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1 Introduction 1/8

1.1 Introduction Stuff and General Things to Note

Stuff to know

This course will need to know more about qubits and determine their probabilities of ending on a quantum state. Some knowledge of linear algebra to help, *NOT* required.

Texts (where one by Elanor and Wolfgang is going to be most used):

- Quantum Computer Science, by David Mermin
- Quantum Computing: A Gentle Introduction, by Elanor Rieffel and Wolfgang Polak

1.2 Polarization of Photons

These are states

$$\hat{y} \Rightarrow |0\rangle$$
$$\hat{x} \Rightarrow |1\rangle$$

Dot product of

$$\vec{A} * \vec{B} = AB\cos\theta$$
$$\vec{A} * \hat{i} = A * 1 * \cos\theta = A\cos\theta$$

Generally speaking, $\vec{A} = (\hat{i} * \vec{A}) * \hat{i} + (\hat{j} + \vec{A}) * \hat{j}$

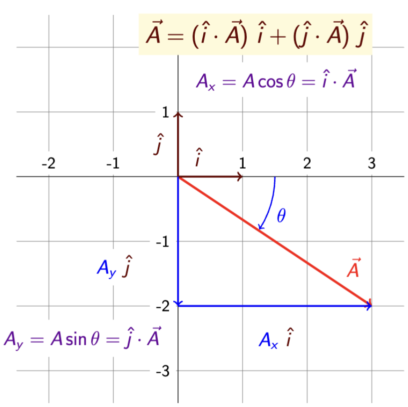


Figure 1: Some introductory material on real vectors in the plane

2 Polarization 1/10

$$\vec{\varepsilon}(t) = \vec{E}\cos(\omega t + \phi)$$

Where ω is the angular frequency and ϕ is the phase shift

$\vec{E}$ Is the most important variable
Where it is defined as:

$$\vec{E} = E_x * \hat{i} + E_y\hat{j} \doteq \begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

Certain direction of the polarization is let through polarized lens Where parallel to polarized lens will go through while those that are perpendicular will not pass through…

For example, if in the y direction nothing will get through, if it is in x direction it will go through

Examples

Passes through $\hat{i}$ Polaroid:

$$\vec{E} = E_x\hat{i} = (\hat{i} * \vec{E}) * \hat{i} = \hat{i} * \hat{i} * \vec{E}$$

Passes through $\hat{j}$ Polaroid:

$$\vec{E} = E_y * \hat{j} = (\hat{j} * \vec{E}) * \hat{j}$$

$$\hat{P} = \cos\theta\hat{i} + \sin\theta\hat{j} \doteq \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

Where $\hat{P}$ Is the orientation of the polaroid vector $(\hat{P} * \vec{E}) * \hat{P}$

Energy of a wave is $\propto$ (proportional) E^2 Fraction of energy that sets through $\hat{P}$.

$$F = (\hat{P} * \vec{E})^2$$

This is the fraction of energy that gets through the polaroid. It is squared as it the same square of the length of the vector
$\hat{p}$ Is the unit vector of E

Where $\hat{p} = \frac{\vec{E}}{|\vec{E}|}$ and

$$F = (\hat{P} * \vec{E})^2 = (\hat{P} * \hat{p})^2$$

Example Question

A linearly polarized wave with a polarization vector of magnitude E_0 making a 60-degree angle with the x-axis impinges on a polarizer that allows only x-polarized light through. What fraction of the energy is transmitted?

Answer: $\hat{P}$ is on the bottom and $\hat{p}$ is on top of big P hat such that it creates an angle of 60 degrees, creates

$$\hat{p} * \hat{P} = |\hat{p}| * |\hat{P}| \cos 60 \deg$$

Which boils down to:

$$F = \cos^2 \theta = \frac{1}{4}$$

Light consisting of photons, a bunch of them combined into light to create a wave. Photons are fundamentally the same, yet if you put a polaroid in $45 \deg$, some will go through, some will not... Some fraction will go through some fraction will not go through. All you know is that there is a **Probability** of going through

Photon polarization $= \hat{p}$ Polaroid orientation $= \hat{P}$

The probability that each photon passes through w.p.

$$(\hat{p} * \hat{P})^2$$

3 Representing Qubits 1/13

Remark 1. Recall that the Probability is:

$$(\hat{P} * \hat{p})^2$$

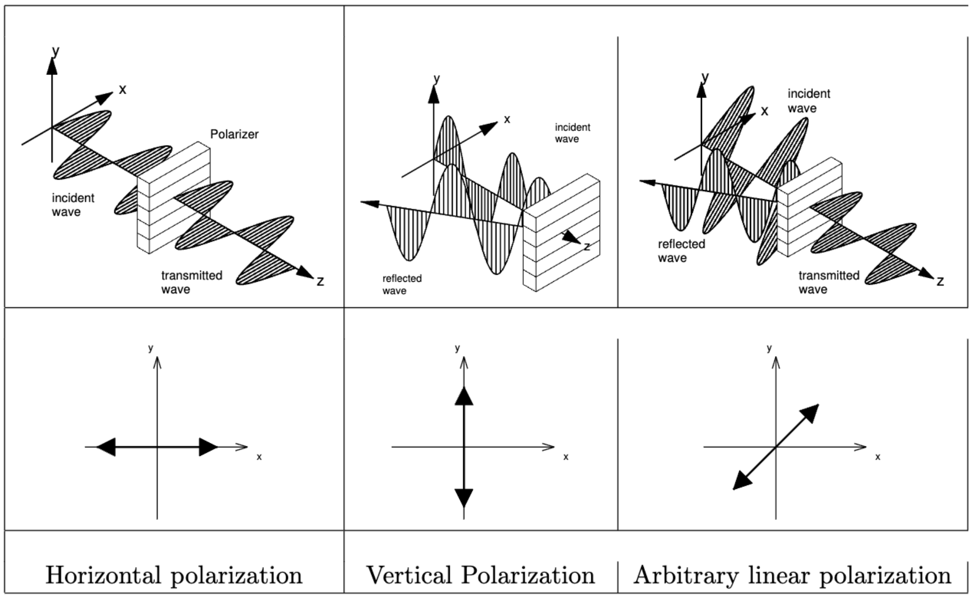


Figure 2: Polarization of electron based on wave direction

$$E_x(t) = E_x \cos(\omega t + \psi)$$

$$E_y(t) = E_y \sin(\omega t + \psi)$$

$E_x = E_y$,

Where Left circular polarization is $\alpha = +\frac{\pi}{2}$

And Right Circular Polarization is $\alpha = -\frac{\pi}{2}$

$$\hat{P}_L = \frac{1}{\sqrt{2}}(i * \hat{i} + 1\hat{j}) \circeq \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$\hat{P}_R = \frac{1}{\sqrt{2}}(-i * \hat{i} + 1\hat{j}) \circeq \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix}$$

Reflects and lets through vectors are perpendicular

Ket:$|v\rangle$ and we can use arrows to show the photon state of polarization

Where each state is vertical and horizontal respectively.

$$0, |0\rangle = |\uparrow\rangle$$

$$1, |1\rangle = |\rightarrow\rangle$$

3 Each of these are qubit states, 0 and 1

Moreover, We can combine them giving us:

$$|\nearrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\rightarrow\rangle)$$

You have to do something to make it a 0 or 1, and we can check by letting all 1's through but not 0's

$$|\uparrow\rangle = \frac{1}{\sqrt{2}}(|\nearrow\rangle + |\nwarrow\rangle)$$

3.1 Complex Numbers

i is defined as:

$$i = \sqrt{-1}$$

Complex numbers are denoted with Z, where

$$Z = a + ib$$

and A and B are real numbers and it is on the complex plane.

This complex plane consists of a as the "x-axis" and ib as the "y-axis" There then exists a length with, r and angle with θ

Moreover,

$$Z = a + ib = re^{i\theta}$$

Because: (We must prove this in our homework)

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Negative representation:

$$Z^* = \overline{Z} = a - ib = re^{-i\theta}$$

Furthermore,

$$|Z| = \sqrt{Z^* Z} = \sqrt{a^2 + b^2} = r^2$$

Which is the real, positive (when a or z are 0)

3.2 Vector Space and Vector Spaces w/ Complex Numbers

Definition

Vector Space is the set of vectors $\{|v\rangle\}$

$$|a\rangle + |b\rangle = |c\rangle = |b\rangle + |a\rangle$$

Which is another vector in vector space, think adding two vectors together to get c

These are also communicative and associative!

Can also give a magnitude to such vectors as well.

There is also a 0 vector, without the ket!

Every vector has an additive inverse for example, $|a\rangle$ and $-|a\rangle$ where adding these two will cancel out and give you the 0 vector

Examples

If $|a\rangle$ is a vector then c, which is an arbitrary complex number, $|a\rangle$ is a vector in space

If $|a\rangle$ and $|b\rangle$ are in vector space, then c$|a\rangle$ + d$|b\rangle$ is in space for all c and d

4 Orthogonal and Perpendicular 1/15

We will discuss linear independence first:

4.1 Linear independence

Definition

We have a set

$$\{|v\rangle\}, i = 1, 2, n$$

This is linearly independent iff, the Linear Combination:

$$|v\rangle = c_1|v_1\rangle + c_2|v_2\rangle \dots c_N|v_N\rangle$$

Example

If

$$c_1|v_1\rangle + c_2|v_2\rangle \cdots + c_N|v_N\rangle = 0$$

and at least one of the c's is not zero (and none of the $|v_i\rangle$, is the zero vector), then:
The set $\{|v_i\rangle\}$ cannot be linearly independent
If you were to move v_1 to the other side

$$|v_1\rangle = \frac{-1}{c_1}(c_2|v_2\rangle + \cdots + c_N|v_N\rangle)$$

meaning that they are not linearly independent (?)

Let's try another example

Example

$\hat{i}, \hat{j}$ a basis for on the board

If we have

$$\vec{v} = 2\hat{i} + \hat{j} = 2\hat{i} + 1\hat{j} \mathrel{\hat{=}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Where $\hat{i}$ is our "x" and $\hat{j}$ is our "y"

Example

A real 2-d vector has components $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ in the usual basis, that is:

$$\vec{v} = 2\hat{i} + \hat{j} \ldots$$

In a different basis $\vec{v_1} = \hat{i}, \vec{v_2} = \hat{i} + \hat{j}$, the components of $\vec{v}$ are:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

This is because,

$$\vec{v} = 1\vec{v_1} + 1\vec{v_2} = 2\hat{i} + \hat{j}$$

Which is able to give us $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ as this is the **components** of the given $\vec{v}$

4.2 Inner Products

Definition

Defining inner product where these two vectors:

$$\vec{v_1}, \vec{v_2}$$

Is given as $\langle v_1 | v_2 \rangle$

Moreover, if we are given the conditions that:

$$|A\rangle = C_1 |v_1\rangle + C_2 |v_2\rangle$$

Then $\langle B | A \rangle$ will equal:

$$C_1 \langle B | v_1 \rangle + C_2 \langle B | v_2 \rangle$$

Moreover, $\langle A | B_1 \rangle = \langle B | A \rangle^* \Rightarrow \langle A | A \rangle$ is real
and

$$\langle A | A \rangle \geq 0$$

only if $|A\rangle = 0$

4.3 Orthogonal Basis

Orthogonal Basis is where $\{|e_i\rangle\}, i = 1, 2, \ldots, N$ And this is true if the inner product,

$$\langle e_i|e_j\rangle = 0 \;\forall\; i \neq j$$

and

$$\langle e_i|e_j\rangle = 1 \;\forall\; i$$

Example

$$|v\rangle = C_1|e_1\rangle + C_2|e_2\rangle + \cdots + C_N|e_N\rangle$$

Where $\langle e_i|v\rangle = C_i$

$$\vec{A} = A_x\hat{i} + A_y\hat{j}$$

Where $A_x = \hat{i} * A$ and $A_y = \hat{j} * A$

Furthermore,

$$|A\rangle = a_1|e_1\rangle + .. + a_n|e_N\rangle$$

$$|B\rangle = b_1|e_1\rangle + .. + b_n|e_N\rangle$$

$$\langle A|B\rangle = (a_1^*\langle e_1| + \cdots + a_N^*\langle e_N|)(b_1^*|e_1\rangle + \cdots + b_N^*|e_N\rangle)$$

But you can always write the inner product as

$$a_1^*b_1 + a_2^*b_2 + \cdots + a_n^*b_n$$

The "bra," $\langle e_1|$ for example, of the "ket", is not something to worry about but rather to expand out

5 Probability 1/17

Recap

Remember that when:

$$|a\rangle = a_1 |e_1\rangle + a_2 |e_2\rangle$$

Then:

$$\langle A|B\rangle = a_1^* \langle e_1|B\rangle + a_2^* \langle e_2|B\rangle$$

Where * is the complex conjugate
Any number multiplied by its complex conjugate is a **positive** number

Moreover, we can see with the given matrices:

$$|A\rangle \doteq \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}$$

$$|B\rangle \doteq \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix}$$

We see that the inner product of the two is:

$$\langle A|B\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix} \begin{pmatrix} b_1 & b_2 & \dots & b_N \end{pmatrix} = a_1^* b_1 + a_n^* b_n$$

Now lets get back to probability:

Recap

Go back with $\hat{p}$ and $\hat{P}$, We know that the probability of the photons getting through is, linear polarization

$$(\hat{p} * \hat{P})^2$$

Definition

Lets all of them through

$$|U\rangle$$

Where $|U\rangle$ is some sort of photon state that always "goes through"

If $|v\rangle$ is like $\hat{p}$ then $|U\rangle$ is similar to $\hat{P}$
And

$$|U_\perp\rangle$$

Is the state where it doesn't "go through"
Where

$$\langle U|U_\perp\rangle = 0$$

Shows us that it must be perpendicular ALWAYS

$$|v\rangle = \alpha\,|U\rangle + \beta\,|U_\perp\rangle$$

Thus, the probability of being $|U\rangle$ after the measurement Is

$$|\,\langle U|v\rangle\,|^2 = |\alpha|^2$$

And probability of being on $|U_\perp\rangle$ is

$$|\,\langle U_\perp|v\rangle\,|^2 = |\beta|^2$$

Therefore,

$$|\alpha|^2 + |\beta|^2 = 1 \Longleftrightarrow |v\rangle$$

Where $|v\rangle$ must be a unit vector

So once you "collapse" the state vector, $|v\rangle$, then it is either $|U\rangle$ or $|U_\perp\rangle$, one part of it is simply gone

Definition

Say we have a quantum computer with 3 qubits, where

$$|w_0\rangle \longrightarrow a\,|000\rangle + b\,|001\rangle + c\,|010\rangle \cdots + \cdots + h\,|111\rangle$$

Therefore, since each coefficient, i.e. a, b, c, etc., have the same probability or different probabilities, you have to arrange gates to make it so that those wrong coefficients are very small or 0

Example

Suppose a photon has polarization state

$$|\Psi\rangle = \sqrt{\frac{1}{3}}\,|{\rightarrow}\rangle + \sqrt{\frac{2}{3}}\,|{\uparrow}\rangle$$

The probability that the photon will be reflected and observed on the "incoming" side of a horizontally oriented polaroid is:

$$\frac{2}{3}$$

As the inner product of:

$$\langle \rightarrow |X\rangle = \sqrt{\frac{1}{3}}$$

Therefore, the P(go through) $= \frac{1}{3}$ and P(reflected) $= \frac{2}{3}$

Example

Suppose

$$|\Psi\rangle = \sqrt{\frac{1}{3}}\,|{\rightarrow}\rangle + \sqrt{\frac{2}{3}}\,|{\uparrow}\rangle$$

The probability that the photon will be reflected and observed on the "incoming" side of a polaroid oriented at +45 deg with respect to the horizontal is: Answer:

$$|\Psi\rangle \doteq \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \sqrt{\frac{2}{3}} \end{pmatrix}$$

Since its reflected, we are interested in the -45 sin state thus

$$|{\searrow}\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Then:

$$|\langle{\searrow}|\psi\rangle|^2 = [\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}]^2 = (\frac{1}{\sqrt{6}} - \sqrt{\frac{2}{6}})^2 = \frac{1}{6} + \frac{1}{3} - \frac{2\sqrt{2}}{6} = \frac{3-2\sqrt{2}}{6}$$

6 Modern Cryptography 1/22

Recap

Recall that

$$|\circlearrowright\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + i\,|\rightarrow\rangle)$$

And that

$$P(\nearrow) = |\,\langle\nearrow \mid \circlearrowright\rangle\,|^2$$

Where their values are:

$$|\circlearrowright\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$$

$$|\nearrow\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$$

Giving us:

$$\langle\nearrow \mid \circlearrowright\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} = \frac{1}{2}(1+i)$$

Therefore,

$$P(\nearrow) = \frac{1}{4}(1+i)(1-i) = \frac{1}{2}$$

If I have a normalized state,

$$|\psi\rangle$$

Then it will be the same as

$$e^{i\theta}\,|\psi\rangle$$

Which is also a normalized same quantum state

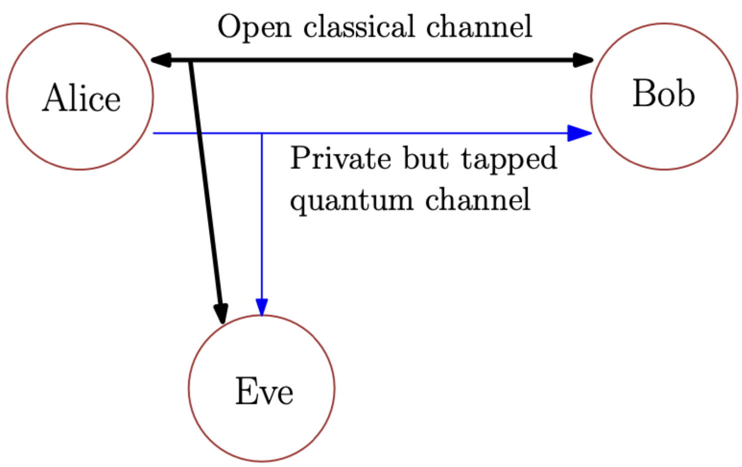
Example

$$a\,|\uparrow\rangle + b\,|\rightarrow\rangle$$

Is not the same as physically

$$a\,|\uparrow\rangle + be^{ie}\,|\rightarrow\rangle$$

Quantum communications or Quantum Key Protocol: Lets say, we want two people to communicate with each other, Alice and Bob. If they want to send a message to each other here is the diagram for that:



Moreover, we have two basis or encodings for these bits: Hadamard or Computation Basis to create a binary string

$$\begin{aligned} |0\rangle &\longrightarrow |\uparrow\rangle \\ |1\rangle &\longrightarrow |\rightarrow\rangle , \end{aligned} \tag{2.5.1}$$

and the "Hadamard basis:"

$$\begin{aligned} |0\rangle &\longrightarrow |\nearrow\rangle \\ |1\rangle &\longrightarrow |\nwarrow\rangle . \end{aligned} \tag{2.5.2}$$

It will only measure in the same basis, meaning that if they had the incorrect basis, then they would have to throw it away if it is incorrect basis

Getting it wrong If, for example, Eve is unaware of what basis Alice and Bob are communicating in, then they would disagree with the final result as Eve has interrupted the communications and changed the string

Example

Suppose Eve is intercepting Alice's qubits, measuring them in whichever of the two bases she guesses, and forwarding the measured qubit on to Bob. For what fraction of the qubits will Alice and Bob get different values, even though they measure in the same basis.

The answer of this is: $\frac{1}{4}$ as there is a $\frac{1}{2}$ chance of getting the right bases and then there is another $\frac{1}{2}$ prob of getting the photons are right after if it is the wrong basis

7 Bloch Sphere 1/24

Recall: Select some subset, i.e. some 100 bits, to check whether there was some sort of interference. If the 100 bits don't match then there was interference and they would discard the message

Example

Suppose Eve is intercepting Alice's qubits, measuring them in whichever of the two bases she guesses, and forwarding the measured qubit on to Bob. How many times must Alice and Bob compare their results when they measured in the same basis to have a 90% of detecting Eve?

Answer: 9 times, this is because the chances of being undetected after, N photons

$$= (\frac{3}{4})^N$$

Where $\frac{3}{4}$ is the chance of NOT being detected
Thus after around 8-9 times, $(\frac{3}{4})^9$, it will be around 90 percent chance of getting caught

Hadamard States:

$$|\nearrow\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

and

$$|\nwarrow\rangle = |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Which leads us to:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Requiring that:

$$|\alpha|^2 + |\beta|^2 = 1$$

We can write $\alpha = r_\alpha e^{i\phi_\alpha}$ where r is a real number and we can write $\beta = r_\beta e^{i\phi_\beta}$ Therefore,

$$|\psi\rangle = r_\alpha e^{i\phi_\alpha}|0\rangle + r_\beta e^{i\phi_\beta}|1\rangle$$

Then we see that

$$|\psi'\rangle = e^{-i\psi_\alpha}|\psi\rangle = r_\alpha|0\rangle + r_\beta e^{i(\psi_\beta - \psi_\alpha)}|1\rangle$$

Therefore if $Z = r_\alpha$ and $X + iY = r_\beta e^{i(\psi_\beta - \psi_\alpha)}$

Then we see that:

$$|\alpha|^2 + |\beta|^2 = 1 = X^2 + Y^2 + Z^2 = 1$$

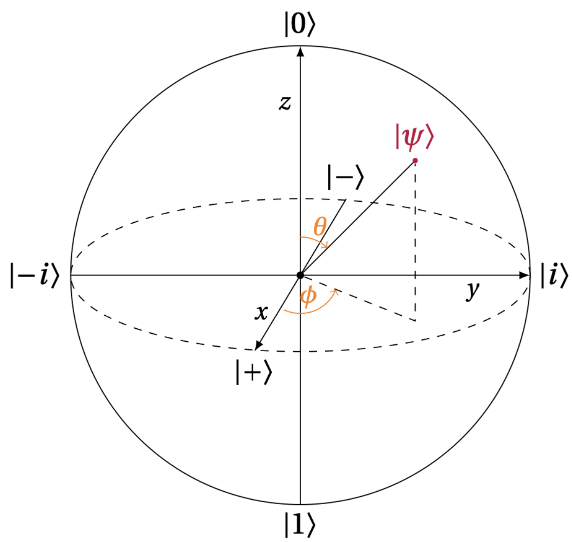


Figure 3: Bloch Sphere

8 Tensor products & Entanglement 1/27

Generic qubit set is defined as

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle$$

8.1 Bloch Sphere

Recall

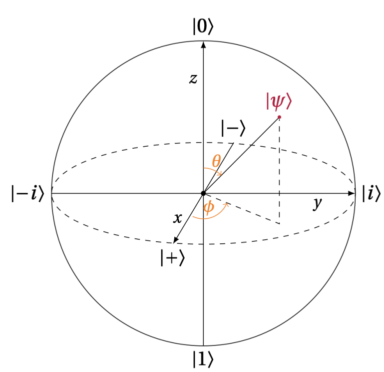
As a reminder for the values:

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)$$

$$|-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle)$$



Example

What states along the positive y-axis of the Bloch sphere?
As a note, the example is pointing at the *positive* y-axis
In this case the answer is:

$$|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)$$

8.2 Tensor Products

Next we want to do a sort of product that creates a tensor product For example, we want something that can go from:

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

To:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Basis states for our case:

$$|00\rangle\,, |01\rangle\,, |10\rangle\,, |11\rangle$$

Which respectively equate to:

$$|0\rangle \otimes |0\rangle\,, |0\rangle \otimes |1\rangle\,, |1\rangle \otimes |0\rangle\,, |1\rangle \otimes |1\rangle$$

Moreover, we can put this in:

$$|\psi\rangle = C_0\,|00\rangle + C_1\,|01\rangle + C_2\,|10\rangle + C_3\,|11\rangle$$

Where **ALL** of these states are Orthogonal
We can also do $|\psi\rangle \otimes |\phi\rangle$
Some rules for these tensor products:

Definition

1. Linearly: $(|u\rangle + |v\rangle) \otimes (|w\rangle + |x\rangle)$
   Will be the same as multiply out like a regular multiplication expansions

2. $(c\,|u\rangle) \otimes |v\rangle = |u\rangle \otimes (c\,|u\rangle) \equiv c\,|u\rangle \otimes |v\rangle$

Definition

The Bell State:

$$\frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

However, this is **NOT** the same as saying that for some sort of variables, a, b, c, d, can do this:

$$(a\,|0\rangle + b\,|1\rangle) \otimes (c\,|0\rangle + d\,|1\rangle)$$

This is because $ad = 0$ and $bc = 0$ must be true

Moreover, $ac = 1$ and $bd = 1$ must be true

Thus, this is impossible, as if, for example if a or d is 0, then ac and bd cannot both be equal to 1

This state is called an **Entangled State**

8.3 Entanglement

Definition

If you can factor into two states then it is not entangled, otherwise it is...

If we take the inner product of some arbitrary quantum states $|a\rangle \otimes |b\rangle$ with $|c\rangle \otimes |d\rangle$

Comes out as:

$$(\langle a| \otimes \langle b|)(|c\rangle \otimes |d\rangle) = \langle a|c\rangle\langle b|d\rangle$$

Another example:

$$w\,\langle 01|11\rangle = \langle 0|1\rangle\langle 1|1\rangle$$

9 Measurement of Multi-Qubit States 1/29

For questions for last lecture

Example

Suppose two quits are in the state

$$|\Psi\rangle = \frac{1}{4}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{\sqrt{7}}{2}|11\rangle$$

What is the prob of finding qubit 1 to be in state $|1\rangle$ and qubit2 to be in state $|0\rangle$?

The answer would be B, as you would find the basis state that you are interested in and then square it

To find the probability of ending up in a state in the subspace after a measurement, you "project" the state Ψ that represents the system before measurement onto the subspace
In other words, it would be the length of the projection squared = absolute value of the project squared

Definition

An example would be having our initial expression as:

$$|\Psi\rangle = a|0\rangle \otimes |0\rangle + b|0\rangle \otimes |1\rangle + c|1\rangle \otimes |0\rangle + d|1\rangle \otimes |1\rangle$$

We get that the projection of this would be, given that we are trying to find the prob. of qubit 2 being $|0\rangle$:

$$Proj_{|\Psi\rangle} = a|0\rangle \otimes |0\rangle + c|1\rangle \otimes |0\rangle$$

Then we want to normalize getting:

$$|\Psi\rangle \longrightarrow \frac{1}{\sqrt{|a|^2 + |c|^2}}(a|0\rangle \otimes |0\rangle + c|1\rangle \otimes |0\rangle)$$

Then the probability of that would be:

$$|a|^2 + |c|^2$$

Example

Suppose two quits are in the state

$$|\Psi\rangle = \frac{1}{4}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{\sqrt{7}}{2}|11\rangle$$

What is the prob of finding qubit 1 to be in state $|1\rangle$
It is $\frac{11}{16}$

Example

Two qubits are in the state:

$$|\Psi\rangle = \sqrt{\frac{2}{11}}(|00\rangle + \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle + 2|11\rangle)$$

Qubit 2 is found to be in state $|1\rangle$. Immediately after the measurement the two-qubit state is?
The answer is:

$$\frac{2}{\sqrt{17}}(\frac{1}{2}|01\rangle + 2|11\rangle)$$

Moving on, we can see that the Bell state,

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|{++}\rangle + |{--}\rangle)$$

10 Entanglement 1/31

Recall

$$|\Psi\rangle = \frac{1}{\sqrt{2}} = (|00\rangle + |11\rangle)$$

Is equivalent to:

$$\frac{1}{\sqrt{2}}(|{+}{+}\rangle + |{-}{-}\rangle)$$

Example

Suppose Eve intercepts Alice's photon from the source that is sending one photo each to Alice and Bob from the entangled pair state

$$|\Psi\rangle = \frac{1}{\sqrt{2}} = (|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|{+}{+}\rangle + |{-}{-}\rangle)$$

Eve measures the photon and sends it on to Alice. What are the chances that she will be detected by Alice and Bob if she doesn't know which of the two bases to measure in?

$\boxed{\text{The answer is } \frac{1}{4}}$

This is because there is a $\frac{1}{2}$ chance of choosing right basis and then there is another $\frac{1}{2}$ chance of getting the right state...

Either Alice's photon is $|00\rangle$ or $|11\rangle$

Analogy

Suppose there is a Cookie factory, and there are conveyer belts in a million miles in each direction

There are cookies are each side that are being pumped out. where Bob and Alice lie on either sides of the million miles.
With our analogy we get:

- Cookie Steams $\longrightarrow |1\rangle$
- Cookie Doesn't Steam $\longrightarrow |0\rangle$
- Cookie Tastes Good $\longrightarrow |-\rangle$
- Cookie Tastes Bad $\longrightarrow |+\rangle$

Moreover, only $\frac{1}{12}$ of the time they both steam, and subsequently taste good
If Alice's cookie steams, then Bob's tastes good
If Bob's cookie steams, then Alice's taste good
Hadamard State

$$|H\rangle = \frac{1}{\sqrt{12}}(3\,|00\rangle + |01\rangle + |10\rangle - |11\rangle)$$

Applying this,

Example

Consider the two photons in the Hardy state,

$$H = \frac{1}{\sqrt{12}}(3\,|00\rangle + |01\rangle + |10\rangle - |11\rangle)$$

with one sent to Alice and one to Bob. What is the probability that Alice and Bob both measure their photons to be in the state $|1\rangle$?
$\boxed{\text{Answer is } \frac{1}{12}}$

But what happens if Alice measures in the computational basis and Bob measures in the Hadamard state? then we get:

$$\begin{aligned}|H\rangle = \frac{1}{\sqrt{12}}(3\,|0\rangle \otimes \frac{1}{\sqrt{2}}[|+\rangle + |-\rangle] + |0\rangle \otimes \frac{1}{\sqrt{2}}[|+\rangle - |-\rangle] \\ + |1\rangle \otimes \frac{1}{\sqrt{2}}[|+\rangle + |-\rangle] - |1\rangle \otimes \frac{1}{2}[|+\rangle - |-\rangle])\end{aligned}$$

Which equates to:

$$\frac{1}{\sqrt{24}}(4\,|0\rangle \otimes |+\rangle + 2\,|0\rangle \otimes |-\rangle + 2\,|1\rangle \otimes |-\rangle)$$

11 Start of Linear Transformations + Recall Day 2/3

Recall

Remembering that we have Bell State, and its equivalency

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|{+}{+}\rangle + |{-}{-}\rangle)$$

An example that for this given entanglement case that is NOT true, is the cookies from mom case. Where if you have something, i.e. burnt cookie, then you know your sister also got a burnt cookie.

This is because they both come from the same source...

However, this is not entirely true for entanglement.

Photons are neither state 1 nor 0 until collapse, where one of them disappears
The experiment that we want to examine is this:

Example

Where we have a factory that pumps out cookie where there is a conveyor belt to Alice (West 1m miles) and to Bob (East 1m miles)
In quantum mechanics, we don't know what is going underneath.
And nothing is really going on that is like cookies

Recall

Moving on we come back to the **Hardy State**:

$$|H\rangle = \frac{1}{\sqrt{12}}(3\,|00\rangle + |01\rangle + |10\rangle - |11\rangle)$$

Where the states mean:

- Cookie Steams $\longrightarrow |1\rangle$
- Cookie Doesn't Steam $\longrightarrow |0\rangle$
- Cookie Tastes Good $\longrightarrow |-\rangle$
- Cookie Tastes Bad $\longrightarrow |+\rangle$

Where $\frac{1}{12}$ the time, the state is $|-\rangle$
Where we can rewrite this Hardy State to be:

$$\frac{1}{\sqrt{6}}(2\,|0\rangle \otimes |+\rangle + |0\rangle \otimes |-\rangle + |1\rangle \otimes |-\rangle)$$

Example

Consider two photons in the Hardy state,

$$H = \frac{1}{\sqrt{12}}(3|00\rangle + |01\rangle + |10\rangle - |11\rangle)$$

With one sent to Alice and one to Bob. What is the probability that Alice and Bob both measure their photons to be in the state $|-\rangle$?
One way could be substitution where

$$|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$$

$$|1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$$

And substitute it into to give us the Hadamard Basis

$$\frac{1}{\sqrt{6}}(2|0\rangle \otimes |+\rangle + |0\rangle \otimes |-\rangle + |1\rangle \otimes |-\rangle)$$

Giving us a really long expression but we only care about $|-\rangle \otimes |-\rangle$
Where the coefficient is 0, therefore, there is a 0 percent chance

$$\frac{1}{\sqrt{12}} = (2|+\rangle \otimes |+\rangle + 2|+\rangle \otimes |-\rangle + 2|-\rangle \otimes |+\rangle + 0|-\rangle \otimes |-\rangle)$$

Therefore, this proves that there is no hidden variables or anything like the cookie example, given that quantum mechanics is correct

11.1 Briefly going over linear transformations

Moving onto linear transformations

Definition

A linear transformation is defined as:

$$A|v\rangle = |v'\rangle$$

And also we have distributive property

$$A(c_1|v_1\rangle + c_2|v_2\rangle) = c_1A|v_1\rangle + c_2A|v_2\rangle$$

$$(CA)|v\rangle = C(\langle A|v\rangle)$$

12 Transformations 2/5

Recall

Transformations

$$A = |u\rangle\langle w|$$

$$A|v\rangle = |u\rangle\langle w|v\rangle = |u\rangle\langle w|v\rangle$$

12.1 Linear Transformations

Example

Consider the state $|\Psi\rangle = c_a|a\rangle + c_b|b\rangle$, with $|a\rangle$ and $|b\rangle$ **orthogonal and normalized**, and let $\hat{p}_b = |b\rangle\langle b|$. What is $\hat{p}_b|\Psi\rangle$

The answer is: $c_b|b\rangle$, this is because

$$|b\rangle\langle b|c_a|a\rangle = c_a|b\rangle\langle b|a\rangle = 0$$

$$|b\rangle\langle b|c_b|b\rangle = c_b|b\rangle\langle b|b\rangle = c_b|b\rangle$$

Therefore, it is only $|c_b|b\rangle\rangle$

There are 2d basis, $|0\rangle$ and $|1\rangle$,
Therefore,

$$|v\rangle = c_0|0\rangle + c_1|1\rangle$$

and

$$A|v\rangle = c_0 A|0\rangle + c_1 A|1\rangle$$

Moreover, if we move on to the higher basis,

$$A = a_{00}|0\rangle\langle 0| + a_{01}|0\rangle\langle 1| + a_{10}|1\rangle\langle 0| + a_{11}|1\rangle\langle 1|$$

Then that means:

$$A|0\rangle = A = a_{00}|0\rangle\langle 0|0\rangle + a_{01}|0\rangle\langle 1|0\rangle + a_{10}|1\rangle\langle 0|0\rangle + a_{11}|1\rangle\langle 1|0\rangle$$

The $\langle 1|0\rangle$ cancels out as they are orthogonal, meaning that we get our final answer of:

$$A|0\rangle = a_{00}|0\rangle + a_{10}|1\rangle$$

Meaning that

$$A|1\rangle = a_{01}|0\rangle + a_{11}|1\rangle$$

Example

Moving onto an example that prof shows,

If we have:

$$|v\rangle = c_0 |0\rangle + c_1 |1\rangle$$

Then:

$$A |v\rangle = A(c_0 |0\rangle + c_1 |1\rangle)$$

Then we can substitute from our previous, example, getting us:

$$A |v\rangle = c_0(a_{00} |0\rangle + a_{10} |1\rangle) + c_1(a_{01} |0\rangle + a_{11} |1\rangle)$$

Which equates to:

$$= (a_{00}c_0 + a_{01}c_1) |0\rangle + (a_{10}c_0 + a_{11} + c_1) |1\rangle = |v'\rangle$$

Since we know that, given some A, we can get:

$$A |v\rangle = |v'\rangle = C'_0 |0\rangle + C'_1 |1\rangle$$

Meaning that we can finally get:

$$c'_0 = a_{00}c_0 + a_{01}c_1$$
$$c'_1 = a_{10}c_0 + a_{11}c_1$$

Example cont.

Finally, we can get the matrix, getting us:

$$= \begin{pmatrix} c'_0 \\ c'_1 \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$$

12.2 Linear Transformations with matrix multiplication

If we have:

$$|v\rangle = c_0 |0\rangle + c_1 |1\rangle + \cdots + c_N |N\rangle$$

And we have:

$$A = a_{00} |0\rangle \langle 0| + a_{01} |0\rangle \langle 1| + \cdots + a_{nn} |n\rangle \langle n|$$

Then this matrix that it produces will be similar to previously mentioned

$$\begin{pmatrix} c'_0 \\ c'_1 \\ \ldots \\ c'_n \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & \ldots & a_{0n} \\ a_{10} & a_{11} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n0} & a_{n1} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \ldots \\ c_n \end{pmatrix}$$

Example

Let's do an example with these transformation

$$A = 2 |01\rangle \langle 00| + i |11\rangle \langle 11| - |00\rangle \langle 10|$$

If it is in the basis: $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, then it should have 16 states

$$A = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}$$

Where the values are the **coefficients** of the states in the expression

12.3 Conjugations or Adjoints

$A^\dagger$ is the adjoint of A

Where:

$$\langle \Psi | A^\dagger | \Phi \rangle = \langle \Phi | A | \Psi \rangle^*$$

Therefore, if we know the matrix for A then we also know the matrix for $A^\dagger$:

$$A^\dagger = \begin{pmatrix} a^*_{00} & a^*_{01} & \ldots & a^*_{0n} \\ a^*_{10} & a^*_{11} & \ldots & a^*_{1n} \\ \ldots & \ldots & \ldots & \ldots \\ a^*_{n0} & a^*_{n1} & \ldots & a^*_{nn} \end{pmatrix}$$

13 Projection Operators & Transforming Quantum States 2/7

13.1 Projection Operators

We will be discussing what a Bra is

$$\langle\Psi|\, and\, |\Phi\rangle = \langle\Psi|\Phi\rangle$$

And where $\langle\Psi|$ acts on $|\Phi\rangle$

Furthermore in this example, we see that:

$$|\Psi\rangle = \begin{pmatrix} c_1 \\ c_2 \\ \dots \\ c_N \end{pmatrix}$$

And we also see that:

$$\langle\Psi| = \begin{pmatrix} c_1^* & c_2^* & \dots & c_N^* \end{pmatrix}$$

Therefore, there is a dual mapping, and we see that here:

$$\langle\Psi| = |\Psi\rangle^\dagger$$

Example

This expression:

$$P_b = |b\rangle\langle b|$$

Is simply a projection on 2 dimensional space and we can see that

$$P_{ab} = |a\rangle\langle b| + |b\rangle\langle b|$$

Looking into a particular example we see:

Example

Consider the state $|\Psi\rangle = c_a|a\rangle + c_b|b\rangle$, with $|a\rangle$ and $|b\rangle$ orthogonal and normalized, and let $\hat{P}_{ab} = |a\rangle\langle a| + |b\rangle\langle b|$. What is $\hat{P}_{ab}$?

The answer is: $|\Psi\rangle$ as it projects on the same space

Another example with computational basis:

Example

Which projection operator should be used to obtain the probability that the value of the first bit in a two qubit system is greater than or equal to the value of the second?

The answer should be $|00\rangle\langle 00| + |10\rangle\langle 10| + |11\rangle\langle 11|$
These would multiply out and only provide the probabilities with these qubit values
A side note: $|00\rangle\langle 10|$ would not be a projector as they need the same values

Suppose we have another qubit state:

$$|\Psi\rangle = \frac{1}{\sqrt{3}}(|00\rangle - |10\rangle + |11\rangle)$$

Where $P_{q_1=+}$ (q_1 part meaning that the first qubit is $+$) $=$

$$|+0\rangle\langle +0| + |+1\rangle + \langle +1|$$

Giving us, after simplification we would get:

$$\begin{aligned}
P_{q_1=+}|\psi\rangle = \frac{1}{\sqrt{3}}\Bigg(&|+0\rangle\underbrace{\langle +0|00\rangle}_{1/\sqrt{2}} + |+1\rangle\cancel{\langle +1|00\rangle} - \underbrace{|+0\rangle\langle +0|10\rangle}_{1/\sqrt{2}} - |+1\rangle\cancel{\langle +1|10\rangle} \\
&+ |+0\rangle\cancel{\langle +0|11\rangle} + \underbrace{|+1\rangle\langle +1|11\rangle}_{1/\sqrt{2}}\Bigg) \\
= \frac{1}{\sqrt{6}}&(|+0\rangle - |+0\rangle + |+1\rangle) \\
= \frac{1}{\sqrt{6}}&|+1\rangle\,.
\end{aligned}$$

13.2 Unitary Transformations

Definition

The defintion of unitary transformation is the inner product of $U|a\rangle$ and $U|b\rangle = \langle a|b\rangle$ for all $|a\rangle$ & $|b\rangle$

So if we do this:

$$(\langle U|a\rangle)^{\dagger} = \langle a|\, U^{\dagger}$$

Which means the top inner product within our defintion, should be equivalent to saying:

$$\langle a|\, U^{\dagger}U\, |b\rangle = \langle a|b\rangle$$

In otherwords,

$$U^{\dagger}U = 1 = I$$

Meaning that it doesn't do anything really...

Example

If we have

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then we have

$$U^{\dagger} = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}$$

Furthermore, we then have:

$$UU^{\dagger} = \begin{pmatrix} aa^* + bb^* & ac^*bd^* \\ ca^* + db^* & cc^*dd^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$