Fundamentals of Electrical Circuits

X

Circuit equations and transients

CIRCUIT EQUATIONS IN TIME DOMAIN AND FREQUENCY P DOMAIN,
TRANSIENTS IN ELECTRICAL CIRCUITS.

CIRCUIT EQUATIONS AND DIFFERENT MATH APPARATUS

What is the same and what is different when we will write circuit equations in the time domain or operational form, or in DC or AC circuits?

Circuit equations, regardless of used mathematical apparatus, are always mathematical formulations of Kirchhoff's laws:

MESH (LOOP) ANALYSIS – KVL

$$\sum_k U_k = 0$$

voltage across R, L, C is qualified by means of current

NODAL ANALYSIS – KCL

$$\sum_k I_k = 0$$

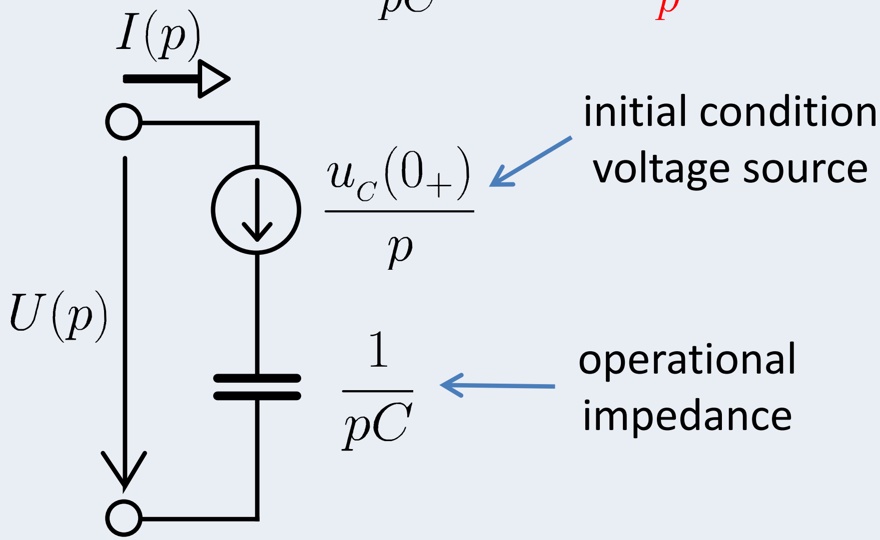
current, passing R, L, C is qualified by means of voltage

RELATIONSHIP BETWEEN VOLTAGE AND CURRENT

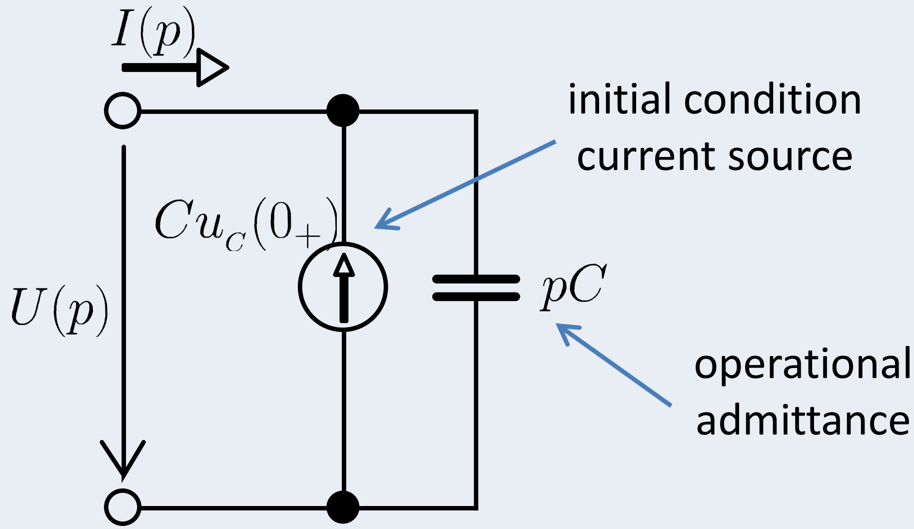
Circuit element	Time domain	DC	AC / Fourier	Laplace
R	$u_R(t) = Ri_R(t)$ $i_R(t) = Gu_R(t)$	$U_R = RI_R$ $I_R = GU_R$	$\mathbf{U}_R = R\mathbf{I}_R$ $\mathbf{I}_R = G\mathbf{U}_R$	$U_R(p) = RI_R(p)$ $I_R(p) = GU_R(p)$
L	$u_L(t) = L\dfrac{\mathrm{d}i_L(t)}{\mathrm{d}t}$ $i_L(t) = \dfrac{1}{L}\displaystyle\int_0^t u_L(\tau)\,\mathrm{d}\tau + i_L(0_+)$	$U_L = 0$ $I_L = \text{any}$ **short**	$\mathbf{U}_L = j\omega L\,\mathbf{I}_L$ $\mathbf{I}_L = \dfrac{1}{j\omega L}\mathbf{U}_L$	$U_L(p) = pL\,I_L(p) - Li_L(0_+)$ $I_L(p) = \dfrac{1}{pL}U_L(p) + \dfrac{i_L(0_+)}{p}$
C	$u_C(t) = \dfrac{1}{C}\displaystyle\int_0^t i_C(\tau)\,\mathrm{d}\tau + u_C(0_+)$ $i_C(t) = C\dfrac{\mathrm{d}u_C(t)}{\mathrm{d}t}$	$U_C = \text{any}$ $I_C = 0$ open circuit	$\mathbf{U}_C = \dfrac{1}{j\omega C}\mathbf{I}_C$ $\mathbf{I}_C = j\omega C\,\mathbf{U}_C$	$U_C(p) = \dfrac{1}{pC}I_C(p) + \dfrac{u_C(0_+)}{p}$ $I_C(p) = pC\,U_C(p) - Cu_C(0_+)$
usage	**transient** analysis *usually DC/ AC* analysis has to be also proceeded	**Steady** state with **DC** source	AC – **steady** state with **sine** wave source Fourier series – **steady** state with **periodical** excitation transform – **pulse** excitation	**General** application contains both **steady** and **transient** components

INDUCTOR AND CAPACITOR EQUIVALENT CIRCUIT DIAGRAMS

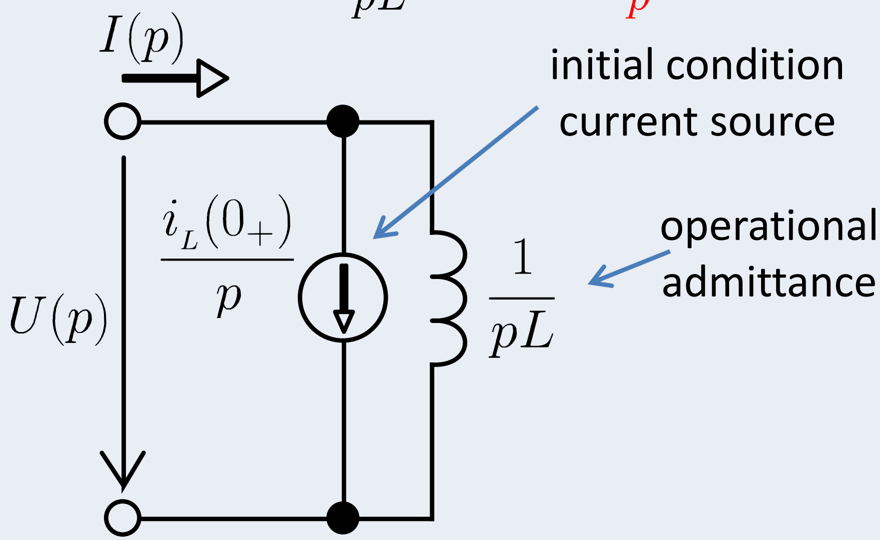
$$U_C(p) = \frac{1}{pC} I_C(p) + \frac{u_C(0_+)}{p}$$



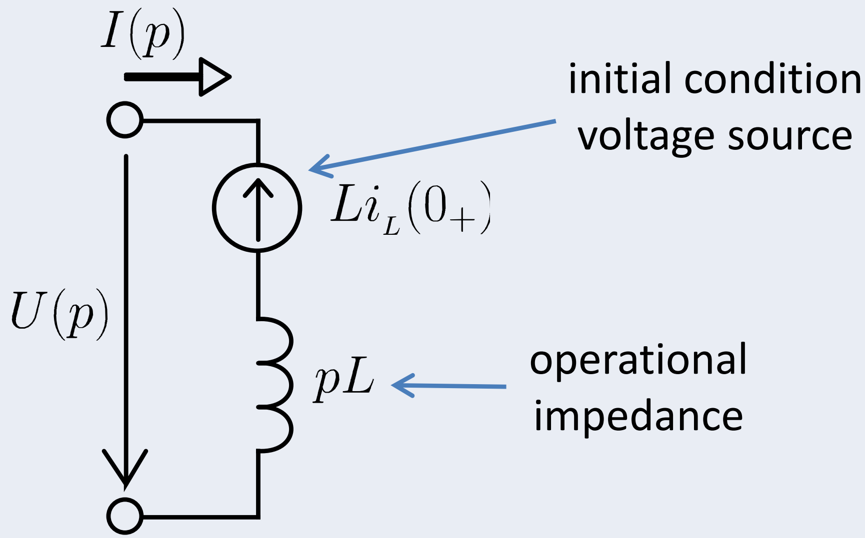
$$I_C(p) = pC\,U_C(p) - C u_C(0_+)$$



$$I_L(p) = \frac{1}{pL} U_L(p) + \frac{i_L(0_+)}{p}$$

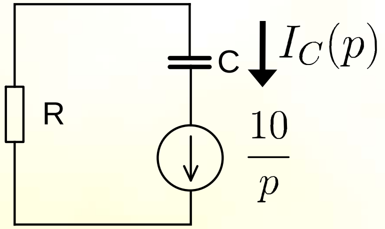


$$U_L(p) = pL\,I_L(p) - L i_L(0_+)$$



The capacitor was charged at 10 V. Find the current passing the capacitor.

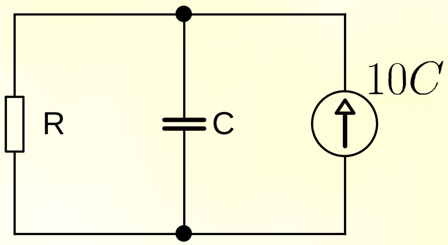
series equivalent circuit



- Ohm's law

$$I_C(p) = -\frac{\frac{u_C(0_+)}{p}}{R + \frac{1}{pC}} = -\frac{10 \cdot C}{1 + pRC}$$

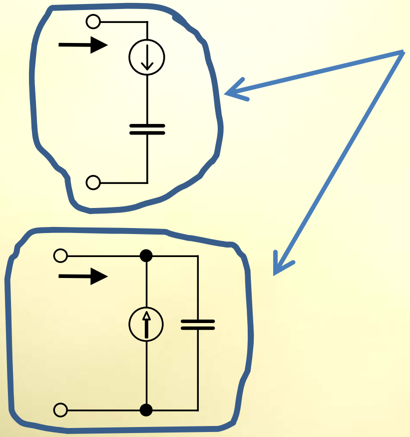
parallel equivalent circuit



- current divider

$$I_C'(p) = C\, u_C(0_+)\, \frac{R}{R + \frac{1}{pC}} = C \cdot 10 \cdot \frac{pRC}{1 + pRC}$$

??? But the current passing capacitor should be the same???

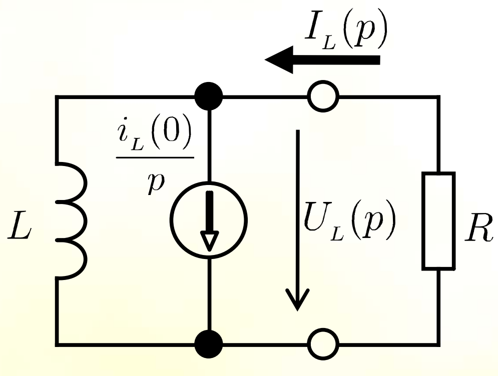


This is actual capacitor!!!

$$I_C(p) = C \cdot 10 \cdot \frac{pRC}{1 + pRC} - 10C = \frac{10C\, pRC - 10C(1 + pRC)}{1 + pRC} = \frac{-10 \cdot C}{1 + pRC}$$

In time $t = 0$ an inductor was passed by the current $i_L(0) = 2$ A.
Find the Laplace transform of current passing the inductor when $t > 0$ and voltage transform at $t > 0$.

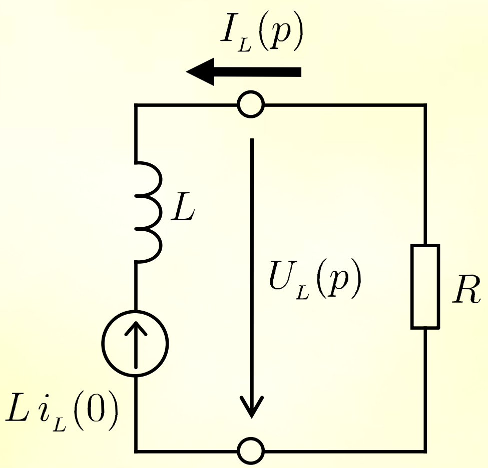
parallel equivalent circuit



$$I_L(p) = -\frac{i_L(0)}{p}\frac{R}{R+pL} + \frac{i_L(0)}{p} = \frac{i_L(0)}{p}\left(1 - \frac{R}{R+pL}\right) = \frac{L\, i_L(0)}{R+pL}$$

$$U_L(p) = -\frac{i_L(0)}{p}\frac{pLR}{pL+R} = \frac{-i_L(0)\, LR}{pL+R}$$

series equivalent circuit

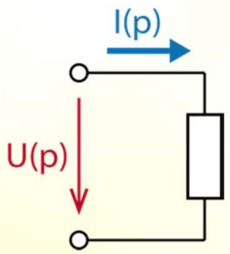


$$I_L(p) = \frac{L\, i_L(0)}{R+pL}$$

$$U_L(p) = L\, i_L(0)\frac{pL}{pL+R} - L\, i_L(0) = \frac{-i_L(0)\, LR}{pL+R}$$

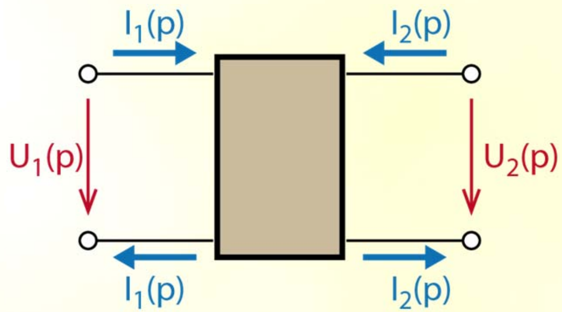
When the **initial conditions are zero,** the operational characteristics will be analogical to those in sinusoidal steady state

- Impedance and admittance, including the input one of two ports



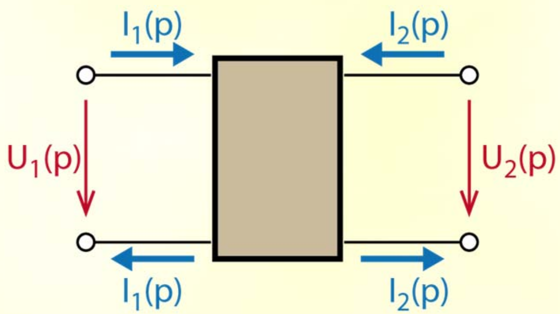
$$Z(p) = \frac{U(p)}{I(p)}$$

$$Y(p) = \frac{I(p)}{U(p)}$$



$$Z_{in}(p) = \frac{U_1(p)}{I_1(p)}$$

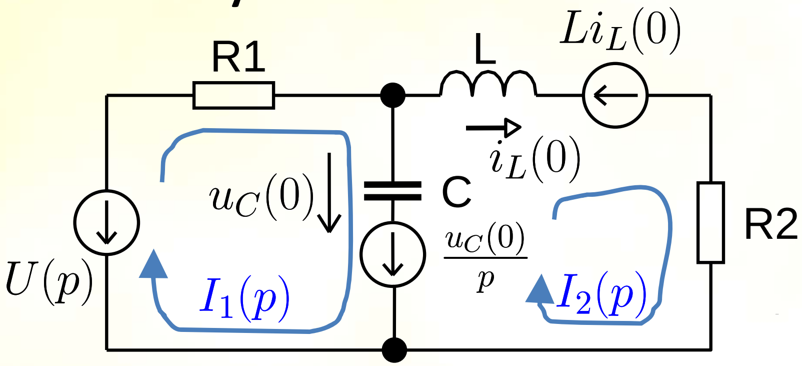
- Transfer function (voltage, current, …)



$$P_U(p) = \frac{U_2(p)}{U_1(p)}$$

$$P_I(p) = \frac{I_2(p)}{I_1(p)}$$

Mesh analysis:



Time domain:

$$R_1 i_1(t) + \frac{1}{C}\int_0^t [i_1(\tau) - i_2(\tau)]\ \mathrm{d}\tau + u_{\mathrm{c}}(0_+) - u(t) = 0$$

$$L\frac{\mathrm{d}i_2(t)}{\mathrm{d}t} + R_2 i_2(t) - u_{\mathrm{c}}(0_+) + \frac{1}{C}\int_0^t [i_2(\tau) - i_1(\tau)]\ \mathrm{d}\tau = 0$$

Laplace:

$$R_1 I_1(p) + \frac{1}{pC}[I_1(p) - I_2(p)] + \frac{u_C(0)}{p} - U(p) = 0$$

$$pLI_2(p) - Li_L(0) + R_2 I_2(p) - \frac{u_C(0)}{p} + \frac{1}{pC}[I_2(p) - I_1(0)] = 0$$

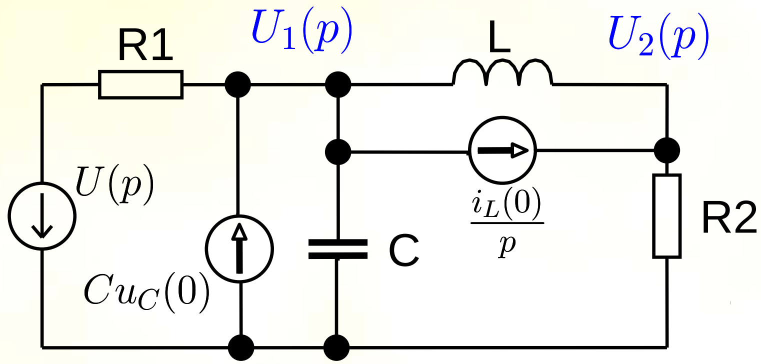
Matrix notation

$$\begin{bmatrix} \overbrace{R_1 + \frac{1}{pC}}^{\text{loop } I_1} & \overbrace{-\frac{1}{pC}}^{\text{branch 12}} \\ \underbrace{-\frac{1}{pC}}_{\text{branch 21}} & \underbrace{pL + R_2 + \frac{1}{pC}}_{\text{loop } I_2} \end{bmatrix} \cdot \begin{bmatrix} I_1(p) \\ I_2(p) \end{bmatrix} = \begin{bmatrix} U(p) - \frac{u_{\mathrm{C}}(0_+)}{p} \\ \frac{u_{\mathrm{C}}(0_+)}{p} + Li_L(0_+) \end{bmatrix}$$

AC:

$$\begin{bmatrix} R_1 + \frac{1}{j\omega C} & -\frac{1}{j\omega C} \\ -\frac{1}{j\omega C} & j\omega L + R_2 + \frac{1}{j\omega C} \end{bmatrix} \cdot \begin{bmatrix} \hat{\mathbf{I}}_1 \\ \hat{\mathbf{I}}_2 \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{U}} \\ 0 \end{bmatrix}$$

Nodal analysis:



Time domain:

$$\frac{u_1(t)-u(t)}{R_1}+\frac{1}{L}\int_0^t [u_1(\tau)-u_2(\tau)]\,\mathrm{d}\tau + C\frac{\mathrm{d}u_1(t)}{\mathrm{d}t}+i_L(0)=0$$

$$\frac{1}{L}\int_0^t [u_2(\tau)-u_1(\tau)]\,\mathrm{d}\tau + \frac{u_2(t)}{R_2}-i_L(0)=0$$

Laplace:

$$\frac{U_1(p)-U(p)}{R_1}+\frac{U_1(p)-U_2(p)}{pL}+pCU_1(p)-Cu_C(0)+\frac{i_L(0)}{p}=0$$

$$\frac{U_2(p)-U_1(p)}{pL}+\frac{U_2(p)}{R_2}-\frac{i_L(0)}{p}=0$$

Matrix notation

$$\begin{bmatrix} \overbrace{\frac{1}{R_1}+pC+\frac{1}{pL}}^{\text{node } U_1} & \overbrace{-\frac{1}{pL}}^{\text{branch between nodes 1, 2}} \\ \underbrace{-\frac{1}{pL}}_{\text{branch between nodes 2, 1}} & \underbrace{\frac{1}{pL}+\frac{1}{R_2}}_{\text{node } U_2} \end{bmatrix} \cdot \begin{bmatrix} U_1(p) \\ U_2(p) \end{bmatrix} = \begin{bmatrix} \frac{U(p)}{R_1}+Cu_\mathrm{C}(0_+)-\frac{i_\mathrm{L}(0_+)}{p} \\ \frac{i_\mathrm{L}(0_+)}{p} \end{bmatrix}$$

AC:

$$\begin{bmatrix} \frac{1}{R_1}+j\omega C+\frac{1}{j\omega L} & -\frac{1}{j\omega L} \\ -\frac{1}{j\omega L} & \frac{1}{j\omega L}+\frac{1}{R_2} \end{bmatrix} \cdot \begin{bmatrix} \hat{\mathrm{U}}_1 \\ \hat{\mathrm{U}}_2 \end{bmatrix} = \begin{bmatrix} \frac{\hat{\mathrm{U}}}{R_1} \\ 0 \end{bmatrix}$$

Be careful – if circuit contains controlled sources,
it is necessary introduce other, more complicated rules
for direct writing of matrices (nulor model)!!!

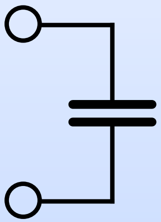
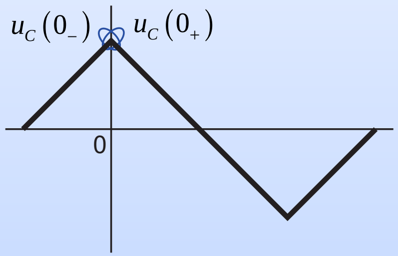
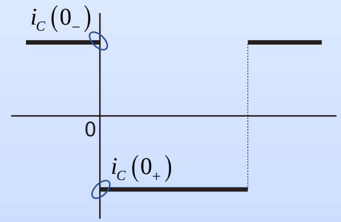
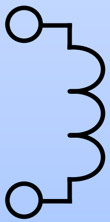
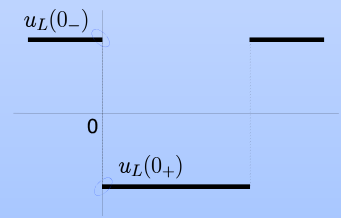
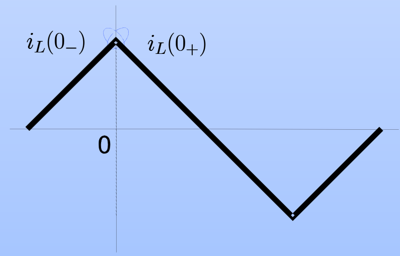
CIRCUIT EQUATIONS – MATRIX NOTATION

- ✓ It is suitable for DC, AC, Laplace
  - ✓ Gaussian elimination solution, Cramer's rule, inverse matrix – very easy with mathematical computer programs (Matlab, Maple, …)
- ✗ Could not be used in time domain

INITIAL CONDITIONS

Energetic initial conditions

– since energy is continuous, energetic circuit variables are also continuous

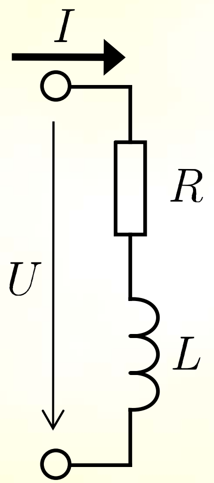
	$q = Cu$ $i = \frac{dq}{dt}$ $i_C(t) = C\frac{du_C(t)}{dt}$	must be $u_C(0_-) = u_C(0_+)$ 	may be $i_C(0_-) \neq i_C(0_+)$ 
	$\Phi_C = Li$ $u(t) = \frac{d\Phi_C}{dt}$	may be $u_L(0_-) \neq u_L(0_+)$ 	must be $i_L(0_-) = i_L(0_+)$ 

- History of capacitor describes charge stored in the capacitor ➡ voltage
- History of inductor describes magnetic flux passing the inductor ➡ electric current

BASIC RULES – SIGNS, …

- **Mesh analysis**
  - Current in the loop, where we currently write the equation has always positive sign
  - Other currents passing circuit element, where we currently evaluate voltage, have positive voltage, when they have same orientation, negative, when they have opposite orientation
  - Voltage sources have positive sign, when the current in the loop, where we currently write the equation, flows into positive source terminal, and negative, when it flows into negative terminal
  - We could not write an equation in loops, passing current sources, but we have to include current sources in closed loops
- **Nodal voltages**
  - Voltage in the node, where we write an equation, has always positive sign
  - Voltages in adjacent nodes, with the exception of voltage sources, have always negative sign (*we suppose, all currents leaves the node, actual orientation results from solution of system of equations; circuit element voltage, is the **difference** of electric potentials*)
  - Voltage sources in adjacent nodes have negative sign, when they are connected by positive terminal to the adjacent node (we subtract them), positive, if they are connected by negative terminal
  - Current of the current source has positive sign, if it leaves the node, negative, it runs into the node
  - When the voltage source is not connected by any terminal with reference node (ground), then we refer it as floating source – we write just one equation for both nodes (where the floating source is connected.

Reduction of number of equations

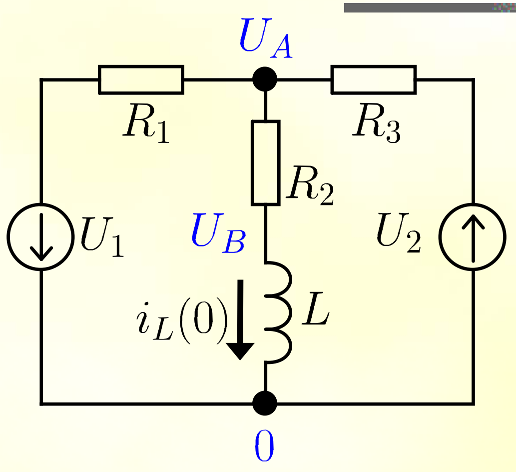


DC $$U = RI$$

AC $$\mathbf{U} = R\,\mathbf{I} + j\omega L\,\mathbf{I} = (R + j\omega L)\,\mathbf{I}$$

Time domain $$u(t) = Ri(t) + L\frac{\mathrm{d}i(t)}{\mathrm{d}t}$$ not possible

Laplace $$U(p) = R\,I(p) + pL\,I(p) - Li_L(0) = (R + pL)\,I(p) - Li_L(0)$$



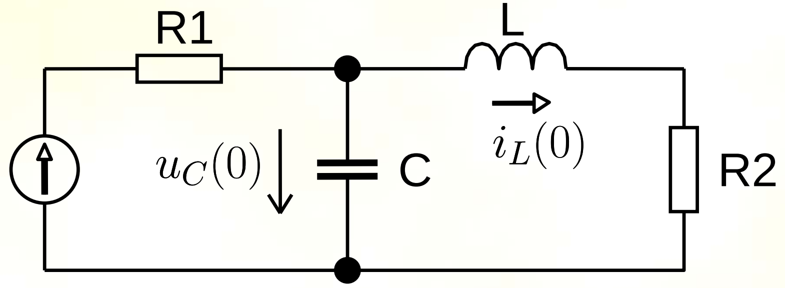
DC $$\frac{U_a - U_1}{R_1} + \frac{U_A}{R_2} + \frac{U_A + U_2}{R_3} = 0$$

AC $$\frac{\mathbf{U}_a - \mathbf{U}_1}{R_1} + \frac{\mathbf{U}_A}{R_2 + j\omega L} + \frac{\mathbf{U}_A + \mathbf{U}_2}{R_3} = 0$$

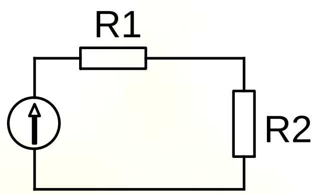
Time domain
$$\frac{u_a(t) - u_1(t)}{R_1} + \frac{u_A(t) - u_B(t)}{R_2} + \frac{u_A(t) + u_2(t)}{R_3} = 0$$
$$\frac{u_b(t) - u_a(t)}{R_2} + \frac{1}{L}\int_0^t u_B(\tau)\,\mathrm{d}\tau + i_L(0) = 0$$

Laplace $$\frac{U_a(p) - U_1(p)}{R_1} + \frac{U_A(p) + Li_L(0)}{R_2 + pL} + \frac{U_A(p) + U_2(p)}{R_3} = 0$$

Example:



DC:



Mesh analysis – 0 equations, voltages across R_1 and R_2 is evaluated by Ohm's law

AC:

$$j\omega L\,\hat{\mathbf{I}}_1 + R_2\hat{\mathbf{I}}_1 + \frac{1}{j\omega C}(\hat{\mathbf{I}}_1 - \hat{\mathbf{I}}) = 0 \qquad \Rightarrow \qquad \hat{\mathbf{I}}_1 = \frac{\hat{\mathbf{I}}}{(j\omega)^2 LC + j\omega R_2 C + 1}$$

Time domain:

$$L\frac{\mathrm{d}i_1(t)}{\mathrm{d}t} + R_2 i_1(t) + \frac{1}{C}\int_0^t \left[i_1(\tau) - i(\tau)\right]\,\mathrm{d}\tau - u_C(0) = 0$$

Solution is obtained in three steps:

1. Initial condition $u_C(0)$ – DC or AC analysis has to be proceeded according to the nature of exciting source before the change
2. Solution of integral-differential equation – solution of the transient
3. To find steady state in the circuit after transient is over we have to proceed DC or AC analysis again

Usage – **transients** – describing currents or voltages in circuit after

- Source was connected
- Source was disconnected
- Circuit layout has been changed
- Some circuit element parameters (resistivity, capacitance, inductance)

DC or AC analysis always has to be proceeded

Laplace: $pL\,I_1(p) - Li_L(0) + R_2 I_1(p) + \frac{1}{pC}\left[I_1(p) - I(p)\right] - \frac{u_C(0)}{p} = 0$

$$\Rightarrow \qquad I_1(p) = \frac{I(p) + Cu_C(0) + pLCi_L(0)}{p^2LC + pR_2C + 1}$$

The solution is obtained in two steps:

1. Initial condition $u_C(0)$ DC or AC analysis has to be proceeded according to the nature of exciting source before the change *(same as time domain)*
2. Solve inverse transform $I_1(p)$ – it contains both transient and steady state after transient dies away

- To find initial conditions, we have to proceed DC or AC analysis as well
- Inverse Laplace transform always corresponds to transient when source $I(p)$ is connected to the circuit (if it is present in the equation), or source is disconnected (then the steady state is „hidden" in the initial conditions)
  - The transient is integral part of Laplace transform solution, the result of time interval restriction - each source is zero when t < 0, because $u(t) = u'(t)1(t)$, $i(t) = i'(t)1(t)$
- The solution of operational circuit equations contains also steady state after transient dies away
- Although operational solution is formally similar to the AC analysis, it is different – „more complete" solution

TRANSIENTS

Order of the transients / circuits

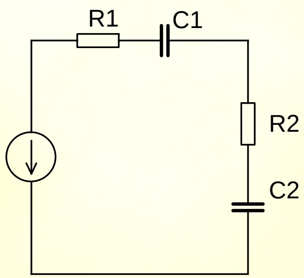
- Each circuit may be described by set of an integral-differential equations
- By elimination of circuit variables *(by successive differentiation of primary circuit equations and by substitution of circuit variables and its derivatives into other equations)* we obtain n^th order differential equation of selected circuit variable

$$a_n \frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + \cdots + a_1 \frac{\mathrm{d}y}{\mathrm{d}t} + a_0 y = x(t)$$

where

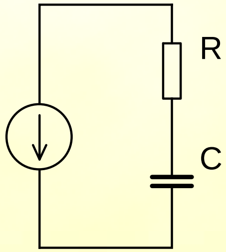
- constants $a_0, a_1, \ldots a_n$ are combinations of *R, L, C, M* parameters of passive circuit elements and *K, R, G, H* parameters of controlled sources
- $x(t)$ is linear combination of independent voltage sources and its derivatives

The order of resulting differential equation (transient) is at most equivalent to the number of energy storing circuit elements (L, C) –number of incompatible inductors and capacitors



Both resistors and capacitors
may be joined
each in one circuit element

$\Rightarrow$



- … or we got in (Laplace) frequency domain an expression, containing the polynomial

$$a_n p^n + a_{n-1} p^{n-1} + \cdots + a_1 p + a_0$$

GENERAL PROCEDURE IN TIME DOMAIN

1. Find energetic initial conditions (steady state before the change)
2. Elimination of circuit variables
3. Find general solution $y_0(t)$ of homogeneous differential equation

$$a_n \frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_{n-1} \frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + \cdots + a_1 \frac{\mathrm{d}y}{\mathrm{d}t} + a_0 y = 0$$

preferably by solution of characteristic equation … or roots of polynomial in Laplace

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0 \qquad a_n p^n + a_{n-1} p^{n-1} + \cdots + a_1 p + a_0 = 0$$

the form of general solution is different depending on kind of roots of characteristic equation:

roots	
Distinct real	$y_0(t) = \sum_{k=1}^{n} A_k e^{\lambda_k t}$
Repeated real (multiplicity m)	$y_{0m}(t) = \left(A_1 + A_2 t + A_3 t^2 + \cdots + A_m t^{m-1}\right) e^{\lambda t}$
Complex conjugated	$\lambda_{1,2} = -\alpha \pm j\omega$ $K_1 e^{\lambda_1 t} + K_2 e^{\lambda_2 t} = e^{-\alpha t} \left(A \sin \omega t + B \cos \omega t\right) =$ $= D \sin\left(\omega t + \psi\right)$

Form does not depend on the kind of exciting sources, just on circuit elements

$\lambda < 0$ ⇨ (asymptotically) stable circuit (when passive, then always)

4. Complete solution of transient has also particular solution $y_p(t)$

$$y(t) = y_o(t) + y_p(t)$$

the form of particular solution depends on the kind of exciting sources (it is steady state after the change)

source	$x(t)$	$y_p(t)$
-	0	0
DC	X_0	Y_0
AC	$X_m \sin(\omega t + \varphi)$	$Y_m \sin(\omega t + \psi)$
periodical	$X_0 + \sum_{k=1}^{\infty} X_{mk} \sin(k\omega t + \varphi_k)$	$Y_0 + \sum_{k=1}^{\infty} Y_{mk} \sin(k\omega t + \psi_k)$