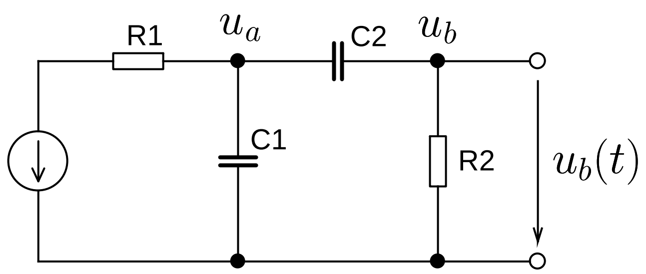
EXAMPLE – SECOND ORDER TRANSIENT – SOLUTION IN TIME AND LAPLACE DOMAINS – COMPARISON OF METHODS

A) Time domain – integral-differential equation

Objective: To the circuit in the figure is at time $t = 0$ connected voltage source $u(t)$. Calculate waveform of the voltage $u_b(t)$.



The number of circuit equations is same both for mesh analysis, and nodal analysis methods. When the task is to find the voltage $u_b(t)$, nodal analysis approach is more suitable, since we need not to evaluate it from a current.

Circuit equations:

$$\frac{u_a(t) - u(t)}{R_1} + C_1 \frac{\mathrm{d}u_a(t)}{\mathrm{d}t} + C_2 \frac{\mathrm{d}}{\mathrm{d}t}\Big(u_a(t) - u_b(t)\Big) = 0 \tag{1}$$

$$C_2 \frac{\mathrm{d}}{\mathrm{d}t}\Big(u_b(t) - u_a(t)\Big) + \frac{u_b(t)}{R_2} = 0 \tag{2}$$

Since we need to calculate the voltage $u_b(t)$, we have to eliminate the variable $u_a(t)$ and its derivatives. Derivative of the voltage $u_a(t)$ can be evaluated from the equation (2).

$$\frac{\mathrm{d}u_a(t)}{\mathrm{d}t} = \frac{\mathrm{d}u_b(t)}{\mathrm{d}t} + \frac{u_b(t)}{R_2 C_2} \tag{3}$$

Equation (1) includes voltage $u_a(t)$ and its derivative. Since from the second equation we can determine just its derivatives, we must to differentiate equation (1). Then it will contain first and second derivative of $u_a(t)$. We should also multiply it by R_1 and put the source voltage $u(t)$ on the right handed side, so the equation (1) we got:

$$R_1(C_1 + C_2)\frac{\mathrm{d}^2 u_a(t)}{\mathrm{d}t^2} + \frac{\mathrm{d}u_a(t)}{\mathrm{d}t} - R_1 C_2 \frac{\mathrm{d}^2 u_b(t)}{\mathrm{d}t^2} = \frac{\mathrm{d}u(t)}{\mathrm{d}t} \tag{4}$$

The second derivative of $u_a(t)$ we get by differentiating of equation (3).

$$\frac{\mathrm{d}^2 u_a(t)}{\mathrm{d}t^2} = \frac{\mathrm{d}^2 u_b(t)}{\mathrm{d}t^2} + \frac{1}{R_2 C_2}\frac{\mathrm{d}u_b(t)}{\mathrm{d}t} \tag{5}$$

By substitution of equations (3) and (5) into (4), we get:

$$R_1R_2C_1C_2\frac{\mathrm{d}^2u_b(t)}{\mathrm{d}t^2}+[R_1C_1+R_1C_2+R_2C_2]\frac{\mathrm{d}u_b(t)}{\mathrm{d}t}+u_b(t)=R_2C_2\frac{\mathrm{d}u(t)}{\mathrm{d}t} \tag{6}$$

This is already second order differential equation of one variable. We can find the solution using method of variation of parameters. Characteristic equation is

$$R_1R_2C_1C_2\lambda^2+[R_1C_1+R_1C_2+R_2C_2]\lambda+1=0. \tag{7}$$

Quadratic equation has two negative roots. To simplify evaluation of this circuit, let $R_1=R_2=R$ and $C_1=C_2=C$.
Then the equation (7) will be rewritten as

$$R^2C^2\lambda^2+3RC\lambda+1=0. \tag{8}$$

The roots of the equation (8) are:

$$\lambda_{1,2}=\frac{-3}{2RC}\pm\sqrt{\frac{9}{(2RC)^2}-\frac{1}{(RC)^2}}=\frac{1}{RC}\cdot\left(\frac{-3}{2}\pm\sqrt{\frac{5}{4}}\right)=\begin{cases}-0.382\cdot\frac{1}{RC}\\-2.618\cdot\frac{1}{RC}\end{cases} \tag{9}$$

General solution is

$$u_b(t)=K_1e^{\frac{-0.382}{RC}t}+K_2e^{\frac{-2.618}{RC}t} \tag{10}$$

Particular solution in the circuit is determined by supplying source[1]. Assume a DC source, $u(t)=U$. Then the particular solution will be

$$u_p=u_b(\infty)=0 \tag{11}$$

Now, set the time $t=0$. From the equation (10) we get

$$u_b(0)=K_1+K_2 \tag{12}$$

Now, we have one equation and two unknown constants K_1 and K_2. To obtain second equation, required for calculation of both constants, we will differentiate equation (10):

$$u_b'(t)=\frac{-0.382}{RC}K_1e^{\frac{-0.382}{RC}t}-\frac{2.618}{RC}K_2e^{\frac{-2.618}{RC}t} \tag{13}$$

Again, set $t=0$ in equation (13):

[1] If the source is DC, we use DC analysis methods (resistive circuit analysis). As far as the source is AC, we would use phasors to find $\mathbf{U}_b$ and here from $u_b(t)$. For periodical non sinusoidal waveforms, we would use Fourier series analysis, etc.

$$u_b'(0_+) = \frac{-0.382}{RC}K_1 - \frac{2.618}{RC}K_2 \tag{14}$$

In the equation (14) on the left side is not energetic, but mathematical initial condition. Mathematical initial condition must be evaluated from the energetic initial conditions, as it is not directly determined by a stored energy. Before the supply source was connected, both capacitors were without any charge

$$u_{C1}(0_-) = u_{C1}(0_+) = 0 \tag{15}$$

$$u_{C2}(0_-) = u_{C2}(0_+) = 0 \tag{16}$$

Using mesh analysis we got equation

$$u_b(0_+) - u_{C2}(0_+) + u_{C1}(0_+) = 0 \tag{17}$$

so

$$u_b(0_+) = u_{C2}(0_+) - u_{C1}(0_+) = 0 \tag{18}$$

To find mathematical initial condition, which we need in the equation (14), we must to return back to circuit equations (1) and (2). Setting $t = 0$ and by substitution of energetic initial conditions we get a set of equations.

$$u_a(0_+) - U + 2RCu_a'(0_+) - RCu_b'(0_+) = 0 \tag{19}$$

$$u_b'(0_+) = u_a'(0_+) - \frac{u_b(0_+)}{RC} \tag{20}$$

Here, the $u_a'(0)$ must be eliminated. Since

$$u_a(0_+) = u_{C1}(0_+) = 0$$

and

$$u_b(0_+) = u_{C1}(0_+) - u_{C2}(0_+) = 0$$

the solution of equations (19) and (20) is

$$u_b'(0_+) = \frac{U}{RC} \tag{21}$$

By substitution of (18) into (12), and (21) into (14) we finally obtain K_1 and K_2.

$K_2 = \frac{-U}{2.236}$, $K_1 = \frac{U}{2.236}$ and

$$\boxed{u_b(t) = \frac{U}{2.236}\left(e^{\frac{-0.382}{RC}t} - e^{\frac{-2.618}{RC}t}\right)}$$

We got it ☺.

B) Laplace transform

a. Circuit equations

The Laplace transform will be much easier:

We can use many different methods, from basic methods of analysis (step by step simplification, Thévenin's equivalent circuit, etc.) to circuit equations, where we can choose both mesh analysis, and nodal analysis, without limitations.

Circuit equations:

$$\frac{U_a(p) - U(p)}{R_1} + pC_1U_a(p) + pC_2\left(U_a(p) - U_b(p)\right) = 0 \tag{22}$$

$$pC_2\left(U_b(p) - U_a(p)\right) + \frac{U_b(p)}{R_2} = 0 \tag{23}$$

It may be rewritten into matrix form:

$$\begin{bmatrix} \frac{1}{R_1} + pC_1 + pC_2 & -pC_2 \\ -pC_2 & pC_2 + \frac{1}{R_2} \end{bmatrix} \begin{bmatrix} U_a(p) \\ U_b(p) \end{bmatrix} = \begin{bmatrix} \frac{U(p)}{R_1} \\ 0 \end{bmatrix} \tag{24}$$

(*the matrix form (24) may be set directly, independently of equations (22) and (23)*).

To solve the equations we can use e.g. Cramer's rule:

$$\begin{aligned} U_b(p) &= \frac{\begin{vmatrix} \frac{1}{R_1} + pC_1 + pC_2 & \frac{U(p)}{R_1} \\ -pC_2 & 0 \end{vmatrix}}{\begin{vmatrix} \frac{1}{R_1} + pC_1 + pC_2 & -pC_2 \\ -pC_2 & pC_2 + \frac{1}{R_2} \end{vmatrix}} = \\ &= \frac{-\frac{U(p)}{R_1}(-pC_2)}{(\frac{1}{R_1} + pC_1 + pC_2)(pC_2 + \frac{1}{R_2}) - (-pC_2)(-pC_2)} = \\ &= \frac{U(p)pC_2}{pC_2 + \frac{1}{R_2} + p^2C_1C_2R_1 + pC_1\frac{R_1}{R_2} + \cancel{p^2C_2^2R_1} + pC_2\frac{R_1}{R_2} - \cancel{p^2C_2^2R_1}} = \\ &= \frac{U(p)pR_2C_2}{P^2R_1R_2C_1C_2 + pR_1C_1 + pR_2C_2 + pR_1C_2 + 1} \end{aligned} \tag{25}$$

If $R_1 = R_2 = R$, $C_1 = C_2 = C$ and supply source is DC, $u(t) = U$,

$$U_b(p) = \frac{\frac{U}{p} pRC}{(pRC)^2 + 3pRC + 1} = \frac{U\frac{1}{RC}}{p^2 + 3p\frac{1}{RC} + \frac{1}{(RC)^2}} = \frac{A}{p - p_1} + \frac{B}{p - p_2} \tag{26}$$

Roots of the polynomial in denominator are

$$p_{1,2} = \frac{-3}{2RC} \pm \sqrt{\frac{9}{(2RC)^2} - \frac{1}{(RC)^2}} = \frac{1}{RC} \cdot \left(\frac{-3}{2} \pm \sqrt{\frac{5}{4}} \right) = \begin{cases} -0.382 \cdot \frac{1}{RC} \\ -2.618 \cdot \frac{1}{RC} \end{cases}$$

Compare it with the roots (9) – they must be the same. The constants A and B in partial fractions can be determined using single pole method (or the general one, if you prefer it)

$$p_1 = -0.382 \frac{1}{RC}$$

$$A = \frac{U\frac{1}{RC}}{\cancel{\left(-0.382\frac{1}{RC} + 0.382\frac{1}{RC}\right)}\left(-0.382\frac{1}{RC} + 2.618\frac{1}{RC}\right)} = \frac{U}{2.236}$$

$$p_2 = -2.618 \frac{1}{RC}$$

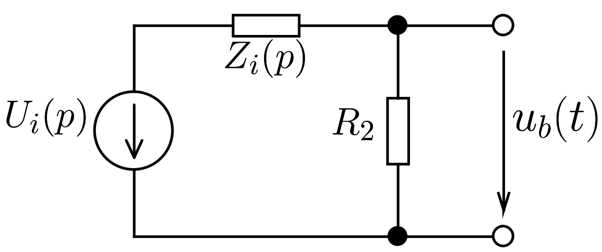
$$B = \frac{U\frac{1}{RC}}{\left(-2.618\frac{1}{RC} + -0.382\frac{1}{RC}\right)\cancel{\left(-2.618\frac{1}{RC} + 2.618\frac{1}{RC}\right)}} = -\frac{U}{2.236}$$

The solution is:

$$\boxed{u_b(t) = \frac{U}{2.236} \left(e^{\frac{-0.382}{RC}t} - e^{\frac{-2.618}{RC}t} \right)}$$

And we have done ☺.

b. Thévenin's equivalent circuit:



Using Laplace transform, we can replace any part of the circuit by its Thévenin's equivalent. It does not matter, it contains the capacitors, and it would even include initial conditions, since they are just simple voltage or current sources in the Laplace domain. In time domain anything like that is not possible. From the viewpoint of the terminals of the resistor R_2, the parameters of Thévenin's equivalent are:

$$U_i = U(p)\frac{\frac{1}{pC_1}}{R_1 + \frac{1}{pC_1}} = U(p)\frac{1}{1 + pR_1C_1} \tag{27}$$

$$Z_i(p) = \frac{R_1 \cdot \frac{1}{pC_1}}{R_1 + \frac{1}{pC_1}} + \frac{1}{pC_2} = \frac{R_1}{1 + pR_1C_1} + \frac{1}{pC_2} \tag{28}$$

Laplace transform of the output voltage $u_b(t)$ can be determined using voltage divider rule:

$$\begin{aligned} U_b(p) &= U_i(p)\frac{R_2}{Z_i(p) + R_2} = \\ &= U(p)\frac{1}{1 + pR_1C_1} \cdot \frac{R_2}{\frac{R_1}{1+pR_1C_1} + \frac{1}{pC_2} + R_2} = \\ &= U(p)\frac{R_2}{R_1 + \frac{1+pR_1C_1}{pC_2} + R_2 + pR_1R_2C_1} = \\ &= U(p)\frac{pR_2C_2}{pC_2R_1 + 1 + pR_1C_1 + pR_2C_2 + p^2R_1R_2C_1C_2} = \\ &= U(p)\frac{p\frac{1}{R_1C_1}}{p^2 + p\frac{R_1C_1+R_2C_2+C_2R_1}{R_1R_2C_1C_2} + \frac{1}{R_1R_2C_1C_2}} \end{aligned} \tag{29}$$

If $R_1 = R_2 = R$, $C_1 = C_2 = C$ and voltage source is DC, $u(t) = U$, then equation (29) will be simplified to

$$U_b(p) = U \cdot \frac{\frac{1}{RC}}{p^2 + p\frac{3}{RC} + (\frac{1}{RC})^2} \tag{30}$$

This is the same equation as (26), so the remaining part of calculation is identical.

That is it ☺.