

# Macroscopic Heat Transfer Model for Additive Manufacturing Testbed Problem

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## 1 Macroscopic model

We model the macroscopic heat transfer in the rectangular domain  $\Omega$

$$\frac{\partial \rho c_p T}{\partial t} = \nabla \cdot (K \nabla T) - \frac{\partial \rho L f_l(T)}{\partial t} \quad \text{in } \Omega \quad (1)$$

$$\frac{\partial T}{\partial t} = \alpha \Delta T - \frac{L}{c_p} \frac{\partial f_l(T)}{\partial t} \quad \text{in } \Omega \quad (2)$$

with boundary conditions

$$(-K \nabla T) \cdot \hat{n} = -q_s + h(T - T_e) + \epsilon \sigma (T^4 - T_e^4) \quad \text{on } \Gamma_{top} \quad (3)$$

$$(-K \nabla T) \cdot \hat{n} = 0 \quad \text{on } \Gamma \setminus \Gamma_{top} \quad (4)$$

where  $K$  is the heat conductivity,  $\alpha$  is thermal diffusivity,  $h$  is the convective heat transfer coefficient,  $\epsilon$  is the thermal radiation coefficient,  $\sigma$  is the Stefan-Boltzmann constant, and  $L$  is the latent heat. The fluid mass fraction  $f_l$  is modeled as

$$f_l(T) = \begin{cases} 1 & T > T_l \\ \frac{T - T_s}{T_l - T_s} & T_s \leq T \leq T_l \\ 0 & T < T_s \end{cases} \quad (5)$$

where  $T_l$  and  $T_s$  are the liquidus and solidus temperature, respectively. The heat source  $q_s$  is modeled as a moving Gaussian

$$q_s(x, t) = \frac{2Q\eta}{\pi r_b^2} \exp\left(-\frac{2(x - V_s t)^2}{r_b^2}\right), \quad (6)$$

where  $Q$  is the source of heat power,  $\eta$  is the absorption coefficient,  $r_b$  is the radius of heat source and  $V_s$  is the scanning speed.

## 2 Discretization

We discretize the domain  $\Omega$  using  $(N + 1) \times (M + 1)$  grid with meshwidth  $h$ .  $Lx = Nh$ ,  $x = ih$ . Let  $T_{ij}^n = T(ih, jh, t_n)$  for  $i = 0, \dots, N$  and  $j = 0, \dots, M$ .

$$\frac{T_{i,j}^n - T_{i,j}^{n-1}}{\Delta t} = \alpha \frac{T_{i,j+1}^n + T_{i,j-1}^n + T_{i-1,j}^n + T_{i+1,j}^n - 4T_{i,j}^n}{h^2} - L_m \frac{\partial f_l(T)}{\partial t} \Big|_{t=t_n} \quad (7)$$

At the top boundary ( $j = 0$ ), we treat the convection and radiation term explicitly,

$$-K \frac{T_{i,-1}^n - T_{i,1}^n}{2h} = -q_s(ih, t_n) + h(T_{i,0}^{n-1} - T_e) + \epsilon \sigma ((T_{i,0}^{n-1})^4 - T_e^4), \quad i = 0, \dots, N \quad (8)$$

$$T_{i,-1}^n = T_{i,1}^n - \frac{2h}{K} (-q_s(ih, t_n) + h(T_{i,0}^{n-1} - T_e) + \epsilon \sigma ((T_{i,0}^{n-1})^4 - T_e^4)) = T_{i,1}^n + U_i^{n-1}, \quad i = 0, \dots, N \quad (9)$$

Other boundaries:

$$T_{i,M+1}^n = T_{i,M-1}^n, \quad i = 0, \dots, N \quad (10)$$

$$T_{-1,j}^n = T_{1,j}^n, \quad j = 0, \dots, M \quad (11)$$

$$T_{N-1,j}^n = T_{N+1,j}^n, \quad j = 0, \dots, M \quad (12)$$

Eventually, the linear system of equation should look like

$$(\mathbf{I} - C\mathbf{L})\mathbf{T}^n + \mathbf{N}(\mathbf{T}^n) = \mathbf{T}^{n-1} + \mathbf{N}(\mathbf{T}^{n-1}) + \text{BC terms} \quad (13)$$

where  $C$  is CFL number,  $\mathbf{L}$  is the discrete 2D laplacian with Neumann BCs and  $\mathbf{N}$  is a nonlinear function due the implicit treatment of latent heat term. If we treat the latent heat term explicitly, we need two starting values initially.

**Bao: 3. Can you find out what  $\mathbf{L}$ ,  $\mathbf{N}$  and BC terms are?**

**Bao: 4. I am afraid we have to solve nonlinear system of equation due to the latent heat term.**

## 3 Numerical Tests

### 3.1 No latent heat term

$$\frac{T_{i,j}^n - T_{i,j}^{n-1}}{\Delta t} = \alpha \frac{T_{i,j+1}^n + T_{i,j-1}^n + T_{i-1,j}^n + T_{i+1,j}^n - 4T_{i,j}^n}{h^2} \quad (14)$$

for top boundary ( $j=0$ )

$$\frac{T_{i,0}^n - T_{i,0}^{n-1}}{\Delta t} = \alpha \frac{T_{i,1}^n + T_{i,-1}^n + T_{i-1,0}^n + T_{i+1,0}^n - 4T_{i,0}^n}{h^2} = \alpha \frac{2T_{i,1}^n + U_i^{n-1} + T_{i-1,0}^n + T_{i+1,0}^n - 4T_{i,0}^n}{h^2} \quad (15)$$

$$\text{BC terms} = CU_i^{n-1}, \quad j = 0, \quad i = 0, \dots, N \quad (16)$$

$$U_i^{n-1} = -\frac{2h}{K} (-q_s(ih, t_n) + h_c(T_{i,0}^{n-1} - T_e) + \epsilon \sigma ((T_{i,0}^{n-1})^4 - T_e^4)), \quad i = 0, \dots, N \quad (17)$$

$$q_s(ih, t_n) = q_0 \exp \left( -\frac{2(ih - V_s t_n)^2}{r_b^2} \right) = q_0 \exp \left( -\frac{2(i - V_s' t_n)^2}{r_b'^2} \right), \quad i = 0, \dots, N \quad (18)$$

### 3.1.1 Test 1

Parameters:  $K = 0.01$ ,  $\rho = 1.0$ ,  $C_p = 1.0$ ,  $Q = 3$ ,  $\eta = 1$ ,  $r_b = 0.2$ ,  $V_s = 0.075$ .  
Up boundary cooling parameters:  $h_c = 0.005$ ,  $\epsilon = 0.005$ ,  $\sigma = 5.67\text{E-}8$ ,  $T_e = 0$ .  
Computation domain:  $L_x = 8$ ,  $L_y = 2$ .

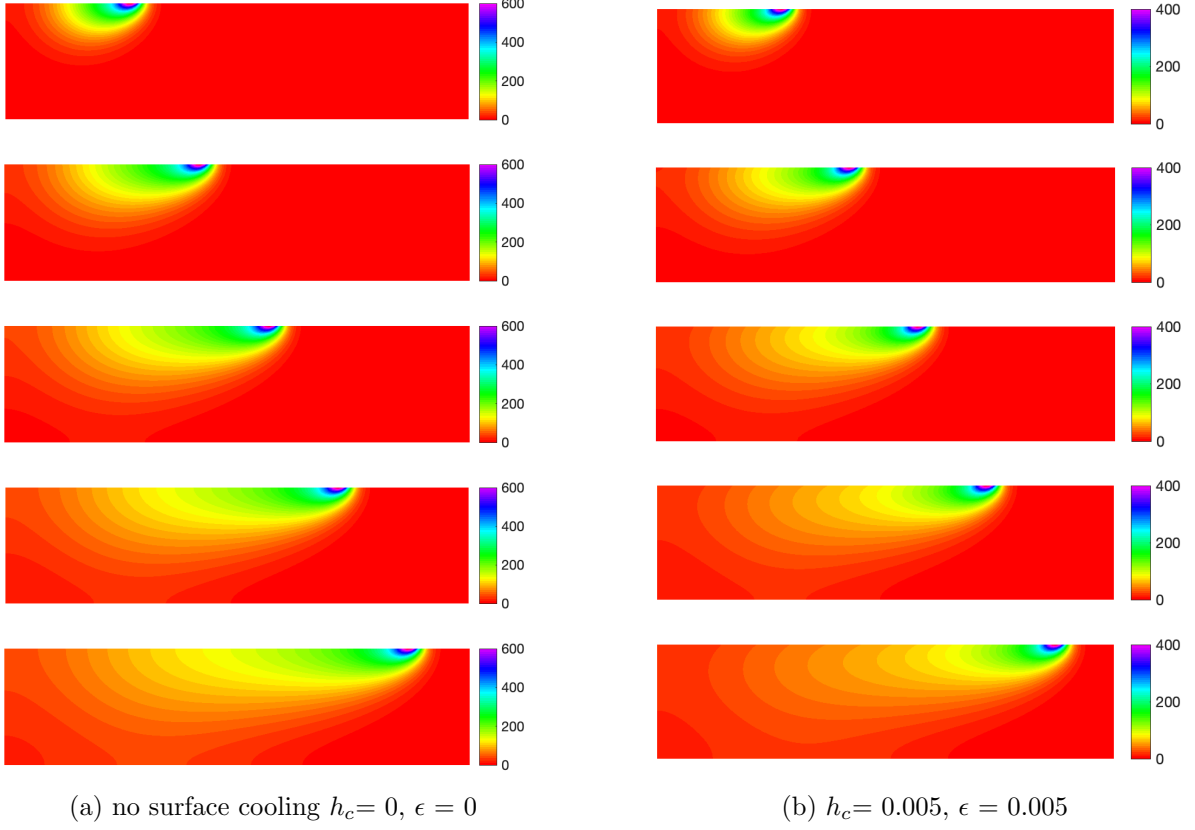


Fig. 1: Temperature distribution without latent heat.

### 3.1.2 Self Convergence Study (with convection and radiation)

Ground truth:  $dt = 0.00625$ ,  $h = L_x/2048$

Trial:  $dt = 0.025, 0.1, 0.4, 1.6$ ;  $h = L_x/1024, L_x/512, L_x/256, L_x/128$

## 3.2 Latent heat term implicit-explicit

Implicit:

$$\frac{\partial f_l(T)}{\partial t} = \frac{f_l^n(T) - f_l^{n-1}(T)}{\Delta t} \quad (19)$$

Explicit:

$$f_l^n(T) = \begin{cases} 1 & T^{n-1} > T_l \\ \frac{T^n - T_s}{T_l - T_s} & T_s \leq T^{n-1} \leq T_l \\ 0 & T^{n-1} < T_s \end{cases} \quad (20)$$

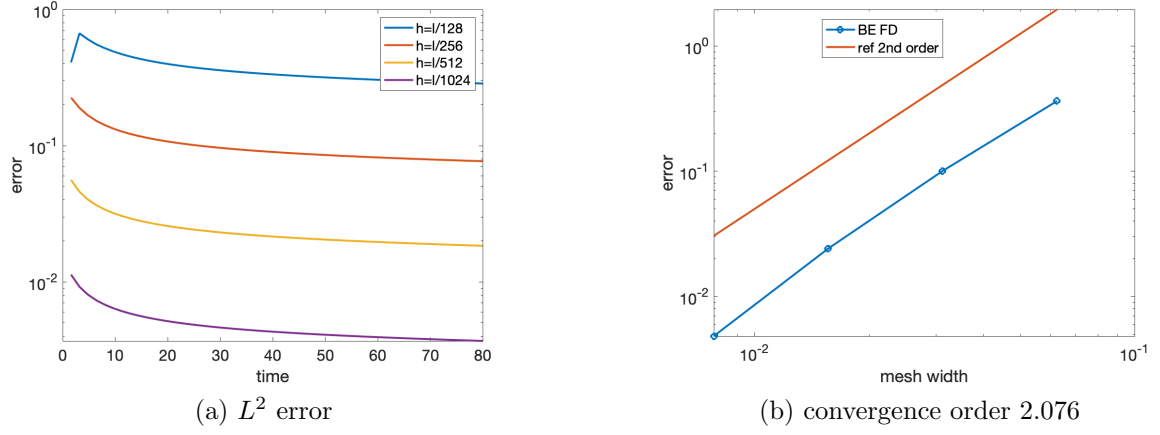


Fig. 2: Self convergence.

$$f_l^{n-1}(T) = \begin{cases} 1 & T^{n-1} > T_l \\ \frac{T^{n-1}-T_s}{T_l-T_s} & T_s \leq T^{n-1} \leq T_l \\ 0 & T^{n-1} < T_s \end{cases} \quad (21)$$

$$\text{Latent heat term} = \begin{cases} 0 & T^{n-1} > T_l \\ -\frac{L_m}{\Delta t} \frac{T^n - T^{n-1}}{T_l - T_s} & T_s \leq T^{n-1} \leq T_l \\ 0 & T^{n-1} < T_s \end{cases} \quad (22)$$

### 3.2.1 Test2

Parameters:  $L=200$ ,  $T_s=40$ ,  $T_l=110$

GMRES solver (Tol =  $1e-9$ ):

For mesh width  $h = L_x/1024, L_x/512, L_x/256, L_x/128$

Iterations per time step without preconditioner: 58, 54, 48, 49, 50

Iterations per time step with preconditioner: 20, 17, 16,... but took longer time to run

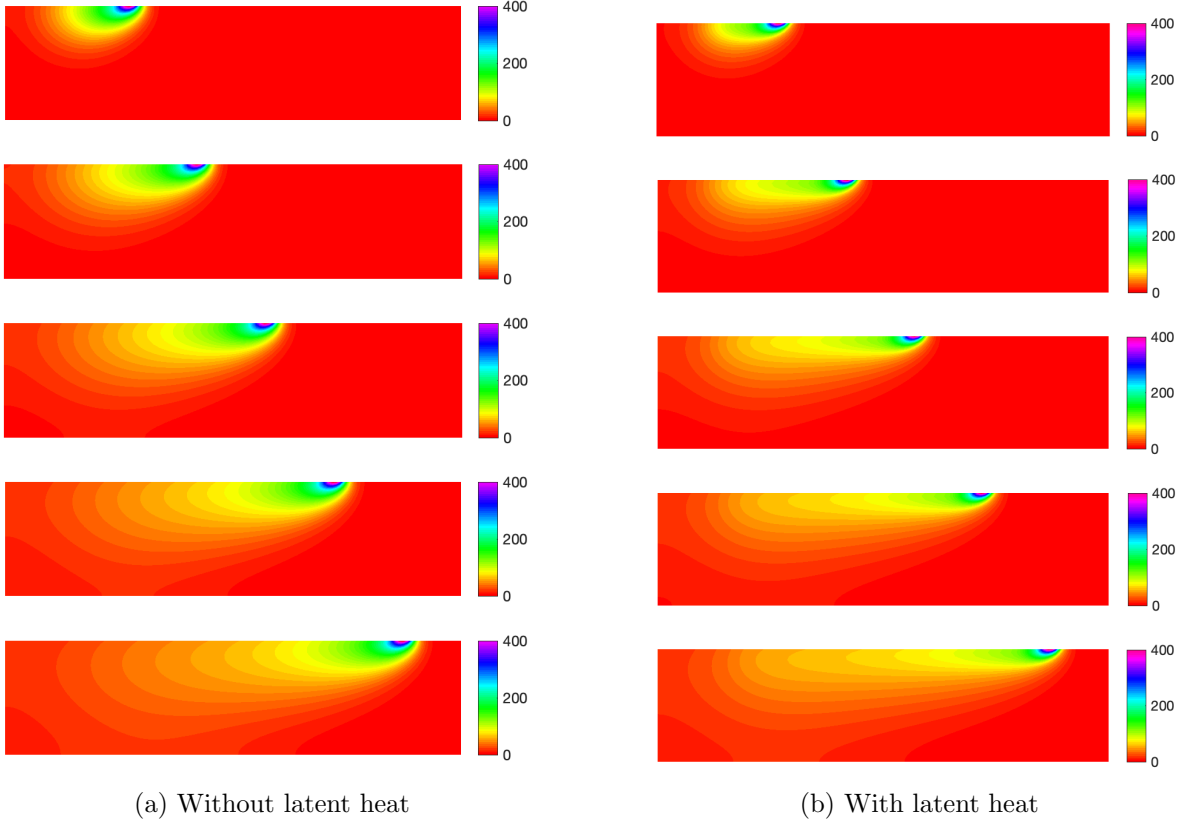


Fig. 3: Temperature distribution with and without latent heat. Surface cooling is included.

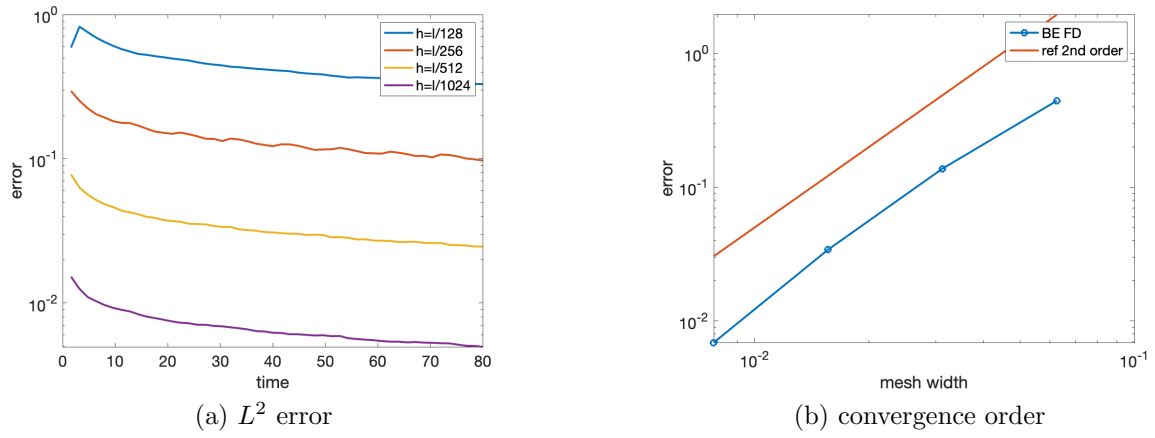


Fig. 4: Self convergence.