Macroscopic Heat Transfer Model for Additive Manufacturing Testbed Problem

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1 Macroscopic model

We model the macroscopic heat transfer in the rectangular domain Ω

$$\frac{\partial \rho c_p T}{\partial t} = \nabla \cdot (K \nabla T) - \frac{\partial \rho L f_l(T)}{\partial t} \quad \text{in } \Omega \tag{1}$$

$$\frac{\partial T}{\partial t} = \alpha \Delta T - \frac{L}{c_p}\frac{\partial f_l(T)}{\partial t} \quad \text{in } \Omega \tag{2}$$

with boundary conditions

$$(-K\nabla T) \cdot \hat{n} = -q_s + h(T - T_e) + \epsilon\sigma(T^4 - T_e^4) \quad \text{on } \Gamma_{top} \tag{3}$$

$$(-K\nabla T) \cdot \hat{n} = 0 \quad \text{on } \Gamma \backslash \Gamma_{top} \tag{4}$$

where K is the heat conductivity, α is thermal diffusivity, h is the convective heat transfer coefficient, ϵ is the thermal radiation coefficient, σ is the Stefan-Boltzmann constant, and L is the latent heat. The fluid mass fraction f_l is modeled as

$$f_l(T) = \begin{cases} 1 & T > T_l \\ \frac{T-T_s}{T_l-T_s} & T_s \leq T \leq T_l \\ 0 & T < T_s \end{cases} \tag{5}$$

where T_l and T_s are the liquidus and solidus temperature, respectively. The heat source q_S is modeled as a moving Gaussian

$$q_s(x,t) = \frac{2Q\eta}{\pi r_b^2} \exp\left(-\frac{2(x - V_s t)^2}{r_b^2}\right), \tag{6}$$

where Q is the source of heat power, η is the absorption coefficient, r_b is the radius of heat source and V_s is the scanning speed.

2 Discretization

We discretize the domain Ω using $(N+1)\times(M+1)$ grid with meshwidth h. Lx = Nh, x = ih. Let $T_{ij}^n = T(ih, jh, t_n)$ for $i = 0,\ldots,N$ and $j = 0,\ldots,M$.

$$\frac{T_{i,j}^n - T_{i,j}^{n-1}}{\Delta t} = \alpha\frac{T_{i,j+1}^n + T_{i,j-1}^n + T_{i-1,j}^n + T_{i+1,j}^n - 4T_{i,j}^n}{h^2} - L_m \frac{\partial f_l(T)}{\partial t}\Big|_{t=t_n} \tag{7}$$

At the top boundary $(j=0)$, we treat the convection and radiation term explicitly,

$$-K\frac{T_{i,-1}^n - T_{i,1}^n}{2h} = -q_s(ih,t_n) + h(T_{i,0}^{n-1} - T_e) + \epsilon\sigma((T_{i,0}^{n-1})^4 - T_e^4),\ i = 0,\ldots,N \tag{8}$$

$$T_{i,-1}^n = T_{i,1}^n - \frac{2h}{K}\left(-q_s(ih,t_n) + h(T_{i,0}^{n-1} - T_e) + \epsilon\sigma((T_{i,0}^{n-1})^4 - T_e^4)\right) = T_{i,1}^n + U_i^{n-1},\ i = 0,\ldots,N \tag{9}$$

Other boundaries:

$$T_{i,M+1}^n = T_{i,M-1}^n,\ i = 0,\ldots,N \tag{10}$$
$$T_{-1,j}^n = T_{1,j}^n,\ j = 0,\ldots,M \tag{11}$$
$$T_{N-1,j}^n = T_{N+1,j}^n,\ j = 0,\ldots,M \tag{12}$$

Eventually, the linear system of equation should look like

$$(\mathbf{I} - C\mathbf{L})\mathbf{T}^n + \mathbf{N}(\mathbf{T}^n) = \mathbf{T}^{n-1} + \mathbf{N}(\mathbf{T}^{n-1}) + \text{BC terms} \tag{13}$$

where C is CFL number, $\mathbf{L}$ is the discrete 2D laplacian with Neumann BCs and $\mathbf{N}$ is a nonlinear function due the implicit treatment of latent heat term. If we treat the latent heat term explicitly, we need two starting values initially.

Bao: 3. Can you find out what L, N and BC terms are?

Bao: 4. I am afraid we have to solve nonlinear system of equation due to the latent heat term.

3 Numerical Tests

3.1 No latent heat term

$$\frac{T_{i,j}^n - T_{i,j}^{n-1}}{\Delta t} = \alpha\frac{T_{i,j+1}^n + T_{i,j-1}^n + T_{i-1,j}^n + T_{i+1,j}^n - 4T_{i,j}^n}{h^2} \tag{14}$$

for top boundary (j=0)

$$\frac{T_{i,0}^n - T_{i,0}^{n-1}}{\Delta t} = \alpha\frac{T_{i,1}^n + T_{i,-1}^n + T_{i-1,0}^n + T_{i+1,0}^n - 4T_{i,0}^n}{h^2} = \alpha\frac{2T_{i,1}^n + U_i^{n-1} + T_{i-1,0}^n + T_{i+1,0}^n - 4T_{i,0}^n}{h^2} \tag{15}$$

$$\text{BC terms} = CU_i^{n-1},\quad j = 0,\ i = 0,\ldots,N \tag{16}$$

$$U_i^{n-1} = -\frac{2h}{K}\left(-q_s(ih,t_n) + h_c(T_{i,0}^{n-1} - T_e) + \epsilon\sigma((T_{i,0}^{n-1})^4 - T_e^4)\right),\ i = 0,\ldots,N \tag{17}$$

$$q_s(ih, t_n) = q_0 \exp\left(-\frac{2(ih - V_s t_n)^2}{r_b^2}\right) = q_0 \exp\left(-\frac{2(i - V_s{'} t_n)^2}{{r_b{'}}^2}\right), \; i = 0, \ldots, N \tag{18}$$

Test1:
Parameters: K = 0.007, $\rho = 1.0$, $C_p = 1.0$, Q = 5, $\eta = 1$, $r_b = 0.15$, $V_s = 0.1$. (Not adding convection so far)
Mesh: h = 0.02, grids (501×101), dt = 0.1

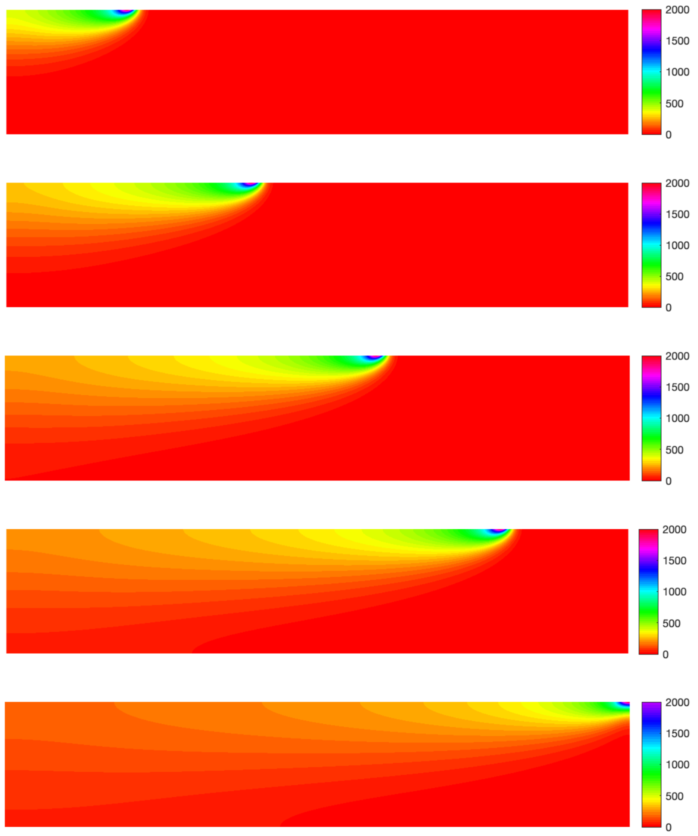

Fig. 1: Temperature distribution. From top to bottom, time is 20, 40, 60, 80, 100

3.2 Latent heat term implicit-explicit

Implicit:

$$\frac{\partial f_l(T)}{\partial t} = \frac{f_l^n(T) - f_l^{n-1}(T)}{\Delta t} \tag{19}$$

Explicit:

$$f_l^n(T) = \begin{cases} 1 & T^{n-1} > T_l \\ \frac{T^n - T_s}{T_l - T_s} & T_s \leq T^{n-1} \leq T_l \\ 0 & T^{n-1} < T_s \end{cases} \tag{20}$$

$$f_l^{n-1}(T) = \begin{cases} 1 & T^{n-1} > T_l \\ \frac{T^{n-1} - T_s}{T_l - T_s} & T_s \leq T^{n-1} \leq T_l \\ 0 & T^{n-1} < T_s \end{cases} \tag{21}$$

$$\text{Latent heat term} = \begin{cases} 0 & T^{n-1} > T_l \\ -\frac{L_m}{\Delta t} \frac{T^n - T^{n-1}}{T_l - T_s} & T_s \leq T^{n-1} \leq T_l \\ 0 & T^{n-1} < T_s \end{cases} \tag{22}$$

Test2:
Parameters: L=0.001, T_s=200, T_l=400

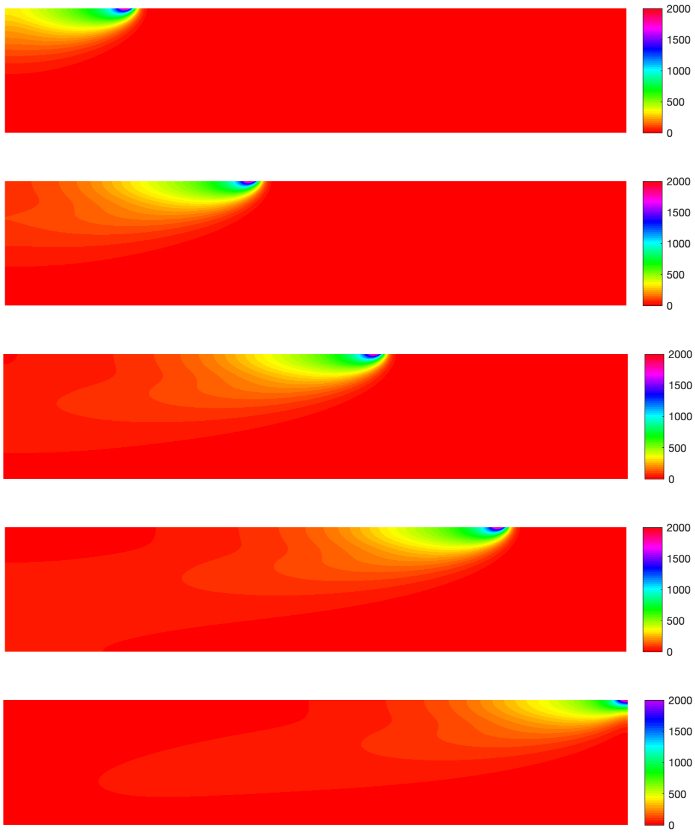

Fig. 2: Temperature distribution with latent heat. From top to bottom, time is 20, 40, 60, 80, 100

3.3 Latent heat term fully implicit

$$f_l^n(T) = \begin{cases} 1 & T^n > T_l \\ \frac{T^n - T_s}{T_l - T_s} & T_s \leq T^n \leq T_l \\ 0 & T^n < T_s \end{cases} \tag{23}$$