Excitation by single pulses, unit impulse and unit step response.

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EXAMPLE

To the input of the circuit, we connected voltage source of the waveform $u_1(t) = 10e^{-500t}\ \mathrm{V}$

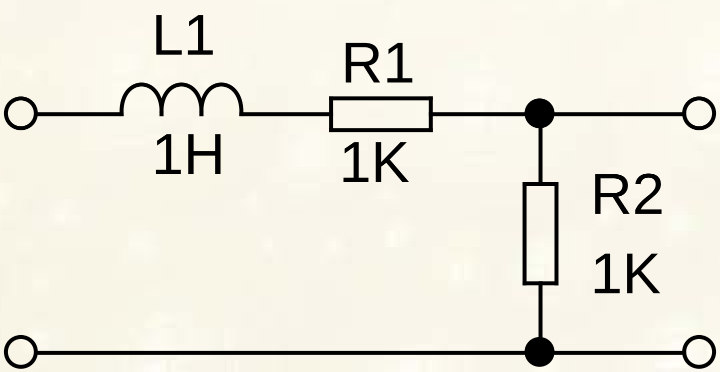
On the output, we measured the waveform $u_2(t) = 6.\dot{6}(e^{-500t} - e^{-2000t})\ \mathrm{V}$

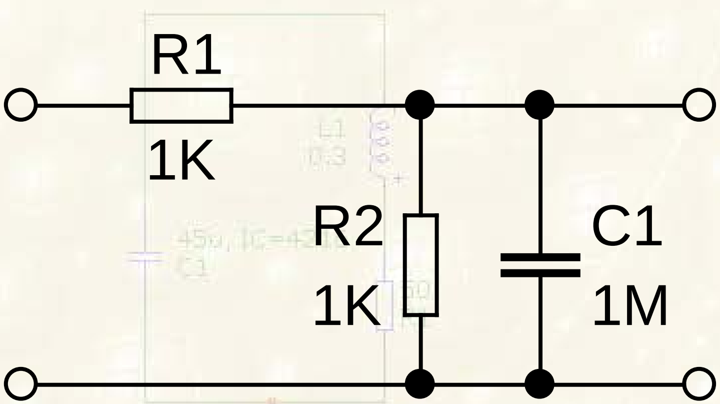
Find transfer function of the circuit. Find suitable circuit diagram.

$$U_1(p) = \frac{10}{p + 500}$$

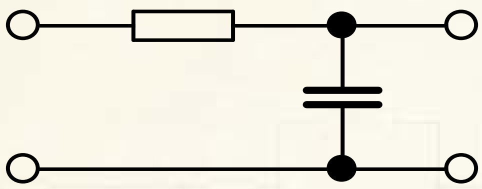
$$U_2(p) = \frac{6.\dot{6}}{p + 500} - \frac{6.\dot{6}}{p + 2000} = \frac{6.\dot{6} \cdot 2000 - 6.\dot{6} \cdot 500}{(p + 500)(p + 2000)} = \frac{10000}{(p + 500)(p + 2000)}$$

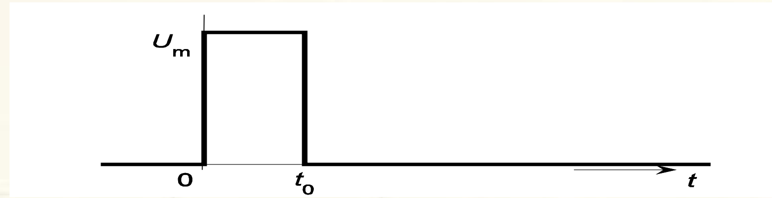
$$P(p) = \frac{U_2(p)}{U_1(p)} = \frac{\frac{10000}{(p+500)(p+2000)}}{\frac{10}{p+500}} = \frac{1000}{p + 2000}$$





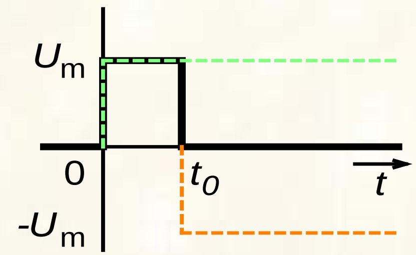
EXAMPLE





The integrating network in the figure is excited by rectangular pulse in second figure. Compute the waveform of output voltage. **The capacitor has zero voltage at the time of connection of source (zero initial condition).**

- To find the solution we will use table of Laplace transforms
- rectangular pulse is superposition of two unit step functions multiplied by U_m



1. $$U_1(p) = \frac{U_m}{p}\left[1 - e^{-pt_0}\right]$$

2. $$P(p) = \frac{1}{1 + pRC}$$

3. $$U_2(p) = U_1(p) \cdot P(p) = \frac{U_m}{p}\frac{1}{1 + pRC}\left[1 - e^{-pt_0}\right]$$

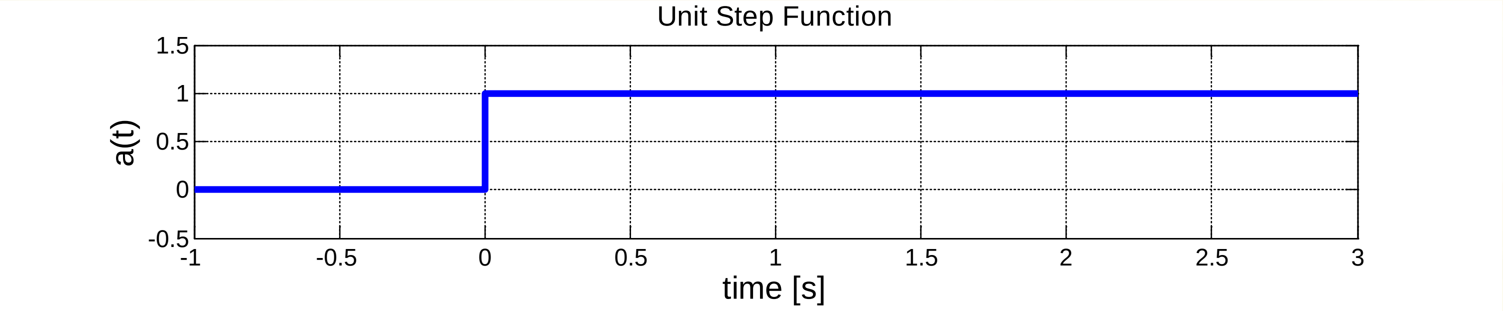
Statement in the square bracket will be temporary omitted (it is information about time delay, transformed later)

$$U_2'(p) = \frac{A}{p} + \frac{B}{p + \frac{1}{RC}} = U_m\left(\frac{1}{p} - \frac{1}{p + \frac{1}{RC}}\right) \quad \Rightarrow \quad u_2'(t) = U_m\left(1 - e^{-\frac{t}{RC}}\right)$$

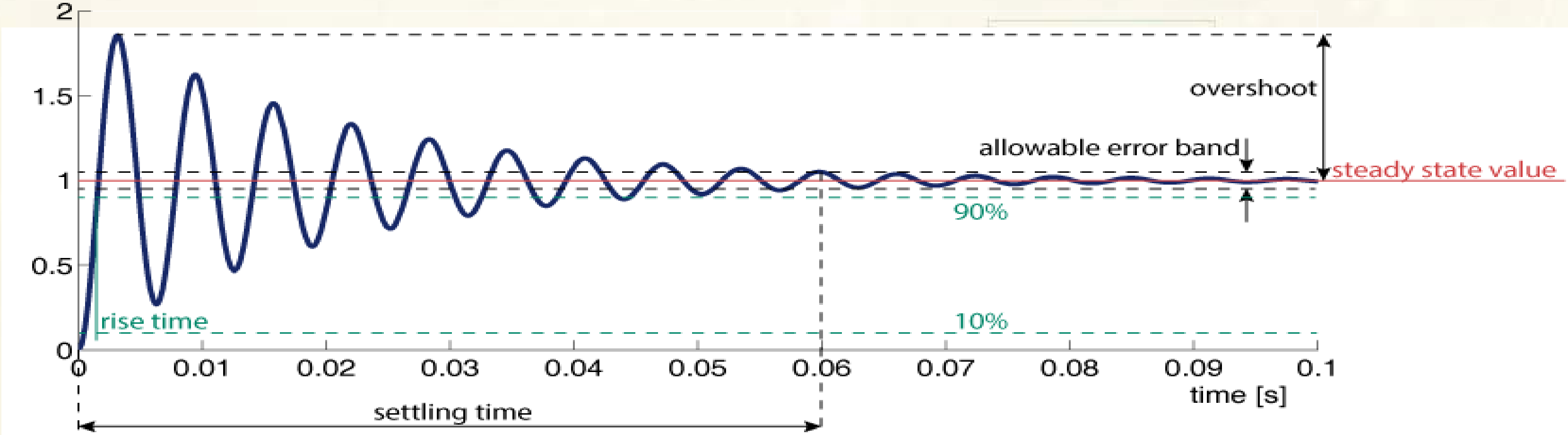
The transform of the square bracket are two unit step functions, the second is time shifted by t_0

$$u_2(t) = U_m\left[(1 - e^{\frac{t}{RC}})1(t) - (1 - e^{-\frac{t-t_0}{RC}})1(t - t_0)\right]$$

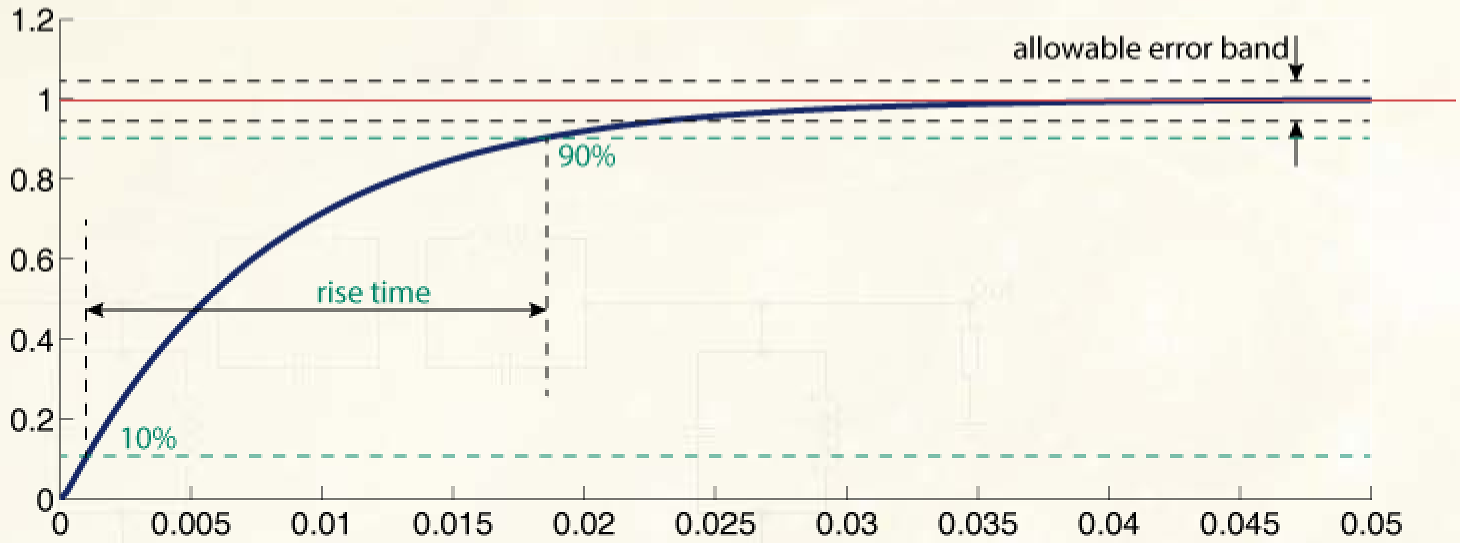
EXCITATION BY THE UNIT STEP FUNCTION



- Consider linear circuit, with zero energetic initial conditions.
- Unit step response is an output of a such circuit, if it is excited by the Unit step function.
- In time domain, it give us important information about rise time, overshoot, settling time.
- In frequency domain, we got information about frequency response of the circuit, and we can calculate response on any excitation.
- In frequency domain, we can assess stability of the circuit.



2nd order response



1^{st} order response

Input – unit step function

$u_1(t) = a(t)$

Output – unit step response $H(t) = u_2(t)$

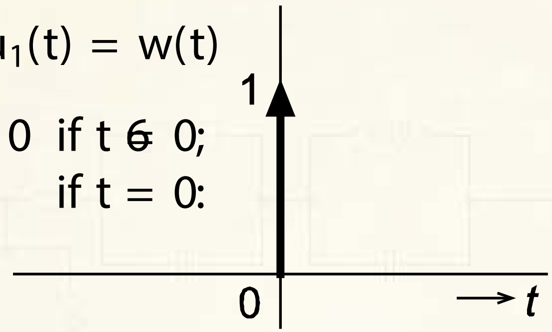
Using Laplace transform:

$A(p) = \frac{1}{p} H(p)$ where $H(p)$ is the transfer function

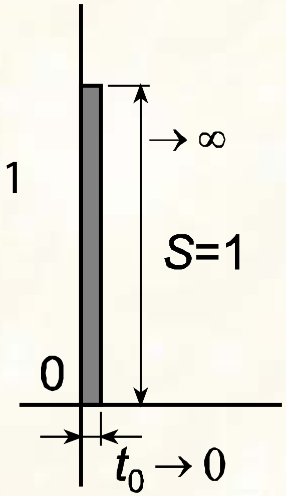
EXCITATION BY THE UNIT IMPULSE FUNCTION

$$u_1(t) = w(t)$$

$$w(t) = \begin{cases} 0 & \text{if } t \neq 0; \\ \infty & \text{if } t = 0. \end{cases}$$



$$\int_{-\infty}^{\infty} \delta(t - t_0) = 1$$



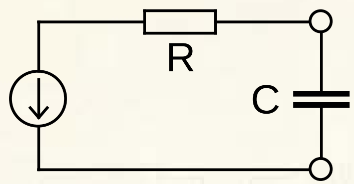
- Consider linear circuit, with zero energetic initial conditions.
- Unit impulse response is an output of a such circuit, if it is excited by the Unit impulse function.
- It give us information about transfer function.
- Connect unit impulse source to the input, measure output response, convert it to Laplace domain, and we get the transfer function

$$W(p) = H(p) = \mathcal{L}\{u_2(t)\}$$

$$w(t) = \frac{da(t)}{dt} \qquad a(t) = \int_{-\infty}^{t} w(t)\, dt$$

Periodical rectangular waveform – integrating circuit

$\tau \ll T$



$\tau = 1\,\mathrm{ms}$ $\quad R = 1\,\mathrm{k\Omega};\ C = 1\,\mu\mathrm{F}$ $\quad U_m = 1\,\mathrm{V}$

$T = 20\,\mathrm{ms} \longrightarrow \omega_0 = 100\pi$

Periodical steady state:

$$u_1(t) = 0.5 + \sum_{k=1}^{\infty} \frac{2}{(2k-1)\pi} \sin((2k-1)\omega_0 t) \qquad U_{01} = 0.5,\ \hat{U}_{1k} = \frac{2}{(2k-1)\pi}$$

$$U_{02} = 0.5,\ \hat{U}_{2k} = \hat{U}_{1k} \frac{1}{1 + j(2k-1)\omega_0 R C} = \frac{2}{(2k-1)\pi} \cdot \frac{1}{1 + j(2k-1)0.1\pi}$$

$$u_2(t) = 0.5 + 0.607\sin(314.16t - 0.304) + 0.154\sin(942.48t - 0.756) + 0.068\sin(1570.79t - 1.004) + \cdots$$

Transient:

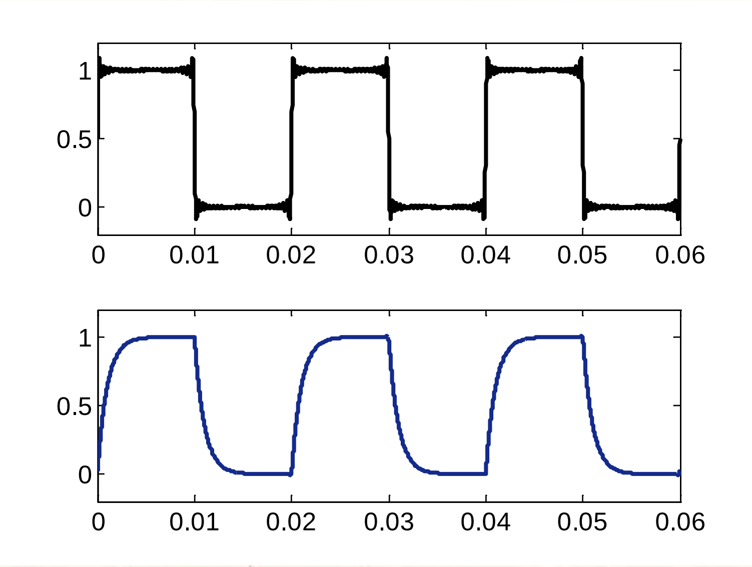
$t \in (0; 0.1)$ $\quad u_2(t) = 1 - e^{-1000t}$

$u_2(0.01) = 0.9999546\,\mathrm{V}$

$t \in (0.1; 0.2)$ $\quad u_2(t) = 0.9999546\, e^{-1000(t-0.1)}$

$u_2(0.02) = 0.0000454\,\mathrm{V}$

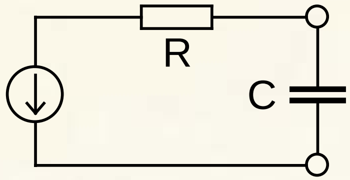
$t \in (0.2; 0.3)$ $\quad u_2(t) = 1 - 0.9999546\, e^{-1000(t-0.2)}$



In this case each change of excitation voltage will be considered as the origin of new transient

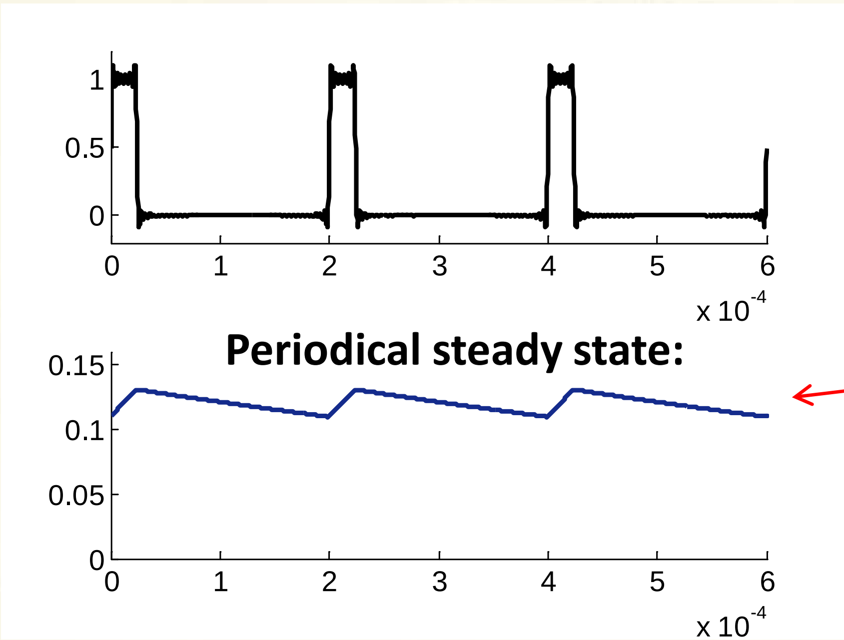
– the *transient voltage* at the end of first half of the period is the *initial condition* of subsequent transient in this case may be omitted

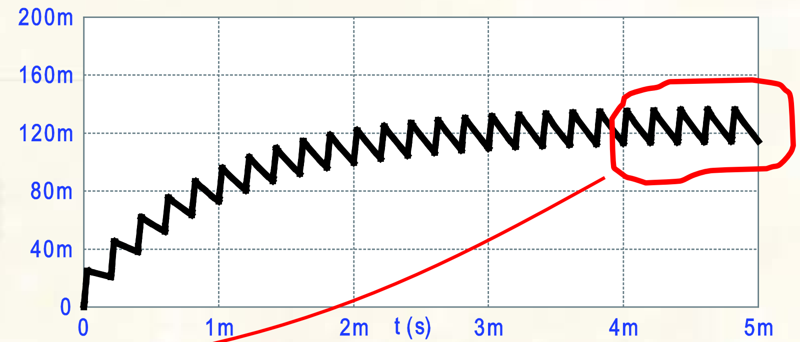
$\tau \gg T$



$\tau = 1\,\mathrm{ms}$ $\quad R = 1\,\mathrm{k\Omega};\ C = 1\,\mu\mathrm{F}$

$T = 0.2\,\mathrm{ms} \longrightarrow \omega_0 = 10000\pi$





Periodical steady state does not resolve transient part, but just steady state part – it is equal to the mean value of rectangular waveform

Exponential trend is equivalent to the „slow“ transient with time constant τ

$$u_1(t) = \sum_{k=-\infty}^{\infty} \hat{U}_{1k} e^{jk\omega_0 t} \quad \Rightarrow \quad U_{01} = \frac{U_m \cdot t_0}{T} = 0.12,\ \hat{U}_{1k} = \frac{U_m}{-jk2\pi}\left[e^{-jk\omega_0 t_0} - 1\right] = \frac{1}{-jk2\pi}\left[e^{-jk0.24\pi} - 1\right]$$

$$U_{02} = U_{01},\ \hat{U}_{2k} = \hat{U}_{1k}\frac{1}{1 + jk\omega_0 RC} = \frac{1}{-jk2\pi}\left[e^{-jk0.24\pi} - 1\right]\cdot\frac{1}{1 + jk10\pi} \quad k = -\infty, \ldots, -1; 1, \ldots, \infty$$

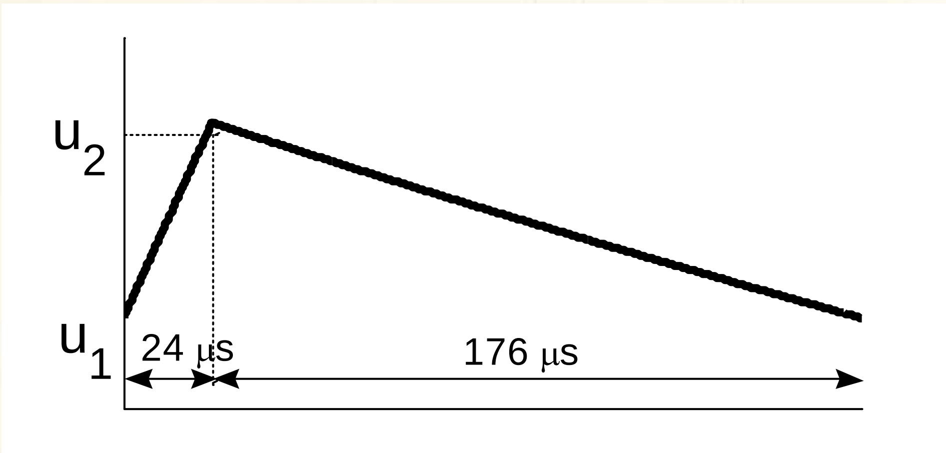
$$t \in (0; 24\,\mu s) \quad u_2(t) = 1 - e^{-1000t} \qquad u_2(24\,\mu s) = 0.0237143\,V$$

$$t \in (24\,\mu s; 200\,\mu s) \quad u_2(t) = 0.0237143\,e^{-1000(t-0.000024)} \qquad u_2(200\,\mu s) = 0.01988723\,V$$

$$t \in (200\,\mu s; 224\,\mu s) \quad u_2(t) = 1 - 0.9801127\,e^{-1000(t-0.0002)} \qquad u_2(224\,\mu s) = 0.04312991\,V$$

$$\ldots$$

Steady state – *how from the transient equation find limit voltages:*



1st part – increasing exponential

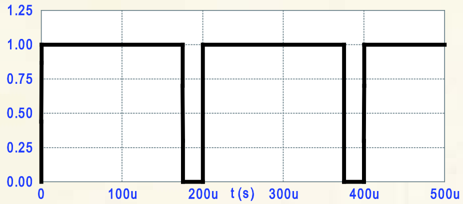
$u_c(0) = u_1;\ \ u_p = 1;\ \ t = 24\,\mu s \qquad u(24\,\mu s) = u_2 = (u_1 - 1)\,e^{-0.024} + 1$

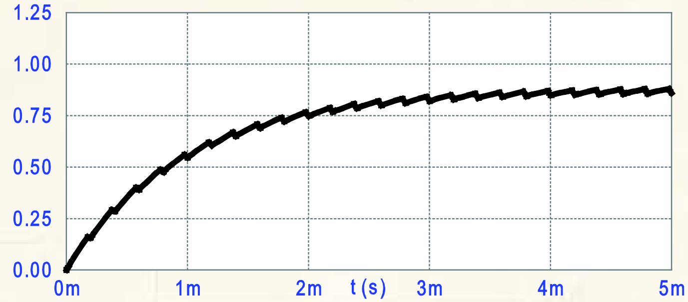
2nd part – decreasing exponential

$u_c(0) = u_2;\ \ u_p = 0;\ \ t = 176\,\mu s \qquad u(176\,\mu s) = u_1 = u_2\,e^{-0.176}$

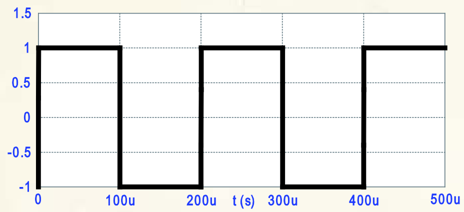
Set of equations

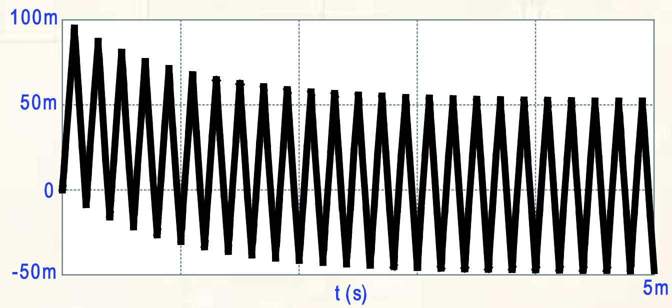
$$\begin{bmatrix} e^{-0.024} & -1 \\ -1 & e^{-0.176} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} e^{-0.024} - 1 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} u_1 &= 0.109711 \\ u_2 &= 0.130824 \end{aligned}$$



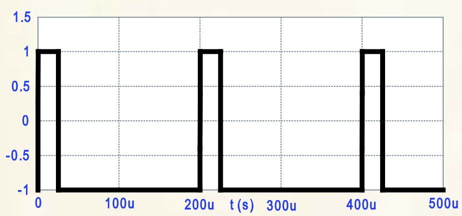


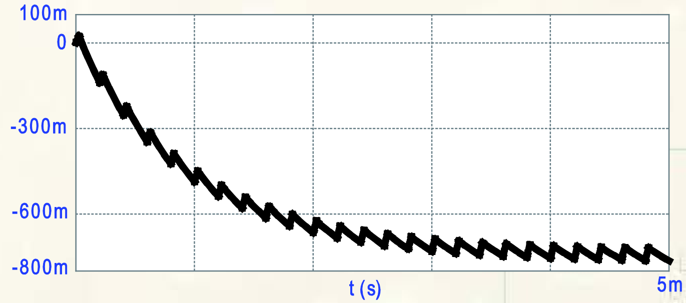
Positive mean value
$U_{\max} = 1$
$U_{\min} = 0$



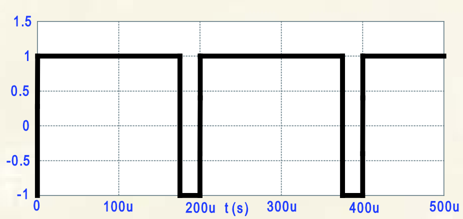


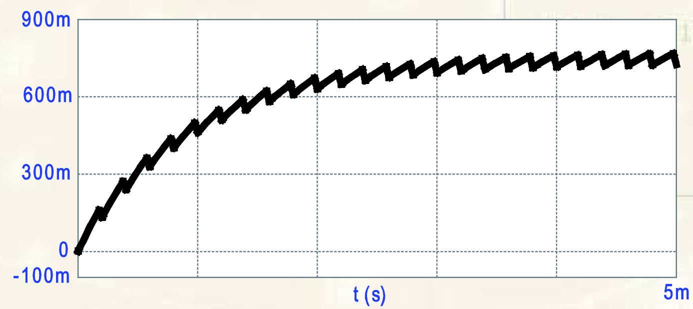
Zero mean value
$U_{\max} = 1$
$U_{\min} = -1$





Negative mean value
$U_{\max} = 1$
$U_{\min} = -1$

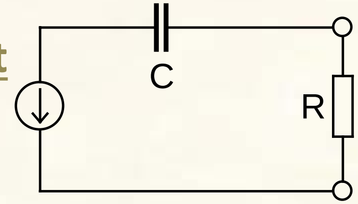




Positive mean value
$U_{\max} = 1$
$U_{\min} = -1$

Periodical rectangular waveform – derivative circuit

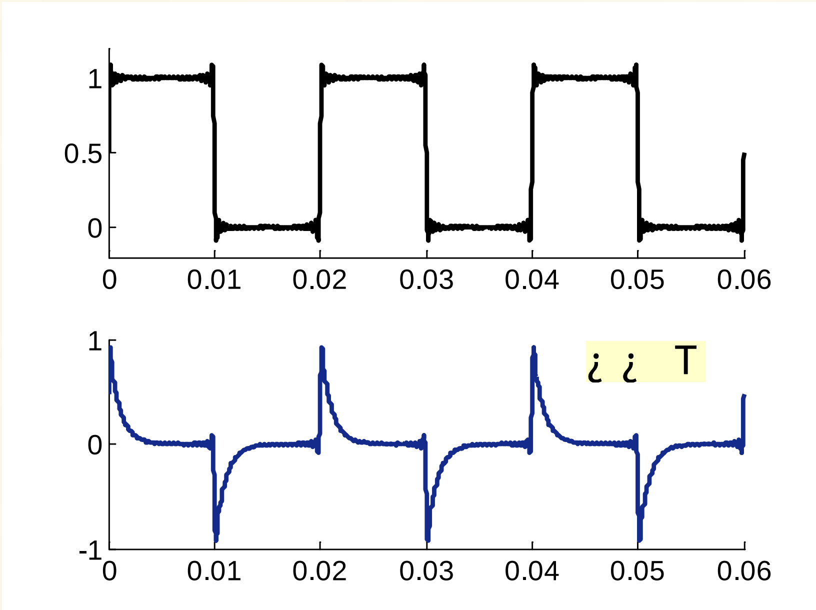
RC circuit exhibits „derivative“ properties, only when $\tau \ll T$



$R = 1\,\mathrm{k\Omega};\ C = 1\,\mu F$

$\tau = 1\,\mathrm{ms}$

$U_m = 1\,V$



$$T = 20\,\mathrm{ms} \longrightarrow \omega_0 = 100\pi$$

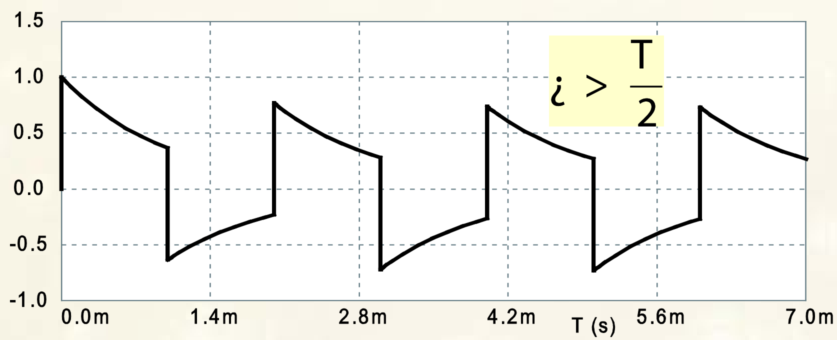
$$u_1(t) = 0.5 + \sum_{k=1}^{\infty} \frac{2}{(2k-1)\pi} \sin(2k-1)\omega_0 t$$

$$U_{01} = 0.5,\ \hat{U}_{1k} = \frac{2}{(2k-1)\pi}$$

$$U_{02} = 0$$

$$\hat{U}_{2k} = \hat{U}_{1k} \frac{j(2k-1)\omega_0 RC}{1 + j(2k-1)\omega_0 RC}$$

$$= \frac{2}{(2k-1)\pi} \cdot \frac{j(2k-1)0.1\pi}{1 + j(2k-1)0.1\pi}$$



$$u_c(t) = [u_c(0) - u_{cp}(0)]\, e^{-\frac{t}{\tau}} + u_{cp}(t)$$

$$i(t) = C \frac{du_c(t)}{dt} = \frac{-C}{\tau} [u_c(0) - u_{cp}(0)]\, e^{-\frac{t}{\tau}}$$

$$u_2(t) = R\, i(t) = \frac{-RC}{\tau} [u_c(0) - u_{cp}(0)]\, e^{-\frac{t}{\tau}}$$

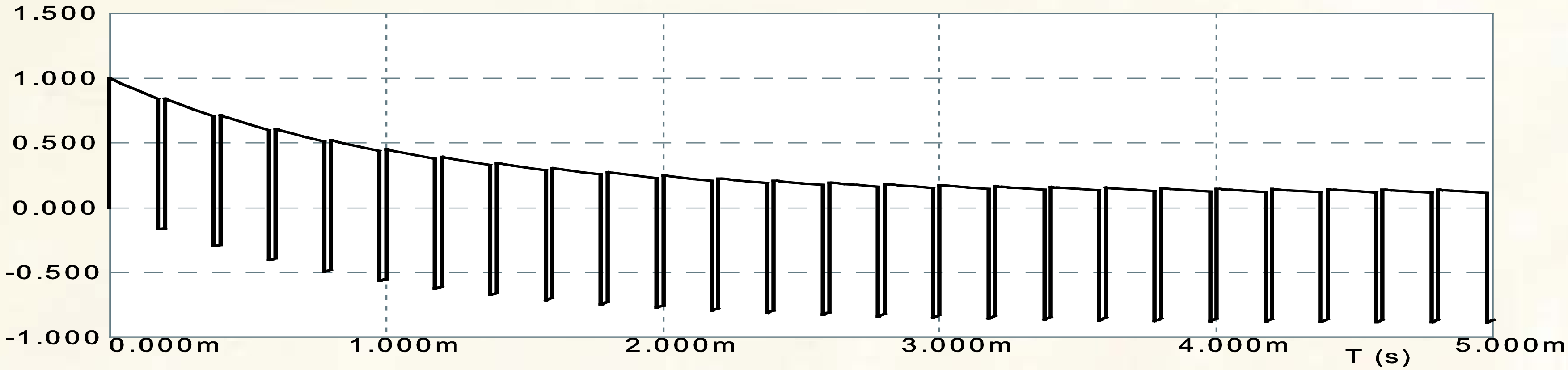
$$= [u_{cp}(0) - u_c(0)]\, e^{-\frac{t}{\tau}} = u_R(0)\, e^{-\frac{t}{\tau}}$$

$u_{cp}(0)$ is particular solution on capacitor – alternatively 1 V (first half period) and 0V

$u_R(0)$ positive on rising edge, negative on falling edge $u_R(0)_n = \pm 1 \mp u_R\left(\frac{T}{2}\right)_{n-1}$

$\tau \gg T$ Derivative circuit

$\tau = 1\,\mathrm{ms} \quad T = 0.2\,\mathrm{ms} \quad t_0 = 176\,\mu\mathrm{s}$



- The voltage is almost rectangular, if the duration of voltage values are: U_m is t_0 and 0 is $T - t_0$, than the waveform has boundary voltages $U_m \frac{t_0}{T}$ and $-U_m(1 - \frac{t_0}{T})$
- The magnitude is still U_m
- „Slow“ exponential transient is boundary waveform
- The waveform of the voltage across capacitor is the same as voltage across capacitor in the integrating circuit above (the sum of both voltages must be in each time instant U_m).