Macroscopic Heat Transfer Model for Additive Manufacturing Testbed Problem

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1 Macroscopic model

We model the macroscopic heat transfer in the rectangular domain Ω

$$\frac{\partial T}{\partial t} = \nabla \cdot (K \nabla T) + L_m \frac{\partial f_l(T)}{\partial t} \quad \text{in } \Omega \tag{1}$$

Qin: 1. From one paper you gave me, I think the equation is following, could you check? I am not sure about the sign of f_l

$$\frac{\partial \rho c_p T}{\partial t} = \nabla \cdot (K \nabla T) - \frac{\partial \rho L f_l(T)}{\partial t} \quad \text{in } \Omega \tag{2}$$

$$\frac{\partial T}{\partial t} = \alpha \Delta T - \frac{L}{c_p} \frac{\partial f_l(T)}{\partial t} \quad \text{in } \Omega \tag{3}$$

where α is thermal diffusivity

with boundary conditions

$$(-K \nabla T) \cdot \hat{n} = q_s + h(T - T_e) + \epsilon \sigma (T^4 - T_e^4) \quad \text{on } \Gamma_{top} \tag{4}$$

Qin : 2.negative sign of qs$(-K \nabla T) \cdot \hat{n} = -q_s + h(T - T_e) + \epsilon \sigma (T^4 - T_e^4) \quad \text{on } \Gamma_{top}$ (5)

$$(-K \nabla T) \cdot \hat{n} = 0 \quad \text{on } \Gamma \backslash \Gamma_{top} \tag{6}$$

where K is the heat conductivity, h is the convective heat transfer coefficient, ϵ is the thermal radiation coefficient, σ is the Stefan-Boltzmann constant, and L is the latent heat. The fluid mass fraction f_l is modeled as

$$f_l(T) = \begin{cases} 1 & T > T_l \\ \frac{T - T_s}{T_l - T_s} & T_s \le T \le T_l \\ 0 & T < T_s \end{cases} \tag{7}$$

where T_l and T_s are the liquidus and solidus temperature, respectively. The heat source q_S is modeled as a moving Gaussian

$$q_s(x,t) = \frac{2Q\eta}{\pi r_b^2} \exp\left(-\frac{2(x - V_s t)^2}{r_b^2}\right), \tag{8}$$

where Q is the source of heat power, η is the absorption coefficient, r_b is the radius of heat source and V_s is the scanning speed.

2 Discretization

Qin: 3. Do we need higher order discretization of time?

We discretize the domain Ω using $(N+1)\times(M+1)$ grid with meshwidth h. Lx = Nh, x = ih. Let $T^n_{ij} = T(ih, jh, t_n)$ for $i = 0, \ldots, N$ and $j = 0, \ldots, M$.

$$\frac{T^n_{i,j} - T^{n-1}_{i,j}}{\Delta t} = \alpha \frac{T^n_{i,j+1} + T^n_{i,j-1} + T^n_{i-1,j} + T^n_{i+1,j} - 4T^n_{i,j}}{h^2} - L_m \left.\frac{\partial f_l(T)}{\partial t}\right|_{t=t_n} \tag{9}$$

At the top boundary ($j = 0$), we treat the convection and radiation term explicitly,

$$-K\frac{T^n_{i,-1} - T^n_{i,1}}{2h} = -q_s(ih, t_n) + h(T^{n-1}_{i,0} - T_e) + \epsilon\sigma((T^{n-1}_{i,0})^4 - T_e^4),\ i = 0, \ldots, N \tag{10}$$

$$T^n_{i,-1} = T^n_{i,1} - \frac{2h}{K}\left(-q_s(ih, t_n) + h(T^{n-1}_{i,0} - T_e) + \epsilon\sigma((T^{n-1}_{i,0})^4 - T_e^4)\right) = T^n_{i,1} + U^{n-1}_i,\ i = 0, \ldots, N \tag{11}$$

Bottom boundaries:

$$T^n_{i,M+1} = T^n_{i,M-1},\ i = 0, \ldots, N \tag{12}$$

$$T^n_{-1,j} = T^n_{1,j},\ j = 0, \ldots, M \tag{13}$$

$$T^n_{N-1,j} = T^n_{N+1,j},\ j = 0, \ldots, M \tag{14}$$

Eventually, the linear system of equation should look like

$$(\mathbf{I} - C\mathbf{L})\mathbf{T}^n - \mathbf{N}(\mathbf{T}^n) = \mathbf{T}^{n-1} + \text{BC terms} \tag{15}$$

where C is CFL number, $\mathbf{L}$ is the discrete 2D laplacian with Neumann BCs and $\mathbf{N}$ is a nonlinear function due the implicit treatment of latent heat term. If we treat the latent heat term explicitly, we need two starting values initially.

Bao: 3. Can you find out what L, N and BC terms are?

Bao: 4. I am afraid we have to solve nonlinear system of equation due to the latent heat term.

3 Numerical Tests

3.1 No latent heat term

$$\frac{T^n_{i,j} - T^{n-1}_{i,j}}{\Delta t} = \alpha \frac{T^n_{i,j+1} + T^n_{i,j-1} + T^n_{i-1,j} + T^n_{i+1,j} - 4T^n_{i,j}}{h^2} \tag{16}$$

for top boundary (j=0)

$$\frac{T^n_{i,0} - T^{n-1}_{i,0}}{\Delta t} = \alpha \frac{T^n_{i,1} + T^n_{i,-1} + T^n_{i-1,0} + T^n_{i+1,0} - 4T^n_{i,0}}{h^2} = \alpha \frac{2T^n_{i,1} + U^{n-1}_i + T^n_{i-1,0} + T^n_{i+1,0} - 4T^n_{i,0}}{h^2} \tag{17}$$

$$\text{BC terms} = CU^{n-1}_i,\quad j = 0,\ i = 0, \ldots, N \tag{18}$$

$$U^{n-1}_i = -\frac{2h}{K}\left(-q_s(ih, t_n) + h(T^{n-1}_{i,0} - T_e) + \epsilon\sigma((T^{n-1}_{i,0})^4 - T_e^4)\right),\ i = 0, \ldots, N \tag{19}$$

3.2 Explicit latent heat term

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