MATH-UA 263 Partial Differential Equations
Recitation Summary

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1 February 2, 2018

Topics: verifying solution to a PDE, dispersion relations, well-posedness, general solution via integration.

1. (EPDE, Exercise 1.6) Verify the general solution of the heat equation on the real line:
$$\begin{cases} u_t = ku_{xx}, \quad -\infty < x < \infty,\, t > 0 \\ u(x,0) = g(x) \end{cases}$$
is given by
$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-s)^2/4kt} g(s)\, ds. \tag{1}$$

2. (PDE p.51-52) Solve the heat equation with the initial condition $u(x,0) = g(x) = e^{-x}$. To do so, use (1) and the integral identity $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx = 1$ to show that the solution is
$$u(x,t) = e^{kt-x}. \tag{2}$$

3. Find the dispersion relation of the linear PDE: $u_t = -u - \delta u_{xx} - u_{xxxx}$, $\delta > 0$.

4. (Well-posed and ill-posed problems) Consider the following PDEs for u and v:
$$(\Box) \begin{cases} u_{tt} - u_{xx} = 0 \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases} \quad \text{and} \quad (\triangle) \begin{cases} v_{tt} - v_{xx} = 0 \\ v(x,0) = 0 \\ v_t(x,0) = \epsilon \sin(\frac{x}{\epsilon}) \end{cases} \tag{3}$$
It is straightforward to check that $u(x,t) = 0$ and $v(x,t) = \epsilon^2 \sin\left(\frac{x}{\epsilon}\right) \sin\left(\frac{t}{\epsilon}\right)$. For small $\epsilon > 0$, note that $(\triangle)$ is a small perturbation to $(\Box)$ in the initial derivative data. Use the notion of "stability with respect to initial data" to argue that $(\triangle)$ is well-posed. (*Hint*: if $\|v_t(x,0) - u_t(x,0)\| \le \epsilon$, then $\|v(x,t) - u(x,t)\| \le \epsilon^2$, for all $t > 0$.)

Similarly, consider

$$(\square)\begin{cases} u_{tt}+u_{xx}=0 \\ u(x,0)=0 \\ u_t(x,0)=0 \end{cases} \quad \text{and} \quad (\triangle)\begin{cases} v_{tt}+v_{xx}=0 \\ v(x,0)=0 \\ v_t(x,0)=\epsilon\sin(\frac{x}{\epsilon}) \end{cases} \tag{4}$$

Note that $u(x,t)=0$ and $v(x,t)=\epsilon^2\sin\left(\frac{x}{\epsilon}\right)\sinh\left(\frac{t}{\epsilon}\right)$. Argue that $(\triangle)$ is an ill-posed problem.

5. Find the general solution of the PDE: $u_{xt}+3u_x=1$.

2 February 9, 2018

Topics: conservations laws, differential operators in polar/spherical coordiates, BVP and eigenvalues/eigenfunctions, method of characteristics.

1. (*1D gas dynamics*) Consider the one dimensional, time-dependent flow of gas under the assumption of constant pressure p. If u deontes the gas velocity, ρ the density and e the energy per unit volume, the basic equations of gas dynamics are

$$u_t+uu_x=0 \tag{5}$$
$$\rho_t+u\rho_x+\rho u_x=0 \tag{6}$$
$$e_t+ue_x+eu_x+pu_x=0. \tag{7}$$

Rewrite the above equations in terms of a conservation of law. In other words, find the quantity (vector) $\boldsymbol{U}$ and flux $\boldsymbol{F}(\boldsymbol{U})$ so that the above equations can be written in the form:

$$\boldsymbol{U}_t+[\boldsymbol{F}(\boldsymbol{U})]_x=\boldsymbol{0}. \tag{8}$$

2. Suppose that $u(x,y)$ satifies the two-dimensional Laplace equation $\nabla^2 u=0$ and u only depends on the distance from the origin $r=\sqrt{x^2+y^2}$, ie, $u(x,y)\equiv v(r,t)$. Show that

$$v_{rr}+\frac{1}{r}v_r=0. \tag{9}$$

3. For what value(s) of λ does the following linear system $(\boldsymbol{A}-\lambda\boldsymbol{I})\boldsymbol{x}=\boldsymbol{0}$ have a unique solution or infinitely many solutions, where

$$\boldsymbol{A}=\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}.$$

Now consider the following two-point *Boundary Value Problem* (BVP) for $0<x<2\pi$:

$$u''(x)+\lambda u=0, \tag{10}$$
$$u(0)=u'(2\pi)=0. \tag{11}$$

For what value(s) of λ does the BVP has a unique solution? Infinitely many solutions?

4. (APDE, §1.2, Exercise 8, p.26) Solve the initial value problem

$$u_t + u_x - 3u = t, \quad x \in \mathbb{R},\ t > 0, \tag{12}$$
$$u(x,0) = x^2, \quad x \in \mathbb{R}. \tag{13}$$

5. (APDE, §1.2, Example 1.10, p.18) Find the general solution to $u_t + 2tu_x = 0$.

3 February 16, 2018

Topics: first-order PDEs and method of characteristics.

1. Revisit APDE §1.2, Exercise 8, solve by two approaches: (1) using a suitable choice of coordinate transformation, and, (2) using the method of characteristics.

2. Solve the following PDE by method of characteristics. Sketch some characteristic curves.

$$\begin{cases} y^2 u_x + u_y = 0, \\ u(x,0) = x^2. \end{cases} \tag{14}$$

3. APDE, §1.2, Example 1.11.

4. Classify and solve the following PDE

$$\begin{cases} 2u_{xx} + 5u_{xt} + 3u_{tt} = 0, \\ u(x,0) = 0,\ u_t(x,0) = xe^{-x^2}. \end{cases} \tag{15}$$

4 February 23, 2018

Topics: Wave equation, d'Alembert's formula, domain of dependence, Heat equation and Cauchy problem.

1. Review the solution of #5 in HW3.

2. (*The hammer blow*, PDE 2.1 Exercise #5,6) Consider the wave equation $u_{tt} = u_{xx}$ on the entire real line $-\infty < x < \infty$ with zero initial position $u(x,0) = 0$ and initial velocity $u_t(x,0) = g(x)$, where $g(x) = 1$ for $|x| < 1$ and $g(x) = 0$ for $|x| \geq 1$. Sketch the solution at time instants $t = \frac{1}{2}, 1, \frac{3}{2}, 2$ and $t = \frac{5}{2}$. What is the maximum displacement $\max_x u(x,t)$?

3. Solve the heat equation on the whole line with given initial data $u(x,0) = \phi(x)$:

   (a) $\phi(x) = 1$ for $|x| < l$, and $\phi(x) = 0$ for $|x| > l$.

   (b) $\phi(x) = x^2$.

   (c) $\phi(x) = e^{3x}$.

5 March 2, 2018

Topics: Duhamel's princple for the wave, heat, and linear advection equations.

1. Review the solution of #4 in HW3.
2. Principle of superposition for breaking down the solution of the inhomogeneous (with sources) equation with nonzero initial data into two subproblems: one homogeneous problem with nonzero initial data, and one inhomogeneous problem with zero initial data (Duhamel's principle).
3. Explain the Duhamel's principle for the wave, heat, and linear advection equations.
4. Exercise 1, §3.4, PDE.
5. Part of #6 in HW4.
6. APDE, §2.5, Exercise #3 and #4.