An Immersed Boundary Method with Divergence-Free Velocity Interpolation

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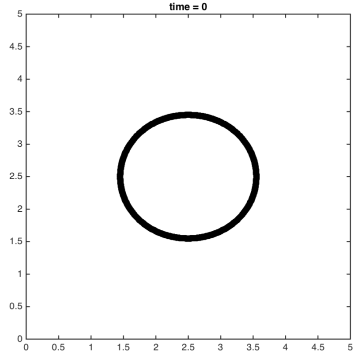
Joint work with Charles Peskin & Aleks Donev

May 11, 2015

A 2D pumping membrane

- A perturbed circular membrane immersed in a viscous incompressible fluid

$$\mathbf{X}(s) = (1 + \epsilon \cos ps)(\cos(s), \sin(s)), \quad p = 2$$



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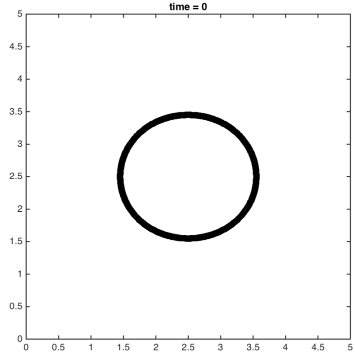
- Lagrangian force density $\mathbf{F} = K(t)\dfrac{\partial^2 \mathbf{X}}{\partial s^2}$, stiffness $K(t) = K_c(1 + \tau \sin(\omega_0 t))$

  $\Longrightarrow$ parametric resonance (Cortez et al. 2004)



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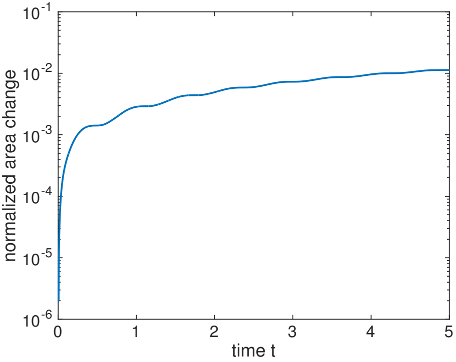
- Standard IB method + collocated-grid fluid solver $\Longrightarrow$ substantial volume loss!



Work on improving volume conservation

- Modified discrete divergence operator (Peskin & Printz 1993)
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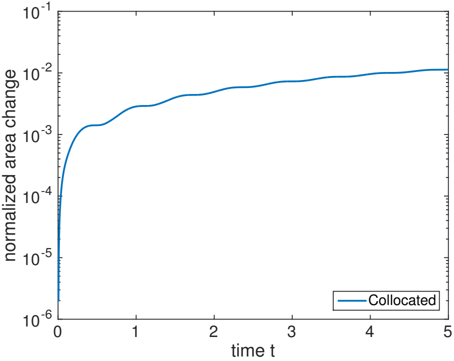
-



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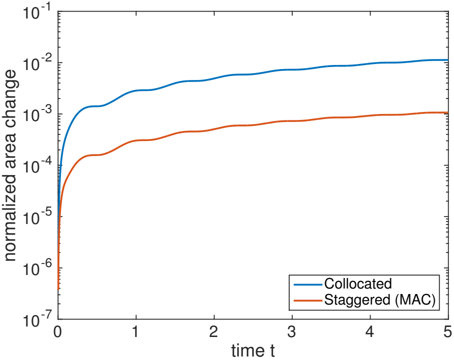
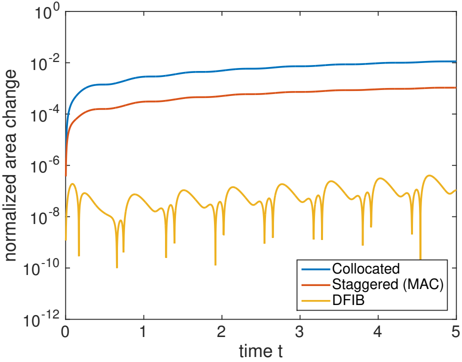


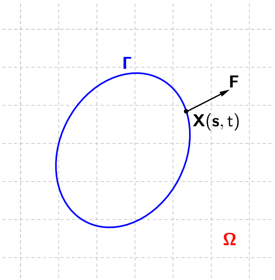
Work on improving volume conservation

- Modified discrete divergence operator (Peskin & Printz 1993)
- Staggered-grid fluid solver (Griffith 2012)
- Divergence-free velocity interpolation & force spreading (DFIB, Peskin)



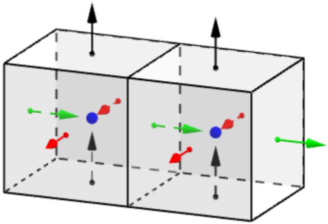
Equations of motion

$$
\begin{aligned}
\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) + \nabla p &= \mu\Delta\mathbf{v} + \mathbf{f} \\
\nabla\cdot\mathbf{v} &= 0 \\
\mathbf{f}(\mathbf{x},t) &= \int_{\Gamma} \mathbf{F}(\mathbf{s},t)\delta(\mathbf{x}-\mathbf{X}(\mathbf{s},t))\,ds \\
\frac{\partial \mathbf{X}}{\partial t} &= \int_{\Omega} \mathbf{v}(\mathbf{x},t)\delta(\mathbf{x}-\mathbf{X}(\mathbf{s},t))\,dx \\
\mathbf{F}(\mathbf{s},t) &= \mathcal{F}(\mathbf{X}(\mathbf{s},t),t)
\end{aligned}
$$



- $\boldsymbol{\Omega}$: fluid domain, $\mathbf{x}$: Eulerian
- $\boldsymbol{\Gamma}$: structure, $\mathbf{s}$: Lagrangian
- $\mathbf{F}$: force density on structure
- Navier-Stokes + fluid-structure coupling via the δ-function

Spatial Discretization



- Periodic domain: $[0, L]^3$

  Meshwidth: $h = L/N$
- Staggered-grid discretization:

  Pressure p: cell-centered

  Velocity $\mathbf{u}$: face-centered

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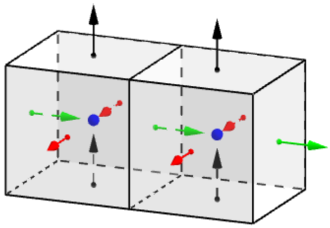

- Periodic domain: $[0, L]^3$

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- Discrete partial derivative of grid function φ:

$$D_\alpha^h \varphi := \frac{\varphi(\mathbf{x} + \frac{h}{2}\mathbf{e}_\alpha) - \varphi(\mathbf{x} - \frac{h}{2}\mathbf{e}_\alpha)}{h}, \quad \alpha = 1, 2, 3.$$

  Discrete gradient: $\mathbf{D}^h \varphi := (D_1^h \varphi,\ D_2^h \varphi,\ D_3^h \varphi)$

  Discrete divergence: $\mathbf{D}^h \cdot \mathbf{u} := \sum_{\alpha=1}^{3} D_\alpha^h u_\alpha,$

  Discrete Laplacian: $L^h \varphi = (\mathbf{D}^h \cdot \mathbf{D}^h)\varphi$

Spatial discretization

$$\boxed{\begin{aligned}
\rho\left(\frac{d\mathbf{u}}{dt}+\mathbf{N}(\mathbf{u})\right)+\mathbf{D}^h p &= \mu L^h\mathbf{u}+\mathbf{f}\\
\mathbf{D}^h\cdot\mathbf{u} &= 0\\
\mathbf{f} &= \mathcal{S}[\mathbf{X}]\mathbf{F}\\
\mathbf{U}_k &= \frac{d\mathbf{X}_k}{dt}=\mathcal{S}^*[\mathbf{X}_k]\mathbf{u}\\
\mathbf{F}_k &= \mathbf{F}(\mathbf{X}_k)=\mathcal{F}(\mathbf{X}_k,t)
\end{aligned}}$$

Lagrangian markers: $\mathbf{X}=\{\mathbf{X}_1,\ldots,\mathbf{X}_M\}$, $\mathbf{F}=\{\mathbf{F}_1,\ldots,\mathbf{F}_M\}$

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- Standard IB force spreading: $\mathbf{f}(\mathbf{x}) = \mathcal{S}[\mathbf{X}]\mathbf{F} = \sum_{k=1}^{M} \mathbf{F}_k \; \delta_h(\mathbf{x} - \mathbf{X}_k) \, \Delta \mathbf{s}_k$

  Standard IB velocity interpolation: $\mathbf{U}_k = \mathcal{S}^*[\mathbf{X}_k]\mathbf{u} = \sum_{\mathbf{x}} \mathbf{u}(\mathbf{x}) \; \delta_h(\mathbf{x} - \mathbf{X}_k) h^3$

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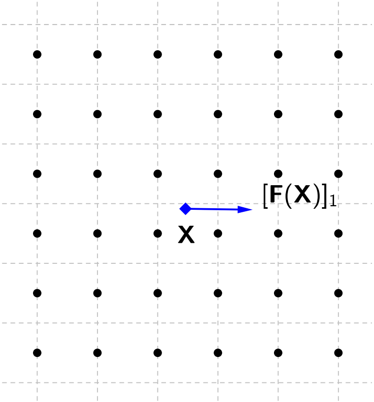
- ▷ Power identity (adjointness): $\sum_{\mathbf{x}} \mathbf{u}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) h^3 = \sum_{k=1}^{M} \mathbf{F}_k \cdot \mathbf{U}_k$

Standard IB force spreading & velocity interpolation



- $\mathbf{u}$ divergence-free in the discrete sense: $\mathbf{D}^h \cdot \mathbf{u} = 0$
- $\mathbf{X}$: any arbitrary point in domain

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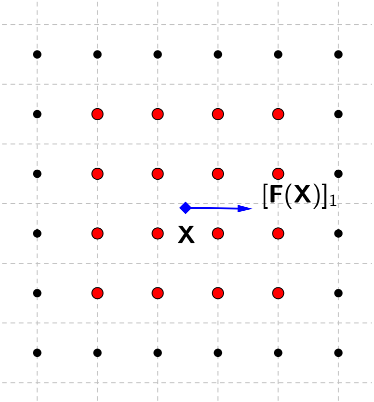

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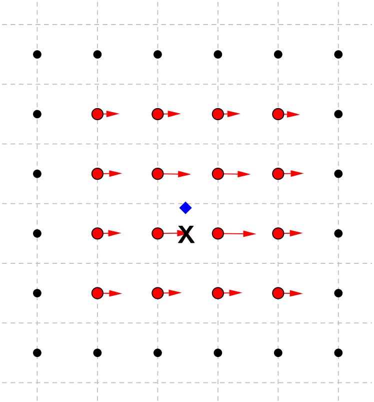

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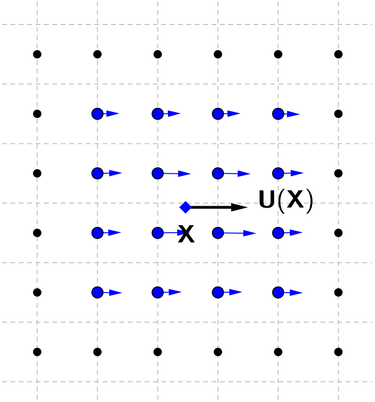

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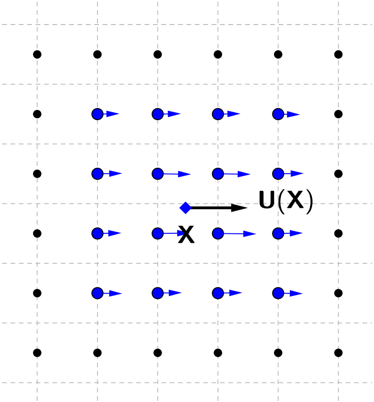

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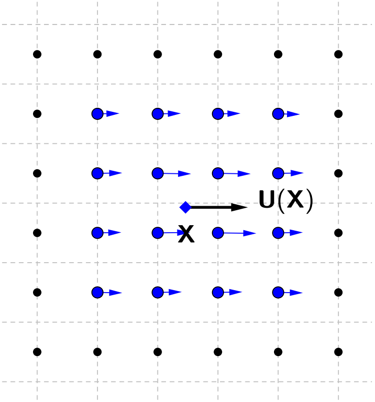

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- $\mathbf{U}(\mathbf{X})$ is not divergence-free in the continuous sense,

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Divergence-free velocity interpolation

- Challenge: given $\mathbf{u}(\mathbf{x})$ that is discretely divergence-free, how to construct $\mathbf{U}(\mathbf{X})$ that is continuously divergence-free?

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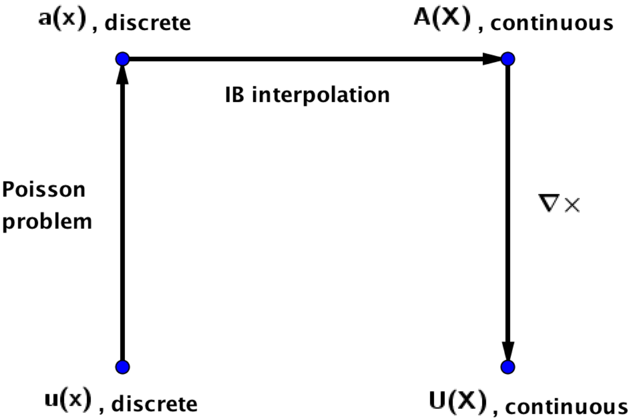

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\end{aligned}
$$

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Impose the gauge condition: $\mathbf{D}^h \cdot \mathbf{a} = 0$

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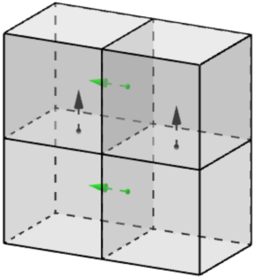
  For example, $[\mathbf{a}(\mathbf{x})]_1 = D_2^h u_3 - D_3^h u_2$



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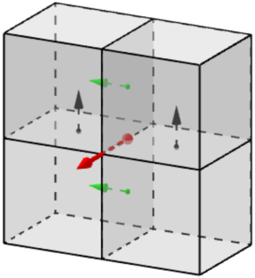

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- Next, interpolate $\mathbf{a}(\mathbf{x})$ to get $\mathbf{A}(\mathbf{X})$ via standard IB interpolation,

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- Finally,

$$\begin{aligned}\mathbf{U}(\mathbf{X}) &= \mathbf{u}_0 + (\nabla \times \mathbf{A})(\mathbf{X}) \\ &= \mathbf{u}_0 + \sum_{\mathbf{x}} \mathbf{a}(\mathbf{x}) \times (\nabla \delta_h)(\mathbf{x} - \mathbf{X}) h^3.\end{aligned}$$

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- By default,

$$\nabla \cdot \mathbf{U} = \nabla \cdot (\mathbf{u}_0 + \nabla \times \mathbf{A}) = 0$$

Divergence-free force spreading

- Goal: seek force spreading $\mathbf{f}$ that preserves the power identity

$$\sum_{\mathbf{x}} \mathbf{u}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) h^3 = \sum_{k=1}^{M} \mathbf{F}_k \cdot \mathbf{U}_k$$

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- can add $\mathbf{D}^h \varphi$ since $\sum_{\mathbf{x}} \mathbf{a}(\mathbf{x}) \cdot \left(\mathbf{D}^h \varphi\right) h^3 = -\sum_{\mathbf{x}} \left(\cancelto{0}{\mathbf{D}^h \cdot \mathbf{a}}\right)(\mathbf{x})\, \varphi(\mathbf{x})\, h^3 = 0$

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- required to add $\mathbf{D}^h \varphi$ for the RHS to be discretely divergence-free

Divergence-free force spreading

- $$(\mathbf{D}^h \times \mathbf{f})(\mathbf{x}) = \sum_{k=1}^{M} (\nabla \delta_h)(\mathbf{x} - \mathbf{X}_k) \times \mathbf{F}_k + \mathbf{D}^h \varphi$$
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- This only determines $\mathbf{f}$ up to a constant. For uniqueness, we use

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A new C^3 6-point IB kernel (Peskin)

$$\delta_h(\mathbf{x}) = \frac{1}{h^3}\phi\left(\frac{x_1}{h}\right)\phi\left(\frac{x_2}{h}\right)\phi\left(\frac{x_3}{h}\right)$$

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- Defining postulates for $\phi(r)$:

$$
\begin{aligned}
&\text{(i)} && \phi(r) \text{ is continuous for all real } r,\\
&\text{(ii)} && \phi(r) = 0 \text{ for } |r| \geq 3,\\
&\text{(iii)} && \sum_{j \text{ even}} \phi(r-j) = \sum_{j \text{ odd}} \phi(r-j) = \frac{1}{2} \text{ for all real } r,\\
&\text{(iv)} && \sum_j (r-j)\,\phi(r-j) = 0,\\
&\text{(v)} && \sum_j (r-j)^2\phi(r-j) = K,\\
&\text{(vi)} && \sum_j (r-j)^3\phi(r-j) = 0,\\
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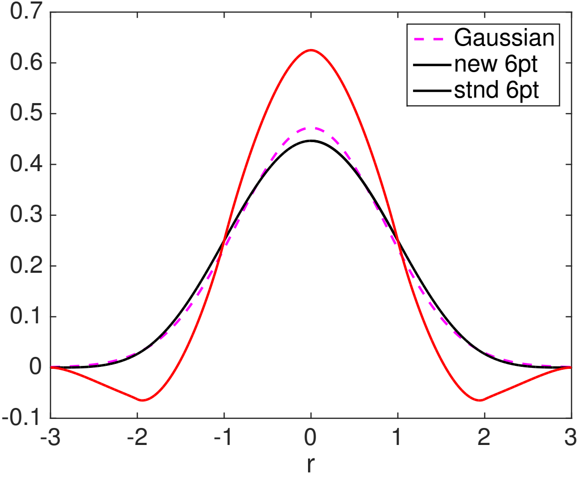
\end{array}$$

- Choose $K = \frac{59}{60} - \frac{\sqrt{29}}{20} \Longrightarrow$ three continuous derivatives

A new C^3 6-point IB kernel (Peskin)



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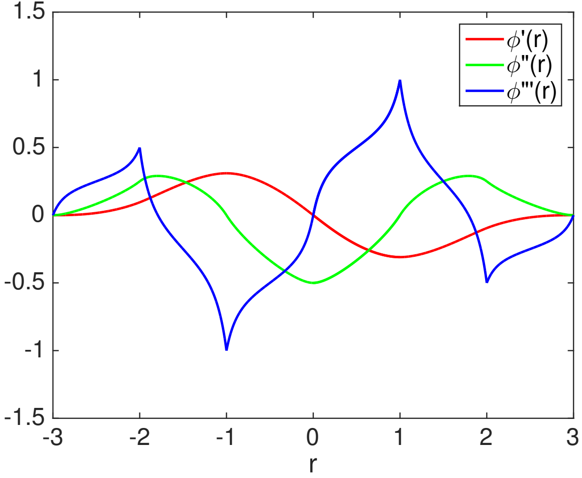
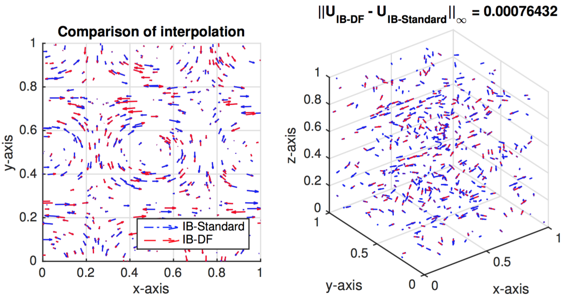


Test divergence-free velocity interpolation

- Taylor vortex:

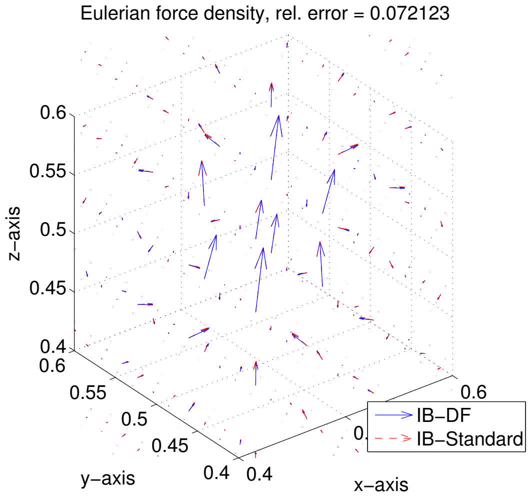
$$
\begin{aligned}
v_1 &= 2\cos(2\pi x)\sin(2\pi y)\sin(2\pi z),\\
v_2 &= -\sin(2\pi x)\cos(2\pi y)\sin(2\pi z),\\
v_3 &= -\cos(2\pi x)\sin(2\pi y)\cos(2\pi z).
\end{aligned}
$$

- $\mathbf{u} = \mathbb{P}\mathbf{v}$, where $\mathbb{P} = I - \mathbf{D}^h(\mathbf{D}^h \cdot \mathbf{D}^h)^{-1}\mathbf{D}^h$.
- interpolate $\mathbf{U}(\mathbf{X})$ at random $\mathbf{X}$ from $\mathbf{u}$



Test divergence-free force spreading

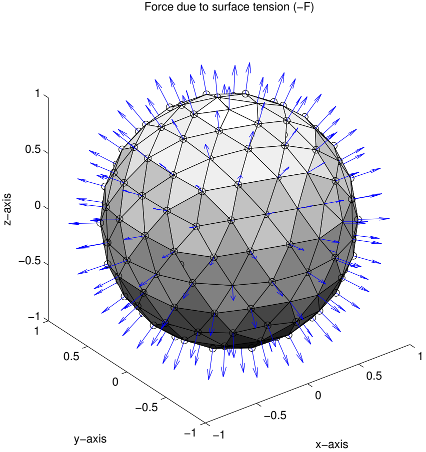
Spread a point force $\mathbf{F} = (1, 1, 1)$ at $X = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$



Test divergence-free force spreading

Spread surface tension on a triangulated sphere:

$$\mathbf{F}_k = -\frac{\partial \mathcal{E}}{\partial \mathbf{X}_k}, \text{ where } \mathcal{E}[\mathbf{X}_1, \cdots \mathbf{X}_M] = \sum_l A_l$$



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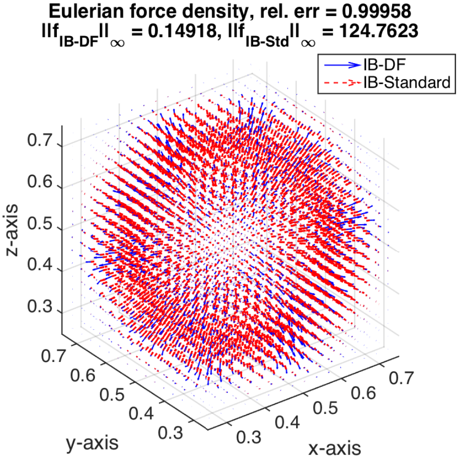
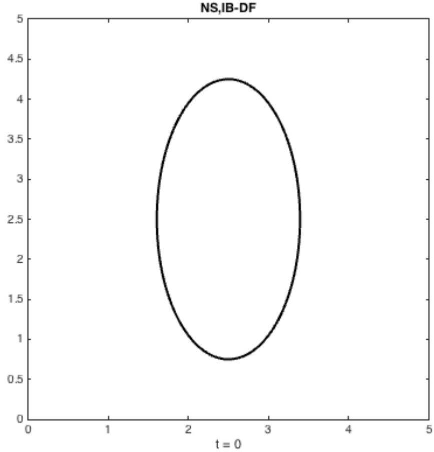


A 2D membrane with surface tension

- Setup: a 2D elliptical membrane immersed in a fluid at rest
- Surface tension: $\mathbf{F} = T\frac{\partial \tau}{\partial s}$, where $\tau = \frac{\partial \mathbf{X}/\partial s}{|\partial \mathbf{X}/\partial s|}$



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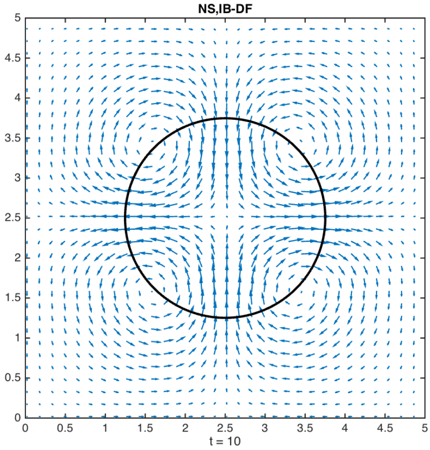
- Parameters: $\rho = 1, \mu = 0.1, N = 128, L = 5, h = L/N, dt = h/4, T = 2$



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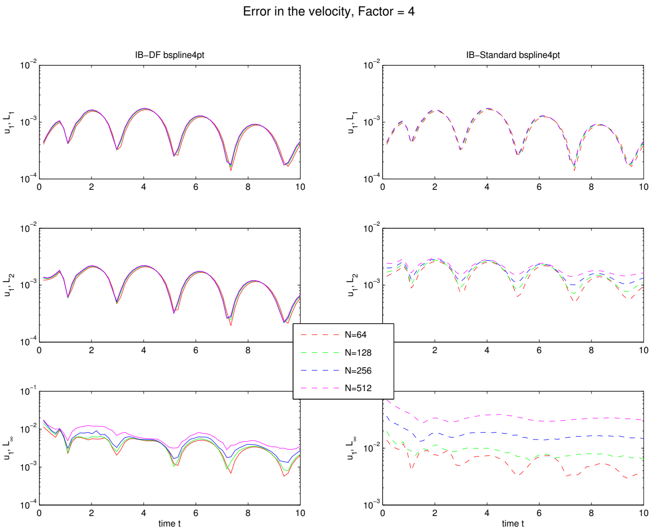

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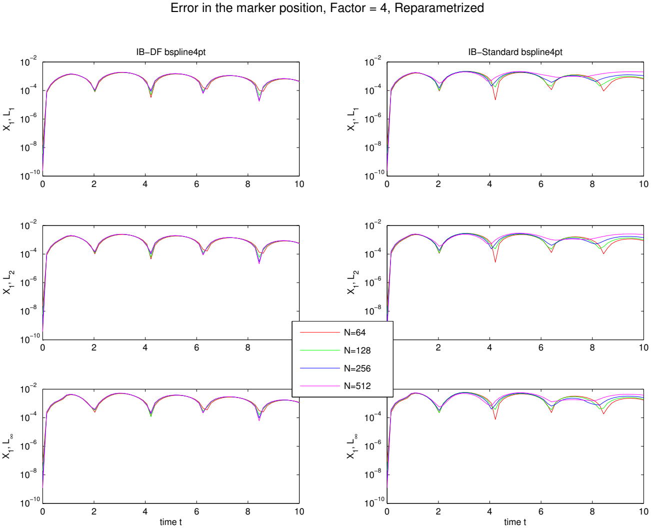

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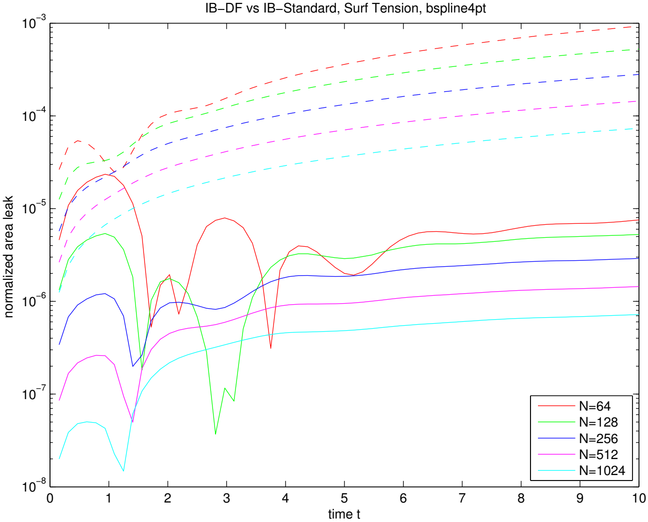

Marker spacing & volume conservation

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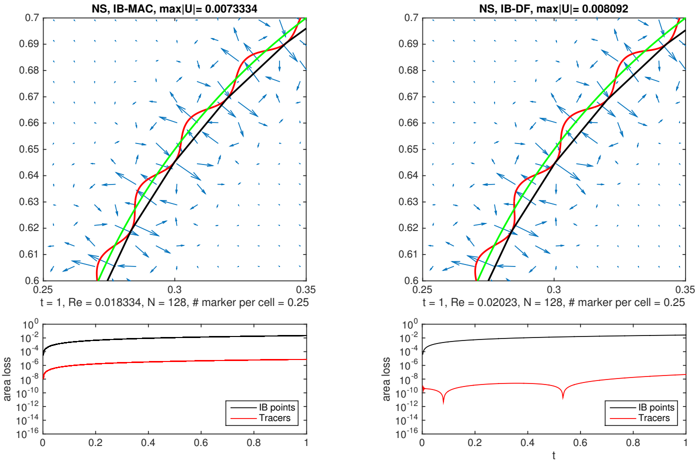
- Investigate the relationship between marker spacing (# of makers per mesh-width) and volume conservation

Marker spacing & volume conservation

- Setup: a 2D circular membrane (equilibrium) immersed in a fluid at rest
- Force density $\mathbf{F} = \dfrac{\partial^2 \mathbf{X}}{\partial s^2}$

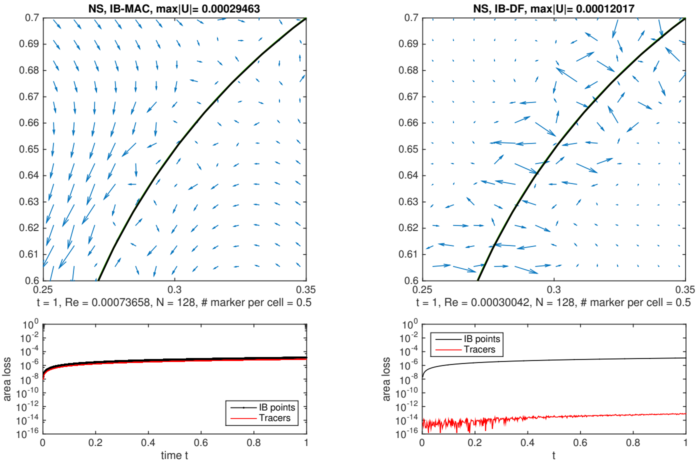
  0.25 marker per meshwidth (4 meshwidth between two IB markers)



Marker spacing & volume conservation

- Setup: a 2D circular membrane (equilibrium) immersed in a fluid at rest
- Force density $\mathbf{F} = \frac{\partial^2 \mathbf{X}}{\partial s^2}$

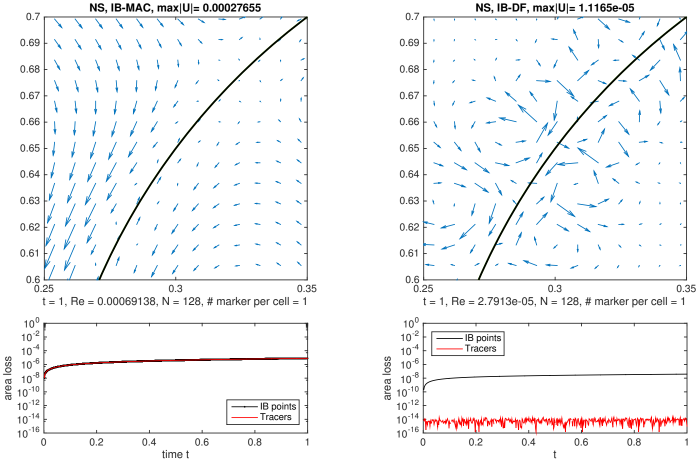
  0.5 marker per meshwidth (2 meshwidth between two IB markers)



Marker spacing & volume conservation

- Setup: a 2D circular membrane (equilibrium) immersed in a fluid at rest
- Force density $\mathbf{F} = \frac{\partial^2 \mathbf{X}}{\partial s^2}$

  1 marker per meshwidth



Marker spacing & volume conservation

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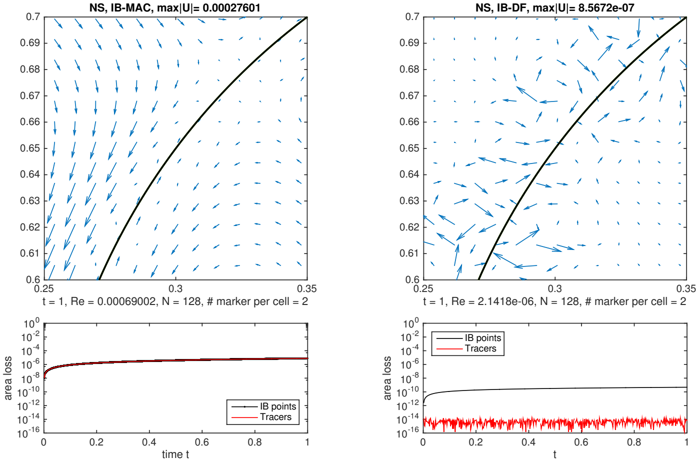
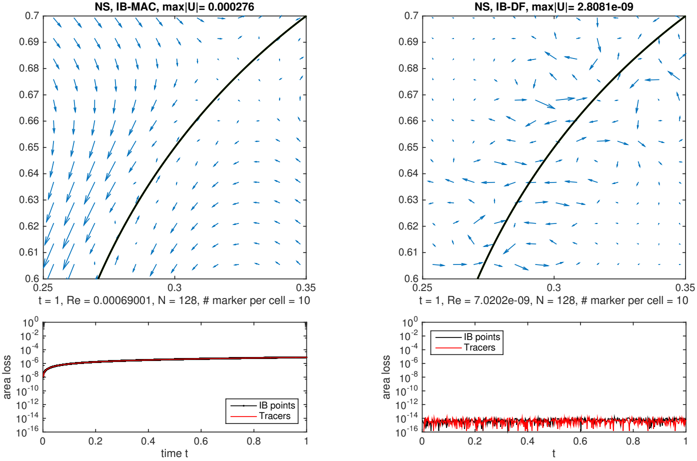
  2 markers per meshwidth



Marker spacing & volume conservation

- Setup: a 2D circular membrane (equilibrium) immersed in a fluid at rest
- Force density $\mathbf{F} = \frac{\partial^2 \mathbf{X}}{\partial s^2}$

  10 markers per meshwidth



In Closing

- ▷ Divergence-free velocity interpolation & force spreading

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- Divergence-free velocity interpolation & force spreading
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- Remark: discretely divergence-free force spreading $\mathbf{f}(\mathbf{x})$

  $\Longrightarrow$ $\mathbf{f}(\mathbf{x})$ includes the pressure gradient that is generated by Lagrangian forces.

  not yet clear how to isolate ∇p from force spreading (advantage or disadvantage?)