$$\forall m, n \in \mathbb{N}, 0 \le x_{m+n} \le x_m + x_n, \quad \text{证明}\{\frac{x_n}{n}\}\text{收敛}$$

两边取 ln，仅需证明$\dfrac{\ln 1+\ln 2+\cdots+\ln n}{n}-\ln n=-1$

$$\frac{\ln 1+\ln 2+\cdots+\ln n}{n}-\ln n=\frac{\sum\limits_{i=1}^{n}\ln i-n\ln n}{n}$$

利用 stolz 定律，仅需证明

$\displaystyle\sum_{i=1}^{n+1}\ln i-(n+1)\ln(n+1)-(\sum_{i=1}^{n}\ln i-n\ln n)=-1$即可

$$\begin{aligned}&\sum_{i=1}^{n+1}\ln i-(n+1)\ln n+1-(\sum_{i=1}^{n}\ln i-n\ln n)\\&=\ln(n+1)-(n+1)\ln(n+1)+n\ln(n)\\&=n\ln(n)-n\ln(n+1)\\&=\ln\frac{1}{(1+\frac{1}{n})^n}\end{aligned}$$

$$e=\lim_{n\to\infty}(1+\frac{1}{n})^n$$

得证