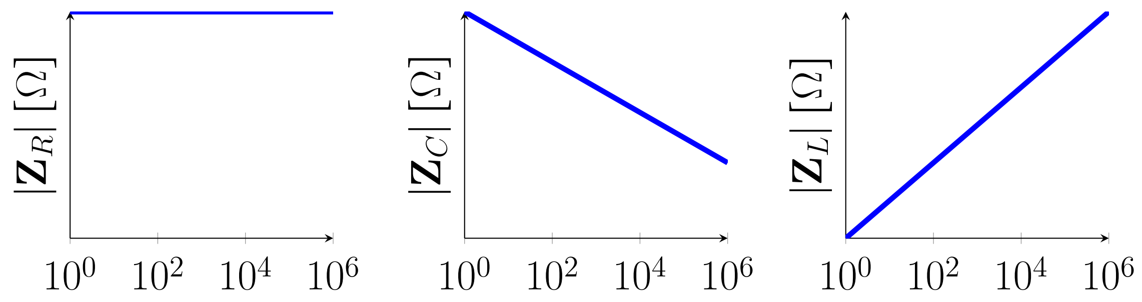
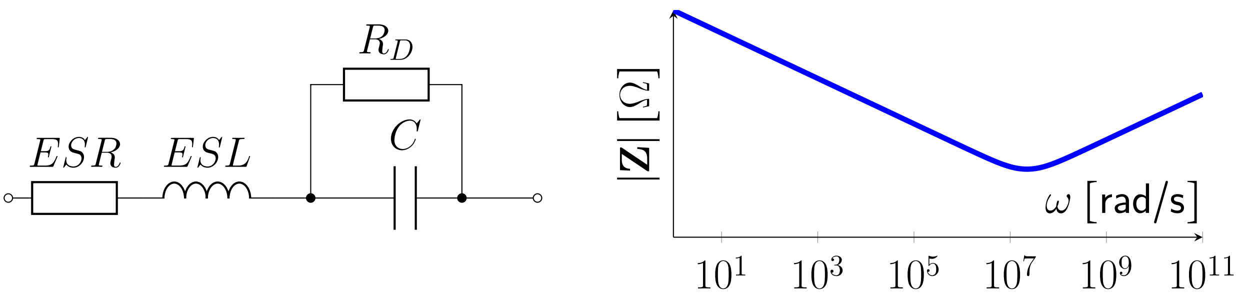



Figure 1 Figure 2 Figure 3

The frequency dependence of the impedance is important, for example, when assessing physical components' applicability for a given application. For example, a physical capacitor, in addition to the required capacitance, always has the inductance and resistance of the supply conductors. In some cases, it is necessary to consider the final resistance of the dielectric. Frequency dependence of the magnitude of the impedance of a practical capacitor with the model according to Fig. 4a is on the Figure .




(a) (b)

Figure 4: Physical capacitor

By comparing Figure with the Figures 2 and 3 it is clear that the physical capacitor acts as a capacitor only for low frequencies. Its impedance module has a (resonant) minimum (equal to $ESR$). Above the frequency at which this minimum is, the impedance modulus no longer decreases, as would be the case with an ideal capacitor, but rather increases – the physical capacitor device acts as an inductor. Therefore, we cannot use components with terminals for high-frequency circuits, and even the maximum size of SMD components is limited (e.g., USB 3 should have a maximum size of 0402). Therefore, in some applications (e.g., decoupling capacitor), we put more capacitors with different values in parallel – they have different minimum frequencies.

## Introductory motivation

Since the capacitor and inductor's impedance is frequency-dependent, it always changes with frequency, and voltages and currents in the circuit change with frequency as well. In electrical circuits, we are most often interested in the frequency dependence of the circuit's output voltage. E.g., the amplifier can amplify the signal only in a limited range of frequencies, where the output voltage is not dampened. Similarly, the speakers are limited in frequency and can not play very low or very high frequencies. On the other hand, some circuits are intentionally designed to transmit a signal only in a limited frequency range – such as filters. For sampling analog signals (conversion from analog voltage to digital form), we need to limit the signal's maximum frequency to half the sampling frequency. For EEG and ECG signals (if we do not do this later in digital processing), we must eliminate isoline fluctuations or 50 Hz interference, etc. Drawing frequency characteristics is, therefore, an essential part of the knowledge of electrical circuit analysis.

## Hodograph

Mapping of complex functions $\mathbf{F} = \mathrm{Re}\{\mathbf{F}\} + j\,\mathrm{Im}\{\mathbf{F}\}$ (or $F(j\omega)$) is called a hodograph. This complex function can be an impedance, admittance, voltage phasor, current phasor, etc. In order that the graph does not depend on the magnitude of the input voltage, we draw the transfer function frequency dependence instead of the voltage, where the transfer function $\mathbf{H} = \frac{\mathbf{V}_2}{\mathbf{V}_1}$ (or we use the notation $H(j\omega) = \frac{V_2(j\omega)}{V_1(j\omega)}$ as well). We can represent this dependence with a hodograph in the complex plane. On the Figure 5 the impedance of the series connection of the resistor and capacitor is displayed, $\mathbf{Z} = R + \frac{1}{j\omega C}$.

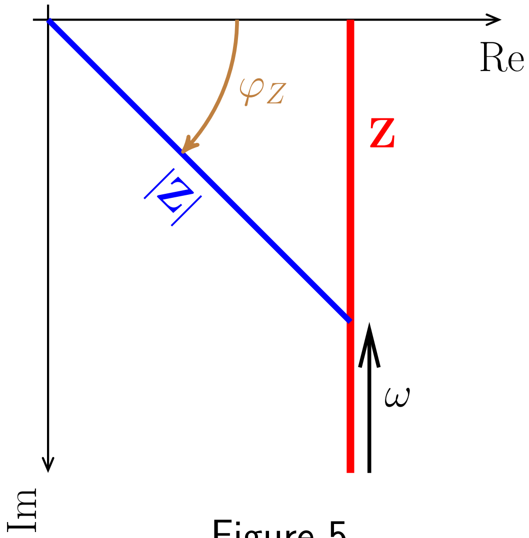



Figure 5

The resistance $R$ is a real part of a complex number. In this example, it is constant. $-j\frac{1}{\omega C}$ is an imaginary part of a complex number. For $\omega = 0$ it is $-\infty j$. For $\omega \to \infty$ it is $0$. In the Gaussian plane, this impedance's representation is the line between the points $(R, -\infty)$, and $(R, 0)$. Each point of the line corresponds to a certain frequency – it is, therefore, the frequency axis. For a given frequency, we read the impedance modulus as the distance of the corresponding point on the line with frequency $\omega$ from the origin, and the phase as the angle between the positive real half-axis and the line representing the impedance modulus $|\mathbf{Z}|$.
The admittance hodograph is more interesting. The inversion of a line in a complex plane is a circle $\mathbf{Y} = \frac{j\omega C}{1+j\omega C}$, see Figure 6.

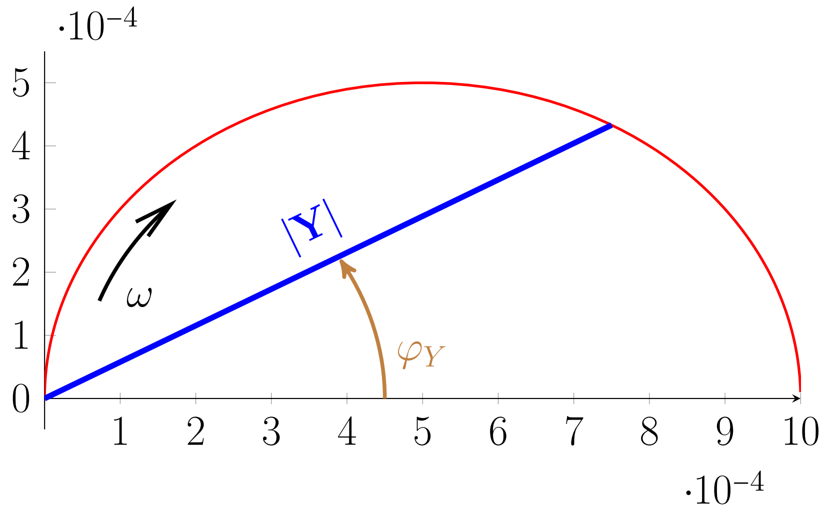



Figure 6

The length of the blue line is the modulus of admittance. The angle that the line makes with the positive real half-axis is the admittance phase.
If we would connect the voltage $V$ with zero phase shift to our RC network terminals, the voltage phasor could be drawn on the real axis. Since $\mathbf{I} = V\mathbf{Y}$, the blue line would represent not only the admittance but with a different scale on the axis also a current phasor; and also, with its own scale also a voltage phasor on the resistor. From the admittance's hodograph, it is, therefore, possible to derive a phasor diagram for any frequency. And since, also, $\mathbf{S} = V^2\mathbf{Y}^*$, it is possible to read (in this case) from the perpendicular to the phasor admittance (in this case) on the real axis the power factor of the circuit.

Similarly, for the RC network on the Figure 7 we can draw the transfer function 1 into the complex plain according to the Figure 8

$$\mathbf{H} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC} \tag{1}$$

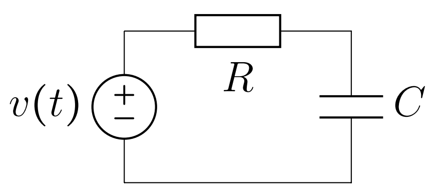



Figure 7

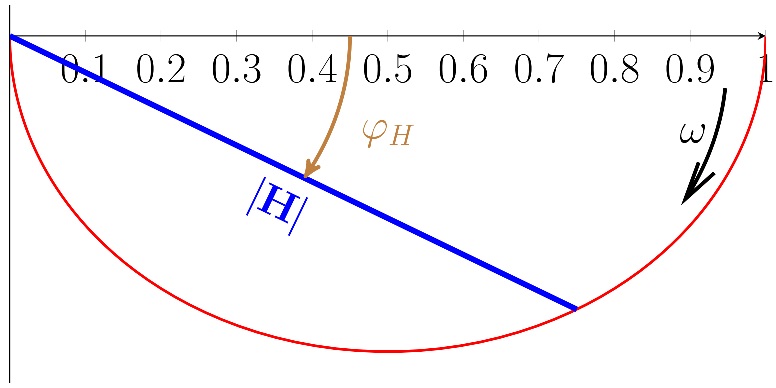



Figure 8

The hodograph of the transfer function is essential mainly for assessing the stability of the circuit according to the Michailov-Leonhard criterion. However, this is not the lecture's topic, so I will not deal with it in more detail here.

## Frequency response

To assess the influence of frequency on the amplitude and phase of a circuit quantity, its exponential shape is more suitable. In the following parts, we will deal with the transfer function of the circuit:

$$\mathbf{H} = \frac{\mathbf{V}_2}{\mathbf{V}_1} = H\, e^{j\varphi_H} \tag{2}$$

E.g., for the RC network on the Figure 7, when we rearrange the equation 1, the transfer function is :

$$\mathbf{H} = \frac{1}{\sqrt{1 + (\omega RC)^2}}\, e^{j\,\arctan(\omega RC)} \tag{3}$$

We draw the modulus and phase in the equation 3 into two separate graphs – as the magnitude and the phase characteristic.

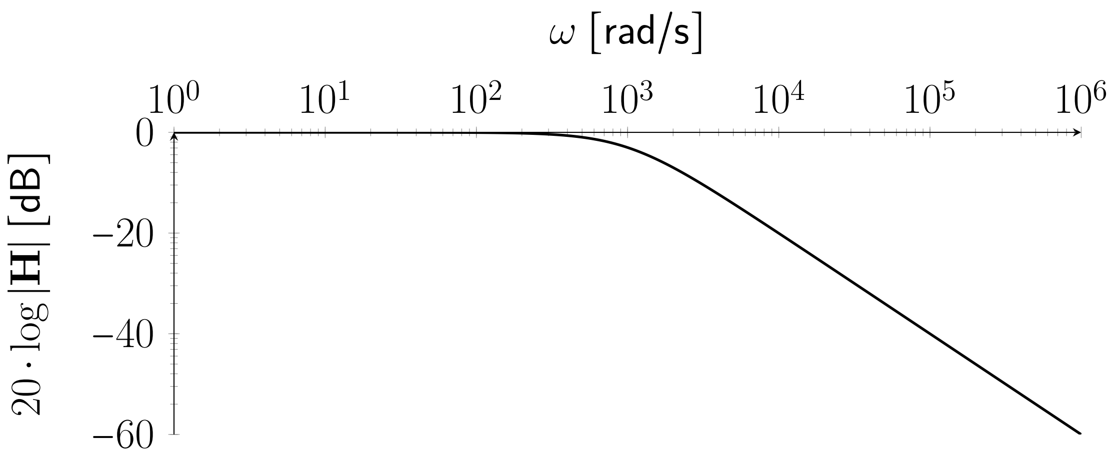



Figure 9: Magnitude frequency characteristic of the RC integrator (Figure 7)

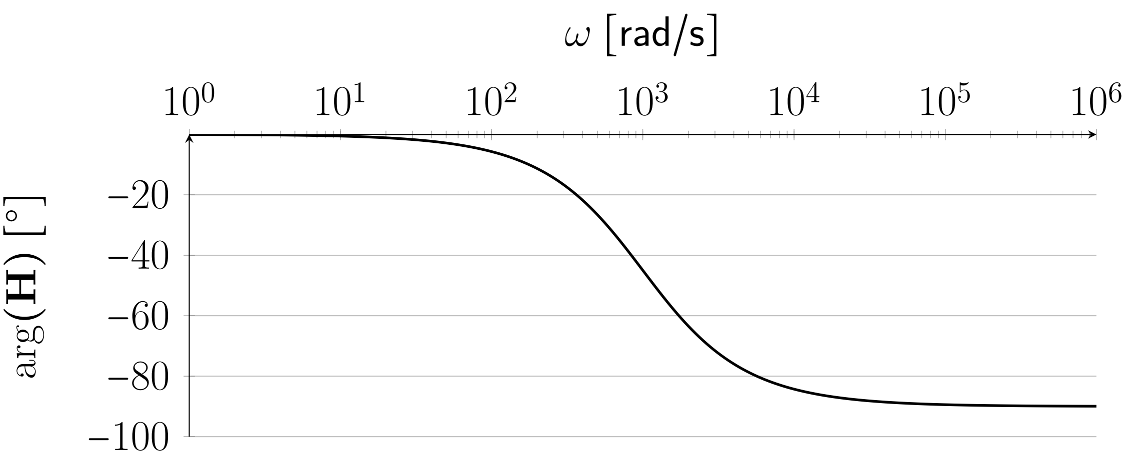



Figure 10: Phase frequency characteristic of the RC integrator (Figure 7)

The magnitude characteristic is a graphical representation of the expression $\frac{1}{\sqrt{1+(\omega RC)^2}}$, phase characteristic is a graphical representation of the expression $\arctan(\omega RC)$.

The following applies to the magnitude characteristic:

1. The horizontal axis represents the frequency and is measured in Hz, or in $\mathrm{rad\,s^{-1}}$
2. The horizontal axis is logarithmic
3. The vertical axis is logarithmic

4. The vertical axis represents the value of $20 \cdot \log |\mathbf{H}|$, and is measured in Decibel [dB]

The reason for using the frequency axis in a logarithmic scale is the possibility to sketch frequency dependence in an extensive range of frequencies. In the case of a linear scale and a broad range of frequencies, it would not be possible to display break frequencies at low frequencies. Similarly, the logarithmic scale on the vertical axis allows us to draw an extensive dynamic range in the order of millions or even more. Small changes would not be visible on a linear scale.

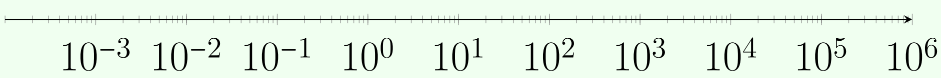
It can take us aback that the multiplicative constant is 20 when the unit is Decibel, and the deci - the decimal unit prefix is 10. The Decibel unit was first used to display power, in which case we use the multiplicative constant 10. There is a relationship between voltage in the scale of maximum values and power $P = \frac{1}{2}\frac{V_m^2}{R}$ – and $\frac{1}{2}20 = 10$.

The following applies to the phase characteristic:

1. The horizontal axis represents the frequency and is measured in Hz, or in rad s$^{-1}$
2. The horizontal axis is logarithmic
3. The vertical axis is linear
4. The vertical axis represents the value of $\arg \mathbf{H}$, thus the phase. It is measured in radians [rad], or degrees [°]

**Definition 1**

**Decade** is the frequency interval $\omega \ldots 10\omega$. Thus, a decade represents a tenfold increase in frequency. Since $\log(10\omega) = 1 + \log\omega$, all decades on the frequency axis are the same size.




## Integrator circuit

RC network in the Figure 7 is known as an RC *integrator*. Why the integrator? Recall the relationship for a capacitor $v(t) = \frac{1}{C}\int_0^t i(\tau)\,\mathrm{d}\tau + v_C(0)$. If we substitute for the current the waveform $i(t) = I_m \sin(\omega t)$, the voltage is $v(t) = \frac{I_m}{\omega C}\sin\left(\omega t - \frac{\pi}{2}\right) + \frac{I_m}{\omega C} + v_c(0)$. When we transform this equation into the frequency domain, it has form $\mathbf{V} = \frac{1}{j\omega C}\mathbf{I}$. Similarly, in the Laplace transform, the transform of the integral is division by the Laplace operator, i.e. $\mathscr{L}\{\int_0^t i(\tau)\,\mathrm{d}\tau\} = \frac{1}{p}I(p)$ where $p$ (or $s$) is Laplace's operator. The transform of the integral in the frequency domain is an expression $\frac{1}{j\omega}$, or $\frac{1}{p}$. Of course, the same as for the relationship between voltage and current on the capacitor, also applies to the relationship between output and input voltage. Thus, $\mathbf{V}_2 = \frac{1}{j\omega}\mathbf{V}_1$ is the transform of the integral of the input voltage $v_1(t)$. For our RC network, however, this only applies if in the expression $\frac{1}{1+j\omega RC}$ we can omit 1, so if $1 \ll \omega RC$. In the frequency response, this only applies at frequencies where the amplitude decreases and at the same time the phase shift is close to $\frac{-\pi}{2}$, thus for high frequencies at least ten times greater, than the break frequency.

## Differentiator

RC network in the Figure 11 is known as an RC *differentiator*. The circuit transfer function is:

$$\mathbf{H} = \frac{R}{\frac{1}{j\omega C} + R} = \frac{j\omega RC}{1 + j\omega RC} = \frac{\omega RC}{\sqrt{1 + (\omega RC)^2}}\, e^{j\left(\frac{\pi}{2} - \arctan(\omega RC)\right)} \tag{4}$$

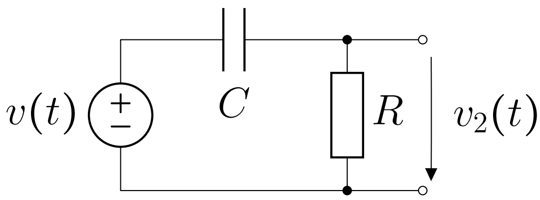



Figure 11: RC differentiator

In the Figure 9 is the magnitude frequency characteristic, and in the Figure 13 is the phase frequency characteristic of the RC differentiator.

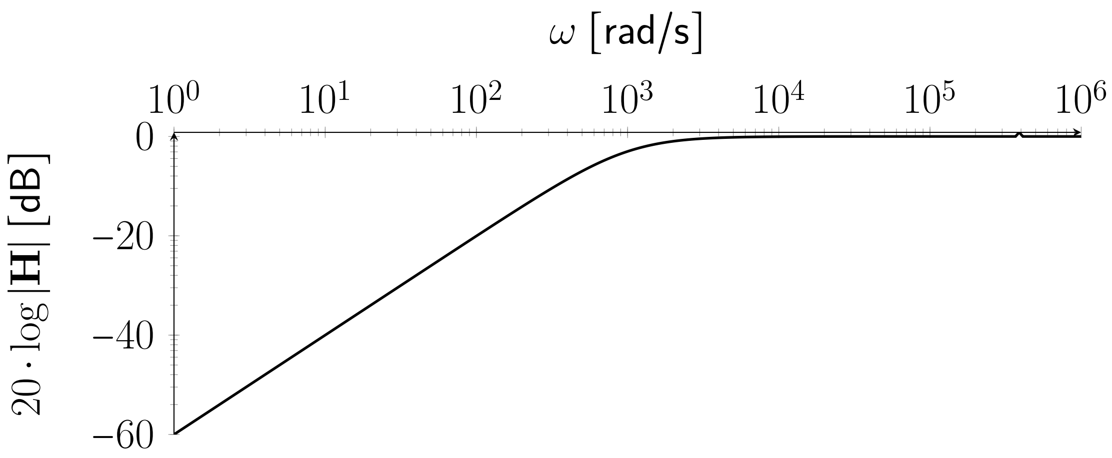



Figure 12: Magnitude frequency characteristic of RC differentiator

A common mistake is to claim that the lowest value of the amplitude characteristic in this example is a value −60 dB. But remember the definition of decade and the logarithmic axis. On the horizontal axis, the same decades for frequency continue between 0.1, 0.01, . . . until $-\infty$. This means that the amplitude characteristic decreases to the level $-\infty$.

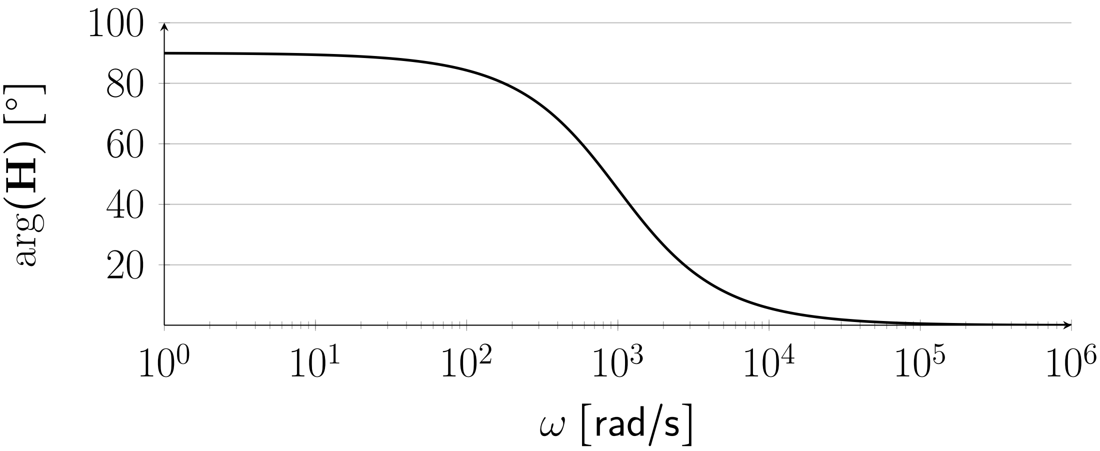
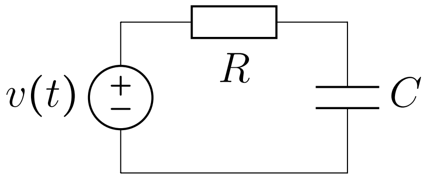



Figure 13: Phase frequency characteristic of RC differentiator

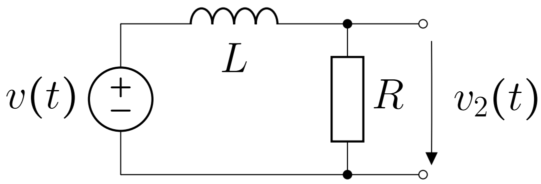
The transform of the derivative in the frequency domain is the expression $j\omega$, or multiplication by the Laplace operator $p$ (or $s$), and the initial condition. Only if $\omega RC \ll 1$ can $\omega RC$ be neglected in the denominator of the equation 4, so the transfer function $\mathbf{H} = j\omega RC$ is then the transform of a derivative. The RC network in Figure 11 operates as a differentiator only at low frequencies, at least ten times (the phase condition) less than the break frequency.

## RC and RL network duality

Let's compare the transfer functions of the following RC and RL networks:



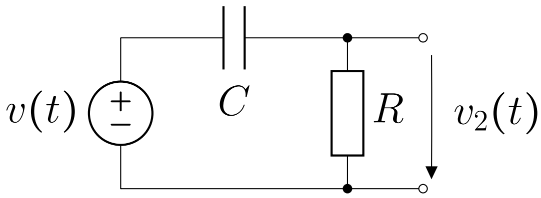

$$\mathbf{H} = \frac{1}{1 + j\omega RC}$$



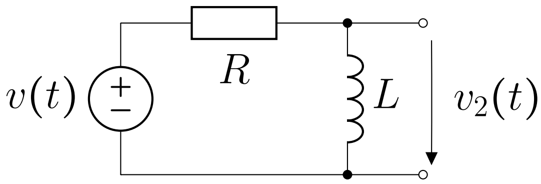

$$\mathbf{H} = \frac{1}{1 + j\omega \frac{L}{R}}$$

If $RC = \frac{L}{R}$, both circuits have identical frequency response according to the Figure 9 and Figure 10. In terms of the frequency response, it is then an identical integrator.

Now compare following two networks:




$$\mathbf{H} = \frac{j\omega RC}{1 + j\omega RC}$$




$$\mathbf{H} = \frac{j\omega \frac{L}{R}}{1 + j\omega \frac{L}{R}}$$

If $RC = \frac{L}{R}$, both circuits have identical frequency response according to the Figure 12 and Figure 13. In terms of the frequency response, it is then an identical differentiator.

## Bode plot

Magnitude and phase characteristics according to equations 3 and 4 are not easy to draw without the use of a computer, especially if the circuits would be more complicated than a simple RC or RL network. Therefore, Hendrik Wade Bode proposed a set of rules for modifying the circuit transfer function and drawing the frequency response of the circuit using several lines – asymptotes.

Let's return to the RC integrator network according to Figure 7 with transfer function $\mathbf{H} = \frac{1}{1+j\omega RC}$ and true magnitude characteristic in the Figure 9. Since we plot the logarithm of the transfer function's module on the vertical axis, we can use the logarithm's properties – the logarithm of the product is the sum, and the logarithm of the quotient is the difference of the logarithms. Is, therefore:

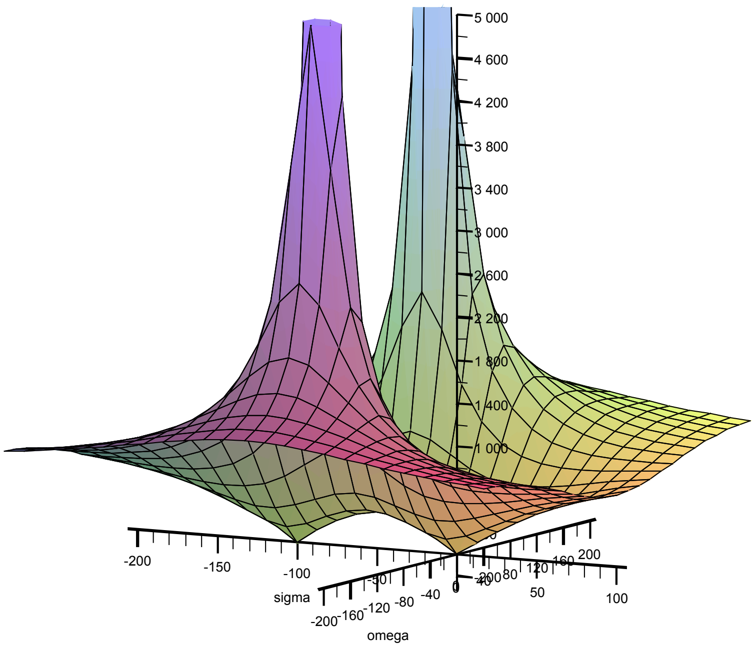
$$20\log\left|\frac{1}{1+j\omega RC}\right| = 20\log 1 - 20\log|1+j\omega RC| \tag{5}$$

$20\log 1 = 0$ and the numerator therefore has no effect on the resulting characteristic. In the denominator we can study two cases:

1. $1 \gg \omega RC$
   In this case, we can ignore the term $\omega RC$ and $1 + \cancel{j\omega RC} \to 1$
   $20\log 1 = 0\,\mathrm{dB}$
   This is a horizontal line with a level 0 dB.
   It's a real number, so the phase is equal to 0
2. $1 \ll \omega RC$
   In this case, we can ignore the term $1$, and $\cancel{1} + j\omega RC \to j\omega RC = \omega RC\, e^{j\frac{\pi}{2}}$
   Magnitude of the term $\omega RC$ changes with frequency. If the frequency changes ten times, it is $20\log(10\omega RC) = 20\log 10 + 20\log(\omega RC) = 20 + \log(\omega RC)$. If the frequency changes a hundred times, it is $20\log(100\omega RC) = 20\log 100 + 20\log(\omega RC) = 40 + \log(\omega RC)$. If

a thousand times, it is $60 + \log(\omega RC)$, etc. The change in amplitude is therefore 20 dB per decade. And because the decades have the same size on the axis, it is a straight line. Ascending in the case of the numerator, decreasing (negative sign) in the case of the denominator. The phase is $\frac{\pi}{2}$ in the case of numerator, $\frac{-\pi}{2}$ in the case, that the term is in denominator.

The first line passes into the second on the so-called *break frequency*, when $1 = \omega RC$, i.e., the imaginary part is as large as the real part (normalized to 1). So the break frequency is $\omega_p = \frac{1}{RC}$. For the numerator we use the symbol $\omega_0$ and we call this frequency *zero*, for the denominator the symbol $\omega_p$ and we call this frequency *pole*. This designation comes from the transfer function in Laplace transform form, where the expression $1 - p \cdot p_0 = 0$, or $1 - p \cdot p_p = 0$ and the numerator thus makes the result of the transfer function equal to 0, while dividing by 0 in the denominator, makes the transfer function going to $\infty$. For example, the RLC with transfer function $H(s) = \frac{sC(sL+R)}{s^2LC+sRC+1}$ has the folowing graph in the complex s-plane:




Frequency response is a sectional view of this graph for a $\sigma = 0$.

So we can write the expression $1 + j\omega RC$ as

$$1 + j\frac{\omega}{\omega_0} \tag{6}$$

for the numerator, or with $\omega_p$ for denominator. $\omega_0 = \frac{1}{RC}$.

The asymptotic magnitude frequency response thus looks like this:

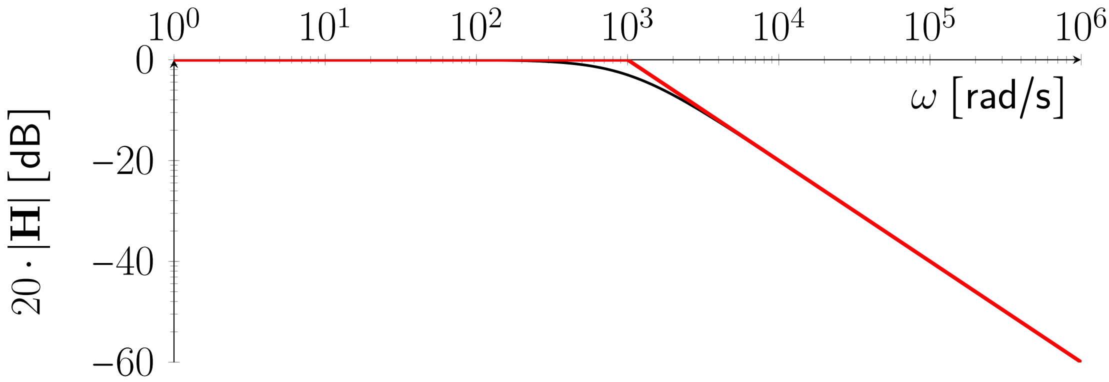



Figure 14: Asymptotic magnitude frequency characteristic of the RC integrator (Magnitude characteristic of the Bode plot)

Asymptotes are marked in red, and the actual magnitude frequency response $\frac{1}{\sqrt{1+(\omega RC)^2}}$ in black.The most significant deviation of the asymptotes from the actual curve is at the break frequency, where the actual transfer is $1 + j\frac{1}{RC}RC = 1 + j$, and its magnitude has the value of $20 \log \sqrt{1+1} = 3\,\text{dB}$. For this reason, for example, the frequency range of the amplifier is referred to as the frequency band at which the output voltage is not attenuated by more than 3 dB.

However, this is not so easy with the asymptotic phase frequency response. Phase frequency characteristic is constant at low and at high frequencies. So if we used only two levels, 0 rad, and $\frac{\pi}{2}$rad, the asymptotic characteristic would be stepwise at the break frequency, which would be too large a deviation from the actual characteristic, which is the $\arctan$ function. Therefore, the asymptotic phase characteristic has two break frequencies, at $0.1\omega_0$ and $10\omega_0$, and the asymptote is increasing (numerator), respectively descending (denominator) line over two decades.

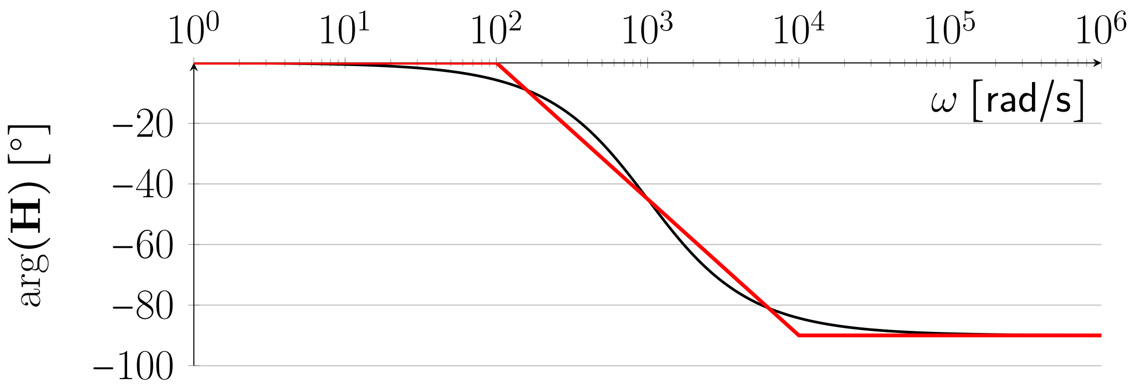



Figure 15: Asymptotic phase frequency characteristic of the RC integrator (Phase characteristic of the Bode plot)

The asymptotic characteristic is marked in red, the actual phase characteristic $-\arctan(\omega RC)$ is marked in black.

The RC differentiator 11 has the transfer function $\mathbf{H} = \frac{j\omega RC}{1+j\omega RC}$. We know that the logarithm of a quotient is the difference of logarithms. That is why we can deal with the numerator and the denominator separately. We already know about the $j\omega RC$ term in the numerator that it is an asymptote that grows with a slope of 20 dB per decade. Since there is no real part, this line grows from $-\infty$ to $+\infty$ and intersects the frequency axis at the frequency $\omega_0 = \frac{1}{RC}$, where $\omega RC = 1$. We draw the numerator and denominator separately and then sum them up graphically:

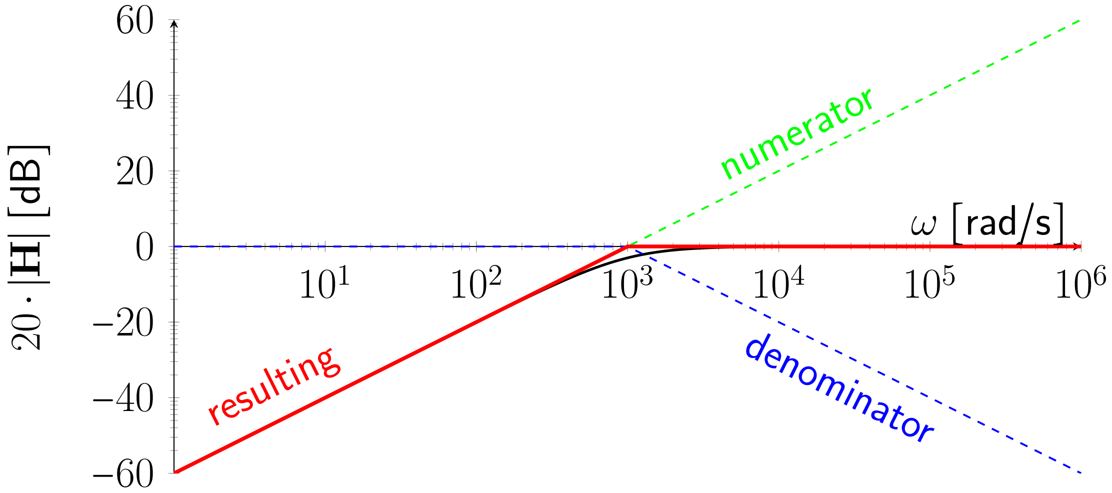



Figure 16: Magnitude characteristic of the Bode plot of the RC differentiator

The phase asymptotic frequency response of RC differentiator is the combination of the term $\frac{\pi}{2}$ in the numerator, and in the case of denominator the same phase response integration cell already had. The graphic sum gives:

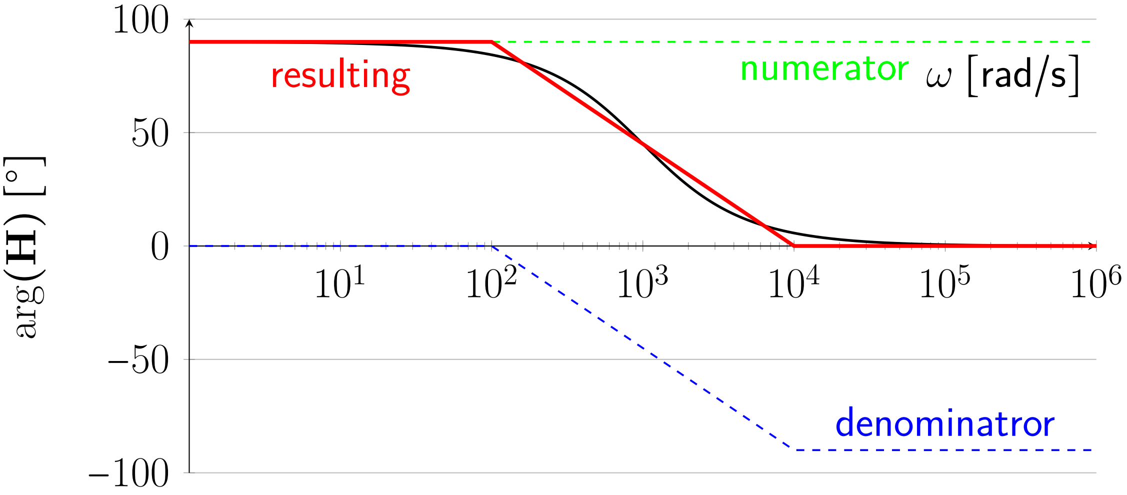
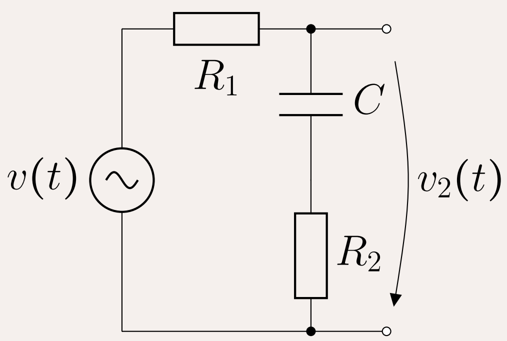



Figure 17: Phase characteristic of the Bode plot of RC differentiator

**Example 1**

Sketch the magnitude and phase characteristic of the Bode plot, labeling all critical slopes and points for the circuit in the figure. $R_1 = 9\,\mathrm{k}\Omega$, $R_2 = 1\,\mathrm{k}\Omega$, $C = 1\,\mu\mathrm{F}$.




The transfer function is:

$$\mathbf{H} = \frac{\frac{1}{j\omega C} + R_2}{R_1 + \frac{1}{j\omega C} + R_2} = \frac{1 + j\omega R_2 C}{1 + j\omega C(R_1 + R_2)}$$

In this case, we had to eliminate the $\frac{1}{j\omega C}$ fraction in both the numerator and denominator. Therefore, we multiplied the numerator and denominator by the term $j\omega C$. This gave us the expression $1 + j\omega RC$, already known from the first example. Further adjustment is therefore not necessary.

The break frequencies in the numerator and denominator are:

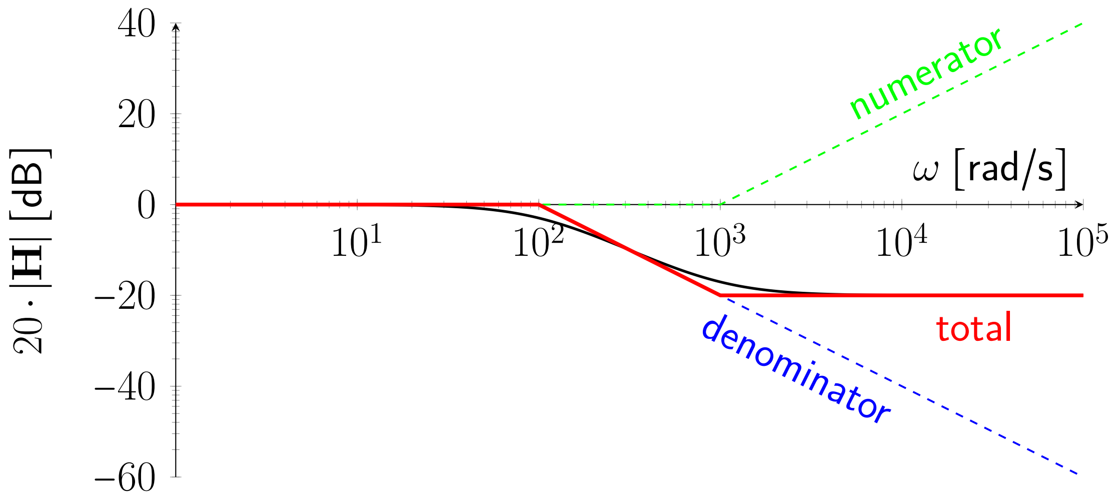
$$\omega_0 = \frac{1}{R_2 C} = \frac{1}{1000 \cdot 10^{-6}} = 1000\,\mathrm{rad\,s^{-1}}$$

$$\omega_p = \frac{1}{(R_1 + R_2) C} = \frac{1}{10000 \cdot 10^{-6}} = 100\,\mathrm{rad\,s^{-1}}$$

Thus the modified transfer function is:

$$\mathbf{H} = \frac{1 + j\frac{\omega}{1000}}{1 + j\frac{\omega}{100}}$$

Magnitude characteristic of the Bode plot:




Thus – the numerator is constant 0 dB until the break frequency $1000\,\mathrm{rad\,s^{-1}}$, then it grows with the slope of 20 dB per decade. Denominator is constant 0 dB until the break frequecy $100\,\mathrm{rad\,s^{-1}}$, and then slope down of −20 dB per decade. We add all asymptotes graphically. The resulting characteristic is constant 0 dB + 0 dB until the frequency $100\,\mathrm{rad\,s^{-1}}$. Between $100\,\mathrm{rad\,s^{-1}}$ and $1000\,\mathrm{rad\,s^{-1}}$ the numerator is still constant 0 dB, but denominator falls down with slope of −20 dB per decade – so the total characteristic is the same. From the frequency $1000\,\mathrm{rad\,s^{-1}}$ denominator still falls down, but the numerator grows with the same slope of 20 dB, and the resulting slope is thus constant.

The level of this constant asymptote can be determined in two ways:

1. The asymptote is contant untill $+\infty$. So we can calculate $\lim\limits_{\omega\to\infty} \mathbf{H} = 0.1$. The asymptote therefore has a level $20\log 0.1 = -20\,\mathrm{dB}$.

2. We can calculate how much decibel decreases the decreasing asymptote ot the total characteristic. It is the line with equation $y = kx + q$, where $k$ is the slope $-20\,\mathrm{dB}$ per decade. There is a level difference in logarithmic coordinates $-20\log\frac{\omega_2}{\omega_1}$, where $\omega_1$ is the initial frequency and $\omega_2$ is the final frequency. In this example $\omega_1 = 100\,\mathrm{rad\,s^{-1}}$ and $\omega_2 = 1000\,\mathrm{rad\,s^{-1}}$, so the drop is by $-20\log 10 = -20\,\mathrm{dB}$.

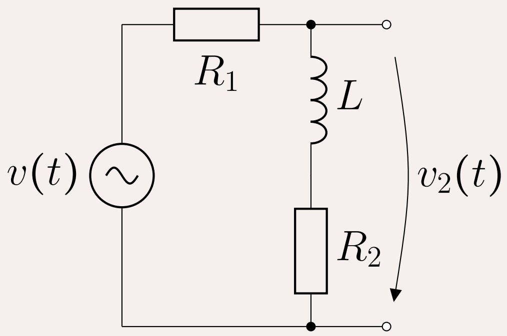
Phase characteristic of the Bode plot:


The phase of the numerator is constant 0 until the frequency $100\,\mathrm{rad\,s^{-1}}$ (one-tenth of the break frequency), and then grows up over two decades (ten times the break frequency) to $\frac{\pi}{2}$, or 90° and then is constant. Denominator is constant 0 until the frequency $10\,\mathrm{rad\,s^{-1}}$ (one-tenth of the break frequency), then falls down over two decades (ten times the break frequency), and then it is constant of $\frac{-\pi}{2}$, or −90°. The resulting characteristic is a combination of numerator and denominator.

The steepness of the rising and falling asymptotes is $\frac{\pi}{4}$ per decade (positive in the numerator, negative in denominator) and the constant asymptote between frequencies $100\,\mathrm{rad\,s^{-1}}$ and $1000\,\mathrm{rad\,s^{-1}}$ is $\frac{-\pi}{4}$, since the falling asymptote declines within one decade with slope of $\frac{-\pi}{4}$ per decade.

**Example 2**

Sketch the magnitude and phase characteristic of the Bode plot, labeling all critical slopes and points for the circuit in the figure. $R_1 = 9\,\mathrm{k\Omega}$, $R_2 = 1\,\mathrm{k\Omega}$, $L = 1\,\mathrm{H}$.




The transfer function is:

$$\mathbf{H} = \frac{j\omega L + R_2}{R_1 + j\omega L + R_2} = \frac{R_2}{R_1 + R_2} \cdot \frac{1 + j\omega\frac{L}{R_2}}{1 + j\omega\frac{L}{R_1+R_2}}$$

To get both in the numerator and denominator 1, we had to normalize the real part of the numerator and denominator to 1 by factor out the term $R_2$ in the numerator and the term $R_1 + R_2$ in the denominator. This gave us a multiplicative constant $\frac{R_2}{R_1+R_2} = \frac{1000}{9000+1000} = 0.1$. The break frequencies are:

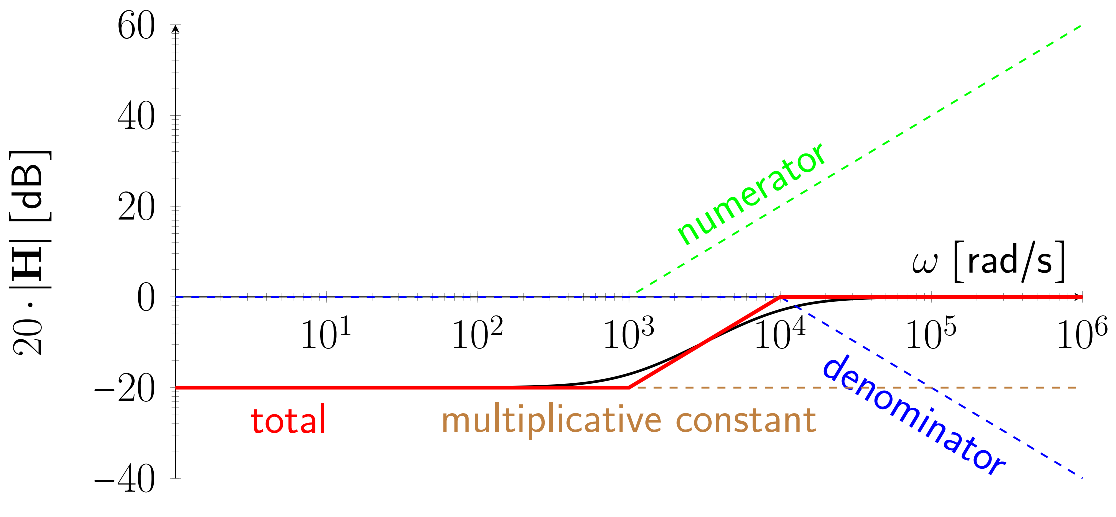
$$\omega_0 = \frac{R_2}{L} = \frac{1000}{1} = 1000\,\mathrm{rad\,s^{-1}}$$

$$\omega_p = \frac{R_1 + R_2}{L} = \frac{10000}{1} = 10\,\mathrm{krad\,s^{-1}}$$

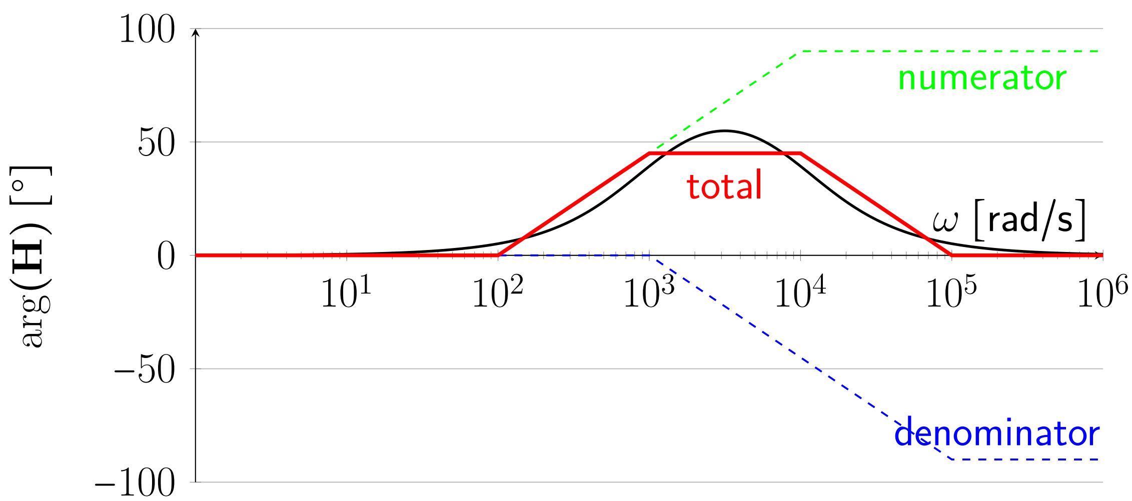
Thus the modified transfer function is:

$$\mathbf{H} = 0.1 \cdot \frac{1 + j\frac{\omega}{1000}}{1 + j\frac{\omega}{10000}}$$

Magnitude characteristic of the Bode plot:

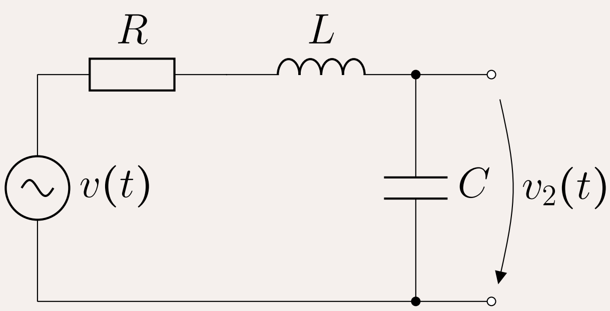



Phase characteristic of the Bode plot:




**Example 3**

Sketch the magnitude and phase characteristic of the Bode plot, labeling all critical slopes and points for the circuit in the figure. $L = 1\,\mathrm{H}$, $C = 1\,\mu\mathrm{F}$. Sketch it for three different resistances $R = 10\,\mathrm{k}\Omega$, $R = 2\,\mathrm{k}\Omega$ and $R = 1\,\Omega$.



The transfer function is:

$$\mathbf{H} = \frac{\frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC + (j\omega)^2 LC} = \frac{\frac{1}{LC}}{(j\omega)^2 + j\omega\frac{R}{L} + \frac{1}{LC}} \tag{7}$$

In denominator is a quadratic equation with variable $j\omega$. Beware, the variable is not the real $\omega$, but it's a complex variable; $j$ is part of it! It is more illustrative with the Laplace transform, where the transfer function would be $H(s) = \frac{\frac{1}{LC}}{s^2+\frac{R}{L}s+\frac{1}{LC}}$, where $s = \sigma + j\omega$ is the Laplace operator. Next, we will try to write the transfer function in the form of the product of the factors, thus, $(s - s_{01})(s - s_{02})$, or $(j\omega - (j\omega)_{01})\,(j\omega - (j\omega)_{02})$. For that reason is necessary to normalize the square power of the variable $(j\omega)$ in the equation 7 to 1. To decompose a quadratic equation as the product of factors, we must calculate the quadratic equation's roots.

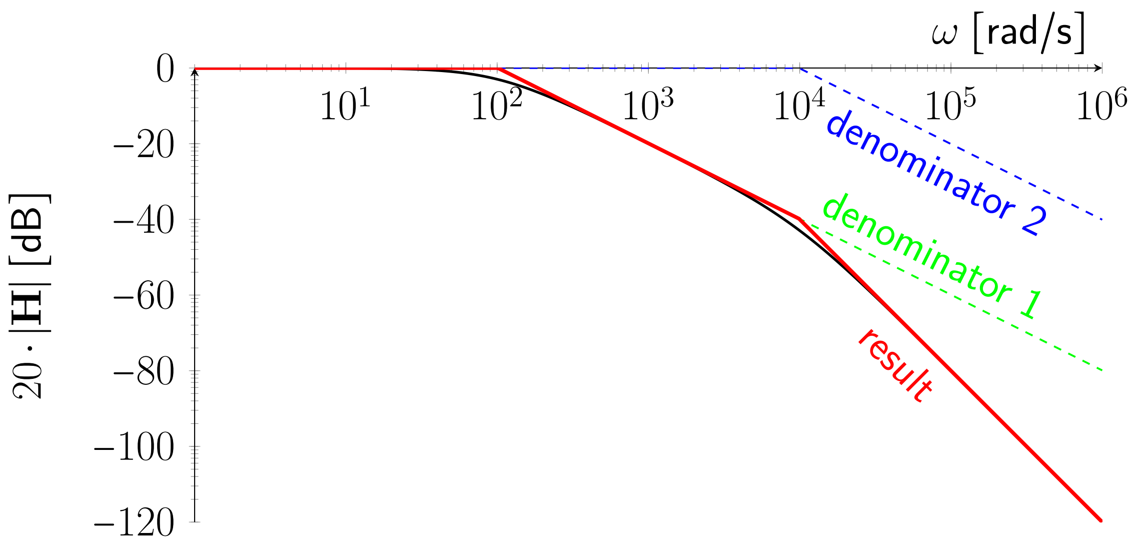
Further simplification depends on the kind of the roots – and therefore the value of the resistance:

1. $R = 10\,\mathrm{k}\Omega$

$$\mathbf{H} = \frac{10^6}{(j\omega)^2 + 10000 j\omega + 10^6} = \frac{10^6}{(j\omega + 101)(j\omega + 9899)}$$
$$= \frac{1}{\left(1 + j\frac{\omega}{101}\right)\left(1 + \frac{\omega}{9899}\right)}$$

The **roots** of the quadratic equation are in this case **real and distinct**, equal approximately to $-101$, and $-9899$. In such a case, we rewrite the quadratic equation as the product of the factors. We factor out the roots from both brackets to get the real part normalized to 1. Both roots will cancel with the constant $10^6$ in the numerator.

Magnitude characteristic of the Bode plot:

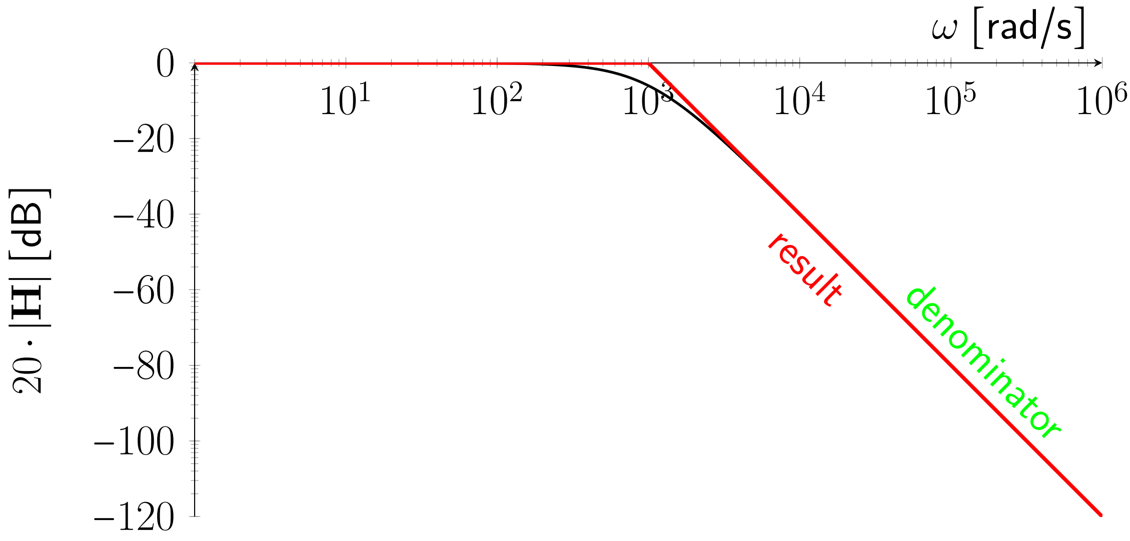



Both terms in the denominator's brackets have the same form as we already had in the RC integrator. The term $1 + j\frac{\omega}{101}$ represents in the logarithmic scale two asymptotes – the constant 0 dB until the break frequancy 101 rad s$^{-1}$, and the the falling asymtote with the slope −20 dB per decade. Also the second term $1 + j\frac{\omega}{9899}$ represents in the logarithmic scale two asymptotes – the constant 0 dB until the break frequency 9899 rad s$^{-1}$, and then the falling asymptote with the slope −20 dB per decade. When we add all asymptotes, we get the resulting response. It is the constant 0 until the break frequency 101 rad s$^{-1}$, then it falls down with the slope −20 dB per decade untill the break frequency 9899 rad s$^{-1}$. Finally, it falls down with the slope −40 dB per decade.

2. $R = 2\,\mathrm{k}\Omega$

$$\mathbf{H} = \frac{10^6}{(j\omega)^2 + 2000j\omega + 10^6} = \frac{10^6}{(j\omega + 1000)^2} = \\ = \frac{1}{\left(1 + j\frac{\omega}{1000}\right)^2}$$

In this case, the roots are **real and equal**, the value is 1000. In denominator, we have now the second power of two same factors. The logarithm in the bode plot leads to $20\log\left(\frac{\omega}{\omega_p}\right)^2 = 40\log\frac{\omega}{\omega_p}$. Thus, the response has just single break frequncy 1000 rad s$^{-1}$ and the asymptote falls down (because the term is in denominator) with the slope −40 dB per decade.




The actual magnitude response deviates from the asymptotic Bode plot by $2 \cdot 3$ dB, which is 6 dB. With the third power, the deviation would be $3 \cdot 3 = 9$ dB, etc.

3. $R = 1\,\Omega$

$$\mathbf{H} = \frac{10^6}{(j\omega)^2 + j\omega + 10^6} = \frac{1}{\left(j\frac{\omega}{1000}\right)^2 + \frac{j\omega}{10^6} + 1} =$$
$$= \frac{1}{\left(j\frac{\omega}{1000}\right)^2 + \frac{1}{1000} \cdot j\frac{\omega}{1000} + 1}$$

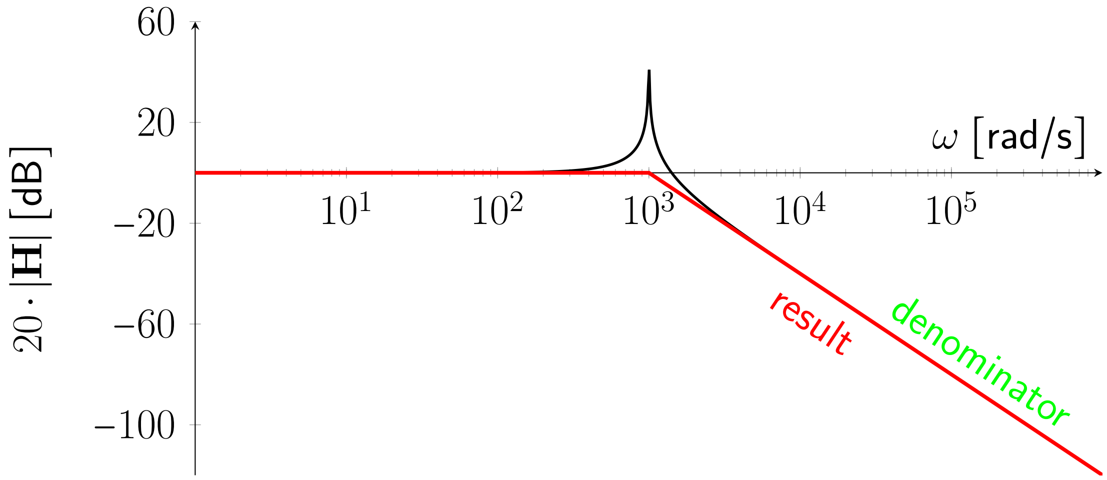
The **roots** are now **complex conjugated**, $(j\omega)_{1,2} = \frac{1}{2} \pm 999.999j$. In the case of complex conjugated root pair we do not factor the quadratic equation as the product of factors, but we divide it by the constant term from the quadratic equation, in this example $10^6$. Generally, the quadratic equation $(j\omega)^2 + b \cdot j\omega + c$ will be modified in the following way:

- Rewrite the term $(j\omega)^2$ into the form $(j\frac{\omega}{\omega_p})^2$ where $\omega_p = \sqrt{c}$ is the break frequency.
- Rewrite the term $b \cdot j\omega$ into the form $\frac{b}{\omega_p} \cdot j\frac{\omega}{\omega_p} = \frac{1}{Q} \cdot j\frac{\omega}{\omega_p}$.

After modification, the quadratic equation has the form: $(j\frac{\omega}{\omega_p})^2 + \frac{b}{\omega_p} \cdot j\frac{\omega}{\omega_p} + 1$.

For examle, if the resistor would be $R = 10\,\Omega$, then the quadratic equation is $(j\omega)^2 + j \cdot 10 \cdot \omega + 10^6$ and we rewrite it into the form $\left(j\frac{\omega}{1000}\right)^2 + \frac{1}{100} \cdot j\frac{\omega}{1000} + 1$.

If we would have quadratic equation $(j\omega)^2 + 100 \cdot j\omega + 2 \cdot 10^6$, the resulting form would be $\left(j\frac{\omega}{1414.214}\right)^2 + \frac{100}{1414.214} \cdot \frac{j\omega}{1414.214} + 1 = \left(j\frac{\omega}{1414.214}\right)^2 + 0.0707 \cdot \frac{j\omega}{1414.214} + 1$.




Asymptotic Bode plot is the same, like in the previous case, when $R = 2\,\mathrm{k}\Omega$ – reducing the resistance does not change the break frequency or the slope of the characteristic (it is 40 dB per decade because there are two reactance elements in the circuit). However, now the actual magnitude response deviates from the asymptotic one at break frequency significantly. In our example, at frequency $\omega = 1000\,\mathrm{rad\,s^{-1}}$ denominator is $\left(j\frac{\omega}{1000}\right)^2 + \frac{1}{1000} \cdot j\frac{\omega}{1000} + 1 = \frac{1}{1000}j$, and so the transfer function is $\mathbf{H} = -1000j$. The peak of the magnitude response is 60 dB. This is a resonant circuit (with a quality factor of $Q = 1000$).

# Summary

In general, when drawing Bode's asymptotic characteristics, follow these steps:

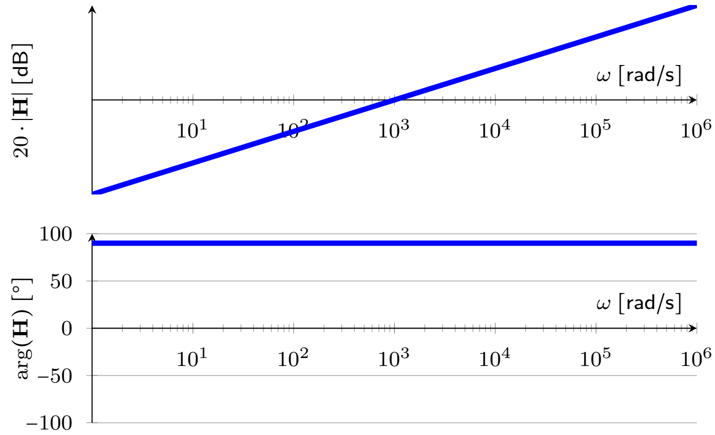
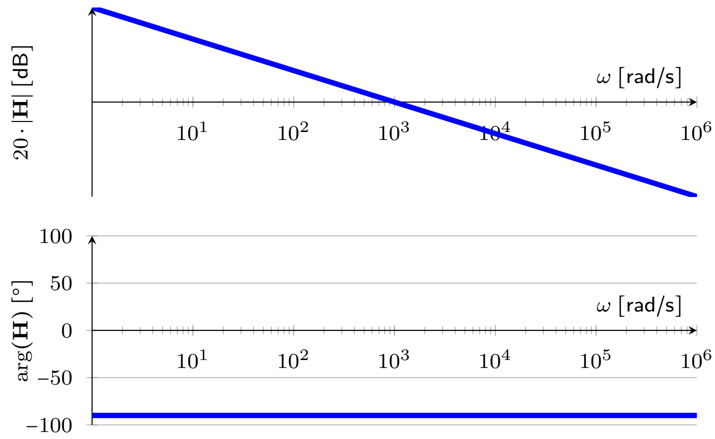
- Express the transfer function of the circuit. We can use elementary methods of analysis like Ohm's law, voltage or current divider, step-by-step simplification, Thévenin / Norton transformation of voltage and current sources... Alternatively, we can use general methods of analysis – nodal analysis or mesh analysis. The transfer function has the general form:

$$\mathbf{H} = \frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{a_N(j\omega)^N + a_{N-1}(j\omega)^{N-1} + a_{N-2}(j\omega)^{N-2} + \cdots + a_0}{b_M(j\omega)^M + b_{M-1}(j\omega)^{M-1} + b_{M-2}(j\omega)^{M-2} + \cdots + b_0} \tag{8}$$

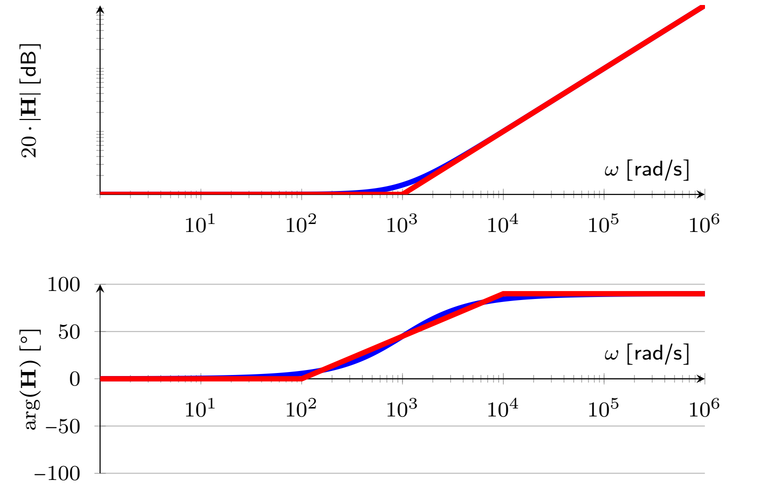
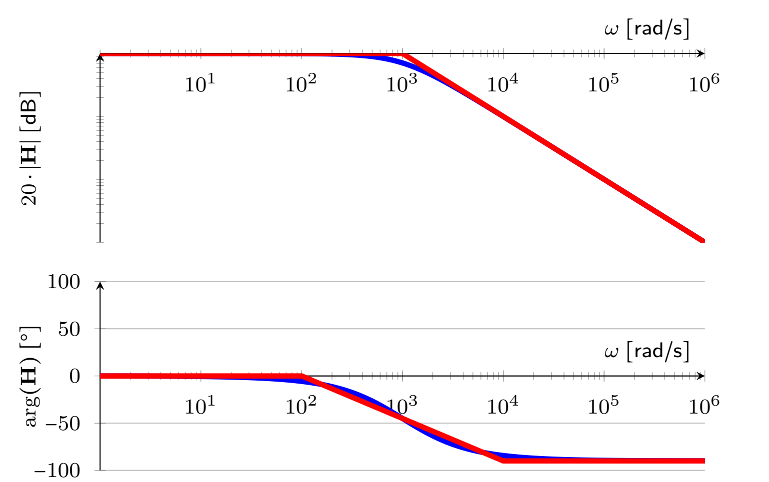
- In the equation 8 factor out coefficient $a_N$ in the numerator, and in the denominator factor out coefficient $b_N$.
- We calculate the roots of polynomials in the numerator and denominator.
- We write the transfer function in the form of the product of the root factors
$$\mathbf{H} = K\frac{\prod_{k=1}^{N}(j\omega - z_k)}{\prod_{k=1}^{M}(j\omega - p_k)} \tag{9}$$
- In the case of a zero root, the numerator or denominator's expression has the form $\cdot j\omega$. In the case of complex conjugated roots, we rearrange their product in the form of the quadratic expression $(j\omega)^2 + aj\omega + b$.
- From all factors both in the numerator and denominator, we factor out the roots $-z_k$ and $-p_k$, so we get a normalized form of the transfer function:
$$\mathbf{H} = K'\frac{\prod_{k=1}^{M}(j\frac{\omega}{-z_k} + 1)}{\prod_{k=1}^{N}(j\frac{\omega}{-p_k} + 1)} \tag{10}$$

Depending on the form of the root factor, we get the following asymptotes:

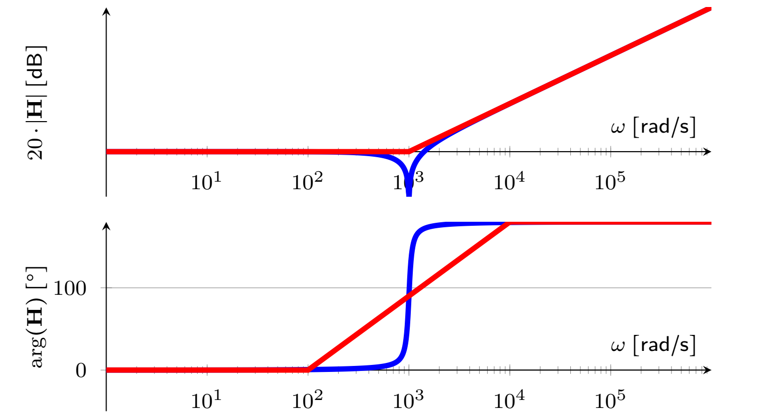
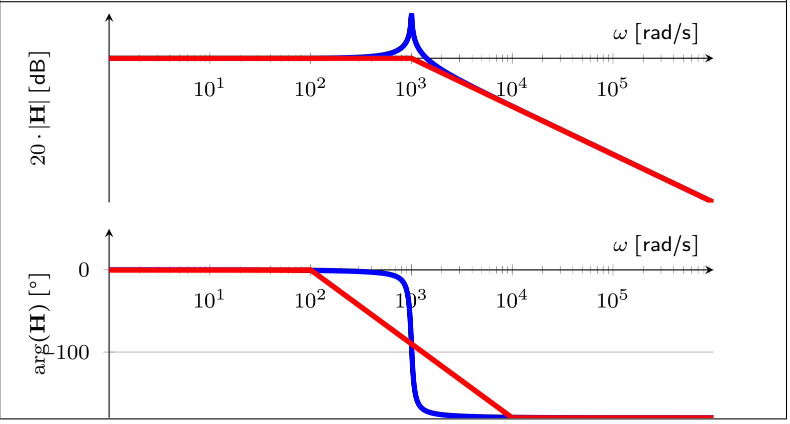
Multiplication constant $K$ – the magnitude response is the constant $20\log K$ dB, the phase is 0.

<table>
<tr>
<td>numerator</td>
<td>$j\dfrac{\omega}{\omega_0}$</td>
<td>
</td>
</tr>
<tr>
<td>denomina-tor</td>
<td>$j\dfrac{\omega}{\omega_p}$</td>
<td>
</td>
</tr>
</table>

The slope of the magnitude asymptote is 20 dB per decade. In the case of repeated roots with multiplicity of $n$, the slope is $n \cdot 20$ dB per decade.

| | | |
|---|---|---|
| numerator | $1 + j\dfrac{\omega}{\omega_0}$ |  |
| denominator | $1 + j\dfrac{\omega}{\omega_0}$ |  |

The slope of rising or descending magnitude asymptote is 20 dB per decade. In the case of repeated roots with the multiplicity of $n$, the slope is $n \cdot 20$ dB per decade. The actual magnitude response deviates from the magnitude Bode plot by 3 dB. In the case of repeated roots with the multiplicity of $n$, it deviates by $n \cdot 3$ dB. Actual characteristics are marked in blue, asymptotic in red.

<table>
<tr>
<td>numerator</td>
<td>$1+a\cdot j\frac{\omega}{\omega_0}+\left(j\frac{\omega}{\omega_0}\right)^2$</td>
<td>
</td>
</tr>
<tr>
<td>denominator</td>
<td>$1+a\cdot j\frac{\omega}{\omega_p}+\left(j\frac{\omega}{\omega_p}\right)^2$</td>
<td>
</td>
</tr>
</table>

The slope of rising or descending magnitude asymptote is 40 dB per decade. In the case of repeated roots with the multiplicity of $n$, the slope is $n \cdot 40$ dB per decade. The actual magnitude response deviates from the magnitude Bode plot by $20 \cdot \log a$ in the numerator, and by $20 \cdot \log \frac{1}{a}$ in denominator. Actual characteristics are marked in blue, asymptotic in red. The actual phase characteristic is steeper with the increasing quality factor of the resonant circuit.