MATH-UA 263 Partial Differential Equations Recitation Summary

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1 February 2, 2018

Topics: verifying solution to a PDE, dispersion relations, well-posedness, general solution via integration.

1. (EPDE, Exercise 1.6) Verify the general solution of the heat equation on the real line:
$$\begin{cases} u_t = ku_{xx}, \quad -\infty < x < \infty,\, t > 0 \\ u(x,0) = g(x) \end{cases}$$
is given by
$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-s)^2/4kt} g(s)\, ds. \tag{1}$$

2. (PDE p.51-52) Solve the heat equation with the initial condition $u(x,0) = g(x) = e^{-x}$. To do so, use (1) and the integral identity $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx = 1$ to show that the solution is
$$u(x,t) = e^{kt-x}. \tag{2}$$

3. Find the dispersion relation of the linear PDE: $u_t = -u - \delta u_{xx} - u_{xxxx}$, $\delta > 0$.

4. (Well-posed and ill-posed problems) Consider the following PDEs for u and v:
$$(\Box) \begin{cases} u_{tt} - u_{xx} = 0 \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases} \quad \text{and} \quad (\triangle) \begin{cases} v_{tt} - v_{xx} = 0 \\ v(x,0) = 0 \\ v_t(x,0) = \epsilon \sin(\frac{x}{\epsilon}) \end{cases} \tag{3}$$
It is straightforward to check that $u(x,t) = 0$ and $v(x,t) = \epsilon^2 \sin\left(\frac{x}{\epsilon}\right) \sin\left(\frac{t}{\epsilon}\right)$. For small $\epsilon > 0$, note that $(\triangle)$ is a small perturbation to $(\Box)$ in the initial derivative data. Use the notion of "stability with respect to initial data" to argue that $(\triangle)$ is well-posed. (*Hint*: if $\|v_t(x,0) - u_t(x,0)\| \le \epsilon$, then $\|v(x,t) - u(x,t)\| \le \epsilon^2$, for all $t > 0$.)

Similarly, consider

$$(\square)\begin{cases} u_{tt}+u_{xx}=0 \\ u(x,0)=0 \\ u_t(x,0)=0 \end{cases} \quad \text{and} \quad (\triangle)\begin{cases} v_{tt}+v_{xx}=0 \\ v(x,0)=0 \\ v_t(x,0)=\epsilon\sin(\frac{x}{\epsilon}) \end{cases} \tag{4}$$

Note that $u(x,t)=0$ and $v(x,t)=\epsilon^2\sin\left(\frac{x}{\epsilon}\right)\sinh\left(\frac{t}{\epsilon}\right)$. Argue that $(\triangle)$ is an ill-posed problem.

5. Find the general solution of the PDE: $u_{xt}+3u_x=1$.

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