MATH-UA 263 Partial Differential Equations
Recitation Summary

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1   February 2, 2018

Topics: verifying solution to a PDE, dispersion relations, well-posedness, general solution via integration.

1. (EPDE, Exercise 1.6) Verify the general solution of the heat equation on the real line:

$$\begin{cases} u_t = ku_{xx}, & -\infty < x < \infty, \, t > 0 \\ u(x,0) = g(x) \end{cases}$$

   is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-s)^2/4kt} g(s)\, ds. \tag{1}$$

2. (PDE p.51-52) Solve the heat equation with the initial condition $u(x,0) = g(x) = e^{-x}$. To do so, use (1) and the integral identity $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx = 1$ to show that the solution is

$$u(x,t) = e^{kt-x}. \tag{2}$$

3. Find the dispersion relation of the linear PDE: $u_t = -u - \delta u_{xx} - u_{xxxx}$, $\delta > 0$.

4. (Well-posed and ill-posed problems) Consider the following PDEs for u and v:

$$(\square)\begin{cases} u_{tt} - u_{xx} = 0 \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases} \quad \text{and} \quad (\triangle)\begin{cases} v_{tt} - v_{xx} = 0 \\ v(x,0) = 0 \\ v_t(x,0) = \epsilon \sin\left(\frac{x}{\epsilon}\right) \end{cases} \tag{3}$$

   It is straightforward to check that $u(x,t) = 0$ and $v(x,t) = \epsilon^2 \sin\left(\frac{x}{\epsilon}\right) \sin\left(\frac{t}{\epsilon}\right)$. For small $\epsilon > 0$, note that $(\triangle)$ is a small perturbation to $(\square)$ in the initial derivative data. Use the notion of "stability with respect to initial data" to argue that $(\triangle)$ is well-posed. (*Hint*: if $\|v_t(x,0) - u_t(x,0)\| \leq \epsilon$, then $\|v(x,t) - u(x,t)\| \leq \epsilon^2$, for all $t > 0$.)

Similarly, consider

$$(\square) \begin{cases} u_{tt} + u_{xx} = 0 \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases} \quad \text{and} \quad (\triangle) \begin{cases} v_{tt} + v_{xx} = 0 \\ v(x,0) = 0 \\ v_t(x,0) = \epsilon \sin(\frac{x}{\epsilon}) \end{cases} \tag{4}$$

Note that $u(x,t) = 0$ and $v(x,t) = \epsilon^2 \sin\left(\frac{x}{\epsilon}\right) \sinh\left(\frac{t}{\epsilon}\right)$. Argue that $(\triangle)$ is an ill-posed problem.

5. Find the general solution of the PDE: $u_{xt} + 3u_x = 1$.

2 February 9, 2018

Topics: conservations laws, differential operators in polar/spherical coordiates, BVP and eigenvalues/eigenfunctions, method of characteristics.

1. (*1D gas dynamics*) Consider the one dimensional, time-dependent flow of gas under the assumption of constant pressure p. If u deontes the gas velocity, ρ the density and e the energy per unit volume, the basic equations of gas dynamics are

$$u_t + uu_x = 0 \tag{5}$$
$$\rho_t + u\rho_x + \rho u_x = 0 \tag{6}$$
$$e_t + ue_x + eu_x + pu_x = 0. \tag{7}$$

Rewrite the above equations in terms of a conservation of law. In other words, find the quantity (vector) $\boldsymbol{U}$ and flux $\boldsymbol{F}(\boldsymbol{U})$ so that the above equations can be written in the form:

$$\boldsymbol{U}_t + [\boldsymbol{F}(\boldsymbol{U})]_x = \boldsymbol{0}. \tag{8}$$

2. Suppose that $u(x,y)$ satifies the two-dimensional Laplace equation $\nabla^2 u = 0$ and u only depends on the distance from the origin $r = \sqrt{x^2 + y^2}$, ie, $u(x,y) \equiv v(r,t)$. Show that

$$v_{rr} + \frac{1}{r}v_r = 0. \tag{9}$$

3. For what value(s) of λ does the following linear system $(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$ have a unique solution or infinitely many solutions, where

$$\boldsymbol{A} = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}.$$

Now consider the following two-point *Boundary Value Problem* (BVP) for $0 < x < 2\pi$:

$$u''(x) + \lambda u = 0, \tag{10}$$
$$u(0) = u'(2\pi) = 0. \tag{11}$$

For what value(s) of λ does the BVP has a unique solution? Infinitely many solutions?

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4. (APDE, §1.2, Exercise 8, p.26) Solve the initial value problem

$$u_t + u_x - 3u = t, \quad x \in \mathbb{R}, \ t > 0, \tag{12}$$

$$u(x, 0) = x^2, \quad x \in \mathbb{R}. \tag{13}$$

5. (APDE, §1.2, Example 1.10, p.18) Find the general solution to $u_t + 2tu_x = 0$.