MATH-UA 263 Partial Differential Equations Recitation Summary

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1 February 2, 2018

Topics: verifying solution to a PDE, dispersion relations, well-posedness, general solution via integration.

1. (EPDE, Exercise 1.6) Verify the general solution of the heat equation on the real line:
$$\begin{cases} u_t = ku_{xx}, & -\infty < x < \infty,\, t > 0 \\ u(x,0) = g(x) \end{cases}$$
is given by
$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-s)^2/4kt} g(s)\, ds. \tag{1}$$

2. (PDE p.51-52) Solve the heat equation with the initial condition $u(x,0) = g(x) = e^{-x}$. To do so, use (1) and the integral identity $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx = 1$ to show that the solution is
$$u(x,t) = e^{kt-x}. \tag{2}$$

3. Find the dispersion relation of the linear PDE: $u_t = -u - \delta u_{xx} - u_{xxxx}$, $\delta > 0$.

4. (Well-posed and ill-posed problems) Consider the following PDEs for u and v:
$$(\square) \begin{cases} u_{tt} - u_{xx} = 0 \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases} \quad \text{and} \quad (\triangle) \begin{cases} v_{tt} - v_{xx} = 0 \\ v(x,0) = 0 \\ v_t(x,0) = \epsilon \sin(\frac{x}{\epsilon}) \end{cases} \tag{3}$$
It is straightforward to check that $u(x,t) = 0$ and $v(x,t) = \epsilon^2 \sin\left(\frac{x}{\epsilon}\right) \sin\left(\frac{t}{\epsilon}\right)$. For small $\epsilon > 0$, note that $(\triangle)$ is a small perturbation to $(\square)$ in the initial derivative data. Use the notion of "stability with respect to initial data" to argue that $(\triangle)$ is well-posed. (*Hint*: if $\|v_t(x,0) - u_t(x,0)\| \leq \epsilon$, then $\|v(x,t) - u(x,t)\| \leq \epsilon^2$, for all $t > 0$.)

Similarly, consider

$$(\square)\begin{cases} u_{tt}+u_{xx}=0 \\ u(x,0)=0 \\ u_t(x,0)=0 \end{cases} \quad \text{and} \quad (\triangle)\begin{cases} v_{tt}+v_{xx}=0 \\ v(x,0)=0 \\ v_t(x,0)=\epsilon\sin(\frac{x}{\epsilon}) \end{cases} \tag{4}$$

Note that $u(x,t)=0$ and $v(x,t)=\epsilon^2\sin\left(\frac{x}{\epsilon}\right)\sinh\left(\frac{t}{\epsilon}\right)$. Argue that $(\triangle)$ is an ill-posed problem.

5. Find the general solution of the PDE: $u_{xt}+3u_x=1$.

2 February 9, 2018

Topics: conservations laws, differential operators in polar/spherical coordiates, BVP and eigenvalues/eigenfunctions, method of characteristics.

1. (*1D gas dynamics*) Consider the one dimensional, time-dependent flow of gas under the assumption of constant pressure p. If u deontes the gas velocity, ρ the density and e the energy per unit volume, the basic equations of gas dynamics are

$$u_t+uu_x=0 \tag{5}$$
$$\rho_t+u\rho_x+\rho u_x=0 \tag{6}$$
$$e_t+ue_x+eu_x+pu_x=0. \tag{7}$$

Rewrite the above equations in terms of a conservation of law. In other words, find the quantity (vector) $\boldsymbol{U}$ and flux $\boldsymbol{F}(\boldsymbol{U})$ so that the above equations can be written in the form:

$$\boldsymbol{U}_t+[\boldsymbol{F}(\boldsymbol{U})]_x=\boldsymbol{0}. \tag{8}$$

2. Suppose that $u(x,y)$ satifies the two-dimensional Laplace equation $\nabla^2u=0$ and u only depends on the distance from the origin $r=\sqrt{x^2+y^2}$, ie, $u(x,y)\equiv v(r,t)$. Show that

$$v_{rr}+\frac{1}{r}v_r=0. \tag{9}$$

3. For what value(s) of λ does the following linear system $(\boldsymbol{A}-\lambda\boldsymbol{I})\boldsymbol{x}=\boldsymbol{0}$ have a unique solution or infinitely many solutions, where

$$\boldsymbol{A}=\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}.$$

Now consider the following two-point *Boundary Value Problem* (BVP) for $0<x<2\pi$:

$$u''(x)+\lambda u=0, \tag{10}$$
$$u(0)=u'(2\pi)=0. \tag{11}$$

For what value(s) of λ does the BVP has a unique solution? Infinitely many solutions?

4. (APDE, §1.2, Exercise 8, p.26) Solve the initial value problem

$$u_t + u_x - 3u = t, \quad x \in \mathbb{R},\, t > 0, \tag{12}$$
$$u(x,0) = x^2, \quad x \in \mathbb{R}. \tag{13}$$

5. (APDE, §1.2, Example 1.10, p.18) Find the general solution to $u_t + 2tu_x = 0$.

3 February 16, 2018

Topics: first-order PDEs and method of characteristics.

1. Revisit APDE §1.2, Exercise 8, solve by two approaches: (1) using a suitable choice of coordinate transformation, and, (2) using the method of characteristics.

2. Solve the following PDE by method of characteristics. Sketch some characteristic curves.

$$\begin{cases} y^2 u_x + u_y = 0, \\ u(x,0) = x^2. \end{cases} \tag{14}$$

3. APDE, §1.2, Example 1.11.

4. Classify and solve the following PDE

$$\begin{cases} 2u_{xx} + 5u_{xt} + 3u_{tt} = 0, \\ u(x,0) = 0,\ u_t(x,0) = xe^{-x^2}. \end{cases} \tag{15}$$

4 February 23, 2018

Topics: Wave equation, d'Alembert's formula, domain of dependence, Heat equation and Cauchy problem.

1. Review the solution of #5 in HW3.

2. (*The hammer blow*, PDE 2.1 Exercise #5,6) Consider the wave equation $u_{tt} = u_{xx}$ on the entire real line $-\infty < x < \infty$ with zero initial position $u(x,0) = 0$ and initial velocity $u_t(x,0) = g(x)$, where $g(x) = 1$ for $|x| < 1$ and $g(x) = 0$ for $|x| \geq 1$. Sketch the solution at time instants $t = \frac{1}{2}, 1, \frac{3}{2}, 2$ and $t = \frac{5}{2}$. What is the maximum displacement $\max_x u(x,t)$?

3. Solve the heat equation on the whole line with given initial data $u(x,0) = \phi(x)$:

   (a) $\phi(x) = 1$ for $|x| < l$, and $\phi(x) = 0$ for $|x| > l$.

   (b) $\phi(x) = x^2$.

   (c) $\phi(x) = e^{3x}$.