Macroscopic Heat Transfer Model for Additive Manufacturing Testbed Problem

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1 Macroscopic model

We model the macroscopic heat transfer in the rectangular domain Ω

$$\frac{\partial \rho c_p T}{\partial t} = \nabla \cdot (K \nabla T) - \frac{\partial \rho L f_l(T)}{\partial t} \quad \text{in } \Omega \tag{1}$$

$$\frac{\partial T}{\partial t} = \alpha \Delta T - \frac{L}{c_p} \frac{\partial f_l(T)}{\partial t} \quad \text{in } \Omega \tag{2}$$

with boundary conditions

$$(-K\nabla T) \cdot \hat{n} = -q_s + h(T - T_e) + \epsilon\sigma(T^4 - T_e^4) \quad \text{on } \Gamma_{top} \tag{3}$$

$$(-K\nabla T) \cdot \hat{n} = 0 \quad \text{on } \Gamma \backslash \Gamma_{top} \tag{4}$$

where K is the heat conductivity, α is thermal diffusivity, h is the convective heat transfer coefficient, ϵ is the thermal radiation coefficient, σ is the Stefan-Boltzmann constant, and L is the latent heat. The fluid mass fraction f_l is modeled as

$$f_l(T) = \begin{cases} 1 & T > T_l \\ \frac{T - T_s}{T_l - T_s} & T_s \le T \le T_l \\ 0 & T < T_s \end{cases} \tag{5}$$

where T_l and T_s are the liquidus and solidus temperature, respectively. The heat source q_S is modeled as a moving Gaussian

$$q_s(x, t) = \frac{2Q\eta}{\pi r_b^2} \exp\left(-\frac{2(x - V_s t)^2}{r_b^2}\right), \tag{6}$$

where Q is the source of heat power, η is the absorption coefficient, r_b is the radius of heat source and V_s is the scanning speed.

2 Discretization

We discretize the domain Ω using $(N+1)\times(M+1)$ grid with meshwidth h. Lx = Nh, x = ih. Let $T^n_{ij} = T(ih, jh, t_n)$ for $i = 0, \dots, N$ and $j = 0, \dots, M$.

$$\frac{T^n_{i,j} - T^{n-1}_{i,j}}{\Delta t} = \alpha \frac{T^n_{i,j+1} + T^n_{i,j-1} + T^n_{i-1,j} + T^n_{i+1,j} - 4T^n_{i,j}}{h^2} - L_m \frac{\partial f_l(T)}{\partial t}\Big|_{t=t_n} \tag{7}$$

At the top boundary ($j = 0$), we treat the convection and radiation term explicitly,

$$-K\frac{T^n_{i,-1} - T^n_{i,1}}{2h} = -q_s(ih, t_n) + h(T^{n-1}_{i,0} - T_e) + \epsilon\sigma((T^{n-1}_{i,0})^4 - T_e^4),\ i = 0, \dots, N \tag{8}$$

$$T^n_{i,-1} = T^n_{i,1} - \frac{2h}{K}\left(-q_s(ih, t_n) + h(T^{n-1}_{i,0} - T_e) + \epsilon\sigma((T^{n-1}_{i,0})^4 - T_e^4)\right) = T^n_{i,1} + U^{n-1}_i,\ i = 0, \dots, N \tag{9}$$

Other boundaries:

$$T^n_{i,M+1} = T^n_{i,M-1},\ i = 0, \dots, N \tag{10}$$

$$T^n_{-1,j} = T^n_{1,j},\ j = 0, \dots, M \tag{11}$$

$$T^n_{N-1,j} = T^n_{N+1,j},\ j = 0, \dots, M \tag{12}$$

Eventually, the linear system of equation should look like

$$(\mathbf{I} - C\mathbf{L})\mathbf{T}^n + \mathbf{N}(\mathbf{T}^n) = \mathbf{T}^{n-1} + \mathbf{N}(\mathbf{T}^{n-1}) + \text{BC terms} \tag{13}$$

where C is CFL number, $\mathbf{L}$ is the discrete 2D laplacian with Neumann BCs and $\mathbf{N}$ is a nonlinear function due the implicit treatment of latent heat term. If we treat the latent heat term explicitly, we need two starting values initially.

Bao: 3. Can you find out what L, N and BC terms are?

Bao: 4. I am afraid we have to solve nonlinear system of equation due to the latent heat term.

3 Numerical Tests

3.1 No latent heat term

$$\frac{T^n_{i,j} - T^{n-1}_{i,j}}{\Delta t} = \alpha \frac{T^n_{i,j+1} + T^n_{i,j-1} + T^n_{i-1,j} + T^n_{i+1,j} - 4T^n_{i,j}}{h^2} \tag{14}$$

for top boundary (j=0)

$$\frac{T^n_{i,0} - T^{n-1}_{i,0}}{\Delta t} = \alpha \frac{T^n_{i,1} + T^n_{i,-1} + T^n_{i-1,0} + T^n_{i+1,0} - 4T^n_{i,0}}{h^2} = \alpha \frac{2T^n_{i,1} + U^{n-1}_i + T^n_{i-1,0} + T^n_{i+1,0} - 4T^n_{i,0}}{h^2} \tag{15}$$

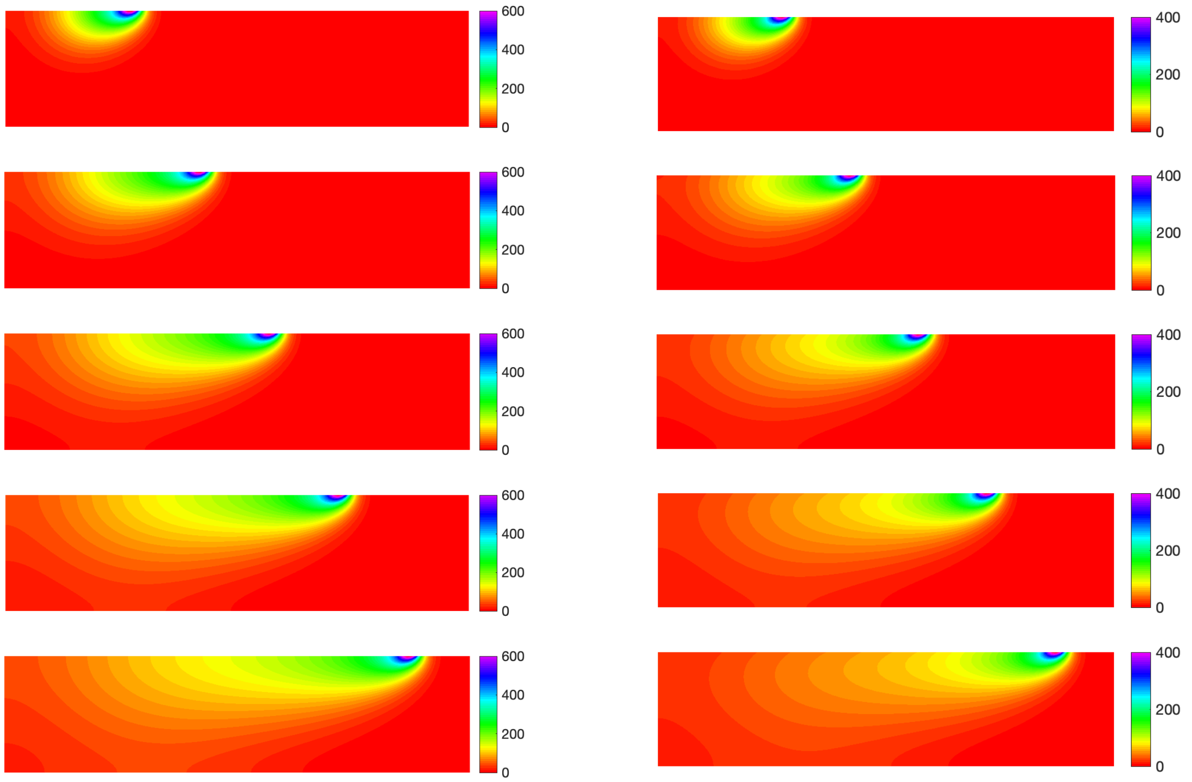
$$\text{BC terms} = CU^{n-1}_i,\quad j = 0,\ i = 0, \dots, N \tag{16}$$

$$U^{n-1}_i = -\frac{2h}{K}\left(-q_s(ih, t_n) + h_c(T^{n-1}_{i,0} - T_e) + \epsilon\sigma((T^{n-1}_{i,0})^4 - T_e^4)\right),\ i = 0, \dots, N \tag{17}$$

$$q_s(ih, t_n) = q_0 \exp\left(-\frac{2(ih - V_s t_n)^2}{r_b^2}\right) = q_0 \exp\left(-\frac{2(i - {V_s}' t_n)^2}{{r_b}'^2}\right),\ i = 0, \dots, N \tag{18}$$

3.1.1 Test 1

Parameters: K = 0.01, ρ = 1.0, C_p = 1.0, Q = 3, η = 1, r_b = 0.2, V_s = 0.075.
Up boundary cooling parameters: h_c= 0.005, ϵ = 0.005, σ = 5.67E-8, T_e = 0.
Computation domain: L_x = 8, L_y = 2.



(a) no surface cooling h_c= 0, ϵ = 0

(b) h_c= 0.005, ϵ = 0.005

Fig. 1: Temperature distribution without latent heat.

3.1.2 Self Convergence Study (with convection and radiation)

Ground truth: dt = 0.00625, h = $L_x/2048$
Trial: dt = 0.025, 0.1, 0.4, 1.6; h = $L_x/1024, L_x/512, L_x/256, L_x/128$

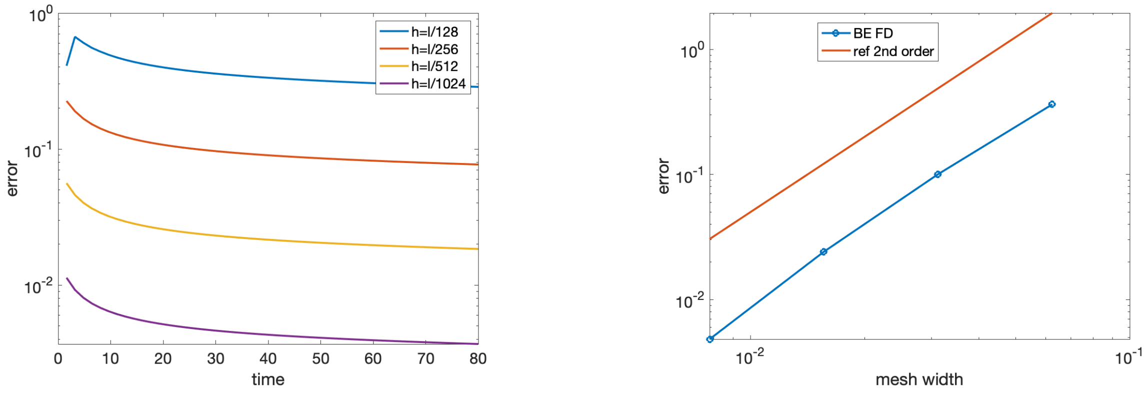
3.2 Latent heat term implicit-explicit

Implicit:

$$\frac{\partial f_l(T)}{\partial t} = \frac{f_l^n(T) - f_l^{n-1}(T)}{\Delta t} \tag{19}$$

Explicit:

$$f_l^n(T) = \begin{cases} 1 & T^{n-1} > T_l \\ \frac{T^n - T_s}{T_l - T_s} & T_s \leq T^{n-1} \leq T_l \\ 0 & T^{n-1} < T_s \end{cases} \tag{20}$$



(a) error

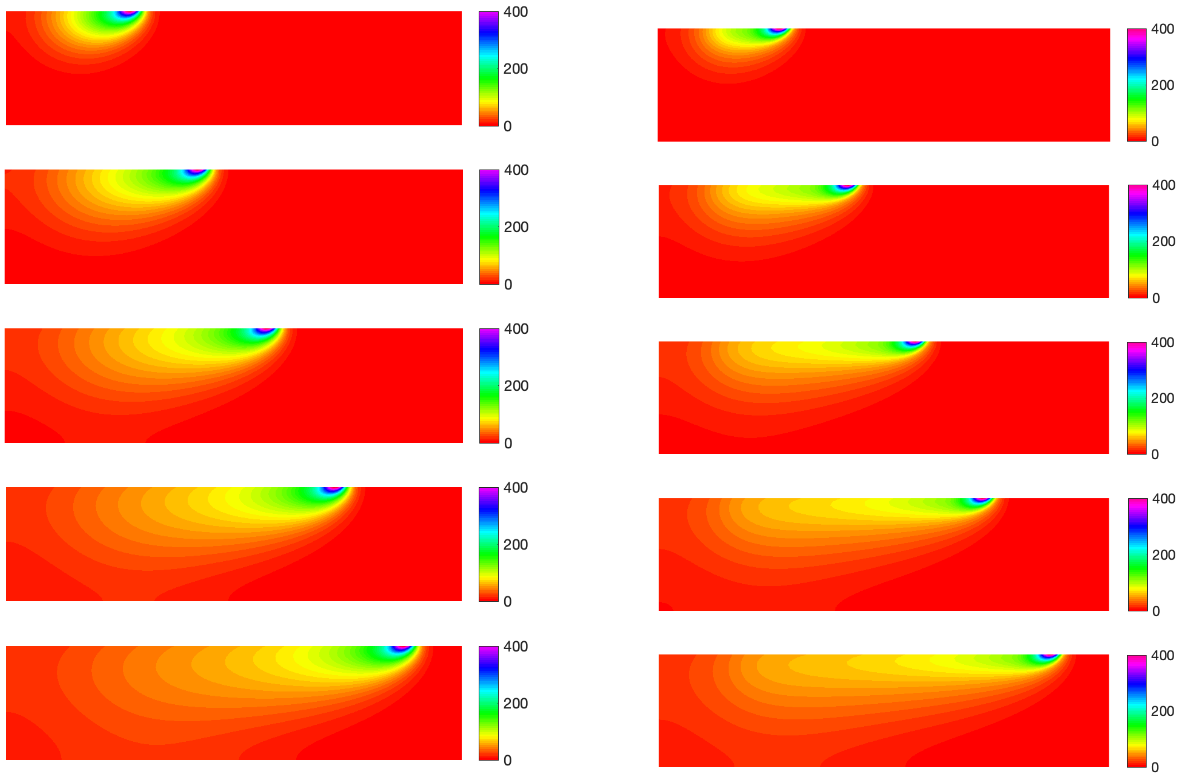
(b) convergence order 2.076

Fig. 2: Self convergence.

$$f_l^{n-1}(T) = \begin{cases} 1 & T^{n-1} > T_l \\ \frac{T^{n-1} - T_s}{T_l - T_s} & T_s \leq T^{n-1} \leq T_l \\ 0 & T^{n-1} < T_s \end{cases} \tag{21}$$

$$\text{Latent heat term} = \begin{cases} 0 & T^{n-1} > T_l \\ -\frac{L_m}{\Delta t}\frac{T^n - T^{n-1}}{T_l - T_s} & T_s \leq T^{n-1} \leq T_l \\ 0 & T^{n-1} < T_s \end{cases} \tag{22}$$

Test2:
Parameters: L=20, T_s=40, T_l=110



(a) Without latent heat

(b) With latent heat

Fig. 3: Temperature distribution with and without latent heat. Surface cooling is included