Fundamentals of Electrical Circuits

XI

2nd order transients

2ND ORDER TRANSIENTS IN CIRCUITS WITH DC EXCITATION (APERIODIC RESPONSE, DAMPED OSCILLATIONS).

- In previous lectures we already learned, when the circuit contains just single storage element (a capacitor or an inductor) and some condition in the circuit changes (the source is connected or disconnected, resistivity or another circuit property changes), the storage element is charged or discharged to new steady state value, the waveform of circuit variables is exponential.
- But what if the circuit contains more storage elements? An example could be defibrillator, automobile ignition system, filters and resonant circuits, …
- Applying KVL or KCL (mesh or nodal analysis), the circuit will be described by system of equations; we may write equations in time domain (integral differential equations), or in complex frequency domain using Laplace transform

Recapitulation – the relationship between voltage and current on basic circuit elements:

resistor	capacitor	inductor
$u(t) = R\,i(t)$	$u(t) = \frac{1}{C}\int_0^t i(\tau)\,\mathrm{d}\tau + u_c(0)$	$u(t) = L\frac{\mathrm{d}i(t)}{\mathrm{d}t}$
$i(t) = \frac{u(t)}{R}$	$i(t) = C\,\frac{\mathrm{d}u(t)}{\mathrm{d}t}$	$i(t) = \frac{1}{L}\int_0^t u(\tau)\mathrm{d}\tau + i_L(0)$

Laplace transform of the terms above:

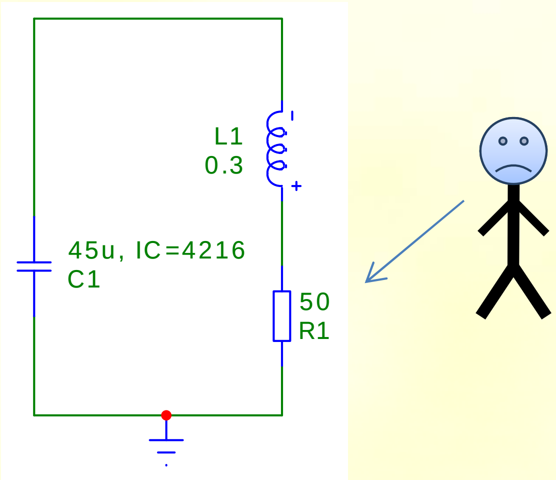
$U(p) = RI(p)$	$U(p) = \frac{I(p)}{pC} + \frac{u_c(0)}{p}$	$U(p) = pLI(p) - Li_L(0)$
$I(p) = \frac{U(p)}{R}$	$I(p) = pC\,U(p) - Cu_C(0)$	$I(p) = \frac{U(p)}{pL} + \frac{i_L(0)}{p}$

For the completeness sake recall phasor terms in frequency domain (that is, AC) – in contrast to two kinds of mathematical description above and despite obvious similarity AC represents sinusoidal ***steady state****, so such method* ***couldn't be used for transient analysis***

$\mathbf{U} = R\,\mathbf{I}$	$\mathbf{U} = \frac{\mathbf{I}}{j\omega C}$	$\mathbf{U} = j\omega L\mathbf{I}$
$\mathbf{I} = \frac{\mathbf{U}}{R}$	$\mathbf{I} = j\omega C\,\mathbf{U}$	$\mathbf{I} = \frac{\mathbf{U}}{j\omega L}$

- Equations in time domain arises directly from mathematical description of physical processes, but it frequently leads to solution of set of integral differential equations – difficult elimination of variables, difficult solution of a system of differential equations.
- Laplace transform simplifies the whole process on the solution of set of linear equations, when the difficulty of operations is similar to that in sinusoidal steady state; but, we have to be familiar with inverse Laplace transform
- On simple example of series RLC circuit we will compare both methods

RLC defibrillator



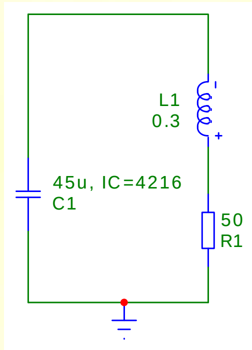
- It is the simplest example of the 2nd order circuit
- What is the waveform of current, which flows through the human body?
- How distinct circuit parameters (L, C, R) affects this waveform?

2ND ORDER TRANSIENT SOLUTION IN TIME DOMAIN

1. **Circuit equations**
   - In contrast to 1st order circuits, in 2nd order circuits is not preferred any method of network analysis
     - ➪ We can choose any method (mesh or nodal analysis)

     For this circuit mesh analysis is more suitable

$$L\frac{\mathrm{d}i(t)}{\mathrm{d}t}+Ri(t)+\frac{1}{C}\int_0^t i(\tau)\,\mathrm{d}\tau-u_C(0)=0$$



Circuit, whereon the decision procedure is illustrated
– RLC defibrillator

2. **separation of variables**

   in this example is only one circuit variable, we omit this step

3. **Differentiate circuit equation** (*if only we did not proceed separation of variables in 2nd step*)

$$\frac{\mathrm{d}^2 i(t)}{\mathrm{d}t^2}+\frac{R}{L}\frac{\mathrm{d}i(t)}{\mathrm{d}t}+\frac{1}{LC}i(t)=0$$

4. **Equation will be solved by method of variation of parameters**

   with zero right-hand side of an equation, thus without sources

$$\lambda^2+\frac{R}{L}\lambda+\frac{1}{LC}=0$$

☛ *The form of general solution of a differential equation depends on the roots of an quadratic equation*

☛ *Recall frequency response; (the same) roots of quadratic equation were solved there*

- Distinct real roots

$$\boxed{i(t) = K_1\, e^{\lambda_1 t} + K_2\, e^{\lambda_2 t} + i_p(t)}$$

Example:

$$\lambda^2 + 4000\lambda + 10^6 = 0 \qquad \Rightarrow \qquad \lambda_{1,2} = -2000 \pm \sqrt{2000^2 - 10^6} = -2000 \pm 1732.1 \;\; \begin{aligned} &= -3732.1 \\ &= -267.9 \end{aligned}$$

$$i(t) = K_1\, e^{-3732.1t} + K_2\, e^{-267.9t} + i_p(t)$$

- Double real roots

$$\boxed{i(t) = (K_1 + K_2\, t)\; e^{\lambda t} + i_p(t)}$$

Example:

$$\lambda^2 + 2000\lambda + 10^6 = 0 \qquad \Rightarrow \qquad \lambda_{1,2} = -1000 \pm \sqrt{1000^2 - 10^6} = -1000 \pm 0 = -1000$$

$$i(t) = (K_1 + K_2 t)\; e^{-1000t} + i_p(t)$$

- Complex conjugate roots $\qquad \lambda_{1,2} = -\alpha \pm j\sqrt{\omega_r^2 - \alpha^2} = -\alpha \pm j\omega$

$$\boxed{i(t) = (K_1 \cos\omega t + K_2 \sin\omega t)\; e^{-\alpha t} + i_p(t)}$$

Example:

$$\lambda^2 + 1000\lambda + 10^6 = 0 \qquad \Rightarrow \qquad \lambda_{1,2} = -500 \pm \sqrt{500^2 - 10^6} = -500 \pm j\sqrt{1000^2 - 500^2} = -500 \pm 866j$$

$$i(t) = (K_1 \cos(866t) + K_2 \sin(866t))\; e^{-500t} + i_p(t)$$

5. **Solve particular solution**

Steady state *in the circuit, after transient dies away*

According to the kind of exciting sources DC / AC / periodical steady state analysis

In the example of RLC defibrillator:

$$u_c(0) = 4216\,\mathrm{V} \qquad i_L(0) = 0\,\mathrm{A}$$

$$\lambda^2 + \frac{50}{0.3}\lambda + \frac{1}{0.3 \cdot 45 \cdot 10^{-6}} = 0 \quad \Rightarrow \quad \lambda = -83.\overline{3} \pm \sqrt{83.\overline{3}^2 - 74074} = -83.\overline{3} \pm 259.1j$$

$$i(t) = (K_1 \cos(259.1\,t) + K_2 \sin(259.1\,t))\; e^{-83.3t} + i(p)$$

$$i_p(t) = 0$$

Now we "just" have to find constants of integration K_1, K_2

- We have just one circuit equation, but two constants of integration K_1, K_2
- It is not enough to set $t = 0$, herewith we got just one equation

6. **Differentiate solution of a differential equation**

form of the solution results from the kind of roots of an quadratic equation

- Distinct real roots

$$i'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left[K_1\, e^{\lambda_1 t} + K_2\, e^{\lambda_2 t} + i_p(t)\right] = K_1\lambda_1 e^{\lambda_1 t} + K_2\lambda_2 e^{\lambda_2 t} + i_p'(t)$$

Example:

$$i'(t) = -3732.1\, K_1\, e^{-3732.1t} - 267.9\, K_2\, e^{-267.9t} + i_p'(t)$$

- Double real roots

$$i'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Big((K_1 + K_2\, t)\; e^{\lambda t} + i_p(t)\Big) = K_2\, e^{\lambda t} + (K_1 + K_2\, t)\, \lambda e^{\lambda t} + i_p'(t)$$

Example:

$$i'(t) = K_2\, e^{-1000t} + (K_1 + K_2 t)\, (-1000) e^{-1000t} + i_p'(t)$$

- Complex conjugate roots $\qquad \lambda_{1,2} = -\alpha \pm j\sqrt{\omega_r^2 - \alpha^2} = -\alpha \pm j\omega$

$$\begin{aligned} i'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left[(K_1 \cos\omega t + K_2 \sin\omega t)\; e^{-\alpha t} + i_p(t)\right] = \\ &= \Big[\omega\,(-K_1 \sin\omega t + K_2 \cos\omega t) - \alpha\,(K_1 \cos\omega t + K_2 \sin\omega t)\Big]\, e^{-\alpha t} + i_p'(t) \end{aligned}$$

Example:

$$i'(t) = \left[866\,(-K_1 \sin(866t) + K_2 \cos(866t)) - 500\,(K_1 \cos(866t) + K_2 \sin(866t))\right]\, e^{-500t} + i_p'(t)$$

But if we substitute for $t = 0$, in differentiated equation, we need to know its value at this time

7. Mathematical initial condition

at this step we return back to primary equation, to which we substitute

- $t = 0$
- energetic initial conditions (initial voltage across capacitors, current, which flowed through inductors)

$$L i'(0) + R i_L(0) + \underbrace{\frac{1}{C}\int_0^0 i(\tau)\,\mathrm{d}\tau}_{=0} - u_C(0) = 0 \qquad \Rightarrow \qquad i'(0) = \frac{u_C(0) - R i_L(0)}{L}$$

Specific form of solution of mathematical initial condition depends on equation (equations), describing the circuit

8. Into solution of a differential equation and its derivative substitute $t = 0$, initial conditions and solve system of linear equations

- Distinct real roots

$$\left.\begin{aligned} i(0) &= K_1 + K_2 + i_p(0) \\ i'(0) &= K_1\lambda_1 + K_2\lambda_2 + i_p'(0) \end{aligned}\right\} \rightarrow \qquad \begin{aligned} K_1 &= \frac{i'(0) - i_p'(0) - \lambda_2\big(i(0) - i_p(0)\big)}{\lambda_1 - \lambda_2} \\ K_2 &= \frac{i'(0) - i_p'(0) - \lambda_1\big(i(0) - i_p(0)\big)}{\lambda_2 - \lambda_1} \end{aligned}$$

- Double real roots

$$\left.\begin{aligned} i(0) &= K_1 + i_p(0) \\ i'(0) &= K_2 + K_1\lambda + i'_p(0) \end{aligned}\right\} \rightarrow \qquad \begin{aligned} K_1 &= i(0) - i_p(0) \\ K_2 &= i'(0) - i'_p(0) - \lambda\big(i(0) - i_p(0)\big) \end{aligned}$$

- Complex conjugate roots

$$\left.\begin{aligned} i(0) &= K_1 + i_p(0) \\ i'(0) &= \omega K_2 - \alpha K_1 + i'_p(0) \end{aligned}\right\} \rightarrow \qquad \begin{aligned} K_1 &= i(0) - i_p(0) \\ K_2 &= \frac{i'(0) - i'_p(0) + \alpha\big(i(0) - i_p(0)\big)}{\omega} \end{aligned}$$

Now we will finish solution of our example of RLC defibrillator:

$$i(t) = (K_1 \cos(259.1\,t) + K_2 \sin(259.1\,t))\; e^{-83.3t} + 0$$

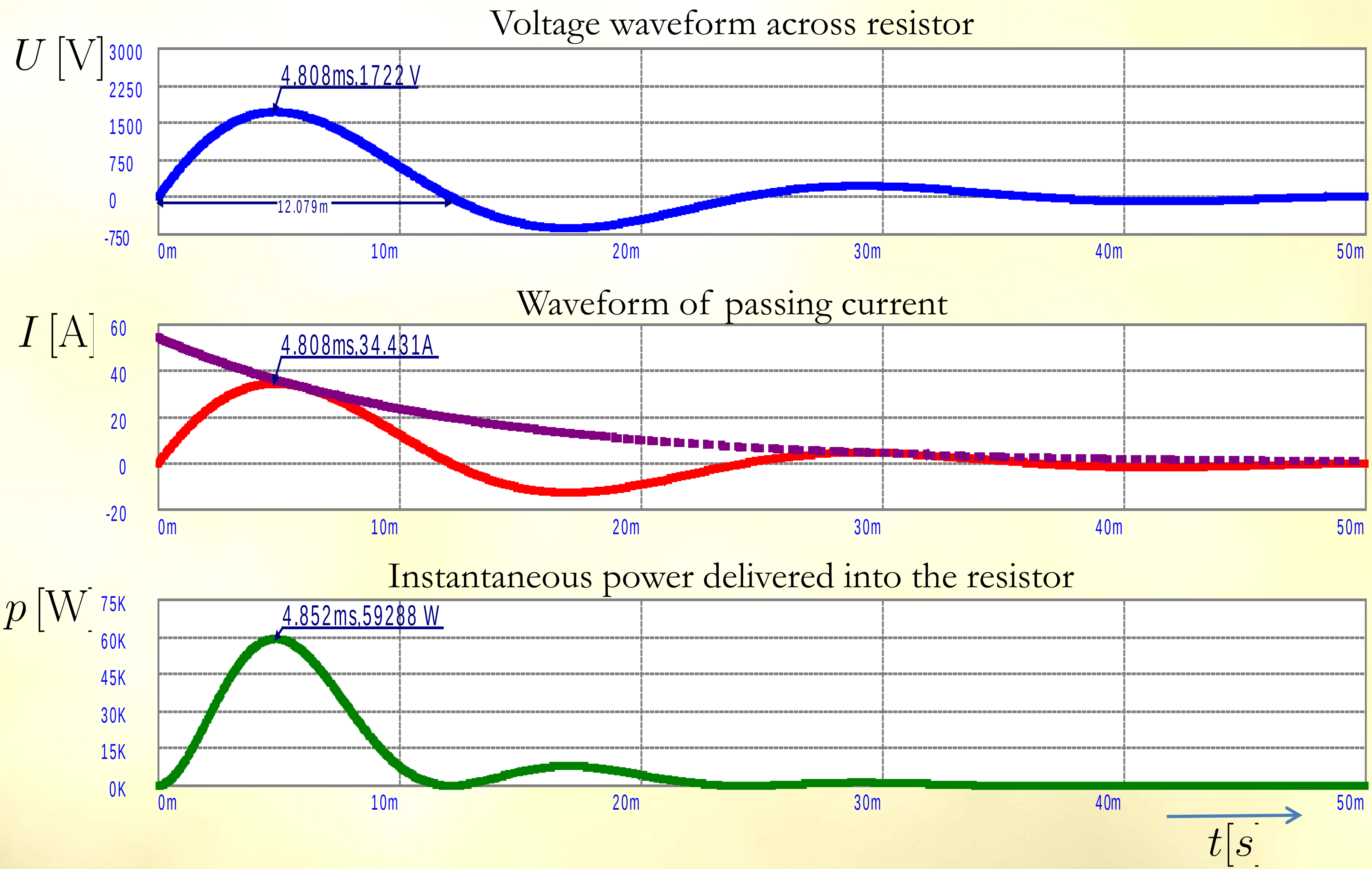
$$i'(t) = 259.1\,(-K_1 \sin 259.1t + K_2 \cos 259.1t) - 83.3\,(K_1 \cos 259.1t + K_2 \sin 259.1t)\Big]\; e^{-83.3t}$$

$$i'(0) = \frac{4216 - 50 \cdot 0}{0.3} = 14053$$

$$\left.\begin{aligned} 0 &= K_1 + 0 \\ 14053 &= 259.1K_2 - 83.3K_1 \end{aligned}\right\} \rightarrow \qquad \begin{aligned} K_1 &= 0 \\ K_2 &= \frac{14053 + 83.3 \cdot 0}{259.1} = 54.24 \end{aligned}$$

$$\underline{\underline{i(t) = 54.24 \sin(259.1\,t) \cdot e^{-83.3t}\ \mathrm{A}}}$$

Waveform of 2nd order transient – RLC defibrillator



WHAT AFFECTS THE WAVEFORM?

We have characteristic equation: $$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$$

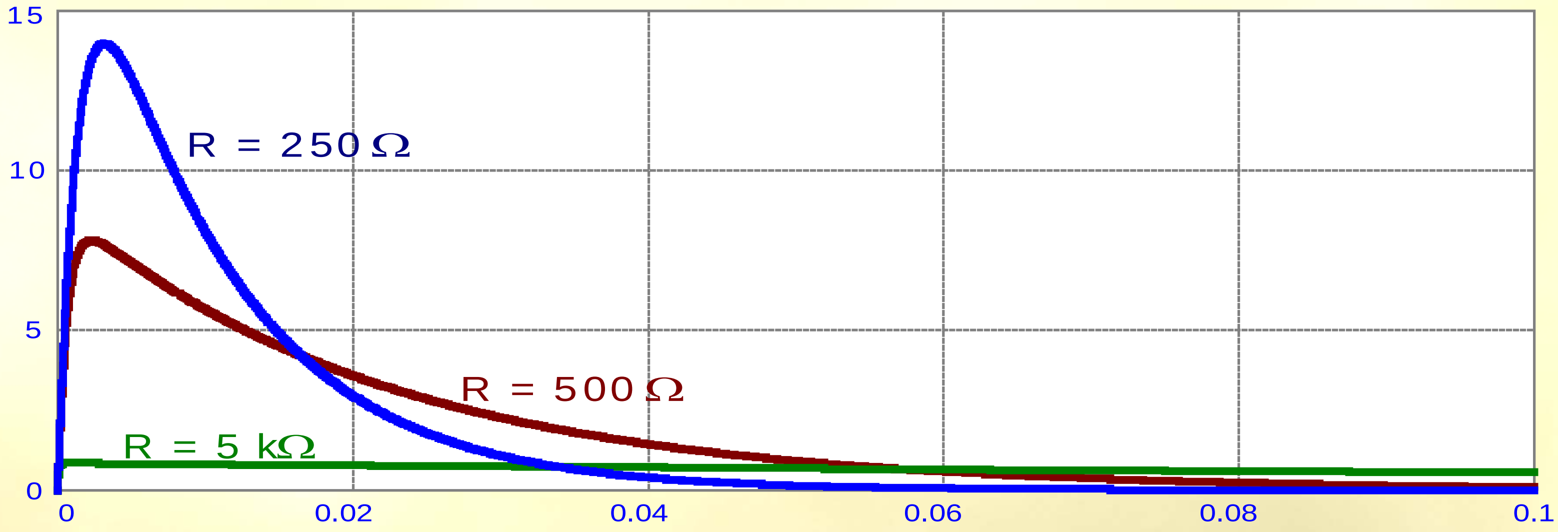
... and its roots $$\lambda_{1,2} = \frac{-R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

- **<u>If:</u>**

$$\left(\frac{R}{2L}\right)^2 > \frac{1}{LC}$$

Aperiodic transient (overdamped response)

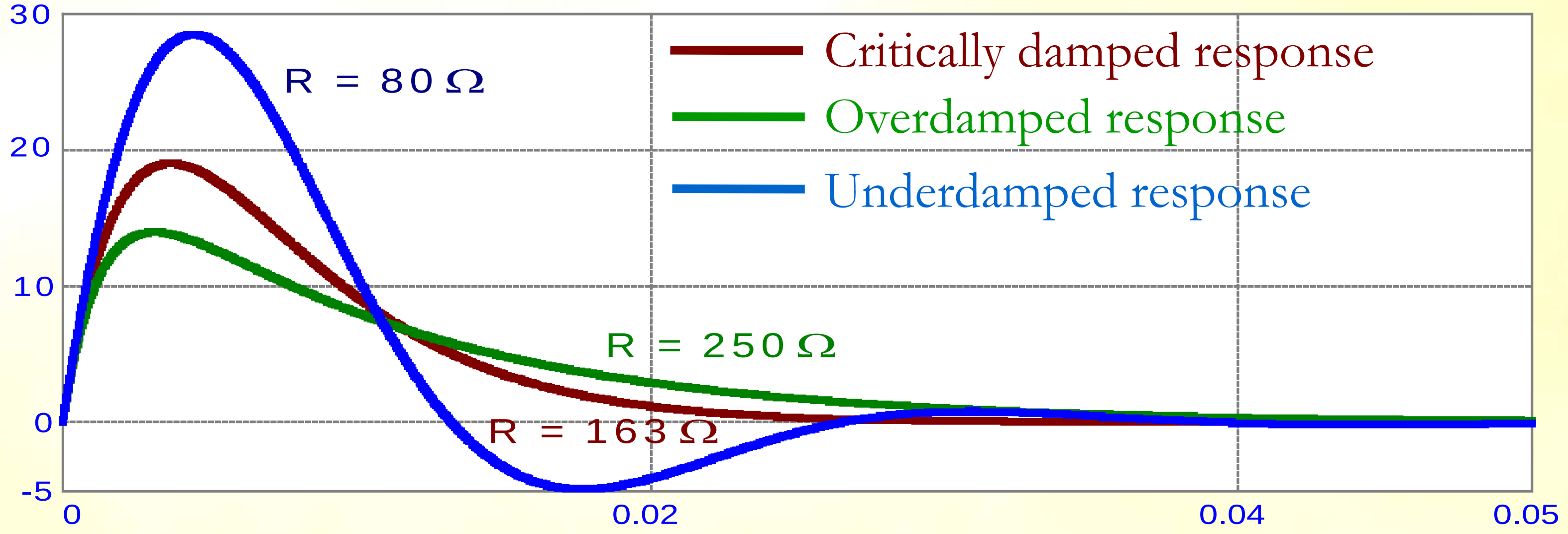
- The bigger the R is, the larger the time constant is
  - The bigger the R is, the transient last longer



$$\left(\frac{R}{2L}\right)^2 = \frac{1}{LC} \quad \Rightarrow \quad R = 2\sqrt{\frac{L}{C}}$$

Limit of aperiodicity (critically damped response)

- Transient last **shortest possible time**



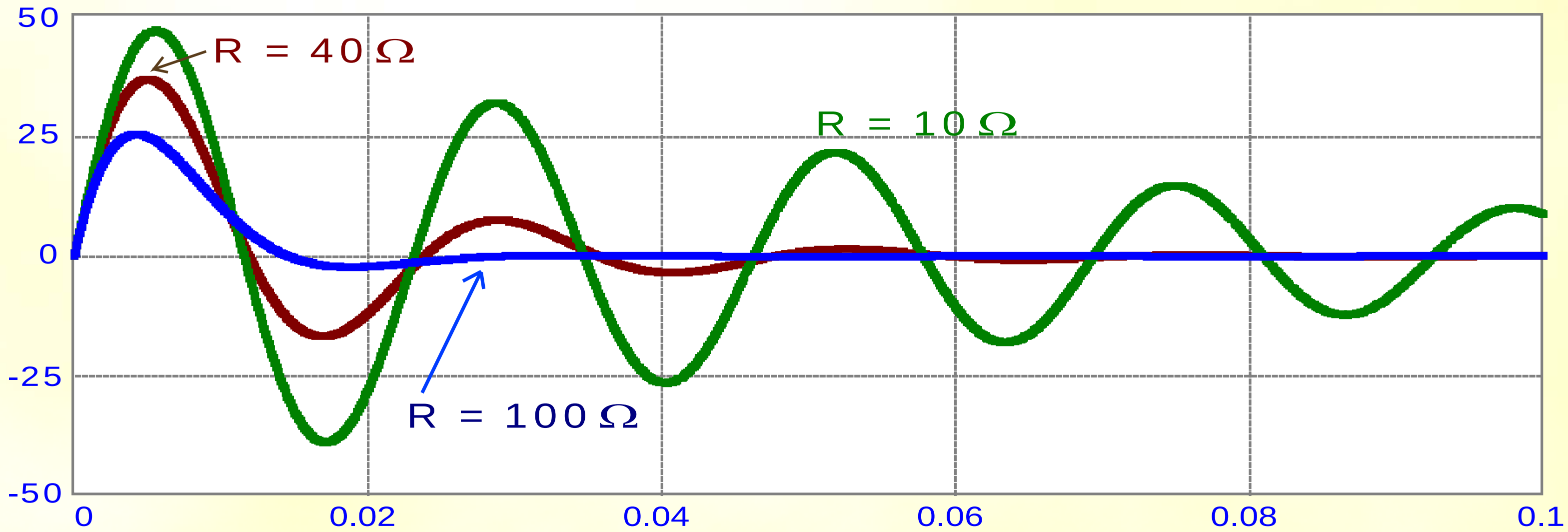
$$\left(\frac{R}{2L}\right)^2 < \frac{1}{LC}$$

Quasi-periodic transient (underdamped response)

- With decreasing R the dumping factor α also decrease
  - **Settling time is increasing**

$$\lambda_{1,2} = \frac{-R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = -\alpha \pm \underbrace{\sqrt{\alpha^2 - \omega_r^2}}_{j\omega}$$

α	Damping factor
ω	Frequency of natural oscillations
ω_r	Resonant frequency

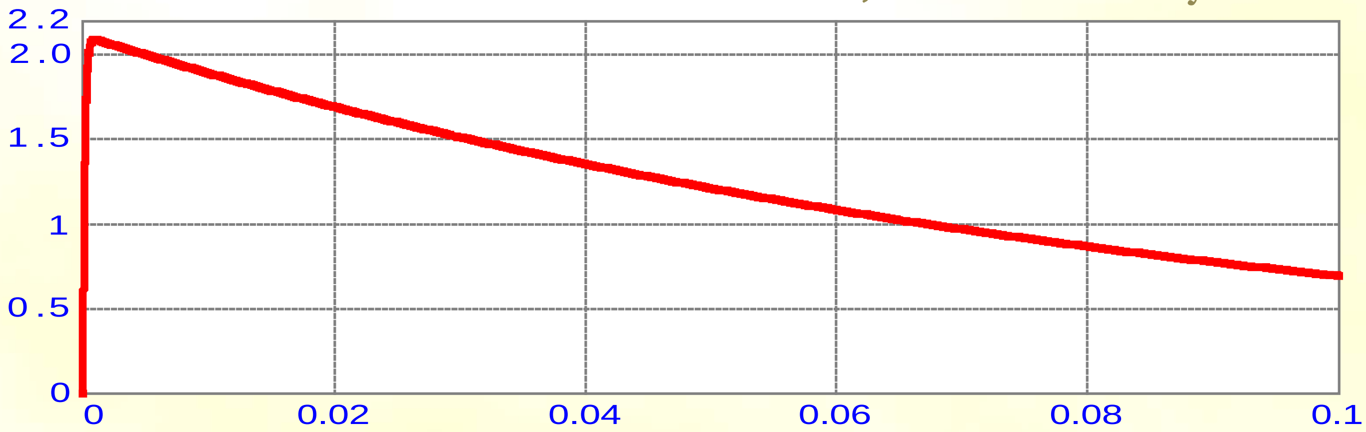


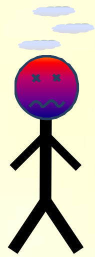
What affects:

R	Damping factor α, but also magnitude of the current I
L	Damping factor α, but also magnitude of the current I (ω in K_2 expression) and frequency of natural oscillations of the circuit (together with resonant frequency)
C	Magnitude of the current I (ω in K_2 expression) and frequency of natural oscillations of the circuit (together with resonant frequency)

- What happens, if the patient will not have the correct resistivity...
  - Typical resistivity of human body while defibrillation is 50 Ω
  - Resistivity of human body exposed to line voltage is indicated in interval of 500 Ω – 10 kΩ, average value is 2 kΩ

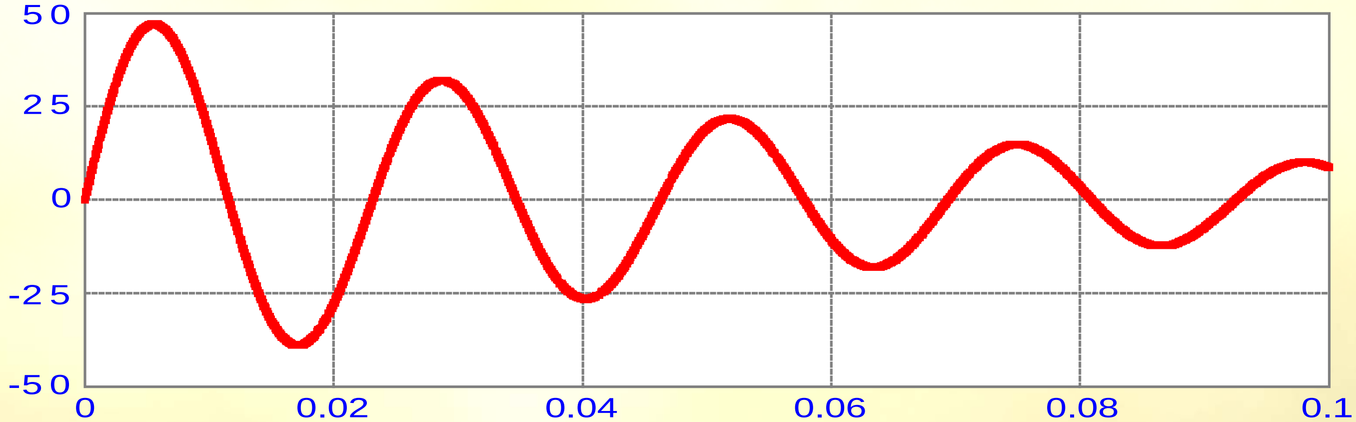
The waveform of the current in the case, the human body has 2 kΩ

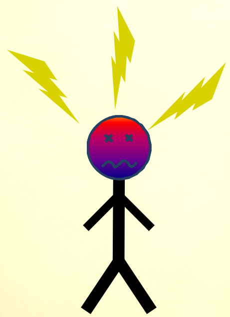




The resistance is too large – overdamped response, large current, flowing through the circuit for long time, skin burn – it is necessary to prevent surface resistance (conductive gel)

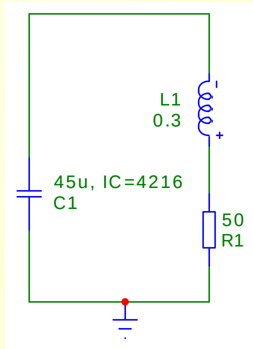
The waveform of the current in the case, the human body has 10 Ω





Too small resistance – underdamped (quasi-periodic) response, repeated stimulation, fibrillation

TRANSIENT ANALYSIS IN LAPLACE DOMAIN



Circuit, whereon the decision procedure is illustrated
– RLC defibrillator

1. **Circuit equations**

$$pLI(p) - Li_L(0) + RI(p) + \frac{I(p)}{pC} - \frac{u_C(0)}{p} = 0$$

2. **Separation of variables / simplification of an equation**
   use matrix of a system of equations can significantly simplify solution

$$I(p)\left(pL + R + \frac{1}{pC}\right) = Li_L(0) + \frac{u_C(0)}{p}$$

Compare with characteristic equation in time domain:

$$I(p) = \frac{pCLi_L(0) + Cu_C(0)}{p^2LC + pRC + 1} = \frac{pi_L(0) + \frac{u_C(0)}{L}}{p^2 + p\frac{R}{L} + \frac{1}{LC}}$$

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$$

3. **Find roots of polynomial in denominator**

$$p_{12} = \frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

4. **Write denominator as product of factors**
   Recall simplifications we did, when we draw frequency responses

5. **Do decomposition into partial fractions and inverse transform**
   without calculation of particular solution, it is part of result of inverse transform

In the example of RLC defibrillator:

$$u_c(0) = 4216\,\mathrm{V} \qquad i_L(0) = 0\,\mathrm{A}$$

$$0.3pI(p) - 0.3\cdot 0 + 50I(p) + \frac{I(p)}{45\cdot 10^{-6}p} - \frac{4216}{p} = 0$$

$$I(p) = \frac{\frac{4216}{0.3}}{p^2 + p\frac{50}{0.3} + \frac{1}{0.3\cdot 45\cdot 10^{-6}}} \quad \Rightarrow \quad p = -83.\overline{3}\pm\sqrt{83.\overline{3}^2 - 74074} = -83.\overline{3}\pm 259.1j$$

$$I(p) = 54.239\cdot\frac{259.1}{(p+83.\overline{3})^2 + 259.1^2}$$

$$\downarrow$$

$$\underline{i(t) = 54.24\sin(259.1\,t)\cdot e^{-83.3t}\,\mathrm{A}}$$

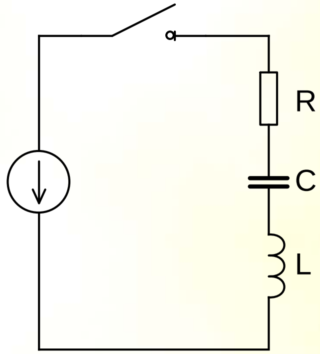
The transient in the circuit of an arbitrary order, with regard to roots of polynomial can have just linear combination of following terms:

roots	
Distinct real	$y_0(t) = \sum_{k=1}^{n} A_k e^{\lambda_k t}$
Repeated root (multiplicity m)	$y_{0m}(t) = \left(A_1 + A_2 t + A_3 t^2 + \cdots + A_m t^{m-1}\right) e^{\lambda t}$
Complex conjugate	$\lambda_{1,2} = -\alpha \pm j\omega$ $K_1 e^{\lambda_1 t} + K_2 e^{\lambda_2 t} = e^{-\alpha t}\left(A\sin\omega t + B\cos\omega t\right) =$ $= D\sin\left(\omega t + \psi\right)$

SERIES RLC CIRCUIT WITH SINUSOIDAL EXCITATION

Now we will study RLC circuit, to which we connect sine wave voltage source

We will assume small damping (small *R*), so that the circuit has quasi-periodic response



$R = 100\ \Omega$
$C = 1\ \mu F$
$L = 1\ H$

$$u(t) = 1\ \sin(1000\,t)$$

$$u_C(0) = 0\,\mathrm{V} \qquad i_L(0) = 0\,\mathrm{A}$$

- Circuit equation and its derivative

$$L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)\,\mathrm{d}\tau + u_C(0) = U_m \sin(\omega_z t + \varphi)$$

$$\frac{\mathrm{d}^2 i(t)}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \frac{1}{LC}i(t) = U_m\omega_z \cos(\omega_z t + \varphi)$$

The response of the circuit to the change are exponentially damped oscillations; they are not related to source waveform in no manner

- Characteristic equation and its roots

The frequency of oscillation is ω_v

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0 \qquad \lambda_{1,2} = -\alpha \pm j\sqrt{\omega_r^2 - \alpha^2} = -\alpha \pm j\omega_v$$

$$\lambda^2 + 100\lambda + 10^6 = 0 \qquad \lambda_{1,2} = -50 \pm j\sqrt{1000^2 - 50^2} = -50 \pm 998.75j$$

- Particular solution

$$u(t) = U_m \sin(\omega_z t + \varphi) \quad \Rightarrow \quad \hat{\mathbf{U}} = U_m \angle\varphi$$

$$\hat{\mathbf{I}} = \frac{\hat{\mathbf{U}}}{R + j(\omega_z L - \frac{1}{\omega_z C})} = I_p \underline{/\psi} \quad \Rightarrow \quad i_p(t) = I_p \sin(\omega_z t + \psi)$$

$$\hat{\mathbf{I}} = \frac{1}{100 + j(1000 - 1000)} = 0.01\,\mathrm{A} \quad \Rightarrow \quad i_p(t) = 0.01 \sin(1000t)$$

In analyzed problem the RLC circuit is in resonance

- Solution of a differential equation

$$i(t) = (K_1 \cos\omega_v t + K_2 \sin\omega_v t)\; e^{-\alpha t} + I_p \sin(\omega_z t + \psi)$$

Whilst the circuit, as a response to the connection of the source, exhibits exponentially dumped oscillations of frequency ω_v,
in steady state sine wave source enforce to the circuit its frequency ω_z
But resulting waveform is superposition of both natural response and forced response

$$i(t) = (K_1 \cos(998.75\,t) + K_2 \sin(998.75\,t))\, e^{-50t} + 0.01 \sin(1000t)$$

When the damping is greater than zero, the frequency of natural oscillations is less than resonance frequency

- Derivative of a solution and mathematical initial condition

$$i'(t) = \Big[\omega_v \left(-K_1 \sin \omega_v t + K_2 \cos \omega_v t\right) - \alpha \left(K_1 \cos \omega_v t + K_2 \sin \omega_v t\right)\Big]\, e^{-\alpha t} + I_p\, \omega_z \cos(\omega_z t + \psi)$$

$$L i'(0) + R i_L(0) + \underbrace{\frac{1}{C}\int_0^0 i(\tau)\,\mathrm{d}\tau}_{=0} + u_C(0) = U_m \sin(\varphi) \qquad \Rightarrow \qquad i'(0) = \frac{-u_C(0) - R i_L(0) + U_m \sin(\varphi)}{L}$$

$$\begin{aligned} i'(t) = \; & \Big[998.75\left(-K_1 \sin(998.75t) + K_2 \cos(998.75t)\right) - 50\left(K_1 \cos(998.75t) + K_2 \sin(998.75t)\right)\Big]\, e^{-50t} \\ & +0.01 \cdot 1000 \cos(1000t) \end{aligned}$$

$$i'(0) = 0$$

- $t = 0$

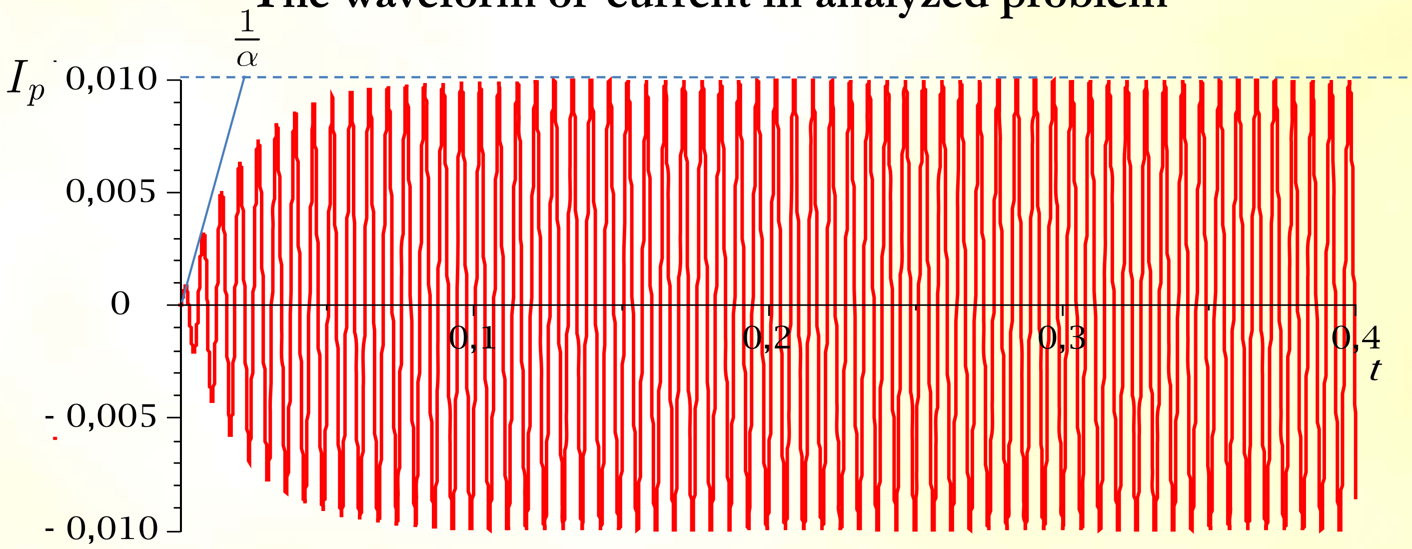
$$\left.\begin{aligned} i(0) &= K_1 + I_p \sin(\psi) \\ i'(0) &= \omega_v K_2 - \alpha K_1 + I_p \omega_z \cos(\psi) \end{aligned}\right\} \rightarrow \qquad \begin{aligned} K_1 &= i(0) - I_p \sin(\psi) \\ K_2 &= \frac{i'(0) - I_p\, \omega_z \cos(\psi) + \alpha\big(i(0) - I_p \sin(\psi)\big)}{\omega_v} \end{aligned}$$

- Solution of a transient:

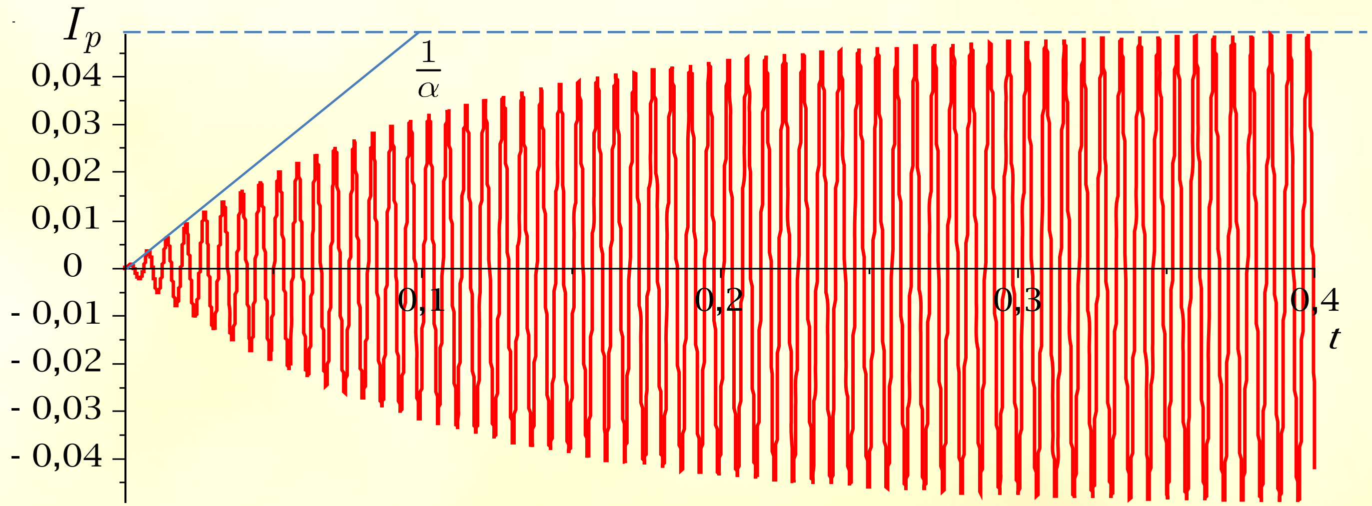
$$i(t) = I_p \left\{ -\left[\frac{1}{\omega_v}\left(\alpha \sin\psi + \omega_z \cos\psi\right) \sin \omega_v t + \sin\psi \cos \omega_v t\right] e^{-\alpha t} + \sin(\omega_z t + \psi) \right\}$$

$$i(t) = 0.01 \left(\frac{-1000}{998.75} \cdot \sin(998.75\, t)\, e^{-50t} + \sin(1000t) \right)$$

The waveform of current in analyzed problem



The waveform of current – same circuit, but R = 20 Ω ⇒ α = 10



The circuit is in resonance – the voltages across L and C have 1000× greater magnitude (so 10, and 50 volts respectively) and its phase shift is +90°, and - 90° (or a quarter of period). The bigger the quality factor is (smaller α), the longer last the transient dies away

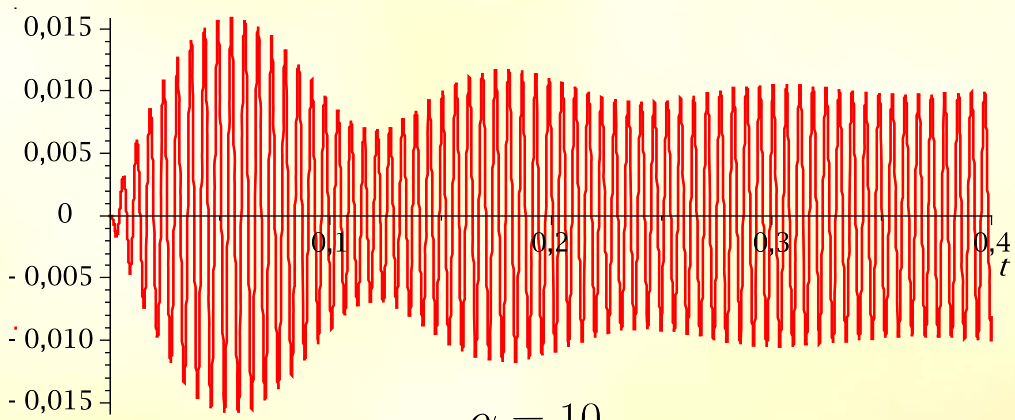
$$Q = \frac{1}{\alpha} \cdot \frac{\omega_r}{2}$$

Beats

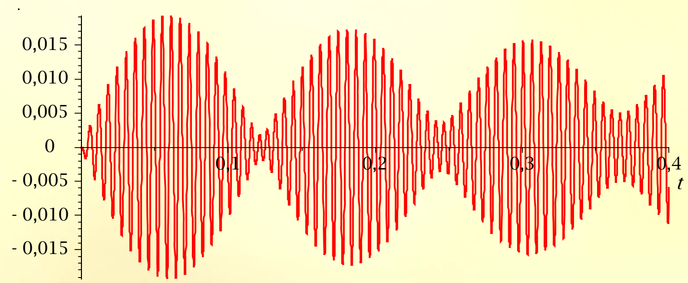
- Consider the case, where the frequency of forced oscillations is near to the resonant frequency
  In such case we may simplify previous result considerably:
  - ✓ Damping factor has to be small, for simplification suppose $\alpha \to 0$
  - ✓ Previous condition implicates the frequency of natural oscillations $\omega_v \to \omega_r$
  - ✓ In resonance the phase shift between source voltage and current $\psi \to 0$
  - ✓ The frequency of the source differs from the resonance frequency by *Δω,* $\omega_z = \omega_r + \Delta\omega$

  Then:

$$
\begin{aligned}
i(t) &= I_p \left\{ -\left[\frac{1}{\omega_v} (\alpha \sin\psi + \omega_z \cos\psi) \sin \omega_v t + \sin\psi \cos\omega_v t \right] e^{-\alpha t} + \sin(\omega_z t + \psi) \right\} \\
&\doteq I_p \left\{ -\sin\omega_r t + \sin(\omega_r + \Delta\omega) t \right\} = 2 I_p \cos\left(\frac{\omega_r t + (\omega_r + \Delta\omega) t}{2} \right) \cdot \sin\left(\frac{(\omega_r + \Delta\omega) t - \omega_r t}{2} \right) \\
&= 2 I_p \cos\left[\left(\omega_r + \frac{\Delta\omega}{2} \right) t \right] \cdot \sin\frac{\Delta\omega t}{2}
\end{aligned}
$$



$\alpha = 10$



$\alpha = 2$

Physical analogy: in acoustic – instrument tuning, used by some musicians, etc.

Recapitulation:

- In any circuit that contain any number of storage elements just 3 kinds of natural response are possible:
    - Overdamped (exponential)
    - Critically damped (the transient last less possible time)
    - Underdamped – exponentially damped oscillations, the natural frequency is the same (when damping $R = 0$) or less than resonant frequency; the circuit exhibits resonant properties, when connected to sinusoidal voltage source

- The response of a 2nd order circuit with two storage elements of the same type (2 inductors or 2 capacitors) cannot be underdamped response (oscillatory), because to rise oscillations storage elements have to interchange stored energy (they are in resonance), two capacitors or inductors can be just charged.

- Damping is affected by resistivity, as well as how long the transient lasts – when the damping (R) is too large, the response just very slowly falls into steady state, when damping is too small, the circuit oscillates – if the circuit contains both inductor and capacitor; two capacitors (inductors) in the circuit with zero damping would be charged immediately, so such transient would last zero time!!!